Modeling Vibrations of a Tennis Racket

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May 12, 2020

Abstract

This project models the vibrations that occur in a tennis racket upon impact by a tennis ball. First, the tennis racket is modeled as a uniform beam clamped at one end, and the natural vibration frequencies are determined using the dynamic fourth-order Euler-Lagrange equation. Next, the vibrations on the racket face are solved numerically using a finite element method for the two dimensional wave equation.

1 Introduction

Prior to the 1970s, tennis rackets had wooden frames and gut strings, and their shape and size were primarily determined by feel and tradition [1]. When aluminum and composite materials became common in manufacturing, racket design changed because these materials were lighter, allowing the rackets themselves to be larger. In the late 1970s, Howard Brody questioned whether the existing tennis racket designs were optimal [2]. In particular, Brody sought to characterize the "sweet spot" of the tennis racket [3].

Brody noted that the "sweet spot" of a tennis racket, colloquially meaning the place on the racket where it "feels best" to hit the ball, was somewhat ambiguous. Instead, Brody defined three different points on the tennis racket that each were optimal in a different way. First the "power region" of the racket is where the coefficient of restitution is greater than a particular threshold, and the location of the maximum is called the "power point". The coefficient of restitution (COR) is defined as the square root of ratio of the bounce height to the initial height of a ball dropped at some point on the racket. The area of the highest coefficient of restitution is where the most energy is returned to the ball. Any energy that goes into deformation of the racket frame is lost because the ball is no longer in contact with the strings by the time the racket comes back to its initial position. For the racket face/strings alone, the strings deform and then shift the energy back to the ball when they return to their regular position. In addition, if the ball deforms less and the strings deform more, then more energy is given back to the ball when in leves. Thus, for the racket face alone, the highest COR occurs in the center of the racket where the strings are "softest". However, because the racket is deformed more when the ball hits farther away from the handle, the highest COR for the whole strings-frame system is along the main axis, but closer to the handle than the center of the racket face is.

For the second potential "sweet spot," the center of percussion is the place on the racket that, when the racket is held during a swing, there is no net force on the hand upon impact by the ball. The net zero force is due to the cancellation of translational velocity and rotational velocity at the point where the racket is held. The center of percussion is well-defined by Brody in his first article on the subject [2]. The third point of interest is the node or nodal region, which is defined as the point where the resulting higher frequency vibrations of the racket are minimized. When the ball strikes the node of the fundamental frequency of the racket, the amplitude of the higher frequencies is minimized. As one moves away from the node, the amplitude of the higher frequencies increases, and there is a phase difference between the frequencies. This puts more energy into the racket and less into the rebound of the ball. At the node, the maximum energy is returned to the ball in the rebound. Brody later experimentally measured the frequencies and corresponding nodes of a standard tennis racket [4]. In his analysis of these three points, Brody also analyzed and experimentally measured the moments of inertia of a tennis racket [5]. Finally, Brody noted that "if the idela tennis racket could be designed, it would have all three of these points located in the center of the stringed area and have a power region and nodal region covering most of the face of the racket."

In his node analysis, Brody did not use any differential equations but rather measured the oscillations, nodes, and frequencies experimentally. This approach is reasonable because a tennis racket is a complicated structure rather than a uniform beam. Despite this, in the first part of this project, we will model a tennis racket as a uniform beam to find the fundamental vibration frequencies. Then in the second part, we will model the stringed face of the racket using the finite element method to evaluate the two dimensional wave equation on an egg shape.

2 The Beam Equation

As a first approximation to what happens to a tennis racket upon impact by a tennis ball, we will model the tennis racket as a beam with uniform density, uniform Young's modulus, and uniform shape (and thus uniform second moment of area) along the beam. The equation describing the behavior of a beam under no tension is

$$\mu \frac{\partial^2 y}{\partial t^2} = EI \frac{\partial^4 y}{\partial x^4} \tag{1}$$

Where μ is the mass per unit length, E is the Young's modulus, I is the second moment of area, y is the transverse displacement, and x is the length along the beam, clamped at $x = 0$ and free at $x = L$. Let the initial position be $f(x)$ and the initial velocity be $g(x)$. Then the initial and boundary conditions are

$$y(0) = 0 \quad y'(0) = 0 \quad y''(L) = 0 \quad y'''(L) = 0$$
$$y(t = 0, x) = f(x) \quad \frac{\partial}{\partial t} y(t = 0, x) = g(x)$$

Let $\gamma = \frac{EI}{\mu}$. Then $\frac{\partial^2 y}{\partial t^2} = -\gamma \frac{\partial^4 y}{\partial x^4}$. We assume a separable solution

$$y(t, x) = w(t)v(x) \tag{2}$$

which, plugging into equation 1 yields

$$\ddot{w}(t)v(x) = -\gamma v''''(x)w(t). \tag{3}$$

We can write

$$-\frac{1}{\gamma}\frac{\ddot{w}(t)}{w(t)} = \frac{v^{(4)}(x)}{v(x)} = \lambda \tag{4}$$

where each must equal a constant λ since each is a function of only one variable.

$$\begin{aligned}
v^{(4)}(x) &= v(x) \\
v(x) = e^{wx} &\implies w^4 e^{wx} = \lambda e^{wx} \\
w^4 = \lambda &\implies w = \pm|\lambda|^{1/4}, \pm|\lambda|^{1/4} i
\end{aligned}$$

Let $\rho = |\lambda|^{1/4}$. Then

$$v(x) = c_1 \cosh(\rho x) + c_2 \sinh(\rho x) + c_3 \cos(\rho x) + c_4 \sin(\rho x). \tag{5}$$

Using the boundary conditions:

$$\begin{aligned}
v(0) = 0 &\implies & c_1 &= -c_3 \\
v'(0) = 0 &\implies & c_2 &= -c_4 \\
v''(L) = 0 &\implies & c_3 &= c_4 \left[\frac{\sin(\rho L) + \sinh(\rho L)}{-\cos(\rho L) - \cosh(\rho L)} \right]
\end{aligned}$$

Let $A = \left[\frac{-\cos(\rho L) - \cosh(\rho L)}{\sin(\rho L) + \sinh(\rho L)} \right]$. Then $c_4 = Ac_3 = -Ac_1$, and $c_2 = -c_4 = -Ac_3 = Ac_1$.

$$v'''(L) = 0 \implies c_1 \left[\sinh(\rho L) + A\cosh(\rho L) - \sin(\rho L) + A\cos(\rho L) \right] = 0$$

If $c_1 = 0$, the solution is trivial, so instead we solve

$$\begin{aligned}
\sinh(\rho L) + 1\cosh(\rho L) - \sin(\rho L) + a\cos(\rho L) &= 0 \\
(\sinh(\rho L) - \sin(\rho L))(\sin(\rho L) + \sinh(\rho L)) - (\cos(\rho L) + \cosh(\rho L))^2 &= 0 \\
-\sin^2(\rho L) + \sinh^2(\rho L) - \cos^2(\rho L) - 2\cos(\rho L)\cosh(\rho L) - \cosh^2(\rho L) &= 0 \\
\cos(\rho L)\cosh(\rho L) &= -1
\end{aligned}$$

This last line can be solved numerically, yielding the first five solutions of

$$\rho = \frac{1}{L}[1.8751, \quad 4.6941, \quad 7.8548, \quad 10.9955, \quad 14.1372].$$

. Let ρ_n denote the n^{th} solution. Now that we know ρ_n, we also know $\lambda_n = \rho_n^4$. The modes of the beam are shown in Figure 1.

We have

$$v_n(x) = \cosh(\rho_n x) + A_n \sinh(\rho_n x) - \cos(\rho_n x) - A_n \sin(\rho_n x) \tag{6}$$

Note that $\gamma > 0$ and $\lambda_n > 0$ for all n. Going back to equation 4, we can solve for $w(t)$:

$$\begin{aligned}
\ddot{(w)}_n(t) &= -\gamma\lambda w_n(t) \\
w_n(t) &= a_n \cos(\sqrt{\gamma\lambda_n} t) + b_n \sin(\sqrt{\gamma\lambda_n} t)
\end{aligned}$$

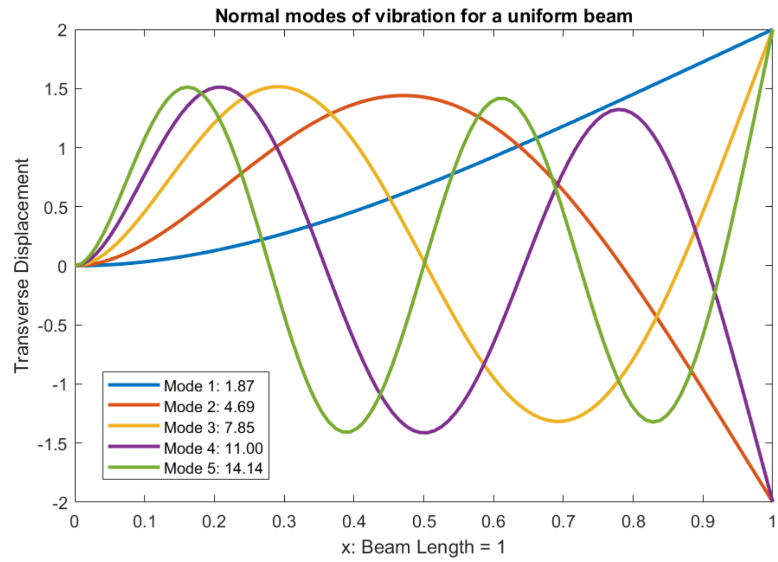


Figure 1: The first five modes of vibration of a cantilevered beam clamped at $x = 0$.

Putting together the solution for $w_n(t)$ and equation 6:

$$y(t,x) = \sum_{n=1}^{\infty} v_n(x)\left(a_n \cos(\sqrt{\gamma\lambda_n}t) + b_n \sin(\sqrt{\gamma\lambda_n}t)\right)$$

The coefficients a_n and b_n can be solved using Fourier series. Let $q = \frac{\pi}{\sqrt{\gamma\lambda_n}}$.

$$a_n = \frac{2}{q}\int_0^q f(x)v_n(x)dx$$
$$b_n = \frac{2}{q\sqrt{\gamma\lambda_n}}\int_0^q g(x)v_n(x)$$

Now we have all the tools for a solution. Images of a solution that is meant to move forward in time are not very interesting, so instead HERE is a video of the solution evaluated for $\gamma = 3$, an initial condition of $f(x) = 0.25x^3$ and an initial velocity of $g(x) = -x$. The values of a_n and b_n for this particular case are shown in Figure 2.

As stated earlier, this does not model a tennis racket very accurately since a tennis racket is certainly not a uniform beam. However, the model agrees with Brody's assessment that the fundamental frequency is dominant, and that there is a node on the face of the racket roughly $\frac{L}{5}$ from the tip that belongs to the next higher mode of oscillation. In particular, this value was found to be $0.22L$ from the tip of the racket in this model. Other than being helping in finding the location of potential "sweet spots," the vibrations of the tennis racket frame do not affect the ball as directly as the vibration on the strings because the first few modes of the racket frame do not have high enough frequencies to transmit energy back to the ball before it leaves contact. Thus, we next turn to analyze the racket face and strings.

The 2D Wave Equation

The face of the racket can be modeled by the two-dimensional wave equation on an egg shape. To begin, we will solve the wave equation using the finite element method on a rectangle, and then we will adapt the domain to be an egg shape instead. The two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \tag{7}$$

where $u(t, \vec{x})$ is the vertical displacement of any given point on the surface from the plane. Let the domain be Ω. The initial conditions and boundary conditions are

$$\begin{aligned} &u(x, y, 0) = u_0(x, y) \quad \partial_t u(x, y, 0) = u_1(x, y) \\ &u(x, y, t) = 0 \text{ for } (x, y) \in \delta\Omega \end{aligned}$$

To get the weak form of the wave equation for the finite element method, we use the Galerkin method. We multiply both sides of equation 7 by a test function $v(x, y)$ and integrate:

$$\int\int_\Omega \frac{\partial^2 u}{\partial t^2} v(x, y) = \int\int_\Omega c^2 \Delta u v(x, y)$$

Using Green's theorem:

$$\int\int_\Omega \frac{\partial^2 u}{\partial t^2} v(x, y) = c^2 \left[\int_{\partial\Omega} v \nabla u \cdot \vec{n} ds - \int\int_\Omega \nabla v \cdot (p \nabla u) \right]$$

Let the test function $v(x, y) = 0$ on $\delta\Omega$, the boundary of the region. Then

$$\int\int_\Omega \frac{\partial^2 u}{\partial t^2} v(x, y) = -c^2 \int\int_\Omega \nabla v \cdot \nabla u$$

$$A(u, v) := \int\int \frac{\partial^2 u}{\partial t^2} v(x, y) dA + c^2 \int\int_\Omega \nabla u \cdot \nabla v dA = 0$$

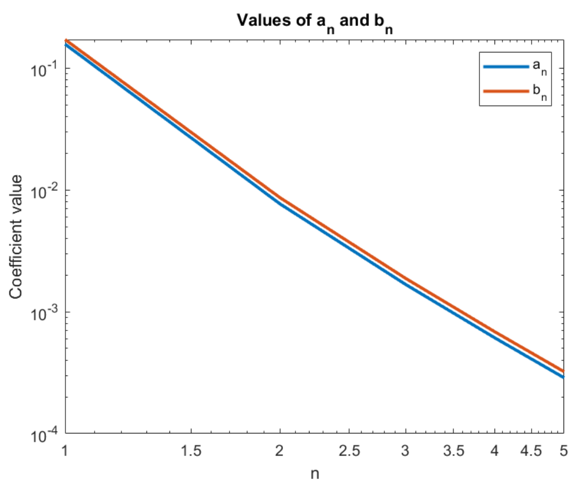


Figure 2: The coefficients of the time-dependent part of the solution to the Euler-Bernoulli dynamic beam equation in order of frequency. Clearly, the lower frequency solutions are much more dominant than the higher frequency solutions.

Any valid solution must satisfy $A(u, v) = 0$ on Ω. We approximate u and v by a trial and test function:

$$U = \sum_{i=1}^{n} U_i(t)\phi_i(x, y) \quad V = \sum_{i=1}^{n} V_i\phi_i(x, y) \tag{8}$$

where $\phi_i(x, y)$ are general basis functions and n is the number of nodes we will use in our approximation. Using these definitions of U and V, we need to satisfy $A(u, v) = 0$:

$$\begin{aligned} A(U(x, y, t), V(x, y)) &= A(U(x, y, t), \sum_{i=1}^{n} V_i\phi_i(x, y,)) \\ &= \sum_{i=1}^{n} V_i A(U(x, y, t), \phi_i(x, y)) \end{aligned}$$

because A is linear in the coefficients V_i. Then since each $A(U(x, y, t), \phi_j(x, y)) = 0$ for each ϕ_j.

$$\begin{aligned} A(U(x, y, t), \phi_i(x, y)) &= \int\int_\Omega \sum_{i=1}^{n} \ddot{U}_i(t)\phi_i(x, y)\phi_j(x, y) + c^2 \int\int_\Omega \sum_{i=1}^{n} U_i(t)\nabla\phi_i(x, y)\nabla\phi_j(x, y) \\ &= \sum_{i=1}^{n} \ddot{U}_i(t) \int\int_\Omega \phi_i(x, y)\phi_j(x, y) + c^2 \sum_{i=1}^{n} U_i(t) \int\int_\Omega \phi_i(x, y)\phi_j(x, y) \end{aligned}$$

Define

$$\begin{aligned} T_{ij} &= \int\int_\Omega \phi_i(x, y)\phi_j(x, y) \\ S_{ij} &= \int\int_\Omega \nabla\phi_i(x, y)\nabla\phi_j(x, y) \end{aligned}$$

We can then write

$$\sum_{i=1}^{n} \ddot{U}_i(t)T_{i,j} + c^2 \sum_{i=1}^{n} U_i(t)S_{i,j} = 0$$

This gives us a set of differential equations. Denote by $S_{[n,n]}$ and $V_{[n,n]}$ the matrices of values $S_{i,j}$ and $V_{i,j}$ for $i, j \in \{1, \ldots, n\}$. Also let

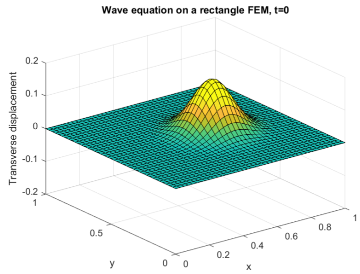
$$\vec{U} = \begin{pmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_n(t) \end{pmatrix}.$$

Then

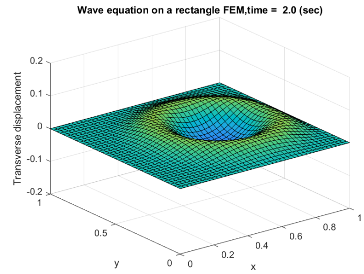
$$T_{[n,n]}\ddot{\vec{U}}(t) = -c^2 S_{[n,n]}\vec{U}(t)$$

We use a finite difference scheme to numerical solve for $U(t)$.

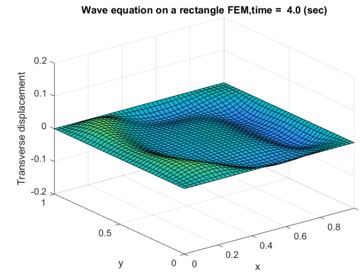
$$\begin{aligned} \ddot{\vec{U}}(t) &\approx \frac{\vec{U}(t + dt) - 2\vec{U}_i(t) + \vec{U}_i(t - dt)}{((dt)^2} \\ \implies \vec{U}_i(t + dt) &= -T^{-1}Sc^2\vec{U}(t)(dt)^2 + 2\vec{U}(t) - \vec{U}(t - dt) \end{aligned}$$



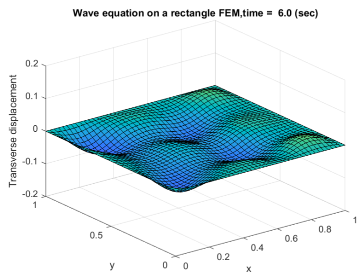
(a) t=0



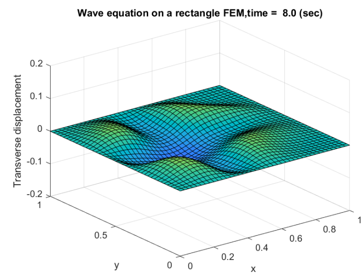
(b) t=2



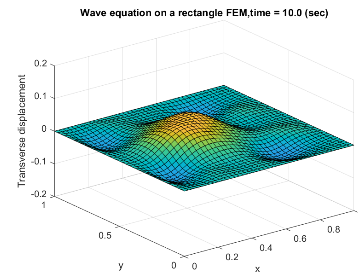
(c) t=4



(d) t=6



(e) t=8



(f) t=10

Figure 3: The numerical solution of the wave equation on a square domain with a Gaussian initial condition centered at $[0.6, 0.4]$ with a covariance of $0.01I$ and zero initial velocity. For a video, see HERE.

Now we have an expression to compute the value of U_i associated with each node for all times, but we need to compute T and S. To do this, we have to define the basis functions and discretize the domain Ω. The method that follows draws heavily from the method used in Ref [6]. Let the domain be discretized into elements that are all triangles. The reference triangle has coordinates $x_{l,1} = (0,0), x_{l,2} = (0,1), x_{l,3} = (0,1)$. We want the local basis functions for the node on each element to be 1 at their respective node and 0 on the other nodes. Let

$$\begin{aligned}\phi_{l,1}(x,y) &= 1 - x - y\\ \phi_{l,2}(x,y) &= x\\ \phi_{l,3}(x,y) &= y\end{aligned}$$

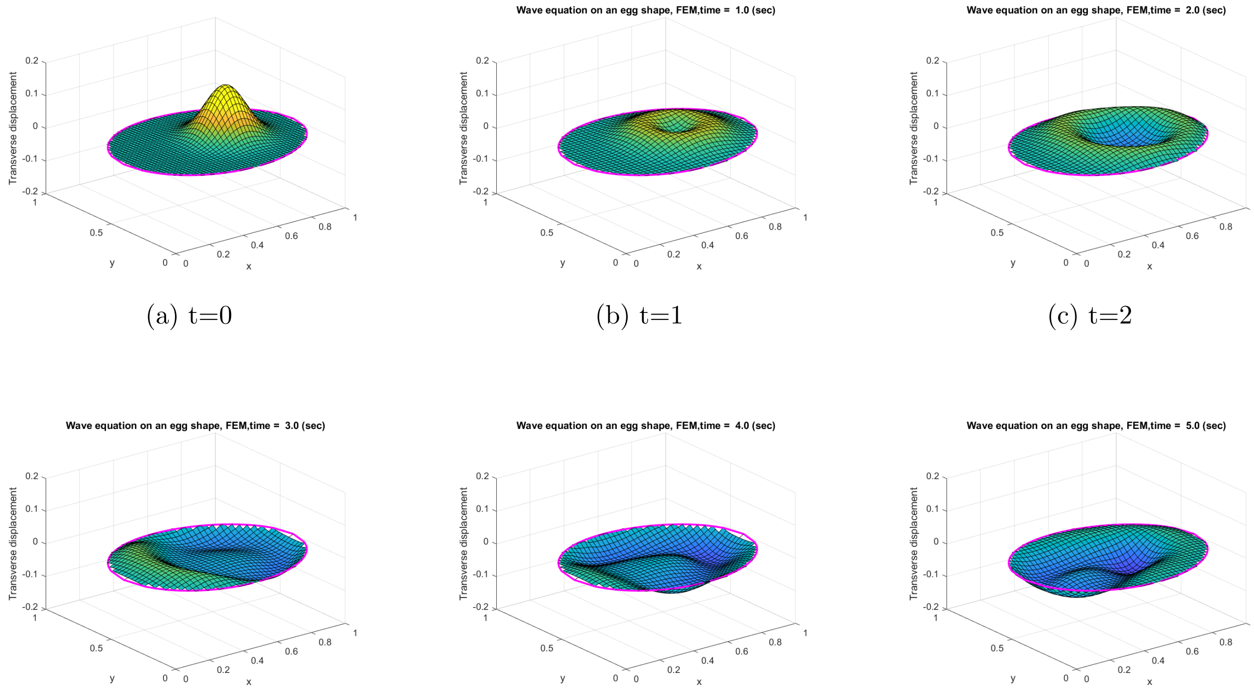
for the reference triangle. For any other triangle, element j, let $\tau_j(x,y)$ be the mapping from the reference triangle to element j. In particular,

$$\tau_j(X,Y) = (1 - X - Y)x_{j,1} + Xx_{j,2} + Yx_{j,3} \tag{9}$$

where $x_{j,i}$ is the point on element j that is the i^{th} point of the triangle with respect to the reference triangle (preserving orientation in the mapping). Now we can define the global basis functions

$$\phi_i(x,y) = \phi_{l,k}(\tau_j^{-1}(x,y)). \tag{10}$$

In other words, the global basis function ϕ_i associated with a node i evaluated at a point (x,y) is 0 if the point does not share an element with node i, and if they do share an element, then it is



(a) t=0 (b) t=1 (c) t=2

(d) t=3 (e) t=4 (f) t=5

Figure 4: The numerical solution of the wave equation on a "egg-shaped" domain described by $\frac{x^2}{a^2} + \frac{y^2}{b^2}(1 + kx) = 1$ where $k = 0.2$ with a Gaussian initial condition centered at $[0.6, 0.4]$ with a covariance of $0.01I$ and zero initial velocity. For a video, see HERE.

equal to the local basis function that node i would have on the reference triangle evaluated at the respective point (x, y) on the reference triangle. When considering only other nodes as points (x, y), it is 0 at every node other than i. Now, instead of computing $T_{i,j}$ and $S_{i,j}$ with integration over the whole domain, we can instead write a sum of integrations over the elements where the integration for a given element is a scaling of an integration over the reference element:

$$T_{ij} = J \int_{E_j} \phi_{l,i}(x, y)\phi_{l,j}(x, y) dA$$

$$S_{ij} = J \int_{E_j} \nabla\phi_{l,i}(x, y)\nabla\phi_{l,j}(x, y) dA$$

where $J = \det(\tau_j(x, y)) = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$ where (x_i, y_i) are the x and y coordinates of point number i on triangle J. The remaining integrals in the expressions for T_{ij} and S_{ij} are always over the reference triangle and its local basis functions, so there are only 6 possibilities for each integral that can be precomputed.

The final step is to enforce the boundary and initial conditions. For the initial conditions, set $U(x, y, 0) = u_0(x, y)$ and $U(x, y, dt) = U(x, y, 0) + (dt)u_1(x, y)$. For the boundary conditions, at each time step set $U(x, y, t) = 0$ if it is on $\partial\Omega$.

The solution for a square domain with a Gaussian initial condition centered at $[0.6, 0.4]$ and covariance of $0.01I$ and zero initial velocity is shown in Figure 3. For a video see HERE.

Obviously, however, tennis rackets are not rectangular, so we have to modify the domain to resemble a tennis racket. We choose an “egg shape,” described by the expression

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}(1 + kx) = 1 \tag{11}$$

where we choose k to be 0.2. The results are shown in Figure 4, and a video can be viewed HERE.

Tennis Racket Upon Impact by a Ball

While the simulations of the solutions over time are nice, in the case of a tennis racket, we only care about the first 5 milliseconds, which is the amount of time the ball spends in contact with the racket. To see how the different racket vibrations affect the ball, the initial conditions need to be adjusted. For the beam equation, the initial conditions would have to be adjusted such that the initial position of the racket was 0 but with an impulse load [7]. For the wave equation, the initial conditions/boundary counditions would incorporate a flat ball-sized disk that deforms the impact point to some distance dependent on momentum over the first 2.5 milliseconds. Ideally, the racket could be designed such that the return time of the fundamental mode of the racket frame and the return of the impact point to 0 displacement take the same amount of time, and are equal to the amount of time the ball spends in contact with the racket.

References

[1] Zhang Hailong Yu, Yong and Shupeng Jing. Features and applications of tennis racket. *Advanced Materials Research*, 2012.

[2] Howard Brody. Physics of the tennis racket. *American Journal of Physics*, (482), 1979.

[3] Howard Brody. Physics of the tennis racket ii: The sweet spot. *American Journal of Physics*, (816), 1981.

[4] Howard Brody. Models of tennis racket impacts. *International Journal of Sports Biomechanics*, pages 293–296, 1987.

[5] Howard Brody. The moment of inertia of a tennis racket. *The Physics Teacher*, (213), 1981.

[6] Finite elements in 2d and 3d. `http://hplgit.github.io/INF5620/doc/pub/sphinx-fem/._main_fem009.html`.

[7] Youn-sik Park Hyungsoon Park. Transient response of an impacted beam and indirect impact force identification using strain measurements. *Shock and Vibration*, (3), 1994.