Edited 2025

Relativistic Momentum

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Overview:

- Constructing 4-vectors
- Common applications of 4-momentum
- Spacetime interval

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1 The 4-vectors framework

4-vectors are 4×1 vectors which

- its components are made of combinations of physical quantities.
- after multiplied by Lorentz transformation matrix, its values change to what should be observed by a moving observer.

Plainly speaking, we want to "pack" physical quantities into a 4×1 vectors such that it satisfies

Physical quantities P, Q, R, S using the values seen by B

$$\begin{pmatrix} P' \\ Q' \\ R' \\ S' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{The Lorentz Transformation, A } 4\times 4 \text{ matrix}} \begin{pmatrix} P \\ Q \\ R \\ S \end{pmatrix}$$

Physical quantities P, Q, R, S using the values seen by A

We have already had the example of **4-position vector** $\vec{\boldsymbol{X}} = \begin{pmatrix} ct & x & y & z \end{pmatrix}$, a 4×1 vector that packs up the time coordinate t and position coordinate x, y, z of an event, and can be used to show the values of the event's coordinate according to different observers.

An event's coordinate t, x, y, z **seen by B**

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

An event's coordinate t, x, y, z **seen by A**

Starting from 4-position vector, we can derive other types of 4-vectors used in relativistics mechanics.

1.1 Velocity 4-vector

The first thing we want to pack into a 4-vector are the observed velocity v_x, v_y, v_z of an object.

$$\vec{\boldsymbol{U}}' = \underbrace{\begin{pmatrix} U_0' \\ U_1' \\ U_2' \\ U_3' \end{pmatrix}} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{pmatrix}} = \boldsymbol{\Lambda}_v \vec{\boldsymbol{U}}$$

U_0', U_1', U_2', U_3' are functions of v_x, v_y, v_z **seen by B**

U_0, U_1, U_2, U_3 are functions of v_x, v_y, v_z **seen by A**

In Newtonian mechanics, velocity of an object can be computed by differentiating its position with respect to time.

$$\vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \sim \frac{\Delta\vec{\boldsymbol{r}}}{\Delta t} = \frac{\text{Change in position}}{\text{Change in time}}$$

The problem when involving relativity is that the change in time (time scale) Δt is different between observers! For example, if we naively "divide" a 4-position vector by the change in time of each observer, the nice Lorentz transform is broken.

$$\begin{pmatrix} \text{Coordinate} \\ \text{change} \\ \text{Seen by B} \end{pmatrix} \sim \begin{pmatrix} c(\Delta t') \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c(\Delta t) \\ \Delta x \end{pmatrix} \sim \begin{pmatrix} \text{Lorentz} \\ \text{Transform} \end{pmatrix} \begin{pmatrix} \text{Coordinate} \\ \text{change} \\ \text{Seen by A} \end{pmatrix}$$

But because $\Delta t' \neq \Delta t$, it is wrong to have

$$\frac{\mathrm{d}\vec{\boldsymbol{X}}'}{\mathrm{d}t'} \sim \frac{1}{\Delta t'} \begin{pmatrix} c(\Delta t') \\ \Delta x' \end{pmatrix} \neq \frac{1}{\Delta t} \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c(\Delta t) \\ \Delta x \end{pmatrix} \sim \boldsymbol{\Lambda} \frac{\mathrm{d}\vec{\boldsymbol{X}}}{\mathrm{d}t}$$

In order to construct a "velocity-like" 4-vector, we should fix to the same time scale $\Delta\tau$ when differentiating, such that

$$\frac{\mathrm{d}\vec{\boldsymbol{X}}'}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} \begin{pmatrix} c(\Delta t') \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\tau} \begin{pmatrix} c(\Delta t) \\ \Delta x \end{pmatrix} = \boldsymbol{\Lambda} \frac{\mathrm{d}\vec{\boldsymbol{X}}}{\mathrm{d}\tau}$$

Although from the principle of relativity - there should not be a preference to any observer - the time scale of every observer are equally "similar". But relative to every object, there is always one observer special to it: its co-moving observer, who

- measures the shortest time difference between stationary events relative to the object.
- measures the largest position difference between two points on the object.

The nice thing about time scale of co-moving observer is that it is always proportional to time scales of other observer. For example, if an observer sees the object moving at velocity v,

$$
\begin{aligned}
\vec{\boldsymbol{X}}' = \begin{pmatrix} ct \\ \cdots \end{pmatrix} &= \begin{pmatrix} \gamma_{-v} & -\gamma_{-v}\beta_{-v} \\ -\gamma_{-v}\beta_{-v} & \gamma_{-v} \end{pmatrix} \begin{pmatrix} c\tau \\ 0 \end{pmatrix} = \boldsymbol{\Lambda}_{-v}\vec{\boldsymbol{X}} \\
&= \begin{pmatrix} \gamma_v \cdot c\tau \\ \cdots \end{pmatrix} \\
\Rightarrow \ t &= \gamma_v \tau \\
&\boxed{\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_v}
\end{aligned}
$$

The relation of their time scale is "clean" that there only involves one relative velocity v. So conventionally, we define the **velocity 4-vector** of an object as the time differentiation with respect to co-moving observer's time scale.

$$
\boxed{\vec{\boldsymbol{U}} \stackrel{\text{def}}{=} \frac{\mathrm{d}\vec{\boldsymbol{X}}}{\mathrm{d}\underline{\tau}}} \longleftarrow \text{Time scale of co-moving observer}
$$

We can tabulate what the 4-velocity vector look like for different observer.

	Seen by co-moving observer	Seen by other observer
Time scale	τ	t
Position coordinate $= \vec{\boldsymbol{X}}$	$\begin{pmatrix} c\tau \\ 0 \end{pmatrix}$	$\begin{pmatrix} ct \\ vt \end{pmatrix}$
4-velocity $= \dfrac{\mathrm{d}\vec{\boldsymbol{X}}}{\mathrm{d}\underline{\tau}} \longleftarrow !!$	$\underline{\underline{\begin{pmatrix} c \\ 0 \end{pmatrix}}}$	$\begin{pmatrix} c \\ v \end{pmatrix} \cdot \dfrac{\mathrm{d}t}{\mathrm{d}\tau} = \underline{\underline{\begin{pmatrix} \gamma_v c \\ \gamma_v v \end{pmatrix}}}$

Side Note:

Note that it is true that there should not be preference in time scale in relativity - Lorentz transformation to 4-vector should stay correct no matter whose time scale is used for differentiation.

$$
\frac{\mathrm{d}\vec{\boldsymbol{X}}'}{\mathrm{d}t} = \boldsymbol{\Lambda}\frac{\mathrm{d}\vec{\boldsymbol{X}}}{\mathrm{d}t} \xLeftrightarrow{\text{Both should work}} \frac{\mathrm{d}\vec{\boldsymbol{X}}'}{\mathrm{d}t'} = \boldsymbol{\Lambda}\frac{\mathrm{d}\vec{\boldsymbol{X}}}{\mathrm{d}t'}
$$

But we specifically mention the time of co-moving observer because it provides a clean way to relate time derivatives from time scales of different observers.

For example if,

- One observer sees the object moving at velocity u, uses time scale t
- The other observer sees the object moving at velocity v, uses time scale t'

Then

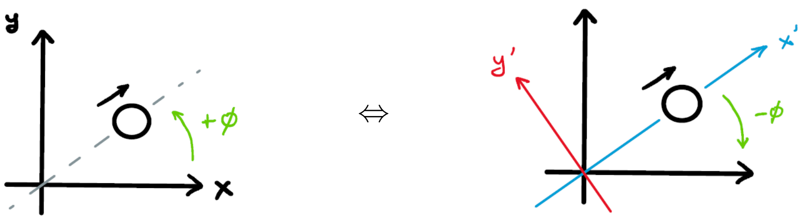
$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\mathrm{d}t'}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\gamma_v}{\gamma_u}$$

1.2 Rotating a 4-Vector

Now we extend the dicussion to motions other than along x-axis. How should we modify the velocity 4-vector if the object is moving along an arbituary direction in the 3D space?

$$\vec{\boldsymbol{v}} = (v, 0, 0) \quad \Rightarrow \quad \vec{\boldsymbol{U}} = \begin{pmatrix} \gamma_v c \\ \gamma_v v \\ 0 \\ 0 \end{pmatrix} \qquad \text{But} \qquad \vec{\boldsymbol{v}} = (v_x, v_y, v_z) \quad \Rightarrow \quad \vec{\boldsymbol{U}} = ?$$

The solution is again, related to transformation. Note that a "horizontal travelling" motion and a "diagonally travelling motion" are physically the same thing - they look different because of the choice of coordinate:



For example, to construct a motion of velocity v along an inclined angle ϕ relative to x-axis,

1. First let the motion be purely along an x'-axis
2. Then roate the x'-y' axes pair by an $-\phi$ angle to become x-y axes.

Such rotation can be done straightforward by the rotation matrix for on x-y plane:

$$\vec{\boldsymbol{U}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\phi) & \sin(-\phi) & 0 \\ 0 & -\sin(-\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_v c \\ \gamma_v v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v c \\ \gamma_v v \cos\phi \\ \gamma_v v \sin\phi \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v c \\ \gamma_v v_{\underline{x}} \\ \gamma_v v_{\underline{y}} \\ 0 \end{pmatrix}$$

i.e. To write a velocity 4-vector in 2D/3D, we only need to multiply cos/sin to the repsective components. γ always takes the magnitude of v and is independent to direction.

2 Momentum 4-Vector

Similar to Newtonian mechanics, we can get a momentum-like 4-vector by multiplying mass to a velocity 4-vector, usually called **4-momentum**:

$$\vec{\boldsymbol{P}} = m\vec{\boldsymbol{U}} = m\frac{\mathrm{d}\vec{\boldsymbol{X}}}{\mathrm{d}\tau}$$

Where m is the **rest mass** of the object, i.e. the mass observed by the co-moving observer to the object. With Lorentz transform, we can find the expression of $\vec{\boldsymbol{P}}$ from other moving observer.

	Seen by co-moving observer	Seen by other observer
4-velocity $= \vec{\boldsymbol{U}}$	$\begin{pmatrix} c \\ 0 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v c \\ \gamma_v v \end{pmatrix}$
4-momentum $= m\vec{\boldsymbol{U}}$	$\begin{pmatrix} mc \\ 0 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} mc \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v mc \\ \gamma_v mv \end{pmatrix}$

2.1 Interpretation

A 4-momentum vector does not only bear the physical observation of the object's momentum, but also its energy. We can analyze by Taylor expansion on the 4-momentum from a moving observer:

- **Time component = Relativistic energy**

$$\begin{aligned}\frac{E}{c} \overset{\text{def}}{=} \gamma_v mc &= \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} \\ &\approx mc\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right) \\ &= \frac{1}{c}\left(\underbrace{mc^2}_{\text{Some energy more fundamental than KE}} + \underbrace{\frac{1}{2}mv^2}_{\text{Newtonian KE}} + \underbrace{\cdots}_{\text{Relativistic correction}}\right)\end{aligned}$$

  The energy-like term mc^2 appears even if the object is at rest ($v = 0$) - It is like some "intrinsic" energy carried by object whenever the object has mass. Therefore it is called the **rest energy**.

- **Position component = Relativistic momentum**

$$\begin{aligned}p \overset{\text{def}}{=} \gamma_v mv &= \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \\ &\approx \underbrace{mv}_{\text{Newtonian momentum}} + \underbrace{mv \cdot \frac{1}{2}\frac{v^2}{c^2} + \cdots}_{\text{Relativistic correction}}\end{aligned}$$

Note 1: Some people (usually the experimentalists) prefer using the *observed mass* $m^* = \gamma_v m$ as the of an moving object, and claim that mass of objects change with velocity, so that they can always stick to the Newtonian formula of momentum:

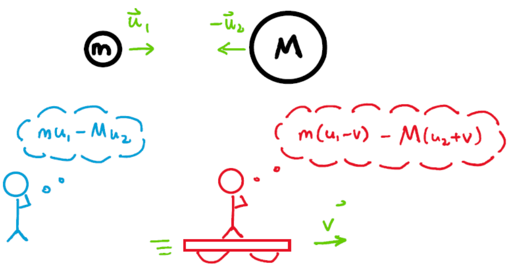
$$(\text{momentum}) = m^* v$$

Note 2: Different texts may have different meanings to their "KE", because we can have

- Newtonian KE $= \frac{1}{2}mv^2$
- Relativistic KE $= (\gamma - 1)mc^2$

2.2 Application of 4-momentum

With special relativity, relative velocity no longer calculates as $\vec{\boldsymbol{v}}_{AC} = \vec{\boldsymbol{v}}_{AB} + \vec{\boldsymbol{v}}_{BC}$, so Newtonian momentum fails to conserved if we switch the frame of reference.



The real conserving quantities are the components in 4-momentum vector, because relations between 4-vector are guarenteed to be correct under different observers.

$$\vec{\boldsymbol{P}}_i = \begin{pmatrix} \gamma mc \\ \gamma mv \end{pmatrix}_i = \begin{pmatrix} \gamma mc \\ \gamma mv \end{pmatrix}_f = \vec{\boldsymbol{P}}_f$$

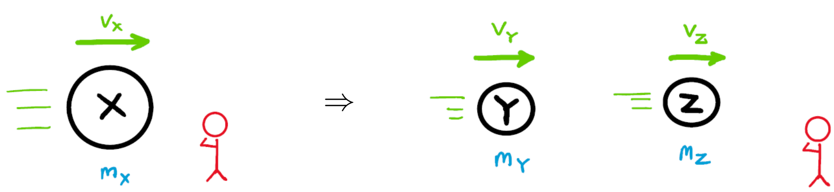
$$\Updownarrow$$

$$\boldsymbol{\Lambda}_u \vec{\boldsymbol{P}}_i = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_u \begin{pmatrix} \gamma mc \\ \gamma mv \end{pmatrix}_i = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_u \begin{pmatrix} \gamma mc \\ \gamma mv \end{pmatrix}_f = \boldsymbol{\Lambda}_u \vec{\boldsymbol{P}}_f$$

Here we can look at different example questions related to 4-momentum conservation.

2.2.1 Decay of Particle

Suppose particle X, with mass m_X, is initially moving at velocity v_X. It suddenly decays into 2 particle Y (mass $= m_Y$) and Z (mass $= m_Z$), with unknown velocity v_Y and v_Z.



To the observer on ground, he can write out the 4-momentum of each particles directly:

$$\vec{\boldsymbol{P}}_X = \begin{pmatrix} \gamma_{v_X} m_X c \\ \gamma_{v_X} m_X v_X \end{pmatrix} \qquad \vec{\boldsymbol{P}}_Y = \begin{pmatrix} \gamma_{v_Y} m_Y c \\ \gamma_{v_Y} m_Y v_Y \end{pmatrix} \qquad \vec{\boldsymbol{P}}_Z = \begin{pmatrix} \gamma_{v_Z} m_Z c \\ \gamma_{v_Z} m_Z v_Z \end{pmatrix}$$

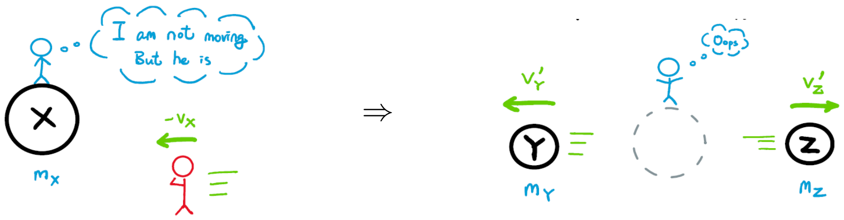
Conservation of 4-momentum is simply $\vec{\boldsymbol{P}}_X = \vec{\boldsymbol{P}}_Y + \vec{\boldsymbol{P}}_Z$, yielding a set of simultaneous equations of v_Y and v_Z:

$$\begin{cases} \gamma_{v_X} m_X = \gamma_{v_Y} m_Y + \gamma_{v_Z} m_Z \\ \gamma_{v_X} m_X v_X = \gamma_{v_Y} m_Y v_Y + \gamma_{v_Z} m_Z v_Z \end{cases}$$

Writing this out is easy, but solving for the v's will be super annoying because they also hide inside the γ's. To reduce the amount of algebra, it is recommended to apply these tricks:

1. **Switch to a co-moving frame**

   We can always choose another observer who is blueco-moving to one of the particle, then the 4-momentum of this particle will contain a 0. i.e. $\vec{\boldsymbol{P}}_X = \begin{pmatrix} m_X c \\ 0 \end{pmatrix}$

   Let the co-moving observer of X sees Y and Z moving at velocity v'_Y and v'_Z. Now the conservation of 4-momentum writes as

   $$\begin{cases} m_X = \gamma_{v'_Y} m_Y + \gamma_{v'_Z} m_Z \\ 0 = \gamma_{v'_Y} m_Y v'_Y + \gamma_{v'_Z} m_Z v'_Z \end{cases}$$

   This system of equation of v'_Y and v'_Z is just easier to solve. Then to retrieve v_Y and v_Z, we can use velocity addition formula.

   $$v_Y = \frac{v'_Y + v_X}{1 + \frac{v'_Y v_X}{c^2}} \qquad \text{and} \qquad v_Z = \frac{v'_Z + v_X}{1 + \frac{v'_Z v_X}{c^2}}$$

2. **Algebraic trick of $\gamma^2 v^2 \equiv c^2(\gamma^2 - 1)$**

   This equality directly comes from the definition of γ.

   $$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \Rightarrow \quad \gamma^2 \left(\sqrt{1 - \frac{v^2}{c^2}} \right)^2 = 1 \quad \Rightarrow \quad c^2(\gamma^2 - 1) = \gamma^2 v^2$$

It is used very frequently when we solve things in relativity because it helps reduce the annoying γv-like terms. For example, on the 2^{nd} equation in the above system:

$$\begin{aligned}
0 &= \gamma_{v'_Y} m_Y v'_Y + \gamma_{v'_Z} m_Z v'_Z \\
(m_Y \underline{\gamma_{v'_Y} v'_Y})^2 &= (m_Z \underline{\gamma_{v'_Z} v'_Z})^2 \\
m_Y^2 c^4 \left(\gamma_{v'_Y}^2 - 1\right) &= m_Z^2 c^4 \left(\gamma_{v'_Z}^2 - 1\right)
\end{aligned}$$

which leaves $\gamma_{v'_Y}$ and $\gamma_{v'_Z}$ to be the only two unknowns in the system of equations.

$$\begin{cases}
m_X = \gamma_{v'_Y} m_Y + \gamma_{v'_Z} m_Z & (1) \\
m_Y^2 \left(\gamma_{v'_Y}^2 - 1\right) = m_Z^2 \left(\gamma_{v'_Z}^2 - 1\right) & (2)
\end{cases}$$

<u>Note</u>: This is exactly the same trick of using the energy-momentum relation $E^2 = m^2c^4 + p^2c^2$ to reduce the problem. But here I express everything in terms of v to avoid using too many symbols like E and p.

The remainings shall be solved by brute force. From (2),

$$\begin{aligned}
m_Y^2 \gamma_{v'_Y}^2 - mY^2 &= m_Z^2 \gamma_{v'_Z}^2 - m_Z^2 \\
m_Y^2 - m_Z^2 &= m_Y^2 \gamma_{v'_Y}^2 - m_Z^2 \gamma_{v'_Z}^2 \\
&= \underline{(m_Y \gamma_{v'_Y} + m_Z \gamma_{v'_Z})}(m_Y \gamma_{v'_Y} - m_Z \gamma_{v'_Z}) \\
&= \underline{m_X}(m_Y \gamma_{v'_Y} - m_Z \gamma_{v'_Z}) \\
m_Y \gamma_{v'_Y} &= \frac{m_Y^2 - m_Z^2}{m_X} + m_Z \gamma_{v'_Z}
\end{aligned}$$

Substitute back to (1),

$$m_X = \left(\frac{m_Y^2 - m_Z^2}{m_X} + m_Z \gamma_{v'_Z}\right) + \gamma_{v'_Z} m_Z$$

$$\boxed{\gamma_{v'_Z} = \frac{m_X^2 + m_Z^2 - m_Y^2}{2 m_X m_Z}}$$

and so

$$m_Y \gamma_{v'_Y} = \frac{m_Y^2 - m_Z^2}{m_X} + \frac{m_X^2 + m_Z^2 - m_Y^2}{2 m_X}$$

$$\boxed{\gamma_{v'_Y} = \frac{m_X^2 + m_Y^2 - m_Z^2}{2 m_X m_Y}}$$

The steps to retriving v_Y and v_Z shall be left to you as an exercise.

2.2.2 Relativistic Doppler Effect - Another Derivation

From quantum mechanics, a photon's energy E and momentum p are related to its frequency f and wavelength λ:

$$E = hf \quad \text{and} \quad p = \frac{h}{\lambda}$$

where h is the Planck constant $\approx 6.63 \times 10^{-34}$ m^{2}kgs^{-1}. Also notice that light always travels at speed $c = f\lambda$ in vacuum. So the 4-momentum of a photon in vacuum can be written as

$$\vec{\boldsymbol{P}} = \begin{pmatrix} \frac{E}{c} \\ \pm p \end{pmatrix} = \begin{pmatrix} \frac{hf}{c} \\ \pm\frac{h}{\lambda} \end{pmatrix} = \begin{pmatrix} \frac{hf}{c} \\ \pm\frac{hf}{c} \end{pmatrix}$$

The $\pm$ sign is to show the traveling direction of the photon. We can derive the relativistic Doppler effect formula by Lorentz transform a photon's 4-momentum. Let B be an observer moving at velocity v relative to observer A.

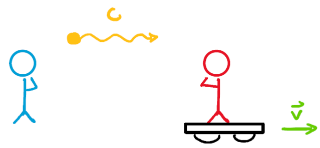
$$\underbrace{\begin{pmatrix} \frac{hf'}{c} \\ \pm\frac{hf'}{c} \end{pmatrix}}_{\text{4-vector made of } f \text{ seen by B}} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \underbrace{\begin{pmatrix} \frac{hf}{c} \\ \pm\frac{hf}{c} \end{pmatrix}}_{\text{4-vector made of } f \text{ seen by A}}$$

$$\begin{aligned} \Rightarrow \quad \frac{f'}{c} &= \gamma_v \frac{f}{c} \mp \gamma_v\beta_v \frac{f}{c} \\ &= \frac{1 \mp \beta_v}{\sqrt{1-\beta_v^2}} \frac{f}{c} \end{aligned}$$

$$\boxed{f' = \sqrt{\frac{1 \mp \beta_v}{1 \pm \beta_v}}\, f}$$

For example, if B is moving at the same direction as the photon relative to A, then $\beta > 0$ and momentum of photon should be taken as $+\frac{hf}{c}$. The photon observed by B is then

$$f' = \sqrt{\frac{1-\beta}{1+\beta}}\, f < f$$

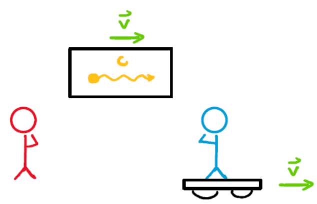


which is smaller than the frequency observed by A, i.e. redshift occurs.

2.2.3 Refractive Index of Moving Medium

This time considers if the photon is travelling in a medium with refractive n instead of vacuum. Wavelength of the photon becomes $\frac{\lambda}{n}$ instead of λ. So the 4-momentum of a photon is

$$\vec{\boldsymbol{P}} = \begin{pmatrix} \frac{E}{c} \\ \pm p \end{pmatrix} = \begin{pmatrix} \frac{hf}{c} \\ \pm\frac{nh}{\lambda} \end{pmatrix} = \begin{pmatrix} \frac{h}{\lambda} \\ \pm\frac{nh}{\lambda} \end{pmatrix}$$



Suppose n is the refractive index observed by the medium's co-moving observer A. We can switch to the moving observer B's frame who sees the medium moving at velocity v by inverse Lorentz transform.

4-vector made of n & λ seen by B

$$\begin{pmatrix} \frac{h}{\lambda'} \\ \pm\frac{n'h}{\lambda'} \end{pmatrix} = \begin{pmatrix} \gamma_{-v} & -\gamma_{-v}\beta_{-v} \\ -\gamma_{-v}\beta_{-v} & \gamma_{-v} \end{pmatrix} \begin{pmatrix} \frac{h}{\lambda} \\ \pm\frac{nh}{\lambda} \end{pmatrix}$$

4-vector made of n & λ seen by A

First component gives the wavelength observed by B:

$$\pm\frac{h}{\lambda'} = \gamma_v\frac{h}{\lambda} \pm \gamma_v\beta_v\frac{nh}{\lambda}$$

$$\boxed{\lambda' = \pm\frac{1}{\gamma_v(1 \pm n\beta_v)}\lambda}$$

Second component gives the refractive index observed by B:

$$\frac{n'h}{\lambda'} = \gamma_v\beta_v\frac{h}{\lambda} \pm \gamma_v\frac{nh}{\lambda}$$

$$n' = \gamma_v(\beta_v \pm n)\frac{\lambda'}{\lambda}$$

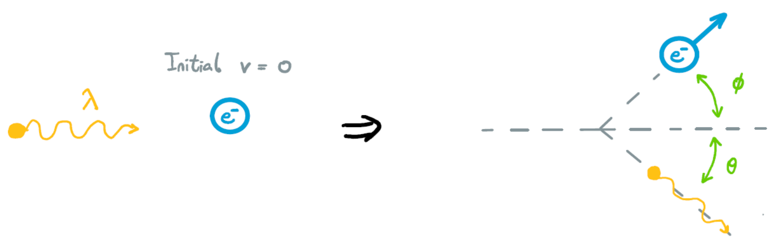
$$= \pm\frac{\beta_v \pm n}{1 \pm n\beta_v}$$

$$\boxed{n' = \frac{n \pm \beta_v}{1 \pm n\beta_v}}$$

$$\approx n + (n^2 - 1)(\pm\beta) - n\beta^2 + \cdots$$

2.2.4 Compton Scattering

Compton scattering was the key experiment showing that light (photon) carries momentum, such that it can collide with electons.



The loss in momentum by the photon can be determined by measuring the photons' change in wavelength (since $p = \frac{h}{\lambda}$). It can usually be found in textbooks:

$$\boxed{\Delta\lambda = \lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta)}$$

Final λ must be larger than initial λ
because momentum must have lost

where m is the mass of electron. Its derivation is just an application to the 2D 4-momentum vector. Recall that we can multiply a rotation matrix to 4-velocity to make it a 2D description. This applies to 4-momentum too.

After the collision,

- Electron moves at velocity v at angle ϕ above horizontal axis:

$$\vec{\boldsymbol{P}}_{e,f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \gamma_v mc \\ \gamma_v mv \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v mc \\ \gamma_v mv\cos\phi \\ \gamma_v mv\sin\phi \end{pmatrix}$$

- Photon's wavelength changed to λ_f, travelling at angle θ below horizontal axis:

$$\vec{\boldsymbol{P}}_{\lambda,f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta) & \sin(-\theta) \\ 0 & -\sin(-\theta) & \cos(-\theta) \end{pmatrix} \begin{pmatrix} \frac{h}{\lambda_f} \\ \frac{h}{\lambda_f} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{h}{\lambda_f} \\ \frac{h}{\lambda_f}\cos\theta \\ -\frac{h}{\lambda_f}\sin\theta \end{pmatrix}$$

The 4-momentum conservation then writes as

$$\begin{array}{ccccccc} \vec{\boldsymbol{P}}_{\lambda,i} & + & \vec{\boldsymbol{P}}_{e,i} & = & \vec{\boldsymbol{P}}_{\lambda,f} & + & \vec{\boldsymbol{P}}_{e,f} \\ \begin{pmatrix} \frac{h}{\lambda_i} \\ \frac{h}{\lambda_i} \\ 0 \end{pmatrix} & + & \begin{pmatrix} mc \\ 0 \\ 0 \end{pmatrix} & = & \begin{pmatrix} \gamma_v mc \\ \gamma_v mv\cos\phi \\ \gamma_v mv\sin\phi \end{pmatrix} & + & \begin{pmatrix} \frac{h}{\lambda_f} \\ \frac{h}{\lambda_f}\cos\theta \\ -\frac{h}{\lambda_f}\sin\theta \end{pmatrix} \end{array}$$

which is a system of 3 equations. The remaining steps are to remove ϕ and v:

1. Move $\vec{\boldsymbol{P}}_{e,f}$ to LHS, then take square on both sides

$$\begin{cases} \left(\dfrac{h}{\lambda_i} + mc - \dfrac{h}{\lambda_f}\right)^2 = \gamma_v^2 m^2 c^2 & (1) \\ \left(\dfrac{h}{\lambda_i} - \dfrac{h}{\lambda_f}\cos\theta\right)^2 = \gamma_v^2 m^2 v^2 \cos^2\phi & (2) \\ \left(\dfrac{h}{\lambda_f}\sin\theta\right)^2 = \gamma_v^2 m^2 v^2 \sin^2\phi & (3) \end{cases}$$

2. Add (2) and (3) to remove ϕ:

$$\begin{aligned} \left(\frac{h}{\lambda_i} - \frac{h}{\lambda_f}\cos\theta\right)^2 + \left(\frac{h}{\lambda_f}\sin\theta\right)^2 &= \gamma_v^2 m^2 v^2 (\cos^2\phi + \sin^2\phi) \\ \left(\frac{h}{\lambda_i}\right)^2 - 2\left(\frac{h}{\lambda_i}\right)\left(\frac{h}{\lambda_f}\right)\cos\theta + \left(\frac{h}{\lambda_f}\right)^2 &= \gamma_v^2 m^2 v^2 \\ &= \gamma_v^2 m^2 c^2 \cdot \frac{v^2}{c^2} \qquad (4) \end{aligned}$$

3. By $\left(1-\frac{v^2}{c^2}\right)\gamma^2 = 1$, We can remove v by (1) subtracts (4)

$$\left(\frac{h}{\lambda_i} + mc - \frac{h}{\lambda_f}\right)^2 - \left[\left(\frac{h}{\lambda_i}\right)^2 - 2\left(\frac{h}{\lambda_i}\right)\left(\frac{h}{\lambda_f}\right)\cos\theta + \left(\frac{h}{\lambda_f}\right)^2\right] = \gamma_v^2 m^2 c^2 - \gamma_v^2 m^2 c^2 \cdot \frac{v^2}{c^2}$$

$$m^2c^2 + 2\left(\frac{h}{\lambda_i}\right)mc - 2\left(\frac{h}{\lambda_f}\right)mc - 2\left(\frac{h}{\lambda_i}\right)\left(\frac{h}{\lambda_f}\right) + 2\left(\frac{h}{\lambda_i}\right)\left(\frac{h}{\lambda_f}\right)\cos\theta = m^2c^2 \cdot \cancelto{1}{\gamma_v^2\left(1-\frac{v^2}{c^2}\right)}$$

$$2hmc\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_f}\right) - 2\left(\frac{h}{\lambda_i}\right)\left(\frac{h}{\lambda_f}\right)(1-\cos\theta) = 0$$

$$2hmc\left(\frac{\lambda_f - \lambda_i}{\lambda_f\lambda_i}\right) = 2\left(\frac{h^2}{\lambda_i\lambda_f}\right)(1-\cos\theta)$$

$$\boxed{\lambda_f - \lambda_i = \frac{h}{mc}(1-\cos\theta)}$$

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3 Spacetime Invariants

(Not yet finish writing. But content will be similar to lecture note.)

— The End —