MULTIVARIATE EXTREME VALUES FOR DYNAMICAL SYSTEMS

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Abstract. We establish a theory for multivariate extreme value analysis of dynamical systems. Namely, we provide conditions adapted to the dynamical setting which enable the study of dependence between extreme values of the components of $\mathbb{R}^d$-valued observables evaluated along the orbits of the systems. We study this cross-sectional dependence, which results from the combination of a spatial and a temporal dependence structures. We give several illustrative applications, where concrete systems and dependence sources are introduced and analysed.

1. Introduction

The study of rare events for dynamical systems is recent but has experienced a vast development in the last decade, partly motivated by applications to climate dynamics, where dynamical systems (such has the Lorenz models) provide accurate description of meteorological phenomena. This development has been anchored in a connection between the observation of rare events, detected by the appearance of extreme values, and the recurrence properties of sensitive regions, under the action of the underlying dynamics. The main idea is that chaotic systems lose memory quickly which makes their orbits behave like random asymptotically independent observations. This strategy has been successfully applied to prove the existence of limit theorems regarding the distribution of the extremal order statistics, point processes, records, as well as ergodic averages of heavy tailed observables.

In some sense, the study of extreme events for dynamical systems has only recently caught up with the state of the art of univariate Extreme Value Theory. However, since the 1980s, many extreme value theorists have moved from the univariate theory to the multivariate context, where one is concerned with extremes in a multivariate random sample, *i.e.*, events for which at least one of the components reaches exceptionally high (or low) values. Of course, focusing on the behaviour of one of the components is the subject of univariate extreme value theory. The main point of multivariate extreme value analysis is understanding the interplay between the extremes in the different components. This insight is of crucial importance in climate dynamics, where the influence between extremal observations of different variables (such as pressure and temperature) as well as their spatial and temporal dependence is vital for predicting extreme weather events. The dependence structure of extreme phenomena is

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pivotal for anticipating compound risks such as in the case of the 2022 European drought addressed in [FPB23] or the study of co-occurring extremes in [MF23] or multivariate rainfall time series in [BN23], just to give some recent examples.

The main goal of this paper is to introduce the first (to our knowledge) theoretical results on multivariate extreme value analysis for dynamical systems, thus closing the gap in theories described above. Unlike independent sequences or the strong mixing extremal properties of the stationary processes from the classical literature, the processes arising from dynamical systems require a more flexible time dependence structure, which we provide here. We will then study the cross-sectional dependence of the vector valued observables evaluated along the orbits of the systems, which will describe the spatial relationships where an extreme observation of one of the components is responsible for the appearance of other extreme observation in another component, but we will also analyse how the short recurrence properties may introduce a source of time dependence, so that the co-occurrence of extremes of different quantities appear slightly out of sync in time. This phenomenon is directly connected with clustering of extremal observations which can be viewed through an extremal index function: here we give a dynamically natural formula for this. Interestingly, while the spatial dependence pertaining to the cross-sectional relations between the components was copiously studied, the literature regarding to the time dependence and the extremal index function is relatively scarce. One of the advantages of this dynamical approach to multivariate extremes is that we can see, in a natural and simple way, how the local (or fast) recurrence properties of the maximal regions contribute to the overall dependence structure of extremes, making it one of the interests of the present paper.

We emphasise that the multivariate extremes perspective here is completely new in the dynamical systems setting, although there have been works considering higher dimensional variables such as [FV18] or multidimensional point processes accounting for the extremes of $\mathbb{R}^d$-valued dynamically defined stationary processes as in [FFMa20, FFT21]. While in the former case, the authors take observables corresponding to the distance to the diagonal, reducing it to a univariate problem, in the latter case the relations between the d-components are not considered.

The paper is organised as follows. In Section 2 we make a brief introduction to Multivariate Extreme Value Analysis, present the pertinent concepts and objects and then, in Section 2.2, we give the main theoretical results which enable its application to dynamical systems. In Section 3, we give some illustrative applications, introducing some mechanisms to create both spatial and temporal dependence, based on particular choices of observables and dynamics so that the recurrence properties of the respective maximal sets give rise to different dependence profiles. We remark that the examples presented and the respective mechanisms are meant to illustrate the potential of the theory and of its applications, so as to make a clear contribution in both the dynamical and the extreme settings: there is ample scope to elaborate further on these, but here we are focused on presenting the main ideas in a simple way.

2. Multivariate extreme value analysis

Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, f)$ be a discrete time dynamical system, where $\mathcal{X}$ is a compact manifold equipped with a norm $\|\cdot\|$, $\mathcal{B}_{\mathcal{X}}$ is its Borel σ-algebra, $f : \mathcal{X} \to \mathcal{X}$ is a measurable map and μ is a f-invariant probability measure, *i.e.*, $\mu(f^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}_{\mathcal{X}}$. Let $\Psi : \mathcal{X} \to \mathbb{R}^d$

be an observable (measurable) function and define the stochastic process $\mathbb{X}_0, \mathbb{X}_1, \ldots$ given by

$$\mathbb{X}_n = \Psi \circ f^n, \qquad \text{for every } n \in \mathbb{N}_0. \tag{2.1}$$

We will use the notation $\mathbb{X}_n = (X_{n1}, \ldots, X_{nd})$ and $\Psi = (\psi_1, \ldots, \psi_s)$ whenever we need to refer to the respective components specifically.

In order to establish notation, we will use blackboard bold for vectors or vector valued functions taking values in $\mathbb{R}^d$. In particular, $\mathbb{0}, \mathbb{1}$ correspond to the vectors in $\mathbb{R}^d$ with all entries equal to 0 and 1, respectively. Operations and functions applied to vectors are to be interpreted componentwise, so for example: for $\mathbb{t}, \mathbb{\tau} \in (0,\infty)^d$ and $c > 0$, we write

$$\mathrm{e}^{-\mathbb{\tau}} := \left(\mathrm{e}^{-\tau_1}, \ldots, \mathrm{e}^{-\tau_d}\right), \quad (\mathbb{\tau} + \mathbb{t})^c := ((\tau_1 + t_1)^c, \ldots, (\tau_d + t_d)^c), \quad \mathbb{\tau}/\mathbb{t} := \left(\frac{\tau_1}{t_1}, \ldots, \frac{\tau_d}{t_d}\right).$$

Our main goal is to study the multivariate extremal behaviour of such stochastic processes arising from chaotic dynamics. For that purpose we introduce the componentwise maxima sequence

$$\mathbb{M}_n := (M_{n1}, \ldots, M_{nd}), \quad \text{where} \quad M_{nj} := \max_{i=0,\ldots,n-1} X_{ij}, \quad \text{for} \quad j = 1, \ldots, d.$$

For an asymptotic frequency vector $\mathbb{\tau} = (\tau_1, \ldots, \tau_d) \in [0, +\infty)^d \setminus \mathbb{0}$, we consider a sequence of normalising vectors $(\mathbb{u}_n(\mathbb{\tau}))_n$ such that $\mathbb{u}_n(\mathbb{\tau}) = (u_{n1}(\tau_1), \ldots, u_{nd}(\tau_d))$ and

$$\lim_{n\to\infty} n\mu(X_{0j} > u_{nj}(\tau_j)) = \lim_{n\to\infty} n\mu\left(\{x \in \mathcal{X} \colon \psi_j(x) > u_{nj}(\tau_j)\}\right) = \tau_j, \quad j = 1, \ldots, d. \tag{2.2}$$

Let $\mathbb{t} \in (0,1)^d$ be such that $\mathbb{t} = \mathrm{e}^{-\mathbb{\tau}}$ (or equivalently $\mathbb{\tau} = -\log \mathbb{t}$). We aim to find a multivariate extreme value distribution function (d.f.) H supported $[0,1]^d$ and such that

$$\lim_{n\to\infty} \mu(\mathbb{M}_n \leq \mathbb{u}_n(-\log \mathbb{t})) = H(\mathbb{t}). \tag{2.3}$$

2.1. **The classical setting.** Let $\hat{\mathbb{X}}_0, \hat{\mathbb{X}}_1, \ldots$ denote an associated i.i.d. sequence of random vectors with $\hat{\mathbb{X}}_0 = \Psi$ (i.e., $\mu(\hat{\mathbb{X}}_i > \mathbb{t}) = \mu(\{x : \Psi(x) > \mathbb{t}\})$). Also define the respective sequence of partial maxima vectors $(\hat{\mathbb{M}}_n)_n$ with components $\hat{M}_{nj}$, $j = 1, \ldots, d$ and multivariate extreme value d.f. $\hat{H}$ analogously as for $\mathbb{X}_n$. Note that by (2.2) and the definition of $\mathbb{t}$ we have that $\hat{H}$ has uniform marginals, namely,

$$\hat{H}_j(t_j) = \lim_{n\to\infty} \mu(\hat{M}_{nj} \leq u_{nj}(-\log t_j)) = \lim_{n\to\infty} \left(\mu\left(\psi_j \leq u_{nj}(-\log t_j)\right)\right)^n = t_j, \quad j = 1, \ldots, d.$$

The main interest of multivariate analysis is understanding the dependence between the various components of the multivariate observations. For this purpose several devices such as dependence functions and intensity measures have been introduced and thoroughly studied (see [FHR11, Seg12], for example). Our main tool here will be the stable tail dependence function associated to a copula (or the D-norm in the terminology of [FHR11], where D is the Pickands dependence function).

We start by introducing the copula of a multivariate d.f. $\mathbb{F}$, following [Hsi89] closely.

Definition 2.1. Let $\mathbb{Y} = (Y_1, \ldots Y_d)$ be a random vector defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with d.f. $\mathbb{F}$. We define the *copula* $C_{\mathbb{F}}$ as a multivariate d.f. supported on $[0,1]^d$ and such that

$$C_{\mathbb{F}}(\mathbb{t}) = \mathbb{P}(F_1(Y_1) \leq t_1, \ldots, F_d(Y_d) \leq t_d),$$

where $(F_j)_j$ denote the marginals of $\mathbb{F}$.

The copula describes how the dependence between the components affects the joint distribution in the sense that we can recover $\mathbb{F}$ from its marginals: $\mathbb{F}(\mathbb{t}) = C_{\mathbb{F}}(F_1(t_1), \ldots, F_d(t_d))$.

Definition 2.2. A copula C is said to be an *extreme value copula* if it is the copula associated to a d.f. arising as a weak limit for the distribution of $\mathbb{M}_n$.

Remark 2.3. Observe that both C_H and $C_{\hat{H}}$ are extreme value copulas and since $\hat{H}$ has uniform marginals then $C_{\hat{H}} = \hat{H}$ (see [Nan94, Section 1]).

In the i.i.d. setting, letting $\mathbb{F}$ denote the d.f. of $\hat{\mathbb{X}}_0$, then $\mathbb{F}_n(\mathbb{t}) := (\mathbb{F}(\mathbb{t}))^n$ is the d.f. of $\hat{\mathbb{M}}_n$ and one can also show that for every $n \in \mathbb{N}$, we have $C_{\mathbb{F}_n}(\mathbb{t}) = \left(C_{\mathbb{F}}(\mathbb{t}^{1/n})\right)^n$ (see [Hsi89, Lemma 2.2]), which eventually leads to the homogeneity property:

$$C_{\hat{H}}(\mathbb{t}^c) = \left(C_{\hat{H}}(\mathbb{t})\right)^c \qquad \text{for all} \quad \mathbb{t} \in [0,1]^d \quad \text{and} \quad c > 0. \tag{2.4}$$

This homogeneity property is often referred to as max-stability and it is easy to see that in the i.i.d. setting the class of extreme value copulas coincides with that of max-stable copulas. Moreover, weak convergence of a sequence of multivariate distributions can be decomposed into two parts, one corresponding to the convergence of the marginals (which is a univariate problem) and the convergence of the copulas (see [Hsi89, Section 3] and references therein). Since the weak convergence of the marginals is a univariate problem, the main interest of multivariate analysis lies in understanding the copulas.

Next we introduce stable dependence functions.

Definition 2.4. The *stable dependence function* associated to an extreme value copula C is defined for $\mathbb{\tau} \in [0,\infty)^d$ as

$$\Gamma(\mathbb{\tau}) := -\log C\left(\mathrm{e}^{-\mathbb{\tau}}\right).$$

Sometimes $\Gamma(\mathbb{\tau})$ is referred to as the *D-norm* of $\mathbb{\tau}$ ([FHR11]). Note that in the i.i.d. setting, since we have uniform marginals, $\hat{\Gamma}(\mathbb{\tau}) = -\log \hat{H}\left(\mathrm{e}^{-\mathbb{\tau}}\right)$. Let $\hat{G}(\mathbb{\tau}) := \mathrm{e}^{-\hat{\Gamma}(\mathbb{\tau})}$. As in the univariate context (see [LLR83, Theorem 1.5.1]), one can show that the existence of the limit $\lim_{n\to\infty} \mu(\hat{\mathbb{M}}_n \leq \mathbb{u}_n(\mathbb{\tau})) = \hat{G}(\mathbb{\tau})$ is equivalent to the existence of the limit

$$\lim_{n\to\infty} n\mu(\mathbb{X}_0 \not\leq \mathbb{u}_n(\mathbb{\tau})) = \lim_{n\to\infty} n\mu\left(\bigcup_{i=1}^d \{X_{0j} > u_n(\tau_j)\}\right) = \hat{\Gamma}(\mathbb{\tau}). \tag{2.5}$$

The homogeneity property (2.4) translates to

$$\hat{\Gamma}(c\mathbb{\tau}) = c\,\hat{\Gamma}(\mathbb{\tau}) \qquad \text{and} \qquad \hat{G}(c\mathbb{\tau}) = (\hat{G}(\mathbb{\tau}))^c, \qquad \text{for all} \quad c > 0. \tag{2.6}$$

Moreover, we have that

$$\max\{\tau_1, \ldots, \tau_d\} \leq \hat{\Gamma}(\mathbb{\tau}) \leq \tau_1 + \cdots + \tau_d. \tag{2.7}$$

where the upper bound corresponds to asymptotic component independence, while the lower bound corresponds to asymptotic perfect association. Note that both bounds correspond to realisable stable dependence functions.

When we consider stationary stochastic processes and drop the independence assumption, a new source of dependence may appear. This temporal dependence is associated to clustering of extremal multivariate observations (corresponding to at least one component taking an extremal value), which is conveniently described by a multivariate extremal index function $\theta(\mathbb{\tau})$

introduced in [Nan94]. Recall that, in the univariate case, the extremal index is a parameter $\theta_j \in [0,1]$ that appears when, for every $\tau_j \geq 0$ and $(u_{nj}(\tau_j))_n$ as in (2.2), we have

$$\lim_{n\to\infty} \mu(M_{nj} \leq u_{nj}(\tau_j)) = \mathrm{e}^{-\theta_j \tau_j}. \tag{2.8}$$

Note that, when this limit exists (which we express by saying that θ_j exists or is well defined) then the marginals of H are such that $H_j(t_j) = t_j^{\theta_j}$. Therefore, assuming that for each $j = 1, \dots, d$, every θ_j is well defined and putting

$$\mathbb{\theta} := (\theta_1, \dots, \theta_d),$$

we have that $C_H(\mathbb{t}^{\mathbb{\theta}}) = H(\mathbb{t})$ and consequently $H\left(\mathrm{e}^{-\mathbb{\tau}}\right) = \mathrm{e}^{-\Gamma(\mathbb{\theta}\mathbb{\tau})} =: G(\mathbb{\tau})$.

Definition 2.5. For every $\mathbb{\tau} \in [0,\infty)^d \setminus \mathbb{0}$, we define the multivariate extremal index function as:

$$\theta(\mathbb{\tau}) := \frac{\log G(\mathbb{\tau})}{\log \hat{G}(\mathbb{\tau})} = \frac{\Gamma(\mathbb{\theta}\mathbb{\tau})}{\hat{\Gamma}(\mathbb{\tau})}. \tag{2.9}$$

In [Nan94, Proposition 3.1], under a distributional mixing assumption inspired by Leadbetter's D condition, denoted there by Δ, it was proved that $\theta(c\mathbb{\tau}) = \theta(\mathbb{\tau})$ for all $c > 0$, which means that both Γ and G share the same homogeneity property of their hat versions stated in (2.6), and the univariate marginal extremal indices θ_j, $j = 1, \dots, d$, can be recovered from the multivariate extremal index function by setting all coordinates of $\mathbb{\tau}$ equal to 0 except the j-th. Moreover, one can check that the bounds of (2.7) also apply to Γ.

As we explain below, motivated by the dynamical applications, we will assume a mixing condition weaker than Leadbetter's, we will use a more tractable formula for the multivariate extremal index function and we will prove that these properties still hold in that context.

The homogeneity property of Γ suggests a reduction to the $(d-1)$-dimensional unit simplex $\mathcal{S}_d = \{\mathbb{\alpha} \in [0,1]^d \colon \alpha_1 + \cdots + \alpha_d = 1\}$: the restriction of Γ to $\mathcal{S}_d$ is often called the *Pickands dependence function* D:

$$\Gamma(\mathbb{\tau}) = (\tau_1 + \cdots + \tau_d) D\left(\alpha_1, \dots, \alpha_{d-1}\right), \quad \text{where} \quad \alpha_j = \frac{\tau_j}{\tau_1 + \cdots + \tau_d}.$$

2.2. **A dynamically adapted approach to the study of multivariate extremes.** Our main goal here is to study the convergence (2.3) for dynamically defined multivariate stochastic processes as in (2.1) and to give a more computable formula for $\theta(\mathbb{\tau})$. Convergence for general stationary stochastic processes was initiated in [Hsi89, Hüs89] under a distributional mixing condition very much akin to the original D condition introduced by Leadbetter in the univariate context. This condition has been used ubiquitously to study extremes of stationary random vectors. The problem is that this condition is not amenable to application in the dynamical setting and therefore we propose a weaker version adapted to this context motivated by our earlier works in the univariate framework, where this weakness is compensated by a condition similar to the $D^{(k)}$ condition from [CHM91], which allows clustering but forbids the concentration of clusters, so that we can still recover the existence of an extremal limit. This is accomplished using an idea introduced in [FFT12] and further elaborated in [FFT15], which essentially says that we may replace the occurrence of an abnormal observation by that consisting of an abnormal observation followed by a block of regular observations, which in the dynamical setting corresponds to replacing balls by annuli. For that purpose, for $q \in \mathbb{N}$ and $\mathbb{\tau} \in [0,\infty)^d$ we define

$$A_n^{(q)}(\mathbb{\tau}) = \left\{\mathbb{X}_0 \not\leq \mathbb{u}_n(\mathbb{\tau}) \cap f^{-1}(\mathbb{M}_q \leq \mathbb{u}_n(\mathbb{\tau}))\right\} = \left\{\mathbb{X}_0 \not\leq \mathbb{u}_n(\mathbb{\tau}), \mathbb{X}_1 \leq \mathbb{u}_n(\mathbb{\tau}), \ldots, \mathbb{X}_q \leq \mathbb{u}_n(\mathbb{\tau})\right\}.$$

We also set $A_n^{(0)}(\mathbb{\tau}) := \{\mathbb{X}_0 \not\leq \mathbb{u}_n(\mathbb{\tau})\}$. Let $B \in \mathcal{B}_{\mathcal{X}}$ be an event. For some $s, \ell \in \mathbb{N}_0$, we define:

$$\mathscr{W}_{s,\ell}(B) = \bigcap_{i=s}^{s+\max\{\ell-1,\ 0\}} f^{-i}(B^c).$$

We will write $\mathscr{W}_{s,\ell}^c(B) := (\mathscr{W}_{s,\ell}(B))^c$. Observe that $\mathscr{W}_{0,n}(A_n^{(0)}(\mathbb{\tau})) = \{\mathbb{M}_n \leq \mathbb{u}_n(\mathbb{\tau})\}$.

We state now the two main conditions on the time dependence structure of the process.

Condition ($Д(\mathbb{u}_n)$)**.** We say that $Д(\mathbb{u}_n)$ holds for the sequence $\mathbb{X}_0, \mathbb{X}_1, \ldots$ if for every $\ell, t, n \in \mathbb{N}$ and $q \in \mathbb{N}_0$,

$$\left|\mu\left(A_n^{(q)}(\mathbb{\tau}) \cap \mathscr{W}_{t,\ell}\left(A_n^{(q)}(\mathbb{\tau})\right)\right) - \mu\left(A_n^{(q)}(\mathbb{\tau})\right)\mu\left(\mathscr{W}_{0,\ell}\left(A_n^{(q)}(\mathbb{\tau})\right)\right)\right| \leq \gamma(q,n,t),$$

where $\gamma(q,n,t)$ is decreasing in t for each q, n and, for every $q \in \mathbb{N}_0$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(n)$ and $\lim_{q\to\infty} \limsup_{n\to\infty} n\gamma(q,n,t_n) = 0$.

The condition above is a mixing type condition obtained from adjusting to the multivariate setting the homonymous condition from our previous work (see [FFT15] for example), which has the advantage of not imposing a uniform bound on q. This means that, as with its univariate version, it will follow easily for systems with sufficiently fast decay of correlations as in all the examples considered below.

We now introduce the corresponding clustering separation condition. For some fixed $q \in \mathbb{N}_0$, consider the sequence $(t_n)_{n\in\mathbb{N}}$, given by condition $Д(u_n)$ and let $(k_n)_{n\in\mathbb{N}}$ be another sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \tag{2.10}$$

Condition ($Д'(\mathbb{u}_n)$)**.** We say that $Д'(\mathbb{u}_n)$ holds for the sequence $\mathbb{X}_0, \mathbb{X}_1, \mathbb{X}_2, \ldots$ if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (2.10) and such that

$$\lim_{q\to\infty} \Delta^{(q)}(\mathbb{u}_n(\mathbb{\tau})) := \lim_{q\to\infty} \limsup_{n\to\infty} n \sum_{j=q+1}^{\lfloor n/k_n \rfloor - 1} \mu\left(A_n^{(q)}(\mathbb{\tau}) \cap f^{-j}\left(A_n^{(q)}(\mathbb{\tau})\right)\right) = 0.$$

The cluster separating condition we give here uses a double limit as the original anti-clustering condition D' from Leadbetter, as opposed to the more general version we used in [FFT21], with a diverging $(q_n)_n$ sequence. This option guarantees a cleaner and easier proof of some of the properties of $\theta(\mathbb{\tau})$ and $\Gamma(\mathbb{\tau})$, without compromising the applications since it will be satisfied in all examples considered below. Note that $Д'(\mathbb{u}_n)$ does not forbid the appearance of clustering, it just imposes the clusters to appear sufficiently well separated in the time line.

We give now a formula for the extremal index function, which together with the dependence conditions above, will allow us to verify that it actually provides an alternative definition, which coincides with the original one and enjoys the same properties:

$$\theta(\mathbb{\tau}) = \lim_{q\to\infty} \lim_{n\to\infty} \frac{\mu\left(A_n^{(q)}(\mathbb{\tau})\right)}{\mu\left(A_n^{(0)}(\mathbb{\tau})\right)}. \tag{2.11}$$

Note that as in condition $Д'(\mathbb{u}_n)$ we use a double limit, which is also the case in [Per97, HV20], for example.

Theorem 2.6. *Let $\mathbb{X}_0, \mathbb{X}_1, \dots$ be a stationary multivariate stochastic process as in* (2.1) *and for $\mathbb{\tau} \in [0,\infty)^d \setminus \mathbb{0}$, let $\mathbb{u}_n(\mathbb{\tau})$ be a sequence such that both* (2.2) *and* (2.5) *hold, for some $\hat{\Gamma}(\mathbb{\tau})$. Assume further that conditions $Д(\mathbb{u}_n)$ and $Д'(\mathbb{u}_n)$ hold and that $\theta(\mathbb{\tau})$ given by* (2.11) *is well defined. Then, we have*

$$\lim_{n\to\infty} \mu(\mathbb{M}_n \le \mathbb{u}_n(\mathbb{\tau})) = \mathrm{e}^{-\theta(\mathbb{\tau})\hat{\Gamma}(\mathbb{\tau})} = \mathrm{e}^{-\Gamma(\theta\mathbb{\tau})} = G(\mathbb{\tau}),$$

where $G(\mathbb{\tau})$ and $\Gamma(\mathbb{\tau})$ satisfy the homogeneity property stated in (2.6)*. Moreover, the marginal univariate extremal indices θ_j, $j = 1,\dots,d$, can be recovered from $\theta(\mathbb{\tau})$ by setting all coordinates of $\mathbb{\tau}$ equal to* 0 *except for the j-th and Γ satisfies* (2.7)*.*

Proof. We follow the proof of [FFT15, Corollary 2.4] closely, whose main steps we recap here since the conditions $Д(\mathbb{u}_n)$ and $Д'(\mathbb{u}_n)$ are somewhat different because they involve double limits. We start by noting that by direct application of [FFT15, Proposition 2.7] we have

$$\left|\mu(\mathbb{M}_n \le \mathbb{u}_n(\mathbb{\tau})) - \mu\left(\mathscr{W}_{0,n}(A_n^{(q)}(\mathbb{\tau}))\right)\right| \le q\mu(A_n^{(0)}(\mathbb{\tau}) \setminus A_n^{(q)}(\mathbb{\tau})) \xrightarrow[n\to\infty]{} 0.$$

Now, from [FFT15, Proposition 2.10] we obtain

$$\left|\mu\left(\mathscr{W}_{0,n}(A_n^{(q)}(\mathbb{\tau}))\right) - \left(1 - \left\lfloor\frac{n}{k_n}\right\rfloor \mu(A_n^{(q)}(\mathbb{\tau}))\right)^{k_n}\right| \le \Upsilon(q,n) := 2k_n t_n \mu(A_n^{(0)}(\mathbb{\tau})) + n\gamma(q,n,t_n)$$
$$+ n \sum_{j=q+1}^{\lfloor n/k_n\rfloor - 1} \mu\left(A_n^{(q)}(\mathbb{\tau}) \cap f^{-j}\left(A_n^{(q)}(\mathbb{\tau})\right)\right).$$

From the definitions of the sequences $(k_n)_n, (t_n)_n$ and conditions $Д(\mathbb{u}_n)$ and $Д'(\mathbb{u}_n)$ we have that $\lim_{q\to\infty}\lim_{n\to\infty}\Upsilon(q,n) = 0$. Moreover, since (2.9) and (2.5) hold, we have that

$$\lim_{q\to\infty}\lim_{n\to\infty}\left(1 - \left\lfloor\frac{n}{k_n}\right\rfloor \mu(A_n^{(q)}(\mathbb{\tau}))\right)^{k_n} = \lim_{q\to\infty}\lim_{n\to\infty}\left(1 - \frac{\mu(A_n^{(q)}(\mathbb{\tau}))}{\mu(A_n^{(0)}(\mathbb{\tau}))}\frac{n\mu(A_n^{(0)}(\mathbb{\tau}))}{k_n}\right)^{k_n} = \mathrm{e}^{-\theta(\mathbb{\tau})\hat{\Gamma}(\mathbb{\tau})}.$$

The homogeneity of $\hat{\Gamma}(\mathbb{\tau})$ can be derived easily from that of $C_{\hat{H}}$ or $\hat{G}$ (see for example [Seg12, Equations (2.4) and (2.7)]). The homogeneity of Γ and G will follow once we show that

$$\theta(c\mathbb{\tau}) = \theta(\mathbb{\tau}) \qquad \text{for all} \quad \mathbb{\tau} \in [0,\infty)^d \setminus \mathbb{0} \quad \text{and} \quad c \in (0,\infty).$$

For that purpose, we start by noting that for $c > 0$ and $\mathbb{\tau}$ as before, one can replace $\mathbb{u}_n(c\mathbb{\tau})$ by $\mathbb{u}_{\lfloor n/c\rfloor}(\mathbb{\tau})$. In fact, for all $i = 1,\dots,d$,

$$\lim_{n\to\infty}\frac{\mu\left(X_{0i} > u_{ni}(c\mathbb{\tau}_i)\right)}{\mu\left(X_{0i} > u_{\lfloor n/c\rfloor i}(\mathbb{\tau}_i)\right)} = \lim_{n\to\infty}\frac{\frac{c}{n}\lfloor\frac{n}{c}\rfloor n\mu\left(X_{0i} > u_{ni}(c\mathbb{\tau}_i)\right)}{cn\frac{1}{n}\lfloor\frac{n}{c}\rfloor\mu\left(X_{0i} > u_{\lfloor n/c\rfloor i}(\mathbb{\tau}_i)\right)} = \frac{c\tau_i}{c\tau_i} = 1.$$

We observe now that this implies that

$$\frac{\left|\mu(\mathbb{X}_0 \not\le \mathbb{u}_n(c\mathbb{\tau})) - \mu(\mathbb{X}_0 \not\le \mathbb{u}_{\lfloor n/c\rfloor}(\mathbb{\tau}))\right|}{\mu(\mathbb{X}_0 \not\le \mathbb{u}_n(c\mathbb{\tau}))} \le \frac{n\sum_{i=1}^d\left|\mu\left(X_{0i} > u_{ni}(c\mathbb{\tau}_i)\right) - \mu\left(X_{0i} > u_{\lfloor n/c\rfloor i}(\mathbb{\tau}_i)\right)\right|}{n\mu(\mathbb{X}_0 \not\le \mathbb{u}_n(c\mathbb{\tau}))}$$
$$\xrightarrow[n\to\infty]{} \frac{0}{\hat{\Gamma}(\mathbb{\tau})} = 0,$$

and therefore $\mu\left(\mathbb{X}_0 \not\leq \mathbb{u}_n(c\mathbb{T})\right) \sim \mu\left(\mathbb{X}_0 \not\leq \mathbb{u}_{\lfloor n/c\rfloor}(\mathbb{T})\right)$. A similar argument leads also shows that $\mu\left(\mathbb{X}_0 \not\leq \mathbb{u}_n(c\mathbb{T}) \cap f^{-1}\left(\mathbb{M}_q \leq \mathbb{u}_n(c\mathbb{T})\right)\right) \sim \mu\left(\mathbb{X}_0 \not\leq \mathbb{u}_{\lfloor n/c\rfloor}(\mathbb{T}) \cap f^{-1}\left(\mathbb{M}_q \leq \mathbb{u}_{\lfloor n/c\rfloor}(\mathbb{T})\right)\right)$. Whence,

$$\begin{aligned}\theta(c\mathbb{T}) &= \lim_{q\to\infty}\lim_{n\to\infty}\frac{\mu\left(\mathbb{X}_0 \not\leq \mathbb{u}_n(c\mathbb{T}) \cap f^{-1}\left(\mathbb{M}_q \leq \mathbb{u}_n(c\mathbb{T})\right)\right)}{\mu(\mathbb{X}_0 \not\leq \mathbb{u}_n(c\mathbb{T}))}\\ &= \lim_{q\to\infty}\lim_{n\to\infty}\frac{\mu\left(\mathbb{X}_0 \not\leq \mathbb{u}_{\lfloor n/c\rfloor}(\mathbb{T}) \cap f^{-1}\left(\mathbb{M}_q \leq \mathbb{u}_{\lfloor n/c\rfloor}(\mathbb{T})\right)\right)}{\mu(\mathbb{X}_0 \not\leq \mathbb{u}_{\lfloor n/c\rfloor}(\mathbb{T}))} = \theta(\mathbb{T}).\end{aligned}$$

For the statement regarding the marginal extremal indices, take w.l.o.g. $\mathbb{T} = (\tau_1, 0, \ldots, 0)$ for some $\tau_1 > 0$ and observe that $\{\mathbb{M}_n \leq \mathbb{u}_n(\mathbb{T})\} = \{M_{n1} \leq u_{n1}(1)\}$, $A_n^{(q)}(\mathbb{T}) = \{X_{01} > u_{n1}(\tau_1), X_{11} \leq u_{n1}(\tau_1), \ldots X_{q1} \leq u_{n1}(\tau_1)\} =: A_{n1}^{(q)}(\tau_1)$, $A_n^{(0)}(\mathbb{T}) = \{X_{01} > u_{n1}(\tau_1)\} := A_{n1}^{(0)}(\tau_1)$. Then by applying [FFT15, Corollary 2.4], the univariate sequence $X_{01}, X_{11}, \ldots$ has a univariate extremal index θ_1, say, that must coincide with $\theta((\tau_1, 0, \ldots, 0))$.

Regarding the bounds in (2.7), the first inequality follows trivially from observing that $\{\mathbb{M}_n \leq \mathbb{u}_n(\mathbb{T})\} \subset \{M_{nj} \leq u_{nj}(\tau_j)\}$ for all $j = 1, \ldots, d$ and that from [FFT15, Corollary 2.4] we have $\mu(M_{nj} \leq u_{nj}(\tau_j)) = \mathrm{e}^{-\theta_j\tau_j}$.

For the second inequality, start by observing that

$$\Gamma(\theta\mathbb{T}) = \theta(\mathbb{T})\hat{\Gamma}(\mathbb{T}) = \lim_{q\to\infty}\lim_{n\to\infty} -\log\left(1 - \left\lfloor\frac{n}{k_n}\right\rfloor \mu(A_n^{(q)}(\mathbb{T}))\right)^{k_n}.$$

Then since $-k_n\log\left(1 - \left\lfloor\frac{n}{k_n}\right\rfloor \mu(A_n^{(q)}(\mathbb{T}))\right) \sim n\mu\left(A_n^{(q)}(\mathbb{T})\right)$ and $A_n^{(q)}(\mathbb{T}) \subset \bigcup_{j=1}^d A_{nj}^{(q)}(\tau_j)$, with the necessary adjustments to [FFT15, Corollary 2.4] we get

$$n\mu\left(A_n^{(q)}(\mathbb{T})\right) \leq \sum_{j=1}^d n\mu\left(A_{nj}^{(q)}(\tau_j)\right) = \sum_{j=1}^d n\mu\left(A_{nj}^{(0)}(\tau_j)\right)\frac{\mu\left(A_{nj}^{(q)}(\tau_j)\right)}{\mu\left(A_{nj}^{(0)}(\tau_j)\right)} \to \sum_{j=1}^d \tau_j\theta_j.$$

□

3. Applications to dynamical systems

In our examples the extremes are realised on sets $\mathcal{Z} = \cup_{i=1}^d \mathcal{Z}_i$. We will assume that $\psi_i(x) = g_i(d(x, \mathcal{Z}_i))$ for a set $\mathcal{Z}_i$, where d is the induced Hausdorff metric. As usual, in order for (2.8) to hold, the g_i should each be one of the three types, see [LFF+16, (4.2.3)–(4.2.5)] for the general case, examples of each are:

$$(1)\ g_i(t) = -\log t;\ (2)\ g_i(t) = t^{-1/\alpha};\ (3)\ g_i(t) = D - t^{1/\alpha}, \tag{3.1}$$

where $\alpha > 0$ and $D \in \mathbb{R}$. We set $U^{(n)}(\mathbb{T}) := \{\mathbb{X}_0 \not\leq \mathbb{u}_n\}$ (note that in this paper we also call this set $A_n^{(0)}(\mathbb{T})$). Moreover, we write $U_i^{(n)}(\tau_i) := \{\mathbb{X}_{0i} > u_{ni}(\tau_i)\}$, so $U^{(n)}(\mathbb{T}) = \cup_{i=1}^d U_i^{(n)}$.

For us to be able to satisfy (2.2) and (2.11), we will need some regularity of our measure and our observables. These will generally involve the scaling of the measures of the sets $U_i^{(n)}(\tau_i)$, $A_n^{(q)}(\mathbb{T})$, and preimages of (parts of) such sets. To keep the setup flexible, we refer to the required properties as 'regularity', but note that there will be concrete examples where these will be satisfied, for example when Ψ is continuous in some $B_\varepsilon(\zeta) \setminus \{\zeta\}$ and μ has a smooth density with respect to Lebesgue at $\zeta \in \mathcal{Z}$.

We also need to check that conditions $Д(\mathbb{u}_n)$ and $Д'(\mathbb{u}_n)$ hold. Here, is where we take advantage of the design of the conditions, which were purposely adapted to the dynamical setting. These conditions are not much different from the ones in the multidimensional setting of [FFMa20, FFT21] and follow in practically the same way.

Condition $Д(\mathbb{u}_n)$ follows easily from sufficiently fast decay of correlations (summable rates are enough). We refer to the discussions in [Fre13, Section 5.1] or [LFF+16, Section 4.4], for non-uniformly expanding systems or to [GHN11, Section 2], for higher dimensional systems with contracting directions.

Condition $Д'(\mathbb{u}_n)$ is typically more involved because depends a lot on the recurrence properties of the maximal set, which often requires a local analysis of the dynamics there. However, if the systems have a strong form of decay of correlations, like uniformly expanding systems do, it can actually be easily checked globally see ([LFF+16, Section 4.2.4]).

Moreover, adapting [FFMa18, Section 5, Theorem 2.C] we can show that if the system admits an induced system for which we can check conditions $Д(\mathbb{u}_n)$ and $Д'(\mathbb{u}_n)$, then the conclusion of Theorem 2.6 applies to both the induced and the original system, which means that we can immediately apply our findings to slowly mixing systems (with non-summable rates), such as Manneville-Pomeau type of maps with indifferent fixed points as well as some quadratic maps, both with measures absolutely continuous with respect to Lebesgue (*acips*). We refer to [FFMa20, Section 4.2] for a list of systems for which we can apply our results.

In many of the examples in this section, we will work out the theory in the general settings just described, for example giving a formula for $\theta(\mathbb{t})$, and then to clarify the phenomena at play and provide more concrete formulae we restrict to the specific example of the doubling map $f : x \mapsto 2x \mod 1$ or the tripling map $f : x \mapsto 3x \mod 1$ on $[0,1]$ with Lebesgue as the invariant measure μ: we use the latter when either we need a fixed point where f is a diffeomorphism in a 2-sided neighbourhood or when we need a map of degree greater than two. Note that these systems satisfies our regularity requirements and $Д(\mathbb{u}_n), Д'(\mathbb{u}_n)$.

3.1. **Points not dynamically linked.**

We start by illustrating the theory with the elementary case of bivariate processes arising from observables where the components are maximised at the same point or at two distinct points with no dynamical link between them, which in particular means that the common point is not periodic.

3.1.1. *Common non-periodic maximal point.*

Let $\mathscr{Z} = \{\zeta\}$ and $\psi_i(x) = g_i(\text{dist}(x, \zeta))$, for $i = 1, 2$, where $\bigcap_{j \geq 0} f^{-j}(\mathscr{Z}) = \emptyset$. Assuming that g_i is as in (3.1) and μ is sufficiently regular then for every $\mathbb{t} = (\tau_1, \tau_2) \in (0, \infty)^2$ there exists $(\mathbb{u}_n(\mathbb{t}))_n$ such that (2.2) holds.

Observe that $\{X_{0i} > u_{ni}(\tau_i)\} = B_{g_i^{-1}(u_{ni}(\tau_i))}(\zeta)$, $i = 1, 2$, where $B_\varepsilon(\zeta)$ denotes a ball of radius ε around ζ and, therefore, whenever $\tau_1 < \tau_2$ we must have that for all n sufficiently large $\{X_{01} > u_{n1}(\tau_1)\} \subset \{X_{02} > u_{n2}(\tau_2)\}$, and vice-versa if $\tau_1 > \tau_2$. It follows that

$$\hat{\Gamma}(\mathbb{t}) = \lim_{n\to\infty} n\mu(\mathbb{X}_0 \not\leq \mathbb{u}_n(\mathbb{t})) = \lim_{n\to\infty} n\mu\left(\bigcup_{i=1}^{2}\{X_{0i} > u_{ni}(\tau_i)\}\right) = \lim_{n\to\infty} n\sum_{i=1}^{2}\mu\left(X_{0i} > u_{ni}(\tau_i)\right)$$
$$- \lim_{n\to\infty} n\mu\left(\bigcap_{i=1}^{2}\{X_{0i} > u_{ni}(\tau_i)\}\right) = \tau_1 + \tau_2 - \min\{\tau_1, \tau_2\} = \max\{\tau_1, \tau_2\}.$$

This means that in this case we have perfect association and $\hat{H}(\mathbb{t}) = C_{\hat{H}}(\mathbb{t}) = \min\{t_1, t_2\}$. Moreover, assuming that the dynamical system has sufficiently fast decay of correlations, we have that condition both conditions $Д(\mathbb{u}_n(\mathbb{\tau}))$ and $Д'(\mathbb{u}_n(\mathbb{\tau}))$ hold, where in fact one can show that $\Delta^{(0)}(\mathbb{u}_n(\mathbb{\tau})) = 0$, which implies that $\theta(\mathbb{\tau}) = 1$ and therefore in this case $\Gamma(\mathbb{\tau}) = \hat{\Gamma}(\mathbb{\tau})$, $G(\mathbb{\tau}) = \mathrm{e}^{-\max\{\tau_1,\tau_2\}}$ and we also have $H(\mathbb{t}) = C_H(\mathbb{t}) = \min\{t_1, t_2\}$. Note that we can also write

$$\Gamma(\mathbb{\tau}) = (\tau_1 + \tau_2)\left(1 - \min\left\{\frac{\tau_1}{\tau_1+\tau_2}, \frac{\tau_2}{\tau_1+\tau_2}\right\}\right),$$

which means that $D(\alpha) = 1 - \min\{\alpha, 1-\alpha\} = \max\{\alpha, 1-\alpha\}$.

To make our example slightly more concrete, suppose that $\mu(B_r(\zeta)) \sim cr^d$, $g_1(t) = -\log t$ and $g_2(t) = t^{-1}$. Then in the above argument we can take $u_{n1}(\tau_1) = \frac{1}{d}(\log(cn) - \log\tau_1)$ and $u_{n2}(\tau_2) = \left(\frac{cn}{\tau_2}\right)^{\frac{1}{d}}$.

3.1.2. *Distinct non-linked maximal points.* For $\zeta_1 \neq \zeta_2$ let $\mathscr{Z}_1 = \{\zeta_1\}$, $\mathscr{Z}_2 = \{\zeta_2\}$ and $\psi_i(x) = g_i(\mathrm{dist}(x, \mathscr{Z}_i))$, for $i = 1, 2$, where $\bigcap_{j\geq 0} f^{-j}(\mathscr{Z}_1 \cup \mathscr{Z}_2) = \emptyset$. Assume that g_i, μ and $(\mathbb{u}_n(\mathbb{\tau}))_n$ are as above.

Observe that $\{X_{0i} > u_{ni}(\tau_i)\} = B_{g_i^{-1}(u_{ni}(\tau_i))}(\zeta_i)$ for $i = 1, 2$ and since ζ_1 and ζ_2 are distinct then for every fixed $\mathbb{\tau}$ and n sufficiently large we have $\{X_{01} > u_{n1}(\tau_1)\} \cap \{X_{02} > u_{n2}(\tau_2)\} = \emptyset$. It follows that

$$\begin{aligned}\hat{\Gamma}(\mathbb{\tau}) &= \lim_{n\to\infty} n\mu(\mathbb{X}_0 \not\leq \mathbb{u}_n(\mathbb{\tau})) = \lim_{n\to\infty} n\mu\left(\bigcup_{i=1}^{2}\{X_{0i} > u_{ni}(\tau_i)\}\right)\\ &= \lim_{n\to\infty} n\sum_{i=1}^{2}\mu\left(X_{0i} > u_{ni}(\tau_i)\right) - \lim_{n\to\infty} n\mu\left(\bigcap_{i=1}^{2}\{X_{0i} > u_{ni}(\tau_i)\}\right) = \tau_1 + \tau_2.\end{aligned}$$

This means that in this case we have asymptotic extremal independence and $\hat{H}(\mathbb{t}) = C_{\hat{H}}(\mathbb{t}) = t_1 \cdot t_2$. Moreover, assuming that the dynamical system has sufficiently fast decay of correlations, we have that condition both conditions $Д(\mathbb{u}_n(\mathbb{\tau}))$ and $Д'(\mathbb{u}_n(\mathbb{\tau}))$ hold, where in fact one can show that $\Delta^{(0)}(\mathbb{u}_n(\mathbb{\tau})) = 0$, which implies that $\theta(\mathbb{\tau}) = 1$ and therefore in this case $\Gamma(\mathbb{\tau}) = \hat{\Gamma}(\mathbb{\tau})$, $G(\mathbb{\tau}) = \mathrm{e}^{-(\tau_1+\tau_2)}$ and we also have $H(\mathbb{t}) = C_H(\mathbb{t}) = t_1 \cdot t_2$. Clearly, $D(\alpha) = 1$, in this case.

3.2. **Distinct linked points.** In this case we will build up a lot of the notation and machinery also required for later cases. Assume $\mathscr{Z}_1 = \{\zeta\}$ and $\mathscr{Z}_2 = \{f(\zeta)\}$.

3.2.1. *Non-periodic case.* Assume first that $\{\zeta, f(\zeta)\} \cap \{\cup_{n\geq 2} f^n(\zeta)\} = \emptyset$. So in particular, given $q \in \mathbb{N}$ and $\mathbb{\tau} = (\tau_1, \tau_2)$, for sufficiently large n, $x \in U_2^{(n)}(\tau_2)$ implies

$$\{f(x), \ldots, f^q(x)\} \cap \left(U_1^{(n)}(\tau_1) \cup U_2^{(n)}(\tau_2)\right) = \emptyset \implies A_n^{(q)}(\mathbb{\tau}) = A_n^{(1)}(\mathbb{\tau}). \tag{3.2}$$

Given $\alpha \in [0, 1]$, define

$$\theta_\zeta(\alpha) := \lim_{n\to\infty} \frac{\mu\left(U_1^{(n)}(\alpha) \setminus f^{-1}U_2^{(n)}(1-\alpha)\right)}{\mu\left(U_1^{(n)}(\alpha)\right)},$$

and observe that by regularity, for $\alpha = \frac{\tau_1}{\tau_1+\tau_2}$,

$$\theta_\zeta(\alpha) := \lim_{n\to\infty} \frac{\mu\left(U_1^{(n)}(\tau_1) \setminus f^{-1}U_2^{(n)}(\tau_2)\right)}{\mu\left(U_1^{(n)}(\tau_1)\right)},$$

so $\theta_\zeta(\alpha)$ represents the asymptotic probability, given $x \in U_1^{(n)}(\tau_1)$, that $f(x) \notin U_2^{(n)}(\tau_2)$, and since n can be assumed large, that $f(x) \notin U_1^{(n)}(\tau_1) \cup U_2^{(n)}(\tau_2)$. We will use this type of idea throughout our examples.

Then $\mathbb{X}_0 = \mathbb{X}_0(x) \nleq \mathbb{u}_n(\tau_1, \tau_2)$ means that $x \in U_1^{(n)}(\tau_1) \cup U_2^{(n)}(\tau_1)$, which implies, for all large n,

(1) with asymptotic probability α we have $X_{01} > u_{n1}(\tau_1)$ (i.e., $x \in U_1^{(n)}(\tau_1)$). Then $X_{11} < u_{n1}(\tau_1)$ with probability 1, but $X_{12} < u_{n2}(\tau_2)$ (i.e., $f(x) \in U_1^{(n)}(\tau_1) \setminus f^{-1}U_2^{(n)}(\tau_2)$) with probability $\theta_\zeta(\alpha, 1-\alpha)$;

(2) with asymptotic probability $1-\alpha$ we have $X_{02} > u_{n2}(\tau_2)$ (i.e., $x \in U_2^{(n)}(\tau_2)$). Then $X_{11} < u_{n1}(\tau_1)$ and $X_{12} < u_{n2}(\tau_2)$ with probability 1 (since $f(x) \notin U_2^{(n)}(\tau_2) \cup U_1^{(n)}(\tau_1)$).

As in (3.2), there are no other entries of importance up to time q. So summing the probabilities gives

$$\theta(\alpha, 1-\alpha) = \alpha\theta_\zeta(\alpha) + 1 - \alpha.$$

To find $\theta_\zeta(\alpha)$, we observe that $U_1^{(n)}(\alpha) = B_{g_1^{-1}(u_{n1}(\alpha))}(\zeta)$ and $U_2^{(n)}(1-\alpha) = B_{g_2^{-1}(u_{n2}(1-\alpha))}(f(\zeta))$. Moreover, if f is conformal then $f^{-1}U_2^{(n)}(1-\alpha)$ is an approximate ball of radius $\frac{1}{|Df(\zeta)|} g_2^{-1}(u_{n2}(1-\alpha))$, so we are left to find the relative (to $U_1^{(n)}(\alpha)$) measure of the annulus

$$A_{\frac{1}{|Df(\zeta)|} g_2^{-1}(u_{n2}(1-\alpha)), g_1^{-1}(u_{n1}(\alpha))},$$

assuming that the inner radius is strictly smaller than the outer radius so that this makes sense. In a setting where μ is an acip with density ρ, f is conformal and the observables are all of the same form, for c_d the volume of the d-dimensional unit ball, the measure of our annulus is asymptotically $\frac{1}{c_d}\left(\frac{\alpha}{n\rho(\zeta)} - \frac{1}{|Df(\zeta)|}\frac{1-\alpha}{n\rho(f(\zeta))}\right)$ (we are assuming that $\frac{1}{|Df(\zeta)|}\frac{1-\alpha}{c_d n\rho(f(\zeta))} < \frac{\alpha}{c_d n\rho(\zeta)}$, if not then $\theta_\zeta(\alpha, 1-\alpha) = 0$). Since $\mu(U_1^{(n)}(\alpha)) \sim \frac{\alpha}{c_d n\rho(\zeta)}$, hence,

$$\theta_\zeta(\alpha) = \max\left\{0, 1 - \frac{\rho(\zeta)}{\rho(f(\zeta))}\frac{1-\alpha}{\alpha}\frac{1}{|Df(\zeta)|}\right\}.$$

Thus

$$\theta(\alpha, 1-\alpha) = \alpha \max\left\{0, 1 - \frac{\rho(\zeta)}{\rho(f(\zeta))}\frac{1-\alpha}{\alpha}\frac{1}{|Df(\zeta)|}\right\} + 1 - \alpha.$$

In the doubling map case with Lebesgue measure, this becomes

$$\theta(\alpha, 1-\alpha) = \begin{cases} 1-\alpha & \text{if } \alpha \le \frac{1}{3}, \\ \frac{1+\alpha}{2} & \text{if } \alpha > \frac{1}{3}, \end{cases} \qquad G(\tau_1, \tau_2) = \begin{cases} e^{-\tau_2} & \text{if } \tau_1 \le \frac{\tau_2}{2}, \\ e^{-\left(\tau_1 + \frac{\tau_2}{2}\right)} & \text{if } \tau_1 \le \frac{\tau_2}{2}. \end{cases}$$

Where G is obtained by adding in the $\hat{H}$ term from Section 3.1.2. Moreover, we see here that $\theta_1 = \theta_2 = 1$, so $\Gamma(\mathbb{T}) = \theta(\mathbb{T})\hat{\Gamma}(\mathbb{T})$ and $D(\alpha) = \Gamma(\alpha, 1-\alpha) = \theta(\alpha, 1-\alpha)$, see Figure 3.1.

3.2.2. *Periodic case.* Next we assume that $\mathscr{Z}_1 = \{\zeta\}$, $\mathscr{Z}_2 = \{f(\zeta)\}$ with $f^2(\zeta) = \zeta$ (and $f(\zeta) \neq \zeta$). As we will see below, in contrast to (3.2), here we need only consider $A_n^{(2)}(\mathbb{\tau})$. Then $\mathbb{X}_0 \nleq \mathbb{u}_n(\tau_1, \tau_2)$ implies, for all large n and α as above,

(1) with asymptotic probability α we have $X_{01} > u_{n1}(\tau_1)$ (i.e., $x \in U_1^{(n)}(\tau_1)$). Then $X_{11} < u_{n1}(\tau_1)$ with probability 1 (since $f(x) \notin U_1^{(n)}(\tau_1)$), but $X_{12} < u_{n2}(\tau_2)$ only if $x \in U_1^{(n)}(\tau_1) \setminus f^{-1}U_2^{(n)}(\tau_2)$; then $X_{21} < u_{n1}(\tau_1)$ only if $x \in U_1^{(n)}(\tau_1) \setminus f^{-2}U_1^{(n)}(\tau_1)$; in total we require the relative probability that $x \in U_1^{(n)}(\tau_1) \setminus \left(f^{-1}U_2^{(n)}(\tau_2) \cup f^{-2}U_1^{(n)}(\tau_1)\right)$;
(2) with asymptotic probability $1-\alpha$ we have $X_{02} > u_{n2}(\tau_2)$ (i.e., $x \in U_2^{(n)}(\tau_2)$). Then $X_{12} < u_{n2}(\tau_2)$ with probability 1 (since $f(x) \notin U_2^{(n)}(\tau_2)$), but $X_{11} < u_{n1}(\tau_1)$ only if $x \in U_2^{(n)}(\tau_2) \setminus f^{-1}U_1^{(n)}(\tau_1)$; then $X_{22} < u_{n2}(\tau_2)$ only if $x \in U_2^{(n)}(\tau_2) \setminus f^{-2}U_2^{(n)}(\tau_2)$; in total we require the relative probability that $x \in U_2^{(n)}(\tau_2) \setminus \left(f^{-1}U_1^{(n)}(\tau_1) \cup f^{-2}U_2^{(n)}(\tau_2)\right)$.

We can see that for given $q \in \mathbb{N}$, for any large n, $A_n^{(q)}(\mathbb{\tau}) = A_n^{(2)}(\mathbb{\tau})$. So setting

$$\theta_\zeta(\alpha) := \lim_{n\to\infty} \frac{\mu\left(U_1^{(n)}(\alpha) \setminus \left(f^{-1}U_2^{(n)}(1-\alpha) \cup f^{-2}U_1^{(n)}(\alpha)\right)\right)}{\mu\left(U_1^{(n)}(\alpha)\right)}$$

$$\theta_{f(\zeta)}(\alpha) := \lim_{n\to\infty} \frac{\mu\left(U_2^{(n)}(1-\alpha) \setminus \left(f^{-1}U_1^{(n)}(\alpha) \cup f^{-2}U_2^{(n)}(1-\alpha)\right)\right)}{\mu\left(U_2^{(n)}(1-\alpha)\right)},$$

we obtain

$$\theta(\alpha, 1-\alpha) = \alpha\theta_\zeta(\alpha) + (1-\alpha)\theta_{f(\zeta)}(\alpha).$$

To compute the annuli here is more involved since we need to incorporate the relative strengths of the derivatives $Df(\zeta)$ and $Df^2(\zeta)$ for θ_ζ and $Df(f(\zeta))$ and $Df^2(f(\zeta))$ for $\theta_{f(\zeta)}$, as well as the relative sizes of α and $1-\alpha$. If f is conformal, for $\theta_\zeta(\alpha, 1-\alpha)$ we consider the relative size of the annulus Lebesgue measure

$$\frac{\alpha}{n\rho(\zeta)} - \max\left\{\frac{1}{|Df(\zeta)|}\frac{1-\alpha}{n\rho(f(\zeta))}, \frac{1}{|Df^2(\zeta)|}\frac{\alpha}{n\rho(\zeta)}\right\},$$

which gives

$$\theta_\zeta(\alpha) = \max\left\{0, 1 - \max\left\{\frac{1}{|Df(\zeta)|}\frac{1-\alpha}{\alpha}\frac{\rho(\zeta)}{\rho(f(\zeta))}, \frac{1}{|Df^2(\zeta)|}\right\}\right\}$$

Similarly,

$$\theta_{f(\zeta)}(\alpha) = \max\left\{0, 1 - \max\left\{\frac{1}{|Df(f(\zeta))|}\frac{\alpha}{1-\alpha}\frac{\rho(f(\zeta))}{\rho(\zeta)}, \frac{1}{|Df^2(\zeta)|}\right\}\right\}.$$

Therefore in this case,

$$\begin{aligned}\theta(\alpha, 1-\alpha) = {}& \alpha \max\left\{0, 1 - \max\left\{\frac{1}{|Df(\zeta)|}\frac{1-\alpha}{\alpha}\frac{\rho(\zeta)}{\rho(f(\zeta))}, \frac{1}{|Df^2(\zeta)|}\right\}\right\} \\ & + (1-\alpha)\max\left\{0, 1 - \max\left\{\frac{1}{|Df(f(\zeta))|}\frac{\alpha}{1-\alpha}\frac{\rho(f(\zeta))}{\rho(\zeta)}, \frac{1}{|Df^2(\zeta)|}\right\}\right\}.\end{aligned}$$

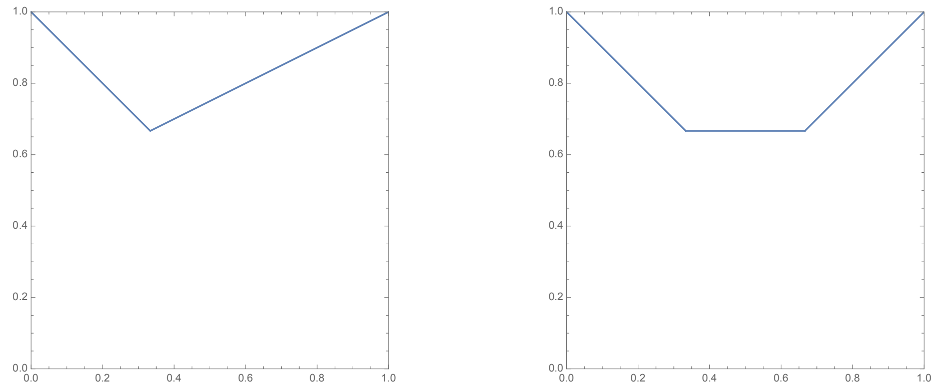


FIGURE 3.1. Graphs of the Pickands dependence functions of the examples in Section 3.2.1 on the left, and Section 3.2.2, on the right.

In the doubling map case with Lebesgue measure (here $\zeta = 1/3$), we compute

$$\theta(\alpha, 1-\alpha) = \begin{cases} \frac{3}{4}(1-\alpha) & \text{if } \alpha \le \frac{1}{3}, \\ \frac{1}{2} & \text{if } \alpha \in (\frac{1}{3}, \frac{2}{3}], \\ \frac{3\alpha}{4} & \text{if } \alpha > \frac{2}{3}, \end{cases} \qquad G(\tau_1, \tau_2) = \begin{cases} e^{-\frac{3}{4}\tau_2} & \text{if } \tau_1 \le \frac{\tau_2}{2}, \\ e^{-\frac{1}{2}(\tau_1+\tau_2)} & \text{if } \tau_1 \in (\frac{\tau_2}{2}, 2\tau_2], \\ e^{-\frac{3\tau_1}{4}} & \text{if } \tau_1 > 2\tau_2. \end{cases}$$

Where we obtained G by using the $\hat{H}$ term from Section 3.1.2. Here we compute $\theta_1 = \theta_2 = 3/4$, so $\Gamma(\mathbb{T}) = \frac{4}{3}\theta(\mathbb{T})\hat{\Gamma}(\mathbb{T})$ (as usual $D(\alpha) = \Gamma(\alpha, 1-\alpha)$), see Figure 3.1.

3.2.3. *Another periodic case.* In order to give an example where $\theta_1 \neq \theta_2$, we assume now that $\mathcal{Z}_1 = \{\zeta\}$, $\mathcal{Z}_2 = \{f(\zeta)\}$ with $f^2(\zeta) = f(\zeta)$ (and $f(\zeta) \neq \zeta$). Here θ_ζ is as in Section 3.2.1, but we also need a $\theta_{f(\zeta)}$ to derive $\theta(\alpha, 1-\alpha) = \alpha\theta_\zeta(\alpha) + (1-\alpha)\theta_{f(\zeta)}$. For an explicit formula, in order to have f being locally diffeomorphic around each of ζ and $f(\zeta)$ we will use the tripling map (so eg $\zeta = 1/6$, $f(\zeta) = 1$). We will not give the full details, but we see that $\theta_\zeta(\alpha) = \max\left\{1, 1 - \frac{1}{3}\frac{1-\alpha}{\alpha}\right\}$ and $\theta_{f(\zeta)}(\alpha) \equiv 2/3$, so

$$\theta(\alpha, 1-\alpha) = \begin{cases} \frac{2}{3}(1-\alpha) & \text{if } \alpha \le \frac{1}{4}, \\ \frac{1}{3} + \frac{2}{3}\alpha & \text{if } \alpha > \frac{1}{4}, \end{cases} \qquad G(\alpha, 1-\alpha) = \begin{cases} e^{-\frac{2}{3}\tau_2} & \text{if } \tau_1 \le \frac{\tau_2}{3}, \\ e^{-\left(\tau_1 + \frac{1}{3}\tau_2\right)} & \text{if } \tau_1 > \frac{\tau_2}{3}. \end{cases}$$

Since $\theta_1 = 1$ and $\theta_2 = 2/3$, we have $\Gamma(\tau_1, \tau_2) = \theta\left(\tau_1, \frac{3}{2}\tau_2\right)\hat{\Gamma}\left(\tau_1, \frac{3}{2}\tau_2\right)$, i.e.,

$$\Gamma(\tau_1, \tau_2) = \begin{cases} \tau_2 & \text{if } \tau_2 \ge 2\tau_1, \\ \tau_1 + \frac{1}{2}\tau_2 & \text{if } \tau_2 < 2\tau_1, \end{cases} \qquad D(\alpha) = \begin{cases} 1-\alpha & \text{if } \alpha \le 1/3, \\ \frac{1+\alpha}{2} & \text{if } \alpha > 1/3. \end{cases}$$

3.3. **Overlapping points.** Here we look at three points $\{\zeta_1, \zeta_2, \zeta_3\}$, then consider $\mathcal{Z}_1 = \{\zeta_1, \zeta_3\}$ and $\mathcal{Z}_2 = \{\zeta_2, \zeta_3\}$.

3.3.1. *Spatial dependence.* We first compute $\hat{\Gamma}$. Let, for $i = 1, 2$,

$$\psi_i(x) = g_i(d(x, \mathcal{Z}_i)),\ x \in \mathcal{X},$$

where g_i is as in (3.1). For $\mathbb{T} = (\tau_1, \tau_2) \in (0, \infty)^2$, choose $u_{ni}(\tau_i) > 0$ such that

$$\mu(U_i^{(n)}(\tau_i)) \sim \frac{\tau_i}{n},$$

where as before $U_i^{(n)}(\tau_i) = \{\psi_i > u_{ni}(\tau_i)\}$.
We assume that for all $\mathbb{\tau}$ and all n large enough, $U_i^{(n)}(\tau_i)$ can be written as a disjoint union

$$U_i^{(n)}(\tau_i) = V_i^{(n)}(\tau_i) \cup \hat{V}_i^{(n)}(\tau_i),$$

where $V_i^{(n)}(\tau_i)$ (resp. $\hat{V}_i^{(n)}(\tau_i)$) is a neighbourhood of ζ_i (resp. ζ_3), and that

$$\mu\left(V_i^{(n)}(\tau_i)\right) \sim p_i \mu\left(V_i^{(n)}(\tau_i)\right),$$

for some $p_i \in (0,1)$, so that

$$\mu\left(V_i^{(n)}(\tau_i)\right) \sim \frac{p_i \tau_i}{n} \text{ and } \mu\left(\hat{V}_i^{(n)}(\tau_i)\right) \sim \frac{(1-p_i)\tau_i}{n}.$$

We define $\hat{V}^{(n)}(\mathbb{\tau}) = \cap_{i=1}^2 \hat{V}_i^{(n)}(\tau_i)$ and we assume that for some $q_1(\mathbb{\tau}) \in (0,1)$,

$$\mu\left(\hat{V}^{(n)}(\mathbb{\tau})\right) \sim q_1(\mathbb{\tau})\mu\left(\hat{V}_1^{(n)}(\tau_1)\right).$$

This implies that $\mu\left(\hat{V}^{(n)}(\mathbb{\tau})\right) \sim q_2(\mathbb{\tau})\mu\left(\hat{V}_2^{(n)}(\tau_2)\right)$ where $q_2(\mathbb{\tau}) = q_1(\mathbb{\tau})\frac{(1-p_1)\tau_1}{(1-p_2)\tau_2}$ and

$$\mu\left(\hat{V}^{(n)}(\mathbb{\tau})\right) \sim \frac{q_1(\mathbb{\tau})(1-p_1)\tau_1}{n} = \frac{q_2(\mathbb{\tau})(1-p_2)\tau_2}{n}.$$

It follows that

$$\hat{\Gamma}(\mathbb{\tau}) = \lim_{n\to\infty} n\mu(\mathbb{X}_0 \not\leq \mathbb{u}_n(\mathbb{\tau})) = \lim_{n\to\infty} n\mu\left(\bigcup_{i=1}^2 \{X_{0i} > u_{ni}(\tau_i)\}\right) = \lim_{n\to\infty} n\sum_{i=1}^2 \mu\left(X_{0i} > u_{ni}(\tau_i)\right)$$
$$- \lim_{n\to\infty} n\mu\left(\bigcap_{i=1}^2 \{X_{0i} > u_{ni}(\tau_i)\}\right) = \tau_1 + \tau_2 - q_1(\mathbb{\tau})(1-p_1)\tau_1 = \tau_1 + \tau_2 - q_2(\mathbb{\tau})(1-p_2)\tau_2.$$

As an example, suppose that $\mu(B_r(\zeta_k)) \sim c_k r^d$ for $k = 1,2,3$ and that the functions g_i are given by $g_i(t) = -\log t$. Then we can choose

$$u_{ni}(\tau_i) = -\frac{1}{d}\left(\log \tau_i + \log \frac{p_i}{c_i} - \log n\right),$$

with $p_i = \frac{c_i}{c_i + c_3}$. Independently of the types of g_i we obtain

$$\hat{\Gamma}(\mathbb{\tau}) = \tau_1 + \tau_2 - c_3 \min_{i=1,2} \frac{\tau_i}{c_i + c_3}.$$

When μ is the Lebesgue measure on $\mathcal{X} = [0,1]$ and all the ζ_i belong to $(0,1)$, we get

$$\hat{\Gamma}(\mathbb{\tau}) = \tau_1 + \tau_2 - \frac{1}{2}\min_{i=1,2} \tau_i.$$

3.3.2. *Non-periodic case.* Suppose that $\zeta_3=f(\zeta_1)=f(\zeta_2)$ and that $\{\cup_{n\geq 1}f^n(\zeta_3)\}\cap\{\zeta_1,\zeta_2,\zeta_3\}=\emptyset$. The computations below show that it suffices to consider $q=1$.

We require some notation for this. First let $U_1^{(n)}(\tau_1)=V_1^{(n)}(\tau_1)\cup\hat V_1^{(n)}(\tau_1)$ where $V_1^{(n)}(\tau_1)$ is the corresponding neighbourhood of ζ_1 and $\hat V_1^{(n)}(\tau_1)$ is that of ζ_3. Similarly write $U_2^{(n)}(\tau_2)=V_2^{(n)}(\tau_2)\cup\hat V_2^{(n)}(\tau_2)$.

Now we define, for $\alpha\in(0,1)$,

$$p_1(\alpha):=\lim_{n\to\infty}\frac{\mu\left(V_1^{(n)}(\alpha)\right)}{\mu\left(U_1^{(n)}(\alpha)\cup U_2^{(n)}(1-\alpha)\right)},\quad p_2(\alpha):=\lim_{n\to\infty}\frac{\mu\left(V_2^{(n)}(\alpha)\right)}{\mu\left(U_1^{(n)}(\alpha)\cup U_2^{(n)}(1-\alpha)\right)},$$

$$p_3(\alpha):=\lim_{n\to\infty}\frac{\mu\left(\hat V_1^{(n)}(\alpha)\cup\hat V_2^{(n)}(1-\alpha)\right)}{\mu\left(U_1^{(n)}(\alpha)\cup U_2^{(n)}(1-\alpha)\right)}.$$

Note that $p_3(\alpha)=1-p_1(\alpha)-p_2(\alpha)$. Moreover, set

$$\theta_1(\alpha):=\lim_{n\to\infty}\frac{\mu\left(V_1^{(n)}(\alpha)\setminus f^{-1}\left(\hat V_1^{(n)}(\alpha)\cup\hat V_2^{(n)}(1-\alpha)\right)\right)}{\mu\left(V_1^{(n)}(\alpha)\right)},$$

$$\theta_2(\alpha):=\lim_{n\to\infty}\frac{\mu\left(V_2^{(n)}(1-\alpha)\setminus f^{-1}\left(\hat V_1^{(n)}(\alpha)\cup\hat V_2^{(n)}(1-\alpha)\right)\right)}{\mu\left(V_2^{(n)}(1-\alpha)\right)}.$$

Then $\mathbb{X}_0\not\leq \mathbb{u}_n(\tau_1,\tau_2)$ implies, for all large n and α as above,

1. with asymptotic probability $p_1(\alpha)$ we have $x\in V_1^{(n)}(\tau_1)$, in which case $X_{12}<u_{n2}(\tau_2)$ with asymptotic probability $\theta_1(\alpha)$ ($X_{11}<u_{n1}(\tau_1)$ with probability 1);
2. with asymptotic probability $p_2(\alpha)$ we have $x\in V_2^{(n)}(\tau_2)$, in which case $X_{11}<u_{n1}(\tau_1)$ with asymptotic probability $\theta_2(\alpha)$ ($X_{12}<u_{n2}(\tau_2)$ with probability 1);
3. with asymptotic probability $p_3(\alpha)$ we have $x\in\hat V_1^{(n)}(\tau_1)\cup\hat V_2^{(n)}(\tau_2)$, in which case $X_{11}<u_{n1}(\tau_1)$ and $X_{12}<u_{n2}(\tau_2)$ with probability 1.

Summing these possibilities we obtain

$$\begin{aligned}\theta(\alpha,1-\alpha)&=p_1(\alpha)\theta_1(\alpha)+p_2(\alpha)\theta_2(\alpha)+1-p_1(\alpha)-p_2(\alpha)\\&=1-p_1(\alpha)\left(1-\theta_1(\alpha)\right)-p_2\left(\alpha\right)\left(1-\theta_2(\alpha)\right).\end{aligned}$$

To obtain a concrete formula, again assume that we have an acip with density ρ, that f is conformal and the observables are all of the same form. Define

$$r_1:=\lim_{n\to\infty}\frac{\mu\left(\hat V_1^{(n)}(1)\right)}{\mu\left(V_1^{(n)}(1)\right)},\ r_2:=\lim_{n\to\infty}\frac{\mu\left(\hat V_2^{(n)}(1)\right)}{\mu\left(V_2^{(n)}(1)\right)}$$

and

$$R(\alpha)=\lim_{n\to\infty}\frac{\mu\left(V_1^{(n)}(\alpha)\right)}{\mu\left(V_2^{(n)}(1-\alpha)\right)}.$$

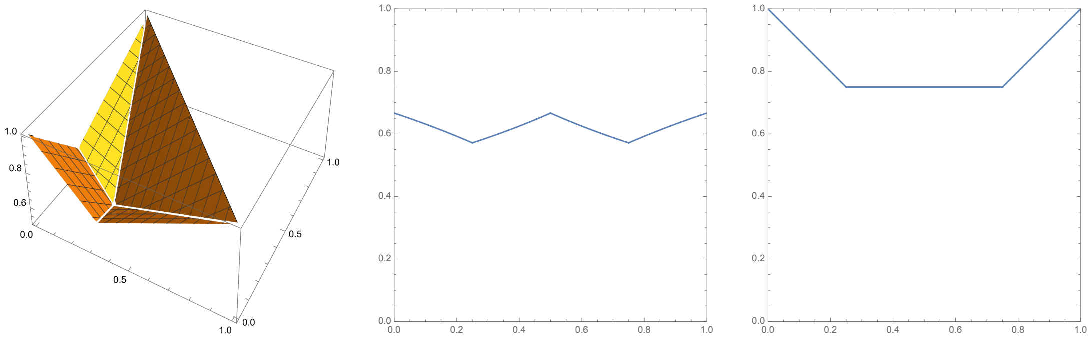


FIGURE 3.2. Graph of the Pickands dependence function of the example in Section 3.3.3 on the left; Graphs of the extremal index function and of the Pickands dependence function of the example in Section 3.3.4, in the middle and on the right, respectively.

Then

$$\theta_1(\alpha) = \max\left\{0, 1 - \frac{1}{|Df(\zeta_1)|} \max\left\{\frac{\rho(\zeta_1)}{\rho(f(\zeta_1))} r_1, \frac{\rho(\zeta_1)}{\rho(f(\zeta_1))} \frac{r_2}{R(\alpha)}\right\}\right\}$$

and

$$\theta_2(\alpha) = \max\left\{0, 1 - \frac{1}{|Df(\zeta_2)|} \max\left\{\frac{\rho(\zeta_2)}{\rho(f(\zeta_1))} r_2, \frac{\rho(\zeta_2)}{\rho(f(\zeta_2))} r_1 R(\alpha)\right\}\right\}.$$

To get a more concrete idea of what is happening here, we consider the case of the doubling map with Lebesgue where we get

$$\theta_1(\alpha) = \max\left\{0, 1 - \frac{1}{2}\max\left\{1, \frac{1-\alpha}{\alpha}\right\}\right\}, \quad \theta_2(\alpha) = \max\left\{0, 1 - \frac{1}{2}\max\left\{1, \frac{\alpha}{1-\alpha}\right\}\right\}.$$

In this case, $p_1(\alpha) = \frac{\alpha}{1+\max\{\alpha,1-\alpha\}}$ and $p_2(\alpha, 1-\alpha) = \frac{1-\alpha}{1+\max\{\alpha,1-\alpha\}}$, so

$$\begin{aligned}\theta(\alpha, 1-\alpha) = 1 &- \frac{\alpha}{1+\max\{\alpha, 1-\alpha\}}\left(1 - \max\left\{0, 1 - \frac{1}{2}\max\left\{1, \frac{1-\alpha}{\alpha}\right\}\right\}\right) \\ &- \frac{1-\alpha}{1+\max\{\alpha, 1-\alpha\}}\left(1 - \max\left\{0, 1 - \frac{1}{2}\max\left\{1, \frac{\alpha}{1-\alpha}\right\}\right\}\right).\end{aligned}$$

Hence we can compute

$$\theta(\alpha, 1-\alpha) = \begin{cases} \frac{3-3\alpha}{4-2\alpha} & \text{if } \alpha \in [0, 1/3], \\ \frac{1}{2-\alpha} & \text{if } \alpha \in (1/3, 1/2], \\ \frac{1}{1+\alpha} & \text{if } \alpha \in (1/2, 2/3], \\ \frac{3\alpha}{2+2\alpha} & \text{if } \alpha \in (2/3, 1], \end{cases} \qquad D(\alpha) = \begin{cases} 1-\alpha & \text{if } \alpha \in [0, 1/3], \\ \frac{2}{3} & \text{if } \alpha \in (1/3, 2/3], \\ \alpha & \text{if } \alpha \in (2/3, 1]. \end{cases}$$

Where we used $G(\underline{\tau}) = e^{-\theta(\underline{\tau})\left(\tau_1+\tau_2-\frac{1}{2}\min\{\tau_1,\tau_2\}\right)}$ with $\hat{\Gamma}$ from Section 3.3.1. In this case $\theta_1 = \theta_2 = 3/4$, so $\Gamma(\underline{\tau}) = \frac{4}{3}\theta(\underline{\tau})\hat{\Gamma}(\underline{\tau}) = \frac{4}{3}\theta(\underline{\tau})\left(\tau_1 + \tau_2 - \frac{1}{2}\min\{\tau_1, \tau_2\}\right)$, see Figure 3.2.

3.3.3. *A trivariate version.* Assume the same dynamical setup as that above, but with three observables, at ζ_1, ζ_2 and $f(\zeta_1) = f(\zeta_2)$ respectively. Now for $\alpha \in [0,1]$, $\beta \in [0, 1-\alpha]$, define

$$\theta_1(\alpha,\beta) := \lim_{n\to\infty} \frac{\mu\left(U_1^{(n)}(\alpha) \setminus f^{-1}U_3(1-\alpha-\beta)\right)}{\mu\left(U_1^{(n)}(\alpha)\right)},$$

$$\theta_2(\alpha,\beta) := \lim_{n\to\infty} \frac{\mu\left(U_2^{(n)}(\alpha) \setminus f^{-1}U_3(1-\alpha-\beta)\right)}{\mu\left(U_2^{(n)}(\alpha)\right)},$$

and

$$p_1(\alpha,\beta) := \lim_{n\to\infty} \frac{\mu\left(U_1^{(n)}(\alpha)\right)}{\mu\left(U_1^{(n)}(\alpha) \cup U_2^{(n)}(\beta) \cup U_3^{(n)}(1-\alpha-\beta)\right)}$$

$$p_2(\alpha,\beta) := \lim_{n\to\infty} \frac{\mu\left(U_2^{(n)}(\beta)\right)}{\mu\left(U_1^{(n)}(\alpha) \cup U_2^{(n)}(\beta) \cup U_3^{(n)}(1-\alpha-\beta)\right)}$$

$$p_3(\alpha,\beta) := \lim_{n\to\infty} \frac{\mu\left(U_3^{(n)}(1-\alpha-\beta)\right)}{\mu\left(U_1^{(n)}(\alpha) \cup U_2^{(n)}(\beta) \cup U_3^{(n)}(1-\alpha-\beta)\right)}.$$

Then $\mathbb{X}_0 \not\leq \mathbb{u}_n(\tau_1,\tau_2,\tau_3)$ implies, for all large n and $\alpha = \frac{\tau_1}{\tau_1+\tau_2+\tau_3}$, $\beta = \frac{\tau_2}{\tau_1+\tau_2+\tau_3}$,

(1) with asymptotic probability $p_1(\alpha,\beta)$ we have $x \in U_1^{(n)}(\tau_1)$, in which case $X_{13} < u_{n3}(\tau_3)$ with asymptotic probability $\theta_1(\alpha,\beta)$ ($X_{11} < u_{n1}(\tau_1)$ and $X_{12} < u_{n2}(\tau_2)$ with probability 1);
(2) with asymptotic probability $p_2(\alpha,\beta)$ we have $x \in U_2^{(n)}(\tau_1)$, in which case $X_{13} < u_{n3}(\tau_3)$ with asymptotic probability $\theta_2(\alpha,\beta)$ ($X_{11} < u_{n1}(\tau_1)$ and $X_{12} < u_{n2}(\tau_2)$ with probability 1);
(3) with asymptotic probability $p_3(\alpha,\beta)$ we have $x \in U_3^{(n)}(\tau_3)$, in which case $X_{11} < u_{n1}(\tau_1)$, $X_{12} < u_{n2}(\tau_2)$ and $X_{13} < u_{n3}(\tau_3)$ with probability 1.

So it is sufficient to consider $q = 1$ and write

$$\theta(\alpha,\beta) = 1 - p_1(\alpha,\beta)\,(1-\theta_1(\alpha,\beta)) - p_2\,(\alpha, 1-\beta)(1-\theta_2(\alpha,\beta))\,.$$

In the acip case where the observables all take the same form,

$$\theta_1(\alpha,\beta) = \max\left\{0, 1 - \frac{1}{|Df(\zeta_1)|}\frac{1-\alpha-\beta}{\alpha}\frac{\rho(\zeta_1)}{\rho(f(\zeta_1))}\right\},$$

$$\theta_2(\alpha,\beta) = \max\left\{0, 1 - \frac{1}{|Df(\zeta_2)|}\frac{1-\alpha-\beta}{\beta}\frac{\rho(\zeta_2)}{\rho(f(\zeta_2))}\right\},$$

hence

$$\theta(\tau_1,\tau_2,\tau_3) = 1 - \alpha\left(1 - \max\left\{0, 1 - \frac{1}{|Df(\zeta_1)|}\frac{1-\alpha-\beta}{\alpha}\frac{\rho(\zeta_1)}{\rho(f(\zeta_1))}\right\}\right)$$
$$- \beta\left(1 - \max\left\{0, 1 - \frac{1}{|Df(\zeta_2)|}\frac{1-\alpha-\beta}{\beta}\frac{\rho(\zeta_2)}{\rho(f(\zeta_2))}\right\}\right).$$

So again in the doubling map case with Lebesgue where the observables are all of the same form,

$$\theta(\tau_1,\tau_2,\tau_3) = 1-\alpha\left(1-\max\left\{0, 1-\frac{1-\alpha-\beta}{2\alpha}\right\}\right) - \beta\left(1-\max\left\{0, 1-\frac{1-\alpha-\beta}{2\beta}\right\}\right).$$

In this case $\theta_1=\theta_2=\theta_3=1$, $\Gamma(\mathbb{T})=\theta(\mathbb{T})\hat{\Gamma}(\mathbb{T})$ and consequently $D(\alpha_1,\alpha_2)$ shares the same expression as θ, with $\alpha_1=\alpha$ and $\alpha_2=\beta$ (it is easy to show that $\hat{\Gamma}(\mathbb{T})=\tau_1+\tau_2+\tau_2$), see Figure 3.2.

3.3.4. *A periodic case.* Suppose that $\zeta_3=f(\zeta_1)=f(\zeta_2)$ and that $f^2(\zeta_1)=f(\zeta_1)$. We use the notation for the neighbourhoods of sets, the p_1,p_2,p_3 and θ_1,θ_2 from Section 3.3.2. However we now also need

$$\theta_3(\alpha) := \lim_{n\to\infty}\frac{\mu\left(\left(\hat{V}_1^{(n)}(\alpha)\cup\hat{V}_2^{(n)}(1-\alpha)\right)\setminus f^{-1}\left(\hat{V}_1^{(n)}(\alpha)\cup\hat{V}_2^{(n)}(1-\alpha)\right)\right)}{\mu\left(\hat{V}_1^{(n)}(\tau_1)\cup\hat{V}_2^{(n)}(\tau_2)\right)}.$$

Then $\mathbb{X}_0 \not\leq \mathbb{u}_n(\tau_1,\tau_2)$ implies, for all large n and α as above,

(1) with asymptotic probability $p_1(\alpha)$ we have $x\in V_1^{(n)}(\tau_1)$, in which case $X_{12}<u_{n2}(\tau_2)$ with asymptotic probability $\theta_1(\alpha)$ ($X_{11}<u_{n1}(\tau_1)$ with probability 1);
(2) with asymptotic probability $p_2(\alpha)$ we have $x\in V_2^{(n)}(\tau_2)$, in which case $X_{11}<u_{n1}(\tau_1)$ with asymptotic probability $\theta_2(\alpha)$ ($X_{12}<u_{n2}(\tau_2)$ with probability 1);
(3) with asymptotic probability $p_3(\alpha)$ we have $x\in\hat{V}_1^{(n)}(\tau_1)\cup\hat{V}_2^{(n)}(\tau_2)$, in which case $X_{11}<u_{n1}(\tau_1)$ and $X_{12}<u_{n2}(\tau_2)$ with probability $\theta_3(\alpha)$.

Summing these possibilities we note that we may consider $q=1$ and obtain

$$\theta(\alpha,1-\alpha)=p_1(\alpha)\theta_1(\alpha)+p_2(\alpha)\theta_2(\alpha)+(1-p_1(\alpha)-p_2(\alpha))\theta_3(\alpha).$$

For our concrete case, note that since ζ_3 has three preimages (including itself), we need a map of degree three or more, so suppose f is the tripling map with Lebesgue as invariant measure. The $p_1(\alpha)$ and $p_2(\alpha)$ are obtained analogously to in Section 3.3.2, and note that in contrast to there, we also need to add $p_3(\alpha)\theta_3(\alpha)=\frac{\max\{\alpha,1-\alpha\}}{(1+\max\{\alpha,1-\alpha\})}\frac{2}{3}$ instead of just $p_3(\alpha)\cdot 1$. We compute

$$\theta(\alpha,1-\alpha)=\begin{cases}\frac{4}{3}\left(\frac{1-\alpha}{2-\alpha}\right) & \text{if } \alpha\in[0,1/4],\\ \frac{1}{2-\alpha} & \text{if } \alpha\in(1/4,1/2],\\ \frac{1}{1+\alpha} & \text{if } \alpha\in(1/2,3/4],\\ \frac{4}{3}\left(\frac{\alpha}{1+\alpha}\right) & \text{if } \alpha\in(3/4,1],\end{cases} \qquad D(\alpha)=\begin{cases}1-\alpha & \text{if } \alpha\in[0,1/4],\\ \frac{3}{4} & \text{if } \alpha\in(1/4,3/4],\\ \alpha & \text{if } \alpha\in(3/4,1].\end{cases}$$

Here $\theta_1=\theta_2=2/3$ so $\Gamma(\mathbb{T})=\frac{3}{2}\theta(\mathbb{T})\hat{\Gamma}(\mathbb{T})=\frac{3}{2}\theta(\mathbb{T})\left(\tau_1+\tau_2-\frac{1}{2}\min\{\tau_1,\tau_2\}\right)$, see Figure 3.2.

3.4. **Note on Gibbs measures.** We can do calculations similar to the above when we have a Gibbs measure μ, absolutely continuous with respect to a conformal measure m and such that $\frac{d\mu}{dm}(\zeta_i)\in(0,\infty)$, see [FFT12, Lemma 3.1] for the implications of this fact and [FFT15, Section 7.3] to see that the condition can be commonly satisfied.

3.5. **A 2d example.** Here we consider the toral automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ induced by $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, i.e. Arnold's cat map. This has a fixed point at 0 and eigenvalues $\lambda, 1/\lambda$ where $\lambda = \frac{3+\sqrt{5}}{2}$. Let μ be Lebesgue measure. We take two small pieces of local unstable manifold $\mathcal{Z}_1 = (-\varepsilon, \varepsilon) \times \{0\}$, $\mathcal{Z}_2 = (\lambda\varepsilon, \lambda^2\varepsilon) \times \{0\}$, written in the local coordinate axes E^u, E^s, with origin at x_0, which correspond to its respective local unstable and stable manifolds. Let $\psi_1(x) = g_1(d(x, \mathcal{Z}_1))$ and $\psi_2(x) = g_2(d(x, \mathcal{Z}_2))$ for g_i as in (3.1). For n large enough, $U_1^{(n)}(\tau_1)$ and $U_2^{(n)}(\tau_2)$ are two disjoint rectangles[1] of widths 2ε and $\lambda(\lambda-1)\varepsilon$, with heights $h_1 \sim \frac{\tau_1}{2\varepsilon n}$ and $h_2 \sim \frac{\tau_2}{(\lambda^2-\lambda)\varepsilon n}$, respectively.

Note that $V_n(\mathbb{\tau}) = \cap_{i=1}^2 U_i^{(n)}(\tau_i) = \emptyset$ and therefore:

$$\begin{aligned}\hat{\Gamma}(\mathbb{\tau}) &= \lim_{n\to\infty} n\mu(\mathbb{X}_0 \not\leq \mathbb{u}_n(\mathbb{\tau})) = \lim_{n\to\infty} n\mu\left(\bigcup_{i=1}^2 \{X_{0i} > u_{ni}(\tau_i)\}\right) \\ &= \lim_{n\to\infty} n\sum_{i=1}^2 \mu\left(X_{0i} > u_{ni}(\tau_i)\right) - \lim_{n\to\infty} n\mu\left(\bigcap_{i=1}^2 \{X_{0i} > u_{ni}(\tau_i)\}\right) = \tau_1 + \tau_2.\end{aligned}$$

We observe that in this invertible case, conditions $Д(\mathbb{u}_n)$ and $Д'(\mathbb{u}_n)$ can be checked with a simple adaptation of the argument used in [CFF+15] for the respective univariate versions.

In order to compute the set $A_n^{(q)}$ (with the right choice of q), we start by noting that the dynamics pushes the set $U_2^{(n)}(\tau_2)$ along the unstable manifold away from both $U_1^{(n)}(\tau_1)$ and $U_2^{(n)}(\tau_2)$, which means that if the orbit enters $U_2^{(n)}(\tau_2)$, then no short returns to $\{\mathbb{X}_0 \leq \mathbb{u}_n(\mathbb{\tau})\}$ are expected. The set $U_1^{(n)}(\tau_1)$ is shrunk in the stable (vertical) direction and stretched in the unstable (horizontal) and overlaps with $U_1^{(n)}(\tau_1)$ immediately in the first iteration. This means that the middle vertical strip of width $2\lambda^{-1}$ (depicted in blue in Figures 3.3 and 3.4) must be removed from $U_1^{(n)}(\tau_1)$. Observe that the right vertical strip (depicted in red in Figures 3.3 and 3.4) falls exactly into the empty space between $U_1^{(n)}(\tau_1)$ and $U_2^{(n)}(\tau_2)$, however after two iterates it stretches completely horizontally across $U_2^{(n)}(\tau_2)$. Now, one of two possible scenarios can occur, either $h_1 < \lambda^2 h_2$ and then the red strip is completely contained in $U_2^{(n)}(\tau_2)$ and must be removed from $A_n^{(q)}$ (Figure 3.3); or not and then only the central part of the red strip that meets $U_2^{(n)}(\tau_2)$ must be removed (Figure 3.4). Note that the dynamics will push the possibly surviving points further and further way from the sets $U_1^{(n)}(\tau_1)$ and $U_2^{(n)}(\tau_2)$. This means that we should take $q = 2$ in this case.

Before we compute the extremal index function we express the turning point $h_1 = \lambda^2 h_2$ as $\tau_1 = \frac{2\lambda\tau_2}{\lambda-1}$ or $\alpha = \frac{2\lambda}{3\lambda-1}$, for $\alpha = \frac{\tau_1}{\tau_1+\tau_2}$.

[1]In fact, for the usual Euclidean metric in the definition of $\mathbb{\Psi}$ these sets have rounded tips. However, since these semidisks have an asymptotically negligible measure of the order $\left(\mu\left(U_i^{(n)}\right)\right)^2$, then to simplify both the computations and the diagrams, we will simply disregard these half discs as if the metric would measure distances only in the stable direction. Therefore, we will assume that the sets $U_i^{(n)}$ are actual rectangles, which asymptotically makes no difference.

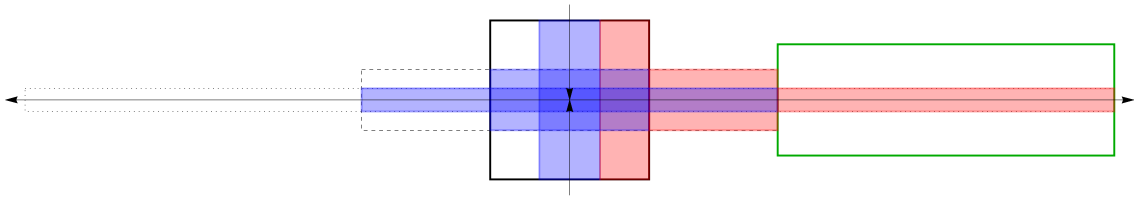


FIGURE 3.3. $U_1^{(n)}(\tau_1)$ is the box outlined with a black line, $U_2^{(n)}(\tau_2)$ is the box outlined in green, the first iterate of $U_1^{(n)}(\tau_1)$ corresponds to the black, dashed box, while the second iterate corresponds to the black, dotted box. $A_n^{(q)}$ is the union of the white parts from $U_1^{(n)}(\tau_1)$ and the whole $U_2^{(n)}(\tau_2)$. In this case, $h_1 < \lambda^2 h_2$. Note that for large n the sets $U_i^{(n)}$ are very thin rectangles with rounded tips, but for pictorial simplicity we disregard the semidisks as mentioned in footnote 1.

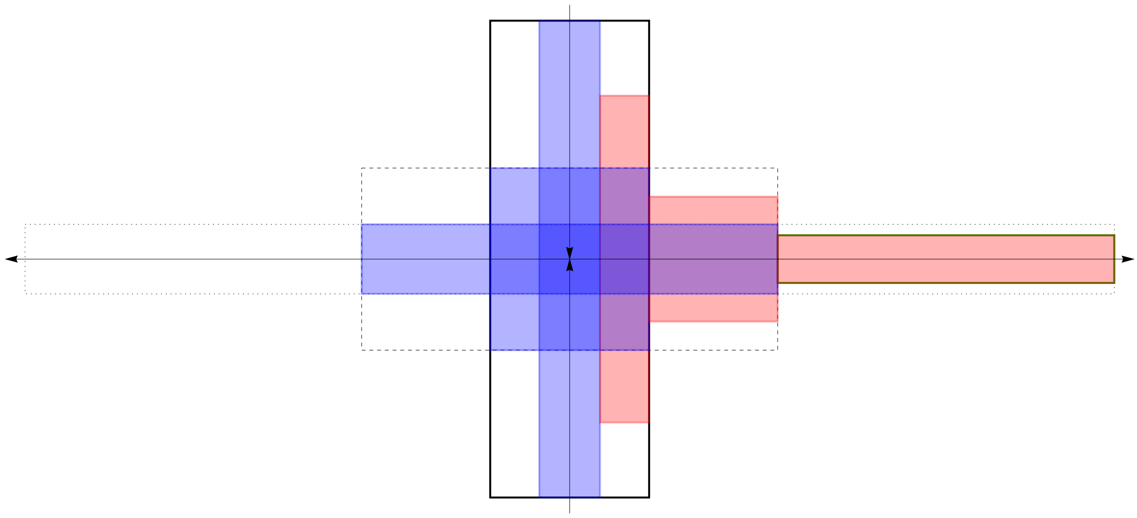
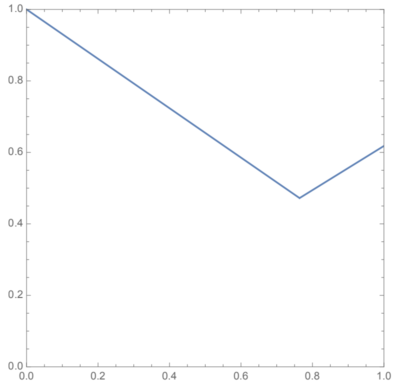


FIGURE 3.4. $U_1^{(n)}(\tau_1)$ is the box outlined with a black line, $U_2^{(n)}(\tau_2)$ is the box green box, the first iterate of $U_1^{(n)}(\tau_1)$ corresponds to the black, dashed box, while the second iterate corresponds to the black, dotted box. $A_n^{(q)}$ is the union of the white parts from $U_1^{(n)}(\tau_1)$ and the whole $U_2^{(n)}(\tau_2)$. In this case, $h_1 > \lambda^2 h_2$. The same comment regarding the shape of $U_i^{(n)}$ as in the caption of Figure 3.3 applies.

We compute $\theta(\mathfrak{r})$, when $\tau_1 \leq \frac{2\lambda\tau_2}{\lambda-1}$ as follows:

$$\theta(\tau_1,\tau_2) = \lim_{n\to\infty} \frac{\mu\left(U_1^{(n)}(\tau_1) \setminus \left(f^{-1}\left(U_1^{(n)}(\tau_1)\right) \cup f^{-2}\left(U_2^{(n)}(\tau_2)\right)\right)\right) + \mu\left(U_2^{(n)}(\tau_2)\right)}{\mu\left(U_1^{(n)}(\tau_1)\right) + \mu\left(U_2^{(n)}(\tau_2)\right)}$$

$$= \lim_{n\to\infty} \frac{(1-\lambda^{-1})\varepsilon h_1 + \varepsilon(\lambda^2-\lambda)h_2}{2\varepsilon h_1 + \varepsilon(\lambda^2-\lambda)h_2} = \lim_{n\to\infty} \frac{\frac{(1-\lambda^{-1})\tau_1}{2n} + \frac{\tau_2}{n}}{\frac{\tau_1}{n} + \frac{\tau_2}{n}} = 1 - \frac{(1+\lambda^{-1})\alpha}{2}.$$



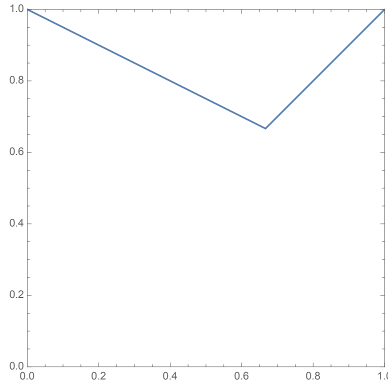


FIGURE 3.5. On the left is the graph of the extremal index function and on the right is the graph of the Pickands D function, both associated to Γ for the cat map example.

Using the same formula, when $\tau_1 > \frac{2\lambda\tau_2}{\lambda-1}$, we have

$$\begin{aligned}\theta(\tau_1,\tau_2) &= \lim_{n\to\infty} \frac{(1-\lambda^{-1})\frac{\varepsilon}{2}h_1 + (1-\lambda^{-1})\frac{\varepsilon}{2}(h_1-\lambda^2 h_2) + \varepsilon(\lambda^2-\lambda)h_2}{2\varepsilon h_1 + \varepsilon(\lambda^2-\lambda)h_2} \\ &= \lim_{n\to\infty} \frac{\frac{(1-\lambda^{-1})\tau_1}{2n} + (1-\lambda^{-1})\left(\frac{\tau_1}{2n} - \frac{\lambda\tau_2}{(\lambda-1)n}\right) + \frac{\tau_2}{n}}{\frac{\tau_1}{n}+\frac{\tau_2}{n}} = \frac{(\lambda-1)\alpha}{\lambda}.\end{aligned}$$

Observe that $\mathbb{\theta} = (\theta_1,\theta_2) = (1-\lambda^{-1},1)$, so

$$\theta(\alpha,1-\alpha) = \begin{cases} 1 - \frac{(1+\lambda^{-1})\alpha}{2} & \text{if } \alpha \in \left[0, \frac{2\lambda}{3\lambda-1}\right], \\ \frac{(\lambda-1)\alpha}{\lambda} & \text{if } \alpha \in \left(\frac{2\lambda}{3\lambda-1}, 1\right], \end{cases} \qquad D(\alpha) = \begin{cases} 1-\frac{\alpha}{2} & \text{if } \alpha\in[0,2/3], \\ \alpha & \text{if } \alpha \in (2/3,1]. \end{cases}$$

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