Physics Enrichment Program HKUST |
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Written by Tony Shing

- EM wave as special soln to Maxwell Equations
- Properties derived from wave nature & polarization
- Energy of EM wave

Soln. to Maxwell Equation without sources

In vacuum, there is no charge ($\rho = 0$) or current ($\vec{J} = 0$)

$\Rightarrow$ The system of equations become simple.

No need to be solved through converting to $V / \vec{A}$

Getting an equation with $\vec{E}$ only

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

(Take Curl to Faraday's Law)

By vector identity

$$\nabla(\underbrace{\nabla \cdot \vec{E}}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\underbrace{\vec{\nabla} \times \vec{B}})$$

Gauss's Law, but charge $\rho = 0$

Ampere's Law but current $\vec{J} = 0$

$$0 - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(0 + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

Getting an equation with $\vec{B}$ only

current $\vec{J} = 0$

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times \left(\underline{0} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

(Take curl to Ampere's Law)

By vector identity

$$\nabla(\underbrace{\nabla \cdot \vec{B}}) - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\underbrace{\nabla \times \vec{E}})$$

Gauss Law for B field

Faraday's Law

$$0 - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$

Therefore in vacuum, $\vec{E}$ & $\vec{B}$ exist as solutions of a 3D wave equation

i.e. Solution to the equation of the form $\nabla^2 \varphi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\varphi$

with wave speed $v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c =$ speed of light

(In linear material, we replace by $\varepsilon_0 \to \varepsilon$, $\mu_0 \to \mu$

and have $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$)

Analytical soln. & interpretation

Recall the general soln to 1D wave Eq. looks like

$$\varphi(x,t) = \underbrace{f(x-vt)}_{\substack{\text{wave travelling in} \\ +x \text{ direction}}} + \underbrace{g(x+vt)}_{\substack{\text{wave travelling in} \\ -x \text{ direction}}}$$

In the 3D case, we promote the position x to vector $\vec{r}$

$$\varphi(\vec{r},t) = \underbrace{f(\vec{k}\cdot\vec{r}-\omega t)}_{\substack{\text{wave travelling in} \\ \vec{k} \text{ direction}}} + \underbrace{g(\vec{k}\cdot\vec{r}+\omega t)}_{\substack{\text{wave travelling in} \\ -\vec{k} \text{ direction}}}$$

which $\vec{k} = (k_x, k_y, k_z)$ and ω are parameters that we

are free to choose, as long as $v = \frac{\omega}{|\vec{k}|}$ is satisfied.

Review from 1D wave Eq.

Recall in 1D case, we can express the general soln

as a combination of sine & cosine after separating x & t part

$$\varphi(x,t) = \sum_{\substack{\text{all } k \text{ allowed} \\ \text{by boundary condition}}} \left[A_k \cos kvt + B_k \sin kvt\right]\left[C_k \cos kx + D_k \sin kx\right]$$

But it is usually easier to do calculation in the complex exponential form

i.e. Make $C_1 e^{i\theta} + C_2 e^{-i\theta} = A\cos\theta + B\sin\theta$

$$\Rightarrow \begin{cases} C_1 + C_2 = A \\ i(C_1 - C_2) = B \end{cases} \quad \text{or} \quad \begin{cases} C_1 = \frac{A - iB}{2} \\ C_2 = \frac{A + iB}{2} \end{cases}$$

(We will always use exponential form from now on)

So the solution is simplified into

$$\varphi(x,t) = \sum_{\text{all allowed } k} \left[A_k' e^{ikvt} + B_k' e^{-ikvt} \right] \left[C_k' e^{ikx} + D_k' e^{-ikx} \right]$$

$$= \sum C_{k\pm\pm} e^{i(\pm kx \pm kvt)}$$

- the terms will be in this form, so we can always start from it.

Promotion to 3D

The solution to 3D wave equation is similar, except that

there are now 3 spatial parts :

$$\varphi(x,y,z,t) = \sum_{\substack{\text{all } k_x\, k_y\, k_z\, \omega \\ \text{with } \sqrt{k_x^2+k_y^2+k_z^2} = \frac{\omega}{v} \\ \text{and allowed by boundary condition}}} C\, e^{\pm ik_x x} \cdot e^{\pm ik_y y} \cdot e^{\pm ik_z z} \cdot e^{\pm i\omega t}$$

$\leftarrow$ try to show this yourself.

$$= \sum_{\vec{k}} C_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{r} \pm \omega t)}$$

By writing $\vec{k} = (k_x, k_y, k_z)$ & $\vec{r} = (x, y, z)$

s.t. $\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$

All the terms will be in this form, so we only need to continue

with $C_{\vec{k}} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ and generalize the result to all $\vec{k}$

$\vec{E}$ & $\vec{B}$ in pairs

Now we have the wave Eq. solution of E & B individually

For convenience we only examine the term with $\vec{k} = (0, 0, k)$

$$\begin{cases} \vec{E}(\vec{r}, t) = \vec{E}_k e^{i(kz-\omega t)} \\ \vec{B}(\vec{r}, t) = \vec{B}_k e^{i(kz-\omega t)} \end{cases}$$

$\vec{E}$ & $\vec{B}$ are vector field, so the coefficient E_k, B_k should also be vectors

$$\vec{E}(\vec{r}, t) = \vec{E}_k e^{i(kz-\omega t)} = \underline{\underline{\begin{pmatrix} E_{k,x} \\ E_{k,y} \\ E_{k,z} \end{pmatrix}}} e^{i(kz-\omega t)}$$

a vector with 3 constants

$$\vec{B}(\vec{r}, t) = \vec{B}_k e^{i(kz-\omega t)} = \underline{\underline{\begin{pmatrix} B_{k,x} \\ B_{k,y} \\ B_{k,z} \end{pmatrix}}} e^{i(kz-\omega t)}$$

a vector with 3 constants

- Gauss Law for $\vec{E}$ & $\vec{B}$:

$$\vec{\nabla} \cdot \vec{E} = \underbrace{\frac{\partial}{\partial x}\left(E_{k,x} e^{i(kz-\omega t)}\right)}_{\text{there is no } x \text{ inside}} \xrightarrow{} 0 + \underbrace{\frac{\partial}{\partial y}\left(E_{k,y} e^{i(kz-\omega t)}\right)}_{\text{there is no } y \text{ inside}} \xrightarrow{} 0 + \frac{\partial}{\partial z}\left(E_{k,z} e^{i(kz-\omega t)}\right)$$

$$0 = E_{k,z} \cdot ik e^{i(kz-\omega t)}$$

$\Rightarrow$ require $E_{k,z} = 0$. Similarly we also have $B_{k,z} = 0$

- Faraday's Law (Will get same results from Ampere's Law)

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{k,x} e^{i(kz-\omega t)} & E_{k,y} e^{i(kz-\omega t)} & 0 \end{vmatrix}$$

$$= ik\left(E_{k,x}\,\hat{y} - E_{k,y}\,\hat{x}\right) e^{i(kz-\omega t)}$$

$$-\frac{\partial \vec{B}}{\partial t} = i\omega\left(B_{k,x}\,\hat{x} + B_{k,y}\,\hat{y}\right) e^{i(kz-\omega t)}$$

By comparing coefficient, we have $\begin{cases} k E_{k,x} = \omega B_{k,y} \\ -k E_{k,y} = \omega B_{k,x} \end{cases}$

Finally, we find that only 2 of the coefficients are free variables.

$$\vec{E}(\vec{r},t) = \begin{pmatrix} E_{k,x} \\ E_{k,y} \\ 0 \end{pmatrix} e^{i(kz-\omega t)}, \quad \vec{B}(\vec{r},t) = \frac{k}{\omega}\begin{pmatrix} -E_{k,y} \\ E_{k,x} \\ 0 \end{pmatrix} e^{i(kz-\omega t)}$$

(In fact, we can check that $\vec{B} = \frac{k\times\vec{E}}{\omega}$)

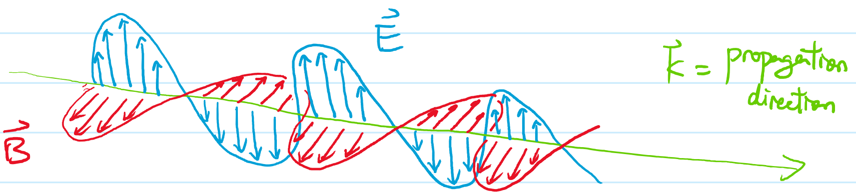
From the solutions we can observe 2 properties :

① $\vec{E}\,/\,\vec{B}\,/\,\vec{k}$ are all $\perp$ to one and other

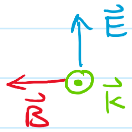
$(\vec{E}\cdot\vec{k} = \vec{B}\cdot\vec{k} = \vec{E}\cdot\vec{B} = 0)$

② $\vec{E}$ & $\vec{B}$'s magnitude are related by $|\vec{B}| = \left|\frac{\vec{k}\times\vec{E}}{\omega}\right| \sim \frac{1}{v}|\vec{E}|$

Therefore, a EM wave is usually depicted as follow in textbook



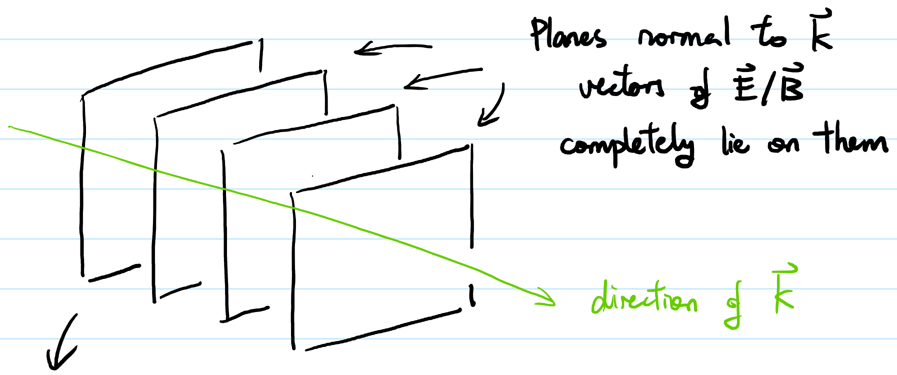
Viewing in front

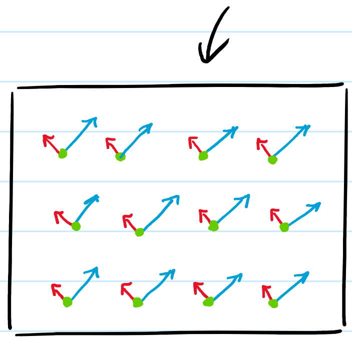


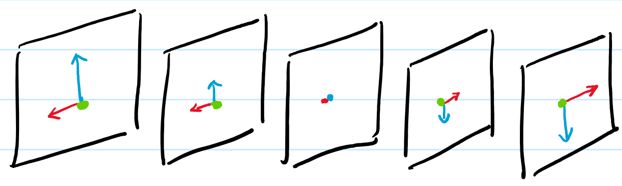
However, this is not entirely accurate because, remember,

$\vec{E}$ and $\vec{B}$ are a distribution of vectors in space

It is more accurate to visualize by cross sections in the space







On each plane, there is a constant distribution of $\vec{E}/\vec{B}$ with $\vec{E} \perp \vec{B}$ & $|\vec{B}| = \left|\frac{\vec{k}\times\vec{E}}{\omega}\right|$

On consecutive planes, the magnitudes vary sinusoidally

Polarization

The general solution of $\vec{E}/\vec{B}$ in vacuum appears with 2 independent coefficients $E_{k,x}$, $E_{k,y}$ suggests there to be freedom of polarization

Note that the coefficients can be complex number without losing any physical meaning, because $a+bi = \sqrt{a^2+b^2}\, e^{i\tan^{-1}(\frac{b}{a})}$

$$\vec{E} = \begin{pmatrix} E_{k,x} \\ E_{k,y} \\ 0 \end{pmatrix} e^{i(kz-\omega t)} = \begin{pmatrix} |E_{k,x}|\, e^{i(kz-\omega t+\phi_x)} \\ |E_{k,y}|\, e^{i(kz-\omega t+\phi_y)} \\ 0 \end{pmatrix}$$

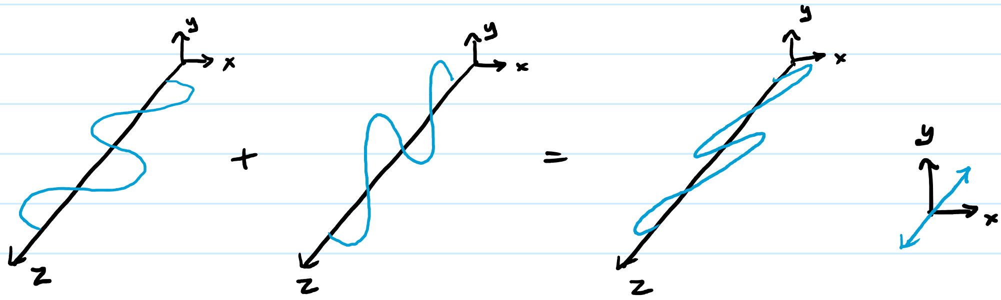
The phase difference in the x/y results in different polarization.

Let $\phi_x = 0$ so ϕ_y is exactly the phase difference.

Plane polarization : $\phi_y = 0$ or $\phi_y = \pi$

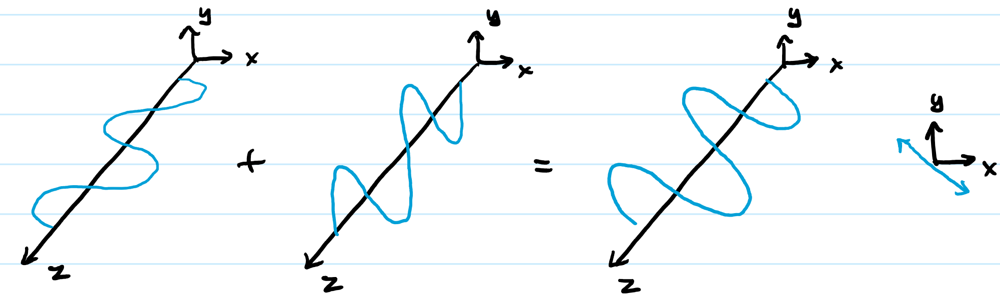
$\phi_y = 0$

In phase



$\phi_y = \pi$

Out of phase

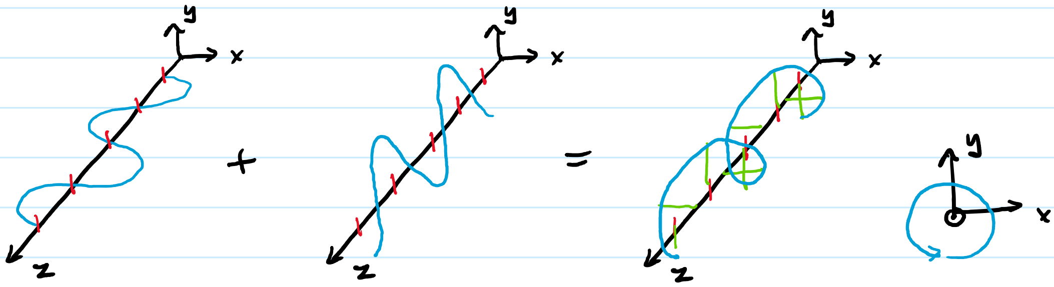


Circular Polarization : $\phi_y = \pm \frac{\pi}{2}$

$\phi_y = -\frac{\pi}{2}$

left - circular

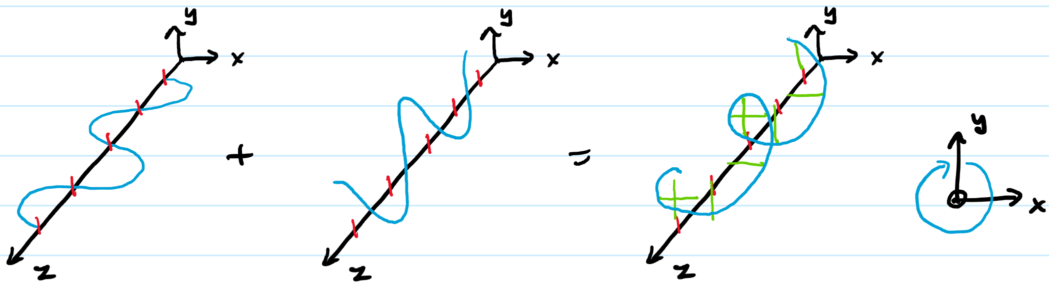
(counter - clockwise from +z)



$\phi_y = \frac{\pi}{2}$

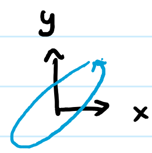
right circular

(clockwise from +z)



For other value of ϕ_y, they are called

elliptic polarization



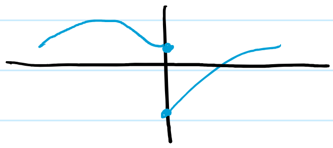
Boundary Condition

When a wave travel onto the boundary between materials

the wave needs to satisfy certain constraints

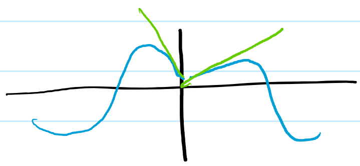
E.g. Constrains on 1D <u>mechanical</u> wave

① Amplitude is continuous



or else the string is broken

② Curve is smooth (i.e. slope is continuous)



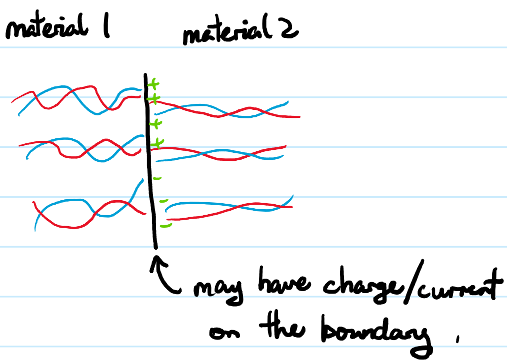
or else tension on both sides are unequal

$\Rightarrow$ momentum not conserved.

Because EM waves are just solutions to the Maxwell's Equations

They should follow the boundary conditions of $\vec{E}/\vec{B}$. i.e.

$$\begin{cases} D_\perp^{left} - D_\perp^{right} = \sigma_f \\ B_\perp^{left} - B_\perp^{right} = 0 \\ E_\parallel^{left} - E_\parallel^{right} = 0 \\ H_\parallel^{left} - H_\parallel^{right} = K_f \end{cases}$$



But for usages in optics, we normally only consider the case

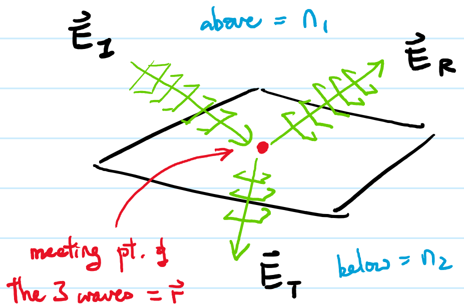
- Materials on both sides are linear
- No charge / current on the boundary

Then the boundary conditions are reduced to :

$$\begin{cases} \varepsilon_{left} E_\perp^{left} - \varepsilon_{right} E_\perp^{right} = 0 \\ B_\perp^{left} - B_\perp^{right} = 0 \\ E_{//}^{left} - E_{//}^{right} = 0 \\ \frac{1}{\mu_{left}} B^{left} - \frac{1}{\mu_{right}} B^{right} = 0 \end{cases}$$

Now we can start deriving for the 3D case of EM wave

We can identify these 3 waves :



Incident wave $\vec{E}_I = \vec{E}_I e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)}$

Reflected wave $\vec{E}_R = \vec{E}_R e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)}$

Transmitted wave $\vec{E}_T = \vec{E}_T e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)}$

(No need to write $\vec{B}$ since $\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$ for each wave)

① Time & Position symmetry

The waves' relations should be independent of whatever $\vec{r}$, t we choose. Therefore it must require

$$e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)} = e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)} = e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)}$$

(The parts that depend on $\vec{r}/t$ must equal)

which then lead to the following :

I Positional symmetry : free to take $\vec{r} = 0$

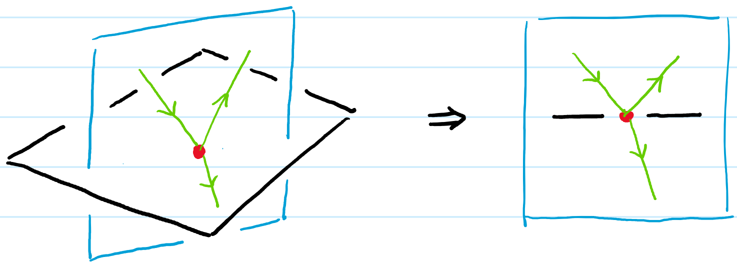
$\Rightarrow \omega_I = \omega_R = \omega_T$ i.e. They must have the same frequency

2 Time symmetry : free to take $t=0$

$$\Rightarrow \vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r}$$

$$\text{or} \quad (\vec{k}_I - \vec{k}_R) \cdot \vec{r} = (\vec{k}_I - \vec{k}_T) \cdot \vec{r} = 0$$

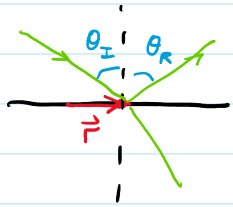
i.e. The 3 waves must be co-planar



We can draw all 3 waves on 1 plane

3 Law of reflection

By $\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r}$



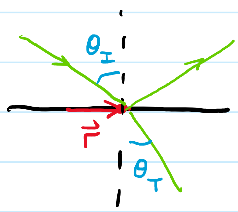
$$\Rightarrow |\vec{k}_I| \cancel{|\vec{r}|} \sin\theta_I = |\vec{k}_R| \cancel{|\vec{r}|} \sin\theta_R$$

$$\because \omega_I = \omega_R \quad \text{and} \quad |\vec{k}_I| = \frac{\omega_I}{v} = \frac{\omega_R}{v} = |\vec{k}_R|$$

We get $\boxed{\theta_I = \theta_R}$

4 Law of refraction

By $\vec{k}_I \cdot \vec{r} = \vec{k}_T \cdot \vec{r}$



$$\Rightarrow |\vec{k}_I| \cancel{|\vec{r}|} \sin\theta_I = |\vec{k}_T| \cancel{|\vec{r}|} \sin\theta_T$$

$$\because \omega_I = \omega_T \quad \text{and} \quad |\vec{k}_I| = \frac{\omega_I}{v_1} = \frac{n_1 \omega_I}{c}$$

$$|\vec{k}_T| = \frac{\omega_T}{v_2} = \frac{n_2 \omega_T}{c}$$

2 sides are of different materials.

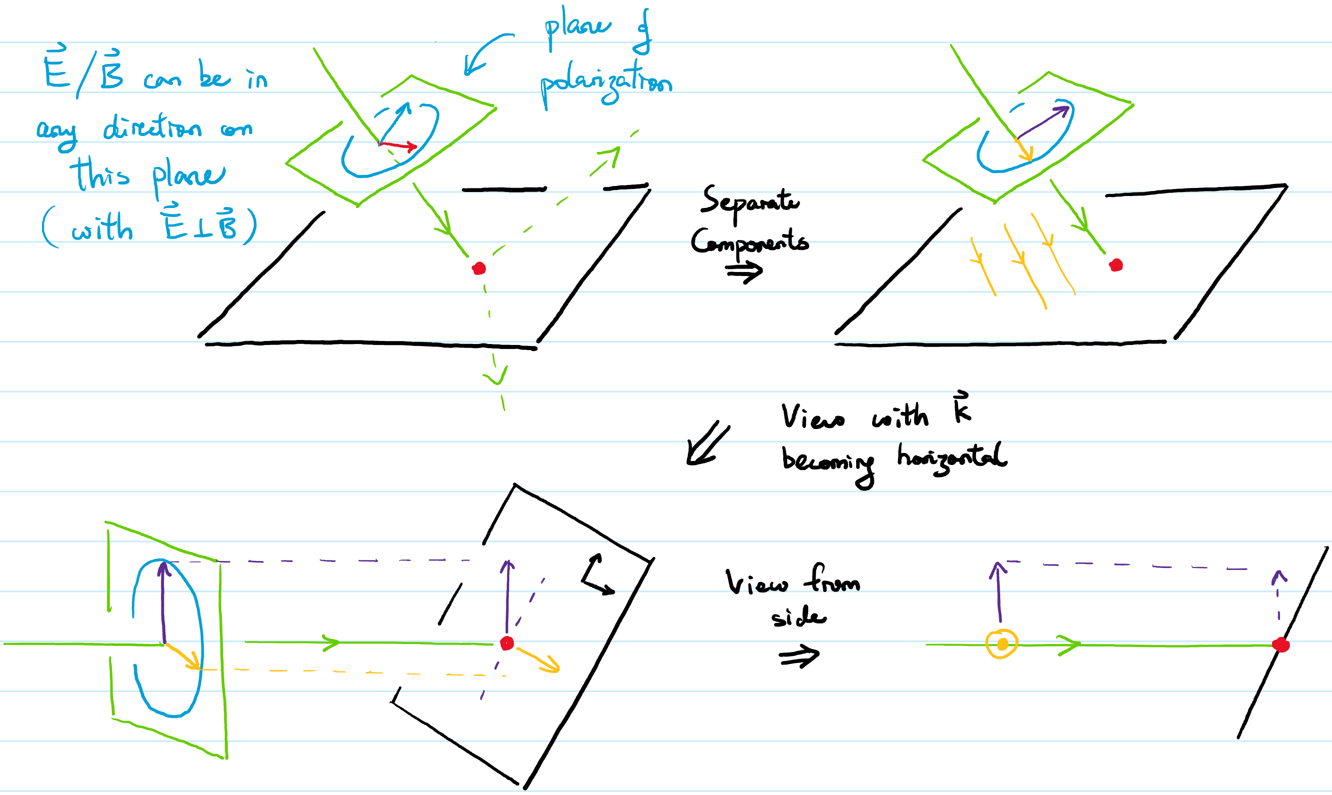
We get $\boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$

② Restriction from $\vec{E}/\vec{B}$'s properties

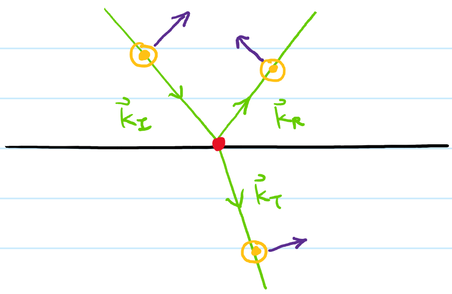
After matching the $e^{i(kr-\omega t)}$ part, we can proceed to matching the amplitudes $\vec{E}_I$, $\vec{E}_R$ & $\vec{E}_T$.

Because they are vectors, we need to match them components by components. But for convenience, we separate the components of $\vec{E}/\vec{B}$ by:

- The component parallel to boundary surface ($\rightarrow$)
- The remaining component on the polarization plane ($\uparrow$)



We do this to all $\vec{E}_I$, $\vec{E}_R$, $\vec{E}_T$ and arrive at this picture

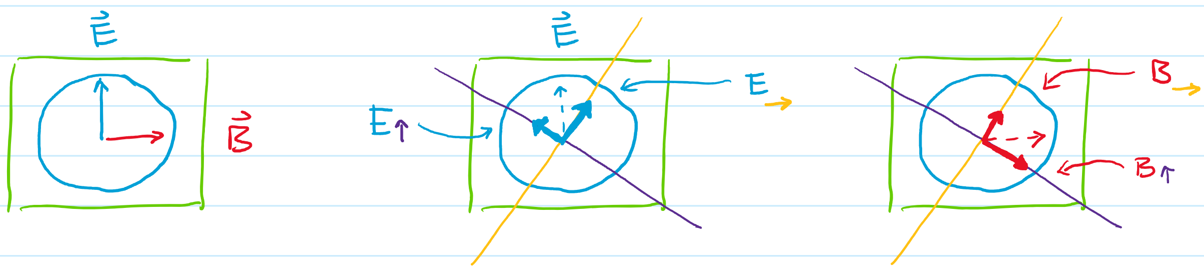


Then on the polarization plane, we can write $\vec{E}/\vec{B}$ as

$$\begin{cases} \vec{E} = \vec{E}_{\rightarrow} + \vec{E}_{\uparrow} \\ \vec{B} = \vec{B}_{\rightarrow} + \vec{B}_{\uparrow} \end{cases}$$

And because $\vec{E} \perp \vec{B}$, we must have

$$\vec{E}_{\rightarrow} \perp \vec{B}_{\uparrow} \quad \& \quad \vec{E}_{\uparrow} \perp \vec{B}_{\rightarrow}$$



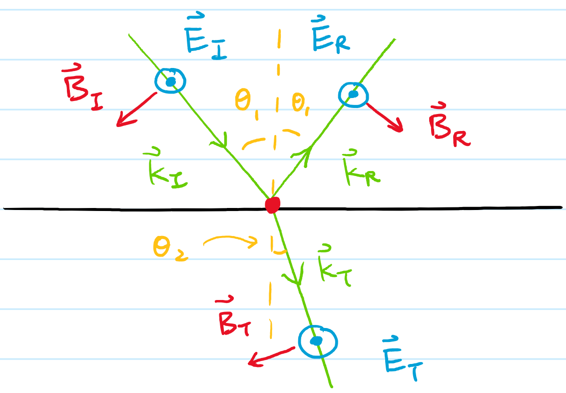
Finally we can discuss the 2 special cases:

$$\begin{cases} \text{TE mode} = \text{Transverse Electric, i.e. } \vec{E} = \vec{E}_{\rightarrow} \text{ only} \\ \qquad (\text{And so by } \vec{E}\perp\vec{B},\ \vec{B} = \vec{B}_{\uparrow} \text{ only}) \\ \text{TM mode} = \text{Transverse Magnetic, i.e. } \vec{B} = \vec{B}_{\rightarrow} \text{ only} \\ \qquad (\text{And so by } \vec{E}\perp\vec{B},\ \vec{E} = \vec{E}_{\uparrow} \text{ only}) \end{cases}$$

[1] TE mode

Let $\vec{E}_I, \vec{E}_R, \vec{E}_T$ all be pointing out of paper

then get direction of $\vec{B}$ by $\vec{k} \times \vec{E}$



The boundary conditions due to Maxwell Equations write as

$$\varepsilon_1 E_\perp^{(1)} - \varepsilon_2 E_\perp^{(2)} = 0 \qquad 0 = 0 \quad \because \text{ no } E \text{ field} \perp \text{boundary surface}$$

$$B_\perp^{(1)} - B_\perp^{(2)} = 0 \qquad B_I \sin\theta_1 + B_R \sin\theta_1 - B_T \sin\theta_2 = 0$$

$$E_\parallel^{(1)} - E_\parallel^{(2)} = 0 \quad \Rightarrow \quad E_I + E_R - E_T = 0 \qquad (\star)$$

$$\frac{1}{\mu_1} B_\parallel^{(1)} - \frac{1}{\mu_2} B_\parallel^{(2)} = 0 \qquad \frac{1}{\mu_1}\left[B_I \cos\theta_1 - B_R \cos\theta_1\right] - \frac{1}{\mu_2} B_T \cos\theta_2 = 0 \qquad (\star)$$

The 2 equations with $(\star)$, together with $|B| = \frac{|k||E|}{\omega} = \frac{n}{c}|E|$

is enough to solve the relationship between $\vec{E}_I, \vec{E}_R, \vec{E}_T$

$$|\vec{E}_R| = \frac{\cos\theta_1 - \beta\cos\theta_2}{\cos\theta_1 + \beta\cos\theta_2} |\vec{E}_I| \qquad \left(\text{where } \beta = \frac{n_2}{n_1} \cdot \frac{\mu_1}{\mu_2}\right)$$

$$|\vec{E}_T| = \frac{2\cos\theta_1}{\cos\theta_1 + \beta\cos\theta_2} |\vec{E}_I|$$

These 2 relations are called "Fresnel Equation in TE mode"

(Caution ! : Most textbook, including wiki, only write $\beta = \frac{n_2}{n_1}$

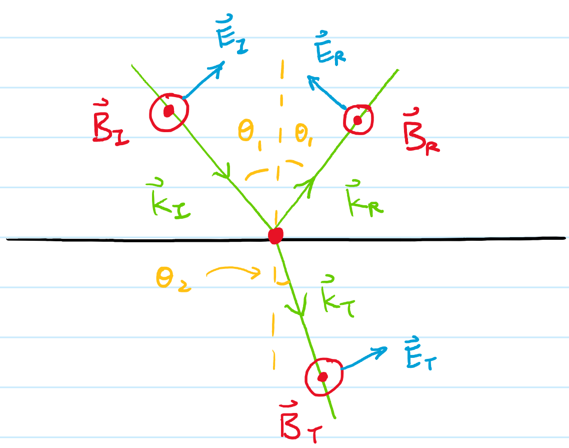
because $\mu \simeq \mu_0$ for most common materials)

[2] TM mode

Let $\vec{B}_I, \vec{B}_R, \vec{B}_T$ all

be pointing out of paper

then get direction of $\vec{E}$ by $\vec{B} \times \vec{k}$



The boundary conditions due to Maxwell Equations write as

$$\varepsilon_1 E_\perp^{(1)} - \varepsilon_2 E_\perp^{(2)} = 0 \qquad \varepsilon_1 [E_I \sin\theta_1 + E_R \sin\theta_1] - \varepsilon_2 E_T \sin\theta_2 = 0$$

$$B_\perp^{(1)} - B_\perp^{(2)} = 0 \qquad 0 = 0 \quad \because \text{ no } B \text{ field } \perp \text{ boundary surface}$$

$$\Rightarrow$$

$$E_\parallel^{(1)} - E_\parallel^{(2)} = 0 \qquad E_I \cos\theta_1 - E_R \cos\theta_1 = E_T \cos\theta_2 \quad (\star)$$

$$\frac{1}{\mu_1} B_\parallel^{(1)} - \frac{1}{\mu_2} B_\parallel^{(2)} = 0 \qquad \frac{1}{\mu_1}[B_I + B_R] - \frac{1}{\mu_2} B_T = 0 \quad (\star)$$

The 2 equations with $(\star)$, together with $|B| = \frac{|\vec{k}||\vec{E}|}{\omega} = \frac{n}{c}|\vec{E}|$

we can find the relationship between $\vec{E}_I, \vec{E}_R, \vec{E}_T$

$$\boxed{\begin{aligned} |\vec{E}_R| &= \frac{\beta\cos\theta_1 - \cos\theta_2}{\beta\cos\theta_1 + \cos\theta_2}|\vec{E}_I| \\ |\vec{E}_T| &= \frac{2\cos\theta_1}{\beta\cos\theta_1 + \cos\theta_2}|\vec{E}_I| \end{aligned} \qquad \left(\text{where } \beta = \frac{n_2}{n_1}\frac{\mu_1}{\mu_2}\right)}$$

These 2 relations are called "Fresnel Equation in TM mode"

(Caution 2 : TE & TM formula are very similar but actually different)

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2 Special Angles

① Critical Angle

This phenomenon appears due to law of refraction

$\Rightarrow$ Occur in all waves

From $n_1 \sin\theta_1 = n_2 \sin\theta_2$, what is happening to the transmitted wave when $\frac{n_1}{n_2}\sin\theta_1 = \sin\theta_2 > 1$?

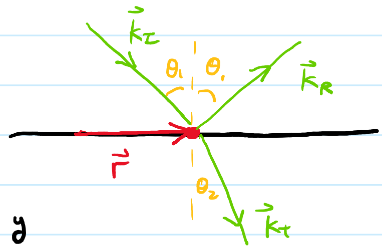
This is totally fine if $\theta_2 = a+ib$ is a complex no.

$$\text{By } e^{i\theta_2} = e^{i(a+ib)} = e^{-b} e^{ia}$$

$$= e^{-b}(\cos a + i \sin a) = \cos\theta_2 + i\sin\theta_2$$

$$\Rightarrow \text{If } e^{-b} > 1, \quad \sin\theta_2 = e^{-b}\sin a > 1$$

So in the transmitted wave



$$\vec{k}_T \cdot \vec{r} = k_T^{(x)} \sin\theta_2 \cdot x + k_T^{(y)} \cos\theta_2 \cdot y$$

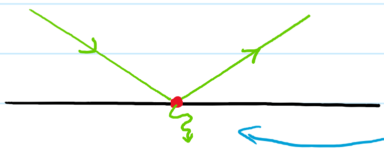
Angle is complex so we cannot use $a \cdot b = |a||b|\cos\theta$

$$= k_T^{(x)} \underbrace{\sin\theta_2}_{\text{real no.} > 1} \cdot x + k_T^{(y)} \underbrace{\left(\sqrt{1-\sin^2\theta_2}\right)}_{\text{complex no.}} \cdot y$$

$$= k_T^{(x)} \cdot p \cdot x + k_T^{(y)} \cdot i g \cdot y$$

$$\vec{E}_T e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \underbrace{\vec{E}_1 e^{i(k_T^{(x)} p x - \omega t)}}_{\text{Sinusoidal wave}} \cdot \underbrace{e^{-(k_T^{(y)} g y)}}_{\text{Decaying amplitude}}$$

Therefore in total internal reflection, the transmitted wave does go through, but only decay very fast such that it is barely visible.



This is also called "Evanescent Wave"

② Brewster Angle

This appear only due to the E/B boundary condition

$\Rightarrow$ Can only be found in light

The reflected wave 's amplitude in TM mode is

$$|\vec{E}_R| = \frac{\beta\cos\theta_1 - \cos\theta_2}{\beta\cos\theta_1 + \cos\theta_2}|\vec{E}_I| \quad \left(\beta = \frac{n_2}{n_1}\frac{\mu_1}{\mu_2}\right)$$

Note that $|\vec{E}_R| = 0$ when $\beta\cos\theta_1 - \cos\theta_2 = 0$

Then by substituting Snell 's Law $\sin\theta_1 = n\sin\theta_2$

$$1 = \sin^2\theta_2 + \cos^2\theta_2 = \left(\frac{1}{n}\sin\theta_1\right)^2 + (\beta\cos\theta_1)^2$$

$$n^2\sec^2\theta_1 = \tan^2\theta_1 + n^2\beta^2$$

$$n^2(\tan^2\theta_1 + 1) = \tan^2\theta_1 + n^2\beta^2$$

$$\tan\theta_1 = \pm n\sqrt{\frac{\beta^2-1}{n^2-1}}$$

(Most textbook only write $\tan\theta_1 = n$ by taking $\mu_1 = \mu_2 = \mu_0$)

While this does not happen to the reflected wave in TE mode

$$|\vec{E}_R| = \frac{\cos\theta_1 - \beta\cos\theta_2}{\cos\theta_1 + \beta\cos\theta_2}|\vec{E}_I| \quad \left(\beta = \frac{n_2}{n_1}\frac{\mu_1}{\mu_2}\right)$$

Requiring $\cos\theta_1 = \beta\cos\theta_2$ & $\sin\theta_1 = n\sin\theta_2$ implies

$$1 = \cos^2\theta_1 + \sin^2\theta_1 = \beta^2\cos^2\theta_2 + n^2\sin^2\theta_2$$

$$\Rightarrow \quad \tan\theta_1 = \pm\frac{\beta}{n}\sqrt{\frac{1-\beta^2}{n^2-1}}$$

But if Brewster angle to exist in TM mode, require $\beta^2 - 1 > 0$

$\therefore$ At the incident angle $\theta_1 = \tan^{-1}\left(n\sqrt{\frac{\beta^2-1}{n^2-1}}\right)$, there is no TM polarization reflected back from the boundary surface.

$\Rightarrow$ The reflected wave is purely of TE polarization.

We can use this to create polarized light source.

Energy of EM Wave

Recall that energy of $E/B \propto (\text{magnitude})^2$ $\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu}B^2\right)$

The energy density carried by a light beam is therefore

$$U = \frac{1}{2}\varepsilon \left|\vec{E} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right|^2 + \frac{1}{2\mu}\left|\vec{B} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right|^2$$

$\hookrightarrow \vec{B} = \frac{\vec{k}\times\vec{E}}{\omega}$

$$= \frac{1}{2}\varepsilon|\vec{E}|^2 + \frac{1}{2\mu}\left|\frac{\vec{k}\times\vec{E}}{\omega}\right|^2$$

$\hookrightarrow$ $\vec{k}$ always $\perp \vec{E}$ so can take out as factor

$$= \frac{1}{2}\varepsilon|\vec{E}|^2 + \frac{1}{2\mu}\left(\frac{k}{\omega}\right)^2|\vec{E}|^2$$

$\hookrightarrow \frac{k}{\omega} = \frac{n}{c} = \sqrt{\mu\varepsilon}$

$$= \varepsilon|E|^2$$

It is also common to see people taking time average and write

$$U_{Avg} = \frac{1}{2}\varepsilon|E|^2 \text{, with } \frac{1}{2} \text{ because } \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right)dt = \frac{1}{2}$$

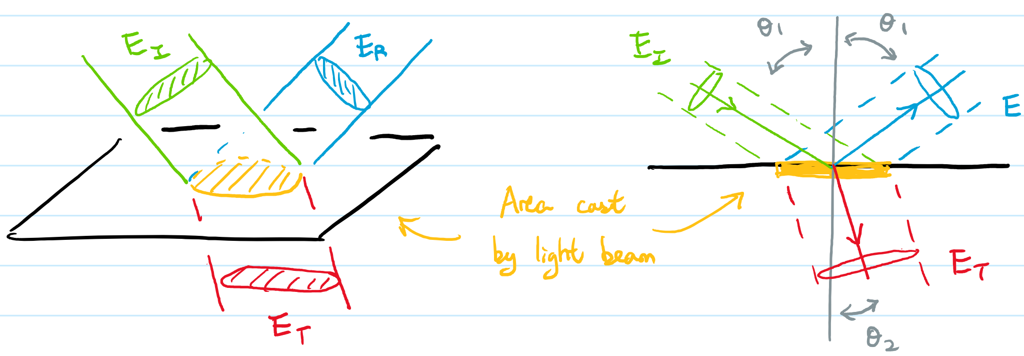
Energy Flux in Reflection / Transmission

The Poynting vector of EM wave is calculated as

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu}\vec{E}\times\left(\frac{\vec{k}\times\vec{E}}{\omega}\right) = \frac{1}{\mu}\frac{n}{c}|E|^2 = \sqrt{\frac{\varepsilon}{\mu}}|E|^2$$

Remind that Poynting vector is the "flux" of energy

So we also need to take of the area



Thus if no energy accumulate on the area, we must have

$$\vec{S}_I \cdot A\cos\theta_1 = \vec{S}_R \cdot A\cos\theta_1 + \vec{S}_T \cdot A\cos\theta_2$$

(in flux) (out flux)

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}|E_I|^2\cos\theta_1 = \sqrt{\frac{\varepsilon_1}{\mu_1}}|E_R|^2\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\mu_2}}|E_T|^2\cos\theta_2$$

One can easily prove that this is true for both TE & TM mode.

Also it is common to see the following definition

Reflection Coefficient : $R = \dfrac{|E_R|^2}{|E_I|^2}$

(Different for TE/TM)

Transmission Coefficient : $T = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}\dfrac{\mu_1}{\mu_2}}\dfrac{\cos\theta_2}{\cos\theta_1}\dfrac{|E_T|^2}{|E_I|^2}$

(Different for TE/TM)

So that energy conservation implies $R + T = 1$