Modeling Vibrations of a Tennis Racket- Discretizations

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1 Discretizations

The equations I seek to solve numerically are:

$$m\frac{\partial^2 y_n}{\partial t^2} = F - (EIs)\frac{\partial^4 y_n}{\partial x^4}$$
$$m_b\frac{d^2 y_b}{dt^2} = -F = -k_b(y_b - y_n)$$

where m is the mass of a segment of length s on the beam of length L, m_b is the mass of the ball, and k_b is the spring constant for the ball. y_b is the ball's position, and y_n is the position of the segment contacted by the ball. Nondimensionalizing y and x by the length of the racket L, the equation becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{F}{mL} - \frac{EI\delta}{L^3 m}\frac{\partial^4 y}{\partial x^2}$$

Here, y is the vertical displacement as a fraction of racket length, x is the fraction of distance along the racket length, and δ is the fractional segment length.

The beam is discretized into n sections, so we evaluated the equation at $n+1$ points. However, to enforce boundadry constraints on the free end of the beam, we create two ghost points at indices $n+2$ and $n+3$. We also assume ghost points of value zero at indices 0 and -1 to enforce the boundary condition at the clamped end. For a clamped end of the beam and a free end of the beam, the discretizations being evaluated numerically for the beam are:

$$y_i'' = \frac{F_i}{mL} - \frac{EIs}{mL^3 s^4}\left[y_{n+2} - 4y_{n+1} + 6y_n - 4y_{n-1} + y_{n+2}\right] \text{ for } i = 3, \ldots, n+1 \tag{1}$$
$$y_1' = 0 \tag{2}$$
$$y_{n+2}' = 2y_{n+1}' - y_n' \tag{3}$$
$$y_{n+3}' = 4y_{n+1}' - 4y_n' + y_{n+1}' \tag{4}$$

NOTE: The mistake I made for a week was confusing the boundary conditions in space with conditions in time.

The third line come from the condition that $y_{n+2} - y_{n+1} = y_{n+1} - y_n$, meaning that $\frac{\partial^2 y_{n+1}}{\partial x^2} = 0$ which yields $y_{n+2} = 2y_{n+1} - y_n$. The fourth line comes from the condition that $\frac{\partial^3 y_{n+1}}{\partial x^3} = 0$. Discretizing:

$$\begin{aligned} y'''_{n+1} &\approx \frac{y'_n - 2y'_{n+1} + y'_{n+2}}{s^2} \\ &\approx \frac{(y_{n+1} - y_{n-1}) - 2(y_{n+1} - y_n)) + (y_{n+3} - y_{n+1})}{2s^2} = \frac{-y_{n-1} - 2y_{n+2} + 2y_n + y_{n+3}}{2s^3} \end{aligned}$$

Combining with $y_{n+2} = 2y_{n+1} - y_n$ and equating to 0, line 4 follows.

For the ball,

$$y''_b = \frac{k_b}{m_b}[r_b - (y_b - y_{ni})]$$

where $r_b = \frac{r}{L}$, with r being the ball's radius and L is the racket length. y_b is the ball's position, and y_{ni} is the position of the point of impact on the racket. k_b is the ball's spring constant, and m_b is the ball's mass. Accordingly, the force is also applied to the point of impact:

$$y''_{ni} + = -\frac{k_b}{mL}[r_b - (y_b - y_{ni})]$$

After release, the ball's acceleration is 0. Release is triggered when the position of the impact point is not 0 AND the position of the ball's center of mass, y_b is greater than the vertical displacement from 0 equal to r_b. In other words, the ball is release from interaction with the racket when the center of mass first returns to the point it was when the edge of the ball impacted the racket.