Assignment 3: The Kepler Problem

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1 Introduction

This model will attempt to show the orbit of Halley's Comet as a function of time, rather than as a function of angle using two different approaches. First is the iteration method which uses the Newton-Raphson method to calculate E, the eccentric anomaly. Second, the differential solution, which uses the 4th order Runge-Kutta approximation on the gravitational differential equation.

2 Part 1: Iterative Solution

2.1 Calculating E, the Eccentric Anomaly

In order to calculate the position of Halley's Comet, you need the true anomaly. In order to calculate the true anomaly, you need the eccentric anomaly. The eccentric anomaly must be found using Newton's method on the equation:

$$M = E - esin(E) = \frac{2\pi(t-T)}{P} \tag{1}$$

Variables: M = mean anomaly, e = eccentricity, E = eccentric anomaly, t = time, T = reference time (since perihelion, P = orbital period

This is done by calculating M using the right side, and then moving it to one side. Then, choosing an arbitrary starting point for E, simply calculating the derivative of the function and using the following equation:

$$E_{n+1} = E_n \frac{f(E_n)}{f'(E_n)} \tag{2}$$

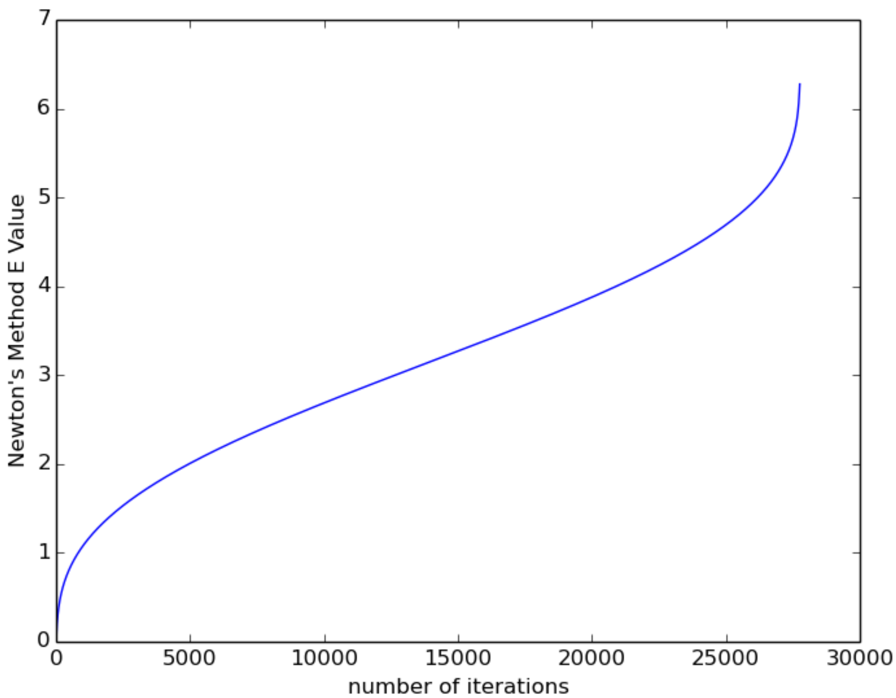

Figure 1: Newton-Raphson value for eccentric anomaly over time

Where n is the number of iterations.

After not too many iterations, the derivatives will converge to find the root of the equation, thus finding the value of the eccentric anomaly. Shown in Figure 1 are the Newton-Raphson method-calculated valued for E vs. the iteration number. The plot shows the evolution of E over the time of the iteration. Since a new E is calculated based on each new time, the E value would be expected to increase like it does in the plot.

2.2 Position of Halley's Comet

Once the eccentric anomaly was calculated, it was fairly straightforward to calculate the true anomaly (v) due to the equation:

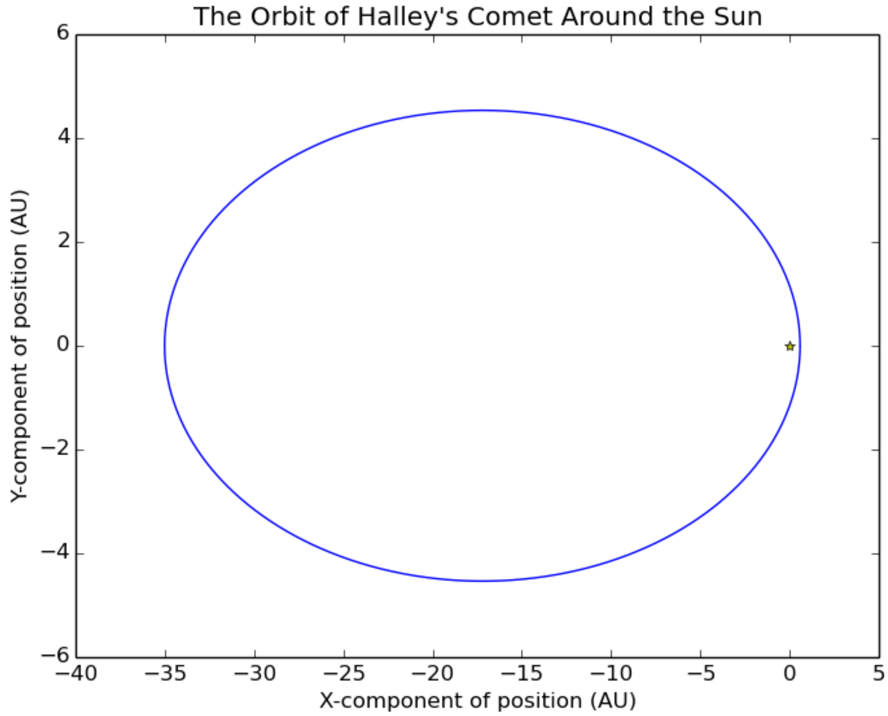

Figure 2: One Orbit of Halley's Comet using the Newton-Raphson Method

$$tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}tan\frac{E}{2} \tag{3}$$

Then, calculating the position is as simple as using the equations:

$$r(v) = \frac{a(1-e^2)}{1+ecos(v)} \tag{4}$$

$$r_x(v) = r(v)cos(v) \tag{5}$$

$$r_y(v) = r(v)sin(v) \tag{6}$$

As shown in Figure 2, the orbit of Halley's comet is quite the perfect ellipse. There are no discontinuous parts or oddly elongated portions.

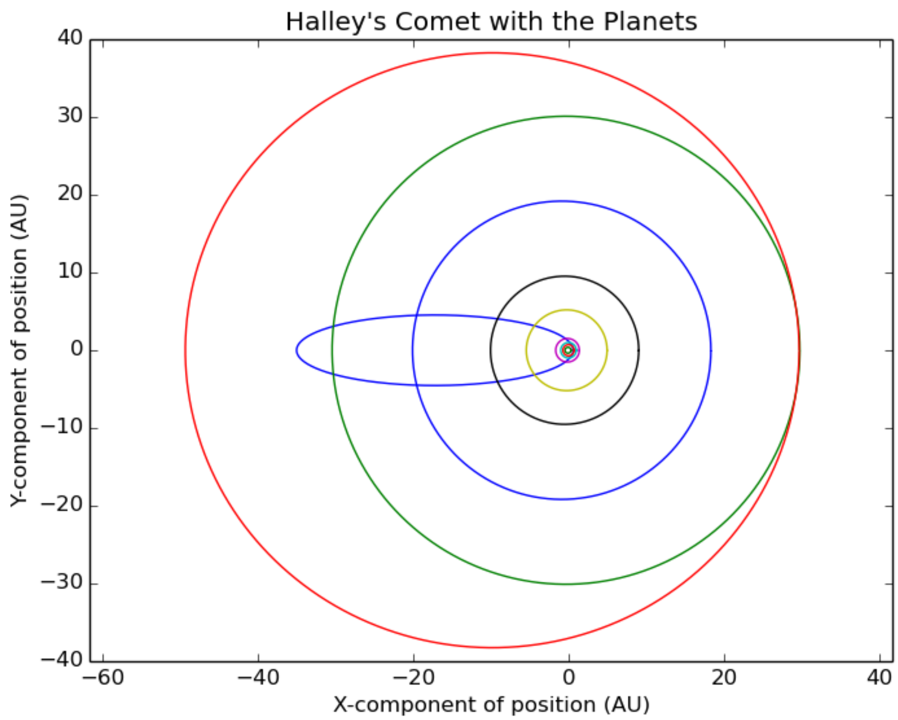

Figure 3: One Orbit of bodies of the solar system using the Newton-Raphson Method

2.3 Expanded Problem: The Whole Solar System

After finding the eccentric anomaly, true anomaly, and position of one body, finding it for the rest of the bodies in the solar system only took the input of the different eccentricities, semi-major axes, and orbital periods of the bodies.

The eccentric anomaly, true anomaly, and then position were calculated for each body at a time, and plotted separately. Then once out of the loop, all plots were shown on one graph as seen in Figure 3.

Clearly, the very eccentric object which extends past Neptune's orbit (green), is Halley's Comet.

The iterative method using the Newton-Raphson procedure to find a zero of a multi-variable function is quite accurate. The error is basically negligible from looking at the shape of orbits in Figure 3 and the value of E converging from Figure 1.

3 Differential Solution

The 4th Order Runge-Kutta Method is quite popular for achieving results in which you want to know the error and control the error involved based on how much or little you know about a system. This section will show how much fall off in accuracy occurs when using Runge-Kutta when compared to methods like Newton-Raphson.

3.1 Initial Parameters

First, the initial parameters of the system are important in feeding the 4th order Runge-Kutta equations. For the bodies in the solar system, the initial point will be taken at perihelion. Therefore:

$$x_0 = r_{perihelion} \tag{7}$$

$$y_0 = 0 \tag{8}$$

$$V_{x0} = 0 \tag{9}$$

$$V_{y0} = 2\pi\sqrt{\frac{2}{r_0} - \frac{1}{a}} \tag{10}$$

3.2 Runge-Kutta Equations

The initial parameters in equations (7)-(10) will be fed into the 4th Order Runge-Kutta equations which are as follows:

$$k1 = \frac{-\mu x}{r^3} \tag{11}$$

$$k2 = \frac{-\mu x + k_1(\frac{h}{2})}{r^3} \tag{12}$$

$$k3 = \frac{-\mu x + k_2(\frac{h}{2})}{r^3} \tag{13}$$

$$k4 = \frac{-\mu x + k_3 h}{r^3} \tag{14}$$

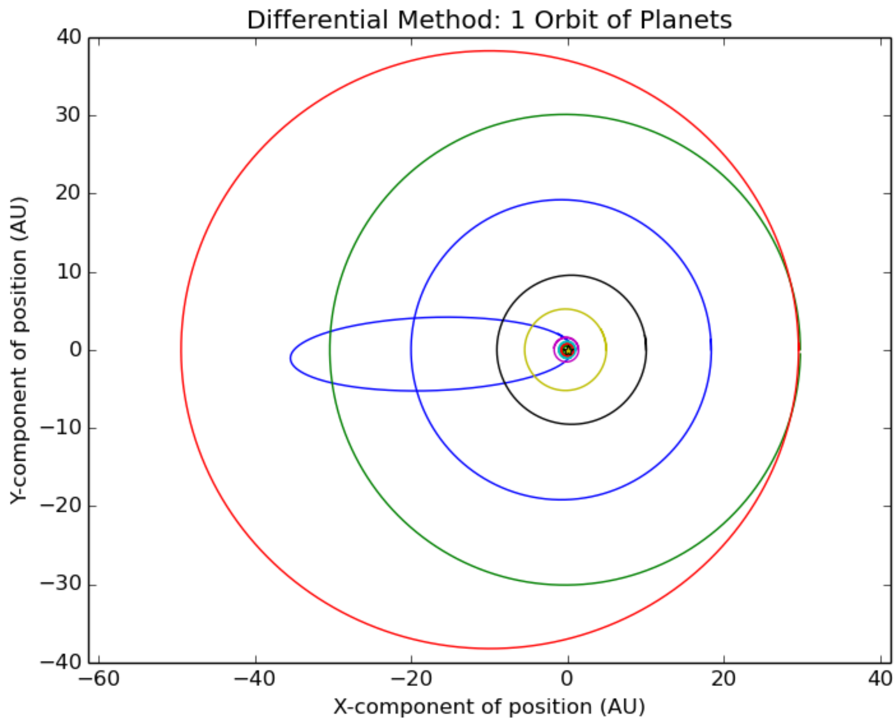

Figure 4: One orbit of Solar System Bodies using Runge-Kutta Method

Where h is the step size (I chose a step size of .001) and $\mu = 4\pi^2$.

Using these equations, the next value for the position and velocity of the body are as follows:

$$x_{next} = x + vh \tag{15}$$

$$v_{next} = v + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{16}$$

3.3 Plots of Bodies Using Differential Method

As shown in Figure 4 and 5, the Runge-Kutta method seems to show some strange trends.

First off, it was very difficult for me to get the orbits to orbit in one exact orbit. This was due to the loop having to have an integral cap as its end

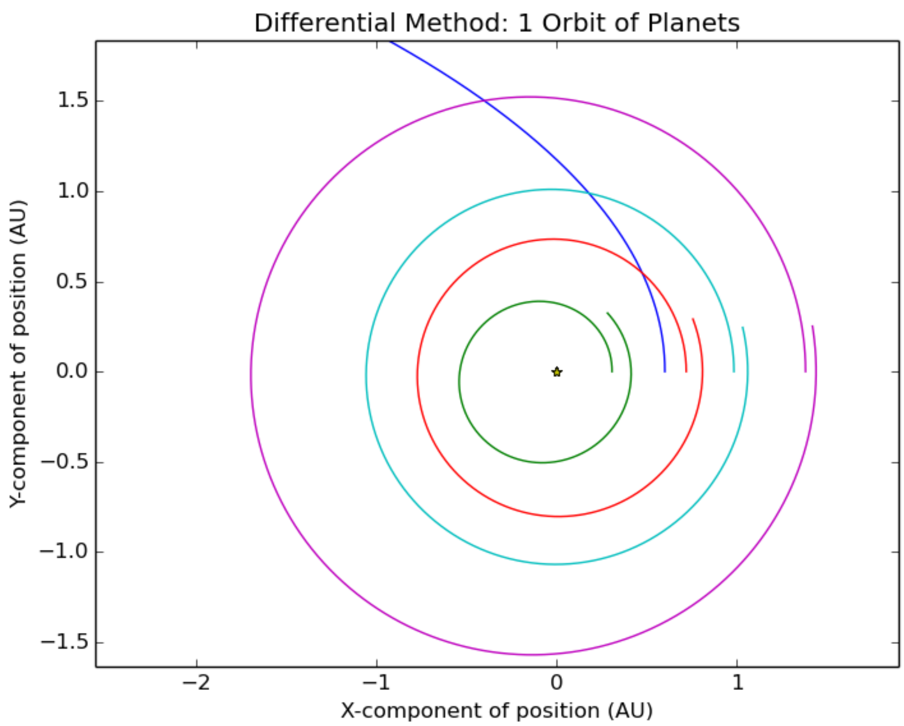

Figure 5: One orbit of the inner planets using Runge-Kutta Method

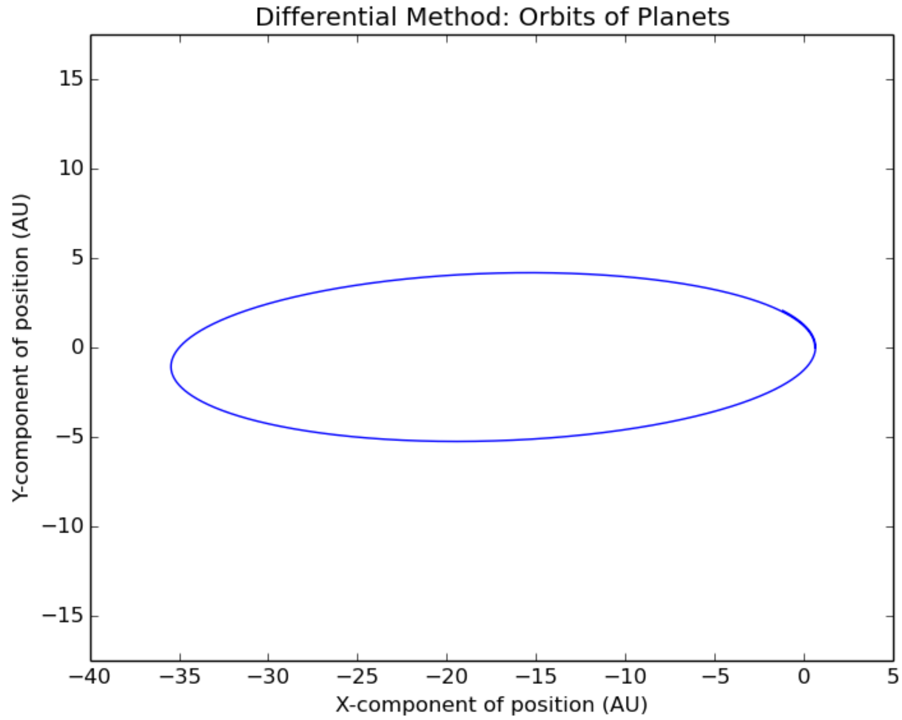

Figure 6: One orbit of Halley's Comet using the differential method

point. With the iterative method, it was not a problem since I could take the orbital period in days, so it would always be integral, since all bodies have a greater than 1 number of days in a year. However, for the differential solution, the units had to be in AU and years in order to simplify the problem as much as possible. Due to this, long period bodies were appearing to orbit a fractional less time than what I expected and smaller bodies orbited a fractional more time than expected. This phenomena can be seen on the plots.

Also of note is the strange warped tendency of the orbit of Halley's comet as seen in Figure 6. This may be due to the number of iterations or the error in the Runge-Kutta method. I am not entirely sure why the orbit is so warped.

3.4 Error of Runge-Kutta Equations

I planned to find the error in the Runge-Kutta equations by taking the difference between the X-values when Y=0 after each orbit. However, since I could not get my loop to loop over exactly one orbit, I ran into a lot of trouble finding this distance. Since I could not simply subtract the end value in the xlist from the starting value in the x list since they were never at the same y value (and differed depending how far away from the sun the body was).

However, the error is quite visible on the graphs, especially on the graphs which show multiple orbits.

4 Part 3: Stability

It is quite clear that the solar system was unstable using the differential method. Here are the graphed results of different number of orbits and how solar system bodies would be impacted. As the number of orbits increases, the likelihood of a planetary body interacting with another increase by a lot, especially in the inner planets.

The solar system collapsing refers to the collapse of the system itself. As shown in Figure 10, 5 orbits seems to be around where the Runge-Kutta method causes the solar system to collapse. All the inner planets' orbits seem to collide or have a high chance of collision. Eventually, the solar system is thrown into chaos as shown in Figures 11 and 12, where 25 orbits are shown, and most bodies have a high chance of colliding with one near to it.

When comparing the Newton-Raphson Method to Runge-Kutta, it is clear which one is more accurate when it comes to results. However, when it comes to controlling the error and ensuring that the results are not modelled to perfectly for what is known about a system, Runge-Kutta is the right solution.

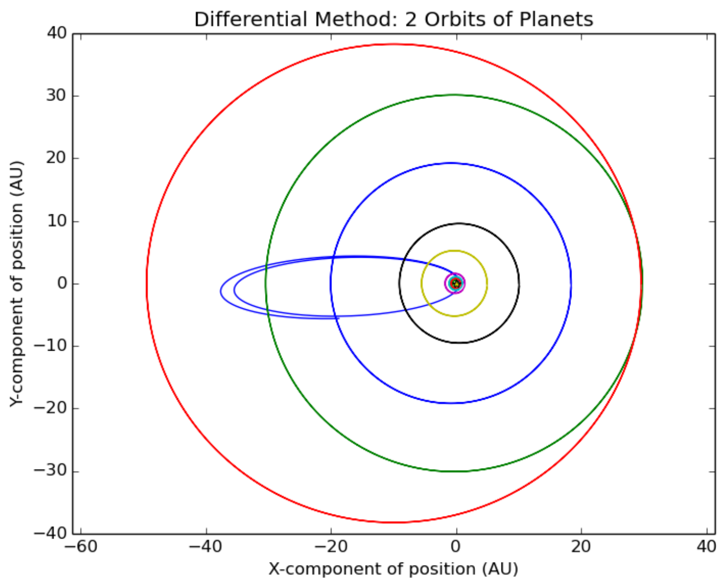

Figure 7: 2 Orbits of the whole solar system

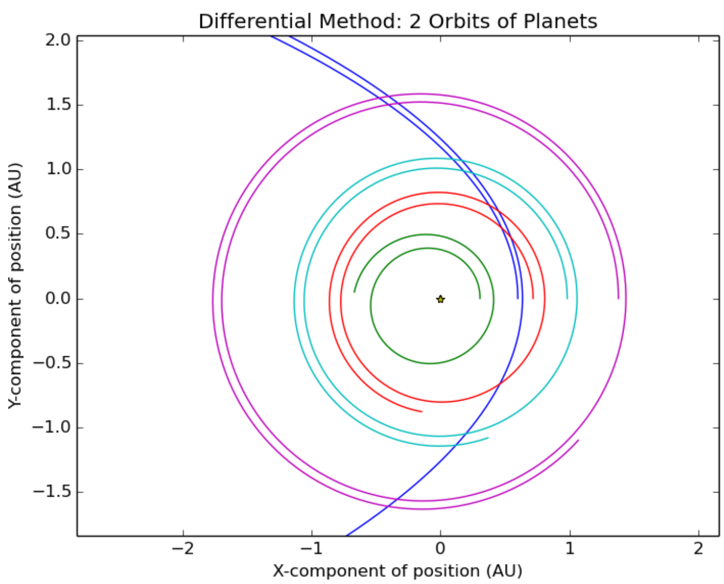

Figure 8: 2 Orbits of the inner solar system

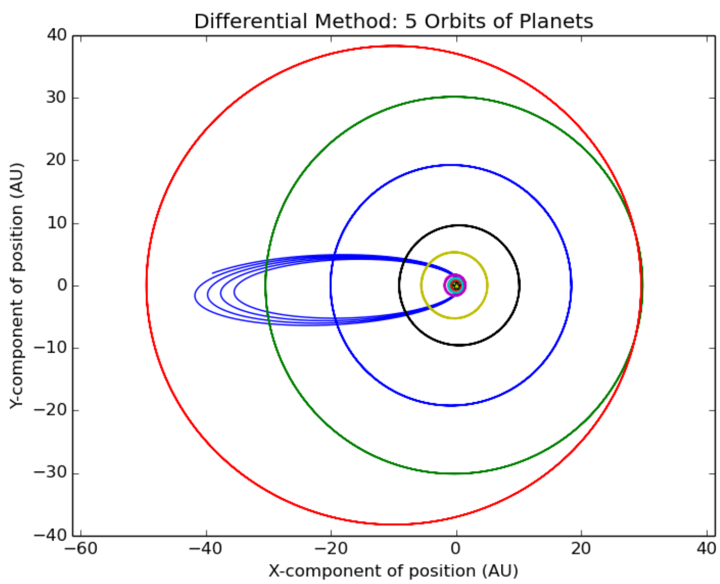

Figure 9: 5 Orbits of the whole solar system

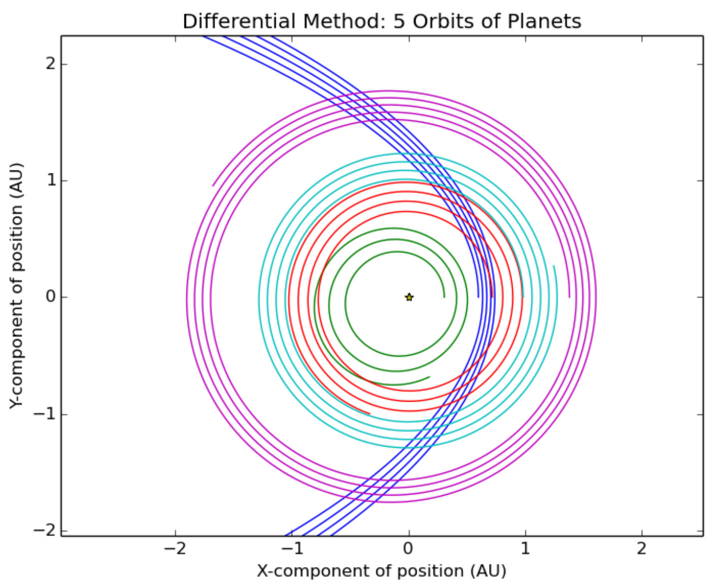

Figure 10: 5 Orbits of the inner solar system

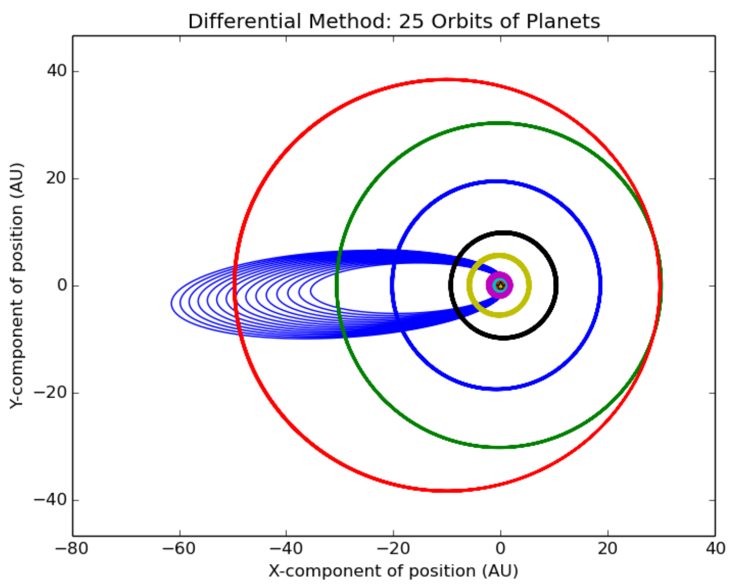

Figure 11: 25 Orbits of the whole solar system

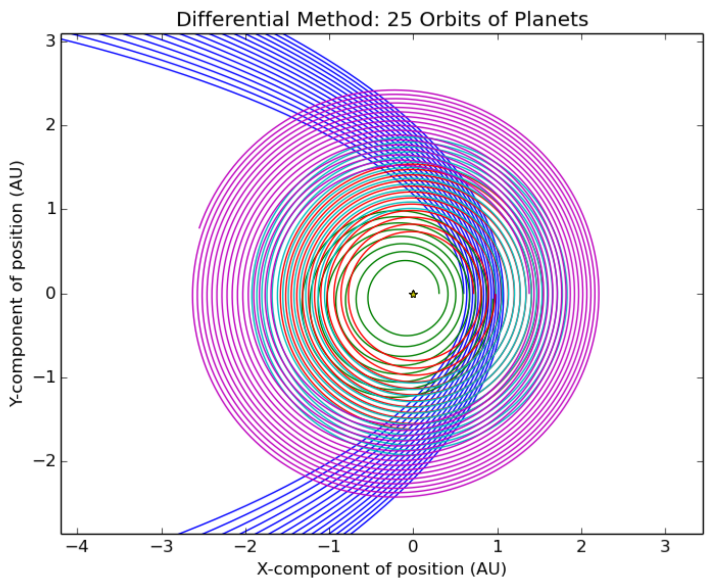

Figure 12: 25 Orbits of the inner solar system