Physics Enrichment Program HKUST
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- Maxwell's Equation in vacuum / material
- Matching boundary condition for $\vec{E}/\vec{B}$
- Extra : Expressing by potentials & Choice of gauge

All the PDEs you need in E&M

① Maxwell Equations in vacuum

If distribution of charge Q/ρ and current $I/\vec{J}$ are known

we may solve $\vec{E}/\vec{B}$ from this system of 4 equations

	Integral form	Differential form
Gauss's Law	$\oiint_{\text{surface}} \vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0}$	$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$
Gauss's Law for B field	$\oiint_{\text{surface}} \vec{B}\cdot d\vec{a} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
Faraday's Law	$\oint_{\text{Loop}} \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_{\text{surface}} \vec{B}\cdot d\vec{a}$	$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$
Ampere's Law	$\oint_{\text{Loop}} \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0\frac{\partial}{\partial t}\iint_{\text{surface}} \vec{E}\cdot d\vec{a}$	$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

② Lorentz force

If we don't know how the charge / current sources move

we have to derive from $\vec{F} = m\vec{a}$ for their trajectories :

$$\vec{F} = q\vec{E} + \int_{\text{line}} I d\vec{\ell} \times \vec{B} \quad \Big| \quad \underset{\text{force per volume}}{\vec{f}} = \rho\vec{E} + \vec{J} \times \vec{B}$$

(Note : Athough current = moving charge, their relations are in the form $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$ which is another PDE and this is already contained in the Ampere's Law)

The above 5 equations are everything. If there is general & analytical solutions to any scenarios, E&M will be trivial.

But sadly we do not. So we learn different methods to avoid PDE

Maxwell Equations in Material

Solving for $\vec{E}$ & $\vec{B}$ in a material is even more complicated because not all charge / current are known. Recall that if we don't know the material properties, Polarization $\vec{P}$ / Magnetization $\vec{M}$ and bound charge / current are all unknown. The best we can write are :

Gauss's Law : Only know free charge's contribution

$$\oiint_{\text{surface}} \vec{D}\cdot d\vec{a} = Q_f \quad \text{or} \quad \vec{\nabla}\cdot\vec{D} = \rho_f$$

Gauss's Law for B field : No change

Faraday's Law : No change

Ampere's Law . Only know free current's contribution

Only know displacement current by free charge

$$\oint_{Loop} \vec{H}\cdot d\vec{l} = I_f + \frac{\partial}{\partial t}\iint_{surface} \vec{D}\cdot d\vec{a} \quad \text{or} \quad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

The system of PDE cannot be further solved unless we know the material's property. For the special case Linear material

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{H} = \frac{1}{\mu}\vec{B}$$

After substitution, it gives the original Maxwell's Equation but with $\varepsilon_0 \to \varepsilon$, $\mu_0 \to \mu$

Ohm's Law

Newton 2nd Law is not much useful to describe particle motion in the material because of collision. So Lorentz force Law may not be used. In replacement we have Ohm's Law

$$I = \frac{V}{R} \quad \Big| \quad \vec{J} = \sigma\vec{E}$$

which describes that E field will create a constant flow of current, rather than accelerating charges indefinitely.

Boundary Condition

Suppose that we successfully solve the Maxwell Equations

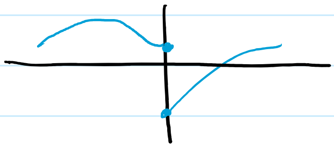
In some cases, very difficult

arriving at some solution of $\vec{E}(\vec{r},t)$ / $\vec{B}(\vec{r},t)$

The final step is to match the boundary condition.

E.g. Boundary condition on 1D mechanical wave

① Amplitude is continuous



or else the string is broken

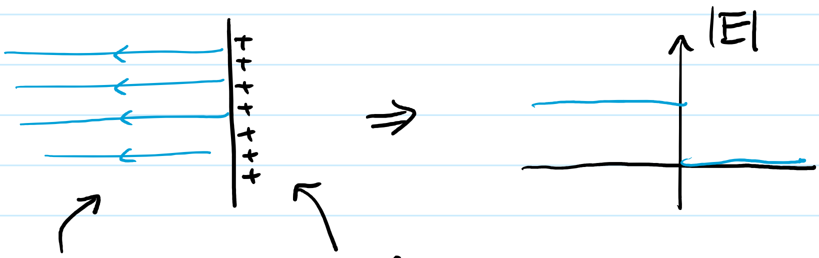
② Curve is smooth (i.e. slope is continuous)

or else tension on both sides are unequal
$\Rightarrow$ momentum not conserved.

☆☆☆ However E/B field need not be continuous

because they can terminate at charge/current

E.g.



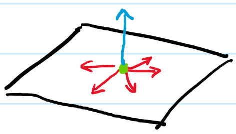
Have E field on this side

No E field on this side

We can use the 4 Maxwell Equations to derive the conditions.

Across a surface, we can separate a vector into

- Direction $\perp$ to the surface
- Directions // to the surface



The $\perp$ Direction

We can draw a very thin box across the interface

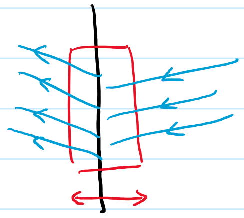
Then by Gauss's Law and Gauss's Law for B field in vacuum:

$$\vec{E}_{\perp}^{left} - E_{\perp}^{right} = \frac{\sigma}{\varepsilon_0}$$

charge density on surface

$$\vec{B}_{\perp}^{left} - B_{\perp}^{right} = 0$$

magnetic charge never exist



flux into the box only concern the components $\perp$ to the box's face

thickness of box can be infinitestimally small

But more commonly the interface is between 2 different material

In this case we can only match the boundary condition of $\vec{D}$ with free charge density σ_f, unless we know the material is linear

nothing changes in $\vec{B}$'s Eq.

$$D_{\perp}^{left} - D_{\perp}^{right} = \sigma_f$$

$$B_{\perp}^{left} - B_{\perp}^{right} = 0$$

if linear material $\longrightarrow$

$$\varepsilon_{left} E_{\perp}^{left} - \varepsilon_{right} E_{\perp}^{right} = \sigma_f$$

$$B_{\perp}^{left} - B_{\perp}^{right} = 0$$

The // directions

We can draw a very thin loop across the interface

Then by Faraday's Law and Ampere's Law in vacuum

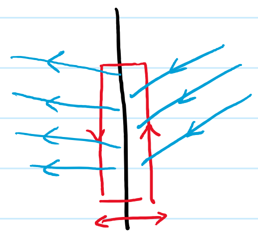
$$E_{/\!/}^{left} - E_{/\!/}^{right} = 0$$

loop area ~ 0 $\Rightarrow$ B's flux ~ 0

$$B_{/\!/}^{left} - B_{/\!/}^{right} = \mu_0 \vec{K} + 0$$

current density on the surface

loop area ~ 0 $\Rightarrow$ E's flux ~ 0



loop integral only concern components // to the loop

loop width can be infinitestimally small

But more commonly the interface is between 2 different materials

In this case we can only match the boundary condition of $\vec{H}$ with free current density K_f, unless we know the material is linear

nothing changes in E's Eq

$$\boxed{\begin{aligned} E_{\parallel}^{left} - E_{\parallel}^{right} &= 0 \\ H_{\parallel}^{left} - H_{\parallel}^{right} &= K_f \end{aligned}} \xrightarrow{\text{if linear material}} \boxed{\begin{aligned} E_{\parallel}^{left} - E_{\parallel}^{right} &= 0 \\ \frac{B_{\parallel}^{left}}{\mu_{left}} - \frac{B_{\parallel}^{right}}{\mu_{right}} &= K_f \end{aligned}}$$

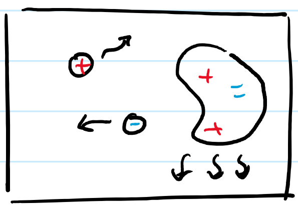
Summary

Ultimate solution to any classical E&M problem

= Solve the set of Maxwell's Equation

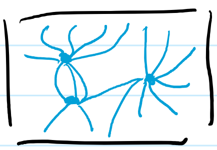
(However we don't really have a convenient expression for every scenarios. In most of the cases we can only arrive at some kinds of series expansion.)

Finally after solving, match the given boundary condition to get the solution to the specific case.

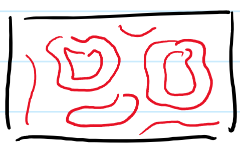


Given some config.

Solve PDE $\Rightarrow$



Solution of $\vec{E}(\vec{r},t)$



Solution of $\vec{B}(\vec{r},t)$

Extra : The Choice of "Gauge"

Recall in electrostatic / magnetostatic, solving $\vec{E}/\vec{B}$ directly from $\rho/\vec{J}$ is annoying. Instead, we turn to solve the potentials $V/\vec{A}$, whose PDEs are in general easier to solve.

When solving the whole set of Maxwell's Equation, we can also try this to simplify the arithmetics.

Scaler potential in electrodynamics

From Faraday's Law $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \neq 0$ in general

i.e. $\vec{E}$ may not be conservative and we cannot write $\vec{E} = -\nabla V$

But since $\vec{B} = \vec{\nabla}\times\vec{A}$ we can write

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{A}) \Rightarrow \nabla\times(\vec{E}+\frac{\partial}{\partial t}\vec{A}) = 0$$

$\Rightarrow$ $\vec{E}+\frac{\partial\vec{A}}{\partial t}$ is a conservative quantity

$\Rightarrow$ Can write $\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\nabla\Phi$

where Φ is some scaler function

Φ is usually called scalar potential, in pair with $\vec{A}$ the vector potential

Then we can substitute $\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t}$ & $\vec{B} = \vec{\nabla}\times\vec{A}$ into Gauss's Law & Ampere's Law :

Gauss :

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t})$$

$$= -\nabla^2\Phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \boxed{\nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\varepsilon_0}}$$

Ampere:

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\varepsilon_0 \frac{\partial E}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0 \vec{J} + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = \mu_0\vec{J} - \mu_0\varepsilon_0 \nabla\left(\frac{\partial}{\partial t}\Phi\right) - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A}$$

$$\Rightarrow \boxed{\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\Phi\right) = -\mu_0\vec{J}}$$

While Faraday's Law and $\vec{\nabla}\cdot\vec{B} = 0$ are automatically satisfied or else the potential cannot be defined.

Now the set of 4 Maxwell Equations has been transformed into a set of 2 Equations about Φ, $\vec{A}$, and after solving together with a given boundary condition, we can compute the field by

$$\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Dependance between Φ & $\vec{A}$

Recall: $\vec{E}/\vec{B}$ and $\Phi/\vec{A}$ are in differential relation.

i.e. $\vec{E} \sim$ derivative of $\Phi/\vec{A}$, $\vec{B} \sim$ derivative of $\vec{A}$

There are infinitely many choices of $\Phi/\vec{A}$ that yield the same $\vec{E}/\vec{B}$ (i.e. the same physics observation)

(Just like if $g(x) = \frac{d}{dx}f(x)$, then $f(x) = \int g(x)\,dx + C$, while C can have infinitely many choices)

But in order to yield the same $\vec{E}$, any choice of $\Phi / \vec{A}$ must satisfy $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$ = the same function.

$\boxed{1}$ Since curl of a conservative field $= 0$

and a conservative field can be written as

the gradient of some scalar function

$$\Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\vec{A} + \underbrace{\vec{\nabla}\Lambda})$$

gradient of any scalar function Λ

Give 0 when taking curl.

$\boxed{2}$ If $\vec{A}$ is changed to $\vec{A}_{new} = \vec{A} + \vec{\nabla}\Lambda$

We have to change Φ to $\Phi_{new} = \Phi - \frac{\partial \Lambda}{\partial t}$

s.t. $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$

$$= -\vec{\nabla}\left(\Phi_{new} + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A}_{new} - \vec{\nabla}\Lambda\right)$$

$$= -\vec{\nabla}\Phi_{new} - \frac{\partial}{\partial t}\vec{A}_{new} - \frac{\partial}{\partial t}(\vec{\nabla}\Lambda) + \frac{\partial}{\partial t}(\vec{\nabla}\Lambda)$$

$$= -\vec{\nabla}\Phi_{new} - \frac{\partial}{\partial t}\vec{A}_{new}$$

i.e. Changing the pair $\Phi / \vec{A}$ to $\Phi_{new} / \vec{A}_{new}$ by

$$\begin{cases} \Phi_{new} = \Phi - \frac{\partial}{\partial t}\Lambda \\ \vec{A}_{new} = \vec{A} + \vec{\nabla}\Lambda \end{cases}$$

has no effect on the result $\vec{E} / \vec{B}$, for any scalar function Λ

Terminologies : Gauge

- $\vec{E}/\vec{B}$ are physically observable (~ force on charge / current)
- For the same $\vec{E}/\vec{B}$ config, there are infinitely many $\Phi/\vec{A}$ pairs of descriptions that correspond to it.
- Any $\Phi/\vec{A}$ pairs of the same $\vec{E}/\vec{B}$ config. can be converted to each other by using some scalar function Λ.

  This conversion = Gauge Transformation
- Λ, as a "free variable", can be chosen freely by us is a Gauge Freedom in the $\Phi/\vec{A}$ description.

Note : For the name origin of "Gauge", see the Wiki page "Introduction to gauge theory"

2 Common choice of gauge

Idea : Certain choices of potentials yield a set of PDE that is easy to solve.

Question 1 : What form of the potentials function can we choose?

Question 2 : How will the Maxwell's Equation become?

① Lorenz Gauge

$\Phi/\vec{A}$ are chosen specifically s.t. they satisfy

$$\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial\Phi}{\partial t} = 0$$

Proving such choice exist :

Let's say we begin with some arbitrary $\Phi / \vec{A}$ and we need to find a Λ that can lead to

$$\vec{\nabla} \cdot \vec{A}_{new} + \mu_0 \varepsilon_0 \frac{\partial \Phi_{new}}{\partial t} = 0$$

$$\Rightarrow \quad \vec{\nabla} \cdot (\vec{A} + \vec{\nabla} \Lambda) + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left(\Phi - \frac{\partial \Lambda}{\partial t} \right) = 0$$

$$\Rightarrow \quad \underline{\nabla^2 \Lambda - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \Lambda = - \vec{\nabla} \cdot \vec{A} - \mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t}} = \text{Some function}$$

This is an inhomogeneous 3D wave equation of $\Lambda(x, y, z, t)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Lambda(x, y, z, t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \Lambda(x, y, z, t) = \text{some known function}$$

$\Rightarrow$ For any starting $\Phi / \vec{A}$, we subst. into this equation, solve it, we can find the Λ that transform it into the new pair $\Phi_{new} / \vec{A}_{new}$ that satisfy $\vec{\nabla} \cdot \vec{A}_{new} - \mu_0 \varepsilon_0 \frac{\partial \Phi_{new}}{\partial t} = 0$

$\hookrightarrow$ But there is no need to solve. All we need to say is that such pair of $\Phi_{new} / \vec{A}_{new}$ exist for any $\vec{E} / \vec{B}$ config.

Nice thing about Lorenz Gauge :

From the Gauss's Law & Ampere's Law in terms of $\Phi / \vec{A}$

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t} = 0$$

$$\nabla^2 \Phi + \frac{\partial}{\partial t} (\underline{\vec{\nabla} \cdot \vec{A}}) = \nabla^2 \Phi + \frac{\partial}{\partial t} \left(- \mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t} \right)$$

$$\boxed{= \left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \right) \Phi = - \frac{\rho}{\varepsilon_0}}$$

$$\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t} - \nabla(\cancel{\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial\Phi}{\partial t}}^{\,0})$$

$$\boxed{= \left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J}}$$

So Φ & $\vec{A}$ can be solved independently in their own inhomogeneous wave equation.

For example, the solution for the boundary condition $\Phi(\vec{r},t) / \vec{A}(\vec{r},t) \to 0$ when $\vec{r}\to\infty$ is

$$\Psi(\vec{r},t) = \iiint \frac{f\left(\vec{r}\,', \boxed{t - \frac{|\vec{r}-\vec{r}\,'|}{c}}\right)}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' + \Psi_c(\vec{r},t)$$

(with $\Psi = \Phi$ or $\vec{A}$, $f = \frac{\rho}{4\pi\varepsilon_0}$ or $\frac{\mu_0\vec{J}}{4\pi}$, $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

Ψ_c = complementary soln. , i.e. soln with $f = 0$)

It actually looks like the potential in static case, e.g.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' \sim \sum \frac{\text{source}}{\text{distance}}$$

Except that now we also need to consider the relativistic effect

i.e. Influence of a source takes time to propagate, and speed of propagation = c = light speed

$$\Phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\iiint\frac{\rho\left(\vec{r}\,', \boxed{t - \frac{|\vec{r}-\vec{r}\,'|}{c}}\right)}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,' \sim \sum \frac{\text{source in the past}}{\text{distance}}$$

For a source at position $\vec{r}'$, its influence take a duration of $\frac{|\vec{r}-\vec{r}'|}{c}$ to arrive $\vec{r}$. So for the potential at time t we need to look at what the source look like at time $t-\frac{|\vec{r}-\vec{r}'|}{c}$

Note that for source at different distance, this time is different.

② Columb Gauge (also called transverse gauge)

$\Phi/\vec{A}$ are chosen specifically s.t. they satisfy

$$\vec{\nabla}\cdot\vec{A} = 0$$

(Together with $\vec{\nabla}\times\vec{A}=\vec{B}$ and a given boundary condition, $\vec{A}$ is completely determinable. Then Φ is also determinable by $\vec{E}$)

Proving such choice exist:

Let's say we begin with some arbitrary $\Phi/\vec{A}$ and we need to find a Λ that can lead to

$$\vec{\nabla}\cdot\vec{A}_{new} = 0$$

$$\Rightarrow \quad \vec{\nabla}\cdot(\vec{A}+\vec{\nabla}\Lambda) = 0$$

$$\Rightarrow \quad \underline{\vec{\nabla}^2\Lambda = -\vec{\nabla}\cdot\vec{A}} = \text{some function}$$

This is a Poisson equation of $\Lambda(x,y,z,t)$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Lambda(x,y,z,t) = \text{some known function}$$

$\Rightarrow$ For any starting $\Phi/\vec{A}$, we subst. into this equation, solve it, we can find the Λ that transform it into the new pair $\Phi_{new}/\vec{A}_{new}$ that satisfy $\vec{\nabla}\cdot\vec{A}_{new}$

↪ But there is no need to solve. All we need to say is that such pair of $\Phi_{new}/\vec{A}_{new}$ exist for any $\vec{E}/\vec{B}$ config.

Nice thing about Columb Gauge

By $\vec{\nabla}\cdot\vec{A} = 0$, Gauss Law turns back to the simple form as in electrostatic.

$$\vec{\nabla}^2\Phi + \frac{\partial}{\partial t}(\cancelto{0}{\vec{\nabla}\cdot\vec{A}}) = \boxed{\vec{\nabla}^2\Phi = -\frac{\rho}{\varepsilon_0}}$$

which means $\Phi(x,y,z,t)$ only depends on the position distribution of charges at time t, not their velocities.

However as a tradeoff, Ampere's Law becomes more complicated

$$\vec{\nabla}^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{A} - \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{A}} + \mu_0\varepsilon_0\frac{\partial\Phi}{\partial t}) = -\mu_0 J$$

$$\Rightarrow \boxed{\vec{\nabla}^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0 J + \underbrace{\mu_0\varepsilon_0\vec{\nabla}\frac{\partial\Phi}{\partial t}}}$$

After applying vector identity & continuity eq

this term $= -\frac{\mu_0}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int\frac{\vec{J}}{|\vec{r}-\vec{r}'|}d^3\vec{r}'$

is called "tranverse current"

i.e. Equation of $\vec{A}$ is still an inhomogeneous wave Eq, while the current source becomes annoying to calculate.