# Thermal Modeling with the Finite Difference Method

## 1D Thermal Diffusion

Consider the one-dimension (1D), transient heat conduction equation in a uniform medium without any heat generating sources. Here the temperature $T$ on varies in the $x$ direction, so $T(t, x)$:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} \tag{1}$$

where $k$ is the thermal conductivity, $\rho$ is the density, and $c_p$ is the heat capacity. These properties will vary with different materials. They can be simplified into a single parameter call the thermal diffusivity:

$$\kappa = \frac{k}{\rho c_p} \tag{2}$$

where $\kappa$ has units of $\mathrm{m}^2/\mathrm{s}$. For typical rocks, $\kappa = 10^{-6}\ \mathrm{m}^2/\mathrm{s}$.

The thermal diffusion above is similar to the 1D hydraulic diffision problem we saw a few weeks ago in the course. Its finite difference solution can be found using:

$$\frac{\partial T(t_i, x)}{\partial t} \approx \frac{T(t_{i+1}, x) - T(t_i)}{\Delta t}, \tag{3}$$

where $\Delta t$ is $t_{i+1} - t_i$. We can also use the 2nd order central difference formula

$$\frac{\partial^2 T(t, x_i)}{\partial x^2} \approx \frac{T(t, x_{i+1}) - 2T(t, x_i) + T(t, x_{i-1})}{\Delta x^2}, \tag{4}$$

We also simplified the notation by using $T_i^n = T(t_n, x_i)$, giving:

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{n+1} - T_i^n}{\Delta t}, \tag{5}$$

$$\frac{\partial^2 T}{\partial x^2} \quad \approx \quad \frac{T^n_{i+1} - 2T^n_i + T^n_{i-1}}{\Delta x^2} \tag{6}$$

We then have

$$\frac{T^{n+1}_i - T^n_i}{\Delta t} = \kappa \frac{T^n_{i+1} - 2T^n_i + T^n_{i-1}}{\Delta x^2} \tag{7}$$

which rearranges into the update equation:

$$T^{n+1}_i = T^n_i + \kappa \frac{\Delta t}{\Delta x^2}\left(T^n_{i+1} - 2T^n_i + T^n_{i-1}\right) \tag{8}$$

## 2D Thermal Diffusion

Here the temperature $T$ on varies in both the $x$ and $y$ direction, so $T(t, x, y)$:

$$\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{9}$$

Using the same method as we did for the 1D problem, we can find the update equation:

$$T^{n+1}_{i,j} = T^n_{i,j} + \kappa \frac{\Delta t}{\Delta x^2}\left(T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j}\right) + \kappa \frac{\Delta t}{\Delta y^2}\left(T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j-1}\right) \tag{10}$$