

# Event-triggered control of nonlinear singularly perturbed systems based only on the slow dynamics

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**Abstract:** We study event-triggered control based on a reduced or simplified model of the plant's dynamics. In particular, we address two time-scale systems and we investigate whether it is possible to synthesize a stabilizing event-triggered controller based only on an approximate model of the slow dynamics given by singular perturbation theory, when the fast one is stable. We highlight specific challenges which arise with the event-triggered implementation: the state of the fast model experiences jumps at transmissions which induces non-trivial difficulties for the stability analysis and the Zeno phenomenon may occur due to the fact that we neglect the fast dynamics. We describe the overall problem as a hybrid singularly perturbed system. We first provide a necessary condition on the triggering condition to avoid the Zeno phenomenon. Afterwards, we propose two strategies which respectively use a dead-zone and a clock variable and which ensure different asymptotic stability properties. The existence of a minimum inter-transmission interval is guaranteed. Our results are illustrated by a physical example.

## 1. INTRODUCTION

Event-triggered control has a great interest in the development of networked control systems because it may allow to significantly reduce the usage of the communication channel. Indeed, although periodic sampling is appealing from the analysis and implementation point of view, it may yield a conservative solution when the communication resources are limited as it may unnecessarily use the network. In event-triggered control, it is the occurrence of an event, typically a variation of the plant's state, which closes the loop. This translates into reducing the resources utilisation compared to the periodic implementation, see e.g. Årzén [1999], Åström and Bernhardsson [1999], Tabuada [2007], Wang and Lemmon [2011], Postoyan et al. [2011b], Heemels et al. [2012], Donkers and Heemels [2012]. Available techniques rely on the knowledge of an accurate model of the plant (which may be affected by uncertainties or external disturbances). However, the controller is often designed in practice based on a *reduced* or *simplified* model of the plant's dynamics which may be obtained by model reduction or by neglecting the fast dynamics. For instance, for the case of two time-scale systems, singular perturbation theory can be used to approximate the slow and the fast dynamics, see Khalil [2002]. In this context, when the origin is stable for the fast model, it is possible to design the controller based only on the slow model, like for linear time-invariant (LTI) systems (see Kokotović et al. [1986]), classes of nonlinear systems (see Khalil [2002]) and linear time-varying sampled data systems with periodic sampling (see Pan and Başar [1994]).

In this paper, we address two time-scale systems and we investigate whether it is possible to synthesize a stabilizing event-triggered controller based only on an approximate

model of the slow dynamics given by singular perturbation theory, when the fast one is stable. We highlight specific challenges which arise with the event-triggered implementation:

- The state of the fast model experiences jumps at transmissions which induces non-trivial difficulties for the stability analysis. It is due to the change of variables we introduce in order to separate the slow and the fast dynamics using the singular perturbation theory. That is not the case for available results on event-triggered control where only the sampling-induced error is reset to zero at each transmission, see e.g. Tabuada [2007]. This characteristic of the problem makes existing results not directly applicable.
- The Zeno phenomenon may occur due to the fact that we neglect the fast dynamics.

We show that the problem we are interested in can be casted as a hybrid singularly perturbed system with the formalism of Goebel et al. [2012]. Unlike Sanfelice and Teel [2011] where such systems are analysed, we *define* the flow and jump sets, we conclude *different* stability properties and we ensure the existence of a minimum inter-transmission interval.

We follow an emulation-like approach to design the event-triggered controllers (see Tabuada [2007]). We first synthesize a stabilizing controller for the approximate slow model obtained by singular perturbation theory, in the absence of communication constraints. Afterwards, we take into account the effect of the network and we propose two event-triggering conditions to ensure asymptotic stability properties for the overall system. The first policy consists of modifying the triggering condition of Tabuada [2007] by including a dead-zone in order to guarantee that all

inter-execution times are bounded from below by a strictly positive constant. We show that this strategy ensures a semiglobal practical stability property. The second strategy consists in merging the event-triggered implementation of Tabuada [2007] with the time-triggered results in Nešić et al. [2009]. The idea is to allow transmissions only after a fixed amount of time  $T^*$  has elapsed since the last control update. In that way, the minimum amount of time between two jumps is lower bounded by the constant  $T^*$  and we guarantee global asymptotic stability properties. This policy relies on an additional assumption compared to the first strategy. Our results are applied to the autopilot control of an F-8 aircraft.

The remainder of the paper is organised as follows. The problem is stated in Section 2. In Section 3, we present the main results. In Section 4, we show that the proposed control strategies are applicable to LTI systems and an illustrative example is provided.

**Notation.** We denote  $\mathbb{R} = (-\infty, \infty)$ ,  $\mathbb{R}_{\geq 0} = [0, \infty)$ ,  $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$ . The Euclidean norm will be denoted as  $|\cdot|$ . We use also the notation  $(x, y)$  to represent the vector  $[x^T, y^T]^T$  for  $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ . A continuous function  $\gamma : [0, \infty) \rightarrow \mathbb{R}_{\geq 0}$  is of class  $\mathcal{K}$  if it is zero at zero, strictly increasing, and it is of class  $\mathcal{K}_\infty$  if in addition  $\gamma(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . A continuous function  $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class  $\mathcal{KL}$  if for each  $t \in \mathbb{R}_{\geq 0}$ ,  $\gamma(\cdot, t)$  is of class  $\mathcal{K}$ , and, for each  $s \in \mathbb{R}_{\geq 0}$ ,  $\gamma(s, \cdot)$  is decreasing to zero. We denote the minimum and maximum eigenvalues of the symmetric positive definite matrix  $A$  as  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  respectively. We use  $\mathbb{I}_n$  to denote the identity matrix of dimension  $n$ .

## 2. PROBLEM STATEMENT

Consider the following nonlinear time-invariant singularly perturbed system

$$\dot{x} = f(x, z, u) \quad (1)$$

$$\epsilon \dot{z} = g(x, z, u) \quad (2)$$

where  $x \in \mathbb{R}^{n_x}$  and  $z \in \mathbb{R}^{n_z}$  are the states,  $u \in \mathbb{R}^{n_u}$  is the control input and  $\epsilon > 0$  is a small parameter. We use singular perturbation theory to approximate the slow and the fast dynamics, see Khalil [2002]. We rely on the following standard assumption.

*Assumption 1.* The equation  $g(x, z, u) = 0$  has  $n \geq 1$  isolated real roots

$$z = h_i(x, u), \quad i = 1, 2, \dots, n \quad (3)$$

where  $h_i$  is continuously differentiable.  $\square$

In that way, the substitution of the  $i$ th root  $z = h(x, u)$  into (1) yields the corresponding approximate slow model

$$\dot{x} = f(x, h(x, u), u). \quad (4)$$

To separate the slow and the fast dynamics, we write the system (1)-(2) with the coordinates  $(x, y)$  where

$$y := z - h(x, u) \quad (5)$$

represents the deviation of  $z$  from the quasi-steady-state manifold  $\{(x, z, u) : z - h(x, u) = 0\}$ . Then, we derive the approximate fast dynamics

$$\frac{dy}{d\tau} = g(x, y + h(x, u), u) \quad (6)$$

where  $\tau := (t - t_0)/\epsilon$  is a new time variable and  $x \in \mathbb{R}^{n_x}$  is treated as a fixed parameter, see Khalil [2002].

In this study, we investigate whether we can design an event-triggered controller based only on the approximate slow model (4) to stabilize the overall system. We follow an emulation-like approach as we first assume that a controller of the form  $u = k(x)$  has been designed to stabilize (4) in the absence of communication constraints. We then implement this controller over a digital platform so that

$$u(t) = k(x(t_i)) \quad \forall t \in [t_i, t_{i+1}]. \quad (7)$$

The sequence of transmission instants  $t_i, i \in \mathbb{Z}_{\geq 0}$  will be defined by the event-triggering condition we will design. We introduce the sampling-induced error  $e_x$ , as in Tabuada [2007],

$$e_x(t) = x(t_i) - x(t) \quad \forall t \in [t_i, t_{i+1}] \quad (8)$$

which is reset to zero at each transmission instant. The state feedback controller (7) is given by

$$u = k(x + e_x). \quad (9)$$

Hence, in view of (5), the variable  $y$  becomes

$$y := z - h(x, k(x + e_x)). \quad (10)$$

We note that the variable  $y$  experiences a jump at each transmission as  $e_x$  is reset to zero after each transmission. Hence, system (1)-(2) in the  $(x, y)$  coordinates becomes

$$\dot{x} = f(x, y + h(x, k(x + e_x)), k(x + e_x)) =: f_x(x, y, e_x) \quad (11)$$

$$\begin{aligned} \epsilon \dot{y} &= g(x, y + h(x, k(x + e_x)), k(x + e_x)) \\ &\quad - \epsilon \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{\partial u}{\partial x} - \frac{\partial h}{\partial u} \frac{\partial u}{\partial e_x} \right) f_x(x, y, e_x) \\ &=: f_y(x, y, e_x), \end{aligned} \quad (12)$$

and we have

$$x(t_{i+1}^+) = x(t_{i+1}) \quad (13)$$

$$\begin{aligned} y(t_{i+1}^+) &= z(t_{i+1}^+) - h(x(t_{i+1}^+), k(x(t_{i+1}^+) + e_x(t_{i+1}^+))) \\ &= z(t_{i+1}) - h(x(t_{i+1}), k(x(t_{i+1}) + 0)) \\ &= y(t_{i+1}) + h(x(t_{i+1}), k(x(t_{i+1}) + e_x(t_{i+1}))) \\ &\quad - h(x(t_{i+1}), k(x(t_{i+1}))) \\ &=: h_y(x(t_{i+1}), y(t_{i+1}), e_x(t_{i+1})). \end{aligned} \quad (14)$$

We model the problem using the hybrid formalism of Goebel et al. [2012] (like in Donkers and Heemels [2012], Postoyan et al. [2011a]). In that way, we obtain

$$\dot{q} = F(q) \quad q \in C, \quad q^+ = G(q) \quad q \in D, \quad (15)$$

where  $q = (x, y, e_x) \in \mathbb{R}^{n_q}$  and

$$F(q) := \begin{pmatrix} f_x(x, y, e_x) \\ \frac{1}{\epsilon} f_y(x, y, e_x) \\ -f_x(x, y, e_x) \end{pmatrix}, \quad G(q) := \begin{pmatrix} x \\ h_y(x, y, e_x) \\ 0 \end{pmatrix}. \quad (16)$$

The sets  $C$  and  $D$  in (15) are defined according to the event-triggering condition which we will synthesize in the following. These sets are closed and represent the flow and jump sets respectively. Typically, the system flows on  $C$  where the triggering condition is not satisfied and

experiences a jump on  $D$  where the triggering condition is verified. When  $q \in C \cap D$ , the system can either jump or flow, the latter only if flowing keeps  $q$  in  $C$ . For more detail on hybrid systems of the form of (15) see Goebel et al. [2012].

**Problem:** Our objective is to define an appropriate triggering condition for system (15) which is equivalent to defining appropriate  $C$  and  $D$  sets to guarantee asymptotic stability properties for system (15). Moreover, we want the triggering condition to only depend on the slow variables  $x$  and  $e_x$  so that we can ignore the fast dynamics when they are stable. It is important to note that the state variable  $y$  experiences a jump on the set  $D$ , see (16), which is not the case for all available results on event-triggered control where only the sampling-induced error (which corresponds to  $e_x$  in (15)) is reset to zero at jumps. This is a characteristic feature of singularly perturbed systems which comes from the definition of the variable  $y$  in (10).

### 3. MAIN RESULTS

#### 3.1 A necessary condition

Before presenting the main results of the paper, we first give a necessary condition the event-triggering condition must satisfy in order to avoid the Zeno phenomenon. Assume that we have designed an event-triggering condition such that the sets  $C$  and  $D$  in (15) are of the form

$$C = \{q : \Gamma(x, e_x) \leq 0\}, \quad D = \{q : \Gamma(x, e_x) = 0\}, \quad (17)$$

where  $\Gamma : \mathbb{R}^{2n_x} \rightarrow \mathbb{R}_{\geq 0}$  is continuous. It is shown in Postoyan et al. [2011b] how various event-triggering conditions lead to a hybrid model with flow and jump sets like in (17). Consider the scenario where  $\Gamma(0, 0) = 0$ . This is the case for the technique in Tabuada [2007] for instance which gives  $\Gamma(x, e) = \gamma(|e_x|) - \sigma\alpha(|x|)$  where  $\alpha, \gamma \in \mathcal{K}$  and  $\sigma \in (0, 1)$ . The problem here is that the Zeno phenomenon may occur as it suffices to have  $x(0, 0) = 0$  and  $e_x(0, 0) = 0$  (and  $y(0, 0) \neq 0$ ) for the system (15) to permanently jump. Indeed, we then have  $q \in C \cap D$  and  $G(q) \in D$ . We cannot allow such solutions in practice. It is therefore mandatory to design triggering conditions  $\Gamma$  such that

$$\Gamma(0, 0) \neq 0. \quad (18)$$

Note that a similar remark has been made in Mazo Jr. and Cao [2012] in a different context, namely for decentralized systems.

#### 3.2 Assumptions

We present the assumptions made on system (15). We will show in Section 4 that all the conditions are satisfied by LTI systems. We first note that the approximate slow and fast models (4) and (6) respectively, are now in view of (11) and (12),

$$\dot{x} = f\left(x, h(x, k(x + e_x)), k(x + e_x)\right) =: f_{x_s}(x, 0, e_x) \quad (19)$$

$$\frac{dy}{d\tau} = g\left(x, y + h(x, k(x + e_x)), k(x + e_x)\right). \quad (20)$$

In that way, we view system (15) as the interconnection of the approximate slow and fast systems above with the  $e_x$ -system. We independently construct Lyapunov functions

for the slow and fast models then we will investigate the overall stability of the original system, like in continuous-time in Khalil [2002]. First, we assume that the slow system (19) is input-to-state stable (ISS) with respect to  $e_x$ .

*Assumption 2.* There exist a smooth function  $V_x : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$  and class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_x, \bar{\alpha}_x$ , a continuously differentiable class  $\mathcal{K}_\infty$  function  $\gamma_1$  and  $\alpha_1 > 0$  such that for all  $(x, e_x) \in \mathbb{R}^{2n_x}$  the following is satisfied

$$\begin{aligned} \underline{\alpha}_x(|x|) &\leq V_x(x) \leq \bar{\alpha}_x(|x|) \\ \frac{\partial V_x}{\partial x} f_{x_s}(x, 0, e_x) &\leq -\alpha_1 V_x(x) + \gamma_1(|e_x|). \end{aligned} \quad (21)$$

□

Condition (21) is similar to (11.39) in Khalil [2002]. We assume the following stability property holds for the fast model (20) like in Khalil [2002].

*Assumption 3.* There exist a smooth function  $V_y : \mathbb{R}^{n_y} \rightarrow \mathbb{R}_{\geq 0}$  and class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_y, \bar{\alpha}_y$  and  $\alpha_2 > 0$  such that for all  $(x, y, e_x) \in \mathbb{R}^{n_q}$

$$\begin{aligned} \underline{\alpha}_y(|y|) &\leq V_y(x, y) \leq \bar{\alpha}_y(|y|) \\ \frac{\partial V_y}{\partial y} g\left(x, y + h(x, k(x + e_x)), k(x + e_x)\right) &\leq -\alpha_2 V_y(x, y). \end{aligned} \quad (22)$$

□

Assumption 3 implies that the origin of the fast dynamics (20) is globally asymptotically stable. Note that Assumption 3 does not imply that the origin of the fast dynamics (20) is globally exponentially stable as the functions  $\underline{\alpha}_y, \bar{\alpha}_y$  can be nonlinear. We impose the following conditions on the interconnections between the slow and fast dynamics (19), (20).

*Assumption 4.* There exist a class  $\mathcal{K}_\infty$  function  $\gamma_2$  and  $\beta_2, \beta_3 > 0$  such that for all  $(x, y, e_x) \in \mathbb{R}^{n_q}$  the following holds

$$\begin{aligned} \frac{\partial V_x}{\partial x} [f_x(x, y, e_x) - f_{x_s}(x, 0, e_x)] &\leq \beta_1 \sqrt{V_x(x) V_y(x, y)} \\ \left[ \frac{\partial V_y}{\partial x} - \frac{\partial V_y}{\partial y} \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{\partial u}{\partial x} - \frac{\partial h}{\partial u} \frac{\partial u}{\partial e_x} \right) \right] f_x(x, y, e_x) &\leq \\ \beta_2 \sqrt{V_x(x) V_y(x, y)} + \beta_3 V_y(x, y) + \gamma_2(|e_x|), \end{aligned} \quad (23)$$

where  $V_x, V_y, \beta_1, \gamma_1$  come from Assumptions 2 and 3. In addition, there exists  $L > 0$  such that, for all  $s \geq 0$

$$\gamma_2 \circ \gamma_1^{-1}(s) \leq Ls. \quad (24)$$

□

Conditions (23) represent the effect of the deviation of the original system (15) from the slow and fast models (19), (20) respectively and are related to (11.43) and (11.44) in Khalil [2002].

*Remark:* It is possible to relax condition (24) by adding a strictly positive constant to the right-hand side of (24). It can then be shown that a practical stability property holds for the event-triggered controlled system with respect to this constant by slightly modifying the proofs of the theorems.

□

Finally, we assume that the dynamics of  $V_y$  along jumps of the states  $x, y$  satisfy the following condition.

*Assumption 5.* There exist  $\lambda_1, \lambda_2 > 0$  such that for all  $q \in \mathbb{R}^{n_q}$

$$V_y(x, h_y(x, y, e_x)) \leq V_y(x, y) + \lambda_1 \gamma_1(|e_x|) + \lambda_2 \sqrt{\gamma_1(|e_x|) V_y(x, y)}, \quad (25)$$

where  $V_x, V_y, \gamma_1$  come from Assumptions 2 and 3 respectively.  $\square$

We are now ready to present the main results of this paper. The proofs are omitted due to space constraints.

### 3.3 Semiglobal practical stabilization

In view of Assumption 2, a first attempt would be to define a triggering condition of the form  $\gamma_1(|e_x|) \geq \sigma \alpha_1 V_x(x)$  where  $\sigma \in (0, 1)$  like in Tabuada [2007]. Unfortunately, we cannot choose this condition as the Zeno phenomenon may occur as discussed in Section 3.1. To overcome this issue, we consider the event-triggering condition below

$$\gamma_1(|e_x|) \geq \max\{\sigma \alpha_1 V_x(x), \rho\}, \quad (26)$$

where  $\rho > 0$  is a design parameter. In view of (17),  $\Gamma(x, e_x) := \gamma_1(|e_x|) - \max\{\sigma \alpha_1 V_x(x), \rho\}$  and  $\Gamma(0, 0) = -\rho \neq 0$ , then the condition (18) is satisfied. Consequently, we define the flow and jump sets of (15) as

$$\begin{aligned} C &= \{q : \gamma_1(|e_x|) \leq \max\{\sigma \alpha_1 V_x(x), \rho\}\} \\ D &= \{q : \gamma_1(|e_x|) = \max\{\sigma \alpha_1 V_x(x), \rho\}\}. \end{aligned} \quad (27)$$

Although this type of triggering conditions has already been used in Donkers and Heemels [2012], Mazo Jr. and Cao [2012], Miskowicz [2006], Otanez et al. [2002] for example, the fact that the state  $y$  experiences jumps has a potentially destabilizing effect and requires to fully modify the stability analysis.

*Theorem 1.* Consider system (15) with the flow and jump sets defined in (27). Suppose that Assumptions 1-5 hold. Then, for any  $\Delta, \rho > 0$ , there exist  $\beta_\Delta \in \mathcal{KL}$ ,  $\gamma_\Delta \in \mathcal{K}$  and  $\epsilon^*(\Delta) > 0$  such that for any  $\epsilon(\Delta) \in (0, \epsilon^*(\Delta))$  and any solution  $\phi = (\phi_x, \phi_y, \phi_{e_x})$  with  $|\phi(0, 0)| \leq \Delta$  and  $\phi_{e_x}(0, 0) = 0$ ,  $\phi$  is complete<sup>1</sup> and it satisfies

$$\begin{aligned} |(\phi_x(t, j), \phi_y(t, j))| &\leq \beta_\Delta(|(\phi_x(0, 0), \phi_y(0, 0))|, t + j) \\ &\quad + \gamma_\Delta(\rho) \quad \forall (t, j) \in \text{dom } \phi. \end{aligned} \quad (28)$$

Moreover, all inter-transmission times are lower bounded by a semiglobal uniform strictly positive constant.  $\square$

The condition that  $\phi_{e_x}(0, 0) = 0$  in Theorem 1 is reasonable as it simply means that the control input is updated at the initial time. Theorem 1 ensures a semiglobal practical stability property for system (15). Indeed, given an arbitrary (large) ball of initial conditions centered at the origin and of radius  $\Delta$  and any constant  $\rho$ , there exists  $\epsilon$  sufficiently small such that  $\phi_x$  and  $\phi_y$  converge towards a neighbourhood of the origin whose ‘size’ can be rendered arbitrarily small by reducing  $\rho$ .

*Remark:* In the proof of Theorem 1, which has been omitted in this version, a semiglobal uniform lower bound on the inter-transmission intervals have been shown to exist. This lower bound has been estimated by the time it takes for  $\gamma_1(|e_x|)$  to evolve from 0 to  $\rho$ .  $\square$

<sup>1</sup> A solution  $\phi$  to (15) is *complete* if its domain  $\text{dom } \phi$  is unbounded. The domain of  $\phi$  is the subset of  $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$  where  $\phi$  is defined, see Goebel et al. [2012] for more detail.

### 3.4 Global asymptotic stabilization

We may want in some cases to ensure a stronger stability property than the one guaranteed by Theorem 1. We thus propose a method to design the event-triggering condition to ensure a global asymptotic stability property under an extra assumption. The idea is to combine the event-triggered technique of Tabuada [2007] with the time-triggered results of Carnevale et al. [2007] such that we allow transmission only after a fixed amount of time  $T^*$  has elapsed since the last jump. We thus augment the original hybrid system (15) with a clock variable  $\tau \in \mathbb{R}_{\geq 0}$  as follows

$$\begin{aligned} \dot{q} &= F(q) & \dot{\tau} &= 1 & (q, \tau) &\in \tilde{C}, \\ q^+ &= G(q) & \tau^+ &= 0 & (q, \tau) &\in \tilde{D}, \end{aligned} \quad (29)$$

where the flow and jump sets are respectively defined as

$$\begin{aligned} \tilde{C} &:= \{(q, \tau) : \gamma_1(|e_x|) \leq \sigma \alpha_1 V_x(x) \text{ or } \tau \in [0, T^*]\} \\ \tilde{D} &:= \left\{ (q, \tau) : \left( \gamma_1(|e_x|) = \sigma \alpha_1 V_x(x) \text{ and } \tau \geq T^* \right) \text{ or } \right. \\ &\quad \left. \left( \gamma_1(|e_x|) \geq \sigma \alpha_1 V_x(x) \text{ and } \tau = T^* \right) \right\}. \end{aligned} \quad (30)$$

While the idea of merging event-triggered and time-triggered techniques is intuitive, the stability analysis is non-trivial as we need to build a common hybrid Lyapunov function for the two approaches. It has to be emphasized that the constant  $T^*$  allows us to directly tune the minimum inter-transmission interval provided it is smaller than the bound given below. This is typically not done in the literature (except for linear systems in Yu and Antsaklis [2012]) where the lower bound on the inter-transmission time is often subject to some conservatism and it cannot be directly selected. We note that the condition (18) which becomes here  $\Gamma(0, 0, 0) \neq 0$  is satisfied with  $\Gamma(x, e_x, \tau) = \min\{\Gamma_1(x, e_x), \Gamma_2(\tau)\}$  and  $\Gamma_1(x, e_x) = \gamma_1(|e_x|) - \sigma \alpha_1 V(x)$  and  $\Gamma_2(\tau) = \tau - T^*$  as  $\Gamma(0, 0, 0) \leq -T^* < 0$ , see (30). Inspired by Carnevale et al. [2007], we make the following additional assumption on system (29).

*Assumption 6.* There exist  $M, N \geq 0$  such that, for all  $(x, y) \in \mathbb{R}^{n_x+n_y}$  and for almost all  $e_x \in \mathbb{R}^{n_x}$

$$\langle \nabla |e_x|, -f_x(x, y, e_x) \rangle \leq M|e_x| + N(\sqrt{V_x(x)} + \sqrt{V_y(x, y)}),$$

where  $V_x, V_y$  come from Assumptions 2 and 3.  $\square$

The constant  $T^*$  in (30) is selected such that  $T^* < \mathcal{T}$ , like in Carnevale et al. [2007], where

$$\mathcal{T} := \begin{cases} \frac{1}{Mr} \arctan(r) & M^2 < \frac{2N^2}{\alpha_1} (\tilde{\gamma}_1 + \tilde{\gamma}_2) \\ \frac{1}{M} & M^2 = \frac{2N^2}{\alpha_1} (\tilde{\gamma}_1 + \tilde{\gamma}_2) \\ \frac{1}{Mr} \operatorname{arctanh}(r) & M^2 > \frac{2N^2}{\alpha_1} (\tilde{\gamma}_1 + \tilde{\gamma}_2) \end{cases} \quad (31)$$

with  $r := \sqrt{\left| \frac{2N^2}{\alpha_1} \frac{\gamma^2}{M^2} - 1 \right|}$ , where  $M, N$  come from Assumption 6 and  $\alpha_1, \tilde{\gamma}_1, \tilde{\gamma}_2$  come from Assumptions 2 and 4 which are respectively assumed to hold with  $\gamma_1(|e_x|) = \tilde{\gamma}_1|e_x|^2$ ,  $\gamma_2(|e_x|) = \tilde{\gamma}_2|e_x|^2$  where  $\tilde{\gamma}_1, \tilde{\gamma}_2 \geq 0$ . We obtain the following result.

*Theorem 2.* Consider system (29) with the flow and jump sets defined in (30) and suppose the following hold.

- (1) Assumptions 1, 3, 5 and 6 hold.

- (2) Assumptions 2 and 4 are satisfied with  $\gamma_1(s) = \tilde{\gamma}_1 s^2$  and  $\gamma_2(s) = \tilde{\gamma}_2 s^2$  with  $\tilde{\gamma}_1, \tilde{\gamma}_2 \geq 0$ , for  $s \geq 0$ .  
(3) The constant  $T^*$  in (30) is such that  $T^* \in (0, T)$ .

Then there exist  $\beta \in \mathcal{KL}$  and  $\bar{\epsilon} > 0$  such that for any  $\epsilon \in (0, \bar{\epsilon})$  and any solution  $\phi = (\phi_x, \phi_y, \phi_{e_x}, \phi_\tau)$  with  $\phi(0, 0) \in \tilde{C} \cup \tilde{D}$  is complete and satisfies

$$|(\phi_x(t, j), \phi_y(t, j))| \leq \beta(|\phi(0, 0)|, t + j) \quad \forall (t, j) \in \text{dom } \phi. \quad (32)$$

The property (32) requires the initial condition to lie in  $\tilde{C} \cup \tilde{D}$ . That condition adds no conservatism. Indeed, it suffices to set the initial condition of  $\tau$  and  $e_x$  to zero which means that the clock variable starts from zero and that the control input is updated at the initial time. We see that Theorem 2 ensures a global asymptotic stability property which is stronger than the conclusions of Theorem 1. However, it requires an addition condition, namely Assumption 6, to hold. The two classes of event-triggered controllers are compared on a physical example at the end of the next section.

#### 4. ILLUSTRATIVE EXAMPLE

We apply the results of Sections 3.3 and 3.4 to the autopilot control of the longitudinal motion of an F-8 aircraft. We borrow the model from Chapter 4 in Kokotović et al. [1986] which is of the form

$$\dot{x} = A_{11}x + A_{12}z + B_1u \quad (33)$$

$$\epsilon \dot{z} = A_{21}x + A_{22}z + B_2u, \quad (34)$$

where  $x \in \mathbb{R}^2$  represents the slow ‘phugoid mode’ and  $z \in \mathbb{R}^2$  represents the fast ‘short period mode’ of the longitudinal motion of an airplane. The parameter  $\epsilon$  is equal to 0.0336 and

$$\begin{aligned} A_{11} &:= \begin{bmatrix} -0.195378 & -0.676469 \\ 1.478265 & 0 \end{bmatrix} & B_1 &:= \begin{bmatrix} -0.023109 \\ -16.945030 \end{bmatrix} \\ A_{12} &:= \begin{bmatrix} -0.917160 & 0.109033 \\ 0 & 0 \end{bmatrix} & B_2 &:= \begin{bmatrix} -0.048184 \\ -3.810954 \end{bmatrix} \\ A_{21} &:= \begin{bmatrix} -0.051601 & 0 \\ 0.013579 & 0 \end{bmatrix} & A_{22} &:= \begin{bmatrix} -0.367954 & 0.438041 \\ -2.102596 & -0.214640 \end{bmatrix}. \end{aligned}$$

We notice that  $A_{22}$  is invertible and Hurwitz with the eigenvalues  $-8.6696 \pm 28.4712i$ . By following similar lines as in Section 2, the approximate slow model (4) is here  $\dot{x} = A_0x + B_0u$ , where  $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$  and  $B_0 := B_1 - A_{12}A_{22}^{-1}B_2$ . The approximate fast model (6) is  $\frac{dy}{d\tau} = A_{22}y$ . The origin of the open-loop system is globally exponentially stable. Nevertheless, the eigenvalues of the approximate slow system are  $-0.0977 \pm 0.9952i$  and, as a result, the overall system solutions exhibit large oscillations and a slow convergence as shown in Figure 1. Hence, we design the controller  $u = Kx$  to improve the closed-loop response. The gain  $K$  is selected to place the eigenvalues of  $\Lambda_s := A_0 + B_0K$  at  $(-2, -3)$  which is possible since the pair  $(A_0, B_0)$  is controllable. We select  $V_x(x) = x^T P_1 x$  for  $x \in \mathbb{R}^{n_x}$  and  $V_y(x, y) = y^T P_2 y$  for  $y \in \mathbb{R}^{n_z}$  as the Lyapunov functions for the slow and fast models respectively, where  $P_1$  and  $P_2$  are the positive definite symmetric matrices

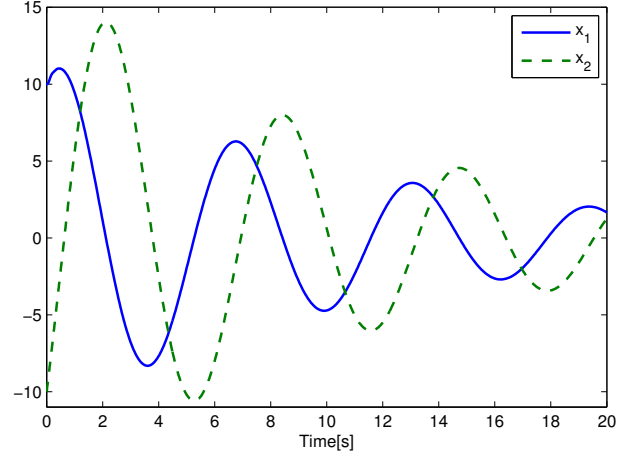


Fig. 1. Open-loop state trajectories of the slow dynamics

such that  $\Lambda_s^T P_1 + P_1 \Lambda_s = -\mathbb{I}_2$  and  $A_{22}^T P_2 + P_2 A_{22} = -\mathbb{I}_2$  (which do exist since  $\Lambda_s$  and  $A_{22}$  are Hurwitz). Hence, Assumptions 2, 3 hold with  $\underline{\alpha}_x(s) = \lambda_{\min}(P_1)s^2$ ,  $\bar{\alpha}_x(s) = \lambda_{\max}(P_1)s^2$ ,  $\gamma_1(s) = 2|P_1 B_0 K|^2 s^2$ ,  $\underline{\alpha}_y(s) = \lambda_{\min}(P_2)s^2$ ,  $\bar{\alpha}_y(s) = \lambda_{\max}(P_2)s^2$  for  $s \geq 0$  and  $\alpha_1 = \frac{1}{2\lambda_{\max}(P_1)}$ ,  $\alpha_2 = \frac{1}{\lambda_{\max}(P_2)}$ . The first two conditions of Assumption 4 are satisfied with  $\beta_1 = 2|P_1 A_{12}| \sqrt{\frac{1}{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}$ ,  $\gamma_2(s) = s^2$  for  $s \geq 0$ ,  $\beta_2 = 2|P_2 \Gamma \Lambda_s| \sqrt{\frac{1}{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}$ , and  $\beta_3 = \frac{2|P_2 \Gamma A_{12}| + |P_2 \Gamma B_0 K|^2}{\lambda_{\min}(P_2)}$ . The third condition is verified with  $L = \frac{1}{2|P_1 B_0 K|^2}$ . Assumption 5 holds with  $\lambda_1 = \frac{|\Gamma_2^2 P_2|}{2|P_1 B_0 K|^2}$  and  $\lambda_2 = \frac{|\Gamma_2^2 P_2|}{|P_1 B_0 K|} \sqrt{\frac{2}{\lambda_{\min}(P_2)}}$ . Assumption 6 is verified with  $M = |B_0 K|$  and  $N = \max\{\frac{|\Lambda_s|}{\lambda_{\min}(P_1)}, \frac{|A_{12}|}{\lambda_{\min}(P_2)}\}$ . Thus, all conditions of Assumptions 1-6 hold.

We then consider the scenario where the controller is implemented over a digital platform and we use the results of Section 3 to design the event-triggering condition. We synthesize the triggering condition (26) with  $\gamma_1 = 1.7795$ ,  $\alpha_1 = 0.3104$  and we set  $\sigma = 0.05$  and  $\rho = 0.0001$ . We take  $\sigma$  small in order to maintain the performance of the continuous-time controller. Second, we apply the technique of Section 3.4 with the same parameter values and  $T^* = 0.0041$  (which has been computed using (31)). The trajectories of the slow state  $x$  in both cases are plotted in Figure 2.

We see that the event-triggered controllers ensure similar performances as in the absence of communication constraints. The strategy in Section 3.3 makes the solutions converge into a ball of radius 0.003 centered at the origin while the state trajectories asymptotically converge to the origin with the technique of Section 3.4. We compare the minimum and the average inter-transmission intervals of the proposed event-triggered strategies which are respectively denoted  $\tau_{\min}$  and  $\tau_{\text{avg}}$ . Table 1 shows the obtained values for 200 initial conditions randomly distributed in the ball centered at the origin of radius 100.

We note that, the event-triggered controllers generate a similar amount of transmissions. Nevertheless, the tech-

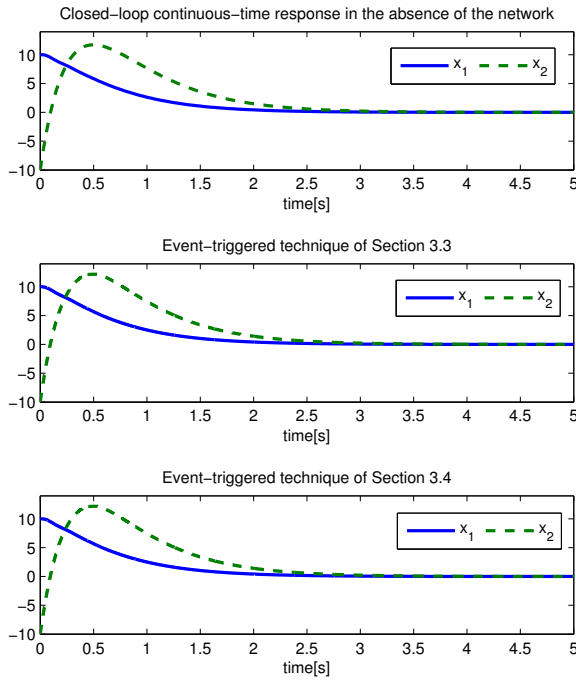


Fig. 2. Closed-loop trajectories of the slow variables

	Section 3.3	Section 3.4
$\tau_{\min}$	$9.2052 \times 10^{-5}$	0.0041
$\tau_{\text{avg}}$	0.0302	0.0301

Table 1. Minimum and average inter-execution times for 200 initial conditions for a simulation time of 2 s.

nique in Section 3.4 exhibits a much larger minimum inter-transmission interval which may be essential in practice.

## 5. CONCLUSION

We have investigated the event-triggered stabilization of nonlinear singularly perturbed systems based only on the slow dynamics. First, we show that semi-global practical stabilization can be obtained using a modified version of the classical triggering condition. Second, under an extra assumption, we prove that global asymptotic stability property can be guaranteed. The presented work will be further extended to the general case where the controller takes into account both the slow and the fast variables.

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