- Single variable function vs Multivariable function
- Partial differentiation (on scalar function)
- Multiple integral (on scalar function)

Multiple variable function

To well-define a function $f(x)$ in advanced math, we actually need to specify its Domain & Range

E.g. Formal notation in math text. :

$$f(x) : \mathbb{R} \rightarrow \mathbb{R}^+$$
$$x \mapsto \frac{1}{|x|}$$

- Domain = The largest set of value that can be x
- Range = The set of all possible value of $f(x)$
- Codomain = Any set that contains the range
- Image = Set of possible values of $f(x)$ to some specific x
  (If x = whole domain, image of x = range)

We can classify the functions by whether their domain / range are made of single numbers / tuple of numbers

① Single Variable Scalar Function

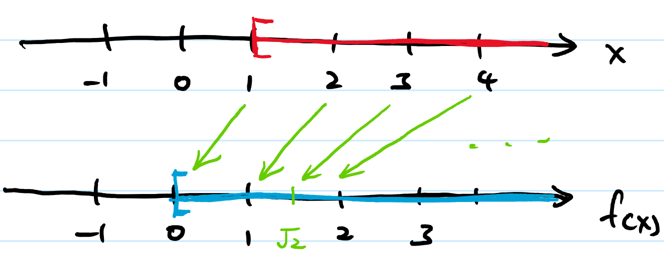
= The functions you already learnt

$\begin{cases} \text{Domain} = \text{Set of single number} \\ \text{Range} = \text{Set of single number} \end{cases}$

E.g. $f(x) = \sqrt{x-1}$

Domain = Any real number ≥ 1

Range = Any real number ≥ 0



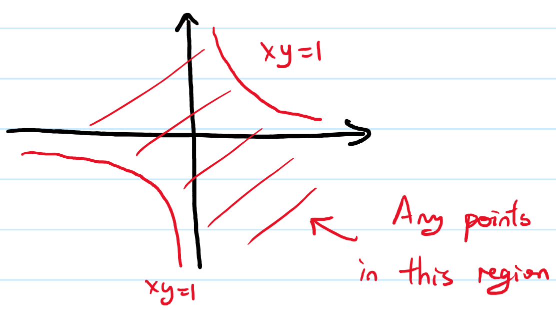
② Multivariable Scalar Function

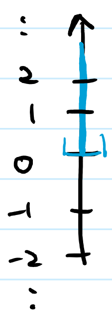
$\begin{cases} \text{Domain} = \text{Set of tuple of number} \\ \text{Range} = \text{Set of single number} \end{cases}$ $\quad x = (1,2,3)$, $\quad f(x) = 5$

E.g. $f(x,y) = \sqrt{1-xy}$

Domain = Any pair of x, y which $xy \leq 1$

Range = Any real number ≥ 0





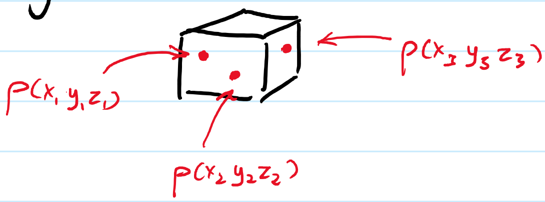
Example in physics

- Gravitational Potential Energy

$$U(x,y,z) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

- Density distribution in object

$$\rho(x,y,z)$$

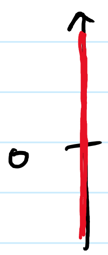


③ Single Variable Vector Function

$$\begin{cases} \text{Domain} = \text{Set of single number} \\ \text{Range} = \text{Set of tuple of number} \end{cases}$$

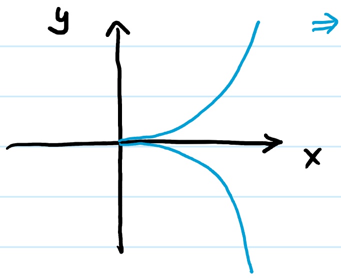
E.g. $\vec{r}(t) = (x(t), y(t)) = (t^2, 3t^3)$

Domain



t can be any real no.

Range



$x = t^2$ & $y = 3t^3$

$\Rightarrow x = \left(\frac{y}{3}\right)^{\frac{2}{3}}$

(x, y) is restricted to any points on this blue line

Examples in physics

$$\vec{s}(t) = (x(t), y(t), z(t))$$

$$\vec{v}(t) = (v_x(t), v_y(t), v_z(t))$$

$$\vec{a}(t) = (a_x(t), a_y(t), a_z(t))$$

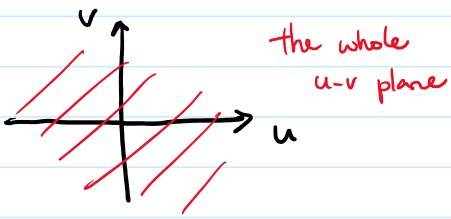
④ Multivariable Vector Function

$$\begin{cases} \text{Domain} = \text{Set of tuple of no.} \\ \text{Range} = \text{Set of tuple of no.} \end{cases}$$

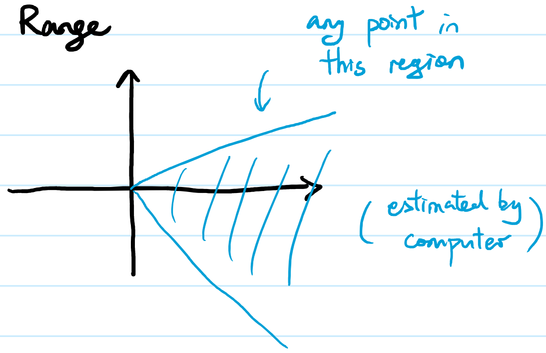
E.g. $\vec{r}(u,v) = (r_1(u,v), r_2(u,v))$

$$= (u^2+v^2, \ u-1-v^2)$$

Domain



Range



Example in physics

Gravitational Force

$$\vec{F}(\vec{r}) = \vec{F}(x,y,z)$$

$$= \boxed{\frac{-GMm}{|\vec{r}|^2}} \cdot \boxed{\frac{\vec{r}}{|\vec{r}|}}$$

Magnitude Unit Vector

$$= \frac{-GMm}{(x^2+y^2+z^2)} \cdot \frac{\boxed{x}\boxed{\hat{x}} + \boxed{y}\boxed{\hat{y}} + \boxed{z}\boxed{\hat{z}}}{\sqrt{x^2+y^2+z^2}}$$

$$= \left[\frac{-GMm\,x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{x} + \left[\frac{-GMm\,y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{y} + \left[\frac{-GMm\,z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{z}$$

Function Composition

- For single variable scalar function

E.g. If $f(x) = \sin x$, $g(x) = e^x$

$$f(g(x)) = \sin(e^x) \quad , \quad g(f(x)) = e^{\sin x}$$

It is the key in chain rule.

$$\frac{d}{dx} f(g(x)) = \frac{d}{d(g(x))} f(g(x)) \cdot \frac{d\,g(x)}{dx}$$

- For multivariable function

We can compose function only if no. of output matches next function's no. of input

E.g.
$$f(p,q) = \sqrt{p+q} \quad (2 \xrightarrow{f} 1)$$
$$\vec{g}(t) = (t-1, t^2) \quad (1 \xrightarrow{g} 2)$$
$$\vec{h}(u,v) = (u^2+v, u-v) \quad (2 \xrightarrow{h} 2)$$

The possible compositions :

$$f(\vec{g}(t)) = \sqrt{(t-1) + (t^2)} \quad (1 \xrightarrow{g} 2 \xrightarrow{f} 1)$$
$$f(\vec{h}(u,v)) = \sqrt{(u^2+v) + (u-v)} \quad (2 \xrightarrow{h} 2 \xrightarrow{f} 1)$$
$$\vec{g}(f(p,q)) = \cdots \quad (2 \xrightarrow{f} 1 \xrightarrow{g} 2)$$
$$\vec{h}(\vec{g}(t)) = \cdots \quad (1 \xrightarrow{g} 2 \xrightarrow{h} 2)$$
$$\vec{h}(\vec{h}(u,v)) = \cdots \quad (2 \xrightarrow{h} 2 \xrightarrow{h} 2)$$

These are NOT allowed :
$$\begin{cases} g(h(u,v)) & 2 \xrightarrow{h} 2 \Rightarrow 1 \xrightarrow{g} 2 \\ h(f(p,q)) & 2 \xrightarrow{f} 1 \Rightarrow 2 \xrightarrow{h} 2 \end{cases}$$

Limit in Multivariable Scalar Function

Recap : Single Variable Function

$\lim\limits_{x \to a} f(x) = L$ means when x close to $a \Rightarrow f(x)$ close to L

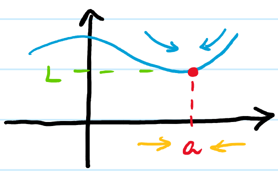
This is similar for multivariable function

$$\lim_{(x_1, x_2 \ldots x_n) \to (a_1, a_2 \ldots a_n)} f(x_1, x_2, \ldots x_n) = L$$

which means when all $(x_1, x_2 \ldots x_n)$ are close to their destination $(a_1, a_2 \ldots a_n)$, $f(\ldots)$ is close to L

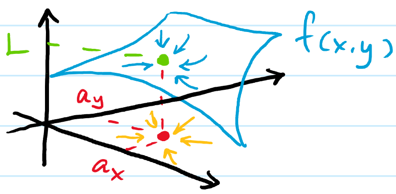
But how to show a limit exists is tricky in multivariable

Single variable



Can approach a limit from 2 directions

Multi variable



There are ∞ possible directions to approach a limit

Limit of multivariable function exist only if limit from ALL directions exist $\Rightarrow$ Proving is usually difficult

But for physics, we basically never deal with strange functions. Calculation can be done like single variable.

E.g. $\lim\limits_{(x,y) \to (\frac{\pi}{2}, \frac{\pi}{2})} \sin x \cos y = \sin\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right) = 0$

Partial Differentiation

Notation : $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, ...

Pronounce simply as " partial x ", " partial y " . etc.

In comparison :
$$\begin{cases} \frac{df}{dt} \rightarrow f \text{ is a function with only 1 input } t \\ \frac{\partial f}{\partial x} \rightarrow f \text{ is a multivariable function, with one of the input being } x \end{cases}$$

Definition

Partial differentiation of $f(x_1, x_2 \dots x_n)$ at $(a_1, a_2 \dots a_n)$

in the x_i 's direction is defined as

$$\frac{\partial}{\partial x_i} f(x_1, x_2 \dots x_i \dots x_n)$$

$$= \lim_{\Delta x_i \to 0} \left[\frac{f(x_1, x_2 \dots x_i + \Delta x_i \dots x_n) - f(x_1, x_2 \dots x_i \dots x_n)}{\Delta x_i} \right]$$

Limit only act on x_i . Other x remains untouched

Equivalently in calculation, only the x_i are differentiated

other input variables are treated as constant

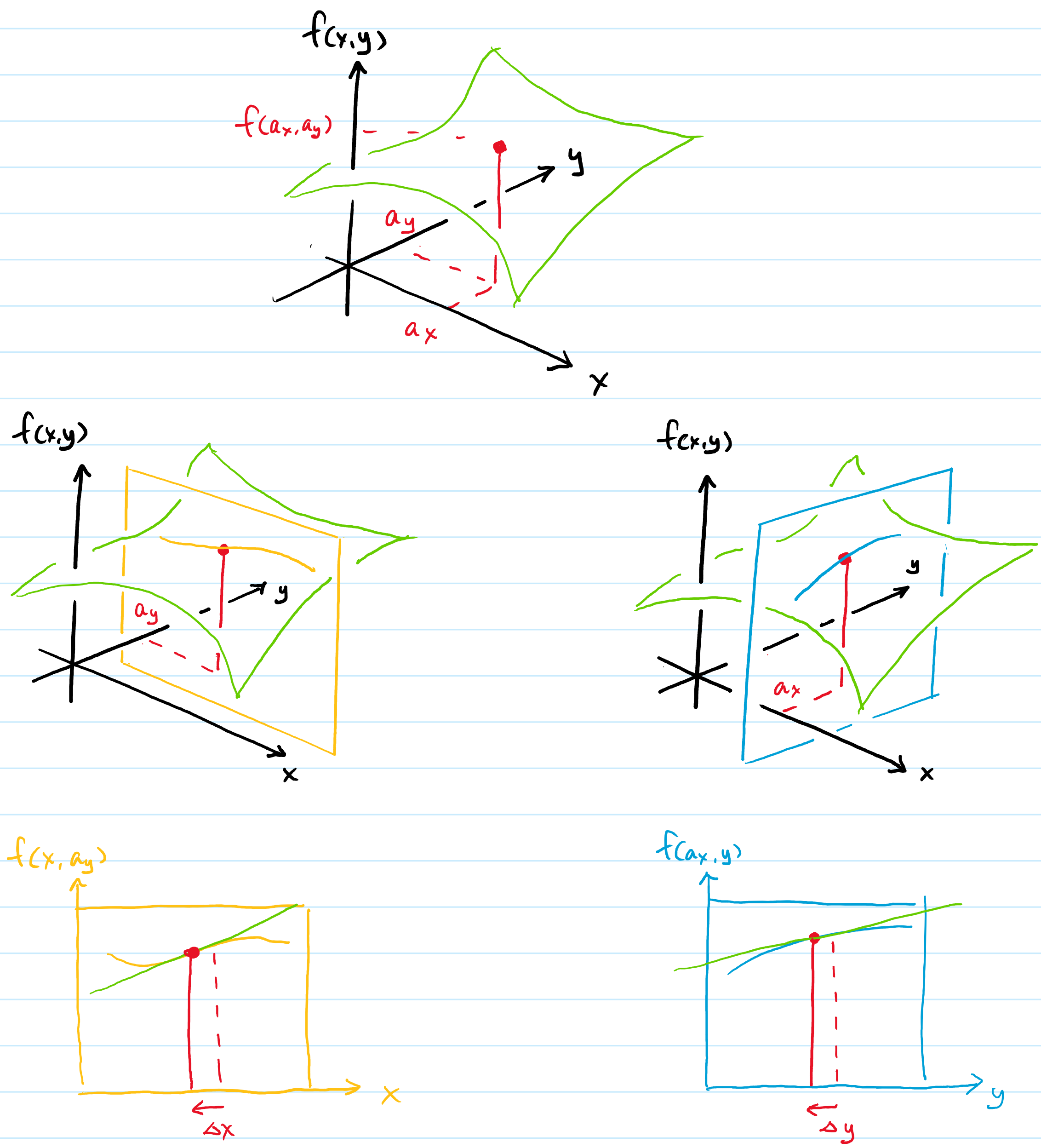
E.g. $f(x, y, z) = x^2 y \sin z$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x \, y \sin z$$

$$\frac{\partial f}{\partial y} = x^2 \cdot 1 \cdot \sin z$$

$$\frac{\partial f}{\partial z} = x^2 y \cdot \cos z$$

Geometrical Interpretation



$\frac{\partial}{\partial x}$ = On the plane of const. y, find slope along x direction

$\frac{\partial}{\partial y}$ = On the plane of const. x, find slope along y direction

$\Rightarrow \frac{\partial}{\partial x_i}$ = Find slope / rate of change of function w.r.t. x_i

Arithmetics

All rules are exactly the same as normal differentiation

Except chain rule, which becomes a mess because of function composition

E.g. Let $\vec{f}(u,v) = (f_1(u,v), f_2(u,v))$ $(2 \xrightarrow{f} 2)$

$g(s,t) = g$ $(2 \xrightarrow{g} 1)$

Try to differentiate $g(\vec{f}(u,v))$ $(2 \xrightarrow{f} 2 \xrightarrow{g} 1)$

$\because$ 2 inputs u, v $\Rightarrow$ 2 possible partial differentiation : $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$

E.g. $\frac{\partial}{\partial u} g(\vec{f}(u,v)) = \frac{\partial}{\partial u} g(f_1(u,v), f_2(u,v))$

$$= \frac{\partial}{\partial s} g(s,t) \cdot \frac{\partial}{\partial u} f_1(u,v) + \frac{\partial}{\partial t} g(s,t) \cdot \frac{\partial}{\partial u} f_2(u,v)$$

$s = f_1(u,v)$ $\qquad$ $t = f_2(u,v)$

To differentiate the u inside f_1
First need to differentiate $s = f_1$
in $g(s,t)$

To differentiate the u inside f_2
First need to differentiate $t = f_2$
in $g(s,t)$

($\frac{\partial}{\partial v}$ is just similar)

Ex : $\begin{cases} f(p,q) = \sqrt{p+q} \\ \vec{g}(t) = (t-1, t^2) \end{cases} \Rightarrow f(\vec{g}(t)) = \sqrt{t^2+t-1}$

Try to do $\frac{d}{dt} f(\vec{g}(t))$ $\begin{cases} \text{① directly on } t \\ \text{② by chain rule} \end{cases}$

Multiple Integral

Definition : Definite multiple integral

$$\iint\cdots\int_{(\text{Some region})} f(x_1, x_2, \ldots x_n)\, dx_1\, dx_2 \ldots dx_n$$

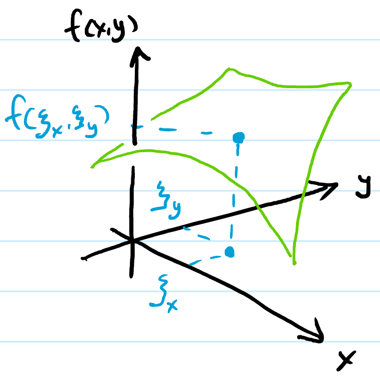
$$= \lim_{\Delta x_1, \Delta x_2 \ldots \Delta x_n \to 0} \sum_{\left(\substack{\text{all division} \\ \text{in the region}}\right)} f(\xi_1, \xi_2, \ldots \xi_n)\, \Delta x_1\, \Delta x_2 \ldots \Delta x_n$$

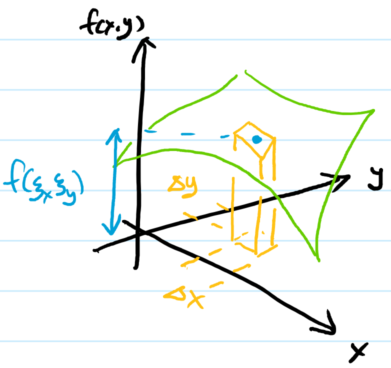
Recall that we have 2 interpretations to what integration is.
Now we can apply them to multiple integral

Special Case 1 : Double Integral – for functions with 2 inputs

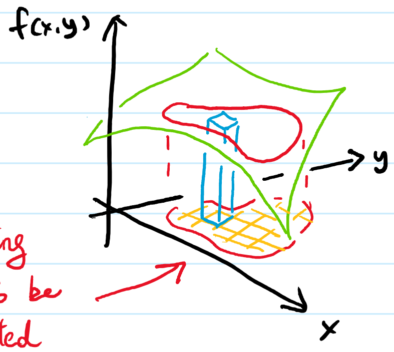
Interpretation 1 : Volume under Surface

$$\iint_{\text{Area}} f(x,y)\, dx\, dy = \lim_{\Delta x, \Delta y \to 0} \sum_{\substack{\text{all division} \\ \text{in area}}} \underline{f(\xi_x, \xi_y)\, \Delta x \Delta y}$$





Each pillar is volume
$= f(\xi_x, \xi_y) \cdot \Delta x \Delta y$

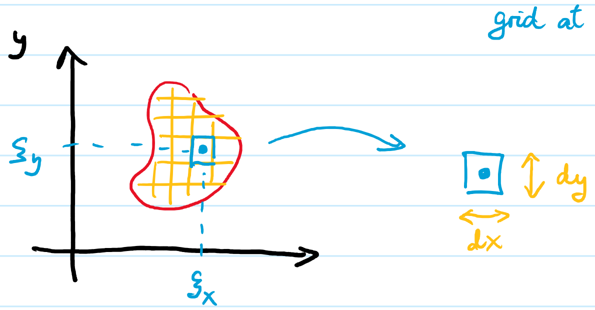


Red ring
= area to be integrated

Double integral
= Sum the volume of all pillars in the red ring

Interpretation 2 : Weighted sum over the area

$$\iint_{\text{Area}} f(x,y)\,dx\,dy = \lim_{\Delta x, \Delta y \to 0} \sum_{\substack{\text{all division} \\ \text{in area}}} \underbrace{f(\xi_x, \xi_y)}_{\text{weight for the grid at } (\xi_x \xi_y)} \underbrace{\Delta x \Delta y}_{\text{area of each grid}}$$



assigned with a weight $f(\xi_x, \xi_y)$

Example : The area mass density $\delta(x,y)$ depends on position (x,y)

$\Rightarrow$ Each small piece □ dy dx at position (ξ_x, ξ_y) has a mass $\delta(\xi_x, \xi_y)\,\Delta x \Delta y$

$\Rightarrow$ Inside a region, total mass = Sum of all small piece

$$= \sum_{\substack{\text{all piece} \\ \text{in the region}}} f(\xi_x, \xi_y)\,\Delta x \Delta y = \iint_{\text{the region}} f(x,y)\,dx\,dy$$

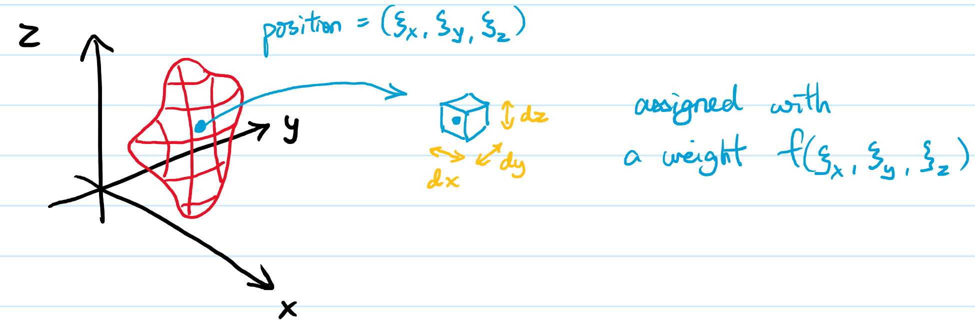
Special Case 2 : Triple Integral – for functions with 3 inputs

Interpretation 1 : ??? under volume

Sorry, we live in a 3D space. Cannot draw 4D objects.

Interpretation 2 : Weighted sum in a volume

$$\iiint_{\text{Volume}} f(x,y,z)\,dx\,dy\,dz = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{\substack{\text{all division} \\ \text{in volume}}} f(\xi_x, \xi_y, \xi_z)\,\Delta x \Delta y \Delta z$$



Similar to double integral, if $\rho(x,y,z)$ is the volume density distribution, total mass in a volume $= \iiint_{\text{the volume}} \rho(x,y,z)\,dx\,dy\,dz$

Calculating Multiple Integral

Step 1: Decide the integration order. i.e. How to divide a region

- Integration is done from inside to outside
- While integrating 1 variable, treat the others as constant

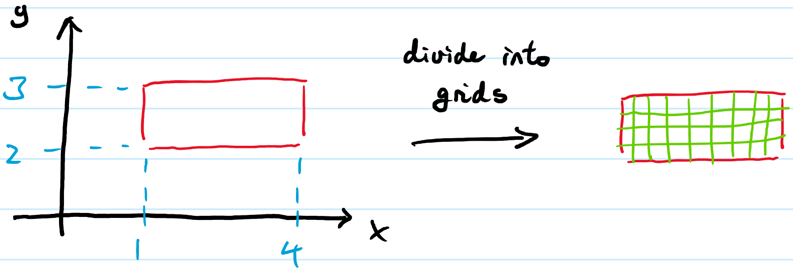
$$\iiint f(x,y,z)\,dx\,dy\,dz = \int\left[\int\left[\int f(x,y,z)\,dx\right]dy\right]dz$$

Step 2: Derive the corresponding upper/lower bounds

- It will be easier if you first draw out the region
- Note if integration order change, bounds can change

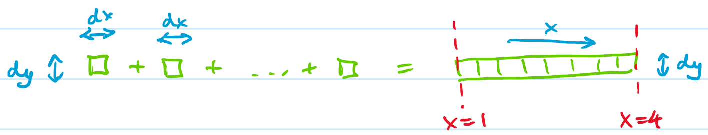
Example 1: Integrate $f(x,y) = x^2y - xy^3$

over the region bounded by $\begin{cases} x=1 \\ x=4 \end{cases}, \begin{cases} y=2 \\ y=3 \end{cases}$



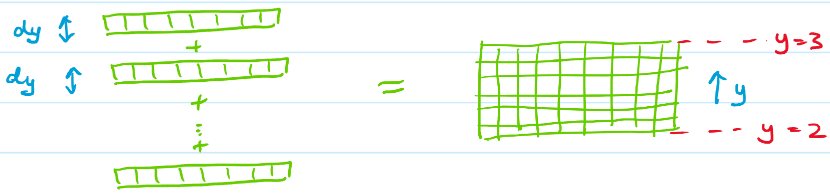
Method 1 : First integrate x, then integrate y

1. Integrate x = Sum all grid of same y coordinate to form horizontal strips



$$\int_{x=1}^{x=4} f(x,y)\,dx$$

2. Integrate y = Sum all horizontal strips to form the region



$$\int_{y=2}^{y=3}\left[\int_{x=1}^{x=4} f(x,y)\,dx\right] dy$$

Before integrating y, needs to remove x by substituting its bounds

$$\int_{y=2}^{y=3}\int_{x=1}^{x=4} x^2y - xy^3 \, dx\,dy$$

$$= \int_{y=2}^{y=3}\left[\frac{x^3}{3}y - \frac{x^2}{2}y^3\right]_{x=1}^{x=4} dy$$

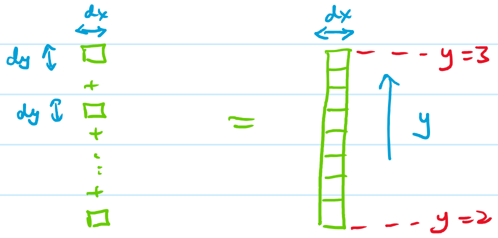
$$= \int_{y=2}^{y=3}\left(\frac{64}{3}y - \frac{16}{2}y^3\right) - \left(\frac{1}{3}y - \frac{1}{2}y^3\right) dy$$

$$= \ldots$$

$$= \frac{-555}{8}$$

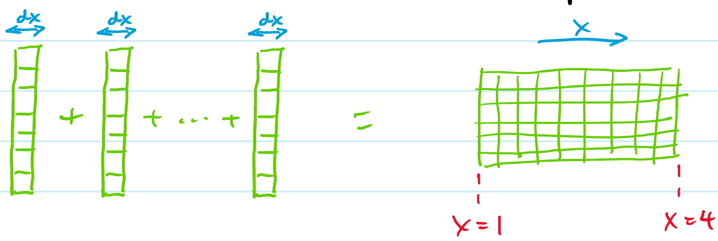
Method 2 : First integrate y, then integrate x

$\boxed{1}$ Integrate y = Sum all grid of same x coordinate to form vertical strips



$$\int_{y=2}^{y=3} f(x,y)\,dy$$

$\boxed{2}$ Integrate x = Sum all vertical strips to form the region



$$\int_{x=1}^{x=4} \left[\int_{y=2}^{y=3} f(x,y)\,dy \right] dx$$

You should get the same result, no matter which order you are integrating

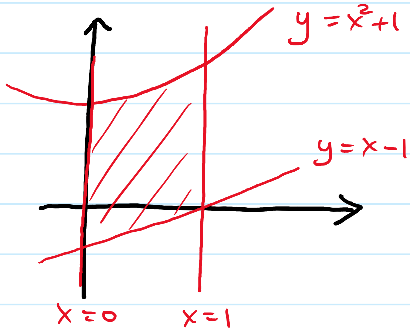
$$\int_{x=1}^{x=4} \int_{y=2}^{y=3} x^2y - xy^3 \,dy\,dx$$

$$= \ldots$$

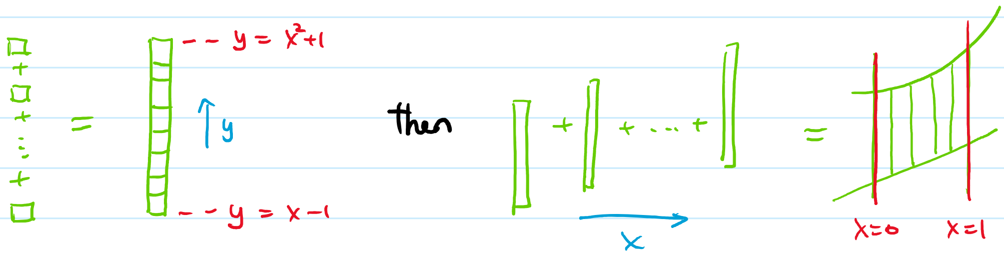
$$= \frac{-555}{8}$$

However if the region is ugly, calculations in some order are simplier than the other way around

Example 2: Integrating over the below region (for whatever $f(x,y)$)



Method 1: First integrate y, then integrate x

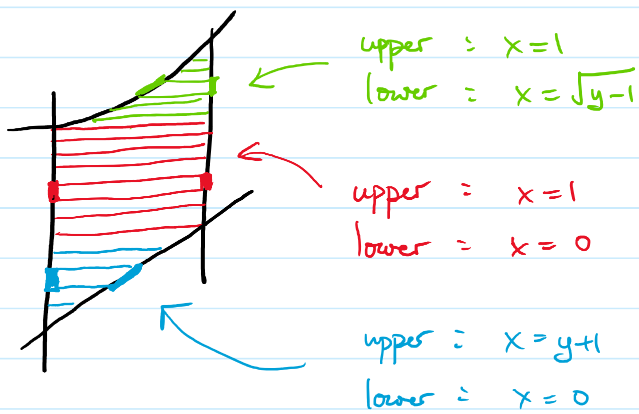


This approach is easy because all vertical strips have the same upper / lower bounds

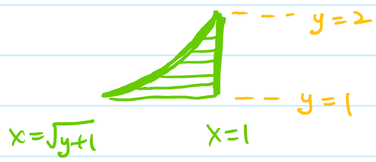
Can compute as: $$\int_{x=0}^{x=1}\int_{y=x-1}^{y=x^2+1} f(x,y)\,dy\,dx$$

Method 2: First integrate x, then integrate y

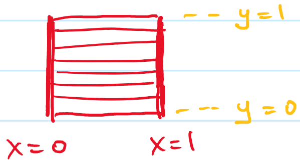
Note that the bounds of horizontal strips are different



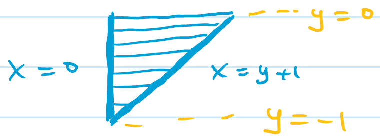
So we need to integrate each region individually



$$\int_{y=1}^{y=2}\int_{x=\sqrt{y+1}}^{x=1} f(x,y)\,dx\,dy$$



$$\int_{y=0}^{y=1}\int_{x=0}^{x=1} f(x,y)\,dx\,dy$$



$$\int_{y=-1}^{y=0}\int_{x=0}^{x=y+1} f(x,y)\,dx\,dy$$

Finally add all the 3 results to get the final ans.

Should get the same value but require a lot more effort.