Newton 2^{nd} Law is a differential equation

$$F(t) = ma(t) = m\frac{d^2}{dt^2}x(t)$$

Need to solve this equation of $x(t)$, which involves $x(t)$'s derivatives

E.g. SHM (Simple harmonic motion)

$$-kx = ma = m\frac{d^2}{dt^2}x$$ has both x and $\frac{d^2}{dt^2}x$

How to solve this kind of equations? What other variants can we solve?

Classification of Differential Equations

① No. of Variables in the wanted function

- Single Variable ⇒ Ordinary Differential Equation (ODE)
- Multivariable ⇒ Partial Differential Equation (PDE)

(not going to discuss here)

② Order = highest derivative in the equation

E.g.

$$\frac{d}{dt}f(t) + f(t) = \ln t$$ highest = 1^{st} derivative ⇒ 1^{st} order

$$\frac{d^2}{dt^2}f(t) + f(t)\frac{d}{dt}f(t) = \sin t$$ highest = 2^{nd} derivative ⇒ 2^{nd} order

$$\left(\frac{d}{dt}f(t)\right)^2 + \left(f(t)\right)^2 = 1$$ highest = 1^{st} derivative ⇒ 1^{st} order

③ **Linearity** = whether any terms contain multiplication between derivatives

E.g. $$\underbrace{\frac{d^2}{dt^2}f(t)}_{\text{power } 1} - \underbrace{e^t\frac{d}{dt}f(t)}_{\text{power } 1} + \underbrace{f(t)}_{\text{power } 1} = 0 \quad \Rightarrow \text{linear}$$

$$\underbrace{\frac{d^2}{dt^2}f(t)}_{\text{power } 1} + \underbrace{\left(\frac{d}{dt}f(t)\right)^2}_{\text{power } 2} = \sin t \quad \Rightarrow \text{non-linear}$$

$$\underbrace{f(t)}_{\text{power } 1}\ \times\ \underbrace{\frac{d}{dt}f(t)}_{\text{power } 1} = 5 \quad \Rightarrow \text{non linear}$$

④ **Constant Coefficients** = whether derivatives are multiplying with constants or function of t

E.g. $$\underbrace{2}_{2}\frac{d^2}{dt^2}f(t) \underbrace{-}_{-1} \frac{d}{dt}f(t) + \underbrace{2}_{2}f(t) = \cos t \quad \text{are all constants}$$

$$\underbrace{(t^2-1)}_{t^2-1}\frac{d^2}{dt^2}f(t) \underbrace{- t}_{-t}\frac{d}{dt}f(t) + \underbrace{2}_{2}f(t) = 0 \quad \text{some are functions of } t$$

⑤ **Homogenity** = whether all terms contain the function or its derivatives

E.g. $$\underbrace{\frac{d^2}{dt^2}f(t)}_{\text{Yes}} + 2\cos t\ \underbrace{f(t)}_{\text{Yes}} = 0 \quad \Rightarrow \text{Homogeneous}$$

$$\underbrace{\frac{d}{dt}f(t)}_{\text{Yes}} + 4\underbrace{f(t)}_{\text{Yes}} = \underbrace{\ln t}_{\text{No!}} \quad \Rightarrow \text{Non-homogeneous}$$

For SHM. problem, the Newton 2^{nd} Law writes

$$F = ma = m\boxed{\frac{d^2}{dt^2}}x(t) = -kx(t)$$

is a 2^{nd} Order Linear Constant Coefficient Homogeneous ODE

1^{st} Order Linear Constant Coefficient Homogeneous ODE

This is the simplest kind of ODE

$$\frac{d}{dt}f(t) + \underline{\lambda} f(t) = 0 \qquad \lambda = \text{some constant}$$

Making use of the fact : $\frac{d}{dt}e^{at} = a \cdot e^{at}$

which is exactly $\frac{d}{dt}f(t) - af(t) = 0$ with $f(t) = e^{at}$

$$\boxed{\Rightarrow \text{Solution of ODE : } f(t) = Ce^{-\lambda t}}$$

C = any constant number

This is the only solution

On the other hand we can also solve it by integration :

$$\frac{1}{f(t)}\frac{df(t)}{dt} + \lambda = 0$$

$$\frac{d}{dt}\left[\ln f(t)\right] = -\lambda$$

$$\ln f(t) = \int -\lambda \, dt = -\lambda t + C$$

$$f(t) = e^{-\lambda t + \underline{C}} \qquad \text{take } C' = e^{C}$$

$$= \underline{C'}e^{-\lambda t}$$

(However basically no other ODE can be solved as simple as this)

E.g. Decay Equation

$$\frac{d}{dt}N(t) = -\underline{k}N(t) \qquad \text{some constant}$$

$N(t)$ = no. of particles

$\frac{d}{dt}N(t)$ = rate of decay in no. of particles $\propto$ no. of particle remains

$\Rightarrow$ General soln : $N(t) = Ce^{-kt}$

We only know C is some constant. What value should we choose?

$\Rightarrow$ By matching a given initial condition

E.g. Given at $t = 0$, no. of particle $= N_0$

Then by substitution $N(0) = Ce^{-k \cdot 0} = C = N_0$

$\Rightarrow$ We arrive a specific soln $N(t) = N_0 e^{-kt}$

Side note: We can derive half life from this soln. form

i.e. At half life $t = \tau_{\frac{1}{2}}$, no. of particles remain

$= \frac{N_0}{2} = \frac{1}{2}$ of original at $t = 0$

$$\Rightarrow N(\tau_{\frac{1}{2}}) = N_0 e^{-k\tau_{\frac{1}{2}}} = \frac{N_0}{2}$$

$$\tau_{\frac{1}{2}} = \frac{\ln 2}{k}$$

What other ODEs can we solve analytically?

- Any linear ODEs
  - ↳ Constant Coefficient
    - ↳ Homogeneous → Just use $e^{\lambda t}$ trick
    - ↳ Non-homogeneous → Undetermined Coefficients

    we will only discuss these
  - ↳ Non-constant Coefficient → More complicate tricks
    E.g. Integrating factor
    Series expansion
    Laplace / Fourier transform
- Non linear ODE — No general method. Only case by case.

2nd Order Linear Constant Coefficient Homogeneous ODE

$$a\frac{d^2}{dt^2}f(t) + b\frac{d}{dt}f(t) + cf(t) = 0$$

a, b, c are some constants

E.g. Basic SHM $m\frac{d^2}{dt^2}x(t) + kx(t) = 0$ $\Rightarrow$ identify $\begin{cases} a = m \\ b = 0 \\ c = k \end{cases}$

To solve this, we can use the $e^{\lambda t}$ trick

$$\Rightarrow \quad a\frac{d^2}{dt^2}e^{\lambda t} + b\frac{d}{dt}e^{\lambda t} + ce^{\lambda t} = 0$$

$$a\lambda^2\cancel{e^{\lambda t}} + b\lambda\cancel{e^{\lambda t}} + c\cancel{e^{\lambda t}} = 0$$

$a\lambda^2 + b\lambda + c = 0$ is a quadratic equation of λ

$$\Rightarrow \quad \lambda_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

So we can take $f(t) = Ce^{\frac{-b+\sqrt{b^2-4ac}}{2a}t}$ or $Ce^{\frac{-b-\sqrt{b^2-4ac}}{2a}t}$

☆☆☆ However this is incompete since for linear ODE superpositions of any of its solutions are also its solution

i.e. If $f_1(t)$, $f_2(t)$ are soln. $C_1f_1(t) + C_2f_2(t)$ is also a soln., for any arbituary constants C_1, C_2

Proof by substitution

Knowing $\begin{cases} a(t)\frac{d^2}{dt^2}f_1(t) + b(t)\frac{d}{dt}f_1(t) + c(t)f_1(t) = 0 \\ a(t)\frac{d^2}{dt^2}f_2(t) + b(t)\frac{d}{dt}f_2(t) + c(t)f_2(t) = 0 \end{cases}$

because $f_1(t)$, $f_2(t)$ are soln. to the ODE. Then

$$a(t)\frac{d^2}{dt^2}[C_1f_1(t) + C_2f_2(t)] + b(t)\frac{d}{dt}[C_1f_1(t) + C_2f_2(t)] + c(t)[C_1f_1(t) + C_2f_2(t)]$$

$$= C_1 \cdot \left[a(t) \frac{d^2}{dt^2} f_1(t) + b(t) \frac{d}{dt} f_1(t) + c(t) f_1(t) \right]$$

$$+ C_2 \cdot \left[a(t) \frac{d^2}{dt^2} f_2(t) + b(t) \frac{d}{dt} f_2(t) + c(t) f_2(t) \right]$$

$$= C_1 \cdot 0 + C_2 \cdot 0$$

$\therefore$ $C_1 f_1(t) + C_2 f_2(t)$ is also a solution

This property can be easily extended to any linear N^{th} order ODE

[1] If a linear ODE is of N^{th}, there will be N independent solution : $f_1(t), f_2(t), \ldots, f_N(t)$ (require rigorous math proof)

[2] The general soln is any linear combination of these N solutions

$$f(t) = C_1 f_1(t) + C_2 f_2(t) + \ldots + C_N f_N(t)$$

with $C_1, C_2, \ldots C_N$ being some constant number

3 Sub-cases of the General Solution

We can derive the general solution further by the value of b^2-4ac

Case 1 : $b^2-4ac > 0$

Both $\lambda_{\pm} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ are real no.

Nothing can be done to further simplify $\boxed{f(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t}}$

Case 2 : $b^2-4ac < 0$

Both $\lambda_{\pm} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ are complex no.

$\Rightarrow$ We can separate the λ into their real/imaginary parts

$$\mathrm{Re}[\lambda_\pm] = \frac{-b}{2a} = p, \quad \mathrm{Im}[\lambda_\pm] = \pm\frac{\sqrt{4ac-b^2}}{2a} = \pm q$$

$\Rightarrow$ Re-label as $\lambda_\pm = p \pm iq$

Then we can make use of the Euler formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

This is just an extension of sin/cos function to complex inputs

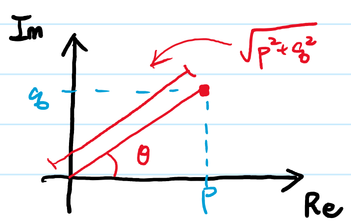
It can be shown by Taylor expansion

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

This allows any complex no. $p+iq$ can be expressed in polar form :



$$\begin{aligned} z &= p + iq \\ &= \sqrt{p^2+q^2}\cos\theta + i\sqrt{p^2+q^2}\sin\theta \\ &= \sqrt{p^2+q^2}(\cos\theta + i\sin\theta) \\ &= \sqrt{p^2+q^2}\, e^{i\theta} \end{aligned}$$

Rewriting $f(t)$ as

$$\begin{aligned} f(t) &= C_1 e^{(p+iq)t} + C_2 e^{(p-iq)t} \\ &= e^{pt}\left[C_1 e^{iqt} + C_2 e^{-iqt}\right] \\ &= e^{pt}\left[C_1(\cos qt + i\sin qt) + C_2(\cos qt - i\sin qt)\right] \\ &= e^{pt}\left[\underline{(C_1+C_2)}\cos qt + \underline{i(C_1-C_2)}\sin qt\right] \\ &= e^{pt}\left[\underline{C_1'}\cos qt + \underline{C_2'}\sin qt\right] \end{aligned}$$

Now we return to an expression without the complex i

It can be further combined into 1 sinusoidal function by taking

$$\begin{cases} C_1' = A\cos\varphi \\ C_2' = -A\sin\varphi \end{cases} \Rightarrow \begin{cases} A = \sqrt{C_1'^2 + C_2'^2} \\ \varphi = \tan^{-1}\left(\frac{-C_2'}{C_1'}\right) \end{cases}$$

Such that

$$f(t) = e^{pt}\left(A\cos\varphi\cos qt - A\sin\varphi\sin qt\right)$$

$$= e^{pt} \cdot A\cos(qt + \varphi)$$

Therefore it is frequent to see 3 kinds of expression

to the solution in the $b^2-4ac<0$ case

$$\boxed{f(t) = \begin{cases} C_1 e^{(p+iq)t} + C_2 e^{(p-iq)t} \\ e^{pt}\left[C_1'\cos qt + C_2'\sin qt\right] \\ e^{pt} \cdot A\cos(qt + \varphi) \end{cases}}$$

Case 3 : $b^2-4ac=0$

This case is problematic because $\lambda_+ = \lambda_- = \frac{-b^2}{2a} = p$

$\Rightarrow$ The general soln. only has 1 independent function Ce^{pt}

But mathematician says if the ODE is of 2^{nd} order,

there must be 2 independent functions in the general soln.

How to find the other function ?

We use the method of Reduction of Order

i.e. Let the other function be $v(t)e^{pt}$ and try to find $v(t)$

[1] Subst. into original ODE

$$a\frac{d^2}{dt^2}\left[v(t)\,e^{pt}\right] + b\frac{d}{dt}\left[v(t)e^{pt}\right] + c\left[v(t)e^{pt}\right] = 0$$

[2] Do product rule for each term

$$\frac{d^2}{dt^2}\left[v(t)\,e^{pt}\right] = \frac{d^2}{dt^2}v(t)\,e^{pt} + 2\frac{d}{dt}v(t)\frac{d}{dt}e^{pt} + v(t)\frac{d^2}{dt^2}e^{pt}$$

$$= e^{pt}\left[\frac{d^2}{dt^2}v(t) + 2p\frac{d}{dt}v(t) + p^2 v(t)\right]$$

$$\frac{d}{dt}\left[v(t)\,e^{pt}\right] = \frac{d}{dt}v(t)\,e^{pt} + v(t)\frac{d}{dt}e^{pt}$$

$$= e^{pt}\left[\frac{d}{dt}v(t) + p\,v(t)\right]$$

Fast trick to do n^{th} derivative : <u>Leibniz Formula</u>

$$d(uv) = \overset{1}{du\cdot v} + \overset{1}{udv}$$

$$d^2(uv) = \overset{1}{d^2u\cdot v} + \overset{2}{2dudv} + \overset{1}{ud^2v}$$

$$d^3(uv) = \overset{1}{d^3u\cdot v} + \overset{3}{3d^2udv} + \overset{3}{3dud^2v} + \overset{1}{ud^3v}$$

$$\vdots$$

The coefficients follows binomial theorem.

$$d^n(uv) = \sum_{r=0}^{n} C^n_r\,(d^r u)(d^{n-r}v)$$

[3] Group terms by derivatives of v(t) and solve

$$a\frac{d^2}{dt^2}v(t) - \underbrace{(2ap+b)}_{=0}\frac{d}{dt}v(t) + \underbrace{(ap^2+bp+c)}_{=0} = 0$$

because $p = \frac{-b}{2a}$ because p is a soln to $ax^2+bx+c=0$

$$\Rightarrow \quad \frac{d^2}{dt^2}v(t) = 0$$

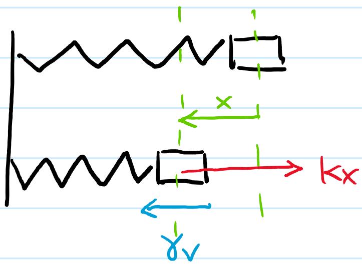
$$\Rightarrow \qquad v(t) = C_1 t + C_2$$

Therefore the other independent function is $(C_1 t + C_2)\, e^{pt}$

Note that it already contain the first function Ce^{pt}

$$\therefore \quad \boxed{f(t) = C_1 \boxed{e^{pt}} + C_2 \boxed{t e^{pt}}}$$

Application : SHM with damping



When a spring mass is compressed by displacement x, it experiences :

- Spring 's force : $-kx$ $(k>0)$
- Damping force $= -\gamma v$ $(\gamma > 0)$

(Assume proportional to velocity
Otherwise difficult to calculate)

Newton 2^{nd} Law writes as :

$$ma = -kx - \gamma v$$

$$\Rightarrow \quad m \frac{d^2}{dt^2} x(t) + \gamma \frac{d}{dt} x(t) + k\, x(t) = 0$$

$$x(t) \sim Ce^{\lambda t} \quad \text{with} \quad \lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

The 3 cases of $\gamma^2 - 4mk \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases}$ correspond

to different physical behaviors

Case 1 : $\gamma^2 - 4mk > 0$ $\Leftrightarrow$ $\lambda_\pm > 0$ $\Leftrightarrow$ Over-Damped

Check the sign of $\lambda_\pm$. Since

$$\gamma^2 > \gamma^2 - 4mk > 0 \qquad (m, k \text{ are +ve number})$$

$$\gamma > \sqrt{\gamma^2 - 4mk}$$

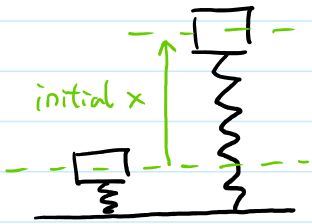
$$0 > \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} = \lambda_+$$

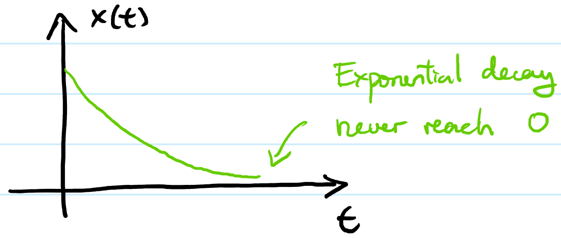
Also $\lambda_- = \dfrac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} < \lambda_+$

$\therefore$ Both λ_+ & λ_- are negative no.

$$\Rightarrow \quad x(t) = C_1 e^{-|\lambda_+|t} + C_2 e^{-|\lambda_-|t}$$

$$= \text{a sum of 2 exponentially decaying function}$$





$\Rightarrow$ "Over-damped" by a too large damping force

s.t. it cannot return to original position

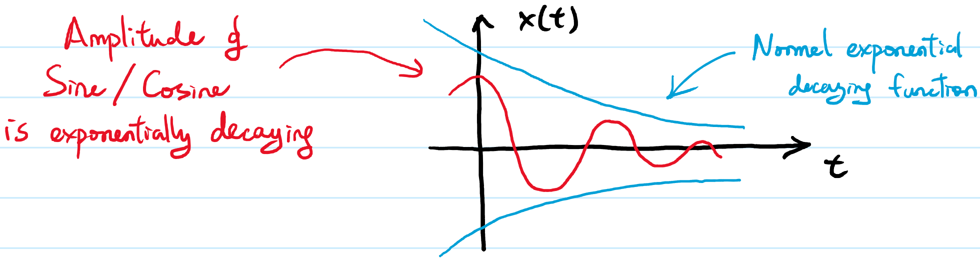
Case 2 : $\gamma^2 - 4mk < 0$ $\Leftrightarrow$ $\lambda_\pm$ complex $\Leftrightarrow$ Under-Damped

Write the soln. in the form

$$x(t) = e^{pt} \cdot A\cos(qt + \varphi)$$

$$= e^{-\frac{\gamma}{2m}t} \cdot A\cos\left(\frac{\sqrt{4mk - \gamma^2}}{2m} t + \varphi\right)$$

$$= (\text{Exponential Decay}) \times (\text{Sinusoidal})$$

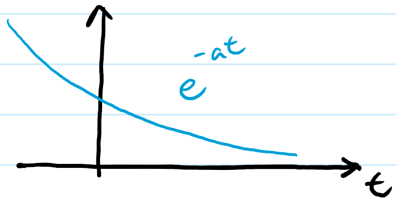


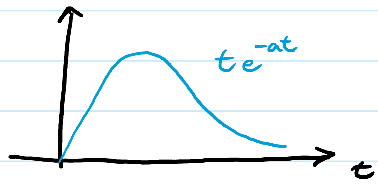
- "Under-damped" because the damping force is not strong enough to stop the vibration.
- In the special case $\gamma = 0$, returns to normal SHM

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}} + \varphi\right)$$

<u>Case 3 : $\gamma^2 - 4mk = 0$</u> $\Leftrightarrow$ $\lambda_+ = \lambda_-$ $\Leftrightarrow$ <u>Critically-Damped</u>

$$x(t) = C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t}$$

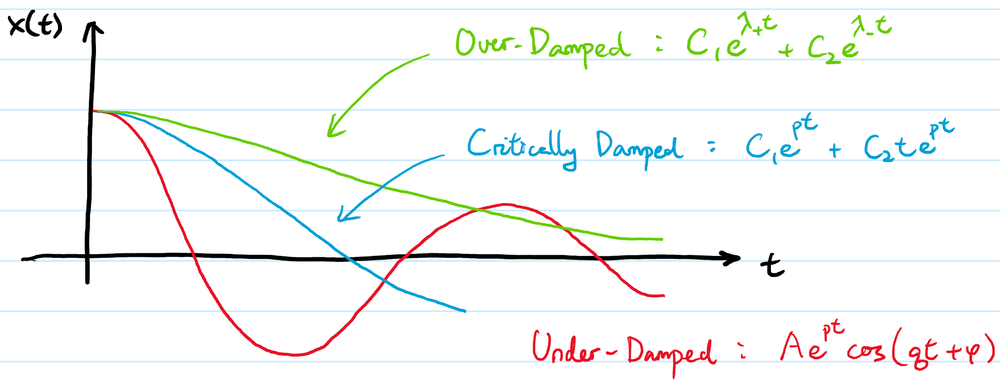




"Critical" because it is in between the other 2 cases

It looks like over-damped case but it can return to original position at when $C_1 + C_2 t = 0$

<u>Summary in 1 graph</u>



2^{nd} Order Linear Constant Coefficient Non-homogeneous ODE

$$a\frac{d^2}{dt^2}f(t) + b\frac{d}{dt}f(t) + cf(t) = \underline{\underline{g(t)}}$$

non-homogeneous term

The general soln. to a non-homogeneous equation can be broken into 2 parts : $f(t) = f_c(t) + f_p(t)$

- $f_p(t)$ = <u>Particular Solution</u>
  = The term to cancel the non-homogeneous term
- $f_c(t)$ = <u>Complementary Solution</u>
  = General soln to its homogeneous counterpart
  i.e. $a\frac{d^2}{dt^2}f_c(t) + b\frac{d}{dt}f_c(t) + cf_c(t) = 0$

This can be shown simply by substitution :

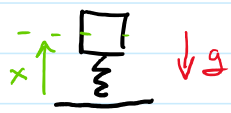
$$a\frac{d^2}{dt^2}\left[f_c(t) + f_p(t)\right] + b\frac{d}{dt}\left[f_c(t) + f_p(t)\right] + c\left[f_c(t) + f_p(t)\right]$$

$$= \underbrace{\left[a\frac{d^2}{dt^2}f_c(t) + b\frac{d}{dt}f_c(t) + cf_c(t)\right]}_{\substack{f_c(t) \text{ is a solution when this } = 0 \\ \text{We already learnt how to solve it}}} + \underbrace{\left[a\frac{d^2}{dt^2}f_p(t) + b\frac{d}{dt}f_p(t) + cf_p(t)\right]}_{\substack{\text{We need a "particular" } f_p(t) \text{ s.t.} \\ \text{this part } = g(t)}}$$

An example of particular solution

A simple mass-spring system under gravity can be described as

$$m\frac{d^2}{dt^2}x(t) = -kx(t) - mg$$



The simplest way to solve it is by grouping mg into x(t)

$$m\frac{d^2}{dt^2}\left[x(t)+\frac{mg}{k}\right] = -k\left[x(t)+\frac{mg}{k}\right]$$

$\frac{mg}{k}$ is a constant. So $\frac{d}{dt}\left(\frac{mg}{k}\right)=0$

Let $y(t) = x(t) + \frac{mg}{k}$. Then we can see that this ODE of y(t)

is the same as a spring-mass system without gravity

$$m\frac{d^2}{dt^2}y(t) = -ky(t)$$

We already know the soln is $y(t) = A\cos\left(\sqrt{\frac{k}{m}}+\varphi\right)$. So

$$x(t) = y(t) - \frac{mg}{k}$$

$$= \underbrace{A\cos\left(\sqrt{\frac{k}{m}}t+\varphi\right)}_{\text{The complementary soln } f_c(t)} - \underbrace{\frac{mg}{k}}_{\text{The particular soln } f_p(t)}$$

The complementary soln $f_c(t)$
i.e. soln when without the term $-mg$

The particular soln $f_p(t)$
i.e the term for canceling $-mg$ in the ODE

Method of Undetermined Coefficients

Finding $f_p(t)$ for *any* g(t) is hard. But in applications

g(t) are usually common functions. In these cases

we can smartly guess what $f_p(t)$ is made of.

Common functions & their derivatives

- Polynomial / Log :
$$\begin{cases} t^n \to t^{n-1} \to \dots t^2 \to t \to 1 & \text{+ve integer power} \\ \ln t \to t^{-1} \to t^{-2} \to \dots & \text{-ve integer power} \\ t^{\frac{1}{2}} \to t^{-\frac{1}{2}} \to t^{-\frac{3}{2}} \to \dots & \text{non integer power} \end{cases}$$

- Trigonometric (especially sin/cos) $\sin t \leftrightarrow \cos t$, $\sin 2t \leftrightarrow \cos 2t$, ...

- Exponential e^{t}, e^{2t}, e^{-t}, ...

Note that a combination of these functions will yield a set of its own derivatives.

E.g. $t^2 \sin t \rightarrow \begin{cases} t\sin t \quad t^2\cos t \\ \sin t \quad t\cos t \quad t^2 \sin t \\ \cos t \end{cases}$

Idea of the method :

$g(t)$ = Combination of $f_p(t)$ & its derivatives

$\Rightarrow \begin{cases} f_p(t) \text{ may contain } g(t) + \text{some other terms} \\ \frac{d}{dt} f_p(t) \text{ may contain } \frac{d}{dt} g(t) + \text{some other terms} \\ \frac{d^2}{dt^2} f_p(t) \text{ may contain } \frac{d^2}{dt^2} g(t) + \text{some other terms} \\ \vdots \end{cases}$

$\Rightarrow$ Try to find a combination that all these terms cancel out

$\Rightarrow$ What are the "some other terms"? Obviously they should be in terms of $\frac{d}{dt} g(t)$, $\frac{d^2}{dt^2} g(t)$, ... so that we can be sure that they will be cancelled.

Calculation Examples

Let the ODE be $a\frac{d^2}{dt^2} f(t) + b\frac{d}{dt} f(t) + cf(t) = g(t)$ (a, b, c = some constants)

Here demonstrates how to find $f_p(t)$ for different $g(t)$

E.g. 1 $\quad g(t) = t^2 + 2t$

- ODE is of 2^{nd} order, so $f_p(t)$ at most contain $\frac{d^2}{dt^2} g(t) \propto 1$
- Derivatives of t^2 already contain derivatives of t

$\Rightarrow$ Guess $f_p(t)$ = Some combination of $t^2, t, 1$

$= At^2 + Bt + C$

and then solve for A, B, C

$$
\begin{aligned}
a \times \left[\frac{d^2}{dt^2} [At^2 + Bt + C] \right] &= [2A] \times a \\
b \times \left[\frac{d}{dt} [At^2 + Bt + C] \right] &= [2At + B] \times b \\
+)\quad c \times [\ At^2 + Bt + C\ ] &= [At^2 + Bt + C] \times c
\end{aligned}
$$

$$a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) = (c \cdot A) t^2 + (c \cdot B + b \cdot 2A) t + (c \cdot C + b \cdot B + a \cdot 2A)$$

$$g(t) = t^2 + 2t + 0$$

By comparing coefficient, we can solve A, B, C by

$$\begin{cases} c \cdot A = 1 \\ c \cdot B + b \cdot 2A = 2 \\ c \cdot C + b \cdot B + a \cdot 2A = 0 \end{cases} \qquad \text{3 Equations, 3 unknowns}$$

E.g. 2 $\quad g(t) = \sin 2t$

- Derivatives of $\sin 2t$ cycle between $\sin 2t$ & $\cos 2t$

$\Rightarrow$ Guess $f_p(t)$ = Some combinations of $\sin 2t$ & $\cos 2t$

$= A \sin 2t + B \cos 2t$

and then solve for A, B

$$a \times \left[\frac{d^2}{dt^2} \left[A \sin 2t + B \cos 2t \right] \right] = \left[-4A \sin 2t - 4B \cos 2t \right] \times a$$

$$b \times \left[\frac{d}{dt} \left[A \sin 2t + B \cos 2t \right] \right] = \left[2A \cos 2t - 2B \sin 2t \right] \times b$$

$$+) \quad c \times \left[A \sin 2t + B \cos 2t \right] = \left[A \sin 2t + B \cos 2t \right] \times c$$

$$a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c \frac{d}{dt} f_p(t) = (-a \cdot 4A - b \cdot 2B + c \cdot A) \sin 2t + (-a \cdot 4B + b \cdot 2A + c \cdot B) \cos 2t$$

$$g(t) = \sin 2t + 0 \cdot \cos 2t$$

By comparing coefficients. we can solve A, B by

$$\begin{cases} -a \cdot 4A - b \cdot 2B + c \cdot A = 1 \\ -a \cdot 4B + b \cdot 2A + c \cdot B = 0 \end{cases}$$

2 Equations 2 unknowns.

E.g. 3 Suppose the ODE is $\frac{d^2}{dt^2} f(t) - \lambda^2 f(t) = e^{\lambda t} = g(t)$

From the homogeneous part $\frac{d^2}{dt^2} f(t) - \lambda^2 f(t) = 0$

we can tell $f_c(t) = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$

So $g(t) = e^{\lambda t}$ is now contained in $f_c(t)$. How to find $f_p(t)$?

$\Rightarrow$ Reduction of order again. Let $f_p(t) = v(t) e^{\lambda t}$

$$\Rightarrow \frac{d^2}{dt^2} \left[v(t) e^{\lambda t} \right] - \lambda^2 \left[v(t) e^{\lambda t} \right] = e^{\lambda t}$$

$$\frac{d^2}{dt^2} v(t) \cdot \cancel{e^{\lambda t}} + 2 \frac{d}{dt} v(t) \cdot \lambda \cancel{e^{\lambda t}} + \cancel{v(t) \cdot \lambda^2 e^{\lambda t}} - \cancel{\lambda^2 v(t) e^{\lambda t}} = \cancel{e^{\lambda t}}$$

$$\frac{d^2}{dt^2} v(t) + 2\lambda \frac{d}{dt} v(t) - 1 = 0$$

some constant

(integrate 1 time) $\frac{d}{dt} v(t) + 2\lambda v(t) = t + C$

arriving at another inhomogeneous 1^{st} order ODE. solving to get

Another constant

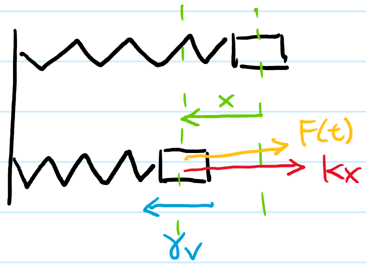
$$v(t) = \frac{t}{2\lambda} + \left(\frac{C}{2\lambda} - \frac{1}{4\lambda^2} \right) + D e^{-2\lambda t}$$

these 2 are already in $f_c(t)$

the true $f_p(t)$

$$\Rightarrow f_p(t) = v(t) e^{\lambda t} = \frac{t}{2\lambda} e^{\lambda t} + \left(\frac{C}{\lambda} - \frac{1}{\lambda^2} \right) e^{\lambda t} + D e^{-\lambda t}$$

Application : Forced SHM



Apply an external force to the spring mass system, s.t. it experiences

- Spring 's force : $-kx$ $(k>0)$
- Damping force $= -\gamma v$ $(\gamma > 0)$
- External force : $F(t)$ (assume time dependent)

Newton 2nd Law writes as :

$$ma = -kx - \gamma v + F(t)$$

$$\Rightarrow \quad m\frac{d^2}{dt^2}x(t) + \gamma\frac{d}{dt}x(t) + kx(t) = F(t)$$

is a non-homogeneous 2nd order ODE

We already know $f_c(t) = \begin{cases} C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} \\ C_1e^{pt} + C_2te^{pt} \\ Ae^{pt}\cos(qt+\varphi) \end{cases}$

And $f_p(t)$ depends on the form of $F(t)$. For example, let

$$F(t) = F_0\cos\omega t$$

We can guess $f_p(t)$ = Combination of $\sin\omega t / \cos\omega t$

$$= A\cos\omega t + B\sin\omega t$$

$$m \times \left[\frac{d^2}{dt^2}\left[A\cos\omega t + B\sin\omega t\right]\right] = \left[-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right] \times m$$

$$\gamma \times \left[\frac{d}{dt}\left[A\cos\omega t + B\sin\omega t\right]\right] = \left[-A\omega\sin\omega t + B\omega\cos\omega t\right] \times \gamma$$

$$+)\quad k \times \left[A\cos\omega t + B\sin\omega t\right] = \left[A\cos\omega t + B\sin\omega t\right] \times k$$

... (skip the steps)

$$\Rightarrow \begin{cases} -A\omega^2 m + B\omega\gamma + kA = F_0 \\ -B\omega^2 m - A\omega\gamma + kB = 0 \end{cases}$$

Solving by substitution to get

$$A = \frac{F_0(m\omega^2 - k)}{(\gamma\omega)^2 + (m\omega^2 - k)^2}, \quad B = \frac{F_0 \gamma\omega}{(\gamma\omega)^2 + (m\omega^2 - k)^2}$$

$$\therefore \; x_p(t) = A\cos\omega t + B\sin\omega t$$

$$= \sqrt{A^2 + B^2}\cos\left[\omega t + \tan^{-1}\left(\frac{B}{A}\right)\right]$$

$$= \frac{F_0}{\sqrt{(\gamma\omega)^2 + (m\omega^2 - k)^2}}\cos\left[\omega t + \tan^{-1}\left(\frac{-\gamma\omega}{m\omega^2 - k}\right)\right]$$

Special Case : When $\omega = \sqrt{\frac{k}{m}}$ = the "natural frequency"

$$\Rightarrow m\omega^2 - k = 0$$

$$\Rightarrow x_p = \frac{F_0}{\gamma\omega}\cos\left[\omega t - \frac{\pi}{2}\right]$$

$$= \frac{F_0}{\gamma}\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}\,t$$

In particular if $\gamma \to 0$ (no damping force)

$$\text{Amplitude} = \frac{F_0}{\gamma}\sqrt{\frac{m}{k}} \to \infty$$

Amplitude will grow to infinitely large = Resonance