Problem:

Students are free to choose some different images for experimenting with the following requirements:

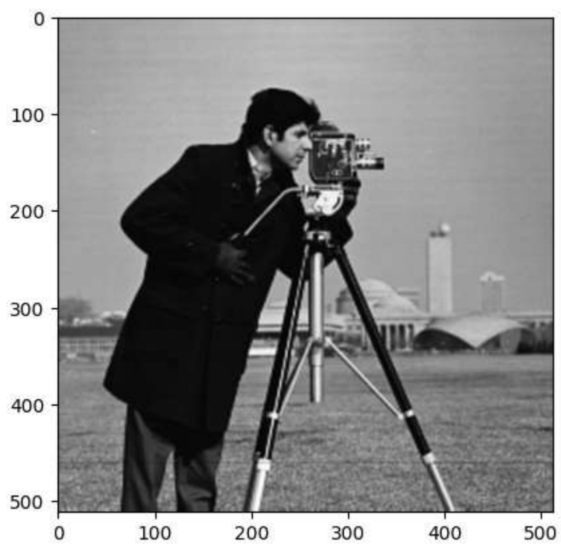
- Load an image.
- Do low-pass filter.
- Do high-pass filter.
- Write report.

Load image

```
import cv2 as cv2
import numpy as np
import matplotlib.pyplot as plt
```

```
image = cv2.imread('input/cameraman.tif',cv2.IMREAD_GRAYSCALE)
```

```
# cv2.imshow(winname='test',mat=image)
plt.imshow(image, cmap='gray')
plt.show()
```



```
image.shape
```

```
(512, 512)
```

Theories

Cross-correlation

Let F be the image, H be the kernel (of size $2k+1$ x $2k+1$), and G be the output image:

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v]F[i+u,j+v]$$

This is called a **cross--correlation** operation:

$$G = H \otimes F$$

Convolution

A **convolution** operation is a cross-correlation where the filter is flipped both horizontally and vertically before being applied to the image:

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v] F[i-u][j-v]$$

. It is written:

$$G = H \star F$$

Fourier Transform

Brief Description

The Fourier Transform is an important image processing tool which is used to decompose an image into its sine and cosine components. The output of the transformation represents the image in the *Fourier* or *frequency domain*, while the input image is the *spatial domain* equivalent. In the Fourier domain image, each point represents a particular *frequency* contained in the spatial domain image.

We are only concerned with digital images, we will restrict this discussion to the Discrete Fourier Transform (DFT)

The DFT is the sampled Fourier Transform and therefore does not contain all frequencies forming an image, but only a set of samples which is large enough to fully describe the spatial domain image. The number of frequencies corresponds to the number of pixels in the spatial domain image, i.e. the image in the spatial and Fourier domain are of the same size.

For a square image of size N x N, the two-dimensional DFT is given by:

$$f(u,v) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f(i,j) e^{-\mathbf{i}2\pi(\frac{ui}{N} + \frac{vj}{N})}$$

where $f(a,b)$ is the image in the spatial domain and the exponential term is the basis function corresponding to each point $F(u,v)$ in the Fourier space. The equation can be interpreted as: **the value of each point $F(u,v)$ is obtained by multiplying the spatial image with the corresponding base function and summing the result**.

In a similar way, the Fourier image can be re-transformed to the spatial domain. The inverse Fourier transform is given by:

$$f(a,b) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) e^{\mathbf{i}2\pi(\frac{ua}{N} + \frac{vb}{N})}$$

with $\frac{1}{N^2}$ is the normalization term in the inverse transformation. This normalization is sometimes applied to the foward transform instead of the inverse transform, but it should not be used for both.

To obtain the result for the above equations, a **double sum** has to be calculated for **each image point**. However, because the Fourier Transform is *separable*, it can be written as

$$F(u,v) = \frac{1}{N} \sum_{b=0}^{N-1} P(u,b) e^{-\mathbf{i}2\pi\frac{vb}{N}}$$

where

$$P(u,b) = \frac{1}{N} \sum_{a=0}^{N-1} f(a,b) e^{-\mathbf{i}2\pi\frac{ua}{N}}$$

The Fourier TRansform produces a complex number valued output image which can be displayed with two images, either with the *real* and *imaginary* or the *magnitude* and *phase*. In **image processing**, ofthen only the magnitude of the Fourier Transform is displayed, as it *contains most of the information of the geometric structure of the spatial domain image*. However, if we want to **re-transform** the Fourier image into the correct spatial domain after some processing in the frequency domain, we must make sure to **preserver both magnitude and phase of the Fourier image**

Frequency Filter

Brief Description

Frequency filters process an image in the frequency domain. The image is Fourier transformed, multiplied with the filter function and then re-transformed into the spatial domain. **Attenuating high frequencies results in a smoother image in the spatial domain, attenuating low frequencies enhances the edges.**

Frequency filtering is based on the Fourier Transform. The operator usually takes an image and a filter function in the Fourier domain__. This image is then multiplied with the filter function in a **pixel-by-pixel fashion**:

$$G(u, v) = F(u, v) * H(u, v)$$

with $H(u, v)$ is the filter function, $F(u, v)$ is the input image in the Fourier domain. To resulting image in the **spatial domain**, $G(u, v)$ has t be re-transformed using the inverse Fourier Transform.

Since the **multiplication in the Fourier space** is **identical** to **convolution in the spatial domain**, all frequency filters can in theory be implemented as a spatial filter. However, in practice, the Fourier domain filter function can only be approximated by the filtering kernel in spatial domain.

Ideal Filter

The most simple low-pass filter is the **ideal low-passs filter**

The ideal low-pass filter suppresses all frequencies higher than the *cut-off frequency* D_0 and leaves smaller frequencies unchaged:

$$H(u, v) = \begin{cases} 1 & if D(u, v) \leq D_0 \\ 0 & if D(u, v) > D_0 \end{cases}$$

with $D(u, v)$ is just a distance from point (u, v) to original point of the frequency plane.

$$D(u, v) = \sqrt{u^2 + v^2}$$

Similarly, the **ideal high-pass filter (IHPF)** is defined as:

$$H(u, v) = \begin{cases} 0 & if D(u, v) \leq D_0 \\ 1 & if D(u, v) > D_0 \end{cases}$$

After the kernel is constructed, we applied to the image in frequency domain, F, to get the filtered image G.

$$G(u, v) = F(u, v) \star H(u, v)$$

with element-wise multiplication.

Finally, the filtered image in spatial domain is calculated as follows:

$$g(x, y) = G^{-1}(u, v)$$

with G^{-1} is the inverse FFT

Butterworth Filter

A two-dimensional ***Butterworth low-pass filter (BLPF)*** of order n is a low-pass filter whose kernel is defined as:

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$

Similarly, the ***Butterworth high-pass filter (IHPF)*** is defined as:

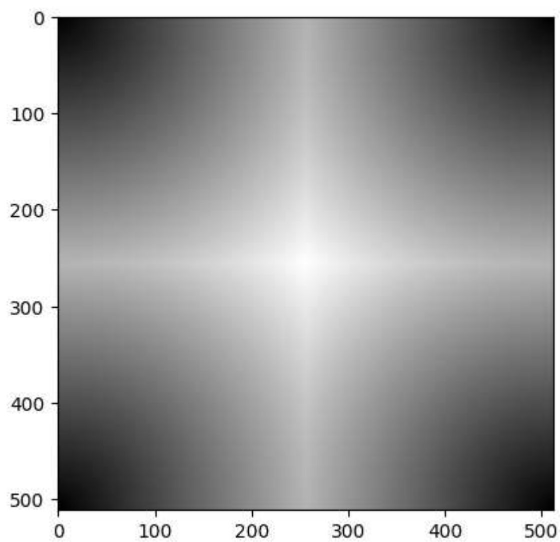
$$H(u, v) = \frac{1}{1 + [D_0/D(u, v)]^{2n}}$$

Implementation

```
def constructDuv(N):
    """Constructs the frequency matrix, D(u,v), of size NxN"""
    u = np.arange(N)
    v = np.arange(N)

    idx = np.where(u>N/2)[0]
    # print(idx)
    u[idx] = u[idx] - N
    idy = np.where(v>N/2)[0]
    # print(idy)
    v[idy] = v[idy] - N

    [V,U] = np.meshgrid(v,u)
    # print(V)
    # print(U)
    D = np.sqrt(U**2 + V**2)

    return D
```

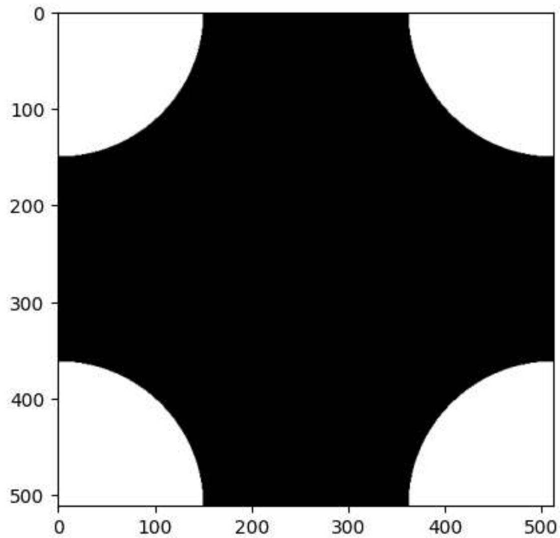
```
# illustration result
D = constructDuv(512)
print(D)
# print(D[4,3])
plt.imshow(D,cmap='gray')
plt.show()
```

```
[[0.         1.         2.         ... 3.         2.         1.        ]
 [1.         1.41421356 2.23606798 ... 3.16227766 2.23606798 1.41421356]
 [2.         2.23606798 2.82842712 ... 3.60555128 2.82842712 2.23606798]
 ...
 [3.         3.16227766 3.60555128 ... 4.24264069 3.60555128 3.16227766]
 [2.         2.23606798 2.82842712 ... 3.60555128 2.82842712 2.23606798]
 [1.         1.41421356 2.23606798 ... 3.16227766 2.23606798 1.41421356]]
```




```
def computeIdealFiltering(D, Do, mode=0):
    """Computes Ideal Filtering based on the cut off frequency (Do).
    If mode=0, it compute Lowpass Filtering otherwise Highpass filtering
    """
    H = np.zeros_like(D)
    if mode==0:
        H = (D<=Do).astype(int)
    else:
        H = (D>Do).astype(int)
    return H
```

```
plt.imshow(computeIdealFiltering(D, 150, 0),cmap='gray')
plt.show()
```




```
import time

def computeIdealFilters(image, F, D, Dos):
    """
    Computes Ideal Filtering for different cut-off frequencies.
    """
    # low-pass filtered images
    LPgs = []

    # High-pass filtered images
    HPgs = []

    # Running Time
    IRunningTime = []

    for Do in Dos:
        starttime = time.time()

        # Computes Lowpass Filtering (ILPF)
        H = computeIdealFiltering(D, Do, 0)

        IRunningTime.append(time.time() - starttime)

        # Compute the filtered images (result in space domain)
        LPgs.append(computeFilteredImage(H, F))

        # Computes Highpass Filtering (IHPF)
        H = computeIdealFiltering(D, Do, 1)

        # Compute the filtered images (result in space domain)
        HPgs.append(computeFilteredImage(H, F))

    return LPgs, HPgs, IRunningTime
```

```
def computeButterworthFiltering(D:np.ndarray, Do, n, mode = 0):
    """Computes Ideal Filtering based on the cut-off frequency and order n.
    """
    H = np.zeros_like(D)
    D = D.astype(float)

    if mode == 0:
        H = 1 / (1 + (D/Do)**(2*n))
    else:
        H = 1 / (1 + (Do/D)**(2*n))
    return H
```

```
def computeButterFilters(image, F, D, Dos, ns):
    """Compute Butterworth Filtering for different cut-off frequencies"""

    # Low-pass filtered images
    LPgs = []

    # High-pass filtered images
    HPgs = []

    # Running time
    BRunningTime = []

    for index,Do in enumerate(Dos):
        startTime = time.time()
        H = computeButterworthFiltering(D, Do, ns[index], 0)
        BRunningTime.append(time.time() - startTime)
        LPgs.append(computeFilteredImage(H, F))

        H = computeButterworthFiltering(D, Do, ns[index], 1)
        HPgs.append(computeFilteredImage(H,F))

    return LPgs, HPgs, BRunningTime
```

```
def computeFilteredImage(H, F):
    """Computes a filtered image based on the given fourier transformed image(F) and filter(H)"""
    G = H * F  #! element-wise
    g = np.real(np.fft.ifft2(G)).astype(int)

    return g
```

```
def visualizeFilteringResults(image, F, LPgs, HPgs, Dos, filterType = 'Ideal', ns=None):
    """Visualizes the filtered images using different cut-off frequencies."""

    fig, axarr = plt.subplots(2,5, figsize=(10,5))

    axarr[0,0].imshow(image, cmap=plt.get_cmap('gray'))
    axarr[0,0].set_title('Original Image')
    axarr[0,0].axes.get_xaxis().set_visible(False)
    axarr[0,0].axes.get_yaxis().set_visible(False)

    axarr[1, 0].imshow(image, cmap=plt.get_cmap('gray'))
    axarr[1, 0].set_title("Original Image")
    axarr[1, 0].axes.get_xaxis().set_visible(False)
    axarr[1, 0].axes.get_yaxis().set_visible(False)

    # Display the results
    for index, g in enumerate(LPgs):

        if filterType=='Ideal':
            lp = 'ILPF (Do=%i)'%Dos[index]
            hp = 'IHPF (Do=%i)'%Dos[index]
        else:
            lp = 'BLPF (Do=%i, n=%i)'%(Dos[index],ns[index])
            hp = 'BHPF (Do=%i, n=%i)'%(Dos[index],ns[index])

        axarr[0,index+1].imshow(LPgs[index], cmap=plt.get_cmap('gray'))
        axarr[0,index+1].set_title(lp)
        axarr[0,index+1].axes.get_xaxis().set_visible(False)
        axarr[0,index+1].axes.get_yaxis().set_visible(False)

        axarr[1,index+1].imshow(HPgs[index], cmap=plt.get_cmap('gray'))
        axarr[1,index+1].set_title(hp)
        axarr[1,index+1].axes.get_xaxis().set_visible(False)
        axarr[1,index+1].axes.get_yaxis().set_visible(False)

    plt.show()
```

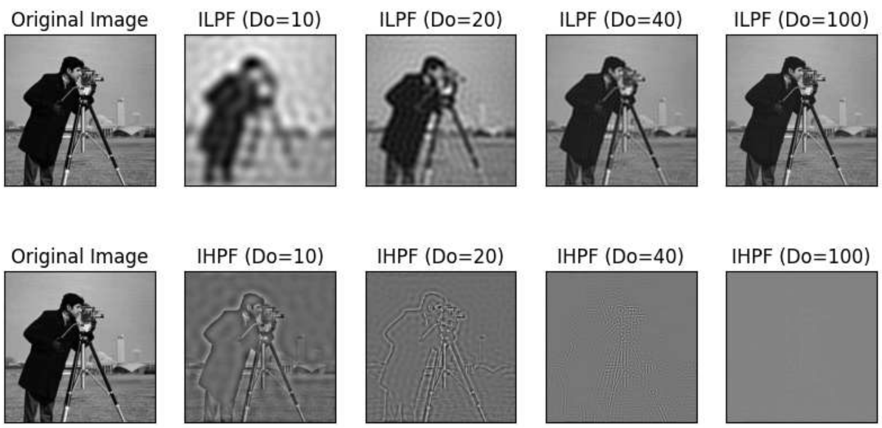
Applying the Filters

```
F = np.fft.fft2(image)
Dos = np.array([10,20,40,100])

# Ideal Filtering
ILPgs, IHPgs, IRunningTime = computeIdealFilters(image, F, D, Dos)

# Butterworth Filtering
ns = np.ones(Dos.shape[0])*2
BLPgs, BHPgs, BRunningTime = computeButterFilters(image, F, D, Dos, ns)
```

```
<ipython-input-13-fbab58d6982d>:10: RuntimeWarning: divide by zero encountered in divide
  H = 1 / (1 + (Do/D)**(2*n))
```

```
print('Ideal Filtering')
visualizeFilteringResults(image, F, ILPgs, IHPgs, Dos, 'Ideal')
```

```
Ideal Filtering
```




```
print('Butterworth Filtering')
visualizeFilteringResults(image, F, BLPgs, BHPgs, Dos, 'Butterworth', ns.astype(int))
```

```
Butterworth Filtering
```