MAX-consensus in open multi-agent systems with gossip interactions

Mahmoud Abdelrahim, Julien M. Hendrickx and W.P.M.H. Heemels

Abstract— We study the problem of distributed maximum computation in an open multi-agent system, where agents can leave and arrive during the execution of the algorithm. The main challenge comes from the possibility that the agent holding the largest value leaves the system, which changes the value to be computed. The algorithms must as a result be endowed with mechanisms allowing to forget outdated information. The focus is on systems in which interactions are pairwise gossips between randomly selected agents. We consider situations where leaving agents can send a last message, and situations where they cannot. For both cases, we provide algorithms able to eventually compute the maximum of the values held by agents.

I. INTRODUCTION

Multi-agent systems involve interacting elements with computing capabilities, also called agents or nodes, who communicate with each other to achieve a collective control task that is more difficult or sometimes even impossible to be performed by an individual agent. This configuration of multi-agent systems has a great benefit to model and solve many problems in different fields of applications including sensor networks [1]–[3], computer networks [4], [5] and social science [6]–[8]. One of the common problems that has been studied in these applications is the consensus of multi-agent systems on aggregate functions such as, e.g., MIN, MAX, SUM and AVERAGE. For instance, in a group of distributed sensors, it can be required to compute the average temperature of a specific region or to elect the sensor with maximum power resource to preserve the communication over a costly link or to reduce energy for a wireless sensor network, see, e.g., [9]–[11].

Most existing results of the literature rely on the assumption that the system composition is static, i.e., the set of agents present in the system does not change after the initial time, see, e.g., [10], [12]–[14] and the references therein.

Mahmoud Abdelrahim is with the Department of Mechanical Engineering, Assiut University, Egypt. Julien Hendrickx is with the ICTEAM institute, Université catholique de Louvain, Louvain-la-Neuve, Belgium. Maurice Heemels is with the Control Systems Technology Group, Department of Mechanical Engineering, Eindhoven University of Technology, the Netherlands. Email addresses: `m.abdelrahim@aun.edu.eg`, `julien.hendrickx@uclouvain.be`, `m.heemels@tue.nl`.

However, this requirement can be difficult to satisfy in some implementation scenarios where new agents can join and/or existing agents can leave the network at any time instant. This phenomenon is known in the literature as "*network churn*" [15], [16], "*dynamic network*" [17]–[19] or "*open multi-agent systems*" [20]–[22]. In this case, the consensus problem on aggregate estimates becomes more challenging to handle compared to the case of static networks. For instance, consider the paradigmatic problem of MAX-consensus with distributed communications and assume that the agent with the largest state value has left the network after all the agents have converged to its state value. In this case, all the existing nodes in the network will then hold outdated information. This scenario cannot occur in static networks, which highlights one of the inherent challenges of open multi-agent systems.

In this paper, we investigate the problem of MAX-consensus in open multi-agent systems with distributed communications. The agents are assumed to be anonymous, do not have global identifiers, and all run the same algorithm. We further assume that interactions only occur via pairwise "gossip" exchanges between randomly selected agents in the sense that, at any (discrete) time instant, (only) two agents are selected randomly to exchange their information, update their MAX estimates and possibly other variables. To cope with the dynamic nature of the network, two different solutions are proposed depending on whether or not it is possible for the leaving agents to announce their departures.

In the case in which announcements are made, our algorithm relies on a variable that describes how "up-to-date" agents are with respect to recent departures, and priority is given to information coming from the most "up-to-date" agents. In the case where agents disappear without sending a last message, our algorithm maintains an estimate of the age of the information, and estimates corresponding to information deemed too old are discarded. We will show that our two approaches ensure that outdated information can be forgotten, and that consensus on the MAX value can be achieved (with high probability) if the system composition stops evolving.

The problem of MAX-consensus in multi-agent systems has been studied in, e.g., [10], [11], [23], [24]. Among existing techniques, the work of [10] has considered MAX-consensus with random gossip interactions between agents, on a *static* network. Compared to existing works of the literature, our result is adapted to the problem of MAX-

consensus in multi-agent systems when the network is *open*, which has not been considered in the previously mentioned works. Our proposed approach encompasses the result of pairwise gossip interaction in static networks in [10] as a particular case.

The remainder of the paper is organized as follows. Notations are given in Section II. The problem is formulated in Section III. In Section IV, we treat the case where leaving agents send a last message, and in Section V, we treat the case where they do not. Numerical simulations are given in Section VI. Conclusions and discussions are provided in Section VII. The proofs of our results are ommited for space reasons, and are available in [25].

II. NOTATION

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geqslant 0} := [0, \infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\mathbb{N}_{\geqslant 1} := \{1, 2, \ldots\}$ and $\mathbb{N}_{\geqslant T} := \{T, T+1, \ldots\}$ for $T \in \mathbb{N}$. We denote by $\mathbf{0}_n$ and $\mathbf{1}_n$ the vectors in $\mathbb{R}^n$ whose all elements are 0 or 1, respectively. We write A^T to denote the transpose of A, and $(x, y) \in \mathbb{R}^{n_x + n_y}$ to represent the vector $[x^T, y^T]^T$ for $x \in \mathbb{R}^{n_x}$ and $y \in \mathbb{R}^{n_y}$. The symbol $\mathbb{I}_n$ stands for the identity matrix of dimension n. For a random variable R, the symbol $\mathbb{E}(R)$ denotes the expectation of R.

III. PROBLEM STATEMENT

Consider a connected time-varying graph $\mathcal{G}(t) = (\mathcal{V}(t), \mathcal{E}(t))$, where $\mathcal{V}(t)$ and $\mathcal{E}(t)$ denote, respectively, the set of existing agents and the set of edges in the graph at time $t \in \mathbb{N}$. The graph $\mathcal{G}(t)$ is dynamic in the sense that new agents can join and/or existing agents can leave at any time t. Hence, the cardinality of $\mathcal{V}(t)$, denoted by $\mathcal{N}(t)$, is not necessarily constant for all $t \in \mathbb{N}$. The agents communicate with each other in a *pairwise* randomized gossip fashion [9]. In other words, at any time instant $t \in \mathbb{N}$, there are three possibilities: (i) an agent joins the system and $\mathcal{N}(t+1) = \mathcal{N}(t) + 1$, (ii) an agent leaves the system and $\mathcal{N}(t+1) = \mathcal{N}(t) - 1$, or (iii) two randomly selected agents $i, j \in \mathcal{V}(t), i \neq j$ communicate with each other (Note that these discrete-time instants may be interpreted as the sampling of an asynchronous process at those times where an event occurs). Joining agents are assumed to know that they join the system. Leaving agents may or may not be able to send one last message (to one other agent) before leaving, which are two cases of interest, which will be discussed in Section IV and V, respectively.

Every agent i has two special states: $x_i \in \mathbb{R}$ is its intrinsic value, which is constant and determined arbitrarily when joining the system, and $y_i(t)$ is its estimated answer at time $t \in \mathbb{N}$ for the MAX value. Our goal is to estimate the maximum intrinsic value of all the agents present in the system, so we would ideally want, when no more agents are joining or leaving the network after time $T \in \mathbb{N}$, that there is a time $T^* \in \mathbb{N} \geqslant T$ such that $y_i(t) = \text{MAX}(T) := \max_{j \in \mathcal{V}(t)} x_j$, for all $i \in \mathcal{V}(t)$ and for $t \in \mathbb{N} \geqslant T^*$. Agents may then have other states that they use to reach this goal.

If the network would be static, i.e., $\mathcal{G}(t)$ is time-invariant, the estimation of the maximum could be achieved in finite time by starting from $y_i(0) = x_i(0)$ for every agent, and setting $y_i(t+1) = y_j(t+1) = \max(y_i(t), y_j(t))$ whenever agents i and j interact at time $t \in \mathbb{N}$, see, e.g., [10]. The main challenge in a dynamic or open network lies with the need for the algorithm to take new agents into account and to eventually discard information related to agents no longer present in the system to ensure that $\max_j x_j$ is eventually recovered once the system composition stops evolving. Classical algorithms such as that in [10] do not guarantee this: outdated values from agents no longer in the system may never be discarded.

Note that an alternative and maybe more natural goal would be to have the $y_i(t), i \in \mathcal{V}(t)$ track $\text{MAX}(t) = \max_{j \in \mathcal{V}(t)} x_j$ sufficiently accurately. This more ambitious goal is left for future studies, see Section VII for further discussions on this issue.

Finally, we chose to make the following assumption for the sake of simplicity of exposition.

Assumption 1. *The graph* $\mathcal{G}(t) = (\mathcal{V}(t), \mathcal{E}(t))$ *is complete for all* $t \in \mathbb{N}$. ☐

This means that every pair of distinct agents in the network can communicate directly with each other. The algorithms we develop would actually also work on general dynamic graphs under suitable connectivity assumptions, but the analysis would be more complex.

IV. DEPARTURES ARE ANNOUNCED

A. Algorithm description

If leaving agents announce their departures (to one other agent), then we can benefit from this knowledge to correct the outdated information. For that purpose, we introduce an auxiliary variable $\kappa_i(t) \in \mathbb{N}$ at each agent, meant to represent the "level of information" available to i about the departures up to time $t \in \mathbb{N}$. It will in general *not be equal to the actual number of departures, nor converge to it*. The algorithm is designed to ensure that those with the largest value κ_i have valid estimates, i.e. their $y_i(t)$ correspond to the x_j of agents present in the system. For this purpose, information coming from agents with higher κ_i will be given priority over information coming from agents with lower values, and it will be made sure that agents with a lower value of κ_i will never have influenced those with a higher value.

The algorithm is summarized as follows. Initially, every existing agent at $t = 0$ sets $y_i(0) = x_i$ and $\kappa_i = 0, i \in \mathcal{V}(0)$, as shown in Algorithm 1.

Algorithm 1 Intialization algorithm

At time $t = 0$, every existing node $i \in \mathcal{V}(0)$ initializes its state as
1: $y_i(0) = x_i$
2: $\kappa_i(0) = 0$

When a new agent n joins the group at time $t \in \mathbb{N}$, it initializes its counter $\kappa_n(t)$ and its estimate $y_n(t)$ according to Algorithm 2.

4754

Algorithm 2 Joining algorithm

Assume at any time $t \in \mathbb{N}_{\geqslant 1}$, a new agent n wants to join, i.e., $\mathcal{V}(t+1) = \mathcal{V}(t) \cup \{n\}$. Agent n initializes its state as
1: $y_n(t) = x_n$
2: $\kappa_n(t) = 0$

If an agent ℓ leaves the system, it sends a last message containing its counter value κ_ℓ to a randomly selected agent m. The reaction of m is governed by Algorithm 3, which can be interpreted as follows: If the counter $\kappa_\ell(t)$ of the leaving agent ℓ is less than $\kappa_m(t)$, then agent m ignores this departure since the information of ℓ is deemed less up-to-date than its own and has not influenced it. On the other hand, if $\kappa_\ell(t) \geqslant \kappa_m(t)$, then ℓ may have influenced m and possibly agents i with values κ_i higher than κ_m, but no larger than κ_ℓ. To ensure that none of the agents with the highest values κ hold the now outdated value x_l, m will reset its y_m to x_m, which is by definition a valid value, and set its κ_m to $\kappa_\ell + 1$, a value above that of all those who could have been influenced by ℓ.

Algorithm 3 Departure algorithm

Assume at time $t \in \mathbb{N}$, agent ℓ leaves, i.e., $\mathcal{V}(t+1) = \mathcal{V}(t) \setminus \{\ell\}$.
Agent ℓ picks a random agent m to inform, and agent m updates its state as follows
1: **if** $\kappa_\ell(t) < \kappa_m(t)$ **then**
2: $\quad y_m(t+1) = y_m(t)$
3: $\quad \kappa_m(t+1) = \kappa_m(t)$
4: **else if** $\kappa_\ell(t) \geqslant \kappa_m(t)$ **then**
5: $\quad y_m(t+1) = x_m$
6: $\quad \kappa_m(t+1) = \kappa_\ell(t) + 1$
7: **end if**

The gossip communication between agents is performed via Algorithm 4 (values not explicitly updated remain constant between t and $t+1$). When $\kappa_i(t) = \kappa_j(t)$, this implies that agents i and j either have not been informed about any departure from the group, i.e., $\kappa_i(t) = \kappa_j(t) = 0$, or have equal information level about the departure of one or more agents. In either case, agents i and j can exchange their information to update their estimate for the MAX value. When $\kappa_i(t) > \kappa_j(t)$, agent i's information about past departures is deemed more up to date. Agent j is then not allowed to transfer its estimate y_j to avoid infecting i with possibly outdated information (unless its estimate is actually its own value, which is by definition valid). Therefore, agent j restarts to $\max(y_i(t), x_j)$ and increments its counter to $\kappa_j(t+1) = \kappa_i(t)$. The case when $\kappa_j(t) > \kappa_i(t)$ is completely symmetric.

B. Eventual Correctness

We now show that the algorithm described in the previous subsection is correct in the sense that, with high probability

Algorithm 4 Gossip algorithm

At each time step $t \in \mathbb{N}$, two agents $i, j \in \mathcal{V}(t)$ are picked randomly (with possibly $i = j$)
1: **if** $\kappa_i(t) = \kappa_j(t)$ **then**
2: $\quad y_i(t+1) = y_j(t+1) = \max(y_i(t), y_j(t))$
3: **else if** $\kappa_i(t) > \kappa_j(t)$ **then**
4: $\quad y_i(t+1) = y_j(t+1) = \max(y_i(t), x_j)$
5: $\quad \kappa_j(t+1) = \kappa_i(t)$
6: **else**
7: $\quad y_i(t+1) = y_j(t+1) = \max(x_i, y_j(t))$
8: $\quad \kappa_i(t+1) = \kappa_j(t)$
9: **end if**

(and even almost surely), it eventually settles on the correct value if arrivals and departures stop.

Remember that $\mathcal{V}(t) := \{1, \ldots, \mathcal{N}(t)\}$ denotes the group of agents present at time t, and let $X := \{x_1, \ldots, x_{\mathcal{N}(t)}\}$ be the set of intrinsic values of nodes in $\mathcal{V}(t)$. Assume that after some time $T \in \mathbb{N}$ no agent leaves and no new agent joins the system, so that $\mathcal{V}(t) = \mathcal{V}(T) = \overline{\mathcal{V}}$, $\mathcal{E}(t) = \mathcal{E}(T) = \overline{\mathcal{E}}$ and $\mathcal{N}(t) = \mathcal{N}(T) = \overline{\mathcal{N}}$ for all $t \in \mathbb{N}_{\geqslant T}$. Then, we need to show that all the currently existing agents $\mathcal{V}(T)$ in the network will successfully reach the correct maximum value.

Definition 1. *We say that an algorithm is eventually correct if for any $T \in \mathbb{N}$ with $\mathcal{G}(t) = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ for all $t \in \mathbb{N}_T$, there exists a $T^* \in \mathbb{N}_{\geqslant T}$ such that $y_i(t) = \max_{j \in \overline{\mathcal{V}}} x_j$ for all $i \in \overline{\mathcal{V}}$ and all $t \in \mathbb{N}_{T^*}$.*

Denote $K(t) := \max_{i \in \mathcal{V}(t)} \kappa_i(t)$, $\mathrm{MAX}(t) = \max_{i \in \mathcal{V}(t)} x_i$ and $X_K(t) := \{x_i : i \in \mathcal{V}(t) \wedge \kappa_i(t) = K(t)\}$. The following lemma, proved in [25], states that at any time $t \in \mathbb{N}$, if the counter value of an agent $j \in \mathcal{V}(t)$ is equal to the maximum value $K(t)$, then its estimate $y_j(t)$ is equal to an intrinsic value $x_i \in X_K(t)$ of one of the agents present in the system at this time t and whose value κ_i is $K(t)$.

Lemma 1. *For all $t \in \mathbb{N}$ and any $j \in \mathcal{V}(t)$, if $\kappa_j(t) = K(t)$ then $y_j(t) \in X_K(t) \subseteq X(t)$.*

Theorem 1. *Suppose that Assumption 1 holds. Then, Algorithm 1-4 is eventually correct.*

The proof of Theorem 1 is available in [25]. It relies on the observations that (i) once agents stop arriving and leaving, the algorithm results in a classical MAX-consensus on the values κ_i, and (ii) once all κ_i are equal, the algorithm behaves as a normal MAX-consensus on the estimate y_i. We then use a result in [10] on convergence of gossip MAX-consensus together with Lemma 1 on the validity of the estimates held by those with the highest κ_i to conclude.

V. DEPARTURES ARE NOT ANNOUNCED

A. Algorithm description

Leaving agents may not always be able to announce their departure, such as in case of unforeseen failures or disconnections. The algorithms in Section IV can no longer

4755

be applied in such a more challenging setting. Therefore, we now propose an alternative algorithm that does not use messages from departing agents. The idea is to have each agent maintain a variable $\mathcal{T}_i$ representing the "age" of its information. This age $\mathcal{T}_i$ is kept at 0 when the agent's estimate $y_i(t)$ of MAX(t) corresponds to (only) its own value x_i, as the validity of its information is then guaranteed. Otherwise $\mathcal{T}_i$ is increased by 1 every time agent i interacts with another agent, as the information gets "older". When an agent i changes its estimate $y_i(t)$ by adopting the estimate $y_j(t)$ of an agent j, it also sets $\mathcal{T}_i(t)$ to the value $\mathcal{T}_j(t)$, which corresponds to the age of the new information it now holds. Finally, when $\mathcal{T}_i(t)$ reaches a threshold $\mathcal{T}^*$, the information $y_i(t)$ is considered too old to be reliable and is discarded; $y_i(t)$ is reset to x_i and $\mathcal{T}_i(t)$ to 0. We defer the discussion on the value of $\mathcal{T}^*$ to Section VII, but already note that it should depend on (bounds of) the system size, or (possibly) change with time.

Formally, the behavior of an agent joining the system is governed by Algorithm 5, while the update of $\mathcal{T}_i(t)$ and the gossip interactions are governed by Algorithms 6 and 7 (where we use $y_i(t^+), \mathcal{T}_i(t^+)$ to denote intermediate values the variables $y_i, \mathcal{T}_i$ may take during the computation leading to their values at $t+1$). Observe that when i and j have the same estimate $y_i(t) = y_j(t)$ they update the age of information to the smallest among $\mathcal{T}_i(t)$ and $\mathcal{T}_j(t)$. Observe also that the algorithms guarantee that $y_i(t) \geqslant x_i$ for every i at all times, since $y_i(t)$ can never decrease except when it is re-initialized at x_i. Finally, there is no algorithm for the departure, since agents are not assumed to be able to take any action when other agents leave as this is not announced.

Algorithm 5 Joining algorithm

Assume at time $t \in \mathbb{N}_{\geqslant 1}$, a new agent n wants to join. Agent n initializes its state as follows

1: $y_n(t) = x_n$
2: $\mathcal{T}_n(t) = 0$

Algorithm 6 UpdateTimer

When agent i calls this procedure:

1: **if** $y_i(t) = x_i$ **then** $\triangleright$ guaranteed validity of estimate
2:   $\mathcal{T}_i(t^+) = 0$
3:   $y_i(t^+) = x_i$
4: **else** $\triangleright$ estimate gets one period older
5:   $\mathcal{T}_i(t^+) = \mathcal{T}_i(t) + 1$
6:   $y_i(t^+) = y_i(t)$
7: **end if**
8: **if** $\mathcal{T}_i(t) = \mathcal{T}^*$ **then** $\triangleright$ Reset if threshold reached
9:   $y_i(t^+) = x_i$
10:   $\mathcal{T}_i(t^+) = 0$
11: **end if**

B. Eventual Correctness

We now discuss the eventual correctness of the algorithm described above. We use the same conventions as in Section

---

Algorithm 7 Gossip algorithm

At each time step t, two agents i, j are picked randomly

1: UpdateTimer(i), UpdateTimer(j)
2: **if** $y_i(t^+) > y_j(t^+)$ **then**
3:   $y_j(t+1) = y_i(t^+)$
4:   $\mathcal{T}_j(t+1) = \mathcal{T}_i(t^+)$
5: **else if** $y_i(t^+) < y_j(t^+)$ **then**
6:   $y_i(t+1) = y_j(t^+)$
7:   $\mathcal{T}_i(t+1) = \mathcal{T}_j(t^+)$
8: **else if** $y_i(t^+) = y_j(t^+)$ **then**
9:   $\mathcal{T}_i(t+1), \mathcal{T}_j(t+1) := \min(\mathcal{T}_i(t^+), \mathcal{T}_j(t^+))$
10: **end if**

IV-B. The first lemma states that outdated values are eventually discarded if agents stop leaving or arriving. Its proof, available in [25] relies on showing that the minimum $\mathcal{T}_i$ among all agents i holding an outdated values keep increasing and eventually reaches the reset threshold.

Lemma 2. *If no arrival or departure takes place after time $T \in \mathbb{N}$, then almost surely there exists a time $T' \in \mathbb{N}$ after which every estimate y_i corresponds to the value of an agent present in the system, i.e., for $t \in \mathbb{N} \geqslant T'$, for all i there exists a $j \in \mathcal{V}(t) = \overline{\mathcal{V}}$ such that $y_i(t) = x_j$. As a consequence, $y_i(t) \leqslant$ MAX$(T) = \max_{j \in \overline{\mathcal{V}}} x_j$ for every $i \in \mathcal{V}(t) = \overline{\mathcal{V}}$ and $t \in \mathbb{N} \geqslant T'$.*

Once all outdated information has been removed, the system behaves essentially as a classical maximum-consensus algorithm provided that no spurious reset occurs, which is the case with a high probability if $\mathcal{T}^*$ is large enough. All agents would then eventually obtain the correct estimate of the maximum. The proof of the following theorem, available in [25], is built on this intuitive idea.

Theorem 2. *For all $\epsilon > 0$, there exists a (sufficiently large) $\mathcal{T}^* \in \mathbb{N}$ such that, if no arrival or departure takes place after time $T \in \mathbb{N}$, then there exists a time $T'' \in \mathbb{N} \geqslant T$ after which $y_i(t) =$ MAX$(T) = \max_{j \in \overline{\mathcal{V}}} x_j$ holds for every $i \in \overline{\mathcal{V}}$ with a probability at least $1 - \epsilon$.*

Observe that the threshold $\mathcal{T}^*$ is subject to a trade-off: a large $\mathcal{T}^*$ indeed decreases the probability of spurious resets, but leads to higher delays before outdated values are discarded.

VI. SIMULATIONS

We demonstrate the application of our algorithms on a group of 25 agents: Initially, the intrinsic states x_i of all agents were assigned to random integer values between 0 and 1000. The largest two values of x_i are found to be $x_9 = 936$ and $x_{13} = 815$. The estimates $y_i(0)$ for all agents are initialized to x_i and all the counters $\kappa_i(0)$ and ages $\mathcal{T}_i(0)$ are initialized to 0. Agent 9 with the highest value, $x_9 = 936$ leaves at $t = 200$. Pairwise interactions between two randomly selected agents take place at every other time.

We have simulated the two algorithms, with two thresholds $\mathcal{T}^*$ for that of Section V, and the results are represented
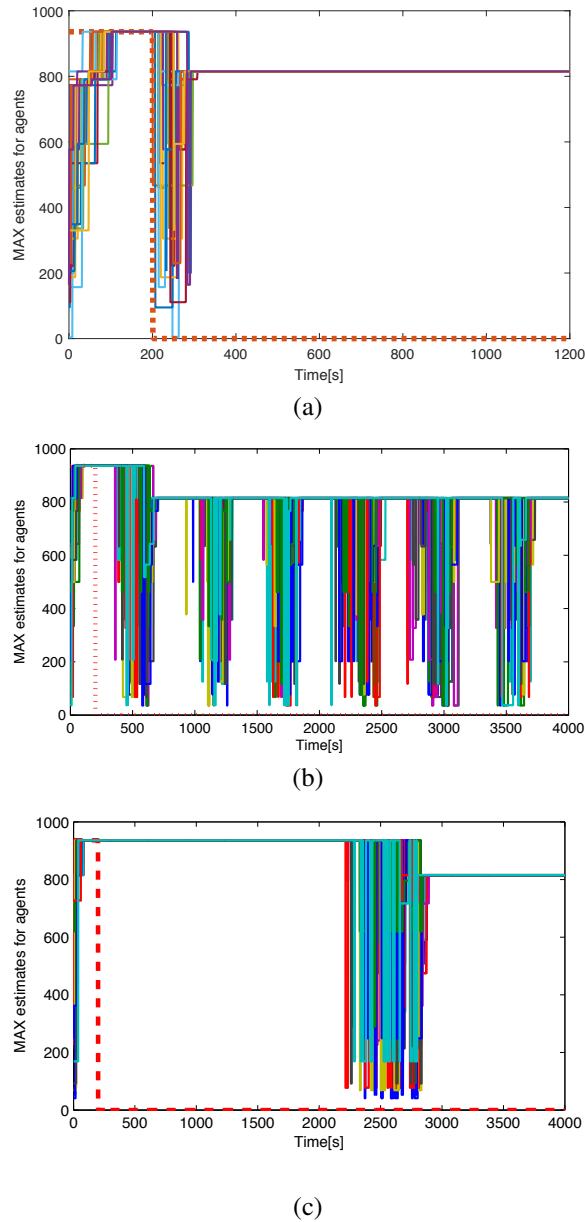
4756

(a)

(b)

(c)

Fig. 1. Evolution with time of the agent estimates $y_i(t)$ for the algorithm of Section IV where departures can be announced (a), and of Section V where departures are not announced, for tresholds $\mathcal{T}^* = 40$ (b) and $\mathcal{T}^* = 200$ (c). (The scale is different in (a)). Departure of the agent with highest value is represented by a dashed line.

in Fig. 1. We note that in the three cases, all the agents first converge to the MAX value of $x_9 = 936$ in a bit more than 100 time steps, before agent 9 leaves the network. After the departure of agent 9 at $t = 200$, we see that the algorithm of Section IV that uses messages from departing agents reconverges to the new maximal value $x_{13} = 815$ in 137 time steps. The performance of the algorithm of Section V without messages from leaving agents are significantly worse. For a threshold $\mathcal{T}^* = 40$, we see that it takes 506 time steps to reconverge to the new maximal value, but the system later suffers from several spurious resets. These are caused by agents reaching the threshold by chance. The probability

of this occurring can be significantly reduced by taking a higher threshold, but this results in an even longer time to react to the departure of 9, as seen in Fig. 1(c) with $\mathcal{T} = 200$. We note that 2439 time-steps are needed in this case to re-obtain the correct value, mostly because it takes very long before the agents abandon their former estimate. This clearly illustrates the trade-off on the threshold value $\mathcal{T}^*$: a too small value will result in spurious resets as soon as some agents "have not heard" about the agent with the highest value for too long. But a too large threshold will result in a significant delay before agents decide that an agent has probably left the system.

We also performed comparisons between the two approaches on the convergence time to reach consensus after the agent with MAX has left the group for other numbers of nodes. The results are summarized in Table I. We take $\mathcal{T}^* = 1.1\mathcal{N}(0)$ with the algorithm of Section V. We observe that when the number of agents increases, the algorithm of Section IV requires proportionally much fewer iterations to reach consensus, as expected and already observed in Fig. 1. Moreover, it also achieves a stronger version of the property of eventual correctness than the algorithm of Section V, as it avoids spurious resets, as discussed above. It does however require the possibility of sending messages when leaving.

Number of nodes	Iterations to reach MAX-consensus	
	Algorithm 1-4	Algorithm 1, 5-7
10	21	64
20	129	162
30	194	599
50	246	1885
100	628	6580

TABLE I

COMPARISON BETWEEN THE TWO TECHNIQUES WITH DIFFERENT NUMBER OF NODES.

VII. DISCUSSION AND CONCLUSION

We have investigated the distributed MAX-consensus problem for *open* multi-agent systems. Two algorithms have been proposed depending on whether the agents who leave the network can inform another existing agent about their departure or cannot. The eventual correctness has been proven for both.

Taking a step back, we see two main challenges in the design of algorithms for open multi-agent systems, as also briefly noted in [20]:

Robustness and dynamic information treatment: The algorithms should be robust to departures and arrivals, in the sense that they should keep updating their estimates to discard outdated information. Moreover, novel information held by arriving agents should be taken into account, and outdated information, for example, related to agents no longer in the system, should eventually be discarded.

Performance in open context: The performance of classical multi-agents algorithms is often measured by the rate at

4757

which they converge to an exact solution or a desired situation (or the time to reach such a situation). This approach is no longer relevant in a context where agents' departures and arrivals keep "perturbing" the system, and possibly the algorithm goal (as is the case here). Rather, efficient algorithms would be those for which the estimated answer remains "close" to some "instantaneous exact solution", according to a suitable metric.

The algorithms we have developed here do answer the first issue of robustness and information treatment for the problem of distributed maximum computation. The characterization and optimization of their performance in an open context, however, remains unanswered at present and could be the topic of further works. We note that the behavior of a gossip averaging algorithm in an open multi-agent system was characterized in [20], but this algorithm was not designed to compute a specific value, as is the case here.

In particular, we observe that both algorithms would suffer from occasional apparently unnecessary resets. This may happen after the departure of an agent that did not have the largest value in the algorithm of Section IV, or when an agent has been isolated for too long from that with the highest value in the algorithm of Section V. We do not know at this stage if these spurious resets can be entirely avoided, especially when leaving agents cannot send a final message. In this case, it is indeed impossible to know for sure whether the agent with the highest value has left or has just not communicated for a while. There are, however, several possibilities to mitigate the damage of these spurious resets and to play on the trade-off between the effect of these perturbation and the speed at which the system reacts. A simple solution could be for example to apply an additional filtering layer when the algorithm requires an important decrease of y_i. In this case, a new estimate $\tilde{y}_i$ would follow y_i except that sharp decrease would be replaced by gradual ones. We also observe that our second algorithm will either only work when the system size is not too large with respect to $\mathcal{T}^*$ (case of a fixed threshold) or eventually work for all size but gradually become slower and slower to react (case of a growing $\mathcal{T}^*$). Whether this can be avoided in a context when leaving agents do not warn others about their departure also remains an open interesting question.

References

[1] R. Olfati-Saber and N. F. Sandell, "Distributed tracking in sensor networks with limited sensing range," *In Proceedings of the 2008 American Control Conference, Washington, U.S.A.*, pp. 3157–3162, 2008.

[2] L. Shi, A. Capponi, K. Johansson, and R. Murray, "Resource optimisation in a wireless sensor network with guaranteed estimator performance," *IET Control Theory and Applications*, vol. 4, no. 5, pp. 710–723, 2010.

[3] O. Demigha, W. Hidouci, and T. Ahmed, "On energy efciency in collaborative target tracking in wireless sensor network: A review," *IEEE Communications Surveys & Tutorials*, vol. 15, no. 3, pp. 1210–1222, 2012.

[4] V. Cerf and R. Kahn, "A protocol for packet network intercommunication," *IEEE Transactions on Communications*, vol. 22, no. 5, pp. 637–648, 1974.

[5] S. Muthukrishnan, B. Ghosh, and M. Schultz, "First and second order diffusive methods for rapid, coarse, distributed load balancing," *Theory of Computing Systems*, pp. 331–354, 1998.

[6] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis, and simulation," *Journal of Artifical Societies and Social Simulation*, vol. 5, no. 3, pp. 1–33, 2002.

[7] V. Blondel, J. Hendrickx, and J. Tsitsiklis, "Continuous-time average-preserving opinion dynamics with opinion-dependent communications," *SIAM Journal on Control and Optimization*, vol. 48, no. 8, pp. 5214–5240, 2010.

[8] J. Liu, N. Hassanpour, S. Tatikonda, and A. Morse, "Dynamic threshold models of collective action in social networks," *In Proceedings of the 51st IEEE Conference on Decision and Control, Hawaii, U.S.A.*, pp. 3991–3996, 2012.

[9] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Transactions on Automatic Control*, vol. 52, no. 6, pp. 2508–2530, 2006.

[10] F. Iutzeler, P. Ciblat, and J. Jakubowicz, "Analysis of max-consensus algorithms in wireless channels," *IEEE Transactions on Signal Processing*, vol. 60, no. 11, pp. 6103–6107, 2012.

[11] S. Giannini, A. Petitti, D. D. Paola, and A. Rizzo, "Asynchronous max-consensus protocol with time delays: Convergence results and applications," *IEEE Transactions on Circuits and Systems-I*, vol. 63, no. 2, pp. 256–264, 2016.

[12] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *In Proceedings of the IEEE*, pp. 215–233, 2007.

[13] J. Hendrickx, A. Olshevsky, and J. Tsitsiklis, "Distributed anonymous discrete function computation," *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2276–2289, 2011.

[14] W. Ren, R. Beard, and E. Atkins, "A survey of consensus problems in multi-agent coordination," *In Proceedings of the 2005 American Control Conference, Portland, U.S.A.*, pp. 1859–1864, 2005.

[15] D. Stutzbach and R. Rejaie, "Understanding churn in peer-to-peer networks," *In Proceedings of the 6th ACM SIGCOMM conference on Internet measurement, Rio de Janeriro, Brazil*, pp. 189–202, 2006.

[16] F. Kuhn, S. Schmid, and R. Wattenhofer, "Towards worst-case churn resistant peer-to-peer systems," *Distributed Computing*, vol. 2, no. 4, pp. 249–267, 2010.

[17] M. Jelasity, A. Montresor, and O. Babaoglu, "Gossip-based aggregation in large dynamic networks," *ACM Transactions on Computer Systems*, vol. 23, no. 3, pp. 219–252, 2005.

[18] F. Kuhn, N. Lynch, and R. Oshman, "Distributed computation in dynamic networks," *In Proceedings of the 42nd ACM symposium on Theory of computing, Massachussetts, U.S.A.*, pp. 513–522, 2010.

[19] C. Dutta, G. Pandurangan, R. Rajaraman, Z. Sun, and E. Viola, "On the complexity of information spreading in dynamic networks," *In Proceedings of the 24th Annual ACM-SIAM Symposium on Discrete Algorithms*, pp. 717–736, 2013.

[20] J. Hendrickx and S. Martin, "Open multi-agent systems: Gossiping with deterministic arrivals and departures," *In Proceedings of the 54th Annual Allerton Conference on Communication, Control, and Computing, Monticello, Illinois, U.S.A.*

[21] T. Huynh, N. Jennings, and N. Shadbolt, "An integrated trust and reputation model for open multi-agent systems," *Autonomous Agents and Multi-Agent Systems*, vol. 13, no. 2, p. 119154, 2006.

[22] I. Pinyol and J. Sabater-Mir, "Computational trust and reputation models for open multi-agent systems: a review," *Artificial Intelligence Review*, vol. 40, no. 1, pp. 1–25, 2013.

[23] B. Nejad, S. Attia, and J. Raisch, "Max-consensus in a max-plus algebraic setting: The case of fixed communication topologies," *In Proceedings of the International Symposium on Information, Communication and Automation Technologies, Sarajevo, Bosnia and Herzegovina*, pp. 1–7, 2009.

[24] S. Zhang, C. Tepedelenlioğlu, M. Banavar, and A. Spanias, "Max consensus in sensor networks: Non-linear bounded transmission and additive noise," *IEEE Sensors Journal*, vol. 16, no. 24, pp. 9089–9098, 2016.

[25] M. Abdelrahim, J. Hendrickx, and W. Heemels, "Max-consensus in open multi-agent systems with gossip interactions (online version)," *arXiv:1709.05793 [cs.MA]*, 2017.