Physics Enrichment Program HKUST
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- Calculate capacitance / inductance of a given structure
  - ⇒ By Gauss Law / Faraday law
- Solve a circuit.
  - Write Kirchoff's Law
  - Solve by ODE / phasor / impedance
- Advanced circuit techniques

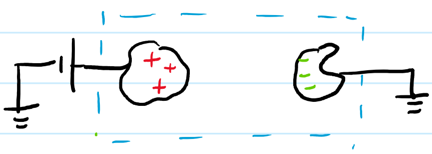
Capacitance

Definition : $C = \dfrac{\text{charge stored}}{\text{potential applied}} = \dfrac{Q}{V}$

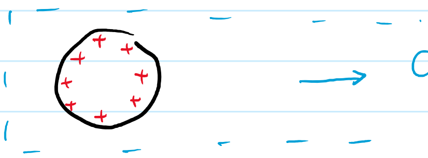
Circuit symbol of capacitor : —||— (parallel plates)

① About Potential

- If you are given 2 objects with opposite charges
  → Take V be the potential difference between 2 objects.



- If you are given only 1 object
  → Take V be the potential difference from infinity



→ Capacitance = potential relative to ∞ per charge added

② Finding capacitance from charge / potential

It is always either of the case :

$\boxed{1}$ Knowing the Q on the objects, find the result V.

(Easier)

Essentially solving Gauss Law

- Solve E by $\nabla \cdot E = -\frac{\rho}{\varepsilon_0}$, then take $-\int E \cdot dl = V$

  (Gauss Law integral form + symmetry claim , image method , ...)

- Solve V directly by $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ (Solve PDE :( )

$\boxed{2}$ Knowing V between the objects , find the Q on each object

(Harder)

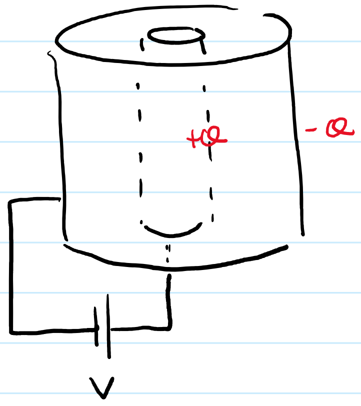
Note that you are not given the V over the entire space

Finding charge distribution is not trivial.

- Argue what V/E distribution by symmetry claims
- Solve V by $\nabla^2 V = 0$ for anywhere without charges

After then we can find Q by Gauss Law

E.g. Cylindrical capacitor



$\boxed{1}$ Assume the charges on inner plate $= Q$

outer plate $= -Q$

$\boxed{2}$ Find $\vec{E}$ by Gauss Law

$$|\vec{E}| = \frac{Q}{2\pi\varepsilon_0 r L} \text{ , point outward}$$

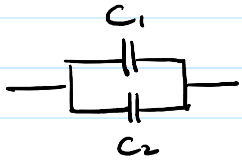
$\boxed{3}$ Find V by $\int \vec{E} \cdot dl = \int_a^b |E| dr$

$$V = \frac{Q}{2\pi\varepsilon_0 L} \ln\left(\frac{b}{a}\right)$$

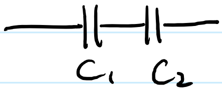
$\boxed{4}$ Capacitance $= \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$

③ Rules for capacitance addition

Parallel : $C_{equiv} = C_1 + C_2$



Series : $\frac{1}{C_{equiv}} = \frac{1}{C_1} + \frac{1}{C_2}$



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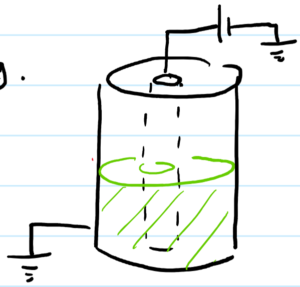

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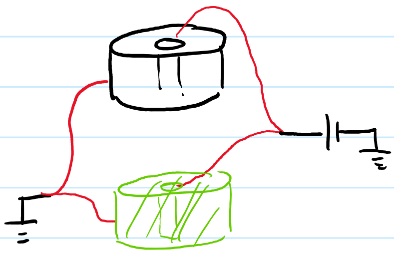


2 capacitors in parallel

fill lower half by dielectric

☆☆ Require V to be the same on both capacitor

⇒ Expect they carry different charges.

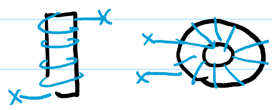
<u>Inductance</u>

Definition : $L = \frac{|\varepsilon|}{\frac{dI}{dt}} = \frac{\text{Induced emf}}{\text{rate of current change}}$

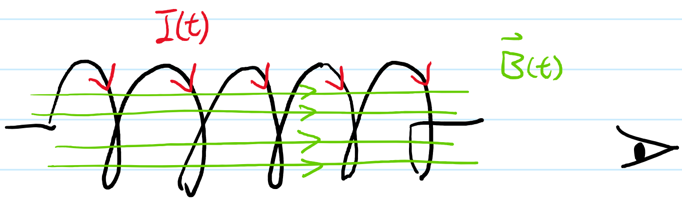
Circuit symbol of inductor =  (solenoid)

There are not many shapes of inductor

Basically you can only find solenoid or toroid

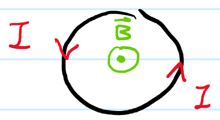


E.g.



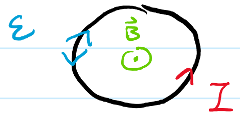
[I] By Ampere's Law, $B(t) = \frac{\mu_0 N I(t)}{L}$

[2] Flux by B field :



$$\Phi_B(t) = B(t) \cdot N \cdot 2\pi r^2 = \frac{\mu_0 N^2 \cdot \pi r^2}{L} I(t)$$

[3] Find emf by Faraday's Law (And direction by Lenz's Law)



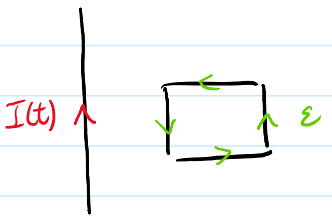
$$|\varepsilon(t)| = \frac{d\Phi_B(t)}{dt} = \frac{\mu_0 N^2 \cdot 2\pi r^2}{L} \frac{dI(t)}{dt}$$

[4] (Self) Inductance $= L = \frac{|\varepsilon(t)|}{\frac{dI(t)}{dt}} = \frac{\mu_0 N^2 \cdot 2\pi r^2}{L}$

2 types of inductance

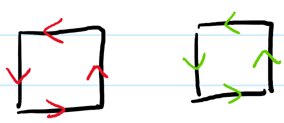
- Self inductance = Induced emf feedback on the original current
- Mutual inductance = Induced emf is created on other circuit, but has no effect on the original current

E.g. 1



Changing $I(t)$ induces ε on RHS ring but has no effect on itself

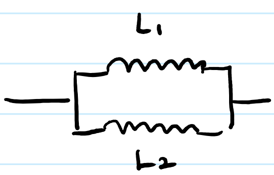
E.g. 2



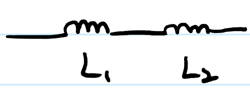
It is called "mutual" because we can have 2 objects induce emf on each other

Rules for inductor addition

Parallel : $L_{equiv} = L_1 + L_2$



Series : $\frac{1}{L_{equiv}} = \frac{1}{L_1} + \frac{1}{L_2}$



Circuit Analysis

Step 1: Simplify circuit, eg. Series / Parallel combination

Step 2: Write Kirchoff 's Law

Step 3: Solve by ODE / phasor / impedance method

ODE: what is taught to physics students (very slow)

phasor: what is taught to people who does not know complex no. (geometrical method)

impedance: most commonly use in practice (fastest)

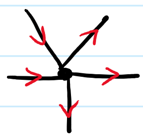
Kirchoff 's Law

① Kirchoff Current Law (KCL)

② Kirchoff Voltage Law (KVL)

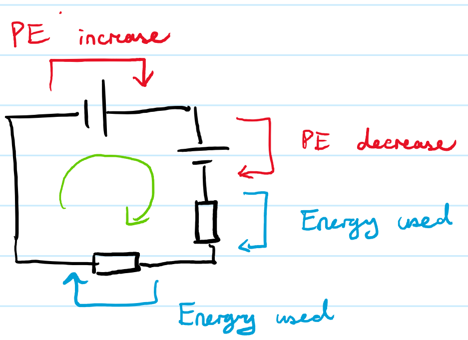
- KCL ≡ conservation of charge at a junction

$$\underbrace{\sum I_{in}}_{\text{rate of charge flow in}} = \underbrace{\sum I_{out}}_{\text{rate of charge flow out}}$$



- KVL ≡ conservation of energy over a loop

$$\sum \Delta V = 0 \quad \text{for any loop}$$



All PE gained should be used up after the charge travel back to its starting point. Or else energy accumulate

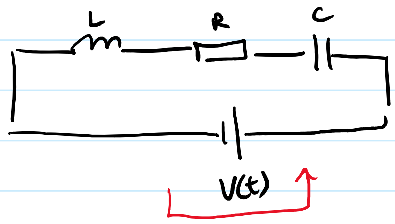
The 3 methods to solve circuit

① ODE

By definition, $I = \frac{dQ}{dt}$ = rate of charge flow through

$$\Rightarrow \begin{cases} C = \frac{Q}{V} & \Rightarrow \quad V_C = \frac{Q}{C} \\ R = \frac{V}{I} & \Rightarrow \quad V_R = IR = R\frac{dQ}{dt} \\ L = \frac{V}{\frac{dI}{dt}} & \Rightarrow \quad V_L = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2} \end{cases}$$

E.g. The easiest circuit = LRC in series



By KVL:

$$V(t) - \frac{Q}{C} - R\frac{dQ}{dt} - L\frac{d^2Q}{dt^2} = 0$$

This is a 2^{nd} order linear inhomogeneous ODE

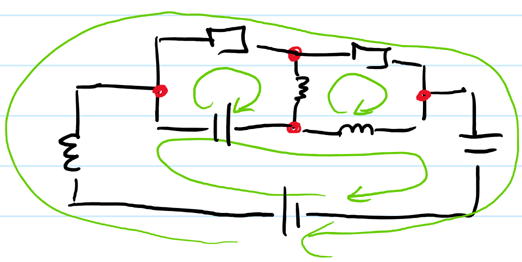
$$Q(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t} + (\text{particular soln})$$

when $\lambda_\pm = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$

But if the circuit gets more complicated

You get a lot more ODEs

E.g.



4 junctions = Can write at most 4 KCL equations

4 loops = Can write at most 4 KVL equations

$\Rightarrow$ A system of ODE with 8 equalities.

(Although they are not all independent of each other)

☆ When the circuit gets more complicated, solving ODE system is too annoying. So no one does it in practice.

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② Phasor = Geometrical method to avoid complex number.

☆ Only applicable for sinusoidal voltage input of pure frequency.

i.e. $V(t) = V_0 e^{i\omega t} = V_1 \cos(\omega t) + V_2 \sin(\omega t)$

ω must be taken as some constant

Given the voltage term being sinusoidal, the ODE can be simplified

$$V_0 e^{i\omega t} = L \frac{d^2}{dt^2} Q + R \frac{d}{dt} Q + \frac{Q}{C}$$

$\boxed{1}$ Let the soln of $Q(t)$ be in the form ;

$$Q(t) = Q_0 e^{i\omega t}$$ (Note that Q_0 can be a complex number)

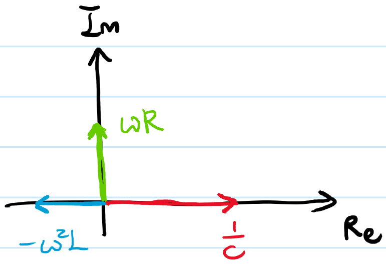
$\boxed{2}$ By substitution, all $e^{i\omega t}$ are cancelled

$$V_0 \cancel{e^{i\omega t}} = L \cdot Q_0 (-\omega^2) \cancel{e^{i\omega t}} + R \cdot Q_0 (i\omega) \cancel{e^{i\omega t}} + \frac{Q_0}{C} \cancel{e^{i\omega t}}$$

$\boxed{3}$ What remains is a pure algebraic equation of Q_0

$$\frac{V_0}{Q_0} = -\omega^2 L + i\omega R + \frac{1}{C}$$

☆☆ Method of phasor = Draw this relation on the complex plane

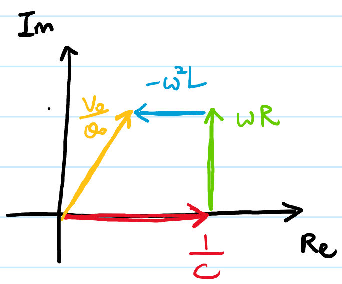


[1] Draw an arrow for each term

$\frac{1}{C}$ = some +ve real no.

$i\omega R$ = some +ve complex no.

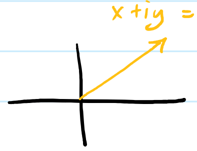
$-\omega^2 L$ = some −ve real no.



[2] By the equality, the vector sum equals $\frac{V_0}{Q_0}$

From the diagram, we can measure $\begin{cases} \text{Magnitude of } \frac{V_0}{Q_0} \\ \text{Angle of } \frac{V_0}{Q_0} \text{ (ie. phase)} \end{cases}$

But of course we can also solve by algebra.



$$\frac{V_0}{Q_0} = \left(\frac{1}{C} - \omega^2 L\right) + i\omega R$$

$$= \sqrt{\left(\frac{1}{C} - \omega^2 L\right)^2 + (\omega R)^2}\; e^{i\tan^{-1}\left(\frac{\omega R}{\frac{1}{C} - \omega^2 L}\right)}$$

$$\therefore\quad Q(t) = Q_0 e^{i\omega t} = \frac{V_0}{\sqrt{\left(\frac{1}{C} - \omega^2 L\right)^2 + (\omega R)^2}}\; e^{i\left[\omega t - \tan^{-1}\left(\frac{\omega R}{\frac{1}{C} - \omega^2 L}\right)\right]}$$

Terminology of lead / lag in phase angle

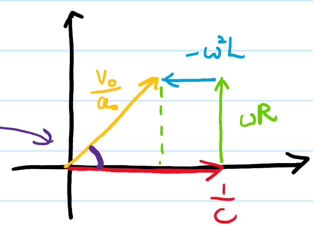
Compare with $V(t) = V_0 e^{i\omega t}$

Q_0 has an extra phase angle $-\tan^{-1}\left(\frac{\omega R}{\frac{1}{C} - \omega^2 L}\right)$

So we say $Q(t)$ is lagging behind $V(t)$

by a phase of $\tan^{-1}\left(\frac{\omega R}{\frac{1}{C} - \omega^2 L}\right)$

negative of this angle



Can also find the phase difference for voltage across each component

- Voltage through C

$$V_c = \frac{Q(t)}{C} = \frac{V_0}{C} \cdot \frac{1}{\sqrt{\cdots}} e^{i[\omega t - \tan^{-1}(\cdots)]}$$

- Voltage through R

$i = e^{i\frac{\pi}{2}}$

$$V_R = R\frac{dQ(t)}{dt} = \frac{V_0 R}{\sqrt{\cdots}} (i\omega) e^{i[\omega t - \tan^{-1}(\cdots)]}$$

$$= \frac{V_0 R\omega}{\sqrt{\cdots}} e^{i[\omega t - \tan^{-1}(\cdots) + \frac{\pi}{2}]}$$

Lead V_c by $\frac{\pi}{2}$

- Voltage through L

$-1 = e^{i\pi}$

$$V_L = L\frac{d^2Q}{dt^2} = \frac{V_0 L}{\sqrt{\cdots}} (-\omega^2) e^{i[\omega t - \tan^{-1}(\cdots)]}$$

$$= \frac{V_0 L\omega^2}{\sqrt{\cdots}} e^{i[\omega t - \tan^{-1}(\cdots) + \pi]}$$

Lead V_c by π

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3 Impedance Method

☆ Only applicable for sinusoidal voltage input of pure frequency.

i.e. $V(t) = V_0 e^{i\omega t} = V_1 \cos(\omega t) + V_2 \sin(\omega t)$

ω must be taken as some constant

Actually the algebaric version of phasor method

Starting from the previous ODE

$$V_0 e^{i\omega t} = L\frac{d^2Q}{dt} + R\frac{dQ}{dt} + \frac{1}{C}\frac{dQ}{dt}$$

1. By letting $Q(t) = Q_0 e^{i\omega t}$, current is

$$I(t) = \frac{dQ}{dt} = i\omega Q_0 e^{i\omega t}$$

2. Rewrite the ODE in terms of $I(t)$

$$V_0 e^{i\omega t} = -\omega^2 L Q_0 e^{i\omega t} + i\omega R Q_0 e^{i\omega t} + \frac{Q_0}{C} e^{i\omega t}$$

$$= i\omega L\, I(t) + R\, I(t) + \frac{1}{i\omega C} I(t)$$

$$V(t) = \left[i\omega L + R + \frac{1}{i\omega C} \right] I(t)$$

(why express by $I(t)$ but not $Q(t)$? Because current can be measured directly)

3. This becomes something looks like Ohm's Law

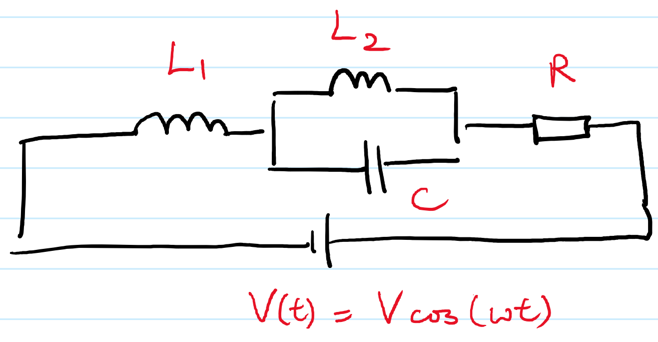
$$V(t) = \left[Z_L + Z_R + Z_C \right] I(t)$$

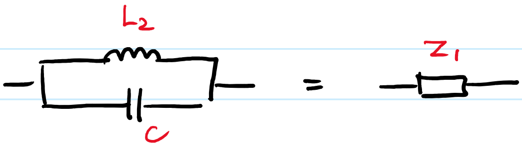
By defining the impedance of each components:

$$\begin{cases} Z_L = i\omega L \\ Z_R = R \\ Z_C = \frac{1}{i\omega C} \end{cases}$$

And the series / parallel addition are the same as resistor.

E.g.



[1] 

with $\frac{1}{Z_1} = \overbrace{\frac{1}{i\omega L_2}}^{\frac{1}{Z_L}} + \overbrace{i\omega C}^{\frac{1}{Z_C}}$

[2] 

with $Z_2 = \overbrace{i\omega L_1}^{Z_L} + \overbrace{Z_1}^{Z_1} + \overbrace{R}^{Z_R}$

After that we can find the current by $V(t) = Z_2 \cdot I(t)$

Question: What if $V(t)$ is not sinusoidal? (not sine/cosine)

[1] Break $V(t)$ into $\underbrace{\sum}\, V(\omega)\, e^{i\omega t}$ by Fourier Transform.

In fact an integral, as frequency can be a continuous spectrum

[2] Each $V(\omega)$ will result in a different current response

i.e. Current will be a function of ω,

$$V(\omega) = Z(\omega)\, I(\omega)$$

Z_C, Z_L are always function of ω

[3] The overall current is the superposition of all $I(\omega)$

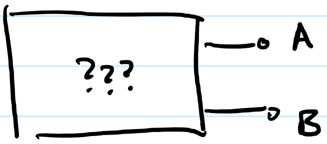
$$I(t) = \sum I(\omega)\, e^{i\omega t}$$

(This is actually the topic of frequency analysis in circuit usually appear in Yr 3/4 of electrical engineering)

Linear circuit blackbox

Situation: Suppose we have a blackbox of circuit

The only things we have access are its 2 terminals



And it only contains linear components.

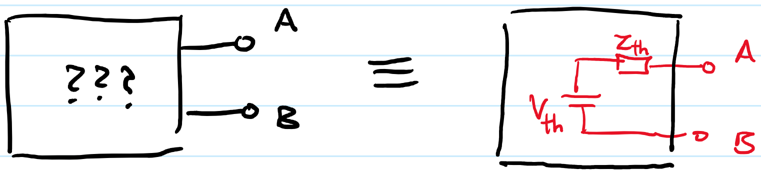
(E.g. L, R, C, voltage/current source)

What properties can we analyse from it?

In circuit theory, there are 2 theorems we may apply:

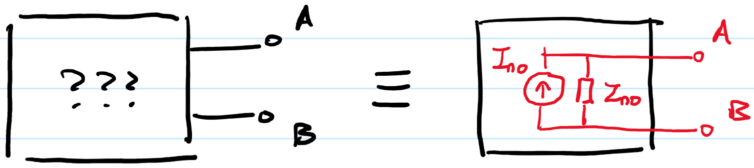
① **Thevenin's theorem**

Any blackbox circuit made of only linear components can be simplified into some voltage source V_{th} in series connection with some impedance Z_{th}.



② **Norton's theorem**

Any blackbox circuit made of only linear components can be simplified into some current source I_{no} in parallel connection with some impedance Z_{no}

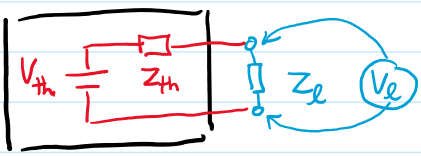


That means without further information, we can only classify these blackbox by 2 parameters. (Either V_{th}/Z_{th} or I_{no}/Z_{no})

- Want to understand what is in the box ⇒ Impossible
- Just use the box as a component in circuit ⇒ Simple

Measure V_{th}/Z_{th} or I_{no}/Z_{no}

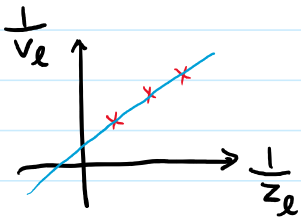
Because the equivalent Thevenin / Norton circuits are so simple, we simply need a voltmeter (oscilloscope) and a few different impedance component. E.g.



← Measure voltage drop across a load impedance Z_ℓ

From above we have
$$V_\ell = \frac{V_{th}}{Z_\ell + Z_{th}} \cdot Z_\ell$$

or
$$\boxed{\frac{1}{V_\ell} = \boxed{\frac{Z_{th}}{V_{th}}} \cdot \frac{1}{Z_\ell} + \boxed{\frac{1}{V_{th}}}}$$



slope $= \frac{Z_{th}}{V_{th}}$

y intercept $= \frac{1}{V_{th}}$

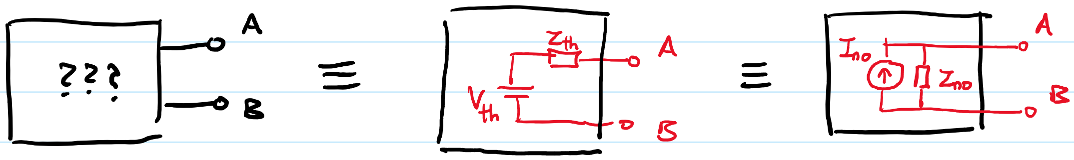
Repeat many times with different Z_ℓ, one should be able to find V_{th} and Z_{th} numerically.

Advanced circuit simplifying techniques

2 very useful tricks so that you do not have to write out Kirchoff 's Law for every node / loop.

① Thevenin - Norton Equivalence

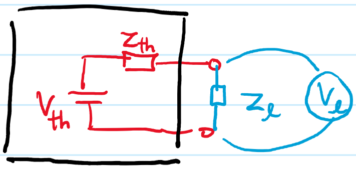
From the equivalent circuit in Thevenin / Norton 's Theorem

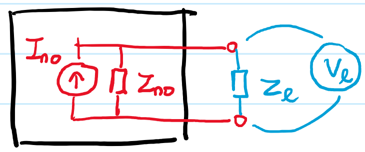


where $\boxed{Z_{th} = Z_{no} \quad \text{and} \quad V_{th} = I_{no} Z_{no} = I_{no} Z_{th}}$

Proof :

Let the box being connected to an impedance Z_ℓ and the voltage across is measured to be V_ℓ





$$\frac{V_\ell}{Z_\ell} = \frac{V_{th}}{Z_{th} + Z_\ell}$$

$$\Rightarrow \quad V_\ell = \frac{V_{th} Z_\ell}{Z_{th} + Z_\ell}$$

$$\frac{V_\ell}{Z_\ell} + \frac{V_\ell}{Z_{no}} = I_{no}$$

$$\Rightarrow \quad V_\ell = \frac{I_{no} Z_{no} Z_\ell}{Z_{no} + Z_\ell}$$

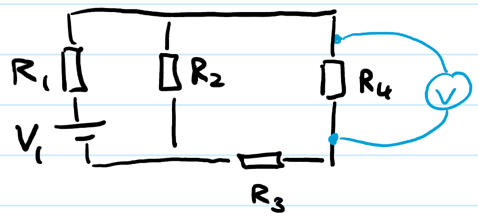
The relationships between V_ℓ and Z_ℓ should be the same since they are equivalent circuit. Also they should be independent to the choice of Z_ℓ. So we must have

$$\frac{V_{th}}{Z_{th}+Z_e} = \frac{I_{no}Z_{no}}{Z_{no}+Z_e}$$

$$\Rightarrow \quad \underline{Z_e\left(V_{th} - I_{no}Z_{no}\right)} + \underline{\left(V_{th}Z_{no} - I_{no}Z_{no}Z_{th}\right)} = 0$$

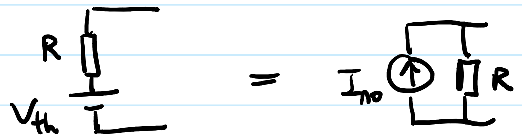
To be independent of Z_e, we must have the coefficients $= 0$

Eg. 1



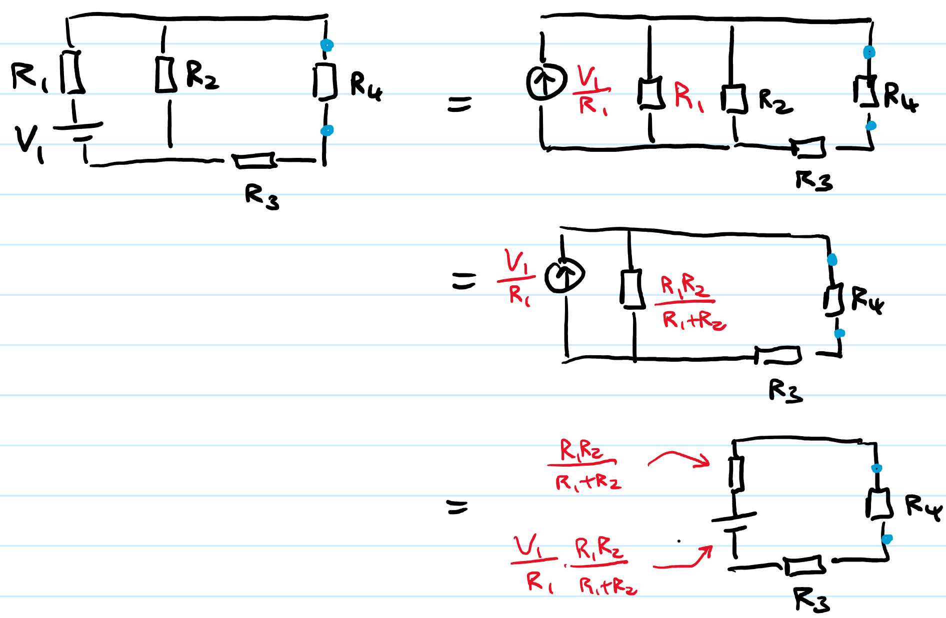
Find voltage across R_4

By Thevenin - Norton Equivalence, these patterns are replacable



with $V_{th} = I_{no}R$

So



At this step we can already tell the voltage across R_4 is

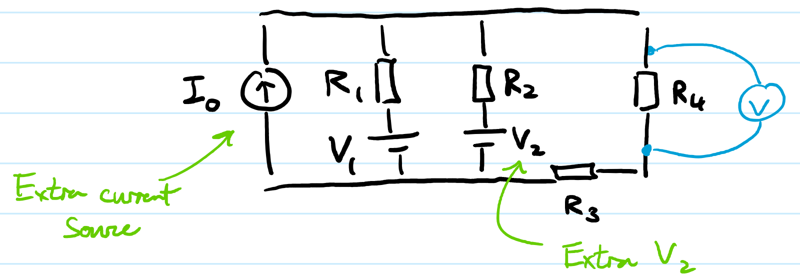
$$V_1 \cdot \frac{R_2}{R_1+R_2} \cdot \frac{R_4}{\frac{R_1R_2}{R_1+R_2} + R_3 + R_4}$$

② Superposition Theorem

In a linear circuit with multiple independent voltage / current source, the voltage / current across any components equal to the sum of the contribution of each source after removing the others by

- If it is a voltage source, replace with short circuit
- If it is a current source, replace with open circuit

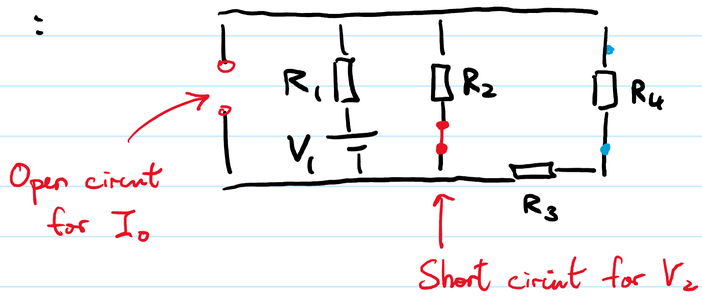
E.g. 2



Find voltage across R_4

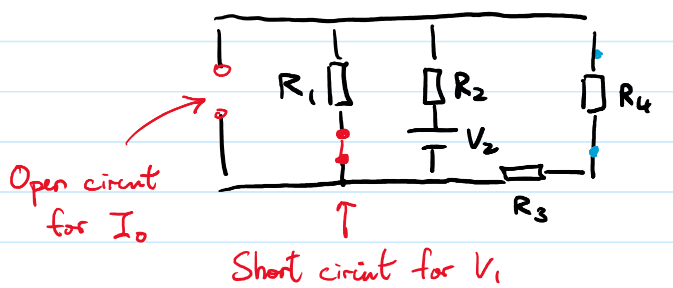
Calculate contribution from each source, after removing others :

By V_1 :



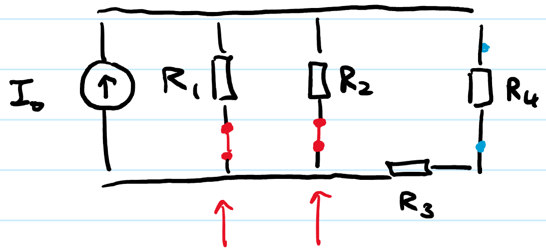
$$\Rightarrow \text{Voltage across } R_4 = V_1 \cdot \frac{R_2}{R_1+R_2} \frac{R_4}{\frac{R_1R_2}{R_1+R_2}+R_3+R_4}$$

By V_2 :



$\Rightarrow$ Voltage across $R_4 = V_2 \cdot \dfrac{R_1}{R_1+R_2} \dfrac{R_4}{\dfrac{R_1R_2}{R_1+R_2}+R_3+R_4}$

By I_o :



Short circuit for V_2 and V_2

$\Rightarrow$ Voltage across $R_4 = I_o \cdot \dfrac{R_1R_2}{R_1+R_2} \cdot \dfrac{R_4}{\dfrac{R_1R_2}{R_1+R_2}+R_3+R_4}$

Finally, total voltage across R_4 sum of all 3 cases

$$= (V_1R_2 + V_2R_1 + I_oR_1R_2) \cdot \frac{1}{R_1+R_2} \frac{R_4}{\dfrac{R_1R_2}{R_1+R_2}+R_3+R_4}$$