Thermal Modeling with the Finite Difference Method

1D Thermal Diffusion

Consider the one-dimension (1D), transient heat conduction equation in a uniform medium without any heat generating sources. Here the temperature T on varies in the x direction, so $T(t, x)$:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where k is the thermal conductivity, ρ is the density, and c_p is the heat capacity. These properties will vary with different materials. They can be simplified into a single parameter call the thermal diffusivity:

$$\kappa = \frac{k}{\rho c_p} \tag{2}$$

where κ has units of m^2/s. For typical rocks, $\kappa = 10^{-6}$ m^2/s.

The thermal diffusion above is similar to the 1D hydraulic diffision problem we saw a few weeks ago in the course. Its finite difference solution can be found using:

$$\frac{\partial T(t_i, x)}{\partial t} \approx \frac{T(t_{i+1}, x) - T(t_i)}{\Delta t}, \tag{3}$$

where Δt is $t_{i+1} - t_i$. We can also use the 2nd order central difference formula

$$\frac{\partial^2 T(t, x_i)}{\partial x^2} \quad \approx \quad \frac{T(t, x_{i+1}) - 2T(t, x_i) + T(t, x_{i-1})}{\Delta x^2}, \tag{4}$$

We also simplified the notation by using $T_i^n = T(t_n, x_i)$, giving:

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{n+1} - T_i^n}{\Delta t}, \tag{5}$$

$$\frac{\partial^2 T}{\partial x^2} \quad \approx \quad \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} \tag{6}$$

We then have

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \kappa \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} \tag{7}$$

which rearranges into the update equation:

$$T_i^{n+1} = T_i^n + \kappa \frac{\Delta t}{\Delta x^2} \left(T_{i+1}^n - 2T_i^n + T_{i-1}^n \right) \tag{8}$$

2D Thermal Diffusion

Here the temperature T on varies in both the x and y direction, so $T(t, x, y)$:

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{9}$$

Using the same method as we did for the 1D problem, we can find the update equation:

$$T_{i,j}^{n+1} = T_{i,j}^n + \kappa \frac{\Delta t}{\Delta x^2} \left(T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n \right) + \kappa \frac{\Delta t}{\Delta y^2} \left(T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n \right) \tag{10}$$