Contents

Intuition behind the construction of $d_{1,2}$ and the Black-Scholes formula

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November 29, 2022

1 Discrete/Continuous Compounding

Discrete compounding below

$$\begin{aligned} PV =& \frac{FV}{\left(1+\frac{r}{n}\right)^{nt}} = FV\left(1+\frac{r}{n}\right)^{-nt} \\ FV =& PV\left(1+\frac{r}{n}\right)^{nt} \end{aligned} \tag{1}$$

Continuous Compounding below

$$\begin{aligned} FV =& PVe^{rt} \\ PV =& \frac{FV}{e^{rt}} = FVe^{-rt} \end{aligned} \tag{2}$$

2 Black-Scholes

Developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes. Premium equals intrinsic (S, K or the value of option if it were exercised today) and extrinsic value (σ, τ). $\Phi(.)$ is the cumulative density function for normal distribution. Call Premium is a function of five parameters (no dividend) $c(S, K, r, \sigma, \tau)$, but because K, r, σ are constants in Black-Scholes, we just write $c(S, \tau)$. Note that r risk-free rate (10-yr treasury, ticker:USGG10YR). σ is the constant volatility of the underlying. $\tau = T - t$ time to maturity, if time to exp is 80 days, then $\tau = T - t = 80/365 = 21.92\%$. The basis for the Black-Scholes formula is simply the current and discounted intrinsic value of the call option $S - K \exp^{-r\tau}$, adjusted for the probabilities $\Phi(.)$.

If $S < K \exp^{-r\tau}$, then premium would have a negative value, which cannot happen and so option is not exercised. Thus, $F(.)$ is introduced to prevent premium from falling below zero. The greater the amount by which S is less than $K \exp^{-r\tau}$, the more $\Phi(.)$ approach zero. And when $\Phi(.)$ is 0, then premium is also zero. Some brokers show the estimated probability of option expiring ITM. This is usually $\Phi(d_2)$.

Expected value of contingent receipt of stock is $S \exp^{r\tau} \Phi(d_1)$. Discounted value is $S\Phi(d_1)$. Similarly, expected value of of contingent exercise payment is $-K\Phi(d_2)$. Discounted value is $-K \exp^{-r\tau} \Phi(d_2)$.

$+S \exp^{r\tau} \Phi(d_1)$ is the amount that will likely be received on selling the stock at expiration. $-KF(d_2)$ is the payment that will likely be made to purchase the stock when the call option is exercised at expiration. The premium is just the DISCOUNTED difference between the two terms.

It is assumed that during the remaining life of option stock grows continuously at risk-free rate $S \exp^{r\tau}$. At maturity, the intrinsic value is $S \exp^{r\tau} - K$. Discounting to present time $S - K \exp^{-r\tau}$ forms the basis for the Black-Scholes.

$$
\begin{aligned}
d_2 =& \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\
d_1 =& d_2 + \sigma\sqrt{\tau} = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}
\end{aligned} \tag{3}
$$

Proof that d_1 is can be derived from d_2 is given in 25. Once $d_{1,2}$ are derived, option premium is

$$
\begin{aligned}
c =& \exp^{-r\tau} \left[-K\Phi(d_2) + S \exp^{r\tau} \Phi(d_1)\right] \\
p =& \exp^{-r\tau} \left[+K\Phi(-d_2) - S \exp^{r\tau} \Phi(-d_1)\right]
\end{aligned} \tag{4}
$$

simplifying the above gives

$$
\begin{aligned}
c =& - K \exp^{-r\tau} \Phi(d_2) + S\Phi(d_1) \\
p =& + K \exp^{-r\tau} \Phi(-d_2) - S\Phi(-d_1)
\end{aligned} \tag{5}
$$

3 z-score

Note that in the Black-Scholes formula for the price of call option, given in 5, $\Phi(.)$ denotes the cumulative density function for the normal distribution. The input to the cumulative density function is a z-score, given by generic formula below

$$
z = \frac{x - \mu}{\sigma} \tag{6}
$$

where x is a variable of interest, μ, σ is mean/standard deviation of normal distribution.

Consider now the case for out-of-money call option, so that S is below K. How we calculate the probability of the option being ITM? Well, we can first find the annualised growth rate needed for the current stock price S to equal K (note that this growth rate is on the x-axis of normal distribution and probability is on the y-aixs). For the purposes of the below derivations, we replace $K = X_t$ and $S = X_0$. In continuous terms, we can express this relationship as

$$X_t = X_0 \exp(x) \tag{7}$$

by simple manipulation, we can isolate growth rate as

$$\begin{aligned} \frac{X_t}{X_0} =& \exp(x) \\ \log\left(\frac{X_t}{X_0}\right) =& x \end{aligned} \tag{8}$$

Note that in 8 we have defined our variable of interest, denoted as x. Moving on, what is the mean of normal distribution in this case?

$$f = \log\left(\frac{X_t}{X_0}\right) \tag{9}$$

and the SDE for our function f is of the form

$$dX_t = \mu X_t dt + \sigma X_t dW(t) \tag{10}$$

Ito's lemma

$$\begin{aligned} df =& \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial f}{\partial X_t} dX_t dX_t \\ =& \frac{1}{x} dX_t + \frac{1}{2}\left(-\frac{1}{x^2}\right) dX_t dX_t \end{aligned} \tag{11}$$

We plug the expression for dX_t into 11

$$df = \frac{1}{X_t}\Big(\mu X_t dt + \sigma X_t dW(t)\Big) + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)\Big(\mu X_t dt + \sigma X_t dW(t)\Big)\Big(\mu X_t dt + \sigma X_t dW(t)\Big) \tag{12}$$

and

$$df = \frac{1}{X_t}\Big(\mu X_t dt + \sigma X_t dW(t)\Big) + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)\mu^2 X_t^2 dt^2 + 2\Big(\mu\sigma X_t^2 dt dW(t)\Big) + \sigma^2 X_t^2 dW(t)^2 \tag{13}$$

using informal rules $dtdt = 0$,$dtdW(t) = 0$,$dW(t)dW(t) = dt$, 13 simplifies to

$$df = \mu dt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt \tag{14}$$

grouping two dt terms together

$$df = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t) \tag{15}$$

integrating both sides

$$\begin{aligned} \int df =& \int \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int \sigma dW(t) \\ f =& \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t) \end{aligned} \tag{16}$$

by simple manipulation

$$\begin{aligned} \log\left(\frac{X_t}{X_0}\right) =& \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t) \\ \frac{X_t}{X_0} =& \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right) \\ X_t =& X_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right) \end{aligned} \tag{17}$$

Finally

$$E(X_t) = X_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t\right) \tag{18}$$

because $E[W(t)] = 0$. Notice that both 9 and 16 is the same function f. Specifically, 9 can be expressed in terms of determinstic component $\left(\mu - \frac{1}{2}\sigma^2\right) t$ and stochastic component $\sigma W(t)$ using Ito's Lemma, as shown in 16.

Lastly, note that we change dt to $\tau = T - t$ to reflect option's time value. Under risk neural pricing, all assets grow at risk-free rate r. Therefore, we can change μ in 18 by r. Now, we can plug in the derivations for x from 8, μ from 18, and the time-adjusted volatility $\sigma\sqrt{\tau}$ into the formula for z-score in 6, which would give the following

$$\begin{aligned} z =& \frac{x - \mu}{\sigma} \\ =& \frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \end{aligned} \tag{19}$$

Now, if we would plug z-score from 19 into $\Phi(.)$, then we would get cumulative density function in the range $[-\infty, \log\left(\frac{K}{S}\right)]$ (left side). However, we need cumulative density function in the range $[\log\left(\frac{K}{S}\right), +\infty]$ (right side). Because the normal distribution is symmetric, we can simply multiply 19 by -1.

$$\begin{aligned} z(-1) =& -1\left(\frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\ =& -1\left(\frac{\log K - \log S - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\ =& \frac{\log S - \log K + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ =& \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \end{aligned} \tag{20}$$

So we now have

$$\begin{aligned} z =& -d_2 = \frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ -z =& d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \end{aligned} \tag{21}$$

Hence, $d_2 = -z$ and $\Phi(d_2) = \Phi(-z)$. In words, it is the probability that $S > K$ at expiration and that option is exercised under risk neutral measure where stock grows at risk free rate. In can also be said that the actual probability, rather than the risk neutral probability we calculated above, is actually greater in reality because stocks earn more than risk free rate.
Noting that $\Phi(z)$ in Black-Scholes is the cumulative density function for the normal distribution

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{1}{2}z^2\right) dz \tag{22}$$

Plugging back into 22

$$\Phi(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} \exp\left[-\frac{1}{2}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right] dz \tag{23}$$

Note that the upper limit of integration in 23 is the formula for d_2. For conciseness, we only write d_2.

Thus, 5 can be rewritten as

$$
\begin{aligned}
c = & - K \exp^{-r\tau} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} \exp\left(-\frac{1}{2}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right) dz \\
& + S \exp^{-r\tau} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \exp\left(-\frac{1}{2}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right) dz
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
d_1 =& d_2 + \sigma\sqrt{\tau} \\
=& \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau} \\
=& \frac{\log\left(\frac{S}{K}\right) + r\tau - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau} \\
=& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} - \frac{\frac{1}{2}\sigma\sqrt{\tau}\sigma\sqrt{\tau}}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau} \\
=& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau} + \frac{1}{1}\sigma\sqrt{\tau} \\
=& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} \\
=& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} + \frac{\frac{1}{2}\sigma\sqrt{\tau}\sigma\sqrt{\tau}}{\sigma\sqrt{\tau}} \\
=& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} + \frac{\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \\
=& \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}
\end{aligned}
\tag{25}
$$

4 Ito's

Theorem 1 *Let $f(X(t), t)$ be a smooth function of two variables, and let $X(t)$ be a stochastic process satisfying $dX(t) = \mu dt + \sigma dW(t)$. Then*

where

$$dt = \frac{T}{N} \tag{26}$$

$$Y(t) = e^{X(t)} \tag{27}$$

$$dW(t) = W(t + dt) - W(t) \tag{28}$$

So I know SDE for X_t process, and I know that $Y_t = Y_0 e^{X_t}$, what does SDE for Y_t process looks like?

$$dX_t = \mu X_t dt + \sigma X_t dW(t) \tag{29}$$

$$dY(t) = \tag{30}$$

Hence, any SDE that has no dt-terms is a martingale

$$f(X(t), t) = Y_t = Y_0 e^{X_t} \tag{31}$$