Contents

Intuition behind the construction of $d_{1,2}$ and the Black-Scholes formula

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1 Log-normal

Probability density function PDF shows the probability that a random variable takes on a certain value.

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2} \tag{1}$$

If I want to see the probability of x occuring, just plug it in 1. CDF

$$F(x) = \left(\frac{\log x - \mu}{\sigma}\right) \tag{2}$$

where $x = S/K, \mu = r + 1/2\sigma^2, \sigma = \sigma\sqrt{\tau}$.

2 Discrete/Continuous Compounding

Discrete compounding below

$$\begin{aligned} PV =& \frac{FV}{\left(1 + \frac{r}{n}\right)^{nt}} = FV \left(1 + \frac{r}{n}\right)^{-nt} \\ FV =& PV \left(1 + \frac{r}{n}\right)^{nt} \end{aligned} \tag{3}$$

Continuous Compounding below

$$\begin{aligned} FV =& PV e^{rt} \\ PV =& \frac{FV}{e^{rt}} = FV e^{-rt} \end{aligned} \tag{4}$$

3 Black-Scholes

Developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes. Premium equals intrinsic (S, K or the value of option if it were exercised today) and extrinsic value (σ, τ). $F(.)$ is the cumulative density function for normal distribution. Call Premium is a function of five parameters (no dividend) $c(S, K, r, \sigma, \tau)$. The basis for the Black-Scholes formula is simply the current and discounted intrinsic value of the call option $S - K \exp^{-r\tau}$, adjusted for the probabilities $F(.)$.

If $S < K \exp^{-r\tau}$, then premium would have a negative value, which cannot happen/option is not exercised. Thus, $F(.)$ is introduced to prevent premium from falling below zero. The greater the amount by which S is less than $K \exp^{-r\tau}$, the more $F(.)$ approach zero. And when $F(.)$ is 0, then premium is also zero. Some brokers show the estimated probability of option expiring ITM. This is usually $F(d_2)$.

Expected value of contingent receipt of stock is $S \exp^{r\tau} F(d_1)$. Discounted value is $SF(d_1)$. Similarly, expected value of of contingent exercise payment is $-KF(d_2)$. Discounted value is $-K \exp^{-r\tau} F(d_2)$.

$+S \exp^{r\tau} F(d_1)$ is the amount that will likely be received on selling the stock at expiration. $-KF(d_2)$ is the payment that will likely be made to purchase the stock when the call option is exercised at expiration. The premium is just the DISCOUNTED difference between the two terms $\exp^{-r\tau} [-KF(d_2) + S \exp^{r\tau} F(d_1)]$.

It is assumed that during the remaining life of option stock grows continuously at risk-free rate $S \exp^{r\tau}$. At maturity, the intrinsic value is $S \exp^{r\tau} - K$. Discounting to present time $S - K \exp^{-r\tau}$ forms the basis for the Black-Scholes in 6.

At maturity $\tau = T - t = 0$, K is longer discounted as $\exp^0 = 1$.

- S
- K
- r risk-free rate (10-yr treasury, ticker:USGG10YR)
- σ volatility of the underlying
- $\tau = T - t$ time to maturity, if time to exp is 80 days, then $\tau = T - t = 80/365 = 21.92\%$

$$\begin{aligned} d_2 =& \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ d_1 =& d_2 + \sigma\sqrt{\tau} = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \end{aligned} \tag{5}$$

The derivation of d_1 from d_2 is shown in 24.

To understand the construction of $d_{1,2}$ from 5, see below

Once we have d_1, d_2, we can calculate the premium. If $\tau > 0$, then

$$c = -K \exp^{-r\tau} F(d_2) + SF(d_1) \tag{6}$$

where c is the premium for the call, S is the current value of the stock, K is the exercise price, r is constant risk-free rate (10-yr treasury, ticker:USGG10YR), τ is the remaining time to expiration of the option (% of year), σ is the volatility of the underlying, and $F(.)$ is the cumulative distribution function for normal distribution.

The Black-Scholes formula essentially calculates the current intrinsic value, adjusted for the probability $F(.)$ that the security will be worth more than the strike price at expiration.

If I sell option, I hope it will NOT reach strike and just collect premium

1. short call

- requires the seller to deliver shares at K if is exercised
- breakeven add premium to K, the seller could also at open market
- short call is a bearish trading strategy, seller hopes it will NOT reach strike
- goal of the seller is to make money from premium and see the option expire worthless
- seller has limited profit (premium), unlimited loss (stock can rise to infinity)
- selling a covered call means that seller already has underlying
- selling a naked call means that seller does NOT have underlying, if current stock price is much higher than strike, then seller would first need to buy shares at high market price and then sell at strike to option buyer
- If stock is trading 100 and the investor wants to sell a 110 strike price call option, they can collect a 2 premium to do so. If the stock trades up to 115, they will be forced to buy the stock at 115 and then deliver the stock to the call buyer at the price of 110, losing 5 in the process. But because the option seller received 2 when they sold the call, their net loss is 3. If, however, the stock continues to trade down or never reaches 110, the trader keeps the 2 premium as profit.

2. short put
   - requires the seller to buy shares at K if exercised
   - short call is a bullish trading strategy, seller hopes it will NOT reach strike and hopes S_0 is above K at expiration

3. Six

Stochastic differential equation

$$dx = \mu x dt + \sigma x dW(t) \tag{7}$$

$$(dx)^2 = \mu^2 x^2 dt^2 + 2(\mu\sigma x^2 dt dW(t)) + \sigma^2 x^2 dW(t)^2 \tag{8}$$

$$d\left(\log\left(\frac{x}{x_0}\right)\right) = \frac{1}{x}dx + \frac{1}{2}\left(-\frac{1}{x^2}\right)(dx)^2 \tag{9}$$

now plug 7 and 8 back into 9

$$d\left(\log\left(\frac{x}{x_0}\right)\right) = \frac{1}{x}\mu x dt + \sigma x dW(t) + \frac{1}{2}\left(-\frac{1}{x^2}\right)\mu^2 x^2 dt^2 + 2(\mu\sigma x^2 dt dW(t)) + \sigma^2 x^2 dW(t)^2 \tag{10}$$

$$d\left(\log\left(\frac{x}{x_0}\right)\right) = \mu dt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt \tag{11}$$

grouping two dt terms together

$$d\left(\log\left(\frac{x}{x_0}\right)\right) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t) \tag{12}$$

integrating both sides

$$\log\left(\frac{x}{x_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t) \tag{13}$$

and

$$x(t) = x_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\} \tag{14}$$

4 z-score

$$z = \frac{x-\mu}{\sigma} \tag{15}$$

$$\begin{aligned} K =& S_0 \exp^x \\ \frac{K}{S} =& \exp^x \\ \log\left(\frac{K}{S}\right) =& x \end{aligned} \tag{16}$$

Now that we have determined the variable of interest x, which is the growth rate from K to S, such that the following holds $K = S\exp^x$.

$$z = \frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \tag{17}$$

However, z-score in 17 is the probability from $-\infty$ to $\log\left(\frac{K}{S}\right)$ (on the left side). However, we need from the right side. Because the normal distribution is symmetric, we can simply multiply 17 by -1.

$$\begin{aligned} z(-1) =& -1\left(\frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\ =& -1\left(\frac{\log K - \log S - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\ =& \frac{\log S - \log K + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ =& \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \end{aligned} \tag{18}$$

$$\begin{aligned} d_2 =& -z \\ F(d_2) =& N(-z) \end{aligned} \tag{19}$$

and similarly

$$\begin{aligned} -d_2 =& z \\ F(-d_2) =& N(z) \end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned} z =& -d_2 = \frac{\log\left(\frac{K}{S}\right) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \\ -z =& d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \end{aligned} \tag{21}$$

Recall that $N(z)$ is the cumulative distribution of the standard normal variable

$$\begin{aligned} N_Z(z) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left(-\frac{1}{2}z^2\right) dz \\ N_Z(d_2) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} \exp\left[-\frac{1}{2}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right] dz \end{aligned} \tag{22}$$

Thus, 6 can be rewritten as

$$\begin{aligned} c =& -K \exp^{-r\tau} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} \exp\left(-\frac{1}{2}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right) dz \\ &+ S \exp^{-r\tau} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \exp\left(-\frac{1}{2}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)^2\right) dz \end{aligned} \tag{23}$$

$$\begin{aligned} d_1 =& d_2 + \sigma\sqrt{\tau} \\ =& \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau} \\ =& \frac{\log\left(\frac{S}{K}\right) + r\tau - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau} \\ =& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} - \frac{\frac{\sigma\sqrt{\tau}\sigma\sqrt{\tau}}{2}}{\frac{\sigma\sqrt{\tau}}{1}} + \sigma\sqrt{\tau} \\ =& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau} + \frac{1}{1}\sigma\sqrt{\tau} \\ =& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} \\ =& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} + \frac{\frac{1}{2}\sigma\sqrt{\tau}\sigma\sqrt{\tau}}{\sigma\sqrt{\tau}} \\ =& \frac{\log\left(\frac{S}{K}\right) + r\tau}{\sigma\sqrt{\tau}} + \frac{\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \\ =& \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \end{aligned} \tag{24}$$