WILD ATTRACTORS AND THERMODYNAMIC FORMALISM.

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Abstract. Fibonacci unimodal maps can have a wild Cantor attractor, and hence be Lebesgue dissipative, depending on the order of the critical point. We present a one-parameter family f_λ of countably piecewise linear unimodal Fibonacci maps in order to study the thermodynamic formalism of dynamics where dissipativity of Lebesgue (and conformal) measure is responsible for phase transitions. We show that for the potential $\phi_t = -t \log |f_\lambda'|$, there is a unique phase transition at some $t_1 \leqslant 1$, and the pressure $P(\phi_t)$ is analytic (with unique equilibrium state) elsewhere. The pressure is majorised by a non-analytic C^∞ curve (with all derivatives equal to 0 at $t_1 < 1$) at the emergence of a wild attractor, whereas the phase transition at $t_1 = 1$ can be of any finite order for those λ for which f_λ is Lebesgue conservative. We also obtain results on the existence of conformal measures and equilibrium states, as well as the hyperbolic dimension and the dimension of the basin of $\omega(c)$.

1. Introduction

The aim of this paper is to understand thermodynamic formalism of unimodal interval maps $f : I \to I$ on the boundary between conservative and dissipative behaviour. For a 'geometric' potential $\phi_t = -t \log |f'|$, the *pressure function* is defined by

$$P(\phi_t) = \sup \left\{ h_\mu + \int \phi_t \, d\mu : \mu \in \mathcal{M}, \int \phi_t \, d\mu > -\infty \right\}, \tag{1}$$

where the supremum is taken over the set $\mathcal{M}$ of f-invariant probability measures μ, and h_μ denotes the entropy of the measure. A measure $\mu_t \in \mathcal{M}$ that assumes this supremum is called an *equilibrium state*. Pressure is a convex and non-increasing function in t and $P(\phi_0) = h_{top}(f)$ is the topological entropy of f. At most parameters t, the pressure function $t \mapsto P(\phi_t)$ is analytic, and there is a unique equilibrium state which depends continuously on t. If the pressure function fails to be analytic at some t, then we speak of a *phase transition* at t, which hints at a qualitative (and discontinuous), rather than quantitative, change in equilibrium states. Refining this further, if the pressure function is C^{n-1} at t, but not C^n, we say that there is an *n-th order phase transition* at t.

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Given a unimodal map f with critical point c, we say that the critical point is *non-flat* if there exists a diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ and $1 < \ell < \infty$ such that for x close to c, $f(x) = f(c) \pm |\phi(x-c)|^\ell$. The value of $\ell = \ell_c$ is known as the *critical order* of c. The metric behaviour of a unimodal map is essentially determined by its topological/combinatorial properties plus its critical order. We give a brief summary of what is known for C^2 unimodal maps with non-flat critical point. A first result is due to Ledrappier [Le] who proved that a measure $\mu \in \mathcal{M}$ of positive entropy is an equilibrium state for $t = 1$ if and only if μ is absolutely continuous w.r.t. Lebesgue (abbreviate acip). This also shows that $t = 1$ is the expected first zero of the pressure function. For simplicity, we assume in the classification below that f is topologically transitive on its dynamical core $[f^2(c), f(c)]$, *i.e.,* there exists a point x_0 such that $\overline{\cup_{n \geqslant 0} f^n(x_0)} = [f^2(c), f(c)])$, except in cases (1) and (5).

(1) If the critical point c of f is attracted to an attracting periodic orbit, then the non-wandering set is hyperbolic on which Bowen's theory [Bo] applies in its entirety. In particular, no phase transitions occur.
(2) If f satisfies the Collet-Eckmann condition, *i.e.,* derivatives along the critical orbit grow exponentially fast, then the pressure is analytic in a neighbourhood of $t = 1$, [BK]; and C^1 for all $t < 1$ except when the critical point is preperiodic, [IT1]. An example of the preperiodic critical point case is the Chebyshev polynomial $x \mapsto 4x(1-x)$ which, as in the much more general work of Makarov & Smirnov [MaS], has a phase transition at $t = -1$. The pressure function is affine for $t \neq -1$ in this case. [1]
(3) If f is non-Collet-Eckmann but possesses an acip μ_{ac}, then there is a first order phase transition at $t = 1$ (*i.e.,* $t \mapsto P(\phi_t)$ is continuous but not C^1). More precisely, $P(\phi_t) = 0$ if and only if $t \geqslant 1$ and the left derivative $\lim_{s \uparrow 1} \frac{d}{ds} P(\phi_s) = -\lambda(\mu_{ac}) < 0$, where $\lambda(\mu_{ac}) = \int \log |f'| \, d\mu_{ac}$ denotes the *Lyapunov exponent* of μ_{ac}, see [IT1, Proposition 1.2].
(4) If f is non-Collet-Eckmann but has an absolutely continuous conservative infinite σ-finite measure, then there is still a phase transition at $t = 1$, but $P(\phi_t)$ is C^1. In fact, $P(\phi_t) = 0$ if and only if $t \geqslant 1$ and the left derivative $\lim_{s \uparrow 1} \frac{d}{ds} P(\phi_s) = 0$. This follows from the proof of [IT1, Lemma 9.2].
(5) If f is infinitely renormalisable, then the critical omega-limit set $\omega(c)$ is a Lyapunov stable attractor, and its basin $\mathrm{Bas} = \{x : \omega(x) \subset \omega(c)\}$ is a second Baire category set of full Lebesgue measure. The best known example is the Feigenbaum-Coullet-Tresser map f_{feig}, for which the topological entropy $h_{top}(f_{\mathrm{feig}}) = 0$, and so $P(\phi_t) \equiv 0$ for all $t \geqslant 0$. More complicated renormalisation patterns can lead to a more interesting thermodynamic behaviour, see Avila & Lyubich [AL], Moreira & Smania [MoS] and Dobbs [D]. However, this thermodynamic behaviour is primarily a topological, rather than a metric, phenomenon, so should be seen as complementary to the results given in this paper.
(6) If f has a wild attractor, then $\omega(c)$ is not Lyapunov stable and attracts a set of full Lebesgue measure, whereas a second Baire category set of points has

[1] Collet-Eckmann maps with "low-temperature phase transitions" were found in [CL], after our paper was first submitted, but which we can include in this revision.

a dense orbit in $[f^2(c), f(c)]$. In [AL, Theorem 10.5] it is asserted that there exists some $t_1 < 1$ such that $P(\phi_t) = 0$ for $t \geqslant t_1$. In this paper we study this fact, as well as further thermodynamic properties of wild attractors, in detail.

A wild attractor occurs for a unimodal map f if it has very large critical order ℓ as well as Fibonacci combinatorics, *i.e.,* the cutting times are the Fibonacci numbers. (The *cutting times* $(S_k)_{k\geqslant 0}$ are the sequence of iterates n at which the image of the central branch of f^n contains the critical point. They satisfy the recursive formula $S_k - S_{k-1} = S_{Q(k)}$ for the so-called *kneading map* $Q : \mathbb{N} \to \mathbb{N}_0$; so Fibonacci maps have kneading map $Q(k) = \max\{k-2, 0\}$, see Section 2 for more precise details.)

Let us parametrise Fibonacci maps by critical order, say

$$\mathrm{Fib}_\ell : [0,1] \to [0,1], \qquad x \mapsto a(\ell)(1 - |2x-1|^\ell),$$

where $a(\ell) \in [0,1]$ is chosen such that Fib_ℓ has Fibonacci combinatorics. The picture is then as follows:

$$\begin{cases} \ell \leqslant 2 & \mathrm{Fib}_\ell \text{ has an acip which is super-polynomially mixing, [LM, BLS],} \\ 2 < \ell < 2+\varepsilon & \mathrm{Fib}_\ell \text{ has an acip which is polynomially mixing with exponent} \\ & \quad \text{tending to infinity as } \ell \to 2, \text{ [KN, RS],} \\ \ell_0 < \ell < \ell_1 & \mathrm{Fib}_\ell \text{ has a conservative } \sigma\text{-finite acim,} \\ \ell_1 < \ell & \mathrm{Fib}_\ell \text{ has a wild attractor [BKNS], with dissipative } \sigma\text{-finite} \\ & \quad \text{acim, [Ma].} \end{cases}$$

For the logistic family (*i.e.,* critical order is 2), Lyubich proved there cannot be a wild attractor, so in particular Fib_ℓ has no wild attractor. In [KN] it was shown that $\ell = 2 + \varepsilon$ still does not allow for a wild attractor for Fib_ℓ. Wild attractors were shown to exist [BKNS] for very large ℓ. The value of ℓ_1 beyond which the existence of a wild attractor is rigorously proven in [BKNS] is extremely large[2], but unpublished numerical simulations by Sutherland et al. suggest that $\ell_1 = 8$ suffices. The region $\ell \in (\ell_0, \ell_1)$ is somewhat hypothetical. It can be shown [B1] that Fib_ℓ has an absolutely continuous σ-finite measure for $\ell > \ell_0$, and it stands to reason that this happens before Fib_ℓ becomes Lebesgue dissipative, but we have no proof that indeed $\ell_0 < \ell_1$, nor that this behaviour occurs on exactly a single interval. The existence of a dissipative σ-finite acim when there is a wild attractor was shown by Martens [Ma], see also [BH, Theorem 3.1].

Within interval dynamics, inducing schemes have become a standard tool to study thermodynamic formalism, [BT1, BT2, PS, S2, BI]. One constructs a *full-branched* Gibbs-Markov induced system (Y, F) whose thermodynamic properties can be understood in terms of a full shift on a countable alphabet. However, precisely in the setting of wild attractors, the set

$$Y^\infty = \{y \in Y : F^n(y) \text{ is well-defined for all } n \geqslant 0\}$$

is dense in Y but of zero Lebesgue measure m. For this reason, we prefer to work with a different induced system, called (Y, F) again, that has branches of arbitrarily short length, but for which Y^∞ is co-countable. By viewing the dynamics under F

[2]For less restrictive Fibonacci-like combinatorics (basically if $k - Q(k)$ is bounded) the existence of wild attractors was proved in [B2].

as a random walk, we can show that transience[3] of this random walk (w.r.t. Lebesgue measure) implies the existence of a Cantor attractor.

Proving transience of (Y, F, m) is very technical due to the severe non-linearity of F for smooth unimodal maps f with large critical order. For this reason, we introduce countably piecewise linear unimodal maps for which induced systems with linear branches can be constructed. This idea is definitely not new, *cf.* the maps of Gaspard & Wang [GW] and Lüroth [Lü, DK] as countably piecewise linear versions of the Farey and Gauss map, respectively.[4] The explicit construction for unimodal maps is new, however. Although we are mostly interested in Fibonacci maps, the method works in far more generality; it definitely suffices if the kneading map $Q(k) \to \infty$ and a technical condition (6) is satisfied. Note that the inducing scheme we will use is somewhat different from that in [BKNS] which was based on preimages of the fixed point. Instead, we will use an inducing scheme based on precritical points, used before in [B1], and we arrive at a two-to-one cover of a countably piecewise interval map $T_\lambda : (0, 1] \to (0, 1]$ defined in Stratmann & Vogt [SV] as follows: For $n \geqslant 1$, let $V_n := (\lambda^n, \lambda^{n-1}]$ and define

$$T_\lambda(x) := \begin{cases} \frac{x-\lambda}{1-\lambda} & \text{if } x \in V_1, \\ \frac{x-\lambda^n}{\lambda(1-\lambda)} & \text{if } x \in V_n, \quad n \geqslant 2. \end{cases} \tag{2}$$

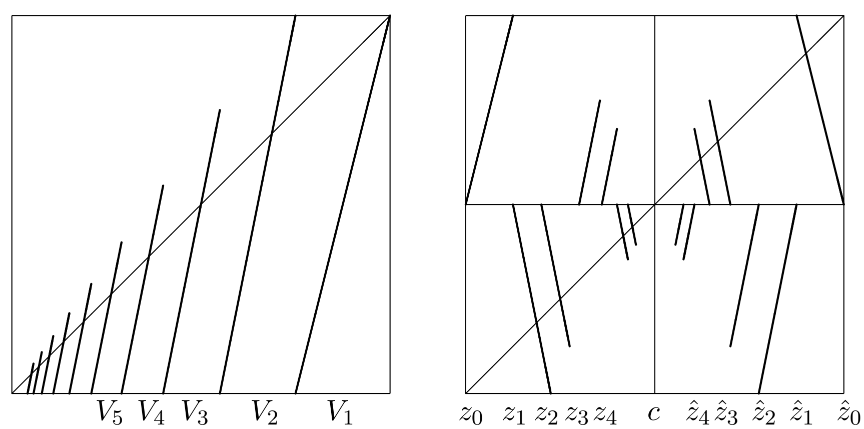

Figure 1. The maps $T_\lambda : [0, 1] \to [0, 1]$ and $F_\lambda : [z_0, \hat{z}_0] \to [z_0, \hat{z}_0]$.

Both the unimodal map f_λ and the induced map F_λ are linear on intervals $W_k = [z_{k-1}, z_k]$ and $\hat{W}_k = [\hat{z}_k, \hat{z}_{k-1}]$ of length $\frac{1-\lambda}{2}\lambda^k$. Here $\hat{x} = 1 - x$ is the symmetric image of a point or set, and $z_k < c < \hat{z}_k$ are the points in $f^{-S_k}(c)$ that are closest to c. We

[3]We discuss transience and (null and positive) recurrence in detail in Section 9.

[4]In fact, considering $-t \log |f'|$ for the Gaspard & Wang map is exactly equivalent to the Hofbauer potential [H1] for the full shift on two symbols.

define $F_\lambda(x) = f^{S_{k-1}}$ if $x \in W_k \cup \hat{W}_k$. The induced map F_λ satisfies

$$\begin{array}{ccc} [z_0, \hat{z}_0] & \xrightarrow{F_\lambda} & [z_0, \hat{z}_0] \\ \pi \downarrow & & \downarrow \pi \\ [0,1] & \xrightarrow{T_\lambda} & [0,1] \end{array} \qquad \pi : x \mapsto \begin{cases} \frac{1-2x}{2(1-z_0)} & \text{if } x \leqslant \frac{1}{2}; \\ \frac{2x-1}{2(1-z_0)} & \text{if } x \geqslant \frac{1}{2}. \end{cases} \tag{3}$$

Note that $\pi^{-1}(V_i) = W_i \cup \hat{W}_i$.

The one-parameter system (Y, F_λ) is of interest both for its own sake, see [BT3, SV], and for the sake of studying (thermodynamic properties of) f itself. Theorem A replaces the somewhat hypothetical picture of smooth Fibonacci maps with precise values of critical orders $\ell = \ell(\lambda)$, where each of the different behaviours occurs. In this non-differentiable setting, the critical order ℓ is defined by the property that $\frac{1}{C}|x-c|^\ell < |f(x) - f(c)| \leqslant C|x-c|^\ell$ for some $C > 0$ and all $x \in [0,1]$.

Theorem A. *The above countably piecewise linear unimodal map f_λ (*i.e., *with $|W_k| = |\hat{W}_k| = \frac{1-\lambda}{2}\lambda^k$ and $\lambda \in (0,1)$) satisfies the following properties:*

(a) The critical order $\ell = 3 + \frac{2\log(1-\lambda)}{\log\lambda}$.
(b) If $\lambda \in (\frac{1}{2}, 1)$, i.e., *$\ell > 5$, then f_λ has a wild attractor.*
(c) If $\lambda \in \left[\frac{2}{3+\sqrt{5}}, \frac{1}{2}\right]$, i.e., *$4 \leqslant \ell \leqslant 5$, then f_λ has no wild attractor, but an infinite σ-finite acim.*
(d) If $\lambda \in (0, \frac{2}{3+\sqrt{5}})$, i.e., *$\ell \in (3,4)$, then f_λ has an acip.*

As above, let $\phi_t = -t\log|f_\lambda'|$ and $\Phi_t = -t\log|F_\lambda'|$ be the *geometric* potentials for the unimodal map f_λ and its induced version F_λ, respectively. (Note that $\Phi_t = \sum_{j=0}^{\tau-1} \phi_t \circ f_\lambda^j$ for inducing time $\tau = \tau(x)$, justifying the name *induced potential.*)

In [BT3], the precise form of the pressure function for $((0,1], T_\lambda, -t\log|T_\lambda'|)$ and therefore also for the system $(Y, F_\lambda, -t\log|F_\lambda'|)$, is given. However, this is of lesser concern to us here, because given $([0,1], f_\lambda)$ with potential $-t\log|f_\lambda'|$, for most results on the induced system (Y, F_λ) to transfer to back to the original system, the correct induced potential on Y is $-\log|F_\lambda'| - p\tau$, where the shift $p\tau$ is determined by a constant p (usually the pressure of $-t\log|f_\lambda'|$) and the inducing time τ where $\tau(x) = S_{k-1}$ whenever $x \in W_k \cup \hat{W}_k$. The fact that the shift by $p\tau$ depends on the interval k increases the complexity of this problem significantly. Results from [BT3] which apply directly are contained in the following theorem.

Theorem B. *Let $Bas_\lambda = \{x \in I : f_\lambda^n(x) \to \omega(c) \text{ as } n \to \infty\}$ be the basin of $\omega(c)$, and let the* hyperbolic dimension *be the supremum of Hausdorff dimensions of hyperbolic sets Λ,* i.e., *Λ is f_λ-invariant, compact but bounded away from c. Then*

$$\dim_{hyp}(f_\lambda) = \dim_H(Bas_{1-\lambda}) = t_1$$

where

$$t_1 := \begin{cases} 1 & \text{if } \lambda \in (0, 1/2], \\ t_2 & \text{if } \lambda \in [1/2, 1), \end{cases} \qquad \text{where} \quad t_2 := -\log 4 / \log[\lambda(1-\lambda)]. \tag{4}$$

For the properties of pressure presented in Theorem D and the related results in Section 7, it is advantageous to use a different approach to pressure, called *conformal pressure* $P_{\mathrm{Conf}}(\phi_t)$, which is the smallest potential shift allowing the existence of a conformal measure for the potential. We refer Sections 5 and 7 for the precise definitions, but in Theorem C we will show that conformal pressure coincides with the (variational) pressure defined in (1). In [BT3], it is shown that t_1 from (4) is the smallest value at which the pressure $P(\Phi_t)$ of the induced system (Y, F_λ, Φ_t) becomes zero. This gives the background information for our third main theorem.

Theorem C. *The countably piecewise linear Fibonacci map f_λ, $\lambda \in (0,1)$, with potential ϕ_t has the following thermodynamical properties.*

(a) The conformal and variation pressure coincide: $P_{\mathrm{Conf}}(\phi_t) = P(\phi_t)$;
(b) For $t < t_1$, there exists a unique equilibrium state ν_t for (I, f_λ, ϕ_t); this is absolutely continuous w.r.t. the appropriate conformal measure n_t. For $t > t_1$, the unique equilibrium state for (I, f_λ, ϕ_t) is ν_ω, the measure supported on the critical omega-limit set $\omega(c)$. For $t = t_1$, ν_ω is an equilibrium state, and if $\lambda \in (0, \frac{2}{3+\sqrt{5}})$ then so is the acip, denoted ν_{t_1};
(c) The map $t \mapsto P(\phi_t)$ is real analytic on $(-\infty, t_1)$. Furthermore $P(\phi_t) > 0$ for $t < t_1$ and $P(\phi_t) \equiv 0$ for $t \geqslant t_1$, so there is a phase transition at $t = t_1$.

Let $\gamma := \frac{1}{2}(1+\sqrt{5})$ be the golden ratio and $\Gamma := \frac{2\log\gamma}{\sqrt{-\log[\lambda(1-\lambda)]}}$. More precise information on the shape of the pressure function is the subject of our fourth main result.

Theorem D. *The pressure function $P(\phi_t)$ of the countably piecewise linear Fibonacci map f_λ, $\lambda \in (0,1)$, with potential ϕ_t has the following shape:*

a) On a left neighbourhood of t_1, there exist $\tau_0 = \tau_0(\lambda), \tau_0' = \tau_0'(\lambda) > 0$ such that

$$P(\phi_t) > \begin{cases} \tau_0 e^{-\pi \frac{\Gamma}{\sqrt{t_1 - t}}} & \text{if } t < t_1 \leqslant 1 \text{ and } \lambda \geqslant \frac{1}{2}; \\ \tau_0'(1-t)^{\frac{\log\gamma}{\log R}} & \text{if } t < 1 \text{ and } \frac{2}{3+\sqrt{5}} \leqslant \lambda < \frac{1}{2}, \end{cases}$$

where $R = \frac{\left(1+\sqrt{1-4\lambda^t(1-\lambda)^t}\right)^2}{4\lambda^t(1-\lambda)^t}$ and $\lim_{t\to 1}\log R \sim 2(1-2\lambda)$ for $\lambda \sim \frac{1}{2}$.
b) On a left neighbourhood of t_1, there exist $\tau_1 = \tau_1(\lambda), \tau_1' = \tau_1'(\lambda) > 0$ such that

$$P(\phi_t) < \begin{cases} \tau_1 e^{-\frac{5}{6}\frac{\Gamma}{\sqrt{t_1 - t}}} & \text{if } t < t_1 \leqslant 1 \text{ and } \lambda \geqslant \frac{1}{2}; \\ \tau_1'(1-t)^{\frac{\lambda\log\gamma}{2t(1-2\lambda)}} & \text{if } t < 1 \text{ and } \frac{2}{3+\sqrt{5}} \leqslant \lambda < \frac{1}{2}. \end{cases}$$

*c) If $\lambda \in (0, \frac{2}{3+\sqrt{5}})$, then $\lim_{s\uparrow t_1} \frac{d}{ds}P(\phi_s) < 0$; otherwise (*i.e.*, if $\lambda \in [\frac{2}{3+\sqrt{5}}, 1)$), $\lim_{s\uparrow t_1} P(\phi_s) = 0$.*

To put these results in context, let us discuss the results of Lopes [Lo, Theorem 3] on the thermodynamic behaviour of the Manneville-Pomeau map $g : x \mapsto x + x^{1+\alpha}$

(mod 1). The pressure function for this family is

$$P(-t\log g') = \begin{cases} \lambda(\mu_{ac})(1-t) + B(1-t)^{1/\alpha} + \text{ h.o.t.} & \text{if } t < 1 \text{ and } \alpha \in (\frac{1}{2}, 1); \\ C(1-t)^{\alpha} + \text{ h.o.t.} & \text{if } t < 1 \text{ and } \alpha > 1; \\ 0 & \text{if } t \geqslant 1, \end{cases}$$

where $B, C > 0$ are constants, and $\lambda(\mu_{ac})$ is the Lyapunov exponent of the non-Dirac equilibrium state (*i.e.,* the acip). Hence the left derivative of the pressure at $t = 1$ when $\alpha \in (1/2, 1)$ is $-\lambda(\mu_{ac})$. Recall that in the acip case, due to Ledrappier's result [Le], $h_{\mu_{ac}} = \lambda(\mu_{ac})$. Note that the transition case $\alpha = 1$ corresponds to the transition from a finite acip (for $\alpha < 1$) to an infinite acim (for $\alpha \geqslant 1$).[5] In the Manneville-Pomeau case there is no transition of Lebesgue measure changing from conservative to dissipative. The phase transition at $t = 1$ is said to be of *first type* if there are two equilibrium states (here an acip and the Dirac measure δ_0); if there is only one equilibrium state, then the phase transition is of *second type*. The exponent $1/\alpha$ is called the *critical exponent of transition*.

For Fibonacci maps, instead of a Dirac measure, there is a unique measure ν_ω supported on the critical ω-limit set; it has zero entropy and Lyapunov exponent. Theorem D paints a similar picture to Lopes' result for Manneville-Pomeau maps. In detail, we have

- a phase transition of first type for $\lambda \in (0, \frac{2}{3+\sqrt{5}})$: the pressure is not C^1 at $t = t_1$. This is precisely the region from Theorem A where f_λ has an acip μ_{ac}, in accordance with the results from [IT1]. According to Ledrappier [Le], $h_\mu = \lambda(\mu_{ac})$ is the Lyapunov exponent, so $\lim_{s\uparrow 1} \frac{d}{ds} P(\phi_s) = -\lambda(\mu_{ac})$. Lebesgue measure is conservative here.
- a phase transition of second type (with unique equilibrium state ν_ω supported on $\omega(c)$) for $\lambda \in (\frac{2}{3+\sqrt{5}}, \frac{1}{2})$: there is some minimal $n \in \mathbb{N}$ such that the n-th left derivative $D^n_- P(\phi_t)|_{t=t_1} < 0$. Thus the pressure function is C^{n-1}, but not C^n, at $t = t_1$ and so there is an n-th order phase transition. Consequently, the critical exponent of transition tends to infinity as $\lambda \nearrow 1/2$. Lebesgue is still conservative here, and also for $\lambda = 1/2$.
- a phase transition of second type for $\lambda \in [1/2, 1)$: the pressure is C^1 with $\frac{d}{dt} P(\phi_t) = 0$ at $t = t_1$. By convexity, also $\frac{d^2}{dt^2} P(\phi_t) = 0$ at $t = t_1$. It is unlikely, but we cannot a priori rule out, that the higher derivatives oscillate rapidly, preventing the pressure function from being C^∞ at $t = t_1$. Lebesgue is dissipative for $\lambda \in (1/2, 1)$.

This paper is organised as follows. In Section 2 we introduce the countably piecewise linear unimodal maps and give conditions under which they produce an induced Markov map that is linear on each of its branches. In Section 3 this is applied to Fibonacci

[5] The asymptotics of $P(t)$ in [Lo, Theorem 3] don't hold for $\alpha = 1$ (personal communication with A.O. Lopes), but since there is no acip, $P(t)$ is differentiable at $t = 1$ with derivative $P'(t) = 0$ as in [IT1]. We don't know the higher order terms in this case. Asymptotics of related systems are obtained in [PFK, BFKP], namely for the Farey map $x \mapsto \frac{x}{1-x}$ if $x \in [0, \frac{1}{2}]$ and $x \mapsto \frac{1-x}{x}$ if $x \in [\frac{1}{2}, 1]$. It is expected that their asymptotics also hold for the Manneville-Pomeau map with $\alpha = 1$. In [BLL], a Manneville-Pomeau-like map with two neutral fixed points, both with $\alpha = 1$, is considered, using a Hofbauer-like potential.

maps, and, using a random walk argument, the existence of an attractor and hence Theorem A is proved. Rather as an intermezzo, Section 4 shows that for countably piecewise linear unimodal maps with infinite critical order, wild attractors do exist beyond the Fibonacci-like combinatorics. In Section 5 we explain how conformal and invariant measures of the induced system relate to conformal and invariant measures of the original system. In Section 6 we discuss the technicalities that the 2-to-1 factor map from (3) poses for invariant and conformal measures; we also prove Theorem B. The properties of the conformal pressure functions (existence, upper/lower bounds and nature of phase transitions) are studied in Section 7; this section contains the proof the main part of Theorem D. In Section 8 we prove the existence and properties of invariant measures that are absolutely continuous w.r.t. the relevant conformal measures. In the final section we present some general theory on countable Markov shifts due to Sarig. This leads up to the proof of Theorem C, and also gives the final ingredient of the proof of Theorem D.

2. The countably piecewise linear model

Let $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout $f : I \to I$ stands for a symmetric unimodal map with unit interval $I = [0, 1]$, critical point $c = \frac{1}{2}$, and $f(0) = f(1) = 0$. For $x \in [0, 1]$, let $\hat{x} = 1 - x$ be the point with the same f-image as x. We use the same notation for sets.

Let us start by some combinatorial notation. For $n \geqslant 1$, the *central branch* of f^n is the restriction of f^n to any of the two largest one-sided neighbourhoods of c on which f^n is monotone. Due to the symmetry, the image of the left and right central branch is the same, and if it contains the critical point, then we say that n is a *cutting time*. We enumerate cutting times as $1 = S_0 < S_1 < S_2 < \dots$ If f has no periodic attractors, S_k is well-defined for all k, and we will denote the point in the left (resp. right) central branch of f^{S_k} that maps to c by z_k (resp. $\hat{z}_k$). These points are called the *closest precritical points* and it is easy to see that the domains of the left (resp. right) central branch of f^{S_k} are $[z_{k-1}, c]$ (resp. $[c, \hat{z}_{k-1}]$).

The difference of two consecutive cutting times is again a cutting time. Hence (see [H1]) we can define the *kneading map* $Q : \mathbb{N} \to \mathbb{N}_0$ by

$$S_k - S_{k-1} = S_{Q(k)}.$$

A kneading map Q corresponds to a sequence of cutting times of a unimodal map if and only if it satisfies

$$\{Q(k+j)\}_{j \geqslant 1} \succeq \{Q(Q^2(k)+j)\}_{j \geqslant 1}, \tag{5}$$

for all $k \geqslant 1$, where $\succeq$ indicates lexicographical order (see [H2]). Note that (5) holds automatically if the kneading map is non-decreasing.

The construction of our unimodal map f proceeds along the following steps:

(I) First fix a kneading map Q such that

$$Q(k+1) > Q(Q^2(k)+1) \tag{6}$$

for every $k \geqslant 2$. This is obviously stronger than (5), but provides a considerable simplification of the proof.

(II) By convention, set $z_{-1} = 0$ and $\hat{z}_{-1} = 1$. For $j \geqslant 0$, choose a strictly increasing sequences of points $z_j \nearrow c = \frac{1}{2}$ and $\hat{z}_j = 1 - z_j \searrow c$. (The points z_j will play the role of the closest precritical points, cf. (IH_j) in the proof of Proposition 1.) Set

$$W_j := (z_{j-1}, z_j),\ \hat{W}_j := (\hat{z}_j, \hat{z}_{j-1}) \text{ and } \varepsilon_j := |W_j| = |\hat{W}_j| > 0.$$

Therefore, $\sum_{j \geqslant 0} \varepsilon_j = \frac{1}{2}$.

(III) Define

$$s_j := \frac{1}{\varepsilon_j} \sum_{i \geqslant Q(j)+1} \varepsilon_i = \frac{|z_{Q(j)} - c|}{|z_j - z_{j-1}|}, \tag{7}$$

for $j \geqslant 1$; these numbers will turn out to be the absolute values of the slopes of $F|_{W_j}$ for the induced map F, see (11) below.

(IV) For $j \geqslant 0$, we define numbers $\kappa_j > 0$ that will represent the slope of $f|_{W_j}$. Let

$$\kappa_0 := \frac{1}{2\varepsilon_0}. \tag{8}$$

(This will give that $f(z_0) = \kappa_0 \cdot (z_0 - z_{-1}) = \frac{1}{2} = c$.) Next, set

$$\kappa_1 := s_1 = \frac{1}{\varepsilon_1} \sum_{i \geqslant 1} \varepsilon_i = \frac{1 - 2\varepsilon_0}{2\varepsilon_1}. \tag{9}$$

(Since inducing time $S_0 = 1$ on W_1, it makes sense that the slopes of f and F on W_1 are the same. In fact, we will have $F|_{W_1} = f^{S_0}|_{W_1} = f|_{W_1}$.) For $j \geqslant 2$, we set inductively

$$\kappa_j := \begin{cases} \frac{s_j}{\kappa_0} \frac{\kappa_{j-1}}{s_{j-1}} & \text{if } Q(j-1) = 0, \\ \frac{s_j \cdot \kappa_{j-1}}{s_{j-1} \cdot s_{Q(j-1)} \cdot s_{Q^2(j-1)+1}} & \text{if } Q(j-1) > 0. \end{cases} \tag{10}$$

(V) Let f be the unique continuous unimodal map such that

$$\begin{cases} f(z_{-1}) = f(\hat{z}_{-1}) = z_{-1} \\ Df|_{W_j} = -Df|_{\hat{W}_j} = \kappa_j, \end{cases}$$

so that $|f(W_i)| = \kappa_i \varepsilon_i$ and each interval $f(W_i)$ is adjacent to $f(W_{i+1})$.

Thus f is completely determined by the choice of Q and points z_j. In Section 3 on Fibonacci combinatorics, we let $z_j \nearrow c$ in a geometric manner, or precisely, $\varepsilon_j = \frac{1-\lambda}{2}\lambda^j$ so that f depends solely on a the single parameter $\lambda \in (0, 1)$. In this section, we will continue with the more general set-up.

The induced map[6] is defined as:

$$F : (z_0, \hat{z}_0) \to (z_0, \hat{z}_0), \qquad F|_{W_j \cup \hat{W}_j} = f^{S_{j-1}}|_{W_j \cup \hat{W}_j} \text{ for } j \geqslant 1. \tag{11}$$

Since the z_j will play the role of the closest precritical points, we will have $f^{S_{j-1}}(z_j) = f^{S_{j-1}}(\hat{z}_j) \in \{z_{Q(j)}, \hat{z}_{Q(j)}\}$, and therefore

$$F(W_j) = F(\hat{W}_j) = \cup_{i > Q(j)} W_i \text{ or } \cup_{i > Q(j)} \hat{W}_i.$$

In Proposition 1, we will prove that $F|_{W_j}$ and $F|_{\hat{W}_j}$ are also linear.

[6] In later sections, the interval on which the induced map is defined will be called Y.

We pose two other conditions on the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$, which will be checked later on for specific examples, in particular the Fibonacci map. Let $x^f = f(x)$ for any point x. For all $j \geqslant 2$:

$$\frac{s_j}{\kappa_j}|c^f - z_j^f| = \frac{s_j}{\kappa_j}\sum_{i=j+1}^{\infty} \kappa_i \varepsilon_i \leqslant \varepsilon_{Q(j)}, \tag{12}$$

and

$$\frac{s_j}{\kappa_j}|c^f - z_j^f| = \frac{s_j}{\kappa_j}\sum_{i=j+1}^{\infty} \kappa_i \varepsilon_i \leqslant \frac{\varepsilon_{Q^2(j)+1}}{s_{Q(j)}} \quad \text{whenever } Q(j) > 0. \tag{13}$$

Proposition 1. *Let f be the map constructed above,* i.e., *assume that* (6)-(13) *hold. Then Q is the kneading map of f, and the induced map F is linear on each set W_j and $\hat{W}_j$, having slope $\pm s_j$.*

Proof. We argue by induction, using the induction hypothesis, for $j \geqslant 2$,

$$\begin{cases} f^{S_{j-1}-1}|_{(c^f, z_{j-1}^f)} \text{ is linear, with slope } \frac{s_j}{\kappa_j}. \\ f^{S_{j-1}}(z_{j-1}) = c. \\ f^{S_{j-1}}(c) \in W_{Q(j)} \text{ or } \hat{W}_{Q(j)}. \end{cases} \tag{IH$_j$}$$

From the first statement, it follows immediately that

$$f^{S_{j-1}}|_{W_j} \text{ is linear, with slope } s_j, \text{ for } j \geqslant 1. \tag{14}$$

From this and the fact that $f^{S_{j-1}}(z_{j-1}) = c$, it follows that

$$f^{S_{j-1}}(z_j) = f^{S_{j-1}}(z_{j-1}) \pm s_j \varepsilon_j = c \pm \sum_{i \geqslant Q(j)+1} \varepsilon_i = z_{Q(j)} \text{ or } \hat{z}_{Q(j)}. \tag{15}$$

Let us prove (IH$_j$) for $j = 2$. It is easily checked that $f(z_0) = f(\hat{z}_0) = c = \frac{1}{2}$, and hence $f(c) \in \hat{W}_0$. $f(z_1) = \frac{1}{2} + \kappa_1 \varepsilon_1 = \frac{1}{2} + \frac{1}{2} - \varepsilon_0 = \hat{z}_0$. So $f^{S_1}(z_1) = c$ and because $c^f \in \hat{W}_0$, $f^{S_1-1}|_{(c^f, z_1^f)} = f|_{(c^f, \hat{z}_0)}$ is also linear, with slope $\kappa_0 = \frac{s_2}{\kappa_2}$. Next we check the position of $f^{S_1}(c)$. By the above formula, and the additional assumption (12),

$$\begin{aligned} f^{S_1}(c) &= f^{S_1}(z_2) - |f^{S_1-1}((c^f, z_2^f))| \\ &= z_{Q(2)} - \frac{s_2}{\kappa_2}|c^f - z_2^f| \geqslant z_{Q(2)} - \varepsilon_{Q(2)} = z_{Q(2)-1}. \end{aligned}$$

Hence $f^{S_1}(c) \in W_{Q(2)}$.

Next assume that (IH$_i$) holds for $i < j$. Using (15) and (IH$_{Q(j-2)}$) subsequently, we get

$$f^{S_{j-1}}(z_{j-1}) = f^{S_{Q(j-1)}} \circ f^{S_{j-2}}(z_{j-1}) = f^{S_{Q(j-1)}}(z_{Q(j-1)}) = c.$$

Because $(c^f, z_{j-1}^f) \subset (c^f, z_{j-2}^f)$, (IH$_{j-1}$) yields that

$$f^{S_{j-2}-1}|_{(c^f, z_{j-1}^f)} \text{ is linear with slope } \frac{s_{j-1}}{\kappa_{j-1}}.$$

By (15) and (IH_{j-1}), its image is the interval $(z_{Q(j-1)}, c_{S_{j-2}}) \subset W_{Q(j-1)}$ or $\hat{W}_{Q(j-1)}$. Now if $Q(j-1) = 0$, then

$$f^{S_{j-1}-1}|_{(c^f, z^f_{j-1})} = f \circ f^{S_{j-2}-1}|_{(c^f, z^f_{j-1})} \text{ is linear with slope } \kappa_0 \frac{s_{j-1}}{\kappa_{j-1}}.$$

By the first part of the definition of κ_j, this slope is equal to $\frac{s_j}{\kappa_j}$. If $Q(j-1) > 0$ then

$$f^{S_{j-1}-1}|(c^f, z^f_{j-1}) = f^{S_{Q^2(j-1)}} \circ f^{S_{Q(j-1)}-1} \circ f^{S_{j-2}-1}|_{(c^f, z^f_{j-1})}.$$

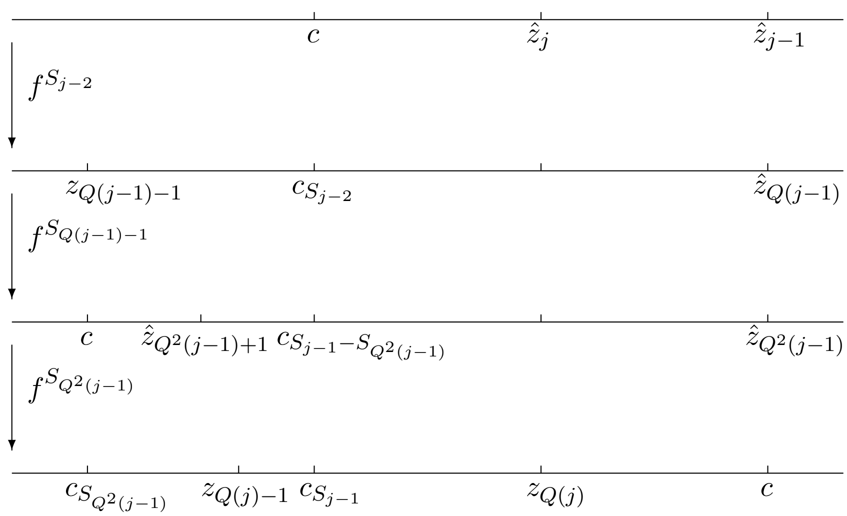

FIGURE 2. Position of various precritical points and their images.

By (14), $f^{S_{Q(j-1)-1}}|_{W_{Q(j-1)}}$ is linear with slope $s_{Q(j-1)}$. Hence $f^{S_{Q(j-1)-1}} \circ f^{S_{j-2}-1}|_{(c^f, z^f_{j-1})}$ is linear with slope $s_{Q(j-1)} \frac{s_{j-1}}{\kappa_{j-1}}$. By (15), its image is the interval

$$(z_{Q^2(j-1)}, c_{S_{j-2}+S_{Q(j-1)-1}}) = (z_{Q^2(j-1)}, c_{S_{j-1}-S_{Q^2(j-1)}}).$$

By (13), the length of this interval is $|c^f - z^f_{j-1}| s_{Q(j-1)} \frac{s_{j-1}}{\kappa_{j-1}} \leqslant \varepsilon_{Q^2(j-1)+1}$, so

$$(z_{Q^2(j-1)}, c_{S_{j-1}-S_{Q^2(j-1)}}) \subset W_{Q^2(j-1)+1} \text{ or } \hat{W}_{Q^2(j-1)+1}.$$

By (14), $f^{S_{Q^2(j-1)}}|_{W_{Q^2(j-1)+1}}$ is also linear, with slope $s_{Q^2(j-1)+1}$. It follows that $f^{S_{Q^2(j-1)}} \circ f^{S_{Q(j-1)-1}} \circ f^{S_{j-2}-1}|_{(c^f, z^f_{j-1})}$ is linear with slope $s_{Q^2(j-1)+1} s_{Q(j-1)} \frac{s_{j-1}}{\kappa_{j-1}}$. The second part of (10) gives that $f^{S_{j-1}-1}|(c^f, z^f_j)$ is linear with slope $\frac{s_j}{\kappa_j}$, as asserted. By (12), the length of the image is $|c^f - z^f_j| \frac{s_j}{\kappa_j} \leqslant \varepsilon_{Q(j)}$. Formula (15) yields $f^{S_{j-1}}(z_j) = z_{Q(j)}$. Hence we obtain

$$z_{Q(j)} > f^{S_{j-1}}(c) \geqslant z_{Q(j)} - \varepsilon_{Q(j)}$$

or

$$\hat{z}_{Q(j)} < f^{S_{j-1}}(c) \leqslant \hat{z}_{Q(j)} + \varepsilon_{Q(j)}.$$

In other words, $f^{S_{j-1}}(c) \in W_{Q(j)}$ or $\hat{W}_{Q(j)}$. This concludes the induction. (Notice that $\frac{|c_{S_{j-1}} - z_{Q(j)}|}{|z_{Q(j)-1} - z_{Q(j)}|} = \frac{1}{\varepsilon_{Q(j)}} \frac{s_j}{\kappa_j} |c^f - z^f_{j-1}|$.) □

3. The Fibonacci case

In this section we prove Theorem A. Let $\varphi_n(x) = j$ if $F^n(x) \in W_j \cup \hat{W}_j$. With respect to the existence of wild attractors and the random walk generated by F, we are in particular interested in the conditional expectation (also called *drift*)

$$\mathbb{E}(\varphi_n - k \mid \varphi_{n-1} = k) = \frac{\sum_{i \geqslant Q(k)+1} (i-k)\varepsilon_i}{\sum_{i \geqslant Q(k)+1} \varepsilon_i} = \frac{\sum_{i \geqslant Q(k)+1} i\varepsilon_i}{\sum_{i \geqslant Q(k)+1} \varepsilon_i} - k. \tag{16}$$

Drift in the setting of Fibonacci maps seems to be used first in [KN]. Note that here that the expectation is with respect to Lebesgue measure.

Proof of Theorem A. We attempt to solve the problem for $\varepsilon_j = |W_j| = |\hat{W}_j| = \frac{1-\lambda}{2}\lambda^j$, so $\sum_{j \geqslant 0} \varepsilon_j = \frac{1}{2}$. By formula (7),

$$\begin{cases} s_1 = \pm \frac{1}{1-\lambda} \\ s_j = \pm \frac{1}{\lambda(1-\lambda)} \text{ for } j \geqslant 2. \end{cases}$$

(Note that the slopes $s_j \geqslant 4$, with the minimum assumed at $\lambda = \frac{1}{2}$.) Using (10), we obtain for the slope $\kappa_j = f'(x)$, $x \in W_j$.

$$\kappa_j = \begin{cases} \frac{1}{1-\lambda} & j = 0, 1; \\ \frac{1}{\lambda} & j = 2; \\ \frac{(1-\lambda)}{\lambda} & j = 3; \\ \frac{(1-\lambda)^3}{\lambda} & j = 4; \\ \frac{\lambda^{2j}(1-\lambda)^{2j}}{\lambda^{10}(1-\lambda)^5} & j \geqslant 5. \end{cases} \tag{17}$$

Let us first check (12) and (13) For simplicity, write $\varepsilon_j = C_1\lambda^j$ and $\kappa_j = C_2\omega^j$ where $\omega = \lambda^2(1-\lambda)^2$. Then

$$\begin{aligned} \frac{s_j}{\kappa_j} \sum_{i=j+1}^{\infty} \kappa_i \varepsilon_i \leqslant \varepsilon_{Q(j)} \quad &\Leftrightarrow \quad \sum_{i=j+1}^{\infty} C_1 C_2 (\lambda\omega)^i \frac{1}{\lambda(1-\lambda)} \frac{1}{C_2 \omega^j} \leqslant C_1 \lambda^{j-2} \\ &\Leftrightarrow \quad \frac{\lambda^{j+1}\omega^{j+1}}{1-\lambda\omega} \frac{1}{\lambda(1-\lambda)\omega^j} \leqslant \lambda^{j-2} \\ &\Leftrightarrow \quad \lambda^4(1-\lambda) \leqslant 1 - \lambda^3(1-\lambda)^2. \end{aligned}$$

This is true for every $\lambda \in (0,1)$. Checking (13) for $Q(j) > 0$, we get

$$\begin{aligned} \frac{s_j}{\kappa_j} \sum_{i=j+1}^{\infty} \kappa_i \varepsilon_i \leqslant \frac{\varepsilon_{Q^2(j)+1}}{s_{Q(j)}} \quad &\Leftrightarrow \quad \frac{\lambda^{j+1}\omega^{j+1}}{1-\lambda\omega} \frac{1}{\lambda(1-\lambda)\omega^j} \leqslant \lambda^{j-3}\lambda(1-\lambda) \\ &\Leftrightarrow \quad \lambda^4 \leqslant 1 - \lambda^3(1-\lambda)^2. \end{aligned}$$

Again, this is true for all $\lambda \in (0,1)$.

Let us compute the order ℓ of the critical point. Indeed, $|Df(x)| = O(\lambda^{2j}(1-\lambda)^{2j})$ and $|x - c| = O(\lambda^j)$ if $x \in W_j$. On the other hand $|Df(x)| = O(|x-c|^{\ell-1})$. Therefore

$$\ell = 1 + \frac{\log\omega}{\log\lambda} = 3 + \frac{2\log(1-\lambda)}{\log\lambda}.$$

Consider (16) again. For $k \geqslant 2$, the *drift* is

$$Dr(\lambda) := \mathbb{E}(\varphi_n - k \mid \varphi_{n-1} = k) = \frac{\sum_{i \geqslant k-1} i\varepsilon_i}{\sum_{i \geqslant k-1} \varepsilon_i} - k = \frac{\lambda}{(1-\lambda)} - 1 = \frac{2\lambda - 1}{1-\lambda}.$$

Hence $\mathbb{E}(\varphi_n - k \mid \varphi_{n-1} = k) > 0$ if $\lambda > 1 - \lambda$, *i.e.,* $\lambda > \frac{1}{2}$. The second moment

$$\frac{\sum_{i \geqslant Q(k)+1} (i-k)^2 \varepsilon_i}{\sum_{i \geqslant Q(k)+1} \varepsilon_i} = \frac{\lambda^2}{(1-\lambda)^2} - 2\frac{\lambda}{1-\lambda} + 1$$

is uniformly bounded, and therefore also the variance. So as in the proof of [BT3, Theorem 1], for $\lambda > \frac{1}{2}$, *i.e.,* a critical order larger than 5, the Fibonacci map f exhibits a wild attractor.

Now we will calculate for what values of λ, f has an infinite σ-finite measure. First take $\lambda < \frac{1}{2}$. Then F (considered as a Markov process) is recurrent, and therefore has an invariant probability measure μ. Let $(A_{i,j})_{i,j}$ be the transition matrix corresponding to F, and let $(v_i)_i$ be the invariant probability vector, *i.e.,* left eigenvector with eigenvalue 1. As F is a Markov map, and F is linear on each state W_k, we obtain $\mu(W_k) = v_k$. So let us calculate this.

$$A_{i,j} = \begin{cases} 0 & \text{if } j \leqslant Q(i), \\ (1-\lambda)\lambda^{j-(Q(i)+1)} & \text{if } j > Q(i), \end{cases}$$

or in matrix form

$$(A_{i,j})_{i,j} = (1-\lambda) \begin{pmatrix} 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \dots & \dots \\ 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \dots & \dots \\ 0 & 1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \dots \\ 0 & 0 & 1 & \lambda & \lambda^2 & \lambda^3 & \dots \\ \vdots & \vdots & 0 & 1 & \lambda & \lambda^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{18}$$

As in [BT3, Theorem 1], this matrix has a unique normalised eigenvector:

$$v_i = \frac{1-2\lambda}{\lambda} \left(\frac{\lambda}{1-\lambda} \right)^i \text{ for } \lambda < \frac{1}{2}. \tag{19}$$

According to [B2, Theorem 2.6], f has a finite measure if and only if

$$\sum_k S_{k-1}\mu(W_k) < \infty. \tag{20}$$

If (20) fails, then f has an absolutely continuous σ-finite measure. This follows because f is conservative, and $\omega(c)$ is a Cantor set [HK]. In the Fibonacci case $S_{k-1} \sim \gamma^{k-1}$, where $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden mean. Since $\mu(W_k) = \beta_i \rho^i$ for $\beta_i \equiv \lambda$, as we saw above, we obtain $\rho > \frac{1}{\gamma}$ if and only if $\frac{1+\sqrt{5}}{2}\frac{\lambda}{1-\lambda} > 1$, *i.e.*, $\lambda > \frac{2}{3+\sqrt{5}}$. This corresponds to the critical order $\ell = 4$. Therefore there exists a σ-finite measure for all $\frac{2}{3+\sqrt{5}} \leqslant \lambda < \frac{1}{2}$, and a finite measure for $0 < \lambda < \frac{2}{3+\sqrt{5}}$. □

Remark 1. *Since $c_{S_k} \in W_{k-1} \cup \hat{W}_{k-1}$ for every $k \geqslant 1$, we obtain $|Df^{S_{Q(k+1)}}(c_{S_k})| = |Df^{S_{k-1}}(c_{S_k})| = \left|Df^{S_{k-1}}|_{W_k}\right| \cdot \left|Df^{S_{k-3}}|_{W_{k-2}}\right| = [\lambda(1-\lambda)]^{-2}$. Therefore $|Df^{S_j}(c_1| \approx \kappa_j[\lambda(1-\lambda)]^{2j} = \lambda^{-10}(1-\lambda)^{-5}$, which is uniformly bounded in j. Therefore the Nowicki-van Strien summability condition (see* [NS]*) fails for all $\lambda \in (0,1)$.*

Remark 2. *As proved in* [BT3, Theorem B]*, F_λ (or equivalently T_λ) is null recurrent w.r.t. Lebesgue when $\lambda = \frac{1}{2}$.*

4. An example of a wild attractor for $k - Q(k)$ unbounded

In [B2] it was shown that smooth unimodal maps for which $k - Q(k)$ is unbounded cannot have any wild attractors, for any large but finite value of the critical order. There are very few results known for unimodal maps with flat critical points (*i.e.*, $\ell = \infty$), although we mention [BM, Z1] and [LS], which deal with Lebesgue conservative Misiurewicz maps and infinitely renormalisable dynamics respectively. The next example serves as a model for a unimodal map with infinite critical order, suggesting that [B2, Theorem 8.1] doesn't hold anymore: There exists countably piecewise linear maps with kneading map $Q(k) = \lfloor rk \rfloor$, $r \in (0,1)$ that have a wild attractor.

Example 1: Consider maps with kneading map

$$Q(k) = \lfloor rk \rfloor$$

for some $r \in (0,1)$ and k large. Here $\lfloor x \rfloor$ indicates the integer part of x. Since Q is non-decreasing, (5) holds and unimodal maps with this kneading map indeed exist.

Let α be such that $\frac{1}{\alpha-1} + \log r > 0$. Take $\varepsilon_k = Ck^{-\alpha}$, where C is the appropriate normalising constant: $C \approx \alpha - 1$. This suffices to compute the expectation from (20), at least for large values of k. But instead of φ_n, we prefer to look at $\log \varphi_n$. It is clear that $\log \varphi_n(x) \to \infty$ if and only if $\varphi_n(x) \to \infty$. So it will have the same consequences. The advantage is that in this way we can keep the second moment bounded.

We will calculate the expectation for large values of k. Therefore we will write rk for $Q(k) + i = \lfloor rk \rfloor + i$ and r^2k for $Q^2(k) + i = \lfloor rk \rfloor + i$, where $i \in \{-1, 0, 1, 2\}$. We will also pass to integrals to simplify the calculations.

$$\begin{aligned}\mathbb{E}(\log\varphi_n - \log k \mid \varphi_{n-1} = k) &= \frac{\sum_{i\geqslant Q(k)+1}\varepsilon_i \log i}{\sum_{i\geqslant Q(k)+1}\varepsilon_i} - \log k \\ &\approx \frac{\int_{rk}^{\infty} t^{-\alpha}\log t dt}{\int_{rk}^{\infty} t^{-\alpha} dt} - \log k \\ &= \frac{1}{\alpha-1} + \log r.\end{aligned}$$

This is positive by the choice of α. For the second moment we get

$$\begin{aligned}\mathbb{E}((\log\varphi_n - \log k)^2 \mid \varphi_{n-1} = k) &= \frac{\sum_{i\geqslant Q(k)+1}(\log i - \log k)^2\varepsilon_i}{\sum_{i\geqslant Q(k)+2}\varepsilon_i} \\ \approx \frac{\int_{rk}^{\infty} t^{-\alpha}(\log t - \log k)^2 dt}{\int_{rk}^{\infty} t^{-\alpha} dt} &= \log^2 r + \frac{2}{\alpha-1}\log r + \frac{2}{(\alpha-1)^2}.\end{aligned}$$

which is uniformly bounded in k. Therefore, the induced map has drift to c, and thus is Lebesgue dissipative.

For the slopes of the induced map, and the original map we get the following:

$$s_j = \frac{1}{\varepsilon_j}\sum_{i\geqslant Q(j)+1}\varepsilon_i \approx j^\alpha \int_{rj}^{\infty} t^{-\alpha} dt = j\frac{r^{1-\alpha}}{\alpha-1},$$

whence

$$\kappa_j \approx \frac{\kappa_{j-1}}{s_{j-1}}\frac{s_j}{s_{[rj]}s_{[r^2j]}} \approx \kappa_{j-1}\frac{(\alpha-1)^2}{r^{5-2\alpha}}\frac{1}{j^2} = O(B^j(j!)^{-2})$$

for $B = \frac{(\alpha-1)^2}{r^{5-2\alpha}}$. Next we check conditions (12) and (13). Because $\frac{\varepsilon_{Q^2(j)+1}}{s_{Q(j)}} \leqslant \varepsilon_{Q(j)}$, it suffices to check (13). For j sufficiently large,

$$\begin{aligned}\frac{s_j}{\kappa_j}\sum_{i=j+1}^{\infty}\kappa_i\varepsilon_i &\approx j\frac{r^{1-\alpha}}{\alpha-1}\frac{(j!)^2}{B^j}\left(\frac{B^{j+1}}{(j+1)!^2}C(j+1)^{-\alpha} + \frac{B^{j+2}}{(j+2)!^2}C(j+2)^{-\alpha} + \cdots\right) \\ &\leqslant \frac{r^{1-\alpha}}{\alpha-1}CB(j+1)^{-\alpha-1}\cdot r^{2-2\alpha} \\ &< C(\alpha-1)r^{-\alpha-2}j^{-\alpha-1} \approx \frac{\varepsilon_{Q^2(j)+1}}{s_{Q(j)}}\end{aligned}$$

Hence, asymptotically there are no restrictions to build a piecewise linear map for this kneading map.

The critical order of this map is infinite. Indeed, the slope on (z_{j-1}, z_j) is $\kappa_j \approx \frac{B^j}{(j!)^2}$. $|c - z_{j-1}| = \sum_{i=j}^{\infty}\varepsilon_j \approx (\alpha-1)\int_j^{\infty} t^{-\alpha}dt = j^{1-\alpha}$. So the critical order ℓ must satisfy

$$\ell j^{(1-\alpha)(\ell-1)} = O\left(\frac{B^j}{(j!)^2}\right).$$

This is impossible for finite ℓ.

5. Projecting thermodynamic formalism to the original system

In order to understand the thermodynamic properties of our systems (I, f_λ) and (Y, F_λ) more deeply, we need the definition of conformal measure. Since we want to use this notion for both of these systems, we define it for general dynamical systems and potentials which preserve the Borel structure (so we implicitly assume our phase space is a topological space).

Definition 1. *Suppose that $g : X \to X$ is a dynamical system and $\phi : X \to [-\infty, \infty]$ is a potential, both preserving the Borel structure. Then a measure m on X is called* ϕ-conformal *if for any measurable set $A \subset X$ on which $g : A \to g(A)$ is a bijection,*

$$m(g(A)) = \int_A e^{-\phi}\, dm.$$

For the geometric potential $\phi_t = -t \log |Df_\lambda|$ of the original system (I, f_λ), we want to determine for which potential shift there is a $(\phi_t - p)$-conformal measure, and potentially an invariant measure equivalent to it. For a general potential ϕ for (I, f_λ), the *induced potential* is defined as

$$\Phi(x) = \sum_{j=0}^{\tau(x)-1} \phi \circ f_\lambda(x),$$

and hence it contains the inducing time in a fundamental way. Even if ϕ is constant (or shifted by a constant amount p), the induced potential is no longer constant (and shifted by τp). More concretely, for potential $\phi_t - p$, the induced potential is $-t \log |F_\lambda'| - \tau p$, where τp is the shift by the scaled inducing time $\tau_i = S_{i-1}$ on $W_i \cup \hat{W}_i$. In Lemma 1 below we prove the connection between a $(\phi_t - p)$-conformal measure for (I, f_λ) and a $(\Phi_t - p\tau)$-conformal measure for (Y, F_λ).

For $n \geqslant 1$ we define the set of *n-cylinders* for F_λ to be the collection of maximal intervals on which F_λ^n is a homeomorphism. It is natural to denote such an n-cylinder by $C_{i_0 \dots i_{n-1}}$, if for each $0 \leqslant k \leqslant n-1$, $F_\lambda^k(C_{i_0 \dots i_{n-1}}) \subset W_{i_k}$ or $F_\lambda^k(C_{i_0 \dots i_{n-1}}) \subset \hat{W}_{i_k}$. The sequence $i_0 \cdots i_{n-1}$ is called the *address* of the n-cylinder. Observe that for each such address there are two n-cylinders: we denote the one to the left of c by $C_{i_0 \cdots i_{n-1}}$ and that on the right by $\hat{C}_{i_0 \cdots i_{n-1}}$, and let $(C \cup \hat{C})_{i_0 \dots i_{n-1}}$ be the union of these. Only certain sequences $i_0 \cdots i_{n-1}$ can be realised as addresses, specifically we require $i_k \leqslant i_{k-1} + 1$ for $1 \leqslant k \leqslant n-1$; we call such addresses *admissible*. Notice that for any $x \in C_{i_0 \dots i_{n-1}}$, $\tau^n(x) = S_{i_0} + \cdots + S_{i_{n-1}}$. Clearly cylinder sets can be defined analogously (without the ambiguity in address) for the map T_λ.

As usual, the original system (I, f) can be connected to the induced system (Y, F) via an intermediate tower construction, say (Δ, f_Δ), defined as follows: The space is the disjoint union

$$\Delta = \bigsqcup_i \bigsqcup_{l=0}^{\tau_i - 1} \Delta_{i,l},$$

where $\Delta_{i,l}$ are copies of W_i and $\hat{W}_i$, and the inducing time $\tau_i = \tau|_{W_i \cup \hat{W}_i} = S_{i-1}$. Points in $\Delta_{i,l}$ are of the form (x, l) where $x \in W_i \cup \hat{W}_i$. The map $f_\Delta : \Delta \to \Delta$ is defined at $(x, l) \in \Delta_{i,l}$ as

$$f_\Delta(x,l) = \begin{cases} (x, l+1) \in \Delta_{i,l+1} & \text{if } l < \tau_i - 1; \\ (F(x), 0) = (0, f^{S_{i-1}}(x)) \in \sqcup_i \Delta_{i,0} & \text{if } l = \tau_i - 1. \end{cases}$$

The projection $\pi : \Delta \to I$, defined by $\pi(x, l) = f^l(x)$ for $(x, l) \in \Delta_{i,l}$, semiconjugates this map to the original system: $\pi \circ f_\Delta = f \circ \pi$. Furthermore, the induced map (Y, F) is isomorphic to the first return map to the *base* $\Delta_0 = \sqcup_i \Delta_{i,0}$.

Lemma 1. *Let Φ_t be the induced potential of ϕ_t, and p be a potential shift.*

(a) A (ϕ_t, p)-conformal measure n_t for (I, f) yields a $(\Phi_t, \tau p)$-conformal measure m_t for (Y, F) by restricting and normalising:

$$m_t(A) = \frac{1}{n_t(Y)} n_t(A) \quad \textit{for every } A \subset Y := \cup_{i \geqslant 1}(W_i \cup \hat{W}_i).$$

(b) A $(\Phi_t, \tau p)$-conformal measure m_t for (Y, F) projects to a (ϕ_t, p)-conformal measure n_t for (I, f): for every i, l and $A \subset W_i$ or $A \subset \hat{W}_i$,

$$n_t(\pi(A, l)) = \frac{1}{M} \int_A \exp\left(lp + \sum_{j=0}^{l-1} \phi_t \circ f^j \right) dm_t,$$

see Figure 3, with normalising constant

$$M := 1 + e^p \sum_{i \geqslant 2} \int_{W_i} e^{-\phi_t} dm_t + e^{2p} \sum_{i \geqslant 3} \int_{W_i} e^{-\phi_t \circ f - \phi_t} dm_t \geqslant 1$$

is (ϕ_t, p)-conformal.

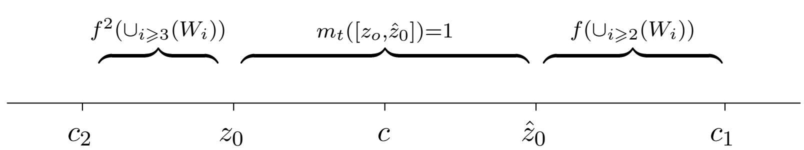

FIGURE 3. Distribution of the conformal mass n_t on $[c_2, c_1]$

In the case that $\phi_t = -t \log |f'|$, then the formula for the normalising constant simplifies to $M = 1 + e^p \sum_{i \geqslant 2} w_i^t \kappa_i^t + e^{2p} \sum_{i \geqslant 3} w_i^t \kappa_i^t \kappa_0^t$ which is finite for all $\lambda \in (0, 1)$, $t > 0$ and $p \in \mathbb{R}$.

(c) The invariant measure μ_t for (Y, F, Φ_t) projects to an invariant measure ν_t provided $\sum_i \tau_i \mu_t(W_i \cup \hat{W}_i) < \infty$ (where in fact $\tau_i = S_{i-1}$), using the formula

$$\nu_t = \frac{1}{\Lambda} \sum_i \sum_{j=0}^{\tau_i - 1} f_*^j \mu_t \quad \textit{for } \Lambda = \sum_i \tau_i \mu_t(W_i \cup \hat{W}_i).$$

Moreover,

$$h(\nu_t) = \frac{h(\mu_t)}{\Lambda} \quad \text{and} \quad \int g \, d\nu_t = \frac{\int G \, d\mu_t}{\Lambda},$$

for any measurable potential g on I and its induced version G on Y.

Remark 3. *Note that the last part of this lemma is just an application of the* Abramov formula*, see for example* [PS, Theorem 2.3] *and* [Z2, Theorem 5.1].

Proof. (a) If n_t is (ϕ_t, p)-conformal for (I, f), it means, as stated in Definition 1, that $n_t(f(A)) = \int_A e^{-\phi_t + p} dn_t$ whenever $f : A \to f(A)$ is one-to-one. Taking $A \subset W_i$ (or $\subset \hat{W}_i$), and applying the above $\tau_i = S_{i-1}$ times gives that $n_t(F(A)) = \int_A e^{-\Phi_t + \tau_i p} dn_t$, so the normalised restriction $m_t = \frac{1}{n_t(Y)} n_t$ is indeed $(\Phi_t, \tau p)$-conformal.

(b) For the second statement, it is straightforward from the definition that if $A \subset W_i$ or $A \subset \hat{W}_i$ and $0 \leqslant l < \tau_i - 1$, then for $B = \pi(A, l)$,

$$\begin{aligned} n_t(f(B)) &= n_t(\pi(A, l+1)) \\ &= \frac{1}{M} \int_A \exp\left((l+1)p - \sum_{j=0}^{l} \phi_t \circ f^j\right) dm_t \\ &= \frac{1}{M} \int_A e^{\phi_t \circ f^l + p} \exp\left(lp - \sum_{j=0}^{l-1} \phi_t \circ f^j\right) dm_t \\ &= \int_B e^{-\phi_t + p} dn_t. \end{aligned}$$

Similarly, if $l = \tau_i - 1$, then

$$\begin{aligned} n_t(f(B)) &= n(F(A)) = \frac{1}{\Lambda} \int_A \exp\left(\tau_i p - \Phi_t\right) dm_t \\ &= \frac{1}{M} \int_A \exp\left(\sum_{j=0}^{\tau_i - 1} (-\phi_t \circ f^j + p)\right) dm_t \\ &= \frac{1}{M} \int_A e^{\phi_t \circ f^l + p} \exp\left(lp - \sum_{j=0}^{l-1} \phi_t \circ f^j\right) dm_t \\ &= \int_B e^{-\phi_t + p} dn_t \end{aligned}$$

This proves the (ϕ_t, p)-conformality. The tricky part is to show that n_t is actually well-defined. Assume that $B = \pi(A, l) = \pi(A', l')$ for two different sets $A \subset W_i$ and $A' \subset W_{i'}$. So we must show that the procedure above gives $n_t(\pi(A, l)) = n_t(\pi(A', l'))$. Assume also that $\tau_i - l \leqslant \tau_{i'} - l'$; then we might as well take B maximal with this property: $B = \pi(W_{i'}, l')$.

Now $B' := f^{\tau_i - l}(B) \subset F(W_i) = (z_{Q(i)}, c)$ or $(c, \hat{z}_{Q(i)})$. It is important to note that the induced map F is not a first return map to a certain region, but $F|_{W_i \cup \hat{W}_i} = f^{S_{i-1}}|_{W_i \cup \hat{W}_i}$ is the first return map to $(z_{Q(i)}, \hat{z}_{Q(i)})$. Since $f^{\tau_{i'} - \tau_i}$ maps B' to $F(W_{i'}) = (z_{Q(i')}, c)$ or

$(c, \hat{z}_{Q(i')})$, the iterate $f^{\tau_{i'}-\tau_i}|'_B$ can be decomposed into an integer number, say $k \geqslant 0$, of applications of F, and B' is in fact a k-cylinder for the induced map. Since m_t is $(\Phi_t, \tau p)$-conformal,

$$m_t(B') = \int_{F^k(B')} \exp\left(\sum_{j=0}^{k-1} (\Phi_t - \tau p) \circ F^{-j}\right) dm_t.$$

Taking an extra $\tau_i - l$ steps backward, we get

$$\begin{aligned} n_t(B) &= \frac{1}{M} \int_{B'} \exp\left(\sum_{j=1}^{\tau_i - l} (\phi_t - p) \circ f^{j-(\tau_i - l)}\right) dm_t \\ &= \frac{1}{M} \int_{F^k(B')} \exp\left(\sum_{j=0}^{k-1} (\Phi_t - \tau p) \circ F^{-j}\right) \exp\left(\sum_{j=1}^{\tau_i - l} (\phi_t - p) \circ f^{j-(\tau_i - l) \circ F^{-k}}\right) dm_t \\ &= \frac{1}{M} \int_{F(W_{i'})} \exp\left(- \sum_{j=1}^{\tau_{i'} - l'} (\phi_t - p) \circ f^{j-(\tau_{i'} - l')}\right) dm_t, \end{aligned}$$

so computing $n_t(B)$ using $\tau_i - l$ or $\tau_{i'} - l'$ both give the same answer.

Now for the normalising constant, since our method of projecting conformal measure only takes the measure of one of the preimages of π in Δ of any set $A \subset I$, we do not sum over all levels of the tower, but just enough so that the image by π covers I, up to a zero measure set. However, modulo a countable set, the core $[c_2, c_1]$ is disjointly covered by $\bigcup_{i \geqslant 1} (W_i \cup \hat{W}_i) \cup \bigcup_{i \geqslant 2} f(W_i) \cup \bigcup_{i \geqslant 2} f^2(W_i)$. This gives

$$\begin{aligned} M &= \sum_{i \geqslant 1} m_t(W_i \cup W_i) + \sum_{i \geqslant 2} \int_{W_i} e^{p - \phi_t} dm_t + \sum_{i \geqslant 3} \int_{W_i} e^{2p - \phi_t - \phi_t \circ f} dm_t \\ &= 1 + \sum_{i \geqslant 2} e^p \int_{W_i} e^{-\phi_t} dm_t + e^{2p} \sum_{i \geqslant 3} \int_{W_i} e^{-\phi_t - \phi_t \circ f} dm_t. \end{aligned}$$

for an arbitrary potential. Using the formulas for the slope $\kappa_i = f'|W_i$ from (17) and the expressing for $\frac{1}{2} w_i = m_t(W_i)$ from (21), we obtain for $\phi_t = -t \log |f'|$:

$$\begin{aligned} M &= 1 + \frac{e^p}{2} \sum_{i \geqslant 2} w_i^t \kappa_i^t + \frac{e^{2p}}{2} \sum_{i \geqslant 3} w_i \kappa_i^t \kappa_0^t \\ &= 1 + \frac{e^p (1 - \lambda^t)}{2} \left(1 + (1-\lambda)^t \lambda^t + (1-\lambda)^{3t} \lambda^{2t} + \sum_{i \geqslant 5} \frac{\lambda^{3ti} (1-\lambda)^{2ti}}{\lambda^{11t} (1-\lambda)^{5t}}\right) \\ &\quad + \frac{e^{2p} (1 - \lambda^t)}{2(1-\lambda)^t} \left((1-\lambda)^t \lambda^t + (1-\lambda)^{3t} \lambda^{2t} + \sum_{i \geqslant 5} \frac{\lambda^{3ti} (1-\lambda)^{2ti}}{\lambda^{11t} (1-\lambda)^{5t}}\right) \\ &= 1 + \frac{e^p (1 - \lambda^t)}{2} \left(1 + (1-\lambda)^t \lambda^t + (1-\lambda)^{3t} \lambda^{2t} + \frac{\lambda^{4t} (1-\lambda)^{5t}}{1 - \lambda^{3t} (1-\lambda)^{3t}}\right) \\ &\quad + \frac{e^{2p} (1 - \lambda^t) \lambda^t}{2} \left(1 + (1-\lambda)^{2t} \lambda^t + \frac{\lambda^{3t} (1-\lambda)^{4t}}{1 - \lambda^{3t} (1-\lambda)^{3t}}\right) < \infty. \end{aligned}$$

(c) The third statement is an Abramov formula, see Remark 3. $\square$

6. The conformal measure and equilibrium state for (Y, F_λ, Φ_t)

In this section we adapt the results for the map T_λ studied in [BT3] to the map F_λ. This also allows us to prove Theorem B.

Proposition 2. *For each $\lambda \in (0,1)$, $t > 0$ and $p = P(\Phi_t)$, the map F_λ has a $(\Phi_t - p)$-conformal measure $\tilde{m}_t$ with*

$$\tilde{m}_t(W_j) = \tilde{m}_t(\hat{W}_j) = \begin{cases} \frac{1-\lambda^t}{2}\lambda^{t(k-1)} & \text{if } \lambda^t \leqslant \frac{1}{2}, \\ \left[(k-1) + \lambda^{-t}(1-\frac{k}{2})\right](\frac{1}{2})^{k+1} & \text{if } \lambda^t \geqslant \frac{1}{2}. \end{cases} \tag{21}$$

If in addition $\lambda^t < \frac{1}{2}$, then F_λ preserves a probability measure $\tilde{\mu}_t \ll \tilde{m}_t$ with

$$\tilde{\mu}_t(W_j) = \zeta_t \frac{1-2\lambda^t}{\lambda^t}\left(\frac{\lambda^t}{1-\lambda^t}\right)^j \quad \text{and} \quad \tilde{\mu}_t(\hat{W}_j) = (1-\zeta_t)\frac{1-2\lambda^t}{\lambda^t}\left(\frac{\lambda^t}{1-\lambda^t}\right)^j \tag{22}$$

for some $\zeta_t \in (0,1)$. Moreover, $\tilde{\mu}_t$ is an equilibrium state for potential Φ_t.

Proof. Recall from (2) and (3) that $T_\lambda \circ \pi = \pi \circ F_\lambda$ for the two-to-one factor map π with $\pi^{-1}(V_j) = W_j \cup \hat{W}_j$. In [BT3, Theorem 2] it is shown that T_λ has a $(\Phi_t - p)$-conformal measure such that for $\psi(t) := \frac{(1-\lambda)^t}{1-\lambda^t}$,

$$m_{t,p}(V_k) = \begin{cases} (1-\lambda^t)\lambda^{t(k-1)} & \text{if } p = \log\psi(t) \text{ and } \lambda^t \leqslant \frac{1}{2}, \\ \left[(k-1) + \lambda^{-t}(1-\frac{k}{2})\right](\frac{1}{2})^k & \text{if } p = \log 4[\lambda(1-\lambda)]^t \text{ and } \lambda^t \geqslant \frac{1}{2}. \end{cases}$$

and an invariant measure (provided $\lambda^t < \frac{1}{2}$) with $\mu_{t,p}(V_k) = \frac{1-2\lambda^t}{\lambda^t}\left(\frac{\lambda^t}{1-\lambda^t}\right)^k$. To obtain $\tilde{m}_t$ and $\tilde{\mu}_t$ we lift these measures by π, distributing the mass to W_j and $\hat{W}_j$ appropriately. Since $F_\lambda(W_j) = F_\lambda(\hat{W}_j) = \cup_{k \geqslant j-1} W_k$ or $\cup_{k\geqslant j-1}\hat{W}_k$, we can distribute the conformal mass evenly. This gives (21).

To obtain (22), first define

$$A^t = (1-\lambda)^t \begin{pmatrix} 1^t & \lambda^t & \lambda^{2t} & \lambda^{3t} & \dots & \dots & \dots \\ 1^t & \lambda^t & \lambda^{2t} & \lambda^{3t} & & & \\ 0 & 1^t & \lambda^t & \lambda^{2t} & \lambda^{3t} & & \\ 0 & 0 & 1^t & \lambda^t & \lambda^{2t} & & \\ \vdots & & 0 & 1^t & \lambda^t & \lambda^{2t} & \dots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{23}$$

That is, the matrix A in (18) with all entries raised to the power t, then $\psi^{-1}(t)A^t$ is a probability matrix and $\frac{m_{t,p}(V_i \cap T_\lambda^{-1}(V_j))}{m_{t,p}(V_i)} = \psi^{-1}(t)A^t_{i,j}$.

Now set $v_j^t = \frac{1-2\lambda^t}{\lambda^t}\left(\frac{\lambda^t}{1-\lambda^t}\right)^j$ (so for $t=1$, this reduces to the value of v_j in (19)) and define

$$\zeta_t = \sum_{\substack{j \geqslant 1 \\ c_{S_{j-1}} < c}} v_j^t,$$

i.e., the proportion of the invariant mass that maps under F_λ to the left of c. Next define μ_t on cylinders $C_{i_0 \cdots i_{n-1}}$ by

$$\mu_t(C_{i_0 \cdots i_{n-1}}) = \zeta_t v_{i_0}^t \prod_{k=1}^{n-1} \psi(t)^{-1} A_{i_{k-1} i_k}^t$$

and similarly $\mu_t(\hat{C}_{i_0 \cdots i_{n-1}}) = \hat{\zeta}_t v_{i_0}^t \prod_{k=1}^{n-1} \psi(t)^{-1} A_{i_{k-1} i_k}^t$.

Since $\psi(t)^{-1} A^t$ is a probability matrix with $A_{k i_0}^t = 0$ if $k > i_0 + 1$, we get for every cylinder set

$$\begin{aligned}
\tilde{\mu}_t(F_\lambda^{-1}(C_{i_0 \cdots i_{n-1}})) &= \sum_{\substack{k \leqslant i_0+1 \\ c_{S_{k-1}} < c}} \tilde{\mu}_t(C \cup \hat{C})_{k i_0 \cdots i_{n-1}} \\
&= \zeta_t \sum_{k \leqslant i_0+1} v_k^t \psi(t)^{-1} A_{k i_0}^t \prod_{j=1}^{n-1} \psi(t)^{-1} A_{i_{j-1} e_j}^t \\
&= \zeta_t v_{i_0}^t \prod_{k=1}^{n-1} \psi(t)^{-1} A_{i_{k-1} i_k}^t = \tilde{\mu}_t(C_{i_0 \cdots i_{n-1}})
\end{aligned}$$

and similarly for $F_\lambda^{-1}(\hat{C}_{e_0 \ldots e_{n-1}})$. This proves F_λ-invariance of $\tilde{\mu}_t$.

The T_λ-invariant measure above is the unique equilibrium state for $-t \log |T_\lambda'|$ provided $\lambda^t < \frac{1}{2}$. Since the factor map π does not affect entropy, and because for any F_λ-invariant measure $\tilde{\nu}$ we have $\int \log |F_\lambda'| d\tilde{\nu} = \int \log |T_\lambda'| d(\tilde{\nu} \circ \pi^{-1})$, it follows that $\tilde{\mu}_t$ is indeed the unique equilibrium state for $(Y, F_\lambda, -t \log |F_\lambda'|)$. □

Proof of Theorem B. Let $x \in [z_0, \hat{z}_0] \setminus \cup_{n \geqslant 0} f^{-n}(c)$ be arbitrary. Since z_k is a closest precritical point, $f^j(W_k \cup \hat{W}_k) \cap [z_k, \hat{z}_k] = \emptyset$ if $0 < j < S_k$. Therefore, if $c \in \omega(x)$ then $F^i(x) \to 0$ along a subsequence. From this we see that hyperbolic sets for F coincide with intersections of hyperbolic sets for f with $[z_0, \hat{z}_0]$, implying that hyperbolic dimension are the same for F and f.

Now for the escaping set, first observe that the intervals $f^j([z_k, c]) = f^j([c, \hat{z}_k])$ for $0 < j \leqslant S_k$ have $f^j(c)$ as boundary point and lengths tending to 0 as $k \to \infty$. Therefore $F^i(x) \to c$ implies that $f^n(x) \to \omega(c)$ which implies that $\omega(x) = \omega(c)$. We next show that $F^i(x) \to c$ if and only if $\omega(x) = \omega(c)$.

Denote by U_n the largest neighbourhood of x on which f^n is monotone, and let R_N be the largest distance between $f^n(x)$ and $\partial f^n(U_n)$. If there is k such that $F^i(x) \notin [z_k, \hat{z}_k]$ infinitely often, then by the Markov property of F, $f^n(U_n) \supset [z_k, c]$ or $[c, \hat{z}_k]$ along a subsequence. This means $R_n \not\to 0$. By [B2], this implies that $\omega(x) \not\subset \omega(c)$.

Therefore $\omega(x) = \omega(c)$ if and only if $F^i(x) \to c$, and hence the escaping set Ω_λ coincides with $\text{Bas}_\lambda \cap [z_0, \hat{z}_0]$. Theorem B therefore follows from [BT3, Theorem C]. □

7. Conformal pressure for (I, f, ϕ_t)

In this section we prove the main part of Theorem D, with the components about existence of conformal measure and upper and lower bounds on conformal pressure in various lemmas. We start by giving the definition of conformal pressure, presented for general dynamical systems.

Definition 2. *For a dynamical system $g : X \to X$ and a potential $\phi : X \to [-\infty, \infty]$, the* conformal pressure *for (X, g, ϕ) is*

$$P_{\text{Conf}}(\phi) := \inf\{p \in \mathbb{R} : \textit{there exists a } (\phi - p)\textit{-conformal measure}\}.$$

The results on the pressure in this section are obtained using $P_{\text{Conf}}(\phi_t)$; in Section 9 we show that the conformal pressure $P_{\text{Conf}}(\phi_t)$ coincides with the (variational) pressure $P(\phi_t)$ from (1). Thus our statements in Theorem D should be read as applying to 'both' quantities. For $P_{\text{Conf}}(\Phi_t)$, we start by quoting the conclusion of Theorems 2 and B of [BT3]: $P_{\text{Conf}}(\Phi_t)$ and $P(\Phi_t)$ coincide, and

$$P_{\text{Conf}}(\Phi_t) = \begin{cases} \log \psi(t) & \text{if } \lambda^t \leqslant \frac{1}{2}; \\ \log[4\lambda^t(1-\lambda)^t] & \text{if } \lambda^t \geqslant \frac{1}{2}. \end{cases}$$

Recall from (4) that $t_2 = -\log 4/\log[\lambda(1-\lambda)]$ is the value of t such that $[\lambda(1-\lambda)]^t = \frac{1}{4}$. Hence $t_2 = t_1$ if $\lambda \geqslant \frac{1}{2}$ and $t_2 < t_1 = 1$ otherwise. We can interpret t_1 as the smallest t such that the pressure of the induced system $P_{\text{Conf}}(\Phi_t) = 0$.

Any $(\Phi_t - p\tau)$-conformal measure m_t must observe the relations (for $\tilde{w}_k^t = m_t(W_k) = m_t(\hat{W}_k)$)

$$\begin{aligned} \tilde{w}_1^t &= (1-\lambda)^t e^{-pS_0} \\ \tilde{w}_2^t &= \lambda^t(1-\lambda)^t e^{-pS_1} \\ \tilde{w}_3^t &= \lambda^t(1-\lambda)^t e^{-pS_2}(1-\tilde{w}_1^t) \\ &\vdots \\ \tilde{w}_j^t &= \lambda^t(1-\lambda)^t e^{-pS_{j-1}}\left(1 - \sum_{k<j-1} \tilde{w}_k^t\right). \end{aligned} \tag{24}$$

Recurrence relations of a similar form were used in [BT3] to prove [BT3, Theorem 2], but our situation here is more complicated since in that setting in the place of each e^{-pS_j} term was simply the constant term $\psi(t)$. The idea now is to find a solution $p = p(t)$ of (24) such that also $H(p,t) := \sum_j \tilde{w}_j^t$ is equal to 1 (this is equivalent to finding a solution set $\{\tilde{w}_j^t\}_j$). Note that in Lemma 3 and Proposition 4 below, we give necessary lower and upper bounds on $p(t)$, without assuming the existence of a solution. Along the way, we will also need to check that $\tilde{w}_k^t > 0$ for all $k \geqslant 1$.

Write

$$\beta := t\log[\lambda(1-\lambda)] \quad \text{and} \quad \beta' := (t-t_2)\log[\lambda(1-\lambda)],$$

so that $e^\beta = \frac{1}{4}$ for $t = t_2$ and $e^\beta = \frac{1}{4}e^{\beta'} > \frac{1}{4}$ for $t < t_2$.

7.1. **Lower bounds on $P_{\mathrm{Conf}}(\phi_t)$.** Let us now compute the asymptotics of $\tilde{w}_k^t$ to show that in this case $p(t)$ has to be positive for $t < t_1$.

Lemma 2. *Fixing $p = 0$, there is a unique solution to (24), denoted by $(\bar{w}_k^t)_{k\in\mathbb{N}}$. It satisfies*

$$\begin{cases} \bar{w}_k^t > 0 \textit{ and } \sum_k \bar{w}_k^t = 1 & \textit{if } t \geqslant t_1; \\ \textit{there exists } k_0 \textit{ such that } \bar{w}_{k_0}^t < 0 & \textit{if } t < t_1. \end{cases}$$

Moreover, for $r_\pm = \frac{1}{2}(1 \pm \sqrt{1-4e^\beta})$,

$$\begin{cases} k_0 = \left\lceil \dfrac{\log \frac{r_+(r_+-\lambda^t)}{r_-(r_--\lambda^t)}}{\log \frac{r_+}{r_-}} \right\rceil + 1 & \textit{if } \ t_2 < t < t_1 = 1 \ (\text{i.e., } \lambda \in (0, \tfrac{1}{2})); \\ k_0 \approx \frac{2(1-\lambda^t)}{1-2\lambda^t} & \textit{if } t \lesssim t_2 \leqslant t_1 = 1 \textit{ for } \lambda \in (0, \tfrac{1}{2}], \\ k_0 \approx \frac{2\pi}{\sqrt{\beta'}} & \textit{if } t \lesssim t_1 \leqslant 1 \textit{ for } \lambda \in [\tfrac{1}{2}, 1). \end{cases}$$

Proof. Subtracting two successive equations in (24), we find that the $\bar{w}_k^t$ satisfy recursive relation

$$\bar{w}_{k+1}^t = \bar{w}_k^t - e^\beta \bar{w}_{k-1}^k.$$

The roots of the corresponding generating equation $r^2 - r + e^\beta = 0$ are $r_\pm = \frac{1\pm\sqrt{1-4e^\beta}}{2}$. It is straightforward to check that

$$\begin{cases} (i)\ r_- < \lambda^t, 1-\lambda^t < r_+ & \text{if } t > 1; \\ (ii)\ r_\pm \in \{\lambda, 1-\lambda\} & \text{if } t = 1; \\ (iii)\ \lambda^t < r_- < r_+ < 1-\lambda^t & \text{if } t_2 < t < 1 \text{ and } \lambda^t < \frac{1}{2}; \\ (iv)\ 1-\lambda^t < r_- < r_+ < \lambda^t & \text{if } t_1 < t < 1 \text{ and } \lambda^t > \frac{1}{2}; \\ (v)\ r_- = r_+ = \frac{1}{2} & \text{if } \begin{cases} t = t_1 < 1 & \text{for } \lambda^t > \frac{1}{2}, \\ t = t_2 & \text{for } \lambda^t \leqslant \frac{1}{2}, \end{cases} \\ (vi)\ r_\pm \text{ are complex conjugate} & \text{if } \begin{cases} t < t_1 < 1 & \text{for } \lambda^t > \frac{1}{2}, \\ t < t_2 \leqslant t_1 = 1 & \text{for } \lambda^t \leqslant \frac{1}{2}. \end{cases} \end{cases}$$

(i)-(iv) In the first four cases, *i.e.*, $r_\pm$ are real and distinct, the recursion combined with the initial values $\bar{w}_1^t = (1-\lambda)^t$ and $\bar{w}_2^t = \lambda^t(1-\lambda)^t$, give the solution

$$\bar{w}_k^t = \frac{(1-\lambda)^t}{\sqrt{1-4e^\beta}} \left[(\lambda^t - r_-)r_+^{k-1} + (r_+ - \lambda^t)r_-^{k-1} \right]. \tag{25}$$

If $t \geqslant 1$, then the coefficients are non-negative, and also if $t_1 < t < 1$. If $t_2 < t < t_1 = 1$, then the coefficient $\lambda^t - r_- < 0$, so there is k_0 such that $\bar{w}_k^t < 0$ for all $k \geqslant k_0$, namely

$$\frac{r_+^2}{r_-^2}\frac{r_+ - \lambda^t}{r_- - \lambda^t} \geqslant \left(\frac{r_+}{r_-}\right)^{k_0} > \frac{r_+}{r_-}\frac{r_+ - \lambda^t}{r_- - \lambda^t}, \tag{26}$$

which results in $k_0 = \left\lceil \frac{\log \frac{r_+(r_+ - \lambda^t)}{r_-(r_- - \lambda^t)}}{\log \frac{r_+}{r_-}} \right\rceil + 1$.

(v) If $t = t_1 < 1$, or when $t = t_2$, then $r_- = r_+ = \frac{1}{2}$, and the general solution is

$$\bar{w}_k^t = \frac{(1-\lambda)^t}{2^k}\ \left(4(1-\lambda^t) + 2k(2\lambda^t - 1)\right).$$

If $\lambda^t \geqslant \frac{1}{2}$ (*i.e.*, $t = t_1 \leqslant 1$), then the coefficient $4(1-\lambda^t) + 2k(2\lambda^t - 1) > 0$ and hence $\bar{w}_k^t > 0$ for all k. If $\lambda^t < \frac{1}{2}$, then the coefficient $4(1-\lambda^t) + 2k(2\lambda^t - 1) < 0$ for all $k \geqslant k_0 = \lceil 2(1-\lambda^t)/(1-2\lambda^t)\rceil + 1$.

(vi) Finally, if $t < t_1 < 1$, or in general when $t < t_2$, then the roots are complex. Together with the initial values $\bar{w}_1^t = (1-\lambda)^t$ and $\bar{w}_2^t = \lambda^t(1-\lambda)^t$, we find the solution

$$\begin{aligned}
\bar{w}_k^t \quad &= \quad \frac{(1-\lambda)^t}{2^k}\left[\left(4\cos\frac{\sqrt{\beta'}}{2} - 4\lambda^t\right)\cos\left(\frac{\sqrt{\beta'}k}{2}\right) + \right. \\
&\qquad \left.\left(4\lambda^t\cos\left(\frac{\sqrt{\beta'}}{2}\right) - 2\cos\left(\sqrt{\beta'}\right)\right)\left(\frac{\sin\frac{\sqrt{\beta'}k}{2}}{\sin\frac{\sqrt{\beta'}}{2}}\right)\right] \\
&= \quad \frac{2(1-\lambda)^t}{2^k}\left[(2\cos\theta - 2\lambda^t)\cos\theta k + (2\lambda^t\cos\theta - 2\cos^2\theta + 1)\frac{\sin\theta k}{\sin\theta}\right],
\end{aligned} \tag{27}$$

for $\theta = \frac{1}{2}\sqrt{\beta'} = \frac{1}{2}\sqrt{(t-t_2)\log[\lambda(1-\lambda)]}$. This is an oscillatory function in k, with an exponential decreasing coefficient 2^{-k}. Recall that $\bar{w}_2^t = \lambda^t\bar{w}_1^t > 0$. First assume that $\lambda \geqslant \frac{1}{2}$, whence $\lambda^t > \frac{1}{2}$. Therefore

$$0 < 2\cos\theta - 2\lambda^t \ll \frac{2\lambda^t\cos\theta - \cos^2\theta + 1}{\sin\theta},$$

so the expression in the square brackets becomes negative when $k_0 \approx \frac{2\pi}{\sqrt{\beta'}}$.

Now set $\lambda < \frac{1}{2}$, and $\lambda^t < \frac{1}{2}$ and moreover assume $t - t_2$ is small. Then $2\lambda^t\cos\theta - 2\cos^2\theta + 1 < 0$, so approximating $\sin\theta k/\sin\theta = k$ for small values of θ, we find the expression in the square brackets becomes negative when $k_0 > \frac{2\cos\theta - 2\lambda^t}{2\cos^2\theta - 1 - 2\lambda^t\cos\theta} \approx \frac{2(1-\lambda^t)}{1-2\lambda^t}$.

Note also that in all cases $\bar{w}_k^t \to 0$, and therefore (24) gives that $1 - \sum_{k<j-1}\bar{w}_k^t \to 0$ as $j \to \infty$. This shows that $\sum_k \bar{w}_k^t = 1$. $\square$

We can now use Lemma 2 to address directly the problem set up in (24): finding a solution $p = p(t)$ to $H(p,t) = 1$ with all summands non-negative.

Lemma 3. *If $\lambda \geqslant \frac{1}{2}$ and $t < t_1 \leqslant 1$ is close to t_1, or if $\lambda < \frac{1}{2}$ and λ^t is sufficiently close to $\frac{1}{2}$, then there is $\tau_0 = \tau_0(\lambda) > 0$ such that $p(t) > \frac{\tau_0}{S_{k_0}}$.*

Proof. Let $\bar{w}_k^t$ be the solution of (24) for $p = 0$ as computed in Lemma 2, while we write $\tilde{w}_k^t = \tilde{w}_k^t(p)$ for the case $p > 0$. We start by showing that, under the assumptions of the lemma, $\bar{w}_{k+1}^t / \bar{w}_k^t \approx \frac{1}{2}$ for $1 \leqslant k \leqslant k_0 - 10$.

- Case 1: $\lambda \geqslant \frac{1}{2}$ and $t < t_1 \leqslant 1$ is close to t_1. In this case, $2\lambda^t \cos\theta - 2\cos^2\theta + 1 = (2\lambda^t - 1)\cos\theta + (1 + 2\cos\theta)(1 - \cos\theta) > 0$ for $0 \leqslant \theta = \frac{1}{2}\sqrt{\beta'} \leqslant \pi/2$. With $\bar{w}_k^t$ as given by (27) and using standard trigonometric formulas, we derive that

$$\begin{aligned}
\frac{\bar{w}_{k+1}^t}{\bar{w}_k^t} &= \frac{1}{2}\left(\cos\theta - \sin\theta \frac{(2\cos\theta - 2\lambda^t)\sin\theta k - \frac{2\lambda^t\cos\theta - 2\cos^2\theta + 1}{\sin\theta}\cos\theta k}{(2\cos\theta - 2\lambda^t)\cos\theta k + \frac{2\lambda^t\cos\theta - 2\cos^2\theta + 1}{\sin\theta}\sin\theta k}\right) \\
&\sim \frac{1}{2}\left(\cos\theta + \frac{\sin\theta}{\tan\theta k}\right) \qquad \text{as } \theta \to 0.
\end{aligned}$$

If $10 \leqslant k \leqslant k_0 - 10$, this reduces to

$$\frac{11}{20} \geqslant \frac{\bar{w}_{k+1}^t}{\bar{w}_k^t} = \frac{1}{2}\left(\cos\theta - \frac{\sin\theta}{\tan\theta k}\right) \geqslant \frac{9}{20} \qquad \text{for small } \theta.$$

- Case 2: $\lambda < \frac{1}{2}$ and λ^t is sufficiently close to $\frac{1}{2}$. In this case $\bar{w}_k^t$ is given by (25), so

$$\begin{aligned}
\frac{\bar{w}_{k+1}^t}{\bar{w}_k^t} &= \frac{(\lambda^t - r_-)r_+^k + (r_+ - \lambda^t)r_-^k}{(\lambda^t - r_-)r_+^{k-1} + (r_+ - \lambda^t)r_-^{k-1}} \\
&= r_+ \cdot \frac{1 + \frac{r_+ - \lambda^t}{\lambda^t - r_-}\left(\frac{r_-}{r_+}\right)^k}{1 + \frac{r_+ - \lambda^t}{\lambda^t - r_-}\left(\frac{r_-}{r_+}\right)^{k-1}} = r_+ \cdot \frac{1 + \frac{r_+ - \lambda^t}{\lambda^t - r_-}\left(\frac{r_-}{r_+}\right)^{k_0}\left(\frac{r_+}{r_-}\right)^{k_0 - k}}{1 + \frac{r_+ - \lambda^t}{\lambda^t - r_-}\left(\frac{r_-}{r_+}\right)^{k_0}\left(\frac{r_+}{r_-}\right)^{k_0 - k - 1}}.
\end{aligned}$$

Using (26), we obtain

$$r_+ \leqslant \frac{\bar{w}_{k+1}^t}{\bar{w}_k^t} \leqslant r_+ \cdot \frac{1 + \left(\frac{r_+}{r_-}\right)^{k_0 - k + 2}}{1 + \left(\frac{r_+}{r_-}\right)^{k_0 - k}} \leqslant r_+ \left(\frac{r_+}{r_-}\right)^2.$$

Since $r_+, r_- \to \frac{1}{2}$ as $\lambda^t \to \frac{1}{2}$, we obtain that $\frac{\bar{w}_{k+1}^t}{\bar{w}_k^t} \approx \frac{1}{2}$ uniformly in k in this case.

The difference between $\bar{w}_k^t$ and $\tilde{w}_k^t$ is $\varepsilon_k = \varepsilon_k(p) = \tilde{w}_k^t(p) - \bar{w}_k^t$. We claim that if $p < 1/S_{k_0}$, then there is K such that

$$|\tilde{w}_k^t - \bar{w}_k^t| =: |\varepsilon_k| \leqslant K\tau_0(1 - e^{-pS_{k-1}})\bar{w}_k^t \quad \text{for all } k \leqslant k_0 - 10. \tag{28}$$

Since $\bar{w}_1^t(e^{-p} - 1) = \varepsilon_1 \leqslant -pS_0\bar{w}_1^t$ and $\bar{w}_2^t(e^{-2p} - 1) = \varepsilon_2 \leqslant -pS_1\bar{w}_2^t$, this claim holds for $k = 1, 2$.

Subtracting two successive equations in (24) gives the recursive relations

$$\tilde{w}_{k+1}^t = e^{-pS_{k-2}}\tilde{w}_k^t - e^{\beta - pS_k}\tilde{w}_{k-1}^t, \tag{29}$$

so for $p=0$ this is $\bar{w}^t_{k+1} = \bar{w}^t_k - e^{\beta}\bar{w}^t_{k-1}$. For ε_k we obtain

$$\begin{aligned} \varepsilon_{k+1} &= \tilde{w}^t_{k+1} - \bar{w}^t_{k+1} \\ &= e^{-pS_{k-2}}\varepsilon_k - e^{\beta}e^{-pS_k}\varepsilon_{k-1} + e^{\beta}(1-e^{-pS_k})\bar{w}^t_{k-1} - (1-e^{-pS_{k-2}})\bar{w}^t_k. \end{aligned}$$

Write $\varepsilon_k = u_k(1-e^{-pS_{k-1}})\bar{w}^t_k$, so $u_1 = u_2 = -1$ and $u_3 \in (-1,0)$. Then we can rewrite the above as

$$\begin{aligned} u_{k+1} &= e^{-pS_{k-2}}\frac{1-e^{-pS_{k-1}}}{1-e^{-pS_k}}\frac{\bar{w}^t_k}{\bar{w}^t_{k+1}}u_k - e^{\beta}e^{-pS_k}\frac{1-e^{-pS_{k-2}}}{1-e^{-pS_k}}\frac{\bar{w}^t_{k-1}}{\bar{w}^t_{k+1}}u_{k-1} \\ &\qquad + \frac{\bar{w}^t_k}{\bar{w}^t_{k+1}}\left(e^{\beta}\frac{\bar{w}^t_{k-1}}{\bar{w}^t_k} - \frac{1-e^{-pS_{k-2}}}{1-e^{-pS_k}}\right) \\ &:= au_k - bu_{k-1} + c. \end{aligned}$$

The numbers a,b,c depend on k, but since $\frac{\bar{w}^t_{k+1}}{\bar{w}^t_k} \approx \frac{\bar{w}^t_k}{\bar{w}^t_{k-1}} \in [0.45, 0.55]$ for all $10 \leqslant k \leqslant k_0 - 10$, and $e^{\beta} \approx \frac{1}{4}$, we have $c \in [0.1, 0.5]$ and $0 < a - b < 0.99$. Therefore the orbit $(u_k)_{k\geqslant 1}$ is bounded, say $|u_k| \leqslant K$ for all k, and in fact positive from the moment that two consecutive terms are positive. In particular, $-1 \leqslant u_k \leqslant K$ for all k, and $|\varepsilon_k| \leqslant K(1-e^{-pS_{k-1}})\bar{w}^t_k$ for all $k \leqslant k_0 - 10$, proving Claim (28). If we now take $p \leqslant \tau_0/S_{k_0}$, then $|\varepsilon_k| \leqslant K\tau_0\gamma^{-11}\bar{w}^t_k$ for $k = k_0 - 10$. Propagating this tiny error (provided τ_0 is small) for another eleven iterates, *i.e.*, eleven recursive steps $\tilde{w}^t_{k+1} = e^{-pS_{k-2}}\tilde{w}^t_k - e^{\beta - pS_k}\tilde{w}^t_{k-1}$, we find that $\tilde{w}^t_{k_0+1} < 0$. This shows that $p(t) > \tau_0/S_{k_0}$. □

Recall that $\gamma = \frac{1}{2}(1+\sqrt{5})$ and $\Gamma = \frac{2\log\gamma}{\sqrt{-\log[\lambda(1-\lambda]}}$.

Proposition 3. *There are $\tau_0 = \tau_0(\lambda)$ and $\tilde{C} = \tilde{C}(\lambda) > 0$ such that*

$$p(t) > \frac{\tau_0}{S_{k_0}} \geqslant \begin{cases} \tau_0 e^{-\pi\Gamma/\sqrt{t_1 - t}} & \text{if } t < t_1 \leqslant 1 \text{ close to } t_1 \text{ and } \lambda \geqslant \frac{1}{2}; \\ \tau_0\tilde{C}(1-t)^{\frac{\log(\gamma)}{\log R}} & \text{if } t < 1 \text{ close to } 1 \text{ and } \lambda < \frac{1}{2}, \end{cases}$$

where $\log R = 2\log(1+\sqrt{1-4\lambda^t(1-\lambda)^t}) - \log[4\lambda^t(1-\lambda)^t] \sim 2(1-2\lambda)$ *as $t \to 1$ and $\lambda \to \frac{1}{2}$.*

Proof. Lemma 3 gives $p(t) > \frac{\tau_0}{S_{k_0}}$. For the second inequality, first assume that $\lambda \geqslant \frac{1}{2}$ and $t < t_1 \leqslant 1$. Using the estimate of k_0 from Lemma 2, and $\beta' = \sqrt{-\log[\lambda(1-\lambda)](t_1-t)}$, we find

$$p(t) \geqslant \frac{\tau_0}{S_{k_0}} \approx \tau_0 e^{-k_0\log\gamma} \geqslant \tau_0 e^{-\frac{\pi\Gamma}{\sqrt{t_1-t}}}.$$

Now for the case $\lambda < \frac{1}{2}$ and $t < 1$, recall from (26) that

$$\frac{\tau_0}{S_{k_0}} \geqslant \tau_0\left(\frac{r_+}{r_-}\right)^{-k_0\frac{\log(\gamma)}{\log(\frac{r_+}{r_-})}} \geqslant \left(\frac{r_+^2}{r_-^2}\cdot\frac{r_+ - \lambda^t}{r_- - \lambda^t}\right)^{-\frac{\log(\gamma)}{\log(\frac{r_+}{r_-})}}.$$

We work out the asymptotics for fixed $\lambda < \frac{1}{2}$ and first order Taylor expansions for $t \approx 1$.

$$\begin{aligned}
4e^{\beta} &= 4\lambda(1-\lambda)\left(1+\log[\lambda(1-\lambda)](t-1)\right) + \text{h.o.t.} \\
\sqrt{1-4e^{\beta}} &= (1-2\lambda)\sqrt{1-\frac{4\lambda(1-\lambda)}{(1-2\lambda)^2}\log[\lambda(1-\lambda)](t-1)+\text{h.o.t.}} \\
&= (1-2\lambda)\left(1-\frac{2\lambda(1-\lambda)}{(1-2\lambda)^2}\log[\lambda(1-\lambda)](t-1)\right)+\text{h.o.t.} \\
R := \frac{r_+}{r_-} &= \frac{(1+\sqrt{1-4e^{\beta}})^2}{4e^{\beta}} = 1+2(1-2\lambda)+\text{h.o.t.} \\
r_+ - \lambda^t &= 1-2\lambda+\text{h.o.t.} \\
r_- - \lambda^t &= \left(\frac{2\lambda(1-\lambda)}{1-2\lambda}\log[\lambda(1-\lambda)] - 2\lambda\log\lambda\right)(t-1)+\text{h.o.t.}
\end{aligned}$$

This gives exponent $\log(\gamma)/\log(R)$ (which is $\sim \log(\gamma)/(2(1-2\lambda))$ as $\lambda \to \frac{1}{2}$) and

$$\frac{r_+^2}{r_-^2}\cdot\frac{r_+-\lambda^t}{r_--\lambda^t} = \frac{(1+4(1-2\lambda))\left(\frac{1-2\lambda}{-\lambda(1-\lambda)\log[\lambda(1-\lambda)]+2\lambda(1-2\lambda)\log\lambda}\right)}{1-t}+\text{h.o.t.}$$

Hence the estimate holds for $0 < \tilde{C} \sim \left(\frac{-\lambda(1-\lambda)\log[\lambda(1-\lambda)]+2\lambda(1-2\lambda)\log\lambda}{(1+4(1-2\lambda))(1-2\lambda)}\right)^{\frac{\log\gamma}{2\log(1-2\lambda)}}$ as $\lambda \to \frac{1}{2}$. □

Lemma 4. *If $p > 0$, then $\tilde{w}_k^t \to 0$ super-exponentially:*

$$\tilde{w}_k^t = \begin{cases} e^{\beta k - pS_{k+1}+\alpha_k} & \text{if } \sum_k \tilde{w}_k^t = 1, \\ e^{\beta - pS_{k-1}+\alpha_k} & \text{otherwise,} \end{cases} \tag{30}$$

where $(\alpha_k)_{k\geqslant 1}$ is a convergent sequence depending on p and t.

Proof. First note that if $\sum_k \tilde{w}_k^t \neq 1$, then the factor $e^{-pS_{k-1}}$ is the only factor in (24) that tends to zero. Hence the final statement of the lemma is immediate. So assume now that $\sum_k \tilde{w}_k^t = 1$, and $\tilde{w}_k^t$ decreases faster than $e^{\beta - pS_{k-1}}$.

Taking a linear combination of two consecutive equations in (24), we obtain

$$e^{pS_{k-1}}\tilde{w}_k^t - e^{pS_k}\tilde{w}_{k+1}^t = e^{\beta}\tilde{w}_{k-1}^t. \tag{31}$$

By setting $\tilde{w}_k^t = e^{\beta k - pS_{k+1}+\alpha_k}$, for some $\alpha_k \in \mathbb{R}$, we rewrite (31) as

$$1 - e^{\beta - pS_{k-1}+\alpha_{k+1}-\alpha_k} = e^{\alpha_{k-1}-\alpha_k}.$$

Abbreviating $\varepsilon_k = \alpha_k - \alpha_{k-1}$, we have

$$1 - e^{\beta - pS_{k-1}-\varepsilon_{k+1}} = e^{\varepsilon_k}.$$

This means that $\varepsilon_k \to 0$ exponentially and hence α_k converges to some limit $\alpha_\infty = \alpha_\infty(p,t)$, exponentially fast in k. Therefore, $\tilde{w}_k^t \to 0$ super-exponentially in k, whenever $p > 0$. □

7.2. **Upper bounds on $P_{\mathrm{Conf}}(\phi_t)$.** We define upper bounds on $p(t)$ using a non-autonomous dynamical system. The following lemma will be applied to this.

Lemma 5. *The map $\eta : r \mapsto 1 - \frac{\xi}{4r}$ has*

$$\begin{cases} \text{one fixed point } \frac{1}{2} & \text{if } \xi = 1; \\ \text{two fixed points } \Theta_\pm = \frac{1}{2}(1 \pm \sqrt{1-\xi}) & \text{if } \xi < 1; \\ \text{no fixed points} & \text{if } \xi > 1. \end{cases}$$

If $\xi < 1$, then the largest fixed point $\frac{1}{2}(1+\sqrt{1-\xi})$ is attracting; if $\xi \leqslant 0$, then the interval $[1, \infty)$ is invariant. If $\xi > 1$, and $\delta = \sqrt{\frac{2(\xi-1)}{3(\sqrt{\xi}+1)}}$ then it takes an orbit at least $\sqrt{\frac{3(\sqrt{\xi}+1)}{2(\xi-1)}}$ iterates to pass through the interval $[\frac{\sqrt{\xi}}{2} - \delta, \frac{\sqrt{\xi}}{2} + \delta]$.

Proof. The first statements follow from straightforward calculus. For the last statement, observe that $\eta'(r) = 1$ for $r = \frac{\sqrt{\xi}}{2}$ and the vertical distance $r - \eta(r) = \sqrt{\xi} - 1$. Furthermore $(r+\delta) - \eta(r+\delta) \leqslant (\sqrt{\xi}-1) + \frac{2}{\sqrt{\xi}}\delta^2$. and $(r-\delta) - \eta(r-\delta) \leqslant (\sqrt{\xi}-1) + \frac{2}{\sqrt{\xi}}\delta^2 + O(\delta^3)$. Hence for ξ sufficiently close to 1, we have $x - \eta(x) \leqslant (\sqrt{\xi}-1) + (3/2)^2\delta^2$ for all $x \in [\frac{\sqrt{\xi}}{2} - \delta, \frac{\sqrt{\xi}}{2} + \delta]$, so it takes an orbit at least $2\delta/[(\sqrt{\xi}-1) + (3\delta/2)^2]$ iterates to pass through this interval. This quantity is maximised for

$$\delta = \sqrt{\frac{2(\sqrt{\xi}-1)}{3}} = \sqrt{\frac{2(\xi-1)}{3(\sqrt{\xi}+1)}}, \tag{32}$$

in which case $(r+\delta) - \eta(r+\delta) \leqslant (1 + \frac{4}{3\sqrt{\xi}})(\sqrt{\xi}-1)$. In this case, it takes at least $\sqrt{\frac{3(\sqrt{\xi}+1)}{2(\xi-1)}}$ iterates to pass through the interval. □

Lemma 6. *Let (u_k) be given by*

$$u_1 = \lambda^t \qquad \text{and} \qquad u_{k+1} = \eta_k(u_k) := 1 - \frac{e^{\beta' - pS_{k-2}}}{4u_k}.$$

There exist constants $\tau_1 = \tau_1(\lambda), \tau_1' = \tau_1'(\lambda)$ (with precise values given in the proof) such that if

$$p > \begin{cases} \tau_1 e^{-\frac{5\Gamma}{6\sqrt{t_1 - t}}} & \text{if } \lambda \geqslant \frac{1}{2},\ t < t_1 \text{ close to } t_1, \quad \Gamma = \frac{2\log\gamma}{\sqrt{-\log[\lambda(1-\lambda)]}} \\ \tau_1'(1-t)^{\frac{\lambda\log\gamma}{2t(1-2\lambda)}} & \text{if } \lambda < \frac{1}{2},\ t < 1 \text{ close to } 1, \end{cases}$$

then $u_k \geqslant \frac{1}{3}$ for all k and $u_k \to 1$ exponentially.

Proof. Let $\xi_k = e^{\beta' - pS_{k-2}}$. The dynamics of the map $\eta_k : r \mapsto r - \frac{\xi_k}{4r}$ depend crucially on whether $\xi_k > 1$ or $\xi_k \leqslant 1$. These cases are roughly parallel to $\lambda \geqslant \frac{1}{2},\ t < t_1$ close to t_1 and $\lambda < \frac{1}{2},\ t < 1$ close to 1. However, if pS_{k-2} is sufficiently large, the factor $e^{-pS_{k-2}}$ turns the first case into the second.

By Lemma 5, if $\xi_k \leqslant 1$, then η_k has an attracting fixed point, tending to 1 as $\xi_k \to 0$. Therefore, once $\beta' - pS_{k-2} \leqslant 0$, and assuming that $u_k \geq \Theta_k$ where $\Theta_k \leqslant \frac{1}{2}$ is the repelling fixed point of η_k, the orbit of u_k will tend to the attracting fixed point which itself moves to 1 at an exponential rate as $k \to \infty$. However, if $\xi_k > 1$, *i.e.*, ξ_k is "before" the saddle node bifurcation that produces the fixed point $\frac{1}{2}$, then u_k will decrease and eventually become negative. The crux of the proof is therefore to show that the "tunnel" between the graph of η_k and the diagonal closes up before the orbit $(u_k)_{k\geqslant 0}$ has moved through this tunnel.

We fix $\xi = e^{\beta'}$ and δ as in (32), and we will choose p so that the repelling fixed point Θ_k is to the left of the tunnel (of width 2δ and centred around the point $x = \sqrt{\xi}/2$ at which $\eta'(x) = 1$), *i.e.*,

$$\Theta_k = \frac{1}{2}\left(1 - \sqrt{1 - \xi e^{-pS_{k-2}}}\right) \leqslant \frac{1}{2}\sqrt{\xi} - \delta = \frac{1}{2}\left(\sqrt{\xi} - G\sqrt{\xi - 1}\right),$$

where we abbreviated $G = \sqrt{\frac{8}{3(\sqrt{\xi}+1)}} < \frac{6}{5}$. Assuming that equality holds, and solving for $e^{-pS_{k-2}}$, we obtain

$$1 - \xi e^{-pS_{k-2}} = (\sqrt{\xi} - 1)\left(\sqrt{\xi} - 1 - 2G\sqrt{\xi - 1} + (\sqrt{\xi} + 1)G^2\right),$$

which can be reduced to $pS_{k-2} = 2(1 + G^2)(\sqrt{\xi} - 1) + o(\sqrt{\xi} - 1)$.

Lemma 5 states that the passage through the tunnel takes at least

$$k = \sqrt{\frac{3(\sqrt{\xi}+1)}{2(\xi - 1)}} = \frac{2}{G\sqrt{\xi - 1}} \leqslant \frac{5}{6}\frac{2}{\sqrt{-\log[\lambda(1-\lambda)](t_2 - t)}}$$

iterates. Note that $2(1 + G^2) = 14/3 < 5$. Choose $\tau_1 = 5\gamma^2(\sqrt{e^{\beta'}} - 1) = 5\gamma^2(\sqrt{\xi} - 1)$ and $p \geqslant \tau_1 e^{-\frac{5\Gamma}{6\sqrt{t_2 - t}}}$. Then

$$p \geqslant \tau_1 e^{-\frac{5}{6}\frac{\Gamma}{\sqrt{t_2 - t}}} \geqslant \tau_1 e^{-k\log\gamma} = 5\gamma^2(\sqrt{\xi} - 1)\gamma^{-k} > \frac{2(1 + G^2)(\sqrt{\xi} - 1)}{S_{k-2}}.$$

Hence $pS_{k-2} > (1 + G^2)(\sqrt{\xi} - 1) + o(\sqrt{\xi} - 1)$ and we conclude that the tunnel closes with fixed point to the left of the tunnel, before u_k passes through it. At (or before) this iterate, u_k starts to increase again, and eventually converge to 1 at an exponential rate.

Now let us assume that $\xi < 1$, so there is a (left) fixed point $\Theta_- = \frac{1}{2}(1 - \sqrt{1 - \xi})$ which for $t = 1$ coincides with $u_1 = \lambda^t$. For $t < 1$, we have $u_1 < \Theta_-$, say $\Theta_1 - u_1 = \varepsilon = \varepsilon(t)$, and Taylor expansion shows that

$$\varepsilon(t) = \frac{1}{2}\left(1 - 2\lambda^t - \sqrt{1 - 4e^{\beta}}\right) = C(1 - t) + O((1 - t)^2)$$

for $C = \lambda\log\lambda - \lambda(1 - \lambda)\frac{\log[\lambda(1-\lambda)]}{1-2\lambda}$. Assume that j is the first iterate such that $u_1 - \sqrt{\varepsilon} \geqslant u_j$. Let $K = \eta'(u_1 - \sqrt{\varepsilon}) \leqslant \frac{e^{\beta}}{(u_1 - \sqrt{\varepsilon})^2} \approx \frac{e^{\beta}}{u_1^2} = \frac{(1-\lambda)^t}{\lambda^t}$. By taking a line with slope K through the point (Θ_-, Θ_-) to approximate the graph of η_j (and this line lies below the graph of η_k on the interval $[u_1 - \sqrt{\varepsilon}, \Theta_-])$, we can estimate $u_j \geqslant u_1 - K^{j-1}\varepsilon$, so $K^{j-1} \geqslant 1/\sqrt{\varepsilon}$.

Due to the inequality $\log K \leqslant t \log\left(\frac{1-\lambda}{\lambda}\right) \leqslant t\left(\frac{1-2\lambda}{\lambda}\right)$, and taking $\tau_1' = (\log 5) C^{\frac{\lambda \log\gamma}{2t(1-2\lambda)}}$, the condition $p > \tau_1'(1-t)^{\frac{\lambda\log\gamma}{2t(1-2\lambda)}}$ implies

$$p > \log 5 \cdot (C(1-t))^{\frac{\log\gamma}{2\log K}} \geqslant (\log 4)(\sqrt{\varepsilon})^{\frac{\log\gamma}{\log K}} \geqslant \frac{\log 4}{K^{(j-1)\frac{\log\gamma}{\log K}}} = \frac{\log 4}{\gamma^{j-1}} \approx \frac{\log 4}{S_{j-1}}.$$

Let Θ_j be the left fixed point of η_j. Thus, given that $p \geqslant (\log 4)/S_{j-1}$, we find that $\Theta_j = \frac{1}{2}(1-\sqrt{1-e^{\beta'-pS_{j-1}}}) > \lambda^t/4$ for t close to 1. Therefore $\Theta_j < u_j$, and u_j will converge to the attracting fixed point Θ_+, which itself converges exponentially to 1. □

Proposition 4. *For the constants τ_1, τ_1' from Lemma 6 we have the following upper bounds for the pressure:*

$$p(t) \leqslant \begin{cases} \tau_1 e^{-\frac{5}{6}\frac{\Gamma}{\sqrt{t_1-t}}} & \text{if } \lambda \geqslant \frac{1}{2},\ t < t_1 \text{ close to } t_1; \\ \tau_1'(1-t)^{\frac{\lambda\log\gamma}{2t(1-2\lambda)}} & \text{if } \lambda < \frac{1}{2},\ t < 1 \text{ close to } 1. \end{cases}$$

Proof. Let $u_k = \frac{\tilde{w}_{k+1}^t}{\tilde{w}_k^t} e^{pS_{k-2}}$, so $u_1 = \lambda^t e^{p(S_{-1}+S_0-S_1)} = \lambda^t > \frac{1}{2}$ (where we set $S_{-1} = 1$ by default). From (29) we have

$$u_{k+1} = 1 - \frac{e^{\beta'-pS_{k-2}}}{4u_k}. \tag{33}$$

For $p > \tau_1 e^{-\frac{5}{6}\frac{\Gamma}{\sqrt{t_1-t}}}$ or $p > \tau_1(1-t)^{\frac{\lambda\log\gamma}{2t(1-2\lambda)}}$ as given in Lemma 6, the iterates u_k are bounded away from zero and $u_k \to 1$ exponentially. Therefore

$$\begin{aligned} u_\infty := \prod_{j\geqslant 1} u_j &= u_1 \cdot \prod_{j\geqslant 2}\left(1 - \frac{e^{\beta'-pS_{j-1}}}{4u_{j-1}}\right) \\ &\geqslant r_2 \cdot \prod_{j\geqslant 3}\left(1-\frac{3e^{\beta'-pS_{j-1}}}{8}\right) > 0 \end{aligned}$$

because $\sum_{j\geqslant 2} 3e^{\beta'-pS_{j-1}}/8 < \infty$ and all terms $e^{\beta'-pS_{j-1}}/8$ are uniformly bounded away from 1. Since

$$\tilde{w}_{k+1}^t = e^{-p(S_{k-2}+S_{k-1}+\cdots+S_{-1})} \cdot \tilde{w}_1^t \cdot \prod_{j=1}^k u_j,$$

it follows that all $\tilde{w}_k^t$ are positive, and as $\tilde{w}_k^t = e^{\beta-pS_{j-1}}\left(1-\sum_{j<k-1}\tilde{w}_j^t\right)$, also

$$H_{k-1}(p,t) := \sum_{j<k-1} \tilde{w}_k^t \leqslant 1$$

for all k.

To prove that $H_k(p) < 1$ for $p > \tau_1 e^{-\frac{5}{6}\frac{\Gamma}{\sqrt{t_1-t}}}$ or $p > \tau_1'(1-t)^{\frac{\lambda\log\gamma}{2t(1-2\lambda)}}$, we will show that $\frac{\partial H_k(p)}{\partial p} < 0$ for these values of p. Observe that $\partial H(p,t)/\partial p$ satisfy the recursive relation:

$$\begin{aligned}
H_1 &= (1-\lambda)^t e^{-p} & \tfrac{\partial H_1}{\partial p} &= -(1-\lambda)^t e^{-p} < 0. \\
H_2 &= (1-\lambda)^t e^{-p} + e^{\beta} e^{-2p} & \tfrac{\partial H_2}{\partial p} &= -(1-\lambda)^t e^{-p} - 2e^{\beta-2p} < \tfrac{\partial H_1}{\partial p}. \\
&\vdots & &\vdots \\
H_j &= H_{j-1} + e^{\beta - pS_{j-1}}(1-H_{j-2}) & \tfrac{\partial H_j}{\partial p} &= \tfrac{\partial H_{j-1}}{\partial p} - e^{\beta-pS_{j-1}} \tfrac{\partial H_{j-2}}{\partial p} \\
& & &\quad - S_{j-1} e^{\beta - pS_{j-1}}(1-H_{j-2}).
\end{aligned}$$

Writing $U_j := \frac{\partial H_j}{\partial p}$ and $q_{j+1} := U_{j-1}/U_{j-2}$, we find $q_4 = 1 + 2e^{\beta-2p} > 1$ and

$$q_{j+1} = \frac{U_{j-1}}{U_{j-2}} = \eta_j(q_j) := 1 - \frac{e^{\beta' - pS_{j-2}}}{4q_j}\left(1 + S_{j-2}\frac{1-H_{j-3}}{U_{j-3}}\right) \geqslant 1 - \frac{e^{\beta'-pS_{j-2}}}{4q_{j-2}},$$

where the final inequality relies on U_{j-3} being negative. This follows from induction, combined with Lemma 6, which implies that $q_k \geqslant \frac{1}{3}$ and $q_k \to 1$ exponentially fast, so $\prod_{i\geqslant 4} q_i > 0$. It follows that

$$\frac{\partial H}{\partial p} = \lim_{j\to\infty} U_j = U_1 \cdot \lim_{j\to\infty} \prod_{i=4}^{j} q_i < 0,$$

and hence $H(p,t) < 1$. □

Remark 4. *The techniques in this proof give no explicit formula for $\frac{\partial H}{\partial p}$ and $\frac{\partial H}{\partial t}$ as $t \nearrow t_1$, so they don't answer the question whether $\frac{dp}{dt} \to 0$ as $t \nearrow t_1$.*

7.3. **Existence and uniqueness of $P_{\mathrm{Conf}}(\phi_t)$.**

Lemma 7. *For all $t < t_1$ there exists $p_u \geqslant p_\ell \geqslant 0$ such that $H(p_\ell, t) = 1$ and $\tilde{w}_i^t(p_\ell) \geqslant 0$ for all i, and $H(p,t) < 1$ for all $p \geqslant p_u$.*

We will show later in this section that in fact $p_u = p_\ell$ for t close to t_1; and then in Section 9 that this is actually true for all t.

Proof. For any $p > 0$, since $\tilde{w}_i^t(p) \leqslant e^{\beta - pS_{k-1}}$, we have $H(p,t) < \infty$. This fact also implies that $H(p,t) < 1$ for all large p, thus proving the existence of p_u.

For each (p,t), define the partial sums $H_j = H_j(p,t) := \sum_{i\leqslant j} \tilde{w}_i^t(p)$. Recall from Lemma 2 that there is some minimal $k_0 \in \mathbb{N}$ such that $\tilde{w}_{k_0}^t(0) = \bar{w}_{k_0}^t < 0$. By the recurrence relations defining $\tilde{w}_k^t(0)$, this means that $H_{k_0-2}(0,t) > 1$. Now we prove the existence of a solution to the equation $H(p,t) = 1$ with all $\tilde{w}_j^t(p) \geqslant 0$ by continuity. For $k \in \mathbb{N}$, let

$$p_k := \inf\left\{p \geqslant 0 : H_j(p', t) < 1 \text{ for each } j \leqslant k \text{ and } p' \geqslant p\right\}.$$

We collect some facts:

- $\sup_k p_k \in (0,\infty)$. Since $H_{k_0-2}(0,t) > 1$ for some $k_0 \in \mathbb{N}$ as shown before, combined with the fact that $(p_k)_{k\geqslant 1}$ is a non-decreasing (which follows immediately from the definition of p_j) gives that $\sup_k p_k > 0$. The finiteness follows from the bound $\tilde{w}_i^t(p) \leqslant e^{-pS_{k-1}}$.

- If $p_k > 0$ then $H_k(p_k, t) = 1$. This follows since each map $p \mapsto H_k(p, t)$ is continuous in p, so by the definition of p_k as an infimum, there must exist a minimal $j \leqslant k$ such that $H_j(p_k, t) = 1$. But our recurrence relation (31) implies that $H_{j+1}(p_k, t) = H_j(p_k, t) + e^{\beta - p_k S_k}(1 - H_{j-1}(p_k, t)) > H_j(p_k, t) = 1$. This must also hold for all p sufficiently close to p_k, so if $j < k$ then this contradicts the definition of p_k.
- $\tilde{w}_j^t(p') \geqslant 0$ for all $j \leqslant k$ and $p' \geqslant p_k$. If this fails, take the minimum such k and note that (31) implies that $H_{j-2}(p', t) > 1$, a contradiction.

Now define $p_\infty := \sup_k p_k$. It follows immediately from this definition that for any $j \in \mathbb{N}$, $H_j(p_\infty, t) < 1$ so $H(p_\infty, t) \leqslant 1$. Note that this also implies that $\tilde{w}_j^t(p_\infty) \geqslant 0$ for all $j \in \mathbb{N}$.

To show that $H(p_\infty, t) = 1$, notice that for $p > 0$ and any $j \in \mathbb{N}$,

$$H(p, t) = H_j(p, t) + \sum_{k>j} \tilde{w}_k^t(p) \geqslant H_j(p, t) - e^\beta \sum_{k>j} e^{-pS_k}.$$

So defining $j_0 \in \mathbb{N}$ such that $p_{j_0} > 0$, let $s(j) := e^\beta \sum_{k>j} e^{-p_{j_0} S_k}$. Then for $p_j \geqslant p_{j_0}$,

$$H(p_j, t) \geqslant H_j(p_j, t) - s(j) = 1 - s(j).$$

So since $s(j) \to 0$ as $j \to \infty$, we have $H(p_j, t) \to 1$ as $j \to \infty$. Therefore, the continuity of $p \mapsto H(p, t)$ on the domain where the sums are bounded implies that $H(p_\infty, t) = 1$. $\square$

Proposition 5. *There is at most one solution $p = p(t)$ to $H(p, t) = 1$ with all $\tilde{w}_k^t > 0$. Moreover, $\frac{\partial H}{\partial p} < 0$, $\frac{\partial H}{\partial t} < 0$, and the map $t \mapsto p(t)$ is analytic with $\frac{dp}{dt} < 0$ on $(t_1 - \varepsilon, t_1)$.*

Proof. The proof uses many of the ideas of the proof of Proposition 4. The previous proof shows that positivity of all $\tilde{w}_k^t$ is equivalent to positivity of the numbers $u_k = \frac{\tilde{w}_{k+1}^t}{\tilde{w}_k^t} e^{pS_{k-1}}$ from (33). Therefore, if $p = p(t)$ is a solution to the problem $\tilde{w}_k^t > 0$ and $H(p, t) = 1$, then the corresponding sequence $(u_k)_k$ is positive. Positivity of $\frac{\partial H(p,t)}{\partial p}$ is equivalent to positivity of an orbit $(v_k)_k$ for a slightly different but larger map, and with an initial value $v_4 > 1 \geqslant u_1$. Therefore, as $(u_k)_k$ is positive, so is $(v_k)_k$, and $0 < \prod_{k \geqslant 1} u_k \leqslant \prod_k v_{k \geqslant 4} = v_\infty$, whence $\frac{\partial H(p,t)}{\partial p} = v_\infty \cdot \frac{\partial H_1(p,t)}{\partial p} < 0$. This shows that there can be at most one solution to $H(p, t) = 1$.

We can use the same technique to estimate $\frac{\partial H(p,t)}{\partial t}$ for $t < t_1$:

$$\begin{aligned}
H_1 &= (1-\lambda)^t e^{-p} & \tfrac{\partial H_1}{\partial t} &= \log(1-\lambda)(1-\lambda)^t e^{-p} < 0. \\
H_2 &= (1-\lambda)^t e^{-p} + e^{\beta} e^{-2p} & \tfrac{\partial H_2}{\partial t} &= \log(1-\lambda)(1-\lambda)^t e^{-p} \\
& & &\quad + \log[\lambda(1-\lambda)] e^{\beta-2p} < \tfrac{\partial H_1}{\partial t}. \\
&\vdots & &\vdots \\
H_j &= H_{j-1} + e^{\beta - pS_{j-1}}(1-H_{j-2}) & \tfrac{\partial H_j}{\partial t} &= \tfrac{\partial H_{j-1}}{\partial t} - e^{\beta-pS_{j-1}} \tfrac{\partial H_{j-2}}{\partial t} \\
& & &\quad + \log[\lambda(1-\lambda)] e^{\beta-pS_{j-1}}(1-H_{j-2}).
\end{aligned}$$

If we now write $U_j = \frac{\partial H_j}{\partial t}$ and $q_{j+1} := U_{j-1}/U_{j-2}$, we find $q_4 = 1 + \frac{\log[\lambda(1-\lambda)]e^{\beta-p}}{\log(1-\lambda)(1-\lambda)^t} > 1$ and

$$q_{j+1} = \frac{U_{j-1}}{U_{j-2}} = 1 - \frac{e^{\beta'-pS_{j-2}}}{4q_j}\left(1 - \log[\lambda(1-\lambda)]\frac{1-H_{j-3}}{U_{j-3}}\right) \geqslant 1 - \frac{e^{\beta'-pS_{j-2}}}{4q_{j-2}},$$

where the final inequality relies on U_{j-3} being negative. The same argument shows that $\frac{\partial H}{\partial t} < 0$ as well. Furthermore, since $H(p,t)$ is analytic in both p and t, the Implicit Function Theorem implies that $t \mapsto p(t)$ is analytic on $(t_1 - \varepsilon, t_1)$ and $\frac{dp}{dt} < 0$. □

8. Invariant measures

Now we look at the invariant measure $\mu_{t,p} \ll m_{t,p}$ for $t < t_1$.

Theorem 1. *Suppose $t < t_1$ and $p > 0$ satisfies $H(p,t) = 1$ with all summands non-negative. Then we have the following:*

(a) There is an F_λ-invariant measure $\mu_t = \mu_{t,p} \ll m_{t,p}$;
(b) The Radon-Nikodym derivative $\frac{d\mu_t}{dm_t}$ is bounded and bounded away from zero;
(c) μ_t projects to an f_λ-invariant probability measure $\nu_t \ll n_t$.

Proof. The solution $\underline{\tilde{w}}^t$ to (24) and $H(p,t) = 1$ gives rise to a probability transition matrix

$$G^t = \begin{pmatrix}
\tilde{w}_1^t & \tilde{w}_2^t & \tilde{w}_3^t & \tilde{w}_4^t & \dots & \dots \\
\tilde{w}_1^t & \tilde{w}_2^t & \tilde{w}_3^t & \tilde{w}_4^t & \dots & \dots \\
0 & \frac{\tilde{w}_2^t}{\sum_{i\geqslant 2}\tilde{w}_i^t} & \frac{\tilde{w}_3^t}{\sum_{i\geqslant 2}\tilde{w}_i^t} & \frac{\tilde{w}_4^t}{\sum_{i\geqslant 2}\tilde{w}_i^t} & \dots & \\
0 & 0 & \frac{\tilde{w}_3^t}{\sum_{i\geqslant 3}\tilde{w}_i^t} & \frac{\tilde{w}_4^t}{\sum_{i\geqslant 3}\tilde{w}_i^t} & & \\
0 & 0 & 0 & \ddots & \ddots & \\
\vdots & & & \ddots & \ddots &
\end{pmatrix}.$$

The left eigenvector $\underline{\tilde{v}}^t = (\tilde{v}_1^t, \tilde{v}_2^t, \dots)$ for eigenvalue 1 represents the invariant measure: $\mu_{t,p}(W_k \cup \hat{W}_k) = \tilde{v}_k^t$. To find it, we start with $v^{(0)} := (1, 0, 0, \dots)$ and iterate $v^{(n)} = v^{(n-1)} G^t$. Since G^t is a stochastic matrix (*i.e.,* nonnegative and with row-sums are 1), each $v^{(n)}$ is non-negative and has $\|v^{(n)}\|_1 = 1$ as well. We prove by induction in n that

$v_j^{(n)}$ decreases super-exponentially in j. We will show that there is $K \in \mathbb{N}$ such that for all $n \geqslant 0$,

$$v_k^{(n)} \leqslant \frac{\tilde{w}_{k-1}^t}{\tilde{w}_{k-2}^t} \quad \text{for all } k \geqslant K. \tag{34}$$

Since $\tilde{w}_k^t = e^{\beta k - \gamma^2 p S_{k+1} + \alpha_k}$ decrease super-exponentially in k as described in (30) we can find K such that $\frac{\tilde{w}_k^t}{\tilde{w}_{k-1}^t} \leqslant \frac{1}{2}\frac{\tilde{w}_{k-1}^t}{\tilde{w}_{k-2}^t}$ for $k \geqslant K$. Clearly (34) holds for $v^{(0)}$. For the inductive step, assume (34) holds for $n-1$. Then for $k \geqslant K$ arbitrary,

$$\begin{aligned} v_k^{(n)} &= \tilde{w}_k^t \left(v_1^{(n-1)} + v_2^{(n-1)} + \sum_{j=3}^{k+1} \frac{v_j^{(n-1)}}{\sum_{i \geqslant j-1} \tilde{w}_i^t} \right) \qquad (35) \\ &= \tilde{w}_k^t \left(v_1^{(n-1)} + v_2^{(n-1)} + \sum_{j=3}^{k} \frac{v_j^{(n-1)}}{\sum_{i \geqslant j-1} \tilde{w}_i^t} \right) + v_{k+1}^{(n-1)} \left(1 - \frac{\sum_{i \geqslant k+1} \tilde{w}_i^t}{\sum_{i \geqslant k} \tilde{w}_i^t} \right) \\ &\leqslant \left(v_1^{(n-1)} + \cdots + v_{k-1}^{(n-1)} + v_k^{(n-1)} \right) \frac{\tilde{w}_k^t}{\tilde{w}_{k-1}^t} + v_{k+1}^{(n-1)} \\ &\leqslant \frac{\tilde{w}_k^t}{\tilde{w}_{k-1}^t} + \frac{\tilde{w}_k^t}{\tilde{w}_{k-1}^t} \leqslant \frac{\tilde{w}_{k-1}^t}{\tilde{w}_{k-2}^t}, \end{aligned}$$

where in the last line we used that $\|v^{(n-1)}\| = 1$ as well the choice of K. This shows that although the unit ball in l^1 is not compact, the sequence $(v^{(n)})_{n \geqslant 0}$ is tight, and hence must have a convergent subsequence. Since G^t is clearly an irreducible aperiodic matrix, $(v^{(n)})_{n \geqslant 0}$ converges; let $\tilde{v}^t$ be the limit. Then $\tilde{v}^t$ is positive and $\|\tilde{v}^t\|_1 = 1$.

The measure μ_t defined by the piecewise constant Radon-Nikodym derivative $h_k := h|_{W_k \cup \hat{W}_k} = \frac{\mu_t(W_k \cup \hat{W}_k)}{m_t(W_k \cup \hat{W}_k)} = \frac{\tilde{v}_k^t}{\tilde{w}_k^t}$ is now easily seen to be invariant. By taking the limit $n \to \infty$ in (35), we find

$$\tilde{v}_k^t = \tilde{w}_k^t \left(\tilde{v}_1^t + \tilde{v}_2^t + \sum_{j=3}^{k+1} \frac{\tilde{v}_j^t}{\sum_{i \geqslant j-1} \tilde{w}_i^t} \right) = \frac{\tilde{w}_k^t}{\tilde{w}_{k-1}^t} \tilde{v}_{k-1}^t + \frac{\tilde{w}_k^t}{\sum_{i \geqslant j-1} \tilde{w}_i^t} \tilde{v}_{k+1}^t,$$

and dividing this by $\tilde{w}_k^t$ shows that $(h_k)_{k \in \mathbb{N}}$ is increasing, and hence bounded away from 0. Now for the upper bound, taking the limit $n \to \infty$ in (34) shows that $\tilde{v}_k^t \to 0$ super-exponentially fast. Take $K \in \mathbb{N}$ such that

$$\sum_{k \geqslant K} \frac{\tilde{w}_k^t}{\tilde{w}_{k-1}^t} < \frac{1}{4}$$

Then by (35):

$$h_k = \frac{\tilde{v}_k^t}{\tilde{w}_k^t} \leqslant \underbrace{\tilde{v}_1^t + \tilde{v}_2^t + \sum_{j=1}^{K} \frac{\tilde{v}_j^t}{\tilde{w}_{j-1}^t}}_{C} + \sum_{j=K+1}^{k-1} \frac{\tilde{w}_j^t}{\tilde{w}_{j-1}^t} h_j + \frac{\tilde{w}_k^t}{\tilde{w}_{k-1}^t} h_k + \frac{\tilde{v}_{k+1}^t}{\tilde{v}_k^t} h_k.$$

This gives

$$h_k \leqslant \frac{C + (\sup_{j<k} h_j) \cdot \sum_{j=K+1}^{k-1} \frac{\tilde{w}_j^t}{\tilde{w}_{j-1}^t}}{1 - \frac{\tilde{w}_k^t}{\tilde{w}_{k-1}^t} - \frac{\tilde{v}_{k+1}^t}{\tilde{v}_k^t}}. \tag{36}$$

If $\frac{\tilde{v}_{k+1}^t}{\tilde{v}_k^t} \leqslant \frac{1}{4}$, then so long as k is sufficiently large, (36) yields $h_k \leqslant 2C + \frac{1}{2}\sup_{j<k} h_j$. Since h_k is an increasing sequence, we conclude that $h_k \leqslant 4C$ and moreover, $h_j \leqslant 4C$ for all $j \leqslant k$. The fact that $\tilde{v}_k^t \to 0$ super-exponentially implies that there are infinitely many k satisfying $\frac{\tilde{v}_{k+1}^t}{\tilde{v}_k^t} \leqslant \frac{1}{4}$, so $h_j \leqslant 4C$ for all $j \in \mathbb{N}$, concluding the upper bound.

Since $\mu_t(W_i \cup \hat{W}_i) = \tilde{v}_k^t$ decreases super-exponentially, $\Lambda := \sum_j S_{j-1}\mu_t(W_j \cup \hat{W}_j) < \infty$ for $t < t_1$, so by Lemma 1, μ_t pulls back to an f_λ-invariant probability measure $\nu_t \ll n_t$. □

9. Thermodynamic formalism for countable Markov shifts

9.1. **Countable Markov shifts.** In previous sections we have computed quantities such as pressure rather directly, which gives a fuller understanding of the underlying properties of our class of dynamical systems. In this section we use the theory of countable Markov shifts, as developed by Sarig, to prove stronger results more indirectly. In particular, we can obtain information about the pressure and equilibrium states for ϕ_t for *all* $t \in \mathbb{R}$.

Let $\sigma : \Sigma \to \Sigma$ be a one-sided Markov shift with a countable alphabet $\mathbb{N}$. That is, there exists a matrix $(t_{ij})_{\mathbb{N}\times\mathbb{N}}$ of zeros and ones (with no row and no column made entirely of zeros) such that

$$\Sigma = \{x \in \mathbb{N}^{\mathbb{N}_0} : t_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{N}_0\},$$

and the shift map is defined by $\sigma(x_0 x_1 \cdots) = (x_1 x_2 \cdots)$. We say that (Σ, σ) is a *countable Markov shift.* We equip Σ with the topology generated by the cylinder sets

$$[e_0 \cdots e_{n-1}] = \{x \in \Sigma : x_j = e_j \text{ for } 0 \leqslant j < n\}.$$

Given a function $\phi \colon \Sigma \to \mathbb{R}$, for each $n \geqslant 1$ we define the *variation* on n-cylinders

$$V_n(\phi) = \sup\{|\phi(x) - \phi(y)| : x, y \in \Sigma,\ x_i = y_i \text{ for } 0 \leqslant i < n\}.$$

We say that ϕ has *summable variations* if $\sum_{n=2}^{\infty} V_n(\phi) < \infty$; clearly summability implies continuity of ϕ. In what follows we assume (Σ, σ) to be topologically mixing (see [S2, Section 2] for a precise definition).

Based on work of Gurevich [Gu1, Gu2], Sarig [S2] introduced a notion of pressure for countable Markov shifts which does not depend upon the metric of the space and which satisfies a Variational Principle. Let (Σ, σ) be a topologically mixing countable Markov shift, fix a symbol e_0 in the alphabet $\mathbb{N}$ and let $\phi \colon \Sigma \to \mathbb{R}$ be a potential of summable variations. We let the *local partition function at* $[e_0]$ be

$$Z_n(\phi, [e_0]) := \sum_{x:\sigma^n x = x} e^{S_n\phi(x)} \chi_{[e_0]}(x)$$

and

$$Z_n^*(\phi, [e_0]) := \sum_{\substack{x:\sigma^n x = x, \\ x:\sigma^k x \notin [e_0] \text{ for } 0<k<n}} e^{S_n\phi(x)} \chi_{[e_0]}(x),$$

where $\chi_{[e_0]}$ is the characteristic function of the 1-cylinder $[e_0] \subset \Sigma$, and $S_n\phi(x)$ is $\phi(x) + \cdots + \phi \circ \sigma^{n-1}(x)$. The so-called *Gurevich pressure* of ϕ is defined by the exponential growth rate

$$P_G(\phi) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, [e_0]).$$

Since σ is topologically mixing, one can show that $P_G(\phi)$ does not depend on e_0. If (Σ, σ) is the full-shift on a countable alphabet then the Gurevich pressure coincides with the notion of pressure introduced by Mauldin & Urbański [MU].

The following can be shown using the proof of [S2, Theorem 3].

Proposition 6 (Variational Principle). *If (Σ, σ) is topologically mixing, $\phi : \Sigma \to \mathbb{R}$ has summable variations and $\phi < \infty$, then*

$$P_G(\phi) = P(\phi).$$

Definition 3. *The potential ϕ is said to be* recurrent *if* [7]

$$\sum_n e^{-nP_G(\phi)} Z_n(\phi) = \infty.$$

Otherwise ϕ is transient*. Moreover, ϕ is called* positive recurrent *if it is recurrent and*

$$\sum_n n e^{-nP_G(\phi)} Z_n^*(\phi) < \infty.$$

If ϕ is recurrent but not positive recurrent, then it is called null recurrent*.*

We use the standard *transfer operator* $(L_\phi v)(x) = \sum_{\sigma y = x} e^{\phi(y)} v(y)$, with dual operator L_ϕ^*. Notice that a measure m is ϕ-conformal if and only if $L_\phi^* m = m$.

The following theorem is [S1, Theorem 1]. Note that the next two theorems were originally proved under stronger regularity conditions (*i.e.,* weak Hölderness) on the potential, but subsequently it was found that these could be relaxed, see for example [S3] Chapters 3 and 4.

Theorem 2. *Suppose that (Σ, σ) is topologically mixing, $\phi : \Sigma \to \mathbb{R}$ has summable variations and $P_G(\phi) < \infty$. Then ϕ is recurrent if and only if there exists $\lambda > 0$ and a conservative sigma-finite measure m_ϕ finite and positive on cylinders, and a positive continuous function h_ϕ such that $L_\phi^* m_\phi = \lambda m_\phi$ and $L_\phi h_\phi = \lambda h_\phi$. In this case $\lambda = e^{P_G(\phi)}$. Moreover,*

(1) if ϕ is positive recurrent then $\int h_\phi \, dm_\phi < \infty$;
(2) if ϕ is null recurrent then $\int h_\phi \, dm_\phi = \infty$.

Moreover the next theorem follows by [S2, Corollary 2]:

[7] The convergence of this series is independent of the cylinder set $[e_0]$, so we suppress it in the notation.

Theorem 3. *Suppose that (Σ, σ) is topologically mixing and $\phi : \Sigma \to \mathbb{R}$ has summable variations and is positive recurrent. Then for the measure $d\mu = h_\phi dm_\phi$ given by Theorem 2, if $-\int \phi \, d\mu < \infty$, then μ is the unique equilibrium state for ϕ.*

We are now ready to apply this theory to our class of dynamical systems. The following proposition contains the main ideas for the proof of Theorem C, but we state and prove it separately to highlight the connection with the results in Section 7.

Proposition 7. *For each $\lambda \in (0,1)$ and any $t \leqslant t_1$,*

(a) there is a unique p such that $H(p,t) = 1$ with all summands non-negative;
(b) this p is the unique value such that there is a $(\Phi_t - p)$-conformal measure.

Proof. We first prove the proposition for the case $t < t_1$, in which case, any p satisfying $H(p,t) = 1$ with all summands non-negative, must be strictly positive. The existence of such a p follows by Lemma 7. By Theorems 1 and 2, for p as in (a) of the proposition, the potential $\Phi_t - \tau p$ is (positive) recurrent. Theorems 1 also implies that $P_G(\Phi_t - \tau p) = 0$. Since $\tau \geqslant 1$, for $\varepsilon > 0$ we always have $P_G(\Phi_t - \tau(p - \varepsilon)) \geqslant P_G(\Phi_t - \tau p) + \varepsilon$: this means that any such p is unique. To summarise, there is one and only one p such that $H(p,t) = 1$ with all non-negative summands and for this p, we have $P_G(\Phi_t - \tau p) = 0$. It is easy to see that such a p yields a $(\Phi_t - \tau p)$-conformal measure.

For the case $t = t_1$, by [BT3, Theorem B], $P_G(\Phi_t) = 0$. Theorem B of that paper guarantees that $p = 0$ is a solution to $H(p,t) = 1$ with all summands positive. The above argument also shows in this case that if there is a solution $p > 0$ to $H(p,t) = 1$ with all summands non-negative, then $P_G(\Phi_t - \tau p) = 0$ and again this can only occur if $p = 0$. To show that there is no negative solution, observe

$$w_j^t = e^\beta e^{-pS_{j-1}} \left(1 - \sum_{k<j-1} w_k^t\right) > e^\beta e^{-pS_{j-1}} w_j.$$

Therefore we must have $p = 0$ as the only solution to $H(p,t) = 1$ with all summands positive. □

9.2. **Proof of Theorem C.**

Proof of Theorem C. We prove parts (a) and (b) simultaneously. First suppose that $t < t_1$. As in the proof of Proposition 7, $\Phi_t - \tau P_{\mathrm{Conf}}(\phi_t)$ is positive recurrent. By Theorem 3, μ_t from Theorem 1 is an equilibrium state for $\Phi_t - \tau P_{\mathrm{Conf}}(\phi_t)$ and hence satisfies

$$h(\mu_t) + \int (\Phi_t - \tau P_{\mathrm{Conf}}(\phi_t)) d\mu_t = 0.$$

Thus the Abramov formula implies that the projected measure ν_t has $h(\nu_t) + \int \phi_t \, d\nu_t = P_{\mathrm{Conf}}(\phi_t)$, so $P(\phi_t) \geqslant P_{\mathrm{Conf}}(\phi_t)$. If $P(\phi_t) > P_{\mathrm{Conf}}(\phi_t)$ then there exists a measure ν (with positive entropy) for which $h(\nu) + \int \phi_t - P_{\mathrm{Conf}}(\phi_t) \, d\nu > 0$. Since any such measure must lift to (Y, F_λ), the Abramov formula and Proposition 6 lead to a contradiction. Hence $P(\phi_t) = P_{\mathrm{Conf}}(\phi_t)$. This also implies that ν_t is the unique equilibrium state for ϕ_t.

For the case $t = t_1$, Proposition 7 implies that $P_{\mathrm{Conf}}(\phi_t) = 0$. This is clearly the same as $P(\phi_t)$, as follows continuity of the pressure. The existence/absence of an equilibrium state here follows as in Theorem A.

Now let $t > t_1$. For each $\lambda \in (0,1)$, [BT3, Theorem A] implies that the Φ_t-conformal measure m_t, if it exists, is dissipative. Hence no finite $\mu_t \ll m_t$ exists. However, just as for smooth Fibonacci maps, $\omega(c)$ supports a unique probability measure ν_ω, which has zero entropy. For each $x \in \omega(c)$ not eventually mapping to c, $Df_\lambda^n(x)$ exists for all n. Moreover, $F_\lambda^k(x) \to c$ so that if $n_k \in \mathbb{N}$ is such that $F_\lambda^k(x) = f^{n_k}(x)$, then $k/n_k \to 0$. The Lyapunov exponent of x under F_λ is $-\log[\lambda(1-\lambda)]$, hence by part c) of Lemma 1, the Lyapunov exponent of x under f_λ is $\lim_{k\to\infty} -\frac{k}{n_k}\log[\lambda(1-\lambda)] = 0$. Therefore ν_ω is an equilibrium state in this case (and in fact also for $t = t_1$). This concludes the proof of (a) and (b).

Now for part c), Lemma 2 implies that $P_{\mathrm{Conf}}(\phi_t) > 0$ for $t < t_1$. Now if $t > t_1$, then $p = 0$ still gives a conformal measure, see (25). This is the smallest (and only) value of p to do so, because if $p < 0$, then $H(p,t)$ no longer converges. Indeed, by taking the linear combinations in (24), we get

$$\tilde{w}_{j+1}^t = e^{-pS_j}\left(\tilde{w}_j^t e^{pS_{j-1}} - e^\beta \tilde{w}_{j-1}^t\right).$$

If $p < 0$, we can no longer assert that $\tilde{w}_j^t$ is decreasing in j, but if $H(p,t)$ converges, then there must be (infinitely many) js such that $\tilde{w}_j^t \leqslant \tilde{w}_{j-1}^t$. If also j is so large that $e^{-pS_{j-1}} > [\lambda(1-\lambda)]^{-t}$, then the equation gives that $\tilde{w}_{j+1}^t < 0$, which is not allowed. (The only other way of creating a conformal measure for f, is by putting Dirac masses on the critical point and its backward orbit. Since $f'(c) = 0$, this enforces no mass on the forward critical orbit. But f' is not defined at $f^{-1}(c) = \{z_0, \hat{z}_0\}$, so this gives no solution.) Therefore $P_{\mathrm{Conf}}(\phi_t) = 0$ for $t \geqslant t_1$.

Now we turn to analyticity. As in for example [IT1], the existence of a unique equilibrium state of positive entropy implies that $p(t) : t \mapsto P(\phi_t)$ is C^1. (We can also use the fact that $p'(t) = -\int \log|Df_\lambda|\, d\nu_t$, which is easily shown to be continuous in t.) It is easy to see that $Dp(t) < 0$ for $t < t_1$. Therefore we have, as in Proposition 5, that $p(t)$ is real analytic on $(-\infty, t_1)$. □

9.3. **Proof of Theorem D.** The following proposition, which should be compared to [IT1, Proposition 1.2], will tell us the shape of the pressure function at t_1. This also gives part (d) of Theorem A.

Proposition 8. *The following are equivalent.*

(a) The left derivative $D_-p(t_1) < 0$;
(b) There exists $K > 0$, $\delta > 0$ so that for all $t \in (t_1 - \delta, t_1)$ there is an equilibrium state ν_t for $-t\log|Df|$ and for the induced version μ_t,

$$\int \tau \, d\mu_t = \sum_k S_{k-1}\mu_t(W_k) \leqslant K.$$

Indeed, when the above holds, there is an equilibrium state ν_{t_1} for ϕ_{t_1} and $\int \tau\, d\mu_{t_1} \leqslant K$.

Proof of Proposition 8. First assume that $K < \infty$ as in item (b) exists. Since $p'(t) = -\int \log|f_\lambda'|\ d\nu_t$, the Abramov formula implies

$$\int \log|f_\lambda'|\ d\nu_t = \frac{\int \log|F_\lambda'|\ d\mu_t}{\int \tau\ d\mu_t} \geqslant -\frac{\log\lambda}{K},$$

uniformly in t, *i.e.,* $D_-p(t_1) \leqslant \frac{\log\lambda}{K} < 0$.

Now let us suppose that $D_-p(t_1) < 0$. As in [IT1, Lemma 4.2], there exists $\eta > 0$ such that any measure $\nu \in \mathcal{M}$ with $h(\nu) - t\lambda(\nu)$ sufficiently close to $p(t)$ has $h(\nu) \geqslant \eta$. Suppose that $(\nu_n)_n$ is a sequence of measures such that $h(\nu_n) - t\lambda(\nu_n) \to p(t)$. For each n, we denote the induced version of ν_n by μ_n. Now applying the Abramov formula and since $h_{top}(F_\lambda) = \log 4$, we obtain for all large n,

$$\eta \leqslant h(\nu_n) = \frac{h(\mu_n)}{\int \tau\ d\mu_n} \leqslant \frac{\log 4}{\int \tau\ d\mu_n},$$

so $\int \tau\ d\mu_n \leqslant (\log 4)/\eta$.

Since $\int \tau\ d\mu_n \leqslant (\log 4)/\eta$ for all large n, for any $\eta' > 0$, there must be some $N \in \mathbb{N}$ such that $\mu_n\left(\cup_{k=1}^N W_k\right) > 1 - \eta'$ for all large n. Notice that the choice of $(\nu_n)_n$, the Abramov formula and the uniform bound on the integral of inducing times implies that

$$h(\mu_n) - \int (\Phi_t - \tau p(t))\ d\mu_n \to 0 \text{ as } n \to \infty.$$

The proof now concludes by a tightness argument. Let μ_∞ be a vague limit of $(\mu_n)_n$, see for example [Bi, Section 28]. This measure is non-zero since $\mu_n\left(\cup_{k=1}^N W_k\right) > 1-\eta'$ for all $n \in \mathbb{N}$. We may assume that it is a probability measure. The Monotone Convergence Theorem implies that $\int \tau\ d\mu_\infty \leqslant (\log 4)/\eta$. Moreover, the continuity of Φ_t and the upper semi-continuity of $-\tau$ implies that μ_∞ is an equilibrium state for $\Phi_t - \tau p(t)$. The fact that the integral of the inducing time is finite implies that we can project μ_∞ to an equilibrium state ν_t for ϕ_t, as required. □

Proof of Theorem D. The lower and upper bounds for the pressure on a left neighbourhood of t_1 stated in (a) and (b) follow from Proposition 3 (with in one case $\tau_0\tilde{C}$ renamed to τ_0') and Propostion 4 respectively. Finally, part (c) follows from Proposition 8. □

9.4. **Recurrence and transience.** We finish the paper with a brief discussion of recurrence/transience in the context of our examples using the definitions given above. Since we can view (Y, F_λ) as a countable Markov shift, by Theorem 2, Proposition 7 and Theorem 1 we have the following results for the system $(Y, F_\lambda, \Phi_t - \tau p)$: note that the precise behaviour at $t = t_1$ is governed by the case $p = 0$ which is discussed in Section 4, see also [BT3]:

- If $\lambda \in (1/2, 1)$ then $(Y, F_\lambda, \Phi_t - \tau p)$ is recurrent iff $t < t_1 < 1$ and $p = P_{\text{Conf}}(\phi_t)$. Whenever the system is recurrent, it is positive recurrent .
- If $\lambda \in (0, 1/2)$ then $(Y, F_\lambda, \Phi_t - \tau p)$ is recurrent iff $t \leqslant t_1 = 1$ and $p = P_{\text{Conf}}(\phi_t)$. Whenever the system is recurrent, it is positive recurrent.
- If $\lambda = 1/2$ then $(Y, F_\lambda, \Phi_t - \tau p)$ is recurrent iff $t \leqslant t_1 = 1$ and $p = P_{\text{Conf}}(\phi_t)$. It is null recurrent for $t = 1$ and positive recurrent if $t < 1$.

For the original system, the Markov shift model is less easy to handle, so we prefer an alternative definition of recurrence. In [IT2] a system (X, f, ϕ) was called recurrent whenever there was a conservative ϕ-conformal measure m_ϕ and transient otherwise. A recurrent system was defined as being positive recurrent if there was an f-invariant probability measure $\mu_\phi \ll m_\phi$, and null-recurrent otherwise. With this in mind, the results of this paper allow use to state:

- If $\lambda \in (1/2, 1)$ then $(I, f_\lambda, \phi_t - p)$ is recurrent iff $t < t_1 < 1$ and $p = P_{\mathrm{Conf}}(\phi_t)$. Whenever the system is recurrent, it is positive recurrent.
- If $\lambda \in (0, 1/2]$ then $(I, f_\lambda, \phi_t - p)$ is recurrent iff $t \leqslant t_1 = 1$ and $p = P_{\mathrm{Conf}}(\phi_t)$. When the system is recurrent and $p = P_{\mathrm{Conf}}(\phi_t)$, it is positive recurrent iff $\lambda \in (0, \frac{2}{3+\sqrt{5}})$.

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