Positional Astronomy

by Fiona Vincent

Fiona Vincent
Positional Astronomy
A Collection of Lectures on
Positional Astronomy for an
undergraduate course at the.
University of St. Andrews. Scotland in 1998

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By Fiona Vincent

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Some words in the original Fiona Vincent's text are not recognized by the dictionaries of the modern Word-processors as the word "centre" , which was widely used in old English writings, is nowadays being replaced by the "Americanism" term "center" , but it has been conserved as in the original text in aims to be respectful with Fiona Vincent's writing.

Index of Chapters

Positional Astronomy: Chapter 00. Introduction.

These pages were created by Fiona Vincent in 1998 to assist the students in the lectures on Positional Astronomy for second-year undergraduate at the University of St.Andrews, Scotland. Now are maintained as a help to the students of St.Andrews as well as a service to the world-wide community. This document was converted to . pdf format by Alfonso Pastor in 2005 as a tribute to Fiona Vincent in order to allow this extraordinary lectures to be loaded and read in computers and eBooks without the possibility of Internet connection.

You are welcome to link to Fiona Vincent´s pages from your own pages, but please do not reproduce any of this material without Fiona Vincent's acknowledgement.

Notes about the pages:

> (1) Many of the equations of the original pages in. html format use Greek letters as symbols.
> (If you try to link to the original pages, if your browser does not distinguish between "a,b" and "α, β" (the Greek letters "*alpha, beta*") then you will not be able to make much sense of the equations.)
> Here in the .pdf version this problem does not exist unless your reader is missing the Greek characters set.

> (2) These pages use "classical" definitions and terminology:
> that is, those used previous to the International Astronomical Union's 2000 resolutions at the site:
> http://syrte.obspm.fr/IAU_resolutions/Resol-UAI.htm

(For more information about time scales and Earth Rotation Models, see USNO Circular No.179) at : http://www.usno.navy.mil/USNO/astronomical-applications/publications/Circular_179.pdf

This course is intended to address the following problems:

1.- How to describe the position of an object in the sky.
2.- Which different coordinate systems are appropriate in different situations.
3.- How to transform between coordinate systems.
4.- What corrections have to be applied.

Objects in the sky appear to be positioned on the **celestial sphere**, an indefinite distance away.

A sphere is a three-dimensional object, but its surface is two-dimensional.

Spherical geometry is carried out on the surface of a sphere: it resembles ordinary (plane) geometry, but it involves new rules and relationships.

Positional Astronomy: Chapter 01

The terrestrial sphere

We will start with a familiar sphere: the Earth (assume for the moment that it *is* spherical), spinning around an axis.

The **North & South Poles** are where this axis meets the Earth's surface. The **equator** lies midway between them.

The equator is an example of a **great circle**: one whose plane passes through the centre of the sphere.

Every great circle has two *poles*. We can define these:

    (a) as the points which are 90° away from the circle, on the surface of the sphere.

    (b) as the points where the perpendicular to the plane of the great circle cuts the surface of the sphere.

These two definitions are equivalent.

The **length of a great-circle arc** on the surface of a sphere is the angle between its end-points, as seen at the centre of the sphere, and is expressed in degrees (not miles, kilometres etc.).

A great circle is a **geodesic** (the shortest distance between two points) on the surface of a sphere, analogous to a straight line on a plane surface.

To describe a location X on the surface of the Earth, we use **latitude** and **longitude** (two coordinates, because the surface is two-dimensional).
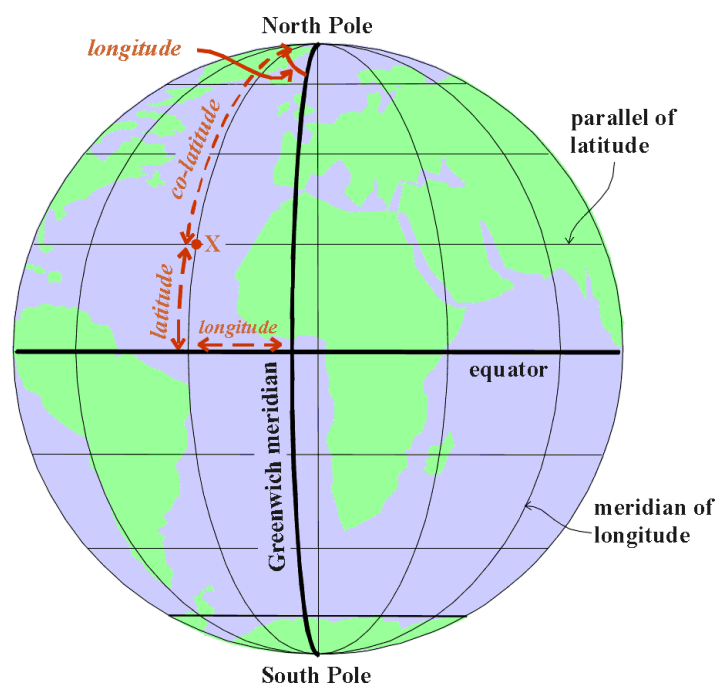
Draw a great circle from pole to pole, passing through location X: this is a **meridian of longitude**.

The **latitude** of X is the angular distance along this meridian from the equator to X, measured from -90° at the South Pole to +90° at the North Pole.

The **co-latitude** of X is the angular distance from the North Pole to X

co-latitude = 90° - latitude.

There is no obvious point of origin for measuring longitude; for historical reasons, the zero-point is the meridian which passes through Greenwich (also called the **Prime Meridian**).
The **longitude** of X is the angular distance along the equator from the Prime Meridian to the meridian through X.
It may be measured east or west 0° to 360°, or both ways 0° to 180°.

Small circles parallel to equator are **parallels of latitude**.
The circumference of a small circle at any given latitude is 360 * cos(latitude) in degrees.

The **length of arc of a small circle** between two meridians of longitude is
    (difference in longitude) * cos(latitude).
The **length of arc of a great-circle** distance is always shorter than the **length of arc of a small circle**, as we shall see in the next section.

The **length of arc of a great- circle** distance is called Orthodromic distance.

Note that a position on the surface of the Earth is fixed using *one fundamental circle* (the equator) and *one fixed point on it* (the intersection with a meridian referred to Greenwich Meridian).

Celestial navigation used at sea (and in the air) involves spherical trigonometry, so the results are in angular measure (degrees).
These must be converted to linear measure for practical use.
We define the **nautical mile** as 1 arc-minute along a great circle on Earth's surface.
This comes out about 15% greater than the normal "statute" mile (6080 feet instead of 5280 feet).

Note: terrestrial coordinates are actually more complicated than this, because the Earth is not really a sphere.
One source where you can find out more about this is the Ordnance Survey's "Guide to coordinate systems in Great Britain".

Exercise: 1

Alderney, in the Channel Islands, has longitude 2°W, latitude 50°N.
Winnipeg, in Canada, has longitude 97°W, latitude 50°N.
How far apart are they, in nautical miles, along a parallel of latitude?

Distance along a parallel of latitude is (difference in longitude) x cos(latitude)
= (97° - 2°) cos(50°) = 61.06°
But 1° = 60 nautical miles.
So the distance is 61.06 x 60 = 3663 nautical miles.
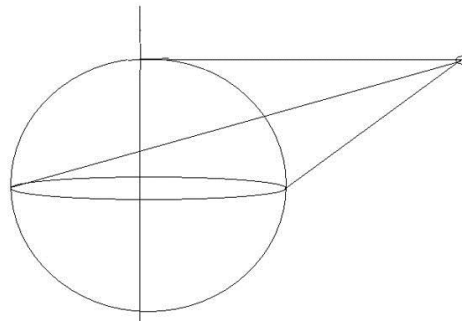
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Positional Astronomy: Chapter 02
Spherical trigonometry

Spherical trigonometry note :

The diagrams on this pages are only correct if the observer is imagined to be at a finite distance from the outside of the sphere, as in this diagram.

If the observer is imagined to be infinitely far away, the diagram should be drawn with the z-axis emerging from the sphere a little way "in front of" the edge, thus:
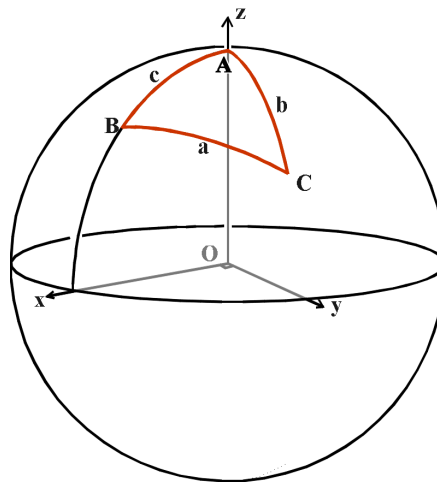
In my opinion, this version, although equally correct, is slightly more difficult to understand. So throughout the rest of these pages, I am using diagrams of the type shown in the next page.

My thanks to Dr Friedrich Firneis of the Austrian Academy of Sciences, to Ryu Izawa of the University of Colorado, USA, and to engineering consultant David Bosher, who provided the diagram above for helpful discussions on this point.

Spherical trigonometry

A great-circle arc, on the sphere, is the analogue of a straight line, on the plane.

Where two such arcs intersect, we can define the **spherical angle** *either* as angle between the tangents to the two arcs, at the point of intersection, *or* as the angle between the planes of the two great circles where they intersect at the centre of the sphere.

(Spherical angle is only defined where arcs of *great* circles meet.)

A **spherical triangle** is made up of three arcs of great circles, all less than 180°.

The sum of the angles is not fixed, but will always be greater than 180°.

If any side of the triangle is exactly 90°, the triangle is called **quadrantal**.

There are many formulae relating the sides and angles of a spherical triangle. In this course we use only two: the *sine rule* and the *cosine rule*.

Consider a triangle ABC on the surface of a sphere with radius = 1.
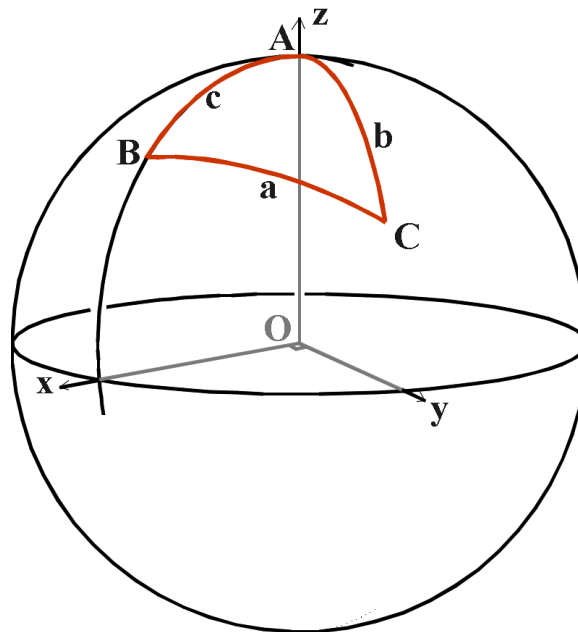
We use the capital letters A, B, C to denote the angles at these corners; we use the lower-case letters a, b, c to denote the *opposite* sides.
(Remember that, in spherical geometry, the side of a triangle is the arc of a great circle, so it is also an angle.)

Now turn the sphere so that A is at the "north pole", and let arc AB define the "prime meridian".

Set up a system of rectangular axes OXYZ:
O is at the centre of the sphere;
OZ passes through A;
OX passes through arc AB (or the extension of it);
OY is perpendicular to both.
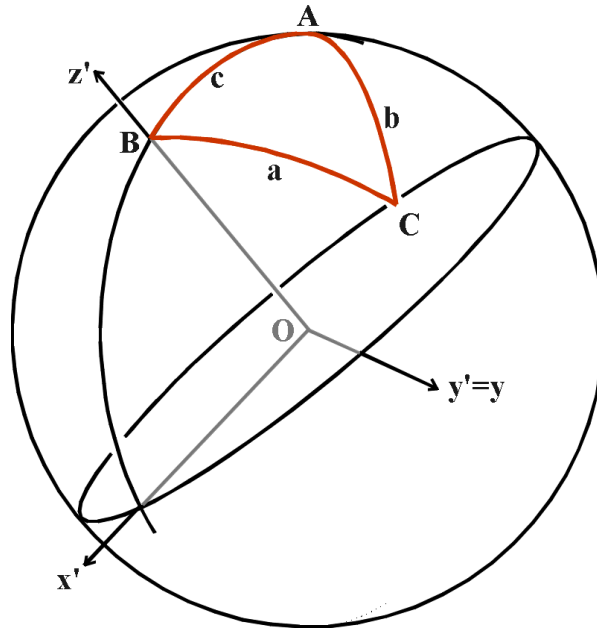
Find the coordinates of C in this system:
x = sin(b) cos(A)
y = sin(b) sin(A)
z = cos(b)

Now create a new set of axes, keeping the y-axis fixed and moving the "pole" from A to B (i.e. rotating the x,y-plane through angle c).
The new coordinates of C are :
x' = sin(a) cos(180-B) = - sin(a) cos(B)
y' = sin(a) sin(180-B) =   sin(a) sin(B)
z' = cos(a)
The relation between the old and new systems is simply a rotation of the x,z-axes through angle c:
x' = x cos(c) - z sin(c)
y' = y
z' = x sin(c) + z cos(c)

That is:

- sin(a) cos(B) = sin(b) cos(A) cos(c) - cos(b) sin(c)
  sin(a) sin(B) = sin(b) sin(A)
        cos(a) = sin(b) cos(A) sin(c) + cos(b) cos(c)

These three equations give us the formulae for solving spherical triangles. The first equation is the **transposed cosine rule**, which is sometimes useful but need not be memorised.

The second equation gives the **sine rule**. Rearrange as:
  sin(a)/sin(A) = sin(b)/sin(B)
Similarly, :   sin(b)/sin(B) = sin(c)/sin(C), etc.

So the sine rule is usually expressed as:
  sin(a)/sin(A) = sin(b)/sin(B) = sin(c)/sin(C)

The third equation gives the **cosine rule**:
  cos(a) = cos(b) cos(c) + sin(b) sin(c) cos(A)
and similarly:
  cos(b) = cos(c) cos(a) + sin(c) sin(a) cos(B)
  cos(c) = cos(a) cos(b) + sin(a) sin(b) cos(C)

Here they are together:

sine rule:
sin(a)/sin(A) = sin(b)/sin(B) = sin(c)/sin(C)

cosine rule:
cos(a) = cos(b) cos(c) + sin(b) sin(c) cos(A)
cos(b) = cos(c) cos(a) + sin(c) sin(a) cos(B)
cos(c) = cos(a) cos(b) + sin(a) sin(b) cos(C)

The cosine rule will solve almost any triangle if it is applied often enough. The sine rule is simpler to remember but not always applicable.
Note that both formulae can suffer from ambiguity:
E.g. if the sine rule yields
  sin(x) = 0.5, then x may be 30° *or* 150°.
Or, if the cosine rule yields
  cos(x) = 0.5, then x may be 60° *or* 300° (-60°).
In this case, there is no ambiguity if x is a *side* of the triangle, as it must be less than 180°, but there could still be uncertainty if an *angle* of the triangle was positive or negative.
So, when applying either formula, check to see if the answer is sensible.
If in doubt, recalculate using the other formula, as a check.
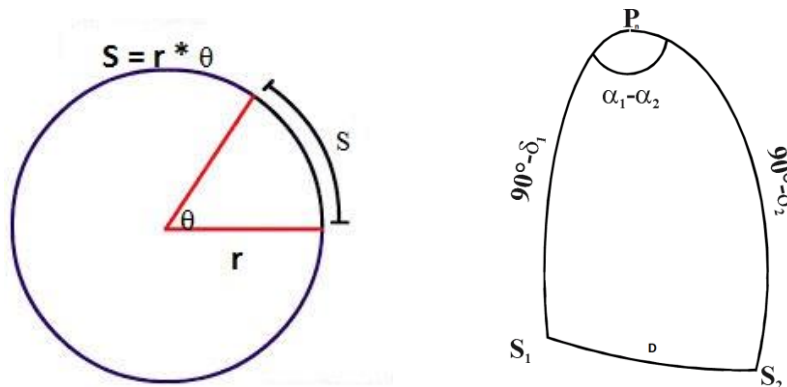
Arc Length = S = r * θ ; *Earth Diameter =* 7917.59 miles

Earth radius = $\frac{1}{2}$ * 7917.59 miles

Distance between two locations in the same meridian = S = r * θ , (θ in radians)

Distance between two locations in the same parallel :

$$D = (\text{longitude}_a - \text{longitude}_b) * \cos (\text{Latitude})$$

Conversion to radians: $\quad 1^{\circ} = \frac{2*\pi}{360} = \frac{\pi}{180}$ radians

$$\theta^{\circ} = \theta^{\circ} * \frac{\pi}{180} = \theta_{\text{rads}} , (\text{ in radians })$$

$$S = \theta^{\circ} * \frac{\pi}{180} = S, (\text{ in radians})$$

Conversion to degrees : $\quad 1rad = \frac{180}{\pi}$ Degrees

Conversion to Nautical Miles:

$S = {}^{\circ} * \left(\frac{\pi}{180}\right) * \left(\frac{7915.6}{2}\right) = S$ in (Nautical Miles) , $1^{\circ} \cong 60$ Nautical Miles

The Nautical Mile is defined as the Arc length subtended by an angle of 1' (one Minute) on a circle of Diameter = 7917.59 Miles ; One Nautical Mile = 1.8532487 Km.

What is the Great Circle distance (Orthodromic) between New York (Lat 40º 40'N , lon 73º 58' 30'' W) to Lisbon (Lat 38º35' , lon 9º 10' W) .

The answer D will be in terms of angle of arc.

To convert from degrees Nautical Miles use 1º= 60 Nautical Miles

$\quad$ cos (D) = sin (40º40') * sin (38º35') + cos (40º40') * cos (38º35') *

$\quad$ cos (73º78'30'' – 9º10') = 0.6587874

$\quad$ acos (0.6587874) = 48.79280046º

$\qquad$ 48.7954º * 60 = 2927.568028 Nautical Miles

Distance between two locations:.

$\quad$ $\text{Lat}_1 = \delta_1$ $\qquad\qquad$ $\text{Lat}_2 = \delta_2$

$\quad$ $\text{lon}_1 = \alpha_1$ $\qquad\qquad$ $\text{lon}_2 = \alpha_2$

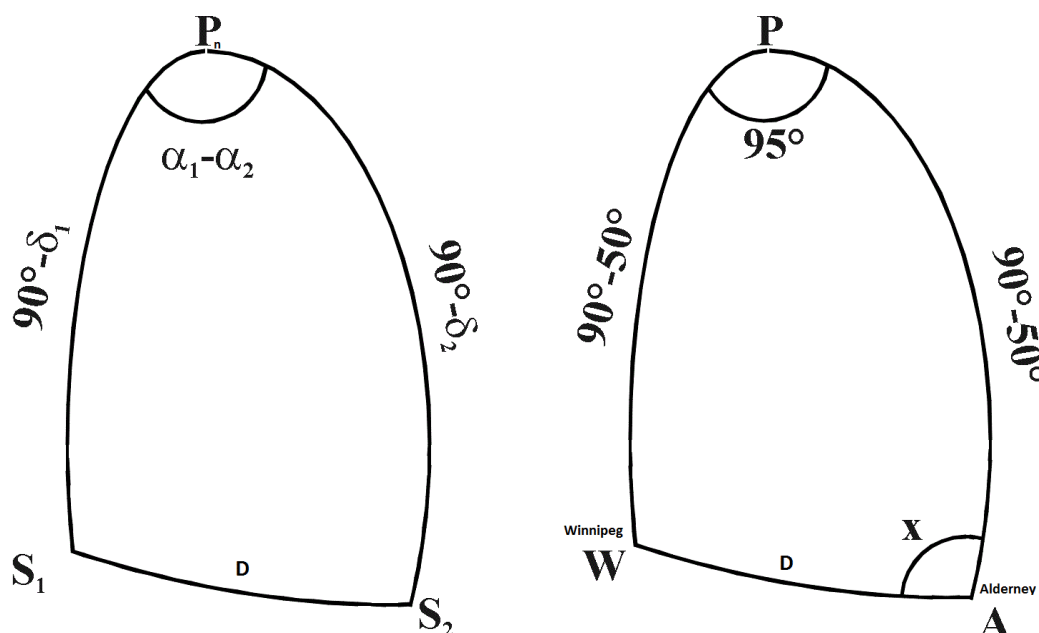$\quad$ $S_1 = \alpha_1 , \delta_1$ $\qquad\qquad$ $S_2 = \alpha_2 , \delta_2$

The standard formula for determining the distance between two locations is:

$$\cos (D) = (\sin (\delta_1) * \sin (\delta_2)) + ((\cos (\delta_1) \cos (\delta_2)) * \cos (\alpha_1 - \alpha_2))$$

The answer will be in terms of the angle of arc D.

To convert to Nautical Miles use the formula: Length of Arc 1º= 60 Nautical Miles

Exercise:02

Alderney, in the Channel Islands, has longitude 2°W, latitude 50°N.
Winnipeg, in Canada, has longitude 97°W, latitude 50°N.
How far apart are they, in nautical miles, along a parallel of latitude?
Distance along a parallel of latitude is : (See Exercise 1)
 D = (difference in longitude) * cos(latitude) = α_1 - α_2 * cos (δ)
In the example: D = (97° - 2°) cos(50°) = 61.06°
1° = 60 nautical miles. So the distance is 61.06 x 60 = 3663 nautical miles.
How far apart are they, in nautical miles, along a great circle arc?
Distance between two places not in the same parallel :
Use the cosine rule :
$$\cos (D) = (\sin (\delta_1) * \sin (\delta_2)) + ((\cos (\delta_1) * \cos (\delta_2)) * \cos (\alpha_1 - \alpha_2)$$
In the example
$\cos (D)$ = (sin (90-50) * sin (90-50)) + ((cos (90-50)*cos (90-50)) * cos D =
          = $\sin^2 (40°)$ + $\cos^2(40°)$ * cos 95° = 0.5508
          D = acos (0.5508) = 56.58°
To convert to Nautical Miles   56.58° * 60 = 3394.8 nautical miles
(This is 7% shorter than the 3663 route along a parallel of latitude).
If you set off from Alderney on a great-circle route to Winnipeg,
in what direction (towards what azimuth) would you head?
Use the sine rule:
sin A / sin (90-50) = sin D / sin (90-50)
so sin x = sin (40°) * sin (95°) / sin (56.58°) = 0.77
          so x = 50.1° or 129.9° .
Common sense says 50.1° (or check using cosine rule to get (90-50)).
Azimuth is measured clockwise from north,
          so azimuth is 360° - 50.1° = 309.9°
(Note that this is 40° north of the "obvious" 270° due-west course.)

Positional Astronomy: Chapter 03
Coordinate systems: the horizontal or "alt-az" system

The location of an object on the sky is fixed by celestial coordinates
analogous to the terrestrial latitude/longitude system.
There are various systems, suitable for different purposes; each system needs
a *fundamental circle* and a *fixed point on it*.

The simplest is the **horizontal system**, which uses the horizon as its
fundamental circle.
The poles of this circle are the **zenith** overhead and the **nadir** underfoot;
these are defined by the local vertical (using a plumb-line or similar).

Draw a **vertical circle** from the zenith to the nadir through object X.
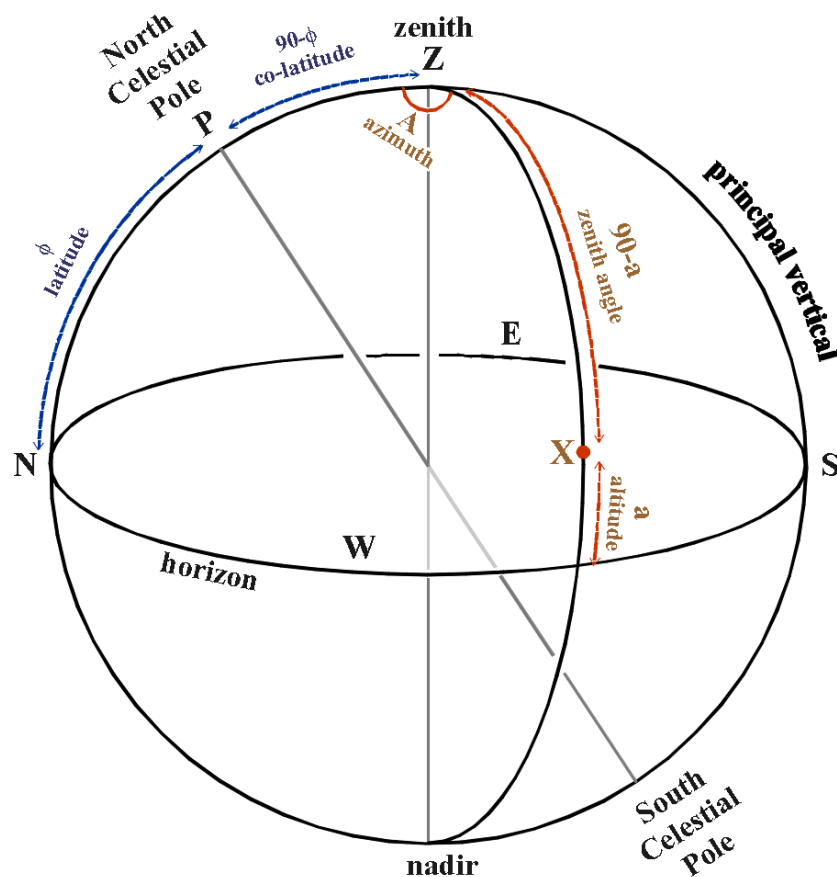
The **altitude (a)** of object X is the angular distance along the vertical circle
from the horizon to X,
measured from -90° at nadir to +90° at zenith.
Alternatively, the **zenith distance** of X is 90° - a.
(Some authors use h instead of a .)
Any two objects with the same altitude
lie on a small circle called a **parallel of altitude**.

To fix a point of origin on horizon, we look at where the spin axis of the Earth intersects the celestial sphere, at the **North** and **South Celestial Poles**. The vertical circle through these is called the **principal vertical**.
Where this intersects the horizon, it gives the north and south **cardinal points** (the north point is the one nearest the North Celestial Pole).
Midway between these are the east and west cardinal points; the vertical circle through these is called the **prime vertical** (not shown on the diagram), at 90° to the principal vertical.

The **azimuth (A)** of object X is the angular distance around the horizon from the north cardinal point to the vertical circle through X, measured 0°-360° westwards (clockwise).

Note that the **altitude of the North Celestial Pole** is equal to **the latitude of the observer**.

Comparison with the terrestrial system:

Terrestrial …………………..… **Alt-Az**

Equator …………………………….... Horizon

North Pole…………………………... Zenith

South Pole……………………….. Nadir

latitude……………………………. Altitude

Co-latitude……………………….Zenith distance

Parallel of latitude………………. Parallel of altitude

Meridian of longitude…………... Vertical circle

Greenwich Meridian……….…… Principal Vertical

Longitude……………………….. Azimuth

Exercise: 3

From St.Andrews, at 6 pm on 1998 February 2nd,
the Moon appeared at altitude +39°, azimuth 196°,
while Saturn is at altitude +34°, azimuth 210°.

How far apart did the two objects appear?
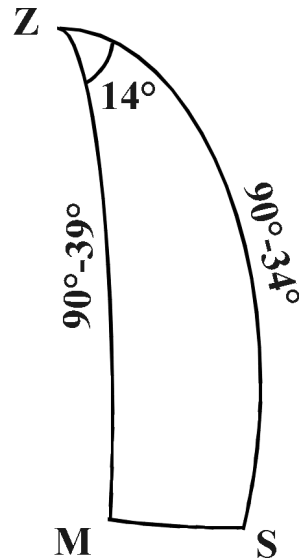
The difference in azimuth is 14°.

Use the cosine rule:
cos MS = cos MZ cos ZS + sin MZ sin ZS cos Z = 0.98
so MS = 12.3°

Which was further east?

The Moon is further east, and higher up, than Saturn.

Positional Astronomy: Chapter 04
## Coordinate systems: the first equatorial or "HA-dec." system

Any coordinates given in the horizontal or alt-az system
depend on the *place* of observation (because the sky appears different from
different points on Earth) and on the *time* of observation (because the Earth
rotates, and each star appears to trace out a circle centred on North Celestial
Pole).

We need a system of celestial coordinates which is fixed on the sky,
independent of the observer's time and place.
For this, we change the *fundamental circle* from the horizon to the *celestial
equator*.

The North Celestial Pole (NCP) and the South Celestial Pole (SCP) lie
directly above North and South Poles of Earth.
The NCP and SCP form the poles of a great circle on celestial sphere,
analogous to the equator on Earth.
It is called the **celestial equator** and it lies directly above the Earth's equator.

Any great circle between the NCP and the SCP is a **meridian**.
The one which also passes through the zenith and the nadir is "the" **celestial
meridian**, or the observer's meridian. (It is identical to the principal vertical.)
This provides our new zero-point; in this case, we use the point where it
crosses the *southern* half of the equator.

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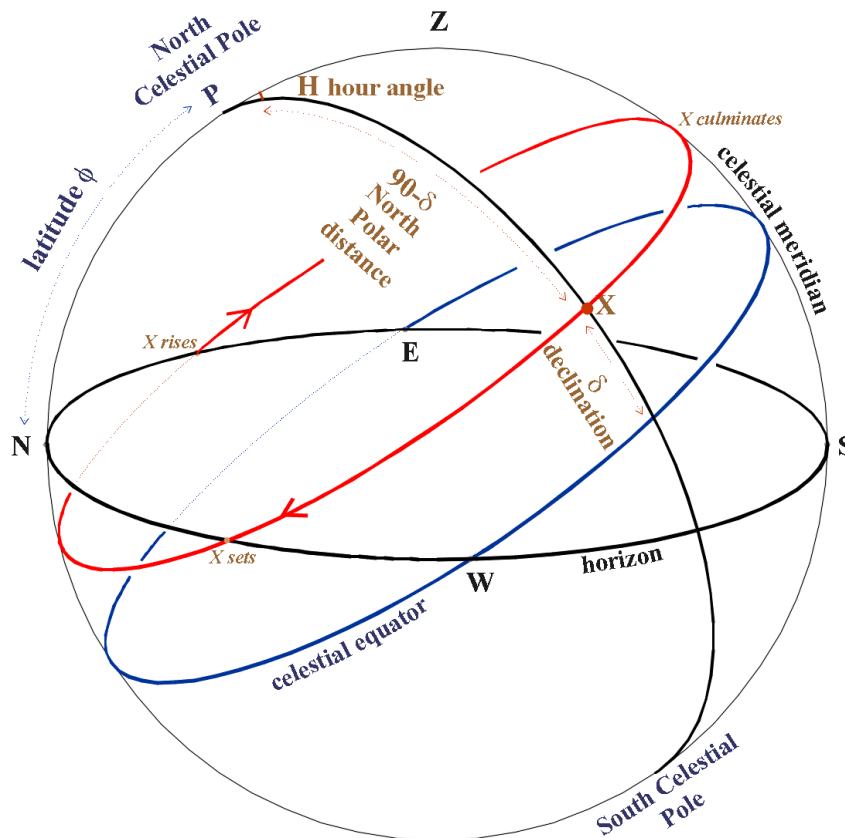
A typical star comes up over the horizon (rises) somewhere in the eastern sector; it moves round to the right, climbing higher in the sky; it reaches its highest point when it's due south, *i.e.* on the meridian; it continues moving right, and sinking lower; and it disappears below the horizon (sets) somewhere in the western sector.
(Note that this is only true in the northern hemisphere;in the southern hemisphere, the star will move to the left, and reach its highest point when it's due north.
In what follows, I assume we are in the northern hemisphere.)
The star's highest point, due south, is called **(upper) transit** or **culmination**.
The star will also cross the meridian again, in the opposite direction, at the lowest point in its daily path.
This is called **lower transit**, and it occurs below the horizon unless the star is **circumpolar**.

Stars close to North Celestial Pole never set; if a star's north polar distance is less than the altitude of the Pole, then that star cannot reach the horizon.
These are defined as **north circumpolar stars**.
Similarly, stars close to the South Celestial Pole will never rise: these are **south circumpolar stars**.
All others are **equatorial stars**, which rise and set.

The division between circumpolar and equatorial stars depends on the altitude of the North Celestial Pole, *i.e.* on the observer's latitude.

To fix the coordinates of an object X on the celestial sphere, draw the meridian through X.
The **declination** δ of X is the angular distance from the celestial equator to X, measured from -90° at the SCP to +90° at the NCP.
Any point on celestial equator has declination 0°.
Alternatively, the **North Polar Distance** of X = 90° - δ.
Any two objects with the same declination lie on a **parallel of declination**.

The **Hour Angle** or HA (H) of object X is the angular distance between the meridian of X and "the" celestial meridian.
It is measured westwards in *hours*, 0h-24h, since the Earth rotates 360° in 24 hours.

time interval	angle
1 hour	15°
1 minute	15'
1 second	15"

An object on the meridian (culminating) has H = 0h.
Its HA then steadily increases as the object moves westwards.
At lower transit, when it is due north (and possibly below the horizon),

 H = 12h.
At

 H = 23h, it is just one hour short of culminating again.

This system is still dependent on the time of observation, but an object's declination generally doesn't change rapidly, and its Hour Angle can be determined quite simply, given the time and the location.
A telescope can be built on an **equatorial mounting**, with its axis pointing at the NCP.
Once it is set on a star, if the telescope rotates about its polar axis at the correct speed (15° per hour), the star will stay in view.

Exercise:04

The most northerly star of the Southern Cross, γ Crucis, has declination -57°.
At what latitude will it just be visible?

The star is at S (just on the horizon), 57° from the equator.
So at this place, it must be 33° from the equator to the zenith.
So it must be 57° from the zenith to the north celestial pole.
So it must be 33° from the pole to the northern horizon.
But the altitude of the north celestial pole is the latitude of the place.
So the latitude is 33°N.
So any observer north of latitude 33°N is unable to see the Southern Cross.
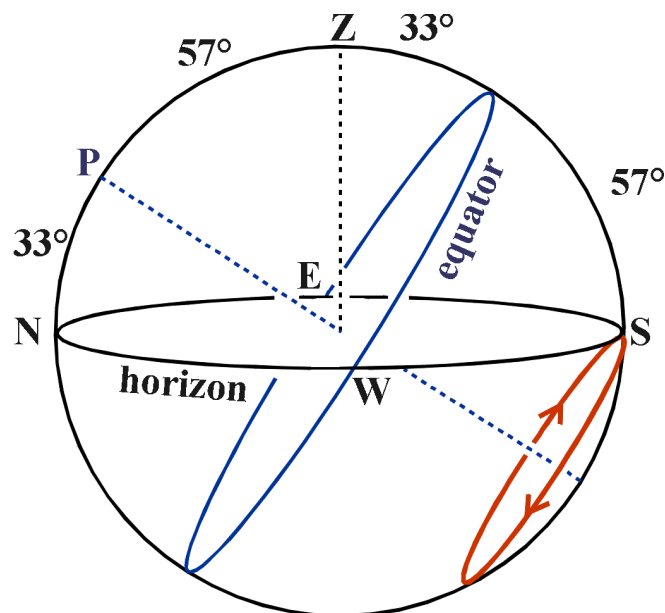
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At what latitude will it pass directly overhead?

The star is at Z, the zenith.
It is 57° from there to the equator,
so at this place it must be 33° from the equator to the horizon.
So P is 57° below the northern horizon.
So the latitude is 57°S.
*Note: as a general rule, if a star of declination x° passes overhead,
then the place has latitude x°.*
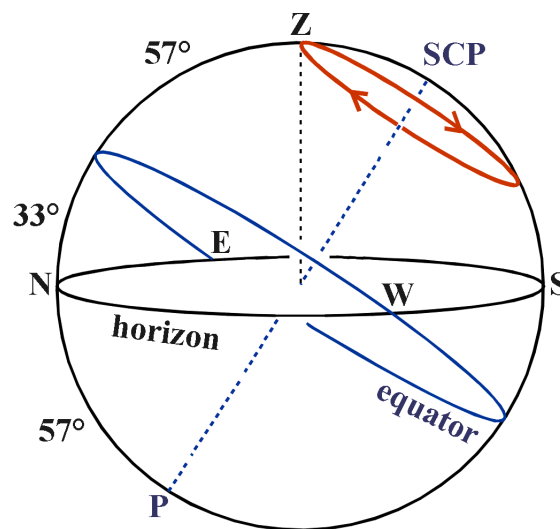
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At what latitudes will it never set?

Suppose the star is at S, just on the southern horizon.
It is 57° from S down to the equator
so it must be 33° from S up to the south celestial pole.
If the SCP is 33° above the southern horizon,
then the NCP must be 33° below the northern horizon.
So the latitude here is -33°, or 33°S.
The star will never set (it will be circumpolar)
for any observer south of 33°S.

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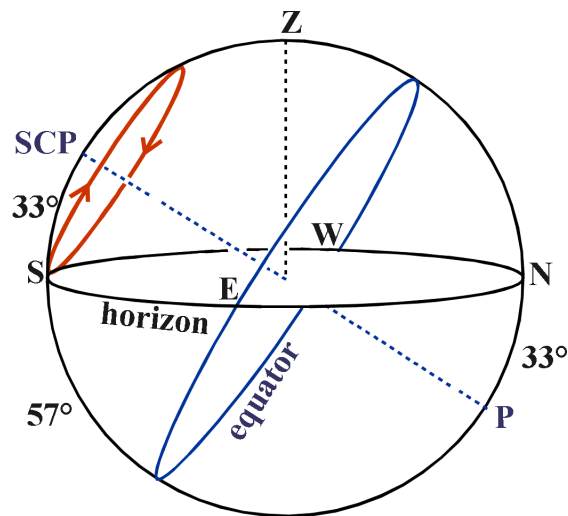
Positional Astronomy: Chapter 05
Coordinate systems: the second equatorial or "RA-dec." system
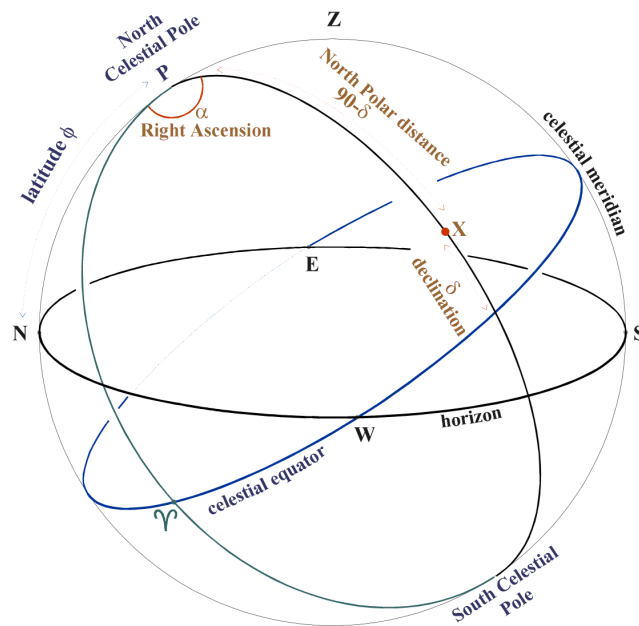
Coordinates in the first equatorial system (HA and declination) still depend on the *time* of observation.

Now we change the *zero-point* for our coordinates.

We choose a fixed point on the celestial equator, called the **vernal equinox**, or the **First Point of Aries**.

The symbol for this is the astrological symbol for Aries: ♈

(The function of this point will become clearer later on.)

The **declination** (δ) of object X is measured in the same way as before.

The **Right Ascension** or RA (α) of object X is the angle along the celestial equator measured eastwards from the vernal equinox to the meridian of X.

Like HA, RA is measured in hours 0-24h, but it goes in the opposite direction.

Comparison of these celestial coordinate systems with the terrestrial system:

terrestrial	alt-az	HA-dec.	RA-dec.
equator	horizon	celestial equator	celestial equator
North Pole	zenith	North Celestial Pole	North Celestial Pole
South Pole	nadir	South Celestial Pole	South Celestial Pole
latitude	altitude	declination	Declination
co-latitude	zenith distance	North Polar Distance	North Polar Distance
parallel of latitude	parallel of altitude	parallel of declination	parallel of declination
meridian of longitude	vertical circle	meridian	Meridian
Greenwich Meridian	Principal Vertical	celestial meridian	vernal equinox
longitude	azimuth	Hour Angle	Right Ascension

The Right Ascension and declination of a star do not normally change over short periods of time; but the Hour Angle changes constantly with time.

Consequently we have to find a way of defining the time.

Exercise:05

The four stars at the corners of the "Great Square of Pegasus" are:

star	R.A.	declination
α And	00h 08m	+29°05'
β Peg	23h 04m	+28° 05'
α Peg	23h 05m	+15° 12'
γ Peg	00h 13m	+15° 11'

Calculate the lengths of the two diagonals of the "Square".
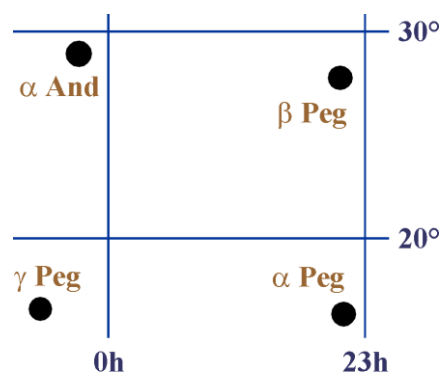
It is necessary to plot the four stars, at least approximately,
to find out which pairs form the diagonals!

Then, to find the length of each diagonal, use the cosine rule:
$$\cos S_1S_2 = \cos S_1P \cos S_2P + \sin S_1P \sin S_2P \cos P$$

This gives α And to α Peg = 20.1°
and β Peg to γ Peg = 20.5°.

Positional Astronomy: Chapter 06
Sidereal Time

Which stars are on your local meridian?
It depends on the time at which you observe.
In fact, it depends on both the *date* and the (clock) time, because the Earth is in orbit around the Sun.
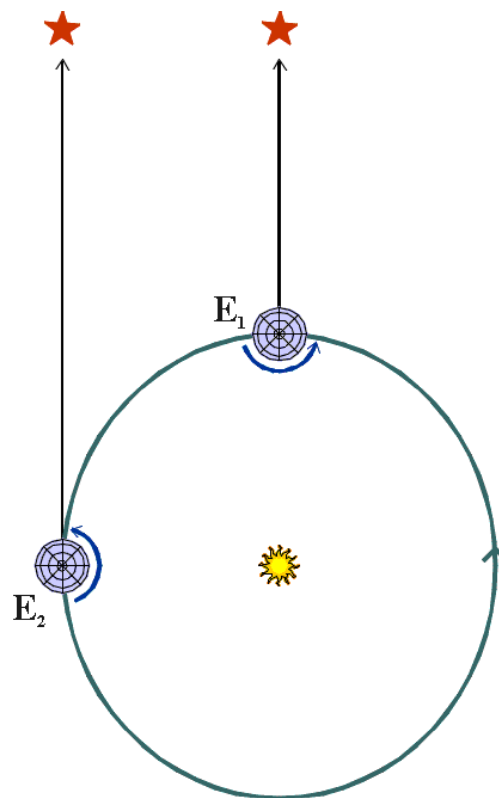
Consider the Earth at position E_1 on the diagram.
The star shown is on the meridian at midnight by the clock.
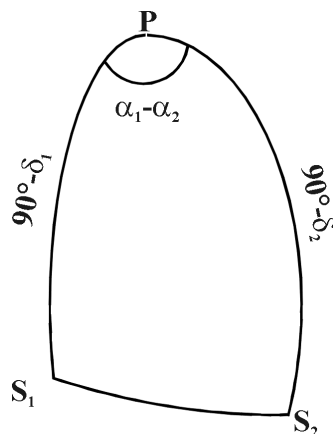But three months later, when the Earth reaches position E_2 , the same star is on the meridian at 6 p.m. by the clock.

Our clocks are set to run (approximately) on **solar time** (sun time).
But for astronomical observations, we need to use **sidereal time** (star time).

Consider the rotation of the Earth relative to the stars.
We define one rotation of Earth as one **sidereal day**, measured as the time between two successive meridian passages of the same star.

Because of the Earth's orbital motion, this is a little shorter than a solar day.
(In one year, the Earth rotates 365 times relative to the Sun,
but 366 times relative to the stars.
So the sidereal day is about 4 minutes shorter than the solar day.)

We define **Local Sidereal Time** (LST) to be 0 hours when the vernal equinox ♈ is on the observer's local meridian.

One hour later, the local Hour Angle (LHA) of the equinox is +1h (by the definition of Hour Angle), and the Local Sidereal Time is 1h.

So at any instant, **Local Sidereal Time = Local Hour Angle of the vernal equinox.**

Here's an alternative definition:

Suppose that LST = 1h.

This means that the vernal equinox has moved 15° (1h) west of the meridian, and now some other star X is on the meridian.

But the Right Ascension of star X is the angular distance from the vernal equinox to X = 1h = LST.

So at any instant, **Local Sidereal Time = Right Ascension of whichever stars are on the meridian**.

And in general, the Local Hour Angle of a star = Local Sidereal Time - RA of the star.

However, at any instant different observers, to the east or west, will have different stars on their local meridians.

We need to choose one particular meridian to act as a reference point; we choose Greenwich.

We define the **Greenwich Hour Angle** of X
as the Hour Angle of X relative to the celestial meridian *at Greenwich*.
Then we can define **Greenwich Sidereal Time** (GST)
as the Greenwich Hour Angle of the vernal equinox.
This gives the important relation
LST = GST - longitude west.

Recall that the Local Hour Angle (LHA) of a star = Local Sidereal Time - RA of the star.
In particular, the Greenwich Hour Angle (GHA) of a star = Greenwich Sidereal Time - RA of the star.
Combining these, we find
LHA(star) = GHA(star) - longitude west.

For a more detailed discussion of Sidereal Time and related topics, see Chapter 2 of USNO Circular No.179.

Exercise:06

At midnight on 1998 February 4th,
Local Sidereal Time at St.Andrews was 8h45m.
St.Andrews has longitude 2°48'W.
What was the Local Hour Angle of Betelgeuse (R.A. = 5h55m) at midnight?

RA of Betelgeuse = 5h 55m
At midnight, LST = 8h 45m
Local Hour Angle = LST - RA
so the Local Hour Angle of Betelgeuse was 8h 45m - 5h 55m = 2h 50m.

At what time was Betelgeuse on the meridian at St.Andrews?

On the meridian, Local Hour Angle = 0,
so if Betelgeuse was on the meridian at St.Andrews,
LST in St.Andrews = RA of Betelgeuse = 5h 55m.
(Recall that LST = RA of stars on local meridian.)

We are told that the LST was 8h 45m at midnight.
But at midnight, Betelgeuse was at Hour Angle 2h 50m,
so it would be on the meridian 2h 50m *before* midnight,
that is, at 21h 10m.
So Betelgeuse was on the meridian in St.Andrews at 21h 10m.

At what time was Betelgeuse on the meridian at Greenwich?

St.Andrews is 2°48' west of Greenwich = 0h 11m (divide by 15).
So Betelgeuse was on the *Greenwich* meridian
11 minutes before it reached the St.Andrews meridian.
i.e. at 20h 59m.

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Positional Astronomy: Chapter 07
Conversion between horizontal and equatorial systems

To convert between the horizontal and equatorial coordinates for an object X, we use a spherical triangle often called *"The" Astronomical Triangle*: XPZ, where Z is the zenith, P is the North Celestial Pole, and X is the object.

The sides of the triangle:
   PZ is the observer's co-latitude = 90°-φ.
   ZX is the zenith distance of X = 90°-a.
   PX is the North Polar Distance of X = 90°-δ.

The angles of the triangle:
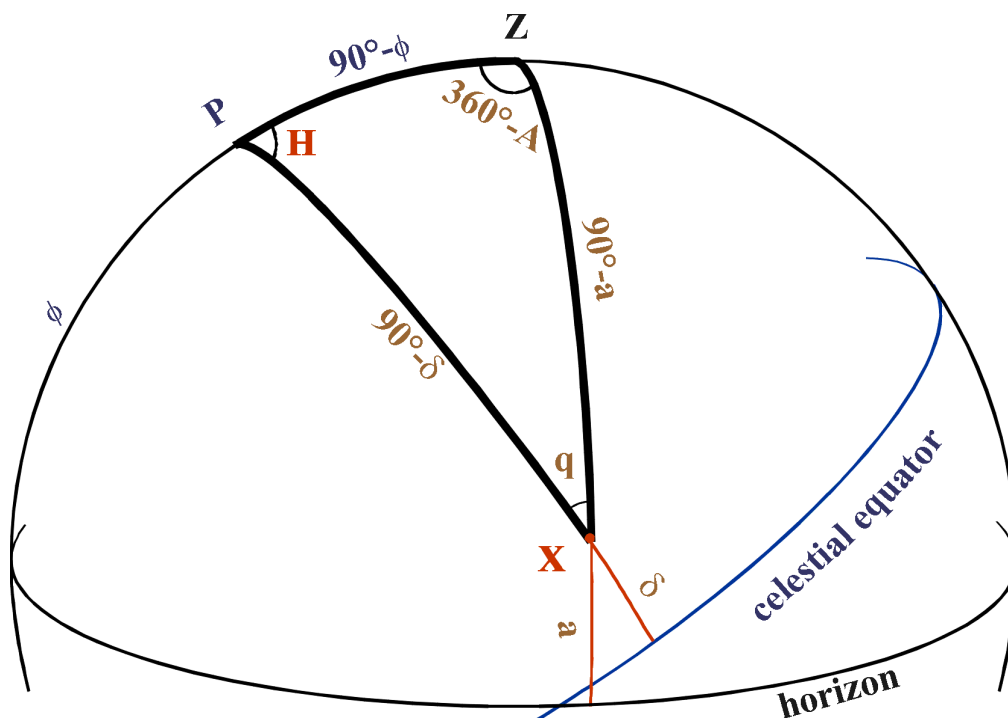   The angle at P is H, the local Hour Angle of X.
   The angle at Z is 360°-A, where A is the azimuth of X.
   The angle at X is q, the **parallactic angle**.

We assume we know the observer's latitude φ and the Local Sidereal Time LST.
(LST may be obtained, if necessary, from Greenwich Sidereal Time and observer's longitude.)

To convert from Equatorial to Horizontal coordinates:

Given RA α and declination δ, we have
Local Hour Angle H = LST - RA, in hours;
convert H to degrees (multiply by 15).
Given H and δ, we require azimuth A and altitude a.

By the cosine rule:
$\cos(90°-a) = \cos(90°-δ) \cos(90°-φ) + \sin(90°-δ) \sin(90°-φ) \cos(H)$
which simplifies to:
$\sin(a) = \sin(δ) \sin(φ) + \cos(δ) \cos(φ) \cos(H)$
This gives us the altitude a.

By the sine rule:
$\sin(360°-A)/\sin(90°-δ) = \sin(H)/\sin(90°-a)$
which simplifies to:
$- \sin(A)/\cos(δ) = \sin(H)/\cos(a)$
i.e. **$\sin(A) = - \sin(H) \cos(δ) / \cos(a)$**
which gives us the azimuth A.

Alternatively, use the cosine rule again:
$\cos(90°-δ) = \cos(90°-φ) \cos(90°-a) + \sin(90°-φ) \sin(90°-a) \cos(360°-A)$
which simplifies to
$\sin(δ) = \sin(φ) \sin(a) + \cos(φ) \cos(a) \cos(A)$
Rearrange to find A:
$\cos(A) = \{ \sin(δ) - \sin(φ) \sin(a) \} / \cos(φ) \cos(a)$
which again gives us the azimuth A.

Here are all the equations together:
$H = t - α$
$\sin(a) = \sin(δ) \sin(φ) + \cos(δ) \cos(φ) \cos(H)$
$\sin(A) = - \sin(H) \cos(δ) / \cos(a)$
$\cos(A) = \{ \sin(δ) - \sin(φ) \sin(a) \} / \cos(φ) \cos(a)$

(Space left intentionally blank for notes)

To convert from Horizontal to Equatorial coordinates:

Given φ, a and A, what are α and δ?

Start by using the cosine rule to get δ, as shown above:
sin(δ) = sin(a)sin(φ) + cos(a) cos(φ) cos(A)

We can now use the sine rule to get H, using the same formula as above:
sin(H) = - sin(A) cos(a) / cos(δ)
Or use the cosine rule instead:
sin(a) = sin(δ)sin(φ) + cos(δ) cos(φ) cos(H)
and rearrange to find H:
cos(H) = { sin(a) - sin(δ) sin(φ) } / cos(δ) cos(φ)

Having calculated H, ascertain the Local Sidereal Time t.
Then the R.A. follows from
α = t − H .

Here are all the equations together:
sin(δ) = sin(a)sin(φ) + cos(a) cos(φ) cos(A)
sin(H) = - sin(A) cos(a) / cos(δ)
cos(H) = { sin(a) - sin(δ) sin(φ)} / cos(δ) cos(φ)
α = t − H

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Exercise: 07

Prove that the celestial equator cuts the horizon at azimuth 90° and 270°,
at any latitude (except at the North and South Poles).
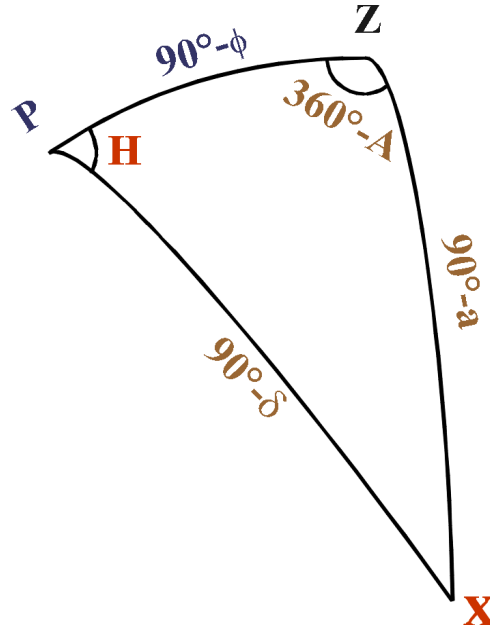
Draw "the" triangle again.
We require the azimuth A of point X,
where X is on the horizon (i.e. a=0)
and also on the equator (i.e. δ=0)

Apply the cosine rule:
cos PX = cos PZ cos XZ + sin PZ sin XZ cos Z
to get 0 = 0 + sin (90-φ) cos A
Since 90°-φ is not 0 (we are not at the Poles),
cos A must be 0
so A = 90° or 270° .

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At what *angle* does the celestial equator cut the horizon, at latitude φ ?
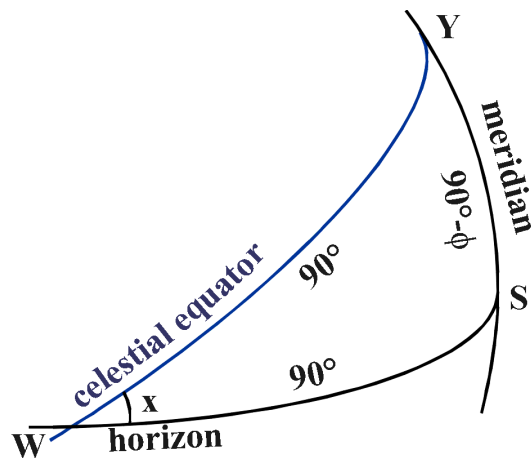
Use the cosine formula:
cos SY = cos SW cos YW + sin SW sin YW cos W
This gives cos (90°-φ) = 0 + cos x
So the angle x is 90°-φ.
The celestial equator cuts the horizon at an angle of 90°-φ

Positional Astronomy: Chapter 08
The Galactic System of Coordinates

The equatorial system of coordinates (Right Ascension and declination) is the one most often used.
But the **galactic system** is sometimes more useful,
e.g. for seeing how objects are distributed with respect to the galactic plane.

Galactic coordinates
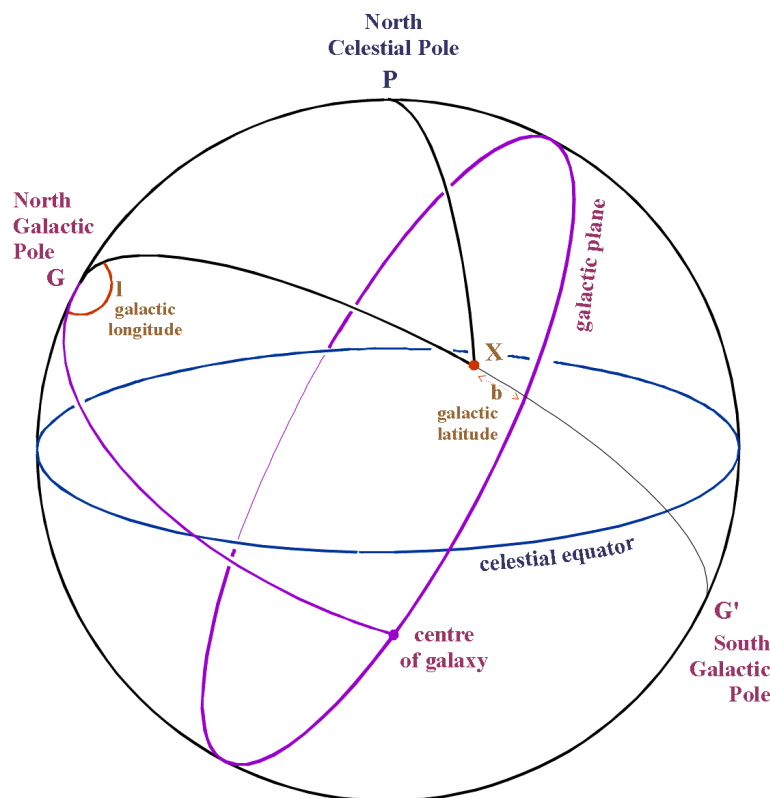
In this system, the *fundamental great circle* is the **galactic equator**,
which is the intersection of the galactic plane with celestial sphere,
with corresponding **galactic poles.**
We define the *North* Galactic Pole as that pole in same hemisphere as the North Celestial Pole.
The positions of the poles were determined by the International Astronomical Union (IAU) in 1959.

To fix the galactic coordinates of object X, draw a great circle between the two galactic poles, passing through X.

The **galactic latitude** (b) of object X is the angular distance on this circle from galactic equator to X, from - 90° at South Galactic Pole to +90° at North Galactic Pole.

The *zero-point* for longitude is the centre of galaxy; again, the position was fixed by the IAU.
The **galactic longitude** (l) of object X is the angular distance around the galactic equator from the centre of the galaxy to the great circle through X, measured eastwards 0-360°.

Although later research may come up with better values for the positions of the galactic poles and the centre of the galaxy, the IAU values will still be used to determine this coordinate system.

To convert between galactic and equatorial coordinates, draw the spherical triangle with points at P (North Celestial Pole), G (North Galactic Pole) and X, and apply the sine and cosine rules.

Exercise:08

The North Galactic Pole is at Right Ascension 12h49m, declination +27°24'. What is the tilt of the galactic plane to the celestial equator?

This one is really easy!

The distance from the North Celestial Pole P to the North Galactic Pole G is just (90°- declination of G) = 62.6°.

So this is also the tilt of the galactic plane to the equator.

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Positional Astronomy: Chapter 09
Ecliptic coordinates

All the objects considered so far have been "fixed stars", which keep almost constant values of Right Ascension and declination.
But bodies *within* the Solar System change their celestial positions.

The most important one to consider is the Sun.
The Sun's *declination* can be found by measuring its altitude when it's on the meridian (at midday).
The Sun's *Right Ascension* can be found by measuring the Local Sidereal Time of meridian transit.
We find that the Sun's RA increases by approximately 4 minutes a day, and its declination varies between +23°26' and -23°26'.
This path apparently followed by Sun is called the **ecliptic**.
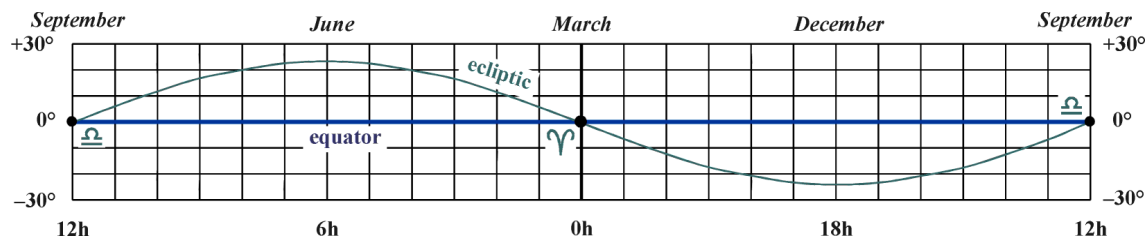
The reason the Sun behaves this way is that the Earth's axis is tilted to its orbital plane.
The angle of tilt is +23°26', which is called the **obliquity of the ecliptic** (symbol ε).
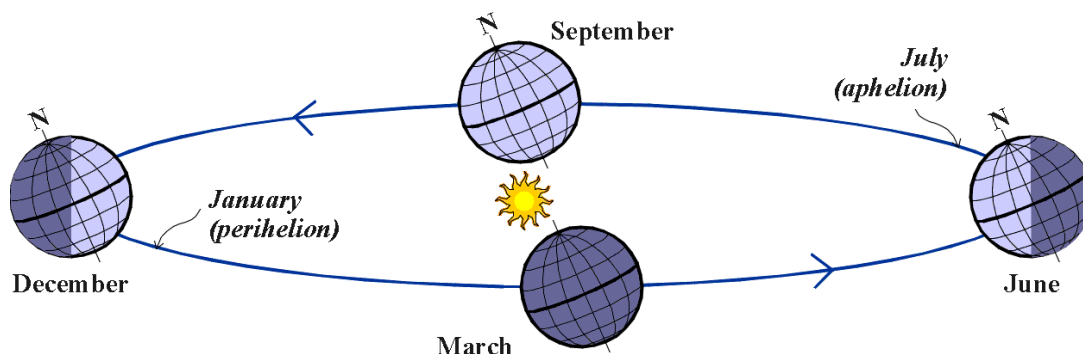
Any two great circles intersect at two **nodes**.
The node where the Sun crosses the equator *from south to north* (the
ascending node) is called the **vernal (or spring) equinox**.
The Sun passes through this point around March 21st each year.
This is the point from which R.A. is measured, so here RA = 0h.
At RA = 12h, the **descending** node is called the **autumnal equinox; t**he Sun
passes through this point around September 23rd each year.
At both these points, the Sun is on the equator, and spends 12 hours above
horizon and 12 hours below.

("Equinox" means "equal night": night equal to day.)

The symbols used for the spring and autumn equinoxes, ♈ *and* ♎,
are the astrological symbols for Aries and Libra.

The most northerly point of the ecliptic is called (in the northern hemisphere)
the **Summer Solstice** (RA = 6h): the Sun passes through this point around
June 21st each year.
The most southerly point is the **Winter Solstice** (RA = 18h); the Sun passes
through this point around December 21st each year.
At the northern Summer Solstice, the northern hemisphere of Earth is tipped
towards Sun, giving longer hours of daylight and warmer weather (despite the
fact that Earth's slightly elliptical orbit takes it *furthest* from the Sun in July!)

Thus the Sun's motion is simple when referred to the ecliptic; also the Moon
and the planets move near to the ecliptic.
So the **ecliptic system** is sometimes more useful than the equatorial system
for solar-system objects.

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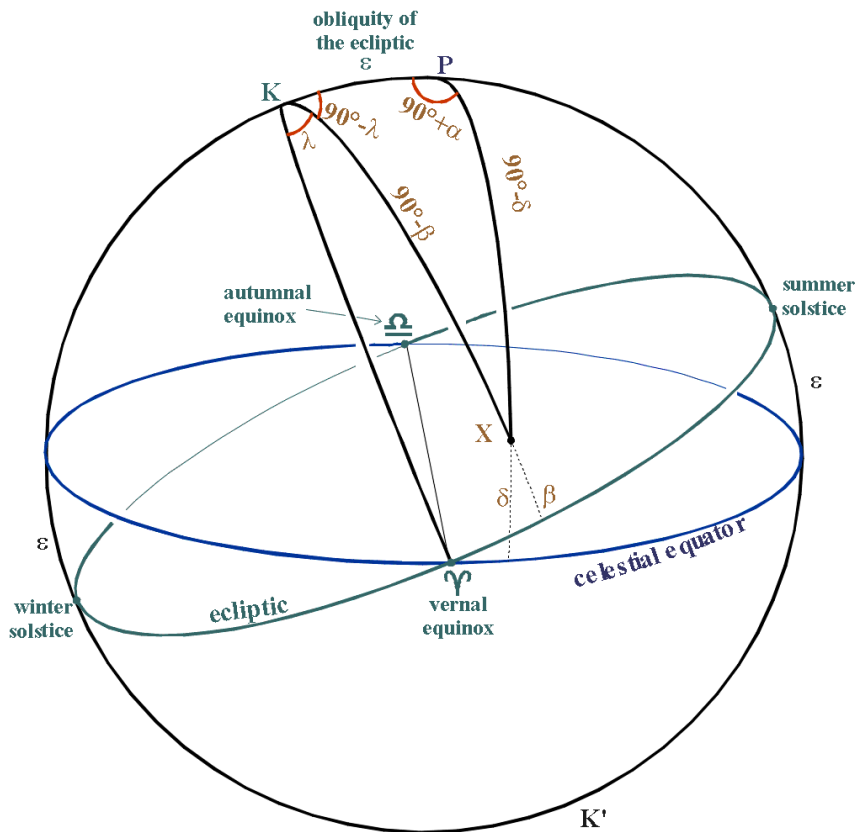
In the ecliptic system of coordinates, the *fundamental great circle* is the **ecliptic**.

The *zero-point* is still the vernal equinox.

Take K as the northern pole of the ecliptic, K' as the southern one.

To fix the ecliptic coordinates of an object X on the celestial sphere, draw the great circle from K to K' through X.

The **ecliptic** (or **celestial**) **latitude** of X (symbol β) is the angular distance from the ecliptic to X, measured from -90° at K' to +90° at K.
Any point on the ecliptic has ecliptic latitude 0°.

The **ecliptic** (or **celestial**) **longitude** of X (symbol λ) is the angular distance along the ecliptic from the vernal equinox to the great circle through X.
It is measured eastwards (like R.A.), but in degrees, 0°-360°.

To convert between ecliptic and equatorial coordinates, use the spherical triangle KPX.

Exercise:9a

The Moon's orbit is tilted at 5°8' to the ecliptic.
What is the lowest latitude from which the Moon may never set (the Moon's "arctic circle")?

Would the Moon *always* be circumpolar, at this latitude?
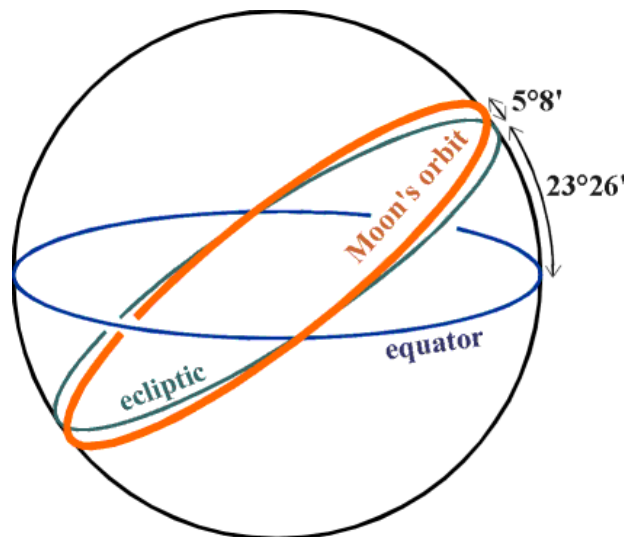
The maximum height of the ecliptic above the equator is $\varepsilon = +23°24'$.
The Moon can get 5°8' above this, i.e. up to +28°32'.
So the Moon's maximum declination could be = +28°32'.
This is when at its "major standstill"

An object of declination δ will be circumpolar at latitude 90°-δ,
When the Moon is at its maximum declination:
$90 - \delta = 90 – 28° 32´ = 61°28'$.latitude

So when the Moon is at its greatest possible declination, it appears circumpolar from any latitude north of 61°28'N.

(The northern tip of Shetland is at latitude 60°52'.)

Would the Moon *always* be circumpolar, at this latitude?

No; only at a "major standstill" (When at its greatest declination)
Sometimes the Moon's orbit will be inclined the other way to the ecliptic, and it will reach a maximum height of only 23°24' - 5°8' = 18°16'.
This is its "minor standstill".(When at its lower declination)
(The interval between major standstills is 18.6 years.)

Exercise:9b

Show that, for any object on the ecliptic:
tan(δ) = sin(α) tan(ε),
α is the object's Right Ascension; δ its declination,
ε is the obliquity of the ecliptic.
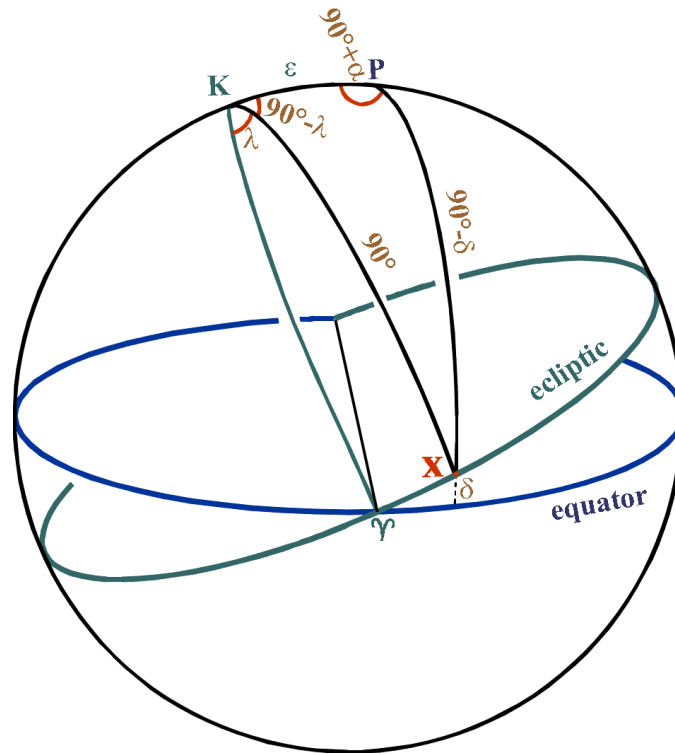
Use the cosine rule :

cos KX = cos PX cos KP + sin PX sin KP cos P

On the ecliptic, latitude β = 0 we have :
cos 90° = cos(90°-δ) cos(ε)
+ sin(90°-δ) sin(ε) cos(90°+α)
i.e. 0 = sin(δ) cos(ε) - cos(δ) sin(ε) sin(α)

Divide throughout by cos(δ) cos(ε) to get
tan(δ) = tan(ε) sin(α)

Positional Astronomy: Chapter 10
The relation between ecliptic and equatorial coordinates

Draw the triangle KPX, where P is the North Celestial Pole, K is the north pole of the ecliptic, and X is the object in question.

Apply the cosine rule:
$\cos(90°-δ) = \cos(90°-β) \cos(ε) + \sin(90°-β) \sin(ε) \cos(90°-λ)$
i.e. $\sin(δ) = \sin(β) \cos(ε) + \cos(β) \sin(ε) \sin(λ)$

Alternatively, apply the same rule to the other corner, and get:
$\cos(90°-β) = \cos(90°-δ) \cos(ε) + \sin(90-δ) \sin(ε) \cos(90°+α)$
i.e. $\sin(β) = \sin(δ) \cos(ε) - \cos(δ) \sin(ε) \sin(α)$

Now try applying the sine rule to the same triangle,
$\sin(90°-β) / \sin(90°+α) = \sin(90°-δ) / \sin(90°-λ)$
i.e. $\cos(λ) \cos(β) = \cos(α) \cos(δ)$
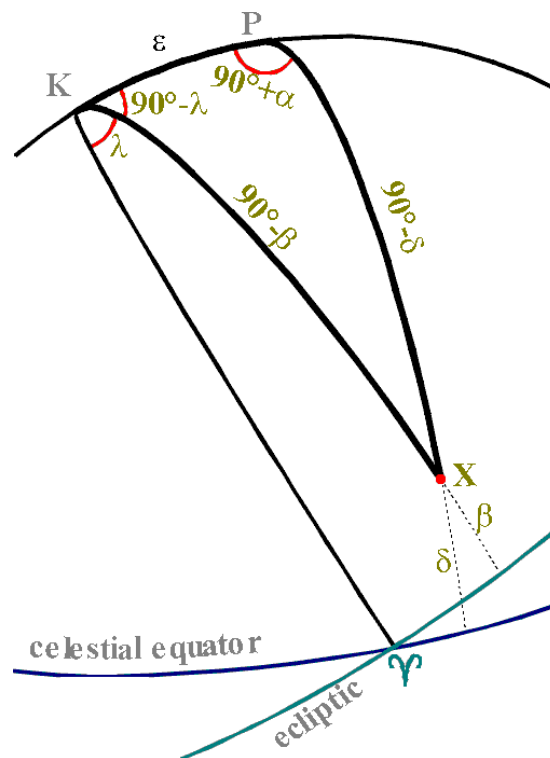
Grouping these three relations together, we have:
$\sin(δ) = \sin(β) \cos(ε) + \cos(β) \sin(ε) \sin(λ)$
$\sin(β) = \sin(δ) \cos(ε) - \cos(δ) \sin(ε) \sin(α)$
$\cos(λ) \cos(β) = \cos(α) \cos(δ)$

Exercise 10

Aldebaran has Right Ascension 4h36m, declination +16°31'.
What are its ecliptic coordinates?

First use
$\sin(\beta) = \sin(\delta) \cos(\epsilon) - \cos(\delta) \sin(\epsilon) \sin(\alpha)$
where α = 4h36m = 69.00°, δ = 16.52°, ϵ = 23.43°
This gives β = -5.45°.

Now use
$\cos(\lambda) \cos(\beta) = \cos(\alpha) \cos(\delta)$
to obtain λ = 69.81°.

So the ecliptic co-ordinates of Aldebaran are
λ = 69.81°, β = -5.45°.

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Positional Astronomy: Chapter 11
The Sun's motion, and its effect on time-keeping

In the ecliptic system, the Sun's coordinates are fairly simple, because its ecliptic latitude is 0° at all times, while its ecliptic longitude constantly increases.

However, the Sun's longitude does *not* increase at a *steady* speed of exactly 360° a year.
And it's important to know the Sun's position, because it's used for normal time-keeping (*solar* time, rather than *sidereal* time).

By Kepler's Second Law, the Earth orbits faster at perihelion and slower at aphelion.
So the Sun appears to move fastest along ecliptic in January and slowest in July.
We can invent an imaginary Sun (the *dynamical mean Sun*) which coincides with the true Sun when the Earth is at perihelion, but moves along the ecliptic at a *constant* speed.
The true Sun appears to move faster than the dynamical mean Sun when the Earth is around perihelion, and slower when the Earth is near aphelion: one cycle per year.

However, an object moving at a constant speed along the *ecliptic* is still moving at a varying speed with respect to the *equator*, since the ecliptic is tilted to the equator.
We invent another imaginary Sun, called simply the **mean Sun**, which moves along the *equator* at constant speed; the dynamical mean Sun appears to lag behind this where the ecliptic is steeply tilted to the equator (around the equinoxes) and catch up where it's nearly parallel (around the solstices): two cycles per year.

Combining these two effects gives the total difference in time between the true Sun and the mean Sun, which is called the **equation of time** (the solid black line on the diagram).

Equation of Time

True Sun ahead of Mean Sun

effect of elliptical orbit

effect of orbital tilt

True Sun behind Mean Sun

January   April   July   October   January
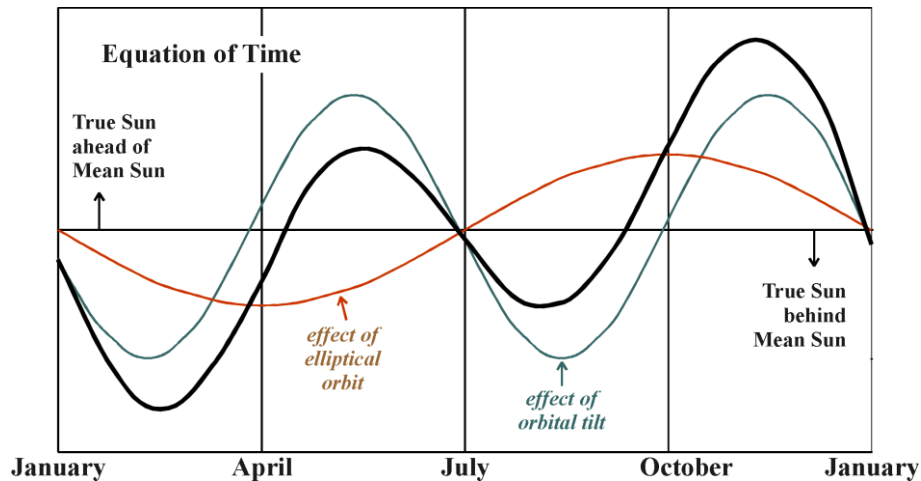
The true Sun is about 14 minutes late, compared to the mean Sun, around 10th February, about 4 minutes early around May 15th, about 6 minutes late around July 25th, and about 16 minutes early around November 5th.

The interval between two meridian transits of the *mean* Sun is the **mean solar day**.
The upper transit of the mean Sun across the local meridian marks midday, **local mean time.**
Greenwich Mean Time (GMT) is defined as midday when the mean Sun crosses the meridian of Greenwich.

Apparent solar time, as measured by the the *true* Sun (*e.g.* on a sundial), may differ from GMT for three reasons.
Firstly, because of the equation of time,
Secondly, because of the longitude of the observer (the further west, the later the Sun will cross the meridian).
Thirdly, because of the "Summer Time".

Britain uses GMT as standard time in winter, adding one hour in summer. Most other countries adopt their own standard time, suitable for their own longitude (large countries may have several time-zones), differing from GMT by a set amount.

In practice, the Earth's rotation is not quite constant.
Time is now regulated by atomic clocks, and called **Coordinated Universal Time (UTC)**, but this is artificially kept within 1 second of GMT by adding a "leap second" when necessary.

Astronomers also use **Terrestrial Time** (TT, formerly called Ephemeris Time, ET) for describing the motions of solar-system bodies.
The difference TT-UTC is called "delta-T".

At any location, local *mean* time and local *sidereal* time agree at the autumn equinox. (Why? Because, at the autumn equinox, the Right Ascension of the mean Sun is 12 hours, and the mean Sun is on the local meridian at 12h, local mean time.)

But sidereal time runs faster than solar time, by one day a year, or approximately 3.94 minutes a day.

After the Exercise about the topic of *time*, we will return to the *position* of an object in the sky.

There are various physical factors which may change the apparent position of an object.

Exercise 11:

On April 1st, what is the Sun's approximate ecliptic longitude?

At the spring equinox, on or around March 21st,
the Sun's ecliptic longitude is exactly 0°.
11 days have elapsed since then,
so the Sun's longitude will be approximately 11°.
(Only approximately,
because the ellipticity of the Earth's orbit means that the Sun does not move around the ecliptic at a steady speed.)

And approximately what is Greenwich Sidereal Time at midnight on April 1st?

GST and GMT are the same at the autumn equinox,
so they are exactly 12 hours out of phase at the spring equinox.
So GST is 12h at midnight on March 21st, approximately.

Eleven days later, on April 1st,
GST will have got ahead of GMT by 11 x 3.94 minutes = 43.3 minutes.
So GST at midnight on April 1st will be approximately 12h43m.

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Positional Astronomy: Chapter 12
The Moon

The Moon always keeps the same face turned towards the Earth.
How much we *see* of that face depends on the direction of the Sun:
we only see the part which is illuminated by sunlight,
as shown in the diagrams below.

Synodic month

diagram not
to scale!
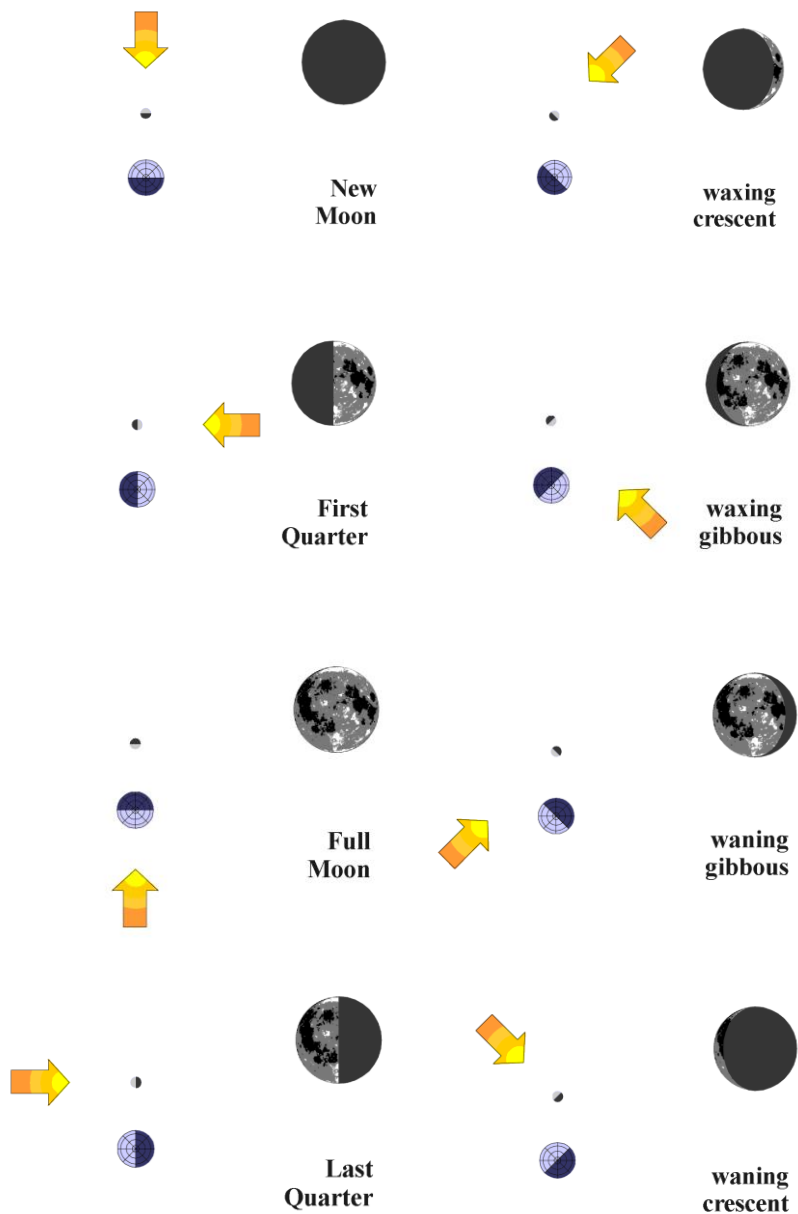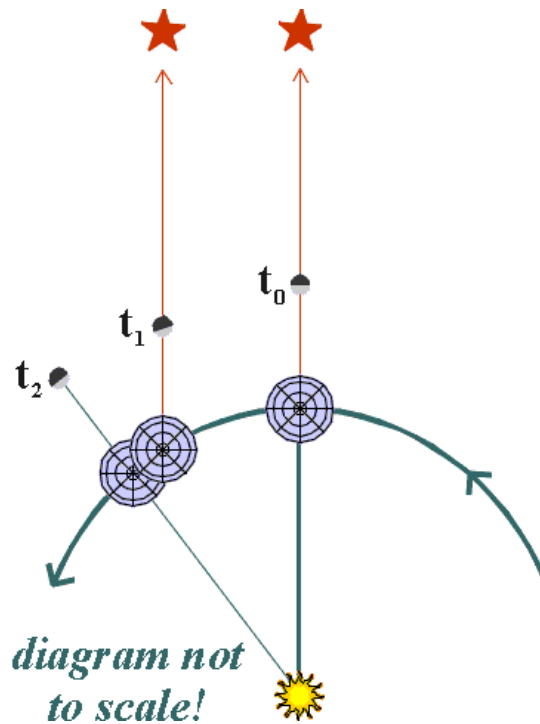
The interval between one New Moon and the next is the **synodic month**;
it averages 29.53 days.
However, because the Earth is orbiting the Sun, the Moon will return to the
same place *relative to the stars* in a shorter interval – the **sidereal month**; this
averages 27.32 days.

Diagram shows the Moon at Full, and lined up with a certain star, at time t_0 .
It is lined up with that star again at time t_1 , after one sidereal month, but it
doesn't reach Full again until time t_2 , after one synodic month.

The Moon's actual motion is extremely complicated: it orbits the Earth in an
elliptical orbit, tipped at an angle to the Earth's own orbital plane (the
ecliptic); and its orbit is constantly being perturbed by the gravitational
influence of the Sun.

This is not the place for the full theory of the Moon's motion.
However, there follows some simple approximations which will help to
determine when the Moon will be or not visible.

1. When will the Moon transit the meridian?

At New Moon, the Moon lies in the same direction as the Sun. *(Owing to the tilt of the Moon's orbit, it does not generally pass directly in front of the Sun.)*
The Moon then moves eastwards, relative to the Sun.
It moves 360° in 29.53 days, at which time it is lined up with the Sun again for the next New Moon.
Thus it moves about 12.2° each day, relative to the Sun; which corresponds to lagging behind the Sun, as it crosses the sky, by about 48.8 minutes of time each day.

So, if you know the "age" of the Moon (that is, how many days since the last New Moon), you can calculate how much later the Moon will cross the sky, compared to the Sun.

The Sun crosses the meridian at noon (you can be more precise than this, if you know your longitude and the Equation of Time).
So you can calculate the time at which the Moon will cross the meridian.

The result will not be very accurate, since the Moon's motion is not uniform, but should be correct to within an hour.

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2. When will the Moon rise and set?

If the Moon were always on the celestial equator, it would always rise 6 hours before transit, and set 6 hours after transit.

We know that the *Sun* does not keep to the celestial equator.
It lies on the equator at the equinoxes, in March and September, but its declination varies between 23.4°N in June and 23.4°S in December.
    At the equinoxes, anywhere in the world, the Sun rises due east, 6 hours before noon, and sets due west, 6 hours after noon.
    But at the summer (June) solstice, in latitudes north of 66.6°N, the Sun never sets at all! (This defines the *Arctic Circle.*)
In latitudes around 58°N, the Sun rises in the *north*-east, about 9 hours before noon, and sets in the *north*-west, about 9 hours after noon.
    Similarly, at the winter (December) solstice, in the Arctic Circle the Sun never rises; in latitudes around 58°N, the Sun rises in the *south*-east, about 3 hours before noon, and sets in the *south*-west, about 3 hours after noon.

Now, the Moon follows roughly the same path as the Sun (ignoring its orbital tilt) but it takes only a month to trace the path which the Sun takes in a year.

The Sun moves about 1° a day (360° in 365.25 days)
The Moon lags behind the Sun by about 12.2° a day, so you can work out the date on which the Sun will be at the point where the Moon now is.

This means you can estimate roughly how long the Moon will be above the horizon.

Having already calculated the time at which it will cross the meridian, you can now estimate its rising and setting times.

This will not be very accurate.
But it should be sufficient to determine, for example, whether a particular night's observing will be affected by moonlight

Exercise : 12a

The Moon was New on December 25th, 2000.
At St.Andrews, on January 1st 2001,
the Sun crossed the meridian at 12:15.
At what time did the Moon cross the meridian?

On January 1st, 7 days had elapsed since New Moon
(the Moon was "7 days old").
So the Moon was lagging the Sun by 7x48.8 minutes = 5h42m.

So it would cross the meridian at 12:15 + 5h42m = 17:57.

In fact it crossed the meridian at 17:24, so our calculation is in error by 33 minutes.
The error is due to:
(a) not knowing the exact *time* of New Moon on December 25th
(b) the Moon's motion not being constant.

.Exercise:12b

Given the data in the previous exercise,
estimate the times of moonrise and moonset
at St.Andrews on January 1st 2001.

We know that the Moon was 7 days old,
so it was 7x12.2° = 85° east of the Sun.
It would take the Sun about 85 days to reach this point:
it would get there on 26th March,
very close to the Spring Equinox.

So the Moon would have been close to the celestial equator,
and it would rise about 6 hours before it transited across the meridian,
and set about 6 hours after.

We have already determined that the Moon crossed the meridian at 17:57.
So it should rise at 17:57 − 6 hours = 11:57
and it should set at 17:57 + 6 hours = 23:57

In fact it rose at 11:56 and set at 23:06.

This approximate calculation has shown us that, on January 1st 2001,
the Moon would be rising about midday,
and setting again around midnight.

Exercise: 12c

Here is an example where you can deduce a great deal
from very little information:

At a point with latitude near 58°N,
the Last-Quarter Moon is seen rising in the north-east.

What time of day is it?
What time of year is it?

If the Moon is at Last Quarter,
it is lagging 18 hours behind the Sun -
alternatively, it is 6 hours *ahead* of the Sun.

The Sun crosses the meridian at noon,
so the Moon must be crossing the meridian at 6 am.

At this latitude, if it is rising in the north-east,
it must be at its most northern point,
so it will be rising about 9 hours before it crosses the meridian.
So it must be rising about **9 pm**.

The Moon is at its most northern point:
the point the Sun only reaches at midsummer
(that is RA = 6h; declination = +23.4°).

And if the Moon is at Last Quarter,
it has travelled three-quarters of the way around the sky from the Sun.
So it is 270° east of the Sun -
alternatively, it is 90° *west* of the Sun.

So the Moon is where the Sun was 90 days ago.
So the Sun must presently be at
(summer solstice + 90 days) = **autumn equinox**.

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Positional Astronomy:Chapter 13
Refraction

The apparent position of an object in the sky may be changed by several
different physical effects. One of these is refraction.The speed of light
changes as it passes through a medium such as air.
We define the **refractive index** of any transparent medium as 1/v,
where v is the speed of light in that medium.

The speed of light in air depends on its temperature and its pressure,
so the refractive index of the air varies in different parts of the atmosphere.

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Make a simple model of the atmosphere as *n* layers of uniform air above a flat Earth, with a different velocity of light v*i* for each layer (*i* from 1 to *n*).
Apply Snell's Law of Refraction at each boundary.
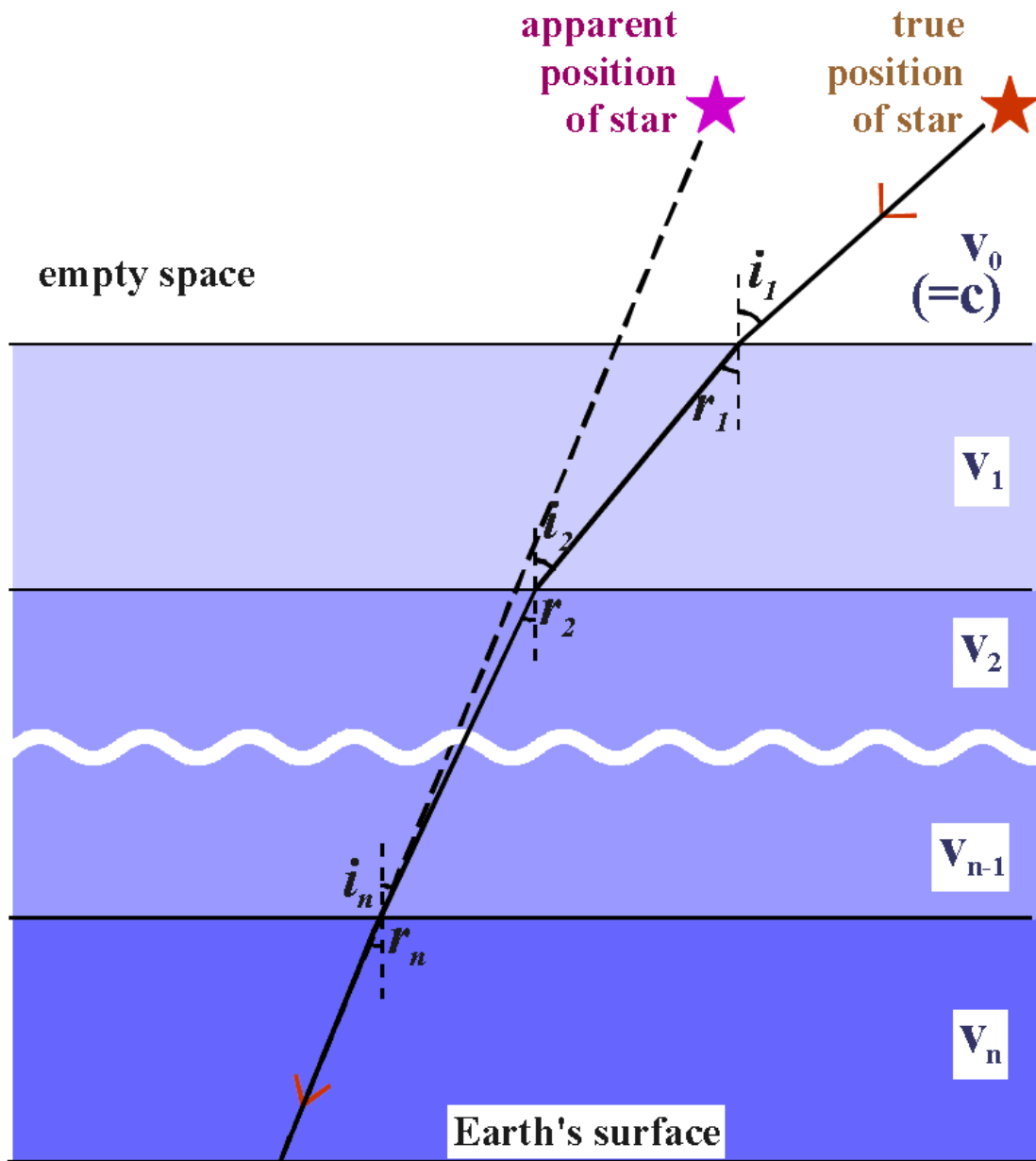
At the first boundary, $\sin(i_1) / \sin(r_1) = v_0 / v_1$.
At the next boundary, $\sin(i_2) / \sin(r_2) = v_1 / v_2$, and so on.

But, by simple geometry, $r_1 = i_2$, $r_2 = i_3$ and so on.

So we have
$$\begin{aligned}
\sin(i_1) &= (v_0 / v_1) \sin(r_1) \\
&= (v_0 / v_1) \sin(i_2) \\
&= (v_0 / v_1) (v_1 / v_2) \sin(r_2) \\
&= (v_0 / v_2) \sin(r_2) \\
&= .... \\
&= (v_0 / v_n) \sin(r_n)
\end{aligned}$$

In other words, the refractive indices of the intervening layers all cancel out. The only thing that matters is the ratio between v_0, which is c, (the speed of light in vacuum) and v_n (the speed in the air at ground level).

Now r_n is the *apparent* zenith distance of the star, z', and i_1 is its *true* zenith distance, z.
So $\sin(z) = (v_0 / v_n) \sin(z')$.

Refraction has no effect if a star is at the zenith (z=0).
But at any other position, the star is apparently *raised*; the effect is greatest at the horizon.

Define the **angle of refraction** R by   **R = z - z'**.
Rearrange this as z = R + z'.
Then $\sin(z) = \sin(R) \cos(z') + \cos(R) \sin(z')$.

We assume R will be small, so, approximately,
$\sin(R) = R$ (in radians), and $\cos(R) = 0$.

Thus, approximately,
    $\sin(z) = \sin(z') + R \cos(z')$.
Divide throughout by sin(z') to get
    $\sin(z)/\sin(z') = 1 + R/\tan(z')$
which is to say,
    $v_0/v_n = 1 + R/\tan(z')$.
So we can write
    $R = (v_0/v_n - 1) \tan(z')$
We write this as
    R = k tan(z')
where $k = (v_0/v_n - 1)$

Here v_0 is c, the velocity of light in a vacuum, which is constant.
But v_n depends on the temperature and pressure of the air at ground level.
At "standard" temperature (0°C = 273K) and pressure (1000 millibars),
    k = 59.6 arc-seconds.
The formula in the *Astronomical Almanac* is
    k = 16.27" P / (273 + T°)
where P is in millibars, and T is in °C.

At large zenith distances, the model is inadequate.
The amount of refraction *near the horizon* is actually determined observationally.
At standard temperature and pressure, refraction at the horizon is found to be 34 arc-minutes. (**Horizon or Horizontal refraction**)

Exercise: 13

A star is at Right Ascension 5h 0m and declination +26°20'.
The latitude φ is +56°20'.
Local Sidereal Time is 5h 0m.
Atmospheric pressure is 1050 millibars,
and the temperature is +5°C.
What is the star's true altitude?

LST = RA, so the star is on the local meridian,
so altitude a = (90°-φ) + δ = 60°.

How much will the star's image be shifted by atmospheric refraction,
and in which direction?

Zenith distance z = 90°-altitude = 30°.
Angle of refraction R = k tan(z')
where k = 16.27" P/(273+T) = 16.27" x 1050 /278 = 61.45".

However, we don't know z', only z.
For a first, approximate answer, take z = z' = 30°.
This gives R = 35.5",
so z' = z+R = 30° 0' 35.5".

Now recalculate R = k tan(z') = 35.5"
(unchanged, so no need to iterate further).

So the star is raised by 35.5".

What will be the star's Right Ascension and declination,
corrected for refraction?

Since the star is on the local meridian, the shift is only in declination.
The apparent altitude is increased by 35.5",
so the apparent declination is similarly increased.
So the apparent coordinates are:
Right Ascension 5h 0m and declination +26°20'35.5".

Positional Astronomy: Chapter 14
Sunrise, sunset and twilight

The time between the object crossing the horizon, and crossing the meridian.
is the **semi-diurnal arc** H:

Since refraction affects zenith angle,
it generally changes both the Right Ascension and declination of an object.
It also affects the time the object appears to rise and set.

The standard formula for the altitude of an object is:
$$\sin(\alpha) = \sin(\delta)\sin(\varphi) + \cos(\delta) \cos(\varphi) \cos(H)$$

If a = 0° (the object is on horizon, either rising or setting),
then this equation becomes:
$$\cos(H) = - \tan(\varphi) \tan(\delta)$$

Knowing the Right Ascension of the object, and its semi-diurnal arc,
we can find the Local Sidereal Time of meridian transit,
and hence calculate its rising and setting times.

However, refraction means that this simplified formula is not accurate,
since the altitude should be, not 0°, but -0°34'.
This is not too important for *stars*, which are rarely observed close to the horizon.
But it makes an important difference in calculating the times of rising and setting of the *Sun*.

Furthermore, "sunrise" and "sunset" generally refer to the moment
when the *top* of the Sun's disc is just on the horizon.
The formula would give us the time of rising or setting
for the *centre* of the Sun's disc.
So we must also allow for the *semi-diameter* of the Sun's disc,
which is 16 arc-minutes.

So sunrise and sunset actually occur when the Sun has altitude -0°50'
(34' for refraction, and another 16' for the semi-diameter of the disc).

Since the atmosphere scatters sunlight, the sky does not become dark instantly
at sunset; there is a period of **twilight**.

Civil twilight, Sun's altitude is -6°. It is still enough light to carry on ordinary activities out-of-doors; this continues until the:

Nautical twilight, Sun's altitude is -12°. It is dark enough to see the brighter stars, but still light enough to see the horizon, enabling sailors to measure stellar altitudes for navigation; this continues until the.

Astronomical twilight, Sun's altitude is -18°. There is still too much light in the sky for making reliable astronomical observations; this continues until the

Astronomical darkness Sun is more than 18° below the horizon. Is when the Astronomicalobservation can best be made.

The same pattern of twilights repeats, in reverse, before sunrise.

In summer, astronomical twilight will last all night, for any place with latitude above 48.6°.

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Exercises: 14 a, b and c

Exercise: 14a

The Sun is at declination -14°.
What will be its hour angle at sunrise
(the moment the top edge of the Sun first appears over the horizon),
at a latitude of +56°20'?

At sunrise, the true altitude of the Sun is a = -0°50'
(allowing for semi-diameter and horizontal refraction).

Use the formula
$\cos(H) = \{ \sin(a) - \sin(\varphi) \sin(\delta) \} / \cos(\varphi) \cos(\delta)$
where φ = +56°20' and δ = -14°.
This gives cos(H) = 0.35,
so H = 69.7° or 290.3°
= 4h39m or 19h21m.

To decide which,
note that the Sun is to the *east* of the meridian at sunrise,
so H = 19h21m.

Exercise: 14b

If the Sun is on the local meridian at 12:03, at what time is sunrise?
and at what time is sunset?

The semi-diurnal arc is 4h39m.
Sunrise is at (12:03 - 4h39m) = 07:24.
Sunset is at (12:03 + 4h39m) = 16:42.

Exercise: 14c

And when will astronomical twilight start and finish?

It is astronomical twilight
if the Sun's altitude is -18° or higher.

Set a = -18° and use the same formula again, to obtain
H = 101.55° = 6h46m.

So twilight starts at (12:03 - 6h46m) = 05:17,
and ends at (12:03 + 6h46m) = 18:49.

Positional Astronomy: Chapter 15
Geocentric or diurnal parallax

Refraction affects the apparent altitude of a star.
But there are other phenomena that affect its apparent position, too.
One of these is parallax.
Refraction decreases the zenith angle, but parallax increases it.

Our observations are made from the surface of the Earth, not its centre.
This is irrelevant when observing distant objects such as stars.
But for closer objects (*e.g.* within the Solar System), a correction must be made.
This is **geocentric parallax**, or **diurnal** (daily) parallax
(since it varies daily as the Earth spins around its axis).
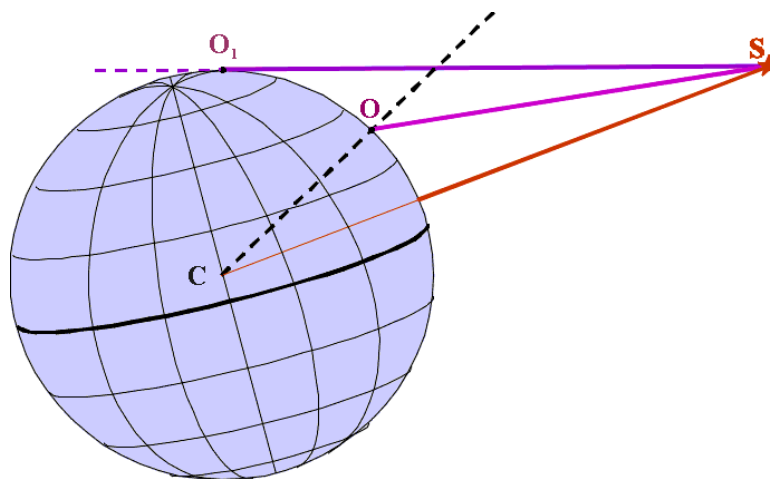
To an observer at O, the zenith angle of object S appears to be z'.
Its true zenith angle, as seen from the centre of the Earth C, is z, which is smaller.
Parallax is greatest for an observer at O_1, where the object appears to be on the horizon.

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We define the **angle of parallax** p by **p = z'-z**.
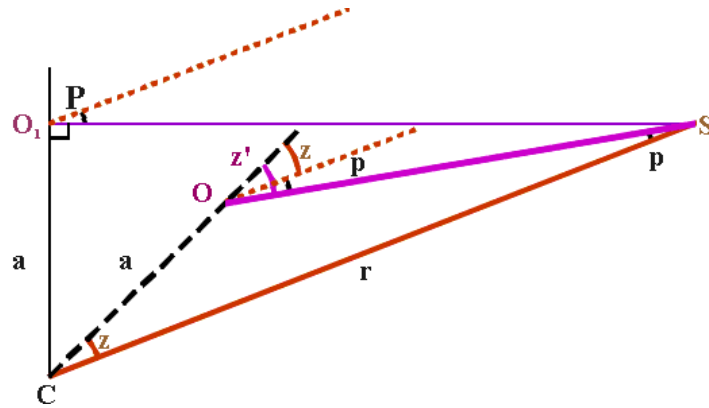
If *a* is the Earth's radius, and *r* is the geocentric distance to the object,
then the *plane* triangle OCS gives:

   $\sin(p) / a = \sin(180°-z') / r = \sin(z') / r$

that is:

   $\sin(p) = (a / r) \sin(z')$

Parallax is greatest at O_1, where z'=90°.
The parallax here is called the **horizontal parallax**, designated by P = 90°-z,
where :

   $\sin(P) = a / r$.

For small angles, we may take:

 P = a / r,
where P is measured in radians.

In the general case, we may replace the term (a/r) by sin(P), and write

   $\sin(p) = \sin(P) \sin(z')$

or, since angles of parallax are generally small,

   p = P sin(z')

Apart from occasional near-earth asteroids, the Moon is the nearest natural
object, with average P around 57 arc-minutes.
So for calculating times of **moonrise and moonset**, we must use an altitude:
0° - 16' [semi-diameter] - 34' [refraction] + 57' [horizontal parallax] = +7'.

Allowing for lunar parallax is essential when predicting occultation of stars by
the Moon (and, of course, solar eclipses).

The Earth is not actually spherical.
For more accurate calculations, we use the **geoid**:
Geoid is a spheroidal solid which closely approximates the Earth's true shape.
For any particular latitude, this gives corrected values for geocentric distance
a and geocentric latitude. L

Exercise: 15a

A minor planet passes very near the Earth,
at a distance of 200,000 km.
What will be its horizontal parallax?
(Take the Earth to be a sphere of radius 6378 km.)

Horizontal parallax P = a/r radians,
where a = 6378 km, r = 200,000 km.
So P = 0.0319 radians = 1.827°

At St.Andrews (latitude +56°20'),
the minor planet is observed to cross the meridian
at an apparent altitude of +35°.
What does its declination appear to be?

Meridian altitude a = (90°-φ) + δ
So the apparent declination is
δ = a - (90°-φ) = +1°20'

What is its true declination, after correcting for geocentric parallax?

Apparent altitude = 35°
so apparent zenith angle z'= 55°.

Angle of parallax p = P sin(z') = 1.497°

Parallax increases the zenith angle.
The true zenith angle z must be less than the observed zenith angle z'.
So the true altitude must be greater than the observed altitude.

So in this case the true declination must be
greater than the observed declination, by 1.497°,
making it +2°50'.

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Exercise:15b

The Moon is at declination -14°.
What will be its hour angle at moonrise
(when the top edge of the Moon first appears over the horizon),
at a latitude of +56°20'?

At moonrise, the true altitude of the Moon is a = +0°7'
(allowing for semi-diameter, horizontal refraction and geocentric parallax).

Use the formula
$\cos(H) = \{ \sin(a) - \sin(\varphi) \sin(\delta) \} / \cos(\varphi) \cos(\delta)$
where φ = +56°20' and δ = -14°.

This gives cos(H) = 0.38,
so H = 67.8° or 292.2°
= 4h31m or 19h29m.

To decide which,
note that the Moon is to the *east* of the meridian,
so H = 19h29m.

(This is 8 minutes later than sunrise,
when the Sun is at the same declination.)

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Exercise:15c

Aldebaran is at Right Ascension 4h36m, declination +16°31'.
At a particular instant, the geocentric coordinates of the Moon
are also Right Ascension 4h36m, declination +16°31'.
Local Sidereal Time at St.Andrews (latitude +56°20') is 4h36m.
What will be the apparent declination of the Moon, after correction for
parallax?
(Take the horizontal parallax of the Moon as 57 arc-minutes.)

Both objects are on the meridian.
So the Moon's altitude a = (90°-φ) + δ = 50.18°
so its true zenith angle z = (90°-a) = 39.82°

The shift due to parallax is p = P sin(z'), where P = 57',
and z' is the apparent zenith angle.
However, we only know z, the true zenith angle.

For a first approximation, take z' = z = 39.82°

Then p = 57' sin(39.82°) = 36.5' = 0.61°.
This would make apparent zenith angle z' = 39.82° + 0.61° = 40.43°.
Re-calculate p = 57' sin(40.43°) = 37.0' = 0.62°.
No need to re-calculate again.

The apparent altitude will be 37' lower,
due to geocentric parallax,
and so the apparent declination will also be 37' lower.

So the Moon's apparent declination will be
+16°31' - 37' = +15°54'.

The semi-diameter of the Moon's disc is 16 arc-minutes.
Will observers at St.Andrews see the Moon occult Aldebaran?

Aldebaran is still at Right Ascension 4h36m, declination +16°31'
(unaffected by geocentric parallax).

The top edge of the Moon will appear to be at declination
+15°54' + 16' = +16°10'

So the top edge of the Moon will appear to pass 21' below Aldebaran:
there will be no occultation.

Positional Astronomy: Chapter 16

Annual parallax

Geocentric or diurnal parallax varies with the daily spinning of the Earth around its axis.
Annual parallax is caused by the Earth's yearly orbit around the Sun.
The Earth shifts by 2a from side to side, where:
 a is the radius of the Earth's orbit (assumed circular) = 1 Astronomical Unit.
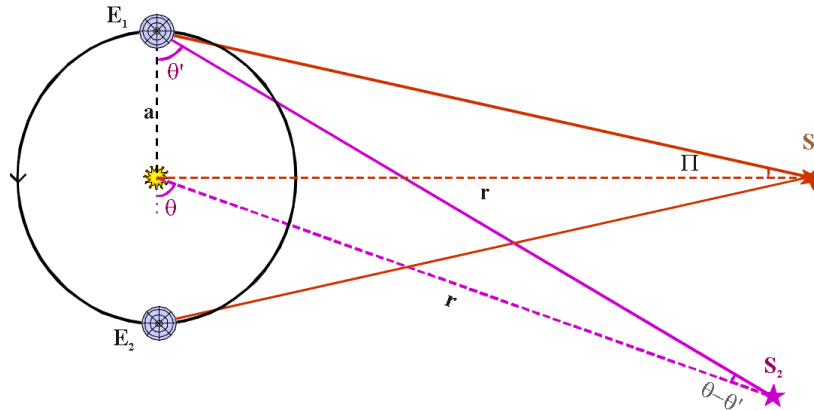
For the star S_1, the maximum shift occurs as the Earth moves from position E_1 to E_2.
If the distance between the Sun and the star S_1 is r, then we define:
 annual parallax as **Π**, where: **tan(Π) = a/r**.

And since a/r is always extremely small, we may write:
Π = a/r (in radians).

If the star is not at S_1, but at some other arbitrary position S_2, then the shift in position as Earth moves from E_1 to E_2 will appear less.
Let the direction from the Sun to the star make an angle θ with the line E_1E_2 .
the star appears at angle θ' from Earth at E_1.
By plane trigonometry,

$$\sin(\theta-\theta')/a = \sin(\theta')/r$$
so
$$\sin(\theta-\theta') = \sin(\theta')\ a/r$$
$$= \sin(\theta')\ \sin(\Pi)$$
Since (θ-θ') is a very small angle, we can replace θ' by θ, and write

$$\theta-\theta' = \Pi \sin(\theta)$$

The apparent shift is *towards the Sun*, and it alters the star's ecliptic longitude
λ (so this is another occasion for using ecliptic coordinates).
If the star is not in the plane of the ecliptic, there is a shift in ecliptic latitude β too.

The star X at (λ, β) is shifted to X' at ($\lambda+\Delta\lambda$, $\beta+\Delta\beta$) along a great circle arc towards the position of the Sun S.
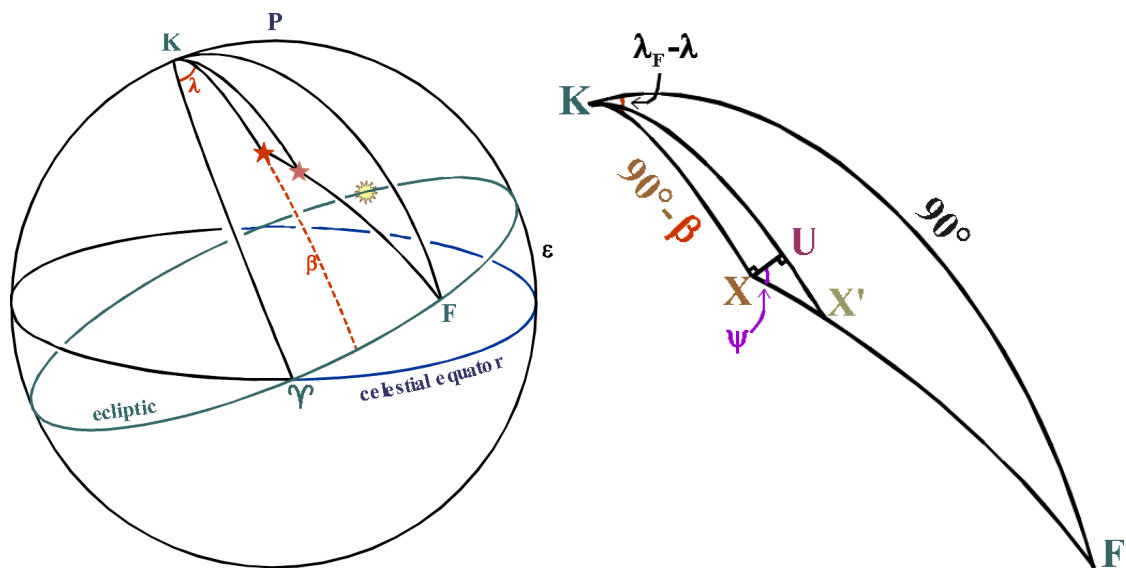
XX' is the parallactic shift $\Pi \sin(\theta)$.

We need to find the shifts $\Delta\lambda$ and $\Delta\beta$.

UX is the arc of a small circle centred on the ecliptic pole K, passing through the star X.

The length of arc UX is $\Delta\lambda \cos(\beta)$ – the shift in longitude.

The length of arc UX' is $-\Delta\beta$ – the shift in latitude.

Consider the tiny triangle UXX' as a *plane* right-angled triangle, and denote the angle at X by the arbitrary symbol ψ:

$UX = XX' \cos(\psi) = \Pi \sin(\theta) \cos(\psi)$

$UX' = XX' \sin(\psi) = \Pi \sin(\theta) \sin(\psi)$

In other words,

$\Delta\lambda \cos(\beta) = \Pi \sin(\theta) \cos(\psi)$

$\Delta\beta = -\Pi \sin(\theta) \sin(\psi)$ *(equations 1)*

To eliminate θ and ψ from these two equations, we use the *spherical* triangle KXS.

First, by the sine rule:

$\sin(90°+\psi/\sin(90°) = \sin(\lambda_S-\lambda)/\sin(\theta)$

i.e. $\sin(\theta) \cos(\psi) = \sin(\lambda_S-\lambda)$ *(expression 2)*

where λ_S is the ecliptic longitude of the Sun.

Then, by the cosine rule:

$\cos(90°) = \cos(\theta) \cos(90°-\beta) + \sin(\theta) \sin(90°-\beta) \cos(90°+\psi)$

i.e. $0 = \cos(\theta) \sin(\beta) - \sin(\theta) \cos(\beta) \sin(\psi)$

So: $\sin(\theta) \sin(\psi) = \cos(\theta) \sin(\beta)/\cos(\beta)$

To get rid of the cos(θ) on the right-hand side of this expression, we use the cosine rule again:

$\cos(\theta) = \cos(90°-\beta) \cos(90°) + \sin(90°-\beta) \sin(90°) \cos(\lambda_S-\lambda)$

i.e. $\cos(\theta) = \cos(\beta) \cos(\lambda_S-\lambda)$

Substituting this expression for cos(θ) in the previous equation:

$\sin(\theta) \sin(\psi) = \cos(\beta) \cos(\lambda_S-\lambda) \sin(\beta) /\cos(\beta)$

i.e. $\sin(\theta) \sin(\psi) = \cos(\lambda_S-\lambda) \sin(\beta)$ *(expression 3)*

Now we can substitute these expressions (2) and (3) in equations (1), to get:

$\Delta\lambda \cos(\beta) = \Pi \sin(\lambda_S-\lambda)$

$\Delta\beta = \Pi \cos(\lambda_S-\lambda) \sin(\beta)$

This is actually the formula for an *ellipse*, of the form:

$x = a \cos(\theta), y = b \sin(\theta)$

where x is the shift parallel to the ecliptic [$\Delta\lambda \cos(\beta)$]

y is the shift perpendicular to the ecliptic [$\Delta\beta$]

and θ is temporary shorthand for [$90° - (\lambda_S-\lambda)$].

So we have a = Π and b = Π sin(β)

In other words, this **parallactic ellipse** has semi-major axis Π, parallel to the ecliptic, and semi-minor axis Π sin(β), perpendicular to the ecliptic.

So, during the year, the star appears to trace out a parallactic ellipse, which is a reflection of the Earth's orbit.

For a star on the ecliptic (β = 0°) it reduces to a straight line;

for a star at the pole of the ecliptic (β = 90°) it becomes a circle.

The *size* of a star's parallactic ellipse yields its distance,

in units of **parsecs** (**par**allax-**sec**onds):

r (in parsecs) = 1 / Π (in arc-seconds),

so a star at 1 parsec would have parallax Π = 1 arc-second.

(In fact, no star is this close.)

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Exercise: 16

A star's true position is
Right Ascension 6h 0m 0s, declination 0° 0' 0",
and it lies at a distance of 25 parsecs.
On the date of the Spring Equinox,
how far will it appear to be shifted by annual parallax,
and in what direction?
First convert from RA and dec. into ecliptic coordinates.

$\sin(\delta) = \sin(\beta)\cos(\varepsilon) + \cos(\beta)\sin(\varepsilon)\sin(\lambda)$

$\sin(\beta) = \sin(\delta)\cos(\varepsilon) - \cos(\delta)\sin(\varepsilon)\sin(\alpha)$

$\cos(\lambda)\cos(\beta) = \cos(\alpha)\cos(\delta)$

where $\alpha = 6h0m = 90°$, $\delta = 0°$

So this star has $\lambda = 90°$, $\beta = -\varepsilon = -23°26'$.

Its distance = 25 pc, so annual parallax $\Pi = (1/25)" = 0.040"$

At the spring equinox, the Sun has ecliptic longitude $\lambda_S = 0°$

so $\lambda_S - \lambda = -90°$.

$\Delta\lambda\cos\beta = \Pi\sin(\lambda_S - \lambda) = 0.040"$

$\cos\beta = 0.918$

so $\Delta\lambda = 0.044"$.

$\Delta\beta = -\Pi\cos(\lambda_S - \lambda)\sin\beta = 0$

So star is shifted 0.044" eastwards, by parallax.

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Positional Astronomy: Chapter 17
Aberration

Early attempts to measure the distances of the stars, by observing their parallactic ellipses, were unsuccessful because the stars are so far away, and their parallaxes are extremely small.
However, another effect was discovered instead: *aberration*.
This is caused by the fact that light moves at a finite velocity, c.

The apparent direction that light comes to us from a star is a combination of its true direction and the direction the Earth is moving.
Stars appear to be shifted slightly in the direction of the Earth's motion.
(This is analogous to the way a person walking through the rain has to hold their umbrella tilted forwards.)
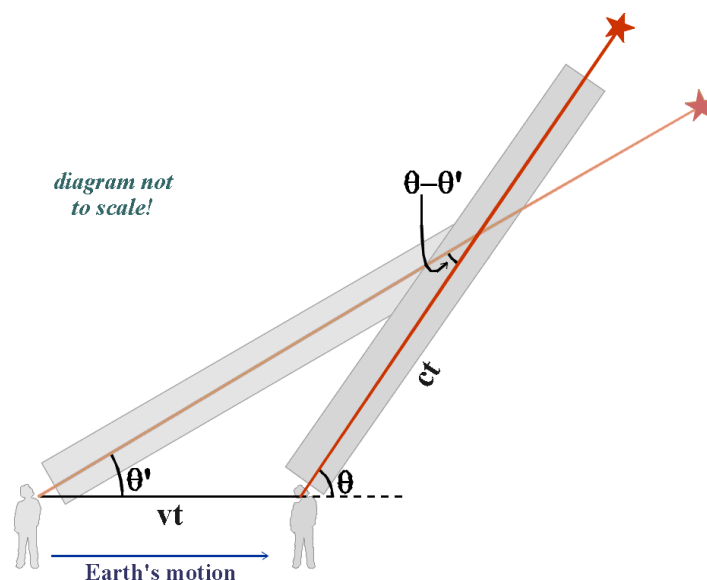
Take the Earth's velocity as v.
During a time-interval t,
Earth moves a distance vt, while light travels a distance ct down the telescope.
By plane trigonometry,
$\sin(\theta-\theta')/vt = \sin(\theta')/ct$
where θ is the true angle between the direction to the star, and the direction the Earth is moving around the Sun,
and θ' is the observed angle.

Since vt is very small compared to ct,
θ' is very nearly equal to θ.
So we may write $\sin(\theta-\theta')/vt = \sin(\theta)/ct$
i.e. $\sin(\theta-\theta') = \sin(\theta)\ v/c$

Because the ratio v/c is very small,
sin(θ-θ') is approximately equal to θ-θ' (in radians),
so we may write:
θ-θ' = sin(θ) v/c = k sin(θ)
where k, the **constant of aberration**, is 20.5 arc-seconds.
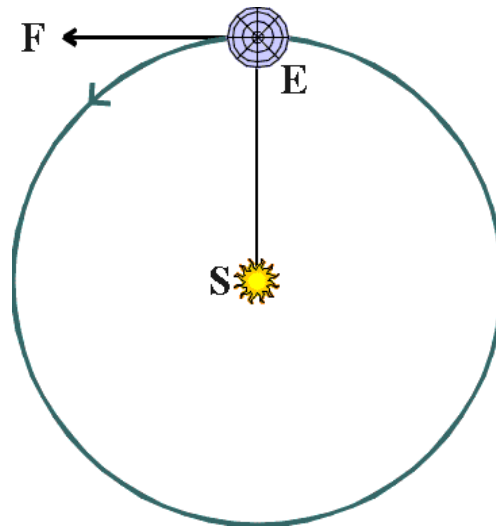
But in which direction is the Earth moving?
Taking the Earth's orbit as circular, the tangent is always at right-angles to the radius.
So the direction of the Earth's motion is always at 90° to the direction of the Sun.
Thus F, the "*apex of the Earth's way*", is on the ecliptic, 90° behind the Sun.
i.e. $\lambda_F = \lambda_S - 90°$.

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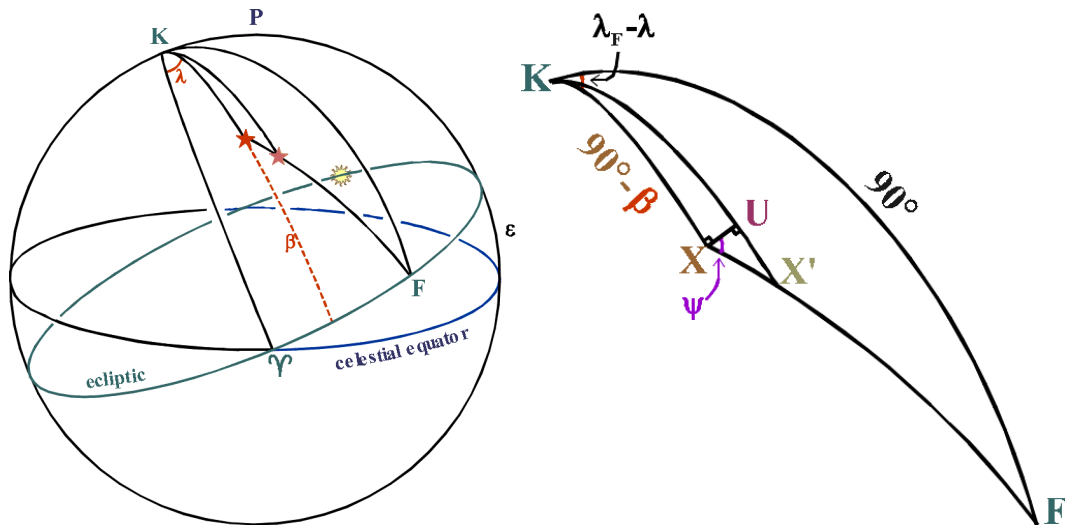
The geometry is very similar to the [parallax problem](#), with the following differences:

(i) we must write λ_F instead of λ_S .

(ii) $\theta-\theta'$ is now the aberrational shift $k \sin(\theta)$,
   not the parallactic shift $\Pi \sin(\theta)$,
   so we replace Π by k.

So we find:

$\Delta\lambda \cos(\beta) = k \sin(\lambda_F-\lambda) = -k \cos(\lambda_S-\lambda)$

$\Delta\beta = -k \cos(\lambda_F-\lambda) \sin(\beta) = - k \sin(\beta) \sin(\lambda_S-\lambda)$

Again this is the formula for an ellipse of the form:

$x = a \cos(\theta), \; y = b \sin(\theta)$

where θ is now temporary shorthand for $(\lambda_S-\lambda)$.

The **aberrational ellipse** has
semi-major axis k, parallel to the ecliptic, and semi-minor axis $k \sin(\beta)$,
perpendicular to the ecliptic.

There are two important differences between the parallactic and aberrational ellipses:

1) The aberrational ellipse is much bigger. (k is 20.5 arc-seconds, whereas parallax is always less than 1 arc-second.)
Also the major axis of the aberrational ellipse is the same for all stars, whereas the major axis of the parallactic ellipse depends on the star's distance.

2) The phase is different. When the Sun has the same longitude as the star, then the *longitude* shift is zero in the parallactic ellipse, but the *latitude* shift is zero in the aberrational ellipse.

So far, we have been assuming that the Earth's orbit is circular, and hence the value of $k = v/c$ is constant; in fact the orbit is elliptical, and this means the velocity v varies with time.
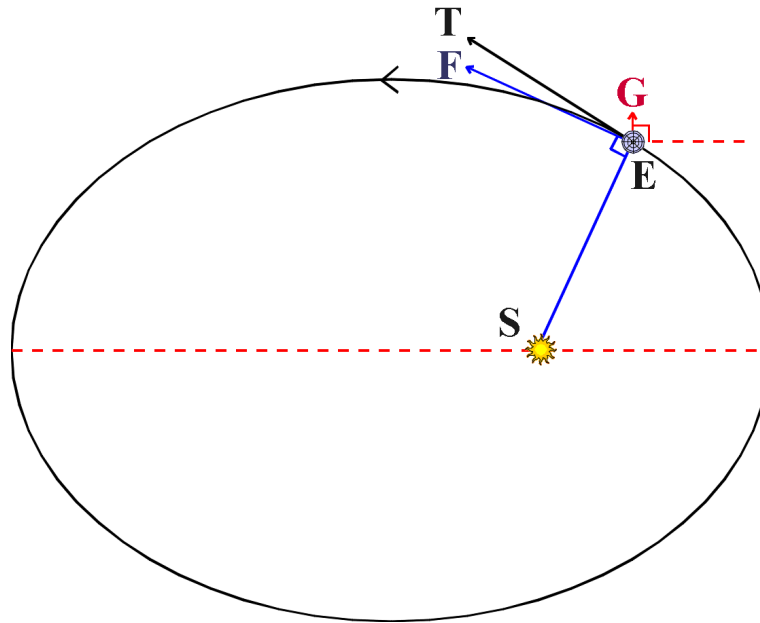
The velocity ET in any elliptical orbit can be resolved into two components:
  EF = h/p, perpendicular to the radius vector,
  EG = eh/p, perpendicular to the major axis of the ellipse.
The values of EF and EG are both constant.
It's the changing *angle* between these two constant components
which causes the orbital velocity to vary (Kepler's Second Law).

Here, EF is the velocity for a circular orbit, as assumed above.
EG adds second-order terms, 0.3 arc-seconds or less,
which are independent of Earth's position,
and depend only on the star's position.

A star itself also has its own **proper motion** across the sky,
but this is always small and generally not known,
so it is often ignored.

However, for objects within the solar system,
the motion *is* usually known, and is too large to ignore.
So astrometric observations of a planet have to be corrected for **light-time**:
During the time between the light is leaving the planet, and being measured on
Earth, the planet may have moved a significant distance.

Annual aberration and light-time are sometimes grouped together
and they are called **planetary aberration**,
in which case annual aberration alone is called **stellar aberration**.

Exercise:17

A star's true position is
Right Ascension 6h 0m 0s, declination 0° 0' 0".
On the date of the Spring Equinox,
how far will it appear to be shifted by aberration,
and in what direction?

First convert from RA and dec. into ecliptic coordinates.
This star has ecliptic longitude $\lambda = 90°$,
and ecliptic latitude $\beta = -\varepsilon = -23°26'$.

At the spring equinox, the Sun has ecliptic longitude $\lambda_S = 0°$
so $\lambda_S - \lambda = -90°$.

$\Delta\lambda \cos\beta = -k \cos(\lambda_S - \lambda) = 0$
so $\Delta\lambda = 0$.

$\Delta\beta = -k \sin(\lambda_S - \lambda) \sin\beta = 0$
where $k = 20.5"$ and $\beta = -23°26'$.
so $\Delta\beta = +8.15"$.

So star is shifted 8.15" northwards, by aberration.

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Positional Astronomy: Chapter 18
Precession

So far, this series of pages has considered how we assign coordinates to any point in the sky and the various physical effects that may alter its apparent position.
But there is a more profound problem with the way we determine coordinates, relative to the celestial equator and the ecliptic, since these are not permanently fixed.
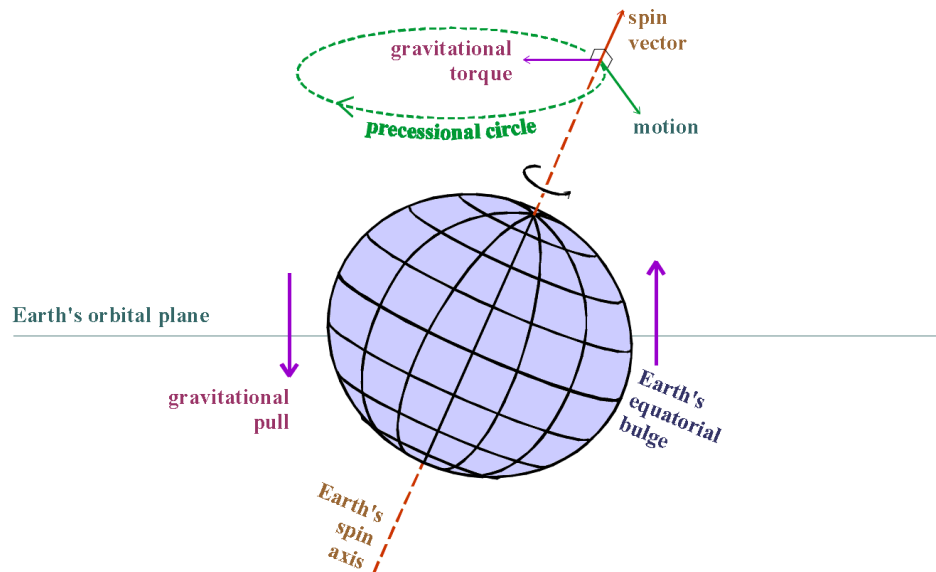
The Earth's axis is tilted to its orbital plane.
The gravitational pull of the Sun and the Moon on the Earth's equatorial bulge tend to pull it back towards the plane of the ecliptic.
Since the Earth is spinning, its axis *precesses*.
The North Celestial Pole traces out a **precessional circle** around the pole of the ecliptic, and this means that the **equinoxes precess** *backwards* around the ecliptic, at the rate of 50.35 arc-seconds per year (around 26,000 years for a complete cycle).

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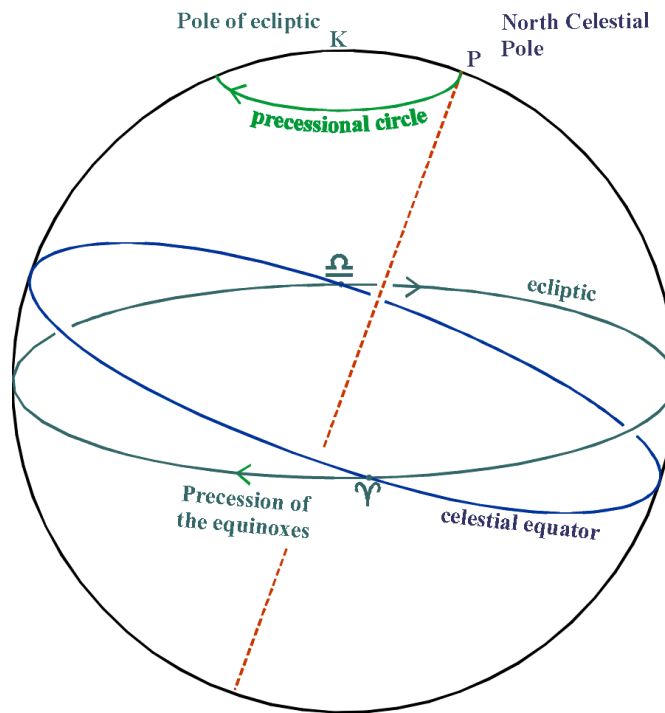
Around 2000 years ago,
the Sun was in the constellation of Aries at the spring equinox,
in Cancer at the summer solstice,
in Libra at the autumn equinox,
and in Capricorn at the winter solstice.
Precession means that all of these have changed,
but we still use the old names
(e.g. the First Point of Aries for the vernal equinox),
and the symbols for the vernal and autumnal equinoxes
are the astrological symbols for Aries and Libra.

Precession is caused by the Sun and the Moon.
However, the Moon does not orbit exactly in the ecliptic plane, but at an
inclination of about 5° to it.
The Moon's orbit precesses rapidly, with the nodes taking 18.6 years to
complete one circuit.
The lunar contribution to **luni-solar precession** adds a short-period, small-
amplitude wobble to the precessional movement of the North Celestial Pole.
This wobble is called **nutation**.

Ignoring nutation, Luni-solar precession simply adds 50.35 arc-seconds per year to the *ecliptic longitude* of every star, leaving the ecliptic latitude unchanged.

This definition assumes the ecliptic itself is unchanging.
In fact, the gravitational pull of the other planets perturbs the Earth's orbit and so it gradually changes the plane of the ecliptic.

If the equator were kept fixed, the movement of the ecliptic would shift the equinoxes forward along the equator by about 0.13 arc-seconds per year.

This is **planetary precession**, which decreases the *Right Ascension* of every star by 0.13 arc-seconds per year, leaving the declination unchanged.

Combining Luni-solar and planetary precessions gives **general precession**. (Lunar nutation and planetary precession also produce slight changes in the obliquity of the ecliptic)

Because of precession, our framework of Right Ascension and declination is constantly changing.
Consequently, it is necessary to state the **equator and equinox** of the coordinate system to which any position is referred.
Certain dates (*e.g.* 1950.0, 2000.0) are taken as **standard epochs**, and used for star catalogues etc.

To point a telescope at an object on a date other than its catalogue epoch, it is necessary to correct for precession.

Recall the formulae relating equatorial and ecliptic coordinates:
$$\sin(\delta) = \sin(\beta) \cos(\varepsilon) + \cos(\beta) \sin(\varepsilon) \sin(\lambda)$$
$$\sin(\beta) = \sin(\delta) \cos(\varepsilon) - \cos(\delta) \sin(\varepsilon) \sin(\alpha)$$
$$\cos(\lambda) \cos(\beta) = \cos(\alpha) \cos(\delta)$$

Luni-solar precession affects the ecliptic longitude λ.

The resulting corrections to Right Ascension and declination can be worked out by spherical trigonometry.
But here we use a different technique.

Consider luni-solar precession first, recalling that it causes λ to increase at a known, steady rate $d\lambda/dt$, while β and ε remain constant.

To find how the declination δ changes with time t, take the first equation and *differentiate* it:

$$\cos(\delta) \, d\delta/dt = \cos(\beta) \sin(\varepsilon) \cos(\lambda) \, d\lambda/dt$$

To eliminate β and λ from this equation, use the third equation:

$$\cos(\delta) \, d\delta/dt = \cos(\alpha) \sin(\varepsilon) \cos(\delta) \, d\lambda/dt$$

i.e. $d\delta/dt = \cos(\alpha) \sin(\varepsilon) \, d\lambda/dt$

To find how the Right Ascension α changes with time, take the second equation and differentiate it:

$$0 = \cos(\varepsilon) \cos(\delta) \, d\delta/dt + \sin(\varepsilon) \sin(\delta) \, d\delta/dt \sin(\alpha) - \sin(\varepsilon) \cos(\delta) \cos(\alpha) \, d\alpha/dt$$

i.e. $\sin(\varepsilon) \cos(\delta) \cos(\alpha) \, d\alpha/dt = d\delta/dt \, [\cos(\varepsilon) \cos(\delta) + \sin(\varepsilon) \sin(\delta) \sin(\alpha)] =$
$\cos(\alpha) \sin(\varepsilon) \, d\lambda/dt \, [\cos(\varepsilon) \cos(\delta) + \sin(\varepsilon) \sin(\delta) \sin(\alpha)]$

Cancelling out sin(ε) and cos(α) from both sides gives:

$$\cos(\delta) \, d\alpha/dt = d\lambda/dt \, [\cos(\varepsilon) \cos(\delta) + \sin(\varepsilon) \sin(\delta) \sin(\alpha)]$$

Dividing through by cos(δ) gives:

$$d\alpha/dt = [\cos(\varepsilon) + \sin(\varepsilon) \sin(\alpha) \tan(\delta)] \, d\lambda/dt$$

So if Δλ is the change in λ in a given time interval Δt, the corresponding changes in α and δ are

$$\Delta\alpha = \Delta\lambda \, [\cos(\varepsilon) + \sin(\varepsilon) \sin(\alpha) \tan(\delta)]$$
$$\Delta\delta = \Delta\lambda \cos(\alpha) \sin(\varepsilon)$$

This is the effect of Luni-solar precession.
We also have to add in the planetary precession, which decreases the RA by a quantity *a*, during the same time interval.

The combination is general precession:

$$\Delta\alpha = \delta\lambda \, [\cos(\varepsilon) + \sin(\varepsilon) \sin(\alpha) \tan(\delta)] - a$$
$$\Delta\delta = \Delta\lambda \cos(\alpha) \sin(\varepsilon)$$

To make this easier to calculate in practice, we introduce two new variables, m and n:

$$m = \Delta\lambda \cos(\varepsilon) - a$$
$$n = \Delta\lambda \sin(\varepsilon)$$

These quantities m and n are almost constant; they are given each year in the *Astronomical Almanac*.
The values for 2000 are approximately:

m = 3.075 seconds of time per year
n = 1.336 seconds of time per year = 20.043 arc-seconds per year

We can now write:

$$\Delta\alpha = m + n \sin(\alpha) \tan(\delta)$$
$$\Delta\delta = n \cos(\alpha)$$

which means that, if you know the equatorial coordinates of an object at one date, you can calculate what they should be at another date, as long as the interval is not too great (20 years or so).

If the object is a star whose proper motion is known, then that should be corrected for as well.

Alternatively, the *Astronomical Almanac* lists **Besselian Day Numbers** throughout the year.

Take a star's equatorial coordinates from a catalogue, and compute various constants from these, as instructed in the *Astronomical Almanac*.
Combine these with the Day Numbers for a given date, to produce the apparent position of the star, corrected for precession, nutation and aberration.
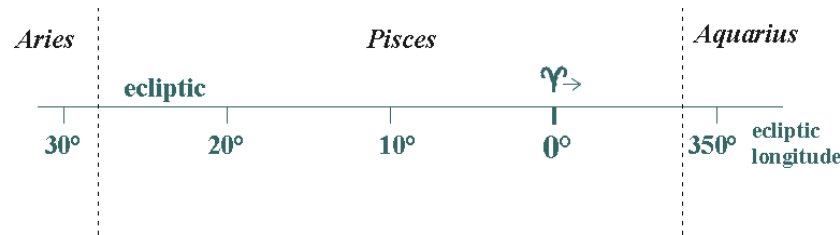
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Exercise: 18a

The Vernal Equinox occurs nowadays when the Sun is in the constellation of Pisces.

Pisces covers a section of the ecliptic from longitude 352° to longitude 28°; at longitude 28° the ecliptic passes into Aries.

How many years would we have to go back, to find the Sun at "the First Point of Aries" at the Vernal Equinox?

The equinoxes regress along the ecliptic at 50.35" per year.

So the " 0° " point moves westwards against the constellations.

It has moved 28° into Pisces, at a rate of 50.35" per year.

Divide 28° by 50.35" to get: 2002 years ago.

Exercise:18b

The coordinates of the Galactic North Pole are given officially as
$\alpha = 12h49m00s$, $\delta = +27°24'00"$,
relative to the equator and equinox of 1950.0.
What should they be,
relative to the equator and equinox of 2000.0?
(For this calculation, take the values of m and n for the year 1975:
m = 3.074s per year;
n = 1.337s per year = 20.049" per year.)
The formulae are:

$\quad \Delta\alpha = m + n \sin(\alpha) \tan(\delta)$

$\quad \Delta\delta = n \cos(\alpha)$

Substitute the coordinates given in the question:

$\quad \alpha_{1950} = 12h49m00s = 192.25°$,

$\quad \delta_{1950} = +27°24'00" = 27.40°$,

to get

$\quad \Delta\alpha = 2.927s$ per year, or +146.348s in 50 years.

$\quad \Delta\delta = -19.59"$ per year, or -979.63" in 50 years.

Add these to the 1950 values of α and δ, to get

$\quad \alpha_{2000} = 12h51m26s$

$\quad \delta_{2000} = +27°07'40"$

Positional Astronomy: Chapter 19
Calendars

The current **standard epoch** for star catalogues *etc*. is J2000.0; the previous one was B1950.0.

In this context, "B" signifies a **Besselian year**, which begins when the mean longitude of the Sun is exactly 280°; this always occurs very close to the start of the calendar year, but not always at the same instant.
For example, "B1950.0" represents the instant 1950 January 0.9235.

"J" signifies a **Julian year**, which is exactly 365.25 days long.
"J2000.0" represents midday on 2000 January 1, and every other Julian year begins at an exact multiple of 365.25 days from then.
In 1984 the International Astronomical Union recommended that star positions should be calculated on the basis of Julian years rather than Besselian ones.

Julian years are named for Julius Caesar, who is credited with the first reform of the calendar.
The year (more accurately, the "tropical year"), is measured from one spring equinox to the next, an interval of 365.2421988 mean solar days.
In the **Julian calendar**, most years have 365 days, with an extra day every fourth year (called a leap-year), thus averaging 365.25 days to a year, with an error of 1 day every 128 years.

By the 16th century, the accumulated error was 10 days. Pope Gregory XIII introduced the **Gregorian calendar**, where century years are only leap-years if they are divisible by 400; thus 1900 was not a leap-year, but 2000 was.
The Gregorian year thus averages 365.2425 days to a year, with an error of 1 day every 3320 years.
The extra 10 days were arbitrarily omitted. 1582 October 4th being followed by 1582 October 15th.
The Gregorian calendar was adopted in different countries at various different dates over the next 350 years.

To avoid complications in calculating calendar dates, astronomers number days in a continuous sequence called the **Julian Date (JD)**.
(This system was devised by the French astronomer Joseph B. Scaliger, in 1582; he named it, not after Julius Caesar, but after his father - who was called Julius Caesar Scaliger.)

The sequence of Julian Days starts on January 1st, 4713 BC, so that all known astronomical records have positive values of JD.

The Julian date changes at *midday*, so that (in European countries) all observations on a particular night have the same Julian date.
Midday on 2000 January 1st was JD 2451545.0.

JD must be given to 5 decimal places for an accuracy of 1 second of time.
A number in this format can cause problems in computing, so many modern applications use the **Modified Julian Date** (**MJD**), where:

 Modified Julian Date (**MJD) =** MJD = JD - 2400000.5

 That means: MJD, like the calendar date, changes at midnight. 0h on 2000. January 1st was MJD 51544.0 .

Exercise:19

When the Gregorian calendar was introduced in 1582,
the accumulated error in the Julian calendar was 10 days.
What was the error by the time the Gregorian calendar was adopted in Turkey, in 1927?

1600 was a leap-year in both calendars,
but 1700, 1800 and 1900 were not leap-years in the Gregorian calendar.
Thus the Julian calendar was 13 days adrift, by 1927.

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Positional Astronomy: Chapter 20

Exercise: 20 Final.

You are lost on a desert island with a sextant, a chronometer, a carrier pigeon,
and your copy of Smart's *Spherical Astronomy.*
Explain how you will save yourself.
(Assume that the chronometer is keeping GMT, and that you know the date.)

Step 1: determine your latitude.
There are (at least) two possible techniques.

1. Measure the altitude of Polaris above the northern horizon, using the sextant.
This is approximately equal to your latitude.
(Polaris, the "North Star", lies very close to the North Celestial Pole.)

There are various problems with this.
   Firstly, if you are in the southern hemisphere, Polaris will be below the horizon!
   Secondly, you need to carry out the measurement in nautical twilight, while it is still
light enough to see the horizon, and Polaris is only a second-magnitude star, so it may
not appear bright enough to measure accurately.
   Thirdly, Polaris does not lie exactly at the North Celestial Pole, so your result could
be nearly 1 degree in error.

2. So, as an alternative: measure the altitude of the Sun at midday, using the sextant.

Knowing the date, calculate the declination of the Sun (it varies sinusoidally, with a
period of 1 year starting at the spring equinox, and an amplitude of 23.4 degrees.)

The midday altitude, when the Sun is on the local meridian, is composed of: the height
of the celestial equator above the southern horizon (equal to the co-latitude) plus the
height of the Sun above the celestial equator (its declination). (If you are in the
southern hemisphere, the celestial equator will be closer to the *northern* horizon; in this
case its distance from the *southern* horizon, the co-latitude, will be greater than 90°.)
Knowing the altitude and the solar declination, calculate the co-latitude and hence the
latitude.

If the sextant can be read to an accuracy of a few arc-minutes, you should correct your
reading for refraction.
The apparent zenith angle of an object z' is greater than its true zenith angle z by the
value $k\ tan(z')$, where k is approximately 1 arc-minute.

Step 2: Determine your longitude.
Again there are (at least) two possible techniques.

1. During nautical twilight, if you can locate a star whose celestial coordinates you know, measure its altitude above the horizon using the sextant, and note the time (GMT) using the chronometer.

Knowing the star's altitude, its declination, and your latitude (previously determined), calculate its Hour Angle by applying the cosine rule to "the" Astronomical Triangle.

Knowing the star's Right Ascension, calculate the local sidereal time of the observation (Local Hour Angle = Local Sidereal Time - Right Ascension).

Knowing the date, calculate the Greenwich Sidereal Time corresponding to the Greenwich Mean Time of the observation.
GST is equal to GMT at the autumn equinox, and GST runs faster than GMT by one day in 365.25 days.

The difference between the Local Sidereal Time (from your observation) and Greenwich Sidereal Time (from the chronometer) is your longitude east or west of Greenwich.

2. Failing a star with known coordinates, use the Sun. Note the time (GMT) when it reaches its greatest altitude: this is midday, Local Apparent Time.

Use the formulae given in Smart's *Spherical Astronomy* to calculate the Equation of Time on that date.
(Or derive it from first principles: allow firstly for the non-uniform motion of the Sun around the ecliptic (Kepler's Second Law); then allow for the fact that the ecliptic is tilted to the equator.)

Add or subtract the Equation of Time to your Local Apparent Time, to obtain Local Mean Time.
The difference between Local Mean Time and GMT is your longitude east or west of Greenwich.

Step 3:
Tear a strip of paper from the title-page of Smart's *Spherical Astronomy* to write a message giving your latitude and longitude.
Launch it by carrier-pigeon and wait to be rescued!

This question formed part of the final exam at UCLA in 1961.
(Trimble, V., "The Observatory" **118**, *32, 1998).*

Bibliography

This course is largely based on the following text-books:

GREEN, Robin M., *Spherical Astronomy* (CUP, 1985)

McNALLY, D., *Positional Astronomy* (Muller, 1974)

SMART, W.M., *Text-book on Spherical Astronomy* (CUP, 1965)

There are other good text-books available; they should all have Library of Congress classifications around class mark QB145.

Also look at:

The Astronomical Almanac

Norton's Star Atlas or *Norton's Star Atlas 2000*

Fiona Vincent

Was brought up in Edinburgh, and studied astronomy at the <u>University of St.Andrews</u>, as an undergraduate and a postgraduate. Having done a variety of jobs, including a spell with the <u>BBC World Service</u> in London. From 1982 to 1994 she held the post of City Astronomer in Dundee, looking after the <u>Mills Observatory</u>, which at the time was Britain's only full-time public observatory.

Fiona Vincent is an honorary member of staff at St.Andrews University, in the <u>School of Physics and Astronomy</u>. Her chief research interest has been in asteroids and other small solar-system objects. From 1996 to 2003 she was an active member of the team taking the St.Andrews mobile <u>planetarium</u> on trips to schools, during which time presented nearly 200 shows. In addition, she provided monthly details about <u>What's in the Sky</u>; Fiona Vincent is no longer updating this site every month, but the site still contains general information about the apparent behaviour of the sun, moon and stars. In February 1998, she gave a short series of lectures on "Positional Astronomy" for second-year students at St.Andrews University; the <u>notes</u> she prepared for these lectures were one of the first available as a source of such information on the Internet.

Fiona Vincent acted as Tutor, or as Study Adviser, on several astronomy courses for the <u>Open University</u>; among others the third-level course on <u>*Astrophysics*</u>, which includes a remote-observing project at the <u>Observatori Astronomic de Mallorca</u>. In 2007 she started training as a keep-fit teacher with <u>Medau Movement</u>, and leading a <u>weekly exercise class</u> for older women in St.Andrews.

Fiona Vincent lives in St.Andrews and she share her name with a <u>starship</u>.

Page last modified 2015 January 26th.

<u>St. Andrews University Astronomy Group Home Page.</u>

Document converted to . pdf format by Alfonso Pastor in 2005 as a tribute to Fiona Vincent in order to allow this extraordinary lectures to be loaded and read in eBooks without the need of an Internet access.

Pdf revised on January 2015