Event-triggered dynamic feedback controllers for nonlinear systems with asynchronous transmissions

Mahmoud Abdelrahim, Romain Postoyan, Jamal Daafouz and Dragan Nešić

Abstract— We study the event-triggered stabilization of non-linear systems using dynamic feedback laws. In particular, we consider the scenario where the plant measurements and the control input are sampled using two independent triggering rules, which leads to asynchronous transmissions. The proposed mechanisms enforce a uniform amount of time between any two transmissions generated by each triggering condition. To that end, we develop and combine time-triggered and event-triggered techniques. We ensure a global asymptotic stability property for the closed-loop system. We present a systematic way to apply the approach to linear time-invariant systems, for which the conditions are formulated as a linear matrix inequality. We compare the obtained results in this particular case with existing techniques in the literature on numerical examples. We emphasize that the presented results are also new in the context of time-triggered control as we explain.

I. INTRODUCTION

Event-triggered control is a sampling paradigm in which a state-dependent criterion is used to trigger transmissions, see [10] and the references therein. The idea is to adapt sampling to the plant state while achieving the control objectives. In that way, the communications between the plant and the controller may be significantly reduced compared to conventional time-triggered setups, which is particularly attractive for networked control systems for instance.

In this paper, we consider nonlinear plants controlled by dynamic feedback laws. We study the scenario where the communications between the plant sensors and the controller, on the one hand, and the controller and the plant actuators, on the other hand, are sampled using two independent rules. The setup is easy to implement in practice as one triggering mechanism is co-located with the sensors, and the other one is implemented at the controller. Since the two triggering mechanisms work independently from each other, this leads to asynchronous transmissions of the plant output and of the control input. Related results concentrate on linear time-invariant (LTI) systems, see [6], [16], to the best of our

knowledge. The purpose of this paper is to present a solution for nonlinear systems.

We focus on the case where only an output of the plant is measured and the closed-loop system is stabilized by a dynamic output-feedback law in the absence of sampling. We then take into account sampling and we design two independent event-triggering conditions to respectively generate the transmission instants of the output measurements and of the control input. The main challenge here is to prevent the occurrence of the Zeno phenomenon, first, because only an output of the plant is measured (and not the full state), second, because each triggering condition only depends on local information, namely the plant output or the control input. These two features may lead to Zeno phenomenon if they are not carefully addressed. To handle these issues, we combine techniques from time-triggered control [12] and event-triggered control [15]. The idea is to wait a fixed amount of time T_y (respectively, T_u) after each transmission of the plant output (respectively, of the control input), and to evaluate an event-triggering condition afterwards. The constants T_y and T_u are obtained by extending the results of [12] to the case of asynchronous time-varying sampling.

The proposed strategy ensures a global asymptotic stability property for the overall system. We show that the method applies to LTI systems as a particular case, for which the required conditions are formulated in terms of a linear matrix inequality (LMI). The feasibility of this LMI condition is guaranteed for any stabilizable and detectable LTI system. We then compare our results with those in [6], [16] on two numerical examples, for which we notice that the amount of transmissions can be reduced with the proposed technique. This work is a generalization of our previous results in [2] where the output measurements and the control input are synchronously updated. This generalization requires a new model, new assumptions as well as a new stability analysis.

The results are also new for time-triggered control, where the constants T_y and T_u mentioned above correspond to the *maximum allowable transmission intervals* for the sampling of the plant output and of the control input, respectively. To the best of our knowledge, asynchronous time-triggered sampling has only been studied in [4], [9]. The authors of [9] consider a different implementation scenario, while the results in [4] are dedicated to a particular class of nonlinear systems where the plant and the controller dynamics are globally Lipschitz, which is relaxed in this paper.

The rest of the paper is organised as follows. Preliminaries are given in Section II. The problem is formally stated in Section III. We give the main results in Section IV.

The application to LTI systems is presented in Section V. Conclusions are provided in Section VI.

II. PRELIMINARIES

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is zero at zero, strictly increasing, and it is of class $\mathcal{K}_\infty$ if in addition $\gamma(s) \to \infty$ as $s \to \infty$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each $t \in \mathbb{R}_{\geq 0}$, $\gamma(\cdot, t)$ is of class $\mathcal{K}$, and, for each $s \in \mathbb{R}_{\geq 0}$, $\gamma(s, \cdot)$ is decreasing to zero. We denote the minimum and maximum eigenvalues of the real symmetric matrix A as $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. We write A^T to denote the transpose of A, and $\mathbb{I}_n$ stands for the identity matrix of dimension n. We write (x, y) to represent the vector $[x^T, y^T]^T$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For a vector $x \in \mathbb{R}^n$, we denote by $|x| := \sqrt{x^T x}$ its Euclidean norm and for a matrix $A \in \mathbb{R}^{n \times m}$, $|A| := \sqrt{\lambda_{\max}(A^T A)}$.

We consider hybrid systems of the following form using the formalism of [8]

$$\dot{x} = F(x) \quad x \in \mathcal{C}, \qquad x^+ \in G(x) \quad x \in \mathcal{D}, \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, F is the flow map, $\mathcal{C}$ is the flow set, G is the jump map, and $\mathcal{D}$ is the jump set. We assume that system (1) satisfies the hybrid basic conditions (see Assumption 6.5 in [8]), which will be the case for the model developed in the next sections. Solutions to system (1) are defined on so-called hybrid time domains. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact hybrid time domain* if $E = \bigcup_{j \in \{0, \ldots, J-1\}} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$ and it is a *hybrid time domain* if for all $(T, J) \in E, E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. A function $\phi : E \to \mathbb{R}^n$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{Z}_{\geq 0}, t \mapsto \phi(t, j)$ is locally absolutely continuous on $I^j := \{t : (t, j) \in E\}$. A hybrid arc ϕ is a solution to system (1) if: (i) $\phi(0, 0) \in \mathcal{C} \cup \mathcal{D}$; (ii) for any $j \in \mathbb{Z}_{\geq 0}$, $\phi(t, j) \in \mathcal{C}$ and $\dot{\phi}(t, j) = F(\phi(t, j))$ for almost all $t \in I^j$; (iii) for every $(t, j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$, $\phi(t, j) \in \mathcal{D}$ and $\phi(t, j+1) \in G(\phi(t, j))$. A solution ϕ to system (1) is *maximal* if it cannot be extended, it is *complete* if its domain, $\operatorname{dom} \phi$, is unbounded, and it is *Zeno* if it is complete and $\sup_t \operatorname{dom} \phi < \infty$, where $\sup_t \operatorname{dom} := \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{Z}_{\geq 0} \text{ such that } (t, j) \in \operatorname{dom} \phi\}$.

III. PROBLEM STATEMENT

Consider the nonlinear plant model

$$\dot{x}_p = f_p(x_p, u), \qquad y = g_p(x_p), \qquad (2)$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input and $y \in \mathbb{R}^{n_y}$ is the measured output. Assume that plant (2) is stabilized by the following dynamic controller

$$\dot{x}_c = f_c(x_c, y), \qquad u = g_c(x_c), \qquad (3)$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state. A common example of feedback laws of the form (3) are observer-based controllers. The functions f_p, f_c are assumed to be continuous and the functions g_p, g_c are assumed to be continuously differentiable.

We study the scenario where the communications between plant (2) and controller (3) are carried out via a digital channel, see Figure 1. In particular, the transmissions of the plant output y to controller (3), and the transmissions of the control input u to plant (2) occur at a priori different sampling instants. We respectively denote the sequences of transmissions of y and of u by $t_i^y, i \in \mathcal{I}_y \subset \mathbb{Z}_{\geq 0}$, and $t_i^u, i \in \mathcal{I}_u \subset \mathbb{Z}_{\geq 0}$. These sequences are defined by two independent triggering conditions, which we design in the following.
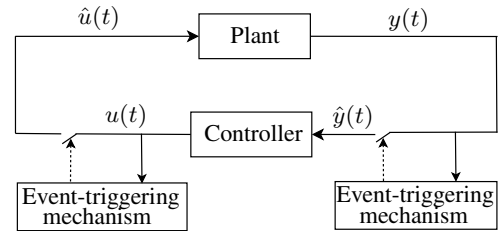

Fig. 1. Event-triggered control schematic.

At each transmission instant $t_i^y, i \in \mathcal{I}_y \subset \mathbb{Z}_{\geq 0}$, the output is sampled and transmitted to the controller. On the other hand, the control input is only broadcasted to the actuators at transmission instants $t_i^u, i \in \mathcal{I}_u \subset \mathbb{Z}_{\geq 0}$. We ignore the possible transmissions delays, but these can be handled like in [15]. In that way, we obtain the impulsive model below

$$
\begin{aligned}
\dot{x}_p &= f_p(x_p, \hat{u}) && \text{for almost all } t \in [t_i^u, t_{i+1}^u] \\
\dot{x}_c &= f_c(x_c, \hat{y}) && \text{for almost all } t \in [t_i^y, t_{i+1}^y] \\
y &= g_p(x_p) \\
u &= g_c(x_c) \\
\dot{\hat{y}} &= 0 && \text{for almost all } t \in [t_i^y, t_{i+1}^y] \\
\dot{\hat{u}} &= 0 && \text{for almost all } t \in [t_i^u, t_{i+1}^u] \\
\hat{y}(t_i^{y+}) &= y(t_i^y) \\
\hat{u}(t_i^{u+}) &= u(t_i^u),
\end{aligned}
\qquad (4)
$$

where $\hat{y}$ and $\hat{u}$ respectively denote the last transmitted values of the plant output and of the control input, which are generated by zero-order-holds between two successive transmission instants. At each transmission instant t_i^y, $\hat{y}$ is reset to the actual value of y. Similarly, $\hat{u}$ is reset to the actual value of u at t_i^{u+}. We define the sampling-induced error $e_y := \hat{y} - y$ and $e_u := \hat{u} - u$, which are reset to 0 at each corresponding transmission instant.

Remark 1. *We can alternatively define the sampling-induced error e_u based on the controller state x_c rather than the control input u, which may help in further reducing the amount of transmissions. In this case, we have $e_u := \hat{x}_c - x_c$, where $\hat{x}_c$ is the value of x_c at the last transmission instant*

5495

of the controller state, which is held constant between two successive updates of controller state. The setup remains the same, the model takes the same form as the one presented below, and the results of Section IV apply to this case. □

We introduce two timers $\tau_y, \tau_u \in \mathbb{R}_{\geq 0}$ to describe the time elapsed since the last transmissions of y and of u, respectively, which have the following dynamics

$$\begin{aligned} \dot{\tau}_y &= 1 \text{ for almost all } t \in [t_i^y, t_{i+1}^y], \quad \tau_y(t_i^{y+}) = 0 \\ \dot{\tau}_u &= 1 \text{ for almost all } t \in [t_i^u, t_{i+1}^u], \quad \tau_u(t_i^{u+}) = 0. \end{aligned} \tag{5}$$

These variables will be useful to define the triggering conditions.

Let $q := (x, e, \tau) \in \mathbb{R}^{n_q}$, where $x := (x_p, x_c) \in \mathbb{R}^{n_x}$, $e := (e_y, e_u) \in \mathbb{R}^{n_e}$, $\tau := (\tau_y, \tau_u) \in \mathbb{R}^2$, $n_x := n_p + n_c$, $n_e := n_y + n_u$, and $n_q := n_x + n_e + 2$. We model the event-triggered controlled system using the hybrid formalism of [8] (like in [2], [7], [13]), for which a jump corresponds to a transmission. Hence, we obtain

$$\begin{aligned} \dot{q} &= F(q) & q \in \mathcal{C}_y \cap \mathcal{C}_u \\ q^+ &\in G(q) & q \in \mathcal{D}_y \cup \mathcal{D}_u, \end{aligned} \tag{6}$$

where $F(q) := (f(x, e), g_y(x, e), g_u(x, e), 1, 1)$ with $f(x, e) := \left(f_p(x_p, g_c(x_c) + e_u), f_c(x_c, g_p(x_p) + e_y) \right)$, $g_y(x, e) := -\frac{d}{dx_p} g_p(x_p) f_p(x_p, g_c(x_c) + e_u)$, and $g_u(x, e) := -\frac{d}{dx_c} g_c(x_c) f_c(x_c, g_p(x_p) + e_y)$. The jump map G is defined as

$$G(q) := \begin{cases} \left\{ (x, 0, e_u, 0, 1) \right\} & q \in \mathcal{D}_y \backslash \mathcal{D}_u \\ \left\{ (x, e_y, 0, 1, 0) \right\} & q \in \mathcal{D}_u \backslash \mathcal{D}_y \\ \left\{ (x, 0, e_u, 0, 1), (x, e_y, 0, 1, 0) \right\} & q \in \mathcal{D}_y \cap \mathcal{D}_u. \end{cases} \tag{7}$$

The sets $\mathcal{C}_y, \mathcal{D}_y$ are defined according to the triggering condition for the plant measurements and the sets $\mathcal{C}_u, \mathcal{D}_u$ are constructed based on the triggering condition for the control input. Solutions to system (6) flow on $\mathcal{C}_y \cap \mathcal{C}_u$, which corresponds to the region of the state space where both triggering conditions are not satisfied. When only the triggering condition of the plant measurements or of the control input is verified, i.e. $q \in \mathcal{D}_y \backslash \mathcal{D}_u$ or $q \in \mathcal{D}_u \backslash \mathcal{D}_y$ respectively, the system experiences a jump according to (7). When both triggering conditions are satisfied at the same instant, i.e. $q \in \mathcal{D}_y \cap \mathcal{D}_u$, solutions experience two successive jumps, in view of (7). This modeling choice is justified by the fact that it ensures that the jump map G is outer semicontinuous (see Definition 5.9 in [8]), as shown in [14], which is one of the hybrid basic conditions (see Assumption 6.5 in [8]). This would not be the case if we would define $G(q)$ as $\{(x, 0, 0, 0, 0)\}$ when $q \in \mathcal{D}_y \cap \mathcal{D}_u$.

Our main objective is to synthesize asynchronous event-triggering rules, i.e. the sets $\mathcal{C}_y, \mathcal{D}_y$ and $\mathcal{C}_u, \mathcal{D}_u$, to ensure a global asymptotic stability property for system (6) as well as the existence of a strictly positive minimum time between two transmissions of each triggering condition.

IV. MAIN RESULTS

We first present the assumptions we make on system (6), we then state the main stability result. Application to time-triggered control is studied at the end of this section.

A. Assumptions

We make the following assumption on system (6), which is inspired by [12].

Assumption 1. *There exist a locally Lipschitz function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, locally Lipschitz positive semi-definite functions $W_y : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}, W_u : \mathbb{R}^{n_u} \to \mathbb{R}_{\geq 0}$, continuous functions $H_y, H_u : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, real numbers $L_{y_1}, L_{y_2}, L_{u_1}, L_{u_2} \geq 0$, $\gamma_y, \gamma_u > 0$, $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, a continuous, positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and continuous, functions $\delta_y : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$, $\delta_u : \mathbb{R}^{n_u} \to \mathbb{R}_{\geq 0}$ such that*

(i) *for all $x \in \mathbb{R}^{n_x}$*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|); \tag{8}$$

(ii) *for all $e \in \mathbb{R}^{n_e}$ and almost all $x \in \mathbb{R}^{n_x}$*

$$\begin{aligned} \langle \nabla V(x), f(x, e) \rangle \leq &-\alpha(|x|) - H_y^2(x) - H_u^2(x) - \delta_y(y) \\ &- \delta_u(u) + \gamma_y^2 W_y^2(e_y) + \gamma_u^2 W_u^2(e_u); \end{aligned} \tag{9}$$

(iii) *for all $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$*

$$\begin{aligned} \langle \nabla W_y(e_y), g_y(x, e) \rangle &\leq L_{y_1} W_y(e_y) + L_{y_2} W_u(e_u) \\ &\quad + H_y(x) \\ \langle \nabla W_u(e_u), g_u(x, e) \rangle &\leq L_{u_1} W_u(e_u) + L_{u_2} W_y(e_y) \\ &\quad + H_u(x). \end{aligned} \tag{10}$$

□

Conditions (8)-(9) imply that the system $\dot{x} = f(x, e)$ is $\mathcal{L}_2$-gain stable from (W_y, W_u) to $(H_y, H_u, \sqrt{\delta_y}, \sqrt{\delta_u})$. This robustness property is useful to interpret the interconnection between the x-system and the e-system. Indeed, condition (9) suggests that the x-system is affected by the e-system through the input $(W_y(e_y), W_u(e_u))$ and similarly the e-system is affected by the x-system via the output $(H_y, H_u, \sqrt{\delta_y}, \sqrt{\delta_u})$. This interpretation was originally proposed in [11] in the context of time-triggered control. Condition (10) is an exponential growth of the e_y-system and of the e_u-system on flows. Contrary to [2], we need to introduce two functions W_y and W_u of the sampling-induced errors, and not a single one. We show in Section V that Assumption 1 can always be ensured for stabilizable and detectable LTI systems for instance.

B. Asynchronous event-triggering conditions

In view of (9), a direct extension of [15] leads to the triggering conditions $\gamma_y^2 W_y^2(e_y) \leq \delta_y(y)$ and $\gamma_u^2 W_u^2(e_u) \leq \delta_u(u)$ which ensure that V in (9) decreases on flows. However, this may lead to Zeno behaviour as shown in [6]. To overcome this issue, we have proposed in [2] to

5496

combine the time-triggering technique in [12] with the event-triggering approach of [15]. We now generalize this approach to address asynchronous transmissions. For that purpose, we need to carefully handle the possible mutual effect of each sampling-induced error on each other, which is reflected by (10).

We propose to transmit the plant output to the controller when the condition below is verified

$$\left(\gamma_y^2 + \frac{L_{u_2}^2}{\kappa_u}\right)W_y^2(e_y) \geq \delta_y(y) \text{ and } \tau_y \geq T_y, \quad (11)$$

where $\gamma_y, L_{u_2}, W_y, \delta_y$ come from Assumption 1, κ_u is any strictly positive constant, and $T_y > 0$ is given below. We recall that τ_y is one of the timers introduced in Section III. The stability analysis (removed due to space reasons) suggests to define the event-triggering part of the mechanism as in (11) and not by $\gamma_y^2 W_y^2(e_y) \leq \delta_y(y)$, which is another substantial difference with the synchronous case in [2]. In view of (11), two successive transmissions of y cannot occur before T_y units of time have elapsed.

Similarly, we only update the control input when the following triggering condition is satisfied

$$\left(\gamma_u^2 + \frac{L_{y_2}^2}{\kappa_y}\right)W_u^2(e_u) \geq \delta_u(u) \text{ and } \tau_u \geq T_u, \quad (12)$$

where $\gamma_u, L_{y_2}, W_u, \delta_u$ come from Assumption 1, κ_y is any strictly positive constant, and $T_u > 0$ is given below. Note that (12) ensures that the inter-transmission times of the control input are lower bounded by T_u. The constants T_y and T_u are selected such that $T_y < \mathcal{T}_y(\gamma_y, L_{y_1}, L_{u_2}, \kappa_y)$ and $T_u < \mathcal{T}_u(\gamma_u, L_{u_1}, L_{y_2}, \kappa_u)$, where

$$\mathcal{T}_y(\gamma_y, L_{y_1}, L_{u_2}, \kappa_y) := \begin{cases} \frac{1}{L_{y_1} r_y}\arctan(r_y) & \lambda_y > L_{y_1} \\ \frac{1}{L_{y_1}} & \lambda_y = L_{y_1} \\ \frac{1}{L_{y_1} r_y}\text{arctanh}(r_y) & \lambda_y < L_{y_1} \end{cases}$$

$$\mathcal{T}_u(\gamma_u, L_{u_1}, L_{y_2}, \kappa_u) := \begin{cases} \frac{1}{L_{u_1} r_u}\arctan(r_u) & \lambda_u > L_{u_1} \\ \frac{1}{L_{u_1}} & \lambda_u = L_{u_1} \\ \frac{1}{L_{u_1} r_u}\text{arctanh}(r_u) & \lambda_u < L_{u_1} \end{cases}$$
$$(13)$$

with $r_y := \sqrt{\left|\left(\frac{\lambda_y}{L_{y_1}}\right)^2 - 1\right|}$, $r_u := \sqrt{\left|\left(\frac{\lambda_u}{L_{u_1}}\right)^2 - 1\right|}$, $\lambda_y := \sqrt{(1+\kappa_y)(\gamma_y^2 + \frac{L_{u_2}^2}{\kappa_u})}$, $\lambda_u := \sqrt{(1+\kappa_u)(\gamma_u^2 + \frac{L_{y_2}^2}{\kappa_y})}$. The constants κ_y and κ_u offer a trade-off between $\mathcal{T}_y$ and $\mathcal{T}_u$ (we omit the arguments of $\mathcal{T}_y$ and $\mathcal{T}_u$ in the sequel for the sake of convenience). Indeed, reducing κ_y generates a decrease of λ_y, and therefore an increase of $\mathcal{T}_y$. On the other hand, this also lead to an increase of λ_u and thus a decrease of $\mathcal{T}_u$. The selection of an optimal pair (κ_y, κ_u) seems to be non-trivial and is out of the scope of this paper, noting that these constants also appear in the event-triggering conditions in view of (11) and (12).

The flow and the jump sets in (6) are now given by

$$\begin{aligned} \mathcal{C}_y &:= \left\{q : \left(\gamma_y^2 + \frac{L_{u_2}^2}{\kappa_u}\right)W_y^2(e_y) \leq \delta_y(y) \text{ or } \tau_y \in [0, T_y]\right\} \\ \mathcal{C}_u &:= \left\{q : \left(\gamma_u^2 + \frac{L_{y_2}^2}{\kappa_y}\right)W_u^2(e_u) \leq \delta_u(u) \text{ or } \tau_u \in [0, T_u]\right\} \\ \mathcal{D}_y &:= \left\{q : \left(\gamma_y^2 + \frac{L_{u_2}^2}{\kappa_u}\right)W_y^2(e_y) \geq \delta_y(y) \text{ and } \tau_y \geq T_y\right\} \\ \mathcal{D}_u &:= \left\{q : \left(\gamma_u^2 + \frac{L_{y_2}^2}{\kappa_y}\right)W_u^2(e_u) \geq \delta_u(u) \text{ and } \tau_u \geq T_u\right\}. \end{aligned}$$
$$(14)$$

We note that both $\hat{y}$ and $\hat{u}$ may be updated at the same time instant, which prevents the existence of a dwell-time for the overall system (6), like in other works on asynchronous event-triggered control, see [6], [16].

C. Stability result

The following theorem presents the ensured global asymptotic stability property for system (6), (14). The proof has been omitted due to space constraints.

Theorem 1. *Suppose that Assumption 1 holds and consider system (6) with the flow and the jump sets in (14), where the constant T_y, T_u are such that $T_y \in (0, \mathcal{T}_y)$ and $T_u \in (0, \mathcal{T}_u)$. There exists $\beta \in \mathcal{KL}$ such that any solution $\phi = (\phi_x, \phi_e, \phi_\tau)$ satisfies, for all $(t, j) \in \text{dom}\,\phi$*

$$|\phi_x(t, j)| \leq \beta(|(\phi_x(0, 0), \phi_e(0, 0))|, t + j). \quad (15)$$

Moreover, if ϕ is maximal, then it is complete. □

Theorem 1 shows that the states of the plant and of the controller, i.e. x, asymptotically converge to the origin. Note that the existence of strictly positive lower bounds on the inter-transmission times of the output measurements and of the control input directly follows from (14).

D. Asynchronous time-triggered implementation

The case where transmissions are time-triggered naturally proceeds from the results above. In this scenario, the flow and the jump sets become

$$\begin{aligned} \mathcal{C}_y &= \{q : \tau_y \in [0, T_y]\}, \quad \mathcal{D}_y = \{q : \tau_y \in [\varepsilon_y, T_y]\} \\ \mathcal{C}_u &= \{q : \tau_u \in [0, T_u]\}, \quad \mathcal{D}_u = \{q : \tau_u \in [\varepsilon_u, T_u]\}, \end{aligned} \quad (16)$$

where $\varepsilon_y \in (0, T_y]$, $\varepsilon_u \in (0, T_u]$, and T_y, T_u are strictly smaller than $\mathcal{T}_y, \mathcal{T}_u$ defined in (13). The constants $\varepsilon_y, \varepsilon_u$ respectively represent the minimum time between two transmissions of each triggering condition. When $\varepsilon_y = T_y$ and $\varepsilon_u = T_u$, the sets in (16) generate periodic transmissions of y and u, respectively. The following corollary extends the results in [12] to asynchronous time-triggered control. Its proof follows the same lines as the proof of Theorem 1, and is also omitted.

Corollary 1. *Suppose that Assumption 1 holds and consider system (6) with the flow and the jump sets in (16). Then, the conclusions of Theorem 1 hold.* □

5497

V. LTI SYSTEMS

We study here the application of the results in Section IV to LTI systems. We consider two approaches commonly used in this context, see Section IV in [10]. The first approach consists of stabilizing the plant by a *direct* dynamic output-based controller, i.e. without an observer. The second strategy is to estimate the plant state by a *Luenberger observer*.

A. Direct dynamic output-based controllers

1) Analysis: Consider the LTI plant model

$$\dot{x}_p = A_p x_p + B_p u, \qquad y = C_p x_p, \tag{17}$$

where $x_p \in \mathbb{R}^{n_p}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ and A_p, B_p, C_p are matrices of appropriate dimensions. Assume that the plant is stabilized by a dynamic controller of the form

$$\dot{x}_c = A_c x_c + B_c y, \qquad u = C_c x_c, \tag{18}$$

where $x_c \in \mathbb{R}^{n_c}$ and A_c, B_c, C_c are matrices of appropriate dimensions. We then take into account the sampling. By following the same lines as in Section IV, we obtain the hybrid model below (recall that $q = (x, e_y, e_u, \tau_y, \tau_u)$)

$$\dot{q} = \begin{pmatrix} \mathcal{A}_1 x + \mathcal{B}_1 e_y + \mathcal{M}_1 e_u \\ \mathcal{A}_2 x + \mathcal{B}_2 e_y + \mathcal{M}_2 e_u \\ \mathcal{A}_3 x + \mathcal{B}_3 e_y + \mathcal{M}_3 e_u \\ 1 \\ 1 \end{pmatrix} \quad q \in \mathcal{C}_y \cap \mathcal{C}_u \tag{19}$$

$$q^+ \in G(q) \qquad\qquad q \in \mathcal{D}_y \cup \mathcal{D}_u,$$

where $G(q)$ is given by (7), $\mathcal{A}_1 := \begin{bmatrix} A_p & B_p C_c \\ B_c C_p & A_c \end{bmatrix}$, $\mathcal{B}_1 := \begin{bmatrix} 0 \\ B_c \end{bmatrix}$, $\mathcal{M}_1 := \begin{bmatrix} B_p \\ 0 \end{bmatrix}$, $\mathcal{A}_2 = \begin{bmatrix} -C_p A_p & -C_p B_p C_c \end{bmatrix}$, $\mathcal{B}_2 := 0$, $\mathcal{M}_2 := -C_p B_p$, $\mathcal{A}_3 := \begin{bmatrix} -C_c B_c C_p & -C_c A_c \end{bmatrix}$, $\mathcal{B}_3 := -C_c B_c$, and $\mathcal{M}_3 = 0$.

The proposition below states that the satisfaction of a linear matrix inequality ensures that the conclusions of Theorem 1 hold in this case. The proof has been omitted due to space constraints.

Proposition 1. *Consider system (19). Suppose that there exist $\varepsilon_x, \varepsilon_y, \varepsilon_u, \mu_y, \mu_u > 0$ and a positive definite symmetric real matrix P such that*

$$\begin{pmatrix} \Sigma & P\mathcal{B}_1 & P\mathcal{M}_1 \\ \mathcal{B}_1^T P & -\mu_y \mathbb{I}_{n_y} & 0 \\ \mathcal{M}_1^T P & 0 & -\mu_u \mathbb{I}_{n_u} \end{pmatrix} \leq 0, \tag{20}$$

where $\Sigma := \mathcal{A}_1^T P + P\mathcal{A}_1 + \varepsilon_x \mathbb{I}_{n_x} + \mathcal{A}_2^T \mathcal{A}_2 + \mathcal{A}_3^T \mathcal{A}_3 + \varepsilon_y \mathcal{C}_1^T \mathcal{C}_1 + \varepsilon_u \mathcal{C}_2^T \mathcal{C}_2$ with $\mathcal{C}_1 := (C_p, 0)^T$ and $\mathcal{C}_2 := (0, C_c)^T$. Take $V(x) = x^T P x$, $W(e_y) = |e_y|$ and $W(e_u) = |e_u|$, for $x \in \mathbb{R}^{n_x}, (e_y, e_u) \in \mathbb{R}^{n_e}$. Then Assumption 1 holds with $\underline{\alpha}(s) = \lambda_{\min}(P)s^2$, $\overline{\alpha}(s) = \lambda_{\max}(P)s^2$, $\alpha(s) = \varepsilon_x s^2$ for $s \geq 0$, $H_y(x) = |\mathcal{A}_2 x|$, $H_u(x) = |\mathcal{A}_3 x|$, $\delta_y(y) = \varepsilon_y |y|^2$, $\delta_u(u) = \varepsilon_u |u|^2$, $L_{y_1} = |\mathcal{B}_2|$, $L_{y_2} = |\mathcal{M}_2|$, $L_{u_1} = |\mathcal{M}_3|$, $L_{u_2} = |\mathcal{B}_3|$, $\gamma_y = \sqrt{\mu_y}$, and $\gamma_u = \sqrt{\mu_u}$. □

Proposition 1 provides a systematic way to satisfy Assumption 1 for LTI systems by casting the required conditions in terms of LMI (20). By using the Schur complement twice (see Section A.5.5 in [5]), the feasibility of LMI (20) is guaranteed if there exists a symmetric, positive definite matrix P such that $\Sigma \leq 0$ and by taking $\mu_y, \mu_u > 0$ sufficiently large. When system (17) is stabilizable and detectable, we can always design a feedback law (18) such that $\mathcal{A}_1$ is Hurwitz, from which we conclude that such a matrix P always exists, hence (20) holds.

2) Example: We consider Example 2 in [6]. The plant and the dynamic controller are given by (17), (18) with $A_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_p = \begin{bmatrix} -1 & 4 \end{bmatrix}$, $A_c = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}$, $B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C_c = \begin{bmatrix} 1 & -4 \end{bmatrix}$. We apply Proposition 1 and we obtain $L_{y_1} = 0, L_{y_2} = 4, L_{u_1} = 0, L_{u_2} = 4$, $\varepsilon_x = 18.0737, \varepsilon_y = 2.08968, \varepsilon_u = 1.8755$ and $\gamma_y = 97.9153, \gamma_u = 100.9371$. In view of (13) (we set $\kappa_y = 0.5, \kappa_u = 0.01$), we have $\mathcal{T}_y = 0.0121, \mathcal{T}_u = 0.0155$. We take $T_y = 0.012, T_u = 0.015$ and we run simulations with the initial condition $(x(0,0), e(0,0)) = (12.5, -12.5, -12.5, 12.5, 0, 0)$, as in [6], and $\tau(0,0) = (0,0)$. The guaranteed lower bounds in [6] on the inter-transmission times of y, u are both equal to 6.5×10^{-9} and the simulated minimum times have been found to be $\tau_{\min}^y = \tau_{\min}^u = 10^{-4}$, while we have $\tau_{\min}^y = 0.012$ and $\tau_{\min}^u = 0.015$ in our case which are larger than those obtained by [6]. Moreover, we ensure a global asymptotic stability property for the closed-loop system, as opposed to a practical property in [6]. A zoom-in of the inter-transmission times is provided in Figure 2 where we notice that the minimum inter-transmission times are equal to the enforced lower bounds T_y, T_u, respectively.
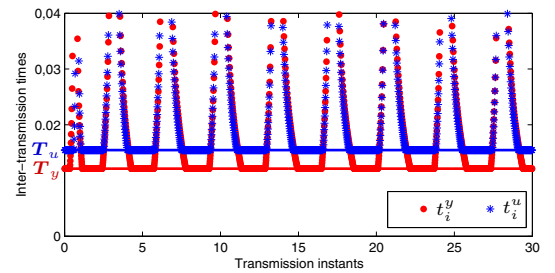

Fig. 2. Inter-transmission times for the example of Section V-A.2.

B. Observer-based controllers

1) Analysis: An alternative approach to stabilize (17) is to design an observer-based feedback law. In this case, the controller takes the form

$$\dot{x}_c = A_p x_c + B_p u + B_c(y - C_p x_c), \qquad u = C_c x_c, \tag{21}$$

where x_c is the estimated state and C_c, B_c are designed such that $A_p + B_p C_c$ and $A_p - B_c C_p$ are Hurwitz (this is always possible when plant (17) is stabilizable and detectable). We take into account sampling and we introduce

5498

the induced error variable. It is convenient in this case to define e_u based on the estimated state x_c, i.e. $e_u := \hat{x}_c - x_c$, where $\hat{x}_c$ denotes the value of x_c at the last transmission instant t_i^u, see Remark 1. By following similar lines as before, we obtain the hybrid model (19) with
$$\mathcal{A}_1 := \begin{bmatrix} A_p & B_p C_c \\ B_c C_p & A_p + B_p C_c - B_c C_p \end{bmatrix}, \mathcal{B}_1 := \begin{bmatrix} 0 \\ B_c \end{bmatrix}, \mathcal{M}_1 :=$$
$$\begin{bmatrix} B_p C_c \\ B_p C_c \end{bmatrix}, \mathcal{A}_2 := \begin{bmatrix} -C_p A_p & -C_p B_p C_c \end{bmatrix}, \mathcal{B}_2 := 0, \mathcal{M}_2 :=$$
$-C_p B_p C_c$, $\mathcal{A}_3 := \begin{bmatrix} -B_c C_p & -(A_p + B_p C_c - B_c C_p) \end{bmatrix}$, $\mathcal{B}_3 := -B_c$, and $\mathcal{M}_3 = -B_p C_c$. Then Proposition 1 holds.

2) Example: We revisit the numerical example in [16] where plant (17) is given by $A_p = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The plant is stabilized by an observer-based controller of the form (21) with $B_c = \begin{bmatrix} 10 & 14 \end{bmatrix}$ and $C_c = \begin{bmatrix} -15 \\ -10 \end{bmatrix}$. By solving (20), we obtain, $L_{y_1} = 0, L_{y_2} = 0, L_{u_1} = 18.0278, L_{u_2} = 17.2047, \varepsilon_x = 27.7148, \varepsilon_y = 143.6239, \varepsilon_u = 0.8865$ and $\gamma_y = 208.8035, \gamma_u = 137.5287$. Then, by using (13) (we set $\kappa_y = 0.1, \kappa_u = 4$), we have $\mathcal{T}_y = 7.2 \times 10^{-3}, \mathcal{T}_u = 4.9 \times 10^{-3}$. We take $T_y = 7 \times 10^{-3}, T_u = 4.5 \times 10^{-3}$ and we run simulations with the initial condition $(x(0,0), e(0,0)) = (2,3,0,0,0,0,0)$, as in [16], and $\tau(0,0) = (0,0)$. The obtained average inter-transmission times are $\tau_{avg}^y = 13 \times 10^{-3}$ and $\tau_{avg}^u = 5 \times 10^{-3}$. The minimum and the average inter-transmission times by Architecture III in [16] are $\tau_{min}^y = 0.61 \times 10^{-3}, \tau_{min}^u = 0.67 \times 10^{-3}, \tau_{avg}^y = 2.2 \times 10^{-3}$, and $\tau_{avg}^u = 2.5 \times 10^{-3}$ which are all smaller than the values obtained with the proposed technique. Figure 3 shows a zoom-in of the inter-transmission times for the plant output y and for the controller state x_c. We notice the interaction between the time-triggering and the event-triggering rules.
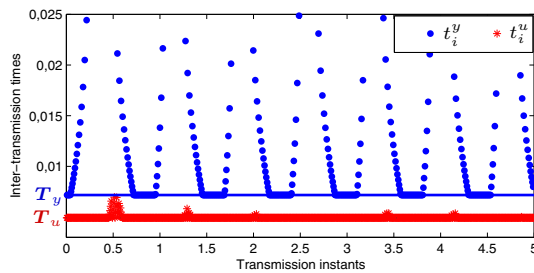

Fig. 3. Inter-transmission times for the example of Section V-B.2

VI. Conclusion

We have investigated the stabilization of nonlinear systems using output feedback event-triggered controllers with asynchronous communications. The proposed technique ensures a global asymptotic stability property for the closed-loop system and enforces strictly positive uniform lower bounds on the inter-transmissions times of each triggering condition. The results are applied to LTI systems as a particular case. The proposed mechanism encompasses the case of asynchronous time-triggering which is also a new result in this context.

This work can be extended in several ways. For instance, the developed results in this paper are dedicated to the disturbance free case. Hence, it is of practical importance to study the robustness of the proposed approach with respect to different exogenous inputs as we have done in [3] for the synchronous implementation scenario. This extension is currently under investigation. Another interesting problem is the joint design of the output feedback law and the event-triggering mechanism for linear systems in the spirit of [1] under asynchronous event-triggered communications.

References

[1] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić. Co-design of output feedback laws and event-triggering conditions for linear systems. *In Proceedings of the 53rd IEEE Conference on Decision and Control, Los Angeles, U.S.A.*, pages 3560–3565, 2014.

[2] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić. Stabilization of nonlinear systems using event-triggered output feedback laws. *In Proceedings of the 21th International Symposium on Mathematics Theory of Networks and Systems, Groningen, The Netherlands*, pages 274–281, 2014.

[3] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić. Input-to-state stabilization of nonlinear systems using event-triggered output feedback controllers. *In Proceedings of the 14th European Control Conference, Linz, Austria*, pages 2185–2190, 2015.

[4] D. Antunes, J. Hespanha, and C. Silvestre. Stability of networked control systems with asynchronous renewal links: An impulsive systems approach. *Automatica*, 49(2):402–413, 2013.

[5] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[6] M.C.F. Donkers and W.P.M.H. Heemels. Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralised event-triggering. *IEEE Transactions on Automatic Control*, 57(6):1362–1376, 2012.

[7] F. Forni, S. Galeani, D. Nešić, and L. Zaccarian. Event-triggered transmission for linear control over communication channels. *Automatica*, 50(2):490–498, 2014.

[8] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[9] W.P.M.H. Heemels, D.P. Borgers, N. van de Wouw, D. Nešić, and A.R. Teel. Stability analysis of nonlinear networked control systems with asynchronous communication: A small-gain approach. *In Proceedings of the 52nd IEEE Conference on Decision and Control, Firenze, Italy*, pages 4631–4637, 2013.

[10] W.P.M.H. Heemels, K.H. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. *In Proceedings of the 51st IEEE Conference on Decision and Control, Maui, U.S.A.*, pages 3270–3285, 2012.

[11] D. Nešić and A.R. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49(10):1650–1667, 2004.

[12] D. Nešić, A.R. Teel, and D. Carnevale. Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems. *IEEE Transactions on Automatic Control*, 54(3):619–624, 2009.

[13] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta. A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 60(4):982–996, 2015.

[14] R.G. Sanfelice, J.J.B. Biemond, N. van de Wouw, and W.P.M.H. Heemels. An embedding approach for the design of state-feedback tracking controllers for references with jumps. *International Journal of Robust and Nonlinear Control*, 24(11):1585–1608, 2014.

[15] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, 2007.

[16] P. Tallapragada and N. Chopra. Event-triggered dynamic output feedback control for LTI systems. *In Proceedings of the 51st IEEE Conference on Decision and Control, Maui, U.S.A.*, pages 6597–6602, 2012.