Solving partial differential equations in two dimensions using finite differences

Review of finite differences in 1D

Earlier in this course when we looked at transient groundwater modeling, you were introduced to the one dimension diffusion equation, which has the basic form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where here we omit the material property coefficient terms to keep things simple. You also learned how to approximate the partial derivatives in this expression using finite differences computed between discrete samples of u. The first order partial derivative has the forward finite difference approximation:

$$\frac{\partial u(t_i)}{\partial t} \approx \frac{u(t_{i+1}) - u(t_i)}{\Delta t}, \tag{2}$$

where Δt is $t_{i+1} - t_i$. You also saw the second order partial derivative has the central finite difference approximation:

$$\frac{\partial^2 u(x_i)}{\partial x^2} \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta x^2}, \tag{3}$$

where Δx is the grid spacing in x. We also simplified the notation by using $u_i = u(x_i)$, giving:

$$\frac{\partial^2 u_i}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \tag{4}$$

Two dimensional Partial Differential Equations

We will expand the finite difference method to two dimension so we can model the behavior of a function u in both the x and y directions: $u = u(x, y)$. Here are three commonly encountered two dimension partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{Laplace's equation} \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \qquad \text{Diffusion equation} \tag{6}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{Wave equation} \tag{7}$$

Notice how these equations are all nearly the same. The differences are in the time dependent terms on the right hand side. Laplace's equation has no time dependence, and can be thought of as the steady state solution to either the diffusion of wave equations. The diffusion equation has a first order time derivative while the wave equation has a second order time derivative.

Here these equations are presented in their generic coefficient-free form. When applying them to real problems, we will have to include terms that account for the material properties of the problem (for example, like the hydraulic properties we saw in the transient groundwater problem). Further, our generic variable u will correspond to some physical behavior, such as pressure, velocity, temperature, electric potential, groundwater head, etc, depending on the particular problem at hand.

Two dimensional Finite Difference Solution

Let's consider Laplace's equation, since it is the simplest of the three equations shown above. The finite difference expression for the second-order partial derivative with respect to x is:

$$\frac{\partial^2 u_{i,j}}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \tag{8}$$

and similarly, the second-order partial derivative with respect to y has the finite difference approximation:

$$\frac{\partial^2 u_{i,j}}{\partial y^2} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}, \tag{9}$$

Here we introduce the notation $u_{i,j} = u(x_i, y_j)$. See Figure 1.

Inserting these into Laplace's equation gives:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0. \tag{10}$$

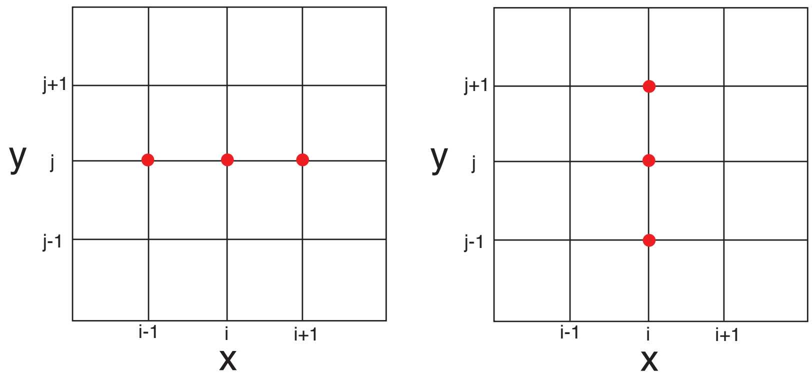

Figure 1: Finite differences in the x and y directions for a two dimensional grid.

If we assume our grid spacing is even in the x and y directions so $\Delta x = \Delta y$, we can factor out the Δx and Δy terms. Rearranging the resulting expression for u_i yields:

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) \tag{11}$$

Here you can see that the central value $u_{i,j}$ is simply the average of all its neighbors! See Figure 2.

In class we will go over how to write an iterative loop that solves Laplace's equation using this averaging equation.

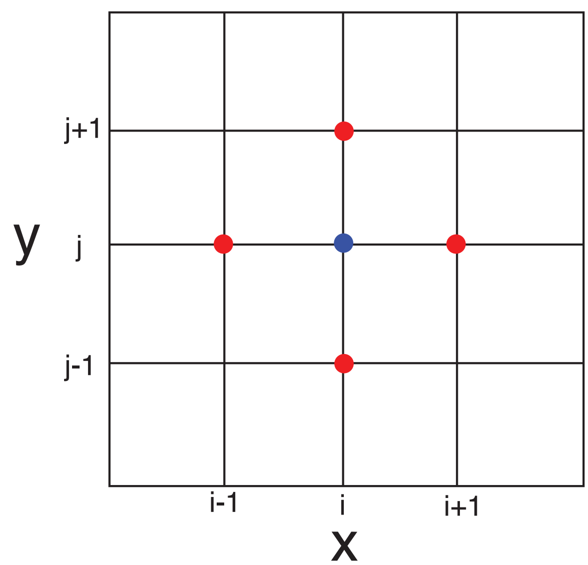

Figure 2: Finite difference stencil for Laplace's equation at point x_i, y_j.