Physics Enrichment Program HKUST |
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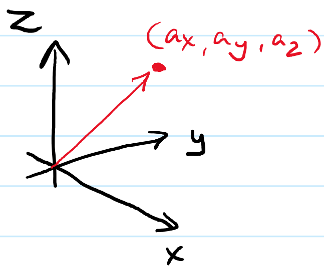
- Limit of Vector functions
- Differentiation $\begin{cases} \text{On vectors} \\ \text{By vectors} \end{cases}$
- Parametrization & Line Integral

Review : Vector Geometry

Vectors ~ Objects with Magnitude & Direction

- Usually visualized as an arrow pointing from origin to some coordinate



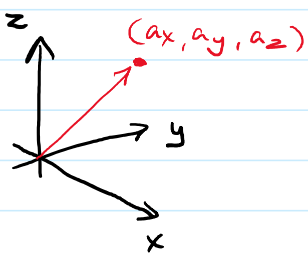
- Can be expressed as row/column matrices

$$(a_x \quad a_y \quad a_z) \qquad \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

① Norm (Magnitude)

Notation : $|\vec{a}|$

Geometrically the length of the vector. Follows Pythagoras Theorem.



$$\Rightarrow \quad |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

② Unit Vectors

Denote by "hat" symbol $\hat{a} \equiv \frac{\vec{a}}{|\vec{a}|}$ = A vector pointing in the same direction but with length (norm) = 1

The 3 "basis" vectors are chosen to be unit vectors

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{pointing in } x \text{ direction}$$

$$\hat{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{pointing in } y \text{ direction}$$

$$\hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{pointing in } z \text{ direction}$$

So any vectors can be written as

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = a_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$$

$a_x, a_y, a_z \rightarrow$ Components of the vector

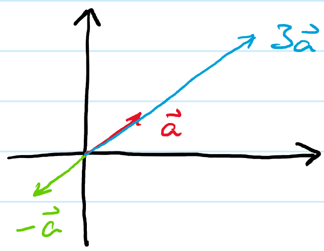
$\hat{x}, \hat{y}, \hat{z} \rightarrow$ Directions of the vector

③ Multiplication by Constant

In matrix form : Multiply each element by the constant

$$k\vec{a} = k \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix}$$

Geometrically as extending / contracting the vector in the same / opposite direction

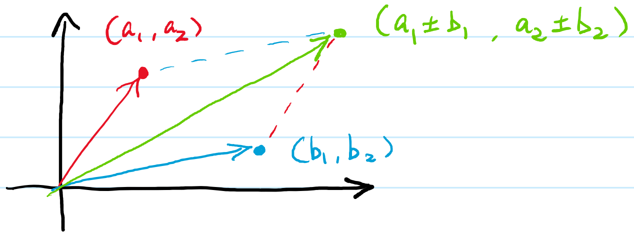


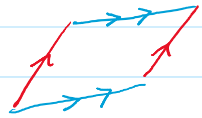
④ Addition / Subtraction

In matrix form : Add / Subtract element by element

$$\vec{a} \pm \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \end{pmatrix}$$

Geometrically, addition can be depicted as parallelogram rule





And subtraction is simply adding a negative vector

⑤ Dot product (Scalar product)

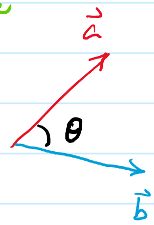
A multiplication between vectors that gives a scalar

geometrical angle between $\vec{a}$, $\vec{b}$

$$\underline{\underline{\text{Def}}}: \quad \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \underline{\underline{\theta}}$$

$$= a_x b_x + a_y b_y + a_z b_z$$

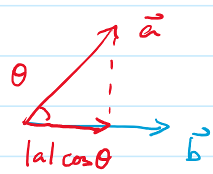
$$= \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$



Dot product can also be visuallized as length of projection

when dotting with a unit vector

$$\vec{a} \cdot \hat{b} = |\vec{a}| \overbrace{\underline{\underline{|\hat{b}|}}}^{=1} \cos \theta = a \cos \theta$$



In particular if it is dotted with a basis unit vectors

we get its component at that direction

$$\text{E.g.} \quad (a_x \quad a_y \quad a_z) \cdot \hat{x}$$

$$= (a_x \; a_y \; a_z)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= a_x = x \text{ component}$$

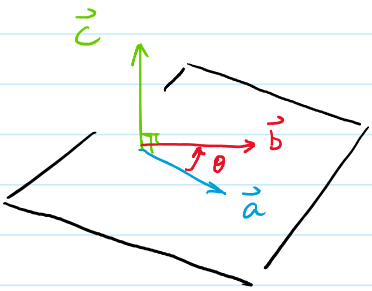
⑥ Cross Product (Vector Product)

A multiplication between vectors that gives a vector

<u>Def</u>: $\vec{a} \times \vec{b} = \vec{c} = -\vec{b} \times \vec{a}$ (Anti-symmetric get minus sign when switching order)

with $|\vec{c}| = |\vec{a}||\vec{b}| \sin\theta$

and direction of $\vec{c} \perp$ plane formed by $\vec{a}$ & $\vec{b}$



(Follow right hand grip rule)

The exact vector can also be computed by determinant

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

E.g. $\vec{a} = \hat{x} + 2\hat{y}$, $\vec{b} = 3\hat{x} - \hat{y}$ (Both are on x-y plane)

$$\Rightarrow \vec{c} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} = -6\hat{z} \quad (\perp \text{ to x-y plane})$$

☆ Note that cross product is only for 3D vectors

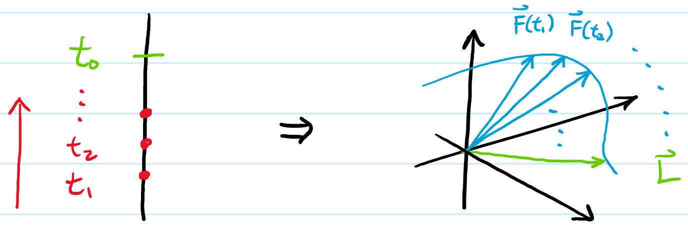
Limit of Vector Functions

Recall that we classify the functions by their no. of inputs/outputs

$$\left\{\begin{matrix}\text{Single Variable}\\ \text{Multivariable}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Scalar}\\ \boxed{\text{Vector}}\end{matrix}\right\} \text{ Functions}$$

Single Variable Vector Functions

$$\lim_{t\to t_0} \vec{F}(t) = \vec{L} \iff$$ When t approaches t_0, $\vec{F}(t)$ approaches $\vec{L}$



Equivalent to having the distance between $\vec{F}(t)$ & $\vec{L}$ diminishing

$$\lim_{t\to t_0} |\vec{F}(t) - \vec{L}| = 0$$

$$\iff \lim_{t\to t_0} \sqrt{(F_x(t) - L_x)^2 + (F_y(t) - L_y)^2 + (F_z(t) - L_z)^2}$$

$$\iff \left\{\begin{matrix} \lim_{t\to t_0} |F_x(t) - L_x| = 0 \\ \lim_{t\to t_0} |F_y(t) - L_y| = 0 \\ \lim_{t\to t_0} |F_z(t) - L_z| = 0 \end{matrix}\right.$$

All 3 components approaching their own limits

Multivariable Vector Function

(similar concept, but hard to draw)

$$\lim_{(x_1, x_2 \dots x_n) \to (a_1, a_2 \dots a_n)} \vec{F}(x_1, x_2 \dots x_n) = \vec{L}$$

When every input approaches their target value

The output approaches the target vector

Differentiation on Vector Functions

On Single Variable Vector Function

$$\frac{d}{dt}\vec{F}(t) = \lim_{\Delta t \to 0}\left[\frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}\right]$$

$$= \left[\lim_{\Delta t \to 0} \frac{F_1(t+\Delta t) - F_1(t)}{\Delta t}\right]\hat{u}_1 + \left[\lim_{\Delta t \to 0} \frac{F_2(t+\Delta t) - F_2(t)}{\Delta t}\right]\hat{u}_2 + \dots$$

$$= \left[\frac{d}{dt}F_1(t)\right]\hat{u}_1 + \left[\frac{d}{dt}F_2(t)\right]\hat{u}_2 + \dots + \left[\frac{d}{dt}F_n(t)\right]\hat{u}_n$$

$$= \begin{pmatrix} \frac{d}{dt}F_1(t) \\ \frac{d}{dt}F_2(t) \\ \vdots \\ \frac{d}{dt}F_n(t) \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix} = \text{Differentiating each components individually}$$

Arithmetics are similar as in single variables

- Addition / Subtraction

$$\frac{d}{dt}\left[\vec{F}(t) \pm \vec{G}(t)\right] = \frac{d}{dt}\vec{F}(t) \pm \frac{d}{dt}\vec{G}(t)$$

- Product Rule

☆ Vectors can multiply each other in 2 different ways

$$\frac{d}{dt}\left[\vec{F}(t) \cdot \vec{G}(t)\right] = \left[\frac{d}{dt}F(t)\right] \cdot \vec{G}(t) + \vec{F}(t) \cdot \left[\frac{d}{dt}\vec{G}(t)\right]$$

$$\frac{d}{dt}\left[\vec{F}(t) \times \vec{G}(t)\right] = \underline{\left[\frac{d}{dt}F(t)\right] \times \vec{G}(t)} + \underline{\vec{F}(t) \times \left[\frac{d}{dt}\vec{G}(t)\right]}$$

Order of "F×G" must be the same because order matters in cross product

(Cannot have quotient rule, obviously)

On Multivariable Vector Function

The major difference is that there is 1 partial D for each input

Functions with m inputs have m partial D

$$\frac{\partial}{\partial x_1}\vec{F}(x_1, x_2 \cdots x_m) = \frac{\partial}{\partial x_1}F_1(x_1, x_2, \cdots x_m)\,\hat{u}_1 + \frac{\partial}{\partial x_1}F_2(x_1, x_2 \cdots x_m)\,\hat{u}_2 + \cdots + \frac{\partial}{\partial x_1}F_n(x_1, x_2 \cdots x_m)\,\hat{u}_n$$

$$\vdots$$

$$\frac{\partial}{\partial x_m}\vec{F}(x_1, x_2 \cdots x_m) = \frac{\partial}{\partial x_m}F_1(x_1, x_2, \cdots x_m)\,\hat{u}_1 + \frac{\partial}{\partial x_m}F_2(x_1, x_2 \cdots x_m)\,\hat{u}_2 + \cdots + \frac{\partial}{\partial x_m}F_n(x_1, x_2 \cdots x_m)\,\hat{u}_n$$

To make the expression easier to read, we can write them as matrices

$$\frac{\partial}{\partial x_1}\vec{F}(x_1, x_2 \ldots x_m) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} \\ \vdots \\ \frac{\partial F_n}{\partial x_1} \end{pmatrix} \quad \cdots \quad \frac{\partial}{\partial x_m}\vec{F}(x_1, x_2 \ldots x_m) = \begin{pmatrix} \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_m} \\ \vdots \\ \frac{\partial F_n}{\partial x_m} \end{pmatrix}$$

Then compress into 1 big matrix :

$$\frac{\partial}{\partial \vec{x}}\vec{F}(\vec{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}$$

n rows for outputting an n components vector

m columns for function with m inputs

$\vec{F}(\ldots) = (F_1, F_2, \ldots F_n)$

$\vec{x} = (x_1, x_2, \ldots x_m)$

(This matrix is called Jacobian matrix)

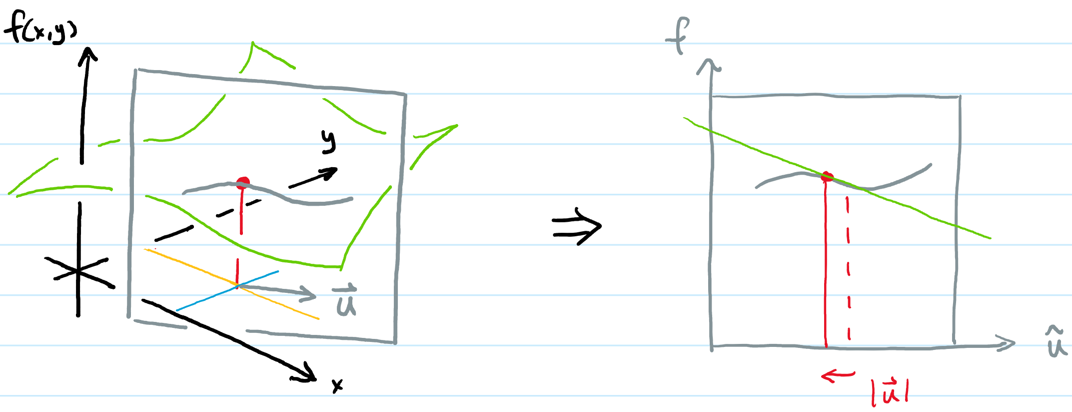
Differentiation by Vector – Gradient

Recall definition of partial differentiation :

$$\frac{\partial}{\partial x} f(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \text{Slope in } x \text{ direction}$$

$$\frac{\partial}{\partial y} f(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \text{Slope in } y \text{ direction}$$

What if we want the slope in arbituary directions ?

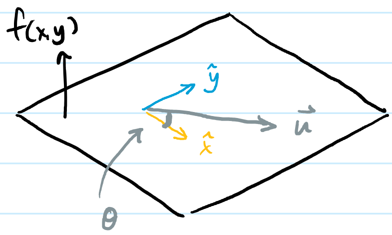


Suppose we want to find the slope along the direction of $\vec{u}$

We know $\vec{u}$ can be decomposed into component form of $\hat{x}/\hat{y}$

$$\vec{u} = u_x \hat{x} + u_y \hat{y}$$

$$= |\vec{u}| \cos\theta \, \hat{x} + |\vec{u}| \sin\theta \, \hat{y}$$



And the unit vector of $\vec{u}$ can be expressed as

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \cos\theta \, \hat{x} + \sin\theta \, \hat{y} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

To find the slope along $\vec{u}$, first vary $f(x,y)$ to $f(x+u_x, y+u_y)$

then divide by $|\vec{u}|$. Finally by limiting $|\vec{u}| \to 0$,

it becomes differentiation.

$$D_{\hat{u}} f(x,y) = \lim_{|\vec{u}|\to 0} \frac{f(x+u_x,\ y+u_y) - f(x,y)}{|\vec{u}|}$$

Just a notation of differenting in $\vec{u}$'s direction

$$= \lim_{|\vec{u}|\to 0} \frac{f(x+|\vec{u}|\cos\theta,\ y+|\vec{u}|\sin\theta) - \boxed{f(x,\ y+|\vec{u}|\sin\theta)}}{|\vec{u}|}$$

Subtract then add back the same term

$$+ \lim_{|\vec{u}|\to 0} \frac{\boxed{f(x,\ y+|\vec{u}|\sin\theta)} - f(x,y)}{|\vec{u}|}$$

$$= \lim_{|\vec{u}|\to 0} \frac{f(\boxed{x+|\vec{u}|\cos\theta},\ y+|\vec{u}|\sin\theta) - f(\boxed{x},\ y+|\vec{u}|\sin\theta)}{\boxed{|\vec{u}|\cos\theta}} \cos\theta$$

$$+ \lim_{|\vec{u}|\to 0} \frac{f(x,\ \boxed{y+|\vec{u}|\sin\theta}) - f(x,\boxed{y})}{\boxed{|\vec{u}|\sin\theta}} \sin\theta$$

$$= \left(\frac{\partial}{\partial x} f(x,y)\right)\cos\theta + \left(\frac{\partial}{\partial y} f(x,y)\right)\sin\theta$$

$$= \underbrace{\left(\frac{\partial}{\partial x} f(x,y) \quad \frac{\partial}{\partial y} f(x,y)\right)}_{\text{This is independent of } \vec{u}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

This is $\hat{u} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$

$\Rightarrow$ Slope in $\vec{u}$ direction $= \left(\begin{array}{c}\text{something independent} \\ \text{of direction}\end{array}\right) \cdot \hat{u}$

$\Rightarrow$ Give a special name to this quantity : Gradient

Notation : $\vec{\nabla} f$, grad f

$\vec{\nabla}$ symbol : pronounce as "Del", formal name = "nabla"

Note 1 : Gradient itself is NOT the slope. It is just a "property" of the function while we can use it to obtain slopes along any direction $\vec{u}$.

Note 2 : Gradient of a scalar function is a vector function.

At every point, the vector points in the direction with max slope.

$\because$ Slope at arbitrary direction $= \vec{\nabla} f \cdot \hat{u}$

$$= |\vec{\nabla} f| |\hat{u}| \cos\theta \leq |\vec{\nabla} f|$$

which reaches the max $= |\vec{\nabla} f|$ when $\theta = 0^\circ / 180^\circ$ (i.e. $\vec{\nabla} f \parallel \hat{u}$)

Gradient in matrix form

Gradient is conventionally written as a row vector

$$\vec{\nabla} f(\vec{x}) = \left(\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \cdots \quad \frac{\partial f}{\partial x_m} \right)$$

For vector function, we can take gradient to each component

If $\vec{F}(\dots)$ outputs n values

$$\left\{ \begin{aligned} \vec{\nabla} F_1(\vec{x}) &= \left(\frac{\partial F_1}{\partial x_1} \quad \frac{\partial F_1}{\partial x_2} \quad \cdots \quad \frac{\partial F_1}{\partial x_m} \right) \\ \vec{\nabla} F_2(\vec{x}) &= \left(\frac{\partial F_2}{\partial x_1} \quad \frac{\partial F_2}{\partial x_2} \quad \cdots \quad \frac{\partial F_2}{\partial x_m} \right) \\ &\vdots \\ \vec{\nabla} F_n(\vec{x}) &= \left(\frac{\partial F_n}{\partial x_1} \quad \frac{\partial F_n}{\partial x_2} \quad \cdots \quad \frac{\partial F_n}{\partial x_m} \right) \end{aligned} \right.$$

Then compress all rows into 1 matrix

$$\vec{\nabla} \vec{F}(\vec{x}) = \underbrace{\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}}_{m \text{ columns for function with } m \text{ inputs}}$$

n rows for outputting an n components vector

(which is again the Jacobian matrix)

Short Summary

Output \ Input	Single Var.	Multi Var.
Scalar	$\frac{df}{dt}$	$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_m} \end{pmatrix}$
Vector	$\begin{pmatrix} \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial t} \\ \vdots \\ \frac{\partial F_n}{\partial t} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}$

Chain Rule in matrix form

Writing chain rule of multivariable function by matrix is much cleaner

E.g. $\vec{g}(\vec{f}(\vec{x}))$ is a function composition of no. of variables ($\underline{n} \to \underline{m} \to \underline{p}$)

$$\vec{x} = (\underbrace{x_1, x_2, \ldots x_n}) $$

$$\vec{f}(\ldots) = (\underbrace{f_1(\ldots), f_2(\ldots), \ldots, f_m(\ldots)})$$

$$\vec{g}(\ldots) = (g_1(\ldots), g_2(\ldots), \ldots, g_p(\ldots))$$

Then the chain rule can be written as

$$\underset{\begin{pmatrix} p \times n \\ \text{matrix} \end{pmatrix}}{\frac{\partial \vec{g}}{\partial \vec{x}}} = \underset{\begin{pmatrix} p \times m \\ \text{matrix} \end{pmatrix}}{\frac{\partial \vec{g}}{\partial \vec{f}}} \cdot \underset{\begin{pmatrix} m \times n \\ \text{matrix} \end{pmatrix}}{\frac{\partial \vec{f}}{\partial \vec{x}}}$$

$$\begin{array}{cc} & \text{}j^{th}\text{ column} \\ i^{th}\text{ row} & \begin{pmatrix} & \vdots & \\ \cdots & \frac{\partial g_i}{\partial x_j} & \cdots \\ & \vdots & \end{pmatrix} \end{array} = \begin{array}{cc} & i^{th}\text{ row} \\ & \begin{pmatrix} & \vdots & & \\ \frac{\partial g_i}{\partial f_1} & \frac{\partial g_i}{\partial f_2} & \cdots & \frac{\partial g_i}{\partial f_m} \\ & \vdots & & \end{pmatrix} \\ & \text{every element} \end{array} \begin{array}{ccc} j^{th}\text{ column} & \begin{pmatrix} \cdots & \frac{\partial f_1}{\partial x_j} & \cdots \\ & \frac{\partial f_2}{\partial x_j} & \\ & \vdots & \\ & \frac{\partial f_m}{\partial x_j} & \end{pmatrix} & \text{every element} \end{array}$$

Or as individual terms

$$\frac{\partial g_i}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial g_i}{\partial f_k} \cdot \frac{\partial f_k}{\partial x_j}$$

$$= \frac{\partial g_i}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_j} + \frac{\partial g_i}{\partial f_2} \cdot \frac{\partial f_2}{\partial x_j} + \cdots + \frac{\partial g_i}{\partial f_m} \cdot \frac{\partial f_m}{\partial x_j}$$

Example

$$\begin{cases} f(p,q) = \sqrt{p+q} \\ \vec{g}(t) = (t-1, t^2) \end{cases} \Rightarrow f(\vec{g}(t)) = \sqrt{t^2+t-1}$$

- Differentiate on t directly : $\frac{d}{dt}\left(\sqrt{t^2+t-1}\right) = \frac{1}{2}\frac{2t+1}{\sqrt{t^2+t-1}}$

- Differentiate by chain rule :

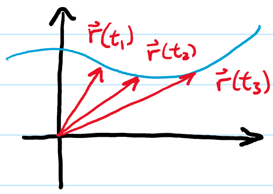
$$\begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \end{pmatrix} = \left.\begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{p+q}} & \frac{1}{2}\frac{1}{\sqrt{p+q}} \end{pmatrix}\right|_{\substack{p=t-1 \\ q=t^2}}$$

$$\begin{pmatrix} \frac{\partial g_1}{\partial t} \\ \frac{\partial g_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\left[f(\vec{g}(t))\right] = \begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial t} \\ \frac{\partial g_2}{\partial t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{p+q}} & \frac{1}{2}\frac{1}{\sqrt{p+q}} \end{pmatrix} \begin{pmatrix} 1 \\ 2t \end{pmatrix} = \frac{1}{2}\frac{2t+1}{\sqrt{t^2+t-1}}$$

Curve Parametrization



Recall single var. vector function

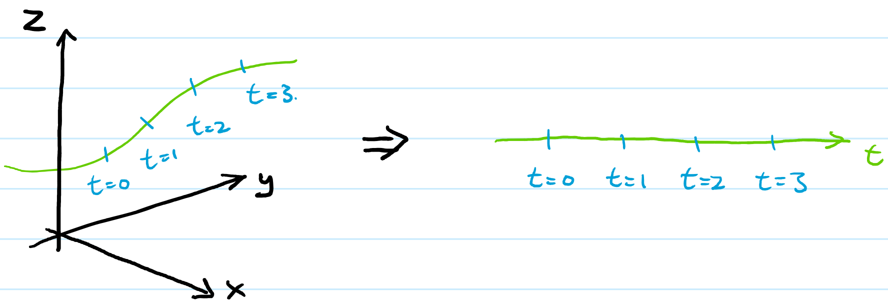
eg. $\vec{r}(t) = [\, x(t),\ y(t)\,]$

This is a curve on a plane. Any point along the curve only needs 1 input (t) to fully locate it

Intuitively: Curve = 1D object = Only 1 free variable

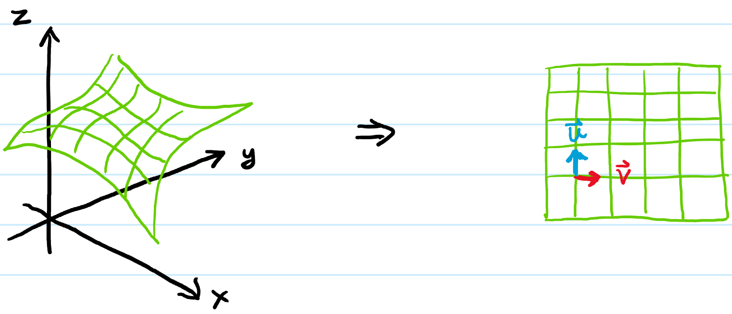
Parametrization = Choose a coordinate system on the object to describe every points, rather than using the environment coordinate ($x/y/z$)

(Note: Parametrization is never unique, because there are infinitely many ways of choosing a coordinate system)



(We can do similar thing to higher dimensional objects but the maths are way more complicated)

E.g.



E.g. Parametrize the curve $y = 3x^{\frac{3}{2}}$

Choice 1 : Let $x = t^2$, then $y = 3(t^2)^{\frac{3}{2}} = 3t^3$

$\Rightarrow$ Parametrize as $[t^2, 3t^3]$

Choice 2 : Let $x = t$, then $y = 3t^{\frac{3}{2}}$

$\Rightarrow$ Parametrize as $[t, 3t^{\frac{3}{2}}]$

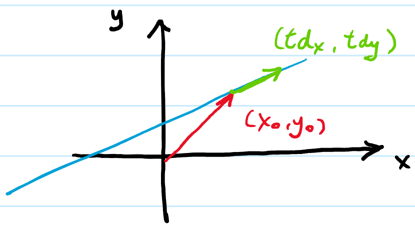
$\vdots$

Obviously there are infinitely more possibilities

Some common parametrization

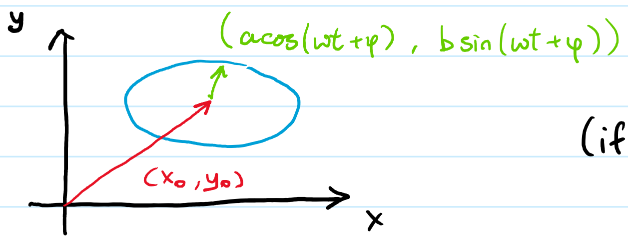
1 Straight line

$$[x(t), y(t)] = [x_0 + td_x, y_0 + td_y]$$



2 Ellipse / Circle

$$[x(t), y(t)] = [x_0 + a\cos(\omega t + \varphi), y_0 + b\sin(\omega t + \varphi)]$$



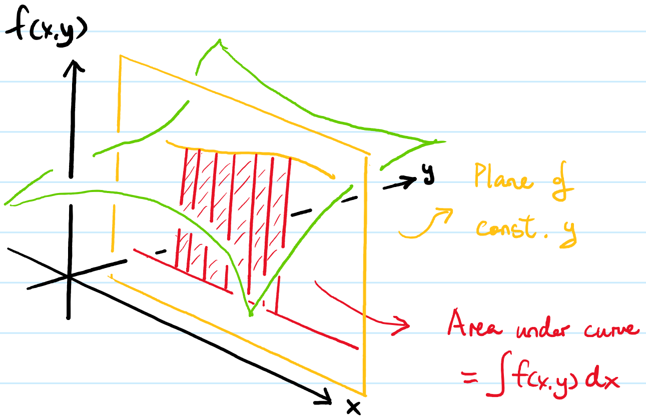
(if $a = b$, it is a circle)

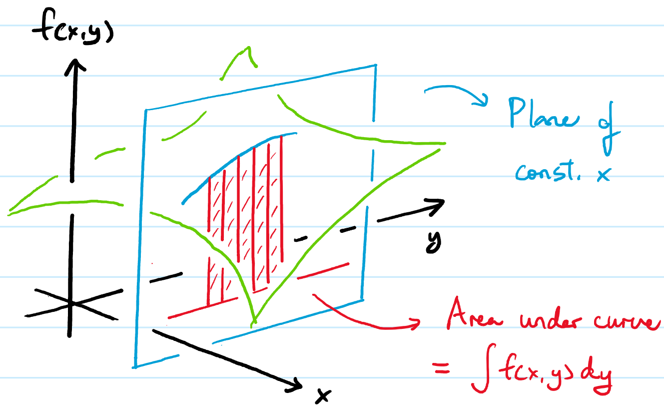
Line Integral on Scalar Function

Recall

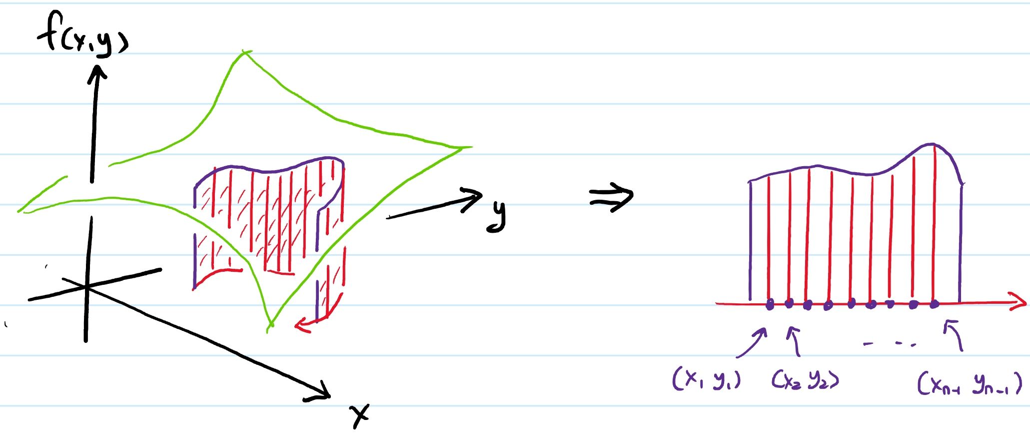
$\int f(x,y)\, dx$ = Integrate along x-axis, at constant y

$\int f(x,y)\, dy$ = Integrate along y-axis, at constant x





What about integrating along an arbituary curve?



Recall integration on single variable function

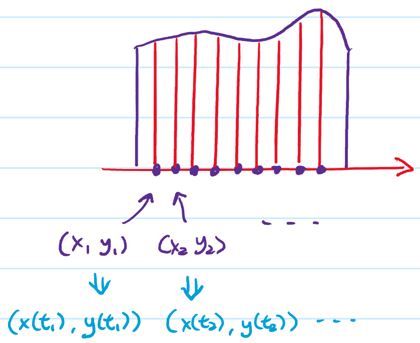
$$\int f(x)\, dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\, \Delta x_i$$

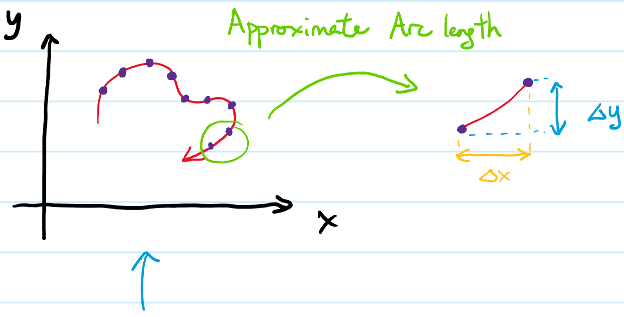
height of slice width of slice

We can write something similar for line integral

$$\int f(x,y)\, d\vec{r} = \lim_{|\Delta \vec{r}_i| \to 0} \sum_{i=1}^{n} f(\xi_{i,x}, \xi_{i,y})\, |\Delta \vec{r}_i|$$

The width of each slice can be estimated by Pythagoras Thm.





parametrize as $(x(t), y(t))$ on x-y plane

Width of interval $|\Delta \vec{r}_i|$

$$= \sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2}$$

$$= |t_{i+1} - t_i| \sqrt{\left[\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}\right]^2 + \left[\frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}\right]^2}$$

$$= |\Delta t_i| \sqrt{\left[\frac{\Delta x_i}{\Delta t_i}\right]^2 + \left[\frac{\Delta y_i}{\Delta t_i}\right]^2}$$

$$\sim dt \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{when } \Delta t_i \to 0$$

So in the line integral definition :

$$\boxed{\int f(x,y)\, d\vec{r}} = \lim_{|\Delta \vec{r}_i| \to 0} \sum_{i=1}^{n} f(\xi_{i,x}, \xi_{i,y}) |\Delta \vec{r}_i|$$

Notation you can find in textbook

$$= \lim_{|\Delta t_i| \to 0} \sum_{i=1}^{n} f(\xi_{i,x}, \xi_{i,y}) |\Delta t_i| \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}$$

$$= \boxed{\int f(x,y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

This is how you really calculate line integral

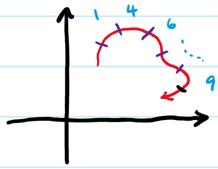
eg. along a line on x-y plane

You have to decide how to parametrize the curve

Weighted Sum Interpretation

This is just like the interpretation in 1D integral

$$\int f(x)\,dx = \lim_{\Delta x_i \to 0} \underbrace{\sum_{i=1}^{n}}_{\text{Sum them all}} \underbrace{f(\xi_{ix}, \xi_{iy})}_{\text{"weight" assigned to the interval}} \underbrace{|\Delta \vec{r}_i|}_{\text{length of the interval } [(x_i, y_i),\ (x_{i+1}, y_{i+1})]}$$



More on notations

[1] Integration range

Single var. integration only requires knowing the upper/lower bounds.

However in line integral we need to describe the whole curve

⇒ Too many words to write under the integral sign

⇒ denote the curve as "C" and define C in text

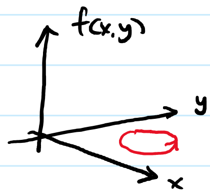
$$\int_C f(x,y)\,d\vec{r} \quad \text{with } C = \text{Curve of xxx} \ldots$$

[2] Loop Integral

If the curve to be integrated along is a closed loop

We draw an extra circle on the integral sign

$$\oint_C f(x,y)\,d\vec{r}$$



This is because integrals over closed loop carry nice properties for a lot of functions and are used in theorems

⇒ Create a new symbol for these specific uses.

Line Integration on Vector Functions

Note that $f(x,y)\,d\vec{r}$ is like a product between number × vector

If $f(x,y)$ is replaced by a vector function, we have 2 possibilities :

Dot Product Line Integral

$$\int_C \vec{F}(x,y) \cdot d\vec{r} = \lim_{|\Delta\vec{r}_i| \to 0} \sum_{i=1}^{n} \vec{F}(\xi_{ix}, \xi_{iy}) \cdot \Delta\vec{r}_i$$

Cross Product Line Integral

$$\int_C \vec{F}(x,y) \times d\vec{r} = \lim_{|\Delta\vec{r}_i| \to 0} \sum_{i=1}^{n} \vec{F}(\xi_{ix}, \xi_{iy}) \times \Delta\vec{r}_i$$

☆ Because order matters in cross product

$$\int_C \vec{F} \times d\vec{r} = -\int_C d\vec{r} \times \vec{F}$$

this form is very common

Calculation

Demonstrate with dot product line integral (Similar for cross product)

$$\int_C \vec{F}(x,y) \cdot d\vec{r} = \lim_{|\Delta\vec{r}_i| \to 0} \sum_{i=1}^{n} \boxed{\vec{F}(\xi_{ix}, \xi_{iy}) \cdot \Delta\vec{r}_i}$$

Notice that this is a summation of all dot product along a curve

Express each dot product by the curve parameter t :

$$\begin{aligned}
\vec{F}(x(t_i), y(t_i)) \cdot \Delta\vec{r}_i &= \vec{F}(x(t_i), y(t_i)) \cdot [\vec{r}(t_{i+1}) - \vec{r}(t_i)] \\
&= \vec{F}(x(t_i), y(t_i)) \cdot \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{t_{i+1} - t_i} (t_{i-1} - t_i) \\
&= \vec{F}(x(t_i), y(t_i)) \cdot \left(\frac{\Delta\vec{r}_i}{\Delta t_i}\right) \Delta t_i \\
&\sim \left(\vec{F}(x,y) \cdot \frac{d\vec{r}}{dt}\right) dt
\end{aligned}$$

$\therefore$ Dot product line integral compute as :

$$\boxed{\int_C \vec{F}(x,y) \cdot d\vec{r}} = \boxed{\int_C \vec{F}(x(t), y(t)) \cdot \frac{d\vec{r}(t)}{dt} \, dt}$$

Notation in textbook

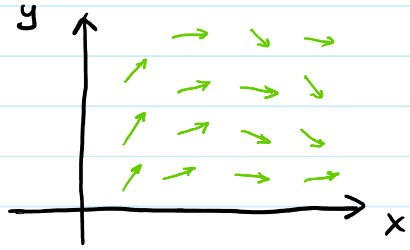
This is how you really calculate
You have to decide how to parametrize the curve

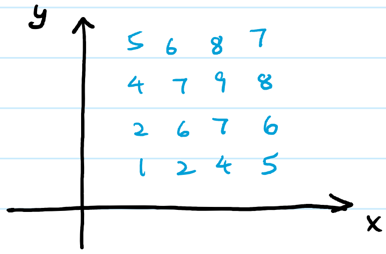
Illustration

The function $\vec{F}(x,y)$ can be plotted as a "field of vectors"

i.e. At each point (x,y), there is a vector (F_x, F_y)

(In scalar function, at each point (x,y) there is a number $f(x,y)$)





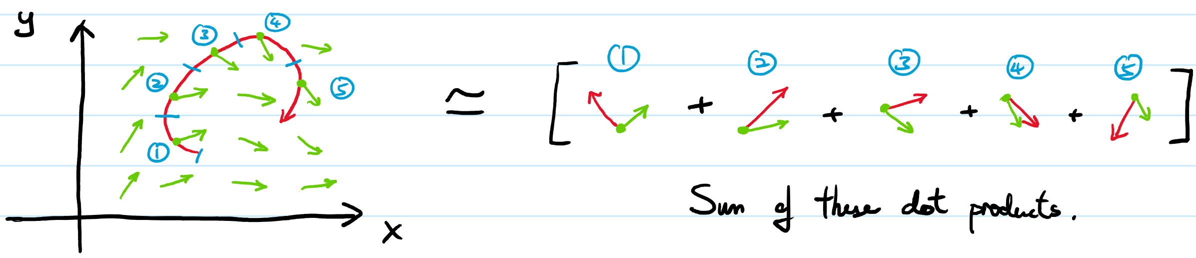
Vector function : Every pt. gives 1 vector

Scalar function : Every pt. gives 1 number

The quantity $\vec{F} \cdot d\vec{r}$ is like,

- Find the vector field on each segment
- Find the vector interval of each segment

Do dot product for each segment, then sum them all

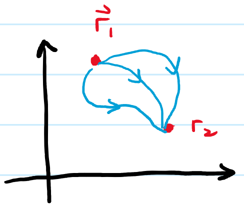


Sum of these dot products.

Gradient Theorem

Theorem : If $f(\vec{r})$ is a continuous scalar function

$\vec{\nabla} f(\vec{r})$ is its gradient vector (field)



Integration always gives the same value

Then $\displaystyle\int_{\text{start from } \vec{r}_1}^{\text{end at } \vec{r}_2} \vec{\nabla} f(\vec{r}) \cdot d\vec{r} = f(\vec{r}_2) - f(\vec{r}_1)$

is independent of the curve to be integrated

Illustration

Recall $\vec{\nabla} f \cdot \hat{u}$ = slope in $\hat{u}$ direction

So $\quad \vec{\nabla} f \cdot d\vec{r} = \vec{\nabla} f \cdot \boxed{\dfrac{d\vec{r}}{|d\vec{r}|}} \, |d\vec{r}|$

← unit vector

$\simeq$ slope × base length

= change in height

= $\vec{\nabla} f \cdot d\vec{r}$

$= \begin{pmatrix} \text{slope in} \\ d\vec{r} \text{ direction} \end{pmatrix} \times \begin{pmatrix} \text{Base} \\ \text{length} \end{pmatrix}$

$= \begin{pmatrix} \text{Change in height} \\ \text{along } d\vec{r} \end{pmatrix}$

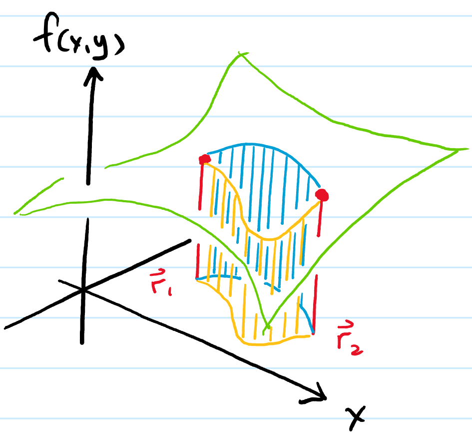
$\Rightarrow$ Sum of all $\vec{\nabla} f \cdot d\vec{r}$ along a curve from $\vec{r}_1$ to $\vec{r}_2$

= Net height change by travelling from $\vec{r}_1$ to $\vec{r}_2$

When the landscape is continuous,

the net height change should be

independent of the path travelled



Application : Conservative Force & Potential

Any vector function $\vec{F}(\vec{r})$ is conservative if it equals to the gradient of some scalar function $U(\vec{r})$

$$\vec{F}(\vec{r}) = -\vec{\nabla} U(\vec{r})$$

Vector field = the force

Have a minus sign in physics convention

Scalar function = the potential energy

☆ The nice property when a force vector field is conservative :

Total W.D. along any path between $\vec{r}_1$, $\vec{r}_2$

$$= \int_{\vec{r}_1 \to \vec{r}_2} \vec{F} \cdot d\vec{r}$$

(Force • displacement in each segment then sum all of them)

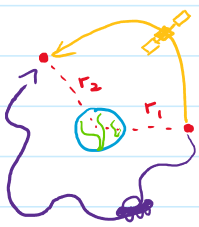
$$= \int_{\vec{r}_1 \to \vec{r}_2} -\vec{\nabla} U \cdot d\vec{r}$$

Gradient Theorem

$$= -\left(U(\vec{r}_2) - U(\vec{r}_1) \right)$$

Only depends on the 2 end points. Independent to the path.

E.g. $\vec{F}(\vec{r}) = \frac{GMm}{|\vec{r}|^2} \hat{r}$ is conservative



That 's why we can define Gravitational potential Energy

$$\Delta U(\vec{r}) = -\int_{\vec{r}_1 \to \vec{r}_2} \vec{F} \cdot d\vec{r} = -\left(\frac{GMm}{|\vec{r}_2|} - \frac{GMm}{|\vec{r}_1|} \right)$$

And we never need to calculate line integral even in 3D

(P.S. To prove that a force field is conservative requires us to show that its curl = 0

i.e. $\vec{\nabla} \times \vec{F} = 0$

However we will not touch curl until E&M)