Thermal Modeling with the Finite Difference Method

1D Thermal Diffusion

Consider the one-dimension (1D) transient heat conduction equation in a uniform medium without any heat generating sources. Here the temperature T only varies in the x direction, so $T(t, x)$:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where k is the thermal conductivity, ρ is the density, and c_p is the heat capacity. These properties will vary with different materials. They can be simplified into a single parameter call the thermal diffusivity:

$$\kappa = \frac{k}{\rho c_p} \tag{2}$$

where κ has units of m^2/s. For typical rocks, κ is around 10^{-6} m^2/s.

The thermal diffusion equation above is similar to the 1D hydraulic diffusion problem we saw a few weeks ago in the course. Its finite difference solution can be found using the *forward in time* finite difference formula:

$$\frac{\partial T(t_n, x)}{\partial t} \approx \frac{T(t_{n+1}, x) - T(t_n)}{\Delta t}, \tag{3}$$

where Δt is $t_{n+1} - t_n$, and the 2nd order central difference formula for the spatial derivative:

$$\frac{\partial^2 T(t, x_i)}{\partial x^2} \quad \approx \quad \frac{T(t, x_{i+1}) - 2T(t, x_i) + T(t, x_{i-1})}{\Delta x^2}, \tag{4}$$

The notation is simplified by using superscripts for the time index and subscripts for the spatial position index: $T_i^n = T(t_n, x_i)$. Thus, the time and space partial derivatives become

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{n+1} - T_i^n}{\Delta t}, \tag{5}$$

and

$$\frac{\partial^2 T}{\partial x^2} \quad \approx \quad \frac{T^n_{i+1} - 2T^n_i + T^n_{i-1}}{\Delta x^2} \tag{6}$$

We then have

$$\frac{T^{n+1}_i - T^n_i}{\Delta t} = \kappa \frac{T^n_{i+1} - 2T^n_i + T^n_{i-1}}{\Delta x^2} \tag{7}$$

which rearranges into the finite difference update equation:

$$T^{n+1}_i = T^n_i + \kappa \frac{\Delta t}{\Delta x^2} \left(T^n_{i+1} - 2T^n_i + T^n_{i-1} \right) \tag{8}$$

This is an explicit finite difference method. It is also known as the forward-in-time, center-in-space, or FTCS finite difference scheme. There are other, more advance and more accurate finite difference methods, but the basic FTCS approach can be used to model a range of useful problems. The key limitation, is that the time step Δt and spatial grid spacing Δx need to be carefully chosen. For the 1D FTCS method to be stable, the time step and grid spacing need to satisfy:

$$\kappa \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \tag{9}$$

This requirement comes from Von Neumann stability analysis. We don't have time to cover its derivation in this course, but a starting point for learning more can be found here: https://en.wikipedia.org/wiki/Von_Neumann_stability_analysis

2D Thermal Diffusion

Here the temperature T on varies in both the x and y direction, so $T(t, x, y)$:

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{10}$$

Using the same method as we did for the 1D problem, we can find the update equation:

$$T^{n+1}_{i,j} = T^n_{i,j} + \kappa \frac{\Delta t}{\Delta x^2} \left(T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j} \right) + \kappa \frac{\Delta t}{\Delta y^2} \left(T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j-1} \right) \tag{11}$$

where now the temperature has two subscripts: i for the x grid index and j for the y grid index.

Assuming $\Delta x = \Delta y$, the 2D FTCS method is stable when the time step and grid spacing satisfy:

$$\kappa \frac{\Delta t}{\Delta x^2} \leq \frac{1}{4} \tag{12}$$