Physics Enrichment Program HKUST |
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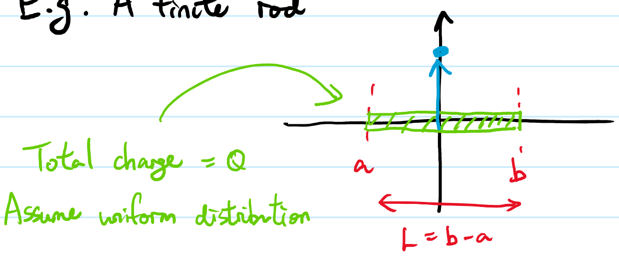
- Basic : Find $\vec{E}$ / V by integration of Columb's Law
- Gauss Law, Flux, Divergence, Divergence Theorem
- Potential by Laplace Equation
- Image Charge method

Basic skill : Find E field / potential at certain point by integral

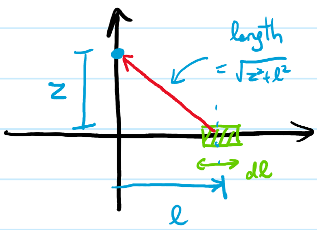
E.g. A finite rod



- What is the E field's magnitude & direction at a given point?
- What is the electrical potential at a given point?

Method : Integral as weighted sum.

i.e. Every small element on the object contributes its E field / potential to the given point = weight.



Charge on each small element of length $d\ell$

$$= \frac{Q}{L} d\ell$$

Distance from the target point $= \sqrt{z^2+\ell^2}$

① Potential

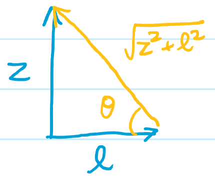
Potential contributed by this segment $= \dfrac{1}{4\pi\varepsilon_0} \dfrac{\frac{Q}{L} d\ell}{\sqrt{z^2+\ell^2}}$ (This is just $\dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{r}$)

$\Rightarrow$ Total potential contributed by the whole object

$$= \int_a^b \frac{1}{4\pi\varepsilon_0} \frac{\frac{Q}{L}\,dl}{\sqrt{z^2+l^2}}$$ integration is over l

② Electric field

$\because$ E field is a vector, also need to consider direction



By the triangle $\cos\theta = \frac{l}{\sqrt{z^2+l^2}}$

$\sin\theta = \frac{z}{\sqrt{z^2+l^2}}$

$\therefore$ E field's vertical component by this element $= (\text{magnitude})\cdot(\sin\theta)$

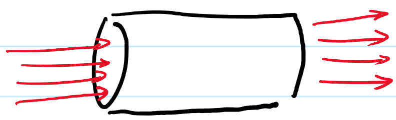
$$= \frac{1}{4\pi\varepsilon_0} \frac{\frac{Q}{L}}{(z^2+l^2)} \cdot \frac{z}{\sqrt{z^2+l^2}}$$

$\Rightarrow$ Total E field's vertical component $= \int_a^b \frac{1}{4\pi\varepsilon_0} \frac{\frac{Q}{L}}{(z^2+l^2)} \cdot \frac{z}{\sqrt{z^2+l^2}}$

Similarly total E field's horizontal component $= \int_a^b \frac{1}{4\pi\varepsilon_0} \frac{\frac{Q}{L}}{(z^2+l^2)} \cdot \frac{l}{\sqrt{z^2+l^2}}$

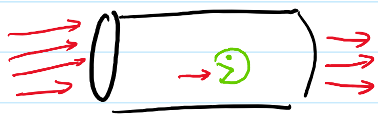
Flux Analogy : A water pipe

- On a normal pipe, water flow in should equal to water flow out



- If we find that water flow in > water flow out

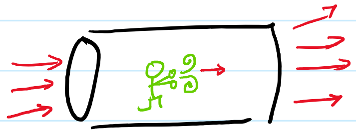
$\Rightarrow$ There must be something in the pipe absorbing water! (a sink)



- If we find that water flow in < water flow out

  $\Rightarrow$ There must be something in the pipe producing water !

(a source)



How to quantitatively tell if there is a source or sink in the pipe ?

↪ We can measure the volume flowing in & flowing out :

In a short interval Δt

velocity at entrance

- Volume flow in $= (v_{in} \cdot \Delta t) \cdot A_{in}$ ← area of the opening at entrance

  Volume flow out $= (v_{out} \cdot \Delta t) \cdot A_{out}$ ← area of the opening at exit

velocity at exit

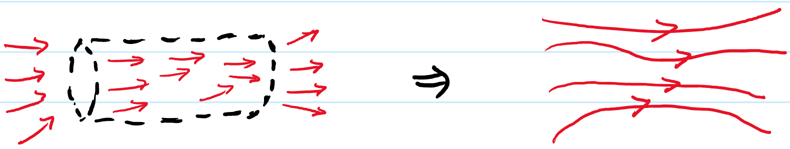
Then define $\Phi = \left(v_{out} A_{out} - v_{in} A_{in} \right)$

$$\begin{cases} \text{if } \Phi > 0 \Rightarrow \text{There is a source} \\ \text{if } \Phi < 0 \Rightarrow \text{There is a sink} \end{cases}$$

Generalize to fields

☆ The flow of water should be continuous

$\Rightarrow$ We can describe the flow in a space by a continuous vector field

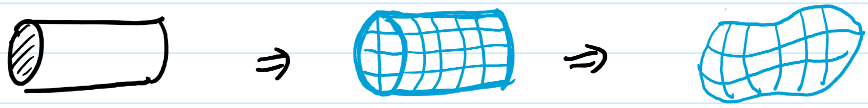


$\vec{F}(\vec{r})$

☆ The water pipe can be of any irregular shape

$\Rightarrow$ We can generalize the pipe into a "close surface"

i.e. - The surface has no holes on it
- You can distinguish the "inner" & "outer" surface



★ In/out flow are not distinguished by the entrance/exit hole

⇒ Any vector that punch through the close surface

$$\begin{cases} \text{from inner to outer surface} = \text{out flow} \\ \text{from outer to inner surface} = \text{in flow} \end{cases}$$

⇒ How to distinguish them mathematically?

↪ Define normal vector on the surface $\vec{S}$

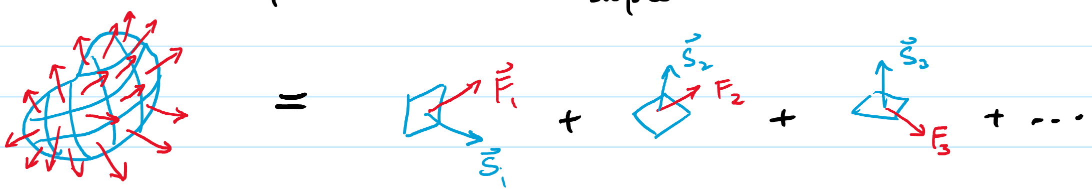
By convention, they point outward from the outer surface

↪ Take the dot product of the field vector on the surface with the normal vector

- If the dot product > 0, field vector is pointing outward
- If the dot product < 0, field vector is pointing inward

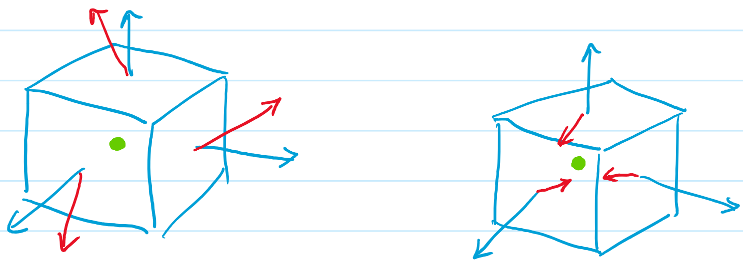
★ Finally, if we define the "Flux" Φ

$$\Phi = \sum_{\substack{\text{all small grid} \\ \text{on the surface}}} \vec{F}_i \cdot \vec{S}_i \rightarrow \iint_{\substack{\text{whole} \\ \text{surface}}} \vec{F}(\vec{r}) \cdot d\vec{S}(\vec{r})$$



We can use the sign of Φ to tell if there are more flow of field lines out of or into the surface

$\Rightarrow$ And then find out if there are source / sink of field lines inside the surface



$\Phi > 0$

$\equiv$ There is a source of field lines

$\equiv$ There is a <u>divergent</u> point.

$\Phi < 0$

$\equiv$ There is a sink of field lines

$\equiv$ There is a <u>convergent</u> point

Calculation Example

Eg. 1 Flux over a sphere

Always first parametrize the sphere surface

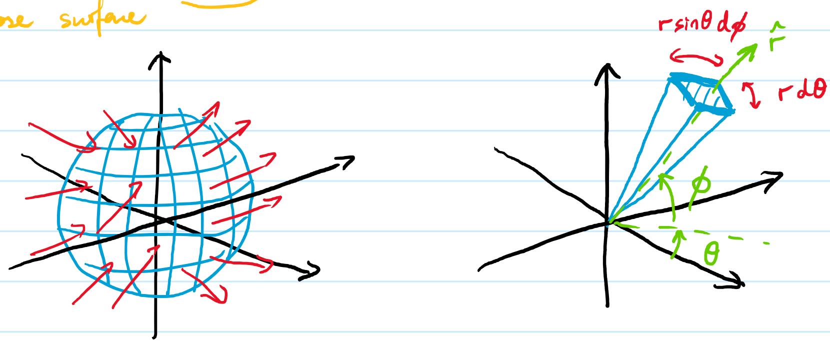
Every small area can be expressed as

$$d\vec{s} = \hat{r}\ r^2 \sin\theta\ d\theta\ d\phi$$

$\Rightarrow$ Total flux through the sphere

Just a convention add a circle for close surface

$$\Phi = \oint\!\!\oint \vec{F}(r,\theta,\phi) \cdot \hat{r}\ r^2 \sin\theta\ d\theta\ d\phi$$



Eg 2 Flux through a plane

You can calculate the flux through a non-close surface

But you need to specify which side the normal vector point to

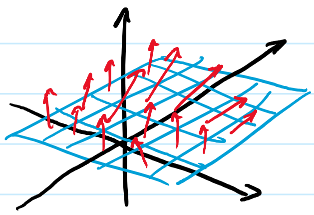
Every small surface of a plane can be expressed as

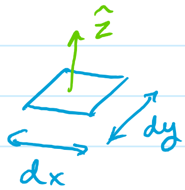
$$d\vec{s} = \hat{z}\, dx\, dy$$

assuming normal vector point to +z direction

⇒ Total flux through the plane

$$\Phi = \iint \vec{F}(x,y,z) \cdot \hat{z}\, dx\, dy$$





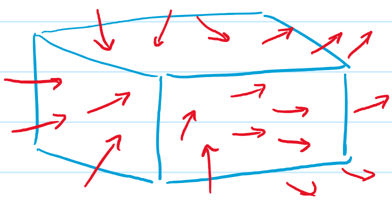
☆☆☆ Just like line integral, the difficulty comes from parametrization

But parametrization of a surface is way more complicated than parametrizing a line.

Divergence

Problem : We have too much freedom to choose the surface

⇒ Cannot be used to show convergent / divergent point if we choose a too big surface



E.g. If flux ≈ 0, we cannot tell if there are really any converge / diverge point, and where they are

Resolution : The close surface should be infinitestimally small

$\Rightarrow$ Consequence : Become the divergence operator

$$\vec{\nabla}\cdot\vec{F} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}F_x \\ F_y \\ F_z\end{pmatrix}$$

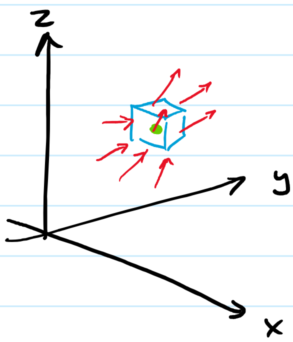
$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

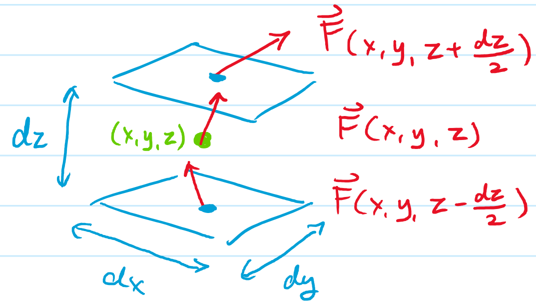
$$= \text{A scalar function}$$

Geometrical representation

- An infinitestimal cube around a point (x, y, z)
- Due to symmetry, only consider the top & bottom surface

$\Rightarrow$ i.e. Surface // xy plane, flux by field's z component only





Total flux through the 2 planes

$$d\Phi = \vec{F}(x,y,z+\tfrac{dz}{2})\cdot\hat{z}\,dx\,dy - \vec{F}(x,y,z-\tfrac{dz}{2})\cdot\hat{z}\,dx\,dy$$

$$= \left(\frac{\vec{F}(x,y,\boxed{z+\frac{dz}{2}}) - \vec{F}(x,y,\boxed{z-\frac{dz}{2}})}{\boxed{dz}}\right)\cdot\hat{z}\,dx\,dy\,dz$$

definition of derivative

$$= \boxed{\frac{\partial}{\partial z}}\vec{F}(x,y,z)\cdot\hat{z}\,(dx\,dy\,dz)$$

only take z component

$$= \frac{\partial}{\partial z}F_z(x,y,z)\quad(dx\,dy\,dz)$$

$$\begin{pmatrix}\text{Flux through} \\ \text{surface // xy plane}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of } \vec{F} \\ \text{along z direction}\end{pmatrix} \times \begin{pmatrix}\text{unit} \\ \text{volume}\end{pmatrix}$$

The other 2 directions are similar. We can add to the total flux

$$\begin{pmatrix}\text{Total flux} \\ \text{through a volume}\end{pmatrix} = \sum_{i=x,y,z} \begin{pmatrix}\text{Flux through the planes} \\ \underline{\underline{\text{normal}}} \text{ to } i \text{ direction}\end{pmatrix}$$

$$= \sum_{i=x,y,z} \begin{pmatrix}\text{Rate of change of } \vec{F} \\ \text{along } i \text{ direction}\end{pmatrix} \times \begin{pmatrix}\text{unit} \\ \text{volume}\end{pmatrix}$$

$$\underbrace{d\Phi_x + d\Phi_y + d\Phi_z}_{\text{total flux}} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) \times (dx\,dy\,dz)$$

$$= (\vec{\nabla} \cdot \vec{F}) \times \underbrace{(dx\,dy\,dz)}_{\text{volume}}$$

$$\boxed{\therefore \text{ Divergence} \sim \frac{\text{Flux through a closed surface}}{\text{Volume under the close surface}} \sim \text{Flux "density" (by volume)}}$$

By this geometrical interpretation, we can directly go to a useful theorem:

Divergence Theorem (state without proof)

must be close surface

Volume integral $\iiint (\cdots)\, dx\,dy\,dz$

$$\underbrace{\boxed{\oiint} \vec{F} \cdot d\vec{S}}_{\text{Total flux}} = \boxed{\iiint} \underbrace{(\vec{\nabla} \cdot \vec{F})}_{\text{Flux per volume}} \boxed{d\tau}$$

Integrate over volume = total flux

Gauss's Law

Gauss's Law is purely an observational statement relating E field & charges.

☆☆☆ Observe Flux of E field $\neq 0$ on the surface

$\Leftrightarrow$ Source / Sink of E field exist within the surface

Charge is the only source of E field. This is THE observation

$\Leftrightarrow$ Charge exist within the surface

Gauss's Law can be written in 2 forms:

Integral form : $\oiint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$

(Flux) ~ (Integrate $\vec{E}$ over a surface) = (Charge within the surface) ~ (charge)

Differential form : $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

(Flux per volume) ~ (Divergence of $\vec{E}$ over the space) = (Charge density over the space) ~ (Charge density)

They can be inter-converted by divergence theorem.

$$\text{LHS} = \oiint \vec{E} \cdot d\vec{s} = \iiint (\vec{\nabla} \cdot \vec{E}) \, d\tau$$

$$\text{RHS} = \frac{Q}{\varepsilon_0} = \iiint \frac{\rho}{\varepsilon_0} \, d\tau$$

Example of using Gauss's Law Integral form

☆ Integral form is nice to use only in very symmetric scenerio

To simplify calculation, we should always choose a surface s.t.

① $\vec{E}$ has constant magnitude on the surface

② $\vec{E}$ form the same angle with the normal vector everywhere on the surface

Only then the surface integral is easy to compute

$$\oint\!\!\oint \vec{E}\cdot d\vec{s} = \oint\!\!\oint |\vec{E}| \cos\theta \, d|\vec{A}|$$

Dot product $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$= |\vec{E}|\cos\theta \oint\!\!\oint d|\vec{A}|$$

Same magnitude & angle everywhere
So you can take them out of integral

$$= |\vec{E}|\cos\theta \cdot (\text{total surface area})$$

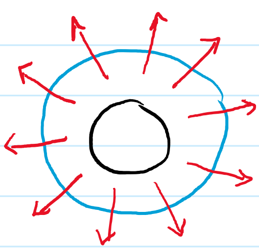
Only then we can use Gauss's Law integral form to find the $\vec{E}$ field from charge

$$|\vec{E}| = \frac{Q}{\varepsilon_0} \cdot \frac{1}{\text{total surface area}} \cdot \frac{1}{\cos\theta}$$

E.g. 1 Solid sphere with uniform charge density ρ, volume $= \frac{4}{3}\pi R^3$

This is a spherical symmetric case. Good surface = Sphere

∵ By rotational symmetry, we must have



[1] E field only point in radial direction

[2] E field 's magnitude does not depends on the latitude or longitude (const. of θ, ϕ)

i.e. same magnitude at the same radial distance from center

These allow us to use Gauss's Law integral form to find $\vec{E}$

$$\oiint \vec{E} \cdot d\vec{S} = |\vec{E}| \cos\theta \text{ (surface area)} = \frac{Q}{\varepsilon_0}$$

$$\underline{|\vec{E}|} = \frac{Q}{\varepsilon_0} \cdot \underline{\frac{1}{4\pi R^2}} \cdot \underline{\frac{1}{\cos 0^\circ}}$$

All we can get is the magnitude

Sphere surface area

E field = Radial
⇒ normal to surface

$$\Rightarrow \vec{E} \text{ as a vector} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}$$

(the final ans)

Add back the direction

<u>Note</u> If the object / charge distribution is not symmetric,

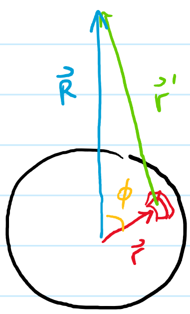
We cannot guarentee that $\vec{E}$ always points in radial direction

And $\vec{E}$'s magnitude may vary with radial distance

Here is how we can find $\vec{E}$ by Columb's Law.

Doing integration by Columb's law is the only solution

E.g. If the charge distribution $\rho(r, \theta, \phi)$ in the sphere is arbituary



- Unit volume in spherical coor. $r^2 \sin\theta \, dr d\theta d\phi$
- Distance of this unit volume from the point of interest

$$r' = \sqrt{R^2 + r^2 - 2Rr\cos\phi}$$

- By symmetry, E field only has radial component

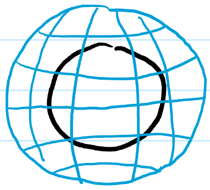
⇒ Extracting radial component by dot product

$$\frac{\vec{r}' \cdot \vec{R}}{|\vec{r}'||\vec{R}|} = \frac{(\vec{R} - \vec{r}) \cdot \vec{R}}{|R|\sqrt{R^2 + r^2 - 2Rr\cos\phi}} = \frac{|\vec{R}| - |\vec{r}|\cos\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}}$$

⇒ Finally the whole integral written as

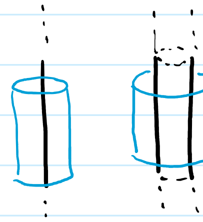
$$E = \iiint_{\text{whole sphere}} \frac{1}{4\pi\varepsilon_0} \frac{\rho(r, \theta, \phi)}{(R^2 + r^2 - 2Rr\cos\theta)} \frac{|R| - |r|\cos\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}} r^2 \sin\phi \, dr d\theta d\phi$$

P.S. There are not many configurations that you can find a good Gaussian surface to use Gauss's Law integral form



Sphere

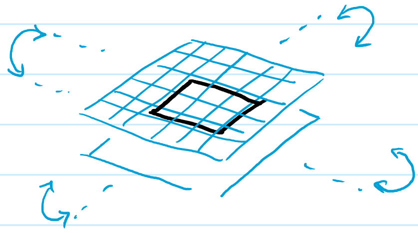
$\Rightarrow$ Sphere



Infinite long rod

Infinite long cylinder

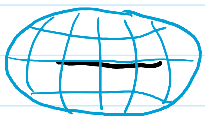
$\Rightarrow$ Cylinder



Infinite large plane

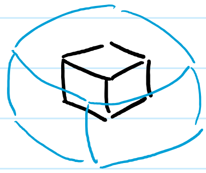
$\Rightarrow$ 2 Infinite large planes on both side that connects at ∞

In most cases, the good Gaussian surfaces are not easy to do calculation



Finite rod

$\Rightarrow$ Ellipsoid

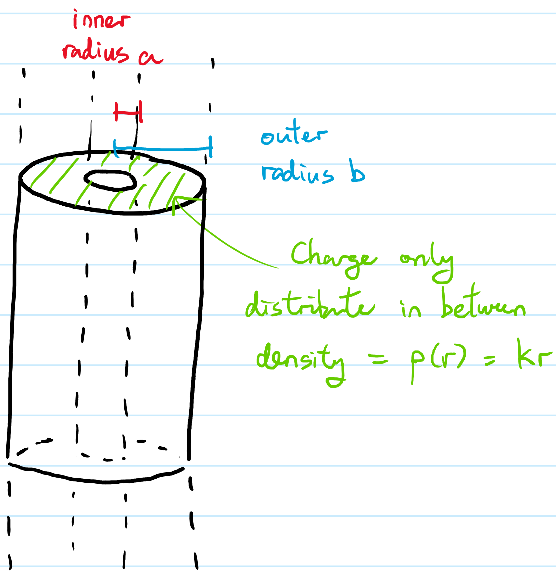


Cube

$\Rightarrow$???

E.g. 2 Infinitely long hollow cylinder with thickness

Charge density $\propto$ radial distance from center. Find $\vec{E}$ everywhere.



Observing the charge distribution can be divided into 3 regions

$$\begin{cases} r \leq a \\ r \leq R \leq b \\ r \geq b \end{cases}$$

$\Rightarrow$ Do Gauss Law 3 times

① $r \leq a$

- Draw a cylinder with radius $R \leq a$
- Charge inside the cylinder $= 0$

$\Rightarrow$ By Gauss's Law $\oint\!\!\oint \vec{E} \cdot d\vec{s} = 0$

$\Rightarrow$ Can claim $\vec{E} = 0$ by symmetry

② $a \leq r \leq b$

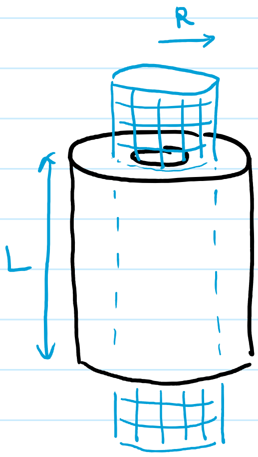
- Draw a cylinder with radius $a \leq R \leq b$
- Within a segment of length L, amount of charge enclosed in the cylinder

$$= \int_a^R \rho \cdot 2\pi r L \, dr = \int_a^R kr \cdot 2\pi r L \, dr = \frac{2\pi k L}{3}(R^3 - a^3)$$

$\Rightarrow$ By Gauss's Law, $\underset{\text{cylinder of radius } R}{\oint\!\!\oint} \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon_0} \cdot \frac{2\pi k L}{3}(R^3 - a^3)$

$\Rightarrow$ Can claim by symmetry

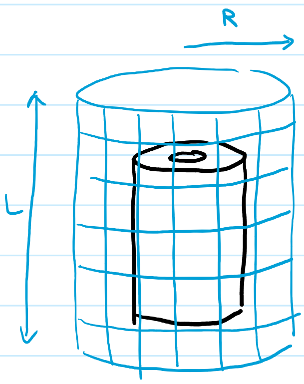
$$|\vec{E}(R)| = \frac{2\pi k L}{3\varepsilon_0}(R^3 - a^3) \, \frac{1}{2\pi R L} \cdot \frac{1}{\cos 0^\circ} = \frac{k}{3\varepsilon_0}\left(R^2 - \frac{a^3}{R}\right)$$



③ $r \geq b$

- Draw a cylinder with radius $R \geq b$
- Within a segment of length L, amount of charge enclosed in the cylinder

$$= \int_a^b \rho \cdot 2\pi r L \, dr = \int_a^b kr \cdot 2\pi r L \, dr = \frac{2\pi k L}{3}(b^3 - a^3)$$



$\Rightarrow$ By Gauss's Law, $\oiint_{\text{cylinder of radius } R} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \cdot \frac{2\pi k L}{3} (b^3 - a^3)$

$\Rightarrow$ Can claim by symmetry

$$|\vec{E}(R)| = \frac{2\pi k L}{3\varepsilon_0}(b^3 - a^3) \frac{1}{2\pi R L} \cdot \frac{1}{\cos 0^\circ} = \frac{k}{3\varepsilon_0 R}(b^3 - a^3)$$

Electric Potential

Observation: E field produced by static charge never form loops

$\Rightarrow$ Static E field is conservative

Mathematic fact: Any conservative field can be expressed as the gradient of some scalar function (potential)

☆ By convention, we write the relation as $\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$

additional -ve sign

And the reverse is simply $V(\vec{r}) = \int_{(\text{Any path } \infty \to \vec{r})} \vec{E}(\vec{r}) \cdot d\vec{\ell}$

Note 1: $\because$ The field is conservative, the integral along any path should give the same value. So usually people just choose a straight line.

Note 2: Remember potential is just a relative concept. This expression is in fact the potential difference between $\vec{r}$ and ∞

Furthermore if we substitute this into the Gauss Law differential form

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}V) = \frac{\rho}{\varepsilon_0}$$

This is called the Laplacian operator

$$\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]V(x,y,z) = -\frac{\rho(x,y,z)}{\varepsilon_0}$$

This is the direct relation between $V(r)$ & $\rho(r)$ and we can use it to find each other

- Knowing $V(r)$ → Differentiate to get $\rho(r)$ — Easy but rarely needed
- Knowing $\rho(r)$ → Need to solve PDE to get $V(r)$ — Painful but almost always the case

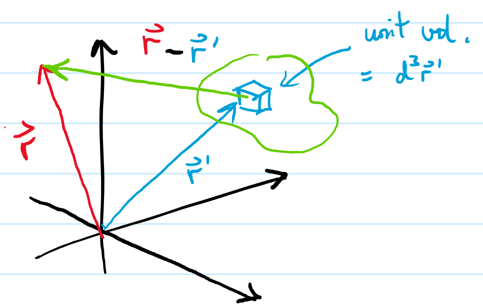
This PDE belongs to the class called Poisson Equation and it is one of the most studied PDE in history.

Its solution depends on the boundary condition. For example in the special case of given $V(\infty) = 0$, it has a familiar solution :

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

integrate to ∞ everywhere

Just a lazy way to write $dx'dy'dz'$



$$\sim \frac{1}{4\pi\varepsilon_0}\sum\frac{\text{charge}}{\text{distance}}$$

$\equiv$ Columb's Law of potential

(This is the fancy form of Columb's Law, expressed using vector)

After solving for $V(\vec{r})$, we can get $\vec{E}$ by differentiation

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

$$= \frac{1}{4\pi\varepsilon_0} \underbrace{\iiint}_{\text{everywhere integrate to } \infty} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \underbrace{\left[\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\right]}_{\text{Just the unit vector of } \vec{r}-\vec{r}\,'} d^3\vec{r}\,'$$

$$\sim \frac{1}{4\pi\varepsilon_0} \sum \frac{\text{charge}}{(\text{distance})^2}$$, Pointing in the direction from source to target

$\equiv$ Just a fancier form of Columb's Law for field

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}$$

Why do we "invent" potential V

The concept of electric potential is introduced not just for fun or for doing visualization of equipotential surface

We use $V(\vec{r})$ because its PDE is easier to solve than solving for $\vec{E}(\vec{r})$'s PDE

- $\vec{\nabla}^2 \boxed{V(\vec{r})} = \frac{-\rho(\vec{r})}{\varepsilon_0}$ — $V(r)$ is a scalar function, so there is only 1 function to solve.
- $\vec{\nabla} \cdot \boxed{\vec{E}(\vec{r})} = -\frac{\rho(\vec{r})}{\varepsilon_0}$ — $\vec{E}(\vec{r})$ is a vector function of 3 components i.e. there are 3 <u>inter-depending</u> functions to solve

Therefore when given $\rho(\vec{r})$ and asked for $\vec{E}(\vec{r})$, it is JUST BETTER to first solve $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$, and then take $\vec{E} = -\nabla V$

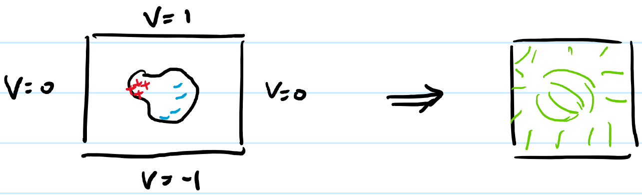
★★★ In practice, the most common problems we need to solve look like this in general :

- Given $\begin{cases} \text{Distribution of charge} \\ \text{Potential at the space boundary} \end{cases}$

Distribution of charge — We can manually place them or model where they should be

Potential at the space boundary — Controlling potential (voltage) is easy

- Solve : Distribution of E field in the space



How does the E field look ?

This is in fact the task of solving the PDE $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

This is an inhomogeneous PDE ! Solving is awful !

Can we try to avoid that ? YES , but not always.

And that is why you are taught these different methods :

- If the configuration is very symmetric

⇒ Can use symmetry argument to convert Gauss Law integral form into just multiplication : $\oiint \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0} \Rightarrow E\cdot(\text{Area})\cos\theta = \frac{Q}{\varepsilon_0}$

- If it is not symmetric enough , but the charges only distribute in a finite size of space AND potential at ∞ is set to 0

⇒ Can write Columb's Law for each charge source , then add / integrate to compute the resultant E field

- If still no ,

  $\Rightarrow$ Sorry , but please solve the PDE from scratch. :(

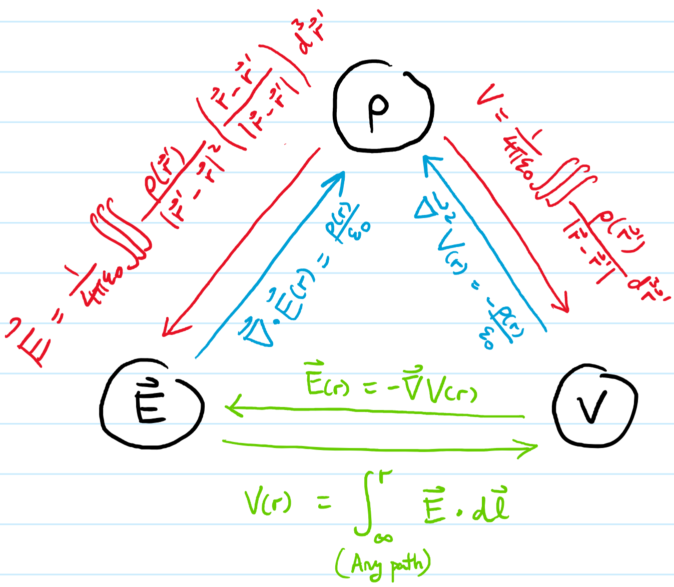
(And as mentioned , prefer first solving $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$, then taking $\vec{E} = -\nabla V$)

Summary

In electrostatic , these 3 quantities are inter-convertible

$\rho(\vec{r})$ $\quad\quad$ $\vec{E}(\vec{r})$ $\quad\quad$ $V(\vec{r})$

i.e. If we know 1 of them, we can solve for other 2



— Gauss's Law

— Columb's Law
(☆ only if $\vec{E}/V = 0$ at ∞)

— By definition.

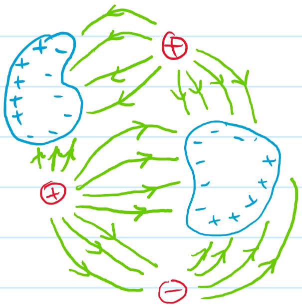
Electrostatic with presense of conductors

Situation : There are charged objects and conductor distributing in the space What is the result $V(\vec{r})$?

(And so on $\vec{E}(\vec{r})$?)

Problem : Induced charge can appear on surface of conductors

- Depends on the distribution of charged objects and shape of conductors



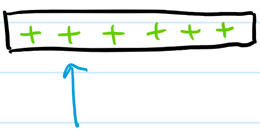
- Measuring the induced charge distribution is basically impossible, because any charge probe will change it

$\Rightarrow$ We cannot use Columb's Law to compute $V(\vec{r})$.

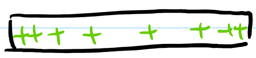
☆ However, conductor always satisfy one property :

Conductor surface = Equipotential surface = const. V

Reason : If there is potential difference, charge will move until potential is equal everywhere



If the charge distribute uniformly this charge experience net force to the left



Charge re-distribute so that net force become 0.

$\Rightarrow$ Potential = constant

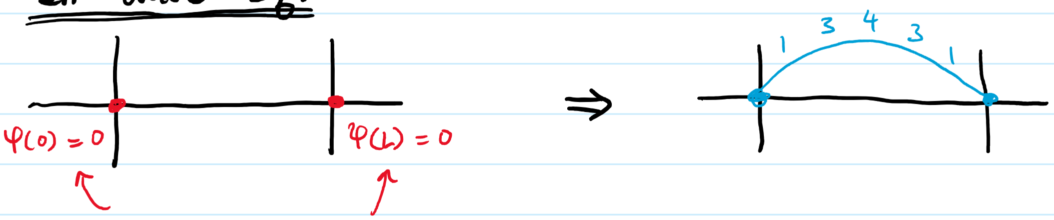
In fact, "Conductor surface = Equipotential" is a boundary condition

☆ By knowing the boundary condition, we can solve the PDE

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ to <u>determine the unique solution $V(\vec{r})$</u>

<u>P.S.</u> Recall that in the note of wave Eq. we have introduced some boundary condition which allow us to solve the PDE of 1D wave Eq. The idea is the same except that we are now solving a 2D/3D distribution

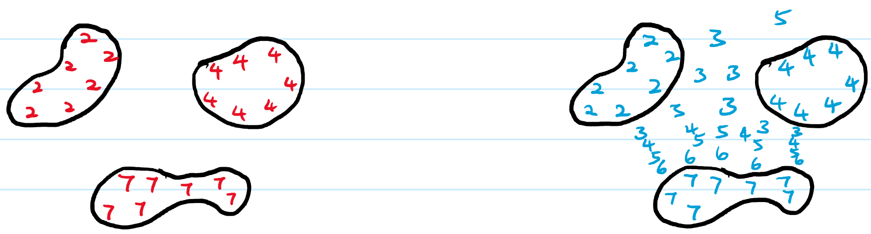
<u>In wave Eq.</u>



If we know the magnitude at the 2 ends

We can solve for the general soln of the waveform by the PDE

<u>In Laplace Eq.</u>



If we know the magnitude of potential on each conductor surface

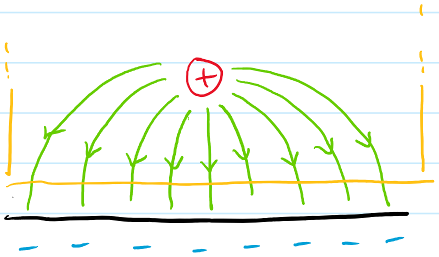
We can solve for the soln of the potential distribution by the PDE

<u>Image Charge Method</u>

Solving PDE is annoying. Are there any shortcut?

Yes, but only in some symmetric configurations.

E.g. 1 Point charge above a metal plate. Find $V(r)$ above the plate

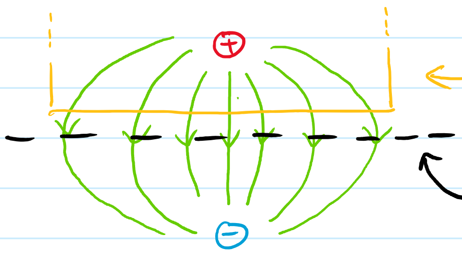


The region we are interested in

Metal plate = Equipotential

Induced charge on the plate

But we know another scenario that looks more or less the same



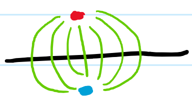
The region we are interested in

Easy to prove that $V = 0$ on this line

☆☆☆ Observe the boundary condition of the yellow regions are the same

$$\begin{cases} V = 0 & \text{on the metal plate / line} \\ V \sim \frac{1}{r} & \text{near } \oplus \text{ and at } \infty \end{cases}$$

$\Rightarrow$ We expect we get the same answer after solving the PDE

But  is obviously much easier to solve

This is just a distribution with 2 point charges.

We can compute $V(r)$ by Columb's law. No need for PDE!

General steps of image charge method :

- Place "virtual charge" at positions outside the region where we want to find $V(r)$.

E.g. We are interested in the $V(r)$ above the metal plate (yellow region). So the virtual charge CANNOT be placed above the metal plate

- The placement of the virtual charge should satisfy:
  1. Potential on the conductor surface are constant
  2. Produce the same E-field / potential distribution as if produced by some induced charge on the conductor surface

- Laplace Eq. guarentee the solution to be unique if the boundary condition is fixed (without proof). Since the solution found by image method also satisfy the given boundary condition, it must be the same solution as if we solve by Laplace Eq.

⇒ Image method help us to avoid solving PDE. But only helpful when the configuration has certain symmetry.

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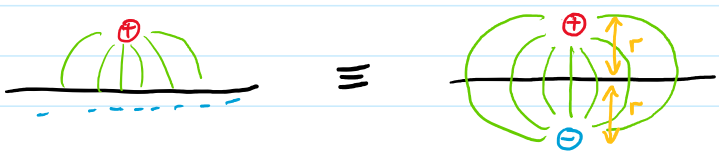
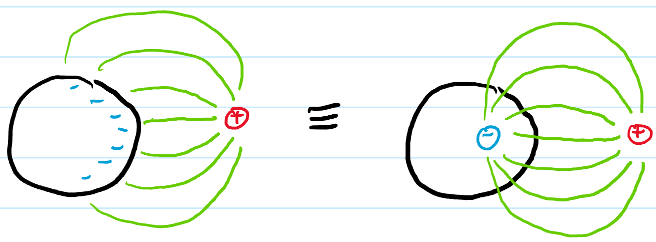
2 most common cases of using image charge

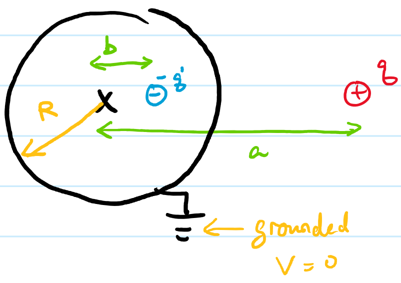
① Metal plane



② Metal sphere



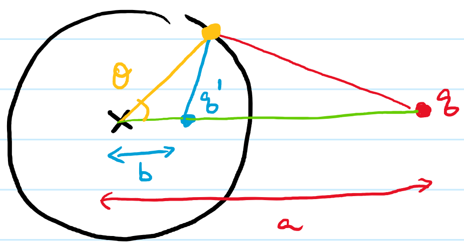
Note that the image charge on the sphere needs to be specifically calculated



If a real charge is placed at distance a from the sphere center, V at surface $= 0$

- Position of image charge from center : $b = \frac{R^2}{a}$
- Magnitude of image charge : $q' = -\frac{b}{a} q$

Proof :



Consider an arbitrary point on the sphere surface. Take the elevation angle be θ

Potential by q : $V = \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}}$

Potential by q' : $V = \frac{1}{4\pi\varepsilon_0} \frac{q'}{\sqrt{R^2 + b^2 - 2Rb\cos\theta}}$

Since the sphere's surface should be equipotential

the total V should be a constant, independent of θ

We have already taken the total V be 0.

Then try to separate terms that depends on θ.

$$\frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{R^2+a^2-2Ra\cos\theta}} + \frac{1}{4\pi\varepsilon_0} \frac{q'}{\sqrt{R^2+b^2-2Rb\cos\theta}} = 0$$

$$\left(\frac{-q'}{q}\right)^2 = \frac{R^2+b^2-2Rb\cos\theta}{R^2+a^2-2Ra\cos\theta}$$

$$\left[1-\left(\frac{q'}{q}\right)^2\right]R^2 + \left[b^2 - \left(\frac{q'}{q}\right)^2 a^2\right] - 2R\cos\theta\left[b - \left(\frac{q'}{q}\right)^2 a\right] = 0$$

Because we require the potential to be independent of θ

$\boxed{1}$ Coefficient of $\cos\theta$ must be 0

$$\Rightarrow \quad b = \left(\frac{q'}{q}\right)^2 a$$

$\boxed{2}$ The remaining terms should also equal to 0

$$\left[1-\left(\frac{q'}{q}\right)^2\right]R^2 + \left[b^2 - a^2\left(\frac{q'}{q}\right)^2\right] = 0$$

$$\left[1-\frac{b}{a}\right]R^2 + \left[b^2 - a^2\cdot\frac{b}{a}\right] = 0$$

$$\left[1-\frac{b}{a}\right]R^2 + a^2\cdot\frac{b}{a}\left[\frac{b}{a}-1\right] = 0$$

$$\left[R^2 - ab\right]\left[1-\frac{b}{a}\right] = 0$$

$$\Rightarrow \text{ either } \underset{\text{(rejected)}}{b = a} \quad \text{or} \quad \boxed{b = \frac{R^2}{a}}$$

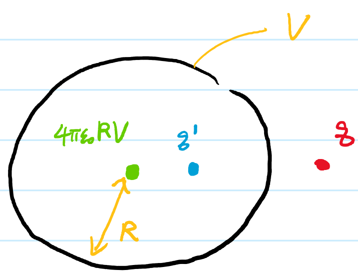
Finally subst. back to get

$$q' = \pm\sqrt{\frac{b}{a}}\,q = -\frac{R}{a}q$$

Obviously we should take $-$ve
$\because$ from the figure, image charge has opposite sign to the real charge

Note: We can choose the sphere surface potential to be any value

By adding an extra charge at the center,



Potential on the surface $= \text{contribution by } \underbrace{\bullet + \bullet}_{\text{total } V = 0 \text{ on surface}} + \bullet$

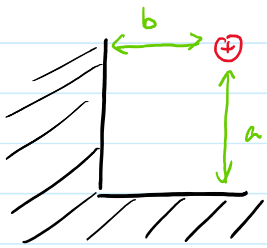
$$= \text{contribution by } \bullet$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{R} \left(4\pi\varepsilon_0 R V \right)$$

$$= V$$

EX. How to add the virtual charge so that potential on the conductor surface is equipotential.

①



②

Hint: Need 3 virtual charges in both cases.

Solution: