Single Variable calculus review

Limit – Naive Definition + How to calculate

Differentiation – Geometrical meaning + How to calculate

Integration – Geometrical meaning + How to calculate

Use of calculus in basic mechanics

Limit :

(Naive) Definition : $\lim_{x \to a} f(x) = L$ $(\neq \pm\infty)$

means $f(x)$ is getting closer to L when x is getting closer to a

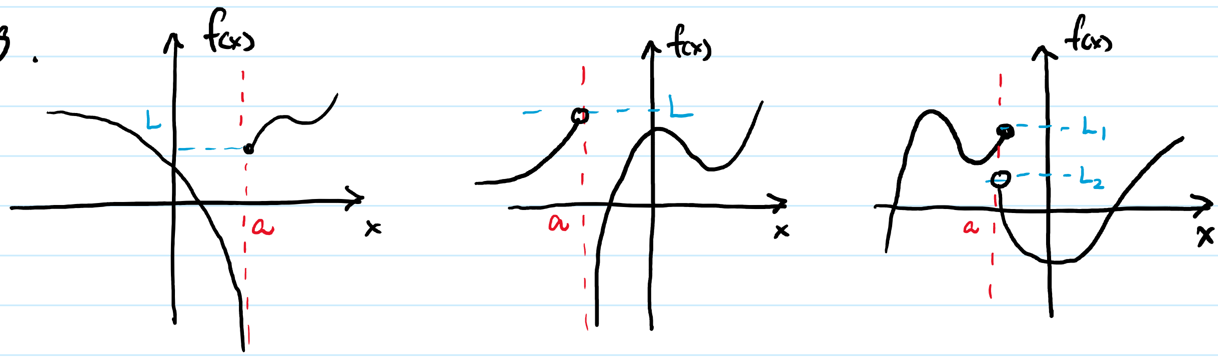
Two sides limit

Right hand limit : $\lim_{x \to a^+} f(x) = L$ $(\neq \pm\infty)$

Left hand limit : $\lim_{x \to a^-} f(x) = L$ $(\neq \pm\infty)$

Existance of limit : A limit "exist" if both sides limit exist & equal

E.g.



RH limit exist
LH limit not exist
∴ limit not exist

RH limit not exist
LH limit exist
∴ limit not exist

Both side limit exist
but they are not equal
∴ limit not exist

Evaluation of limit

(Assume both $f(x)$ & $g(x)$'s limits exist)

Addition / Subtraction $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

Product $\lim_{x\to a}[f(x) \cdot g(x)] = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$

Quotient $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (only if $\lim_{x\to a} g(x) \neq 0$)

Trigonometric Relation

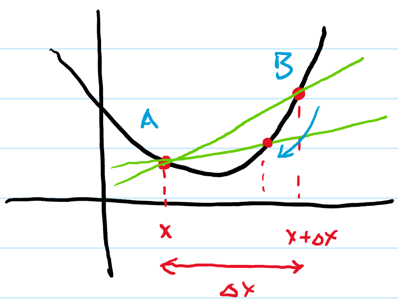
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \cos x = 1$$

Differentiation

Def : $\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{d}{dx} f(x) = f'(x)$ (If this limit exists $f(x)$ is "differentiable" at the given x)

Geometrical Meaning : Slope of tangent line of function



Slope of line AB $= \frac{f(x+\Delta x) - f(x)}{\Delta x}$

when $\Delta x \to 0$, AB become tangent

Evaluation

Addition / Subtraction $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

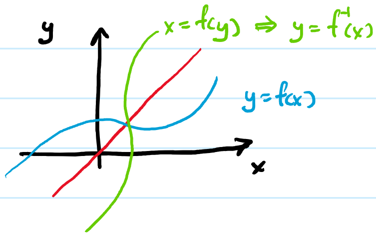
Product Rule $\frac{d}{dx}[f(x) \cdot g(x)] = \left[\frac{d}{dx} f(x)\right] \cdot g(x) + f(x) \cdot \left[\frac{d}{dx} g(x)\right]$

Quotient Rule $\quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{1}{[g(x)]^2}\left[g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)\right]$

(This is just product rule of $\frac{d}{dx}\left[f(x)\cdot\frac{1}{g(x)}\right]$)

Chain rule $\quad \frac{d}{dx}\left[g(f(x))\right] = \frac{d}{d[f(x)]}g(f(x)) \cdot \frac{d}{dx}f(x)$

Inverse function $\quad \frac{d f^{-1}(x)}{dx} = \frac{1}{\left.\frac{d}{dy}f(y)\right|_{y=f^{-1}(x)}}$



Polynomial $\quad \frac{d}{dx}x^n = n x^{n-1}$

Trigonometric $\quad \frac{d}{dx}\begin{cases}\sin x\\ \cos x\\ \tan x\end{cases} = \begin{cases}\cos x\\ -\sin x\\ \sec^2 x\end{cases}$

Exponential $\quad \frac{d}{dx}\begin{cases}e^x\\ \ln x\end{cases} = \begin{cases}e^x\\ \frac{1}{x}\end{cases}$

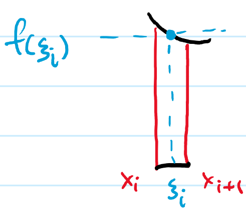
Integration

Def : Definite Integral

① Choose $n-1$ points between $[a, b]$

$a < x_1 < x_2 < \cdots < x_{n-1} < b$ = seperate into total n strips

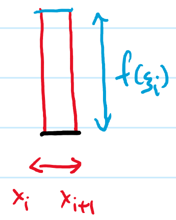
② In each strip, pick an arbitrary point $x = \xi_i$



Approximate the strip's height as $\simeq f(\xi_i)$

③ Approximate area of each strip $\simeq f(\xi_i) \cdot [x_{i+1} - x_i]$

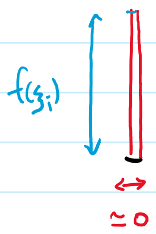
$\Rightarrow$ Total area under curve = Summing all strips



$$= \sum_{i=0}^{n-1} f(\xi_i) \cdot [x_{i+1} - x_i]$$

with $x_0 = a$, $x_n = b$

④ And then take each of $[x_{i+1} - x_i]$ limit to 0



No matter how "arbituary" we choose ξ_i in $[x_i, x_{i+1}]$

the height of the strip is basically equal to $f(\xi_i)$

↳ Area under curve $\simeq$ Summing the area of infinitely many strips that have $\simeq 0$ width

Def : $\displaystyle \int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i$

Def : Indefinite integral

Given a function $f(x)$. If there exists another function $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$

Then $F(x)$ is called antiderivative / primitive of $f(x)$

Indefinite integral = Set of all antiderivative of $f(x)$

= Set of all $F(x)$ s.t. $\frac{d}{dx}F(x) = f(x)$

E.g. $f(x) = x^2 \Rightarrow \int f(x)\,dx = \frac{x^3}{3} + \underline{\underline{C}}$

[arbitrary constant

$\Rightarrow$ Set of antiderivative of x^2

$$= \left\{ \frac{x^3}{3} + 1,\ \frac{x^3}{3} - 2,\ \frac{x^3}{3} + \pi, \ldots \right\}$$

Fundermental Theorem of Calculus

$\Rightarrow$ Differentiation & Integration are opposite operations

$\because$ Definite integration of $f(x)$ over $[a,b]$

$$= \int_a^b f(x)\,dx = F(b) - F(a)$$

Then differentiate both side

Treat b be a free variable, b can be any real no.

$$\frac{d}{db}\int_a^b f(x)\,dx = \frac{d}{db}F(b) - 0$$

a = some constant

$$= \frac{d}{dx}F(x)$$

b is just a symbol representing a free varible, You can replace it with other letters

$$= f(x)$$

Rule of Integration

Addition / Subtraction $\int f(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

Integration by part (correspondant of product rule)

$$\int f(x) \cdot \frac{d}{dx} g(x) \cdot dx = f(x)\, g(x) - \int \frac{d}{dx} f(x) \cdot g(x) \cdot dx$$

$$\int f\, dg = f \cdot g - \int g \cdot df$$

$$\frac{d}{dx}\left(\int f\, dg + \int g\, df\right) = \frac{d}{dx}(f \cdot g)$$

Integration by change of variable (correspondant of chain rule)

$$\int g[f(x)] \cdot \frac{d}{dx} f(x)\, dx = \int g[f(x)] \cdot d[f(x)]$$

$$= \int g[u]\, du \Big|_{u = f(x)}$$

Polynomial $\int x^n\, dx = \frac{x^{n+1}}{n+1}$

Trigonometric $\int \begin{Bmatrix} \sin x \\ \cos x \\ \sec^2 x \end{Bmatrix} dx = \begin{cases} -\cos x \\ \sin x \\ \tan x \end{cases}$

Exponential $\int \begin{Bmatrix} e^x \\ \frac{1}{x} \end{Bmatrix} dx = \begin{cases} e^x \\ \ln|x| \end{cases}$

Substitution of trigonometric function

$\int \sqrt{a^2 - x^2}\, dx \longrightarrow$ Subst. $x = a\sin\theta \rightarrow$ $\sqrt{a^2 - x^2} = a\cos\theta$, $dx = a\cos\theta\, d\theta$

$\int \sqrt{a^2 + x^2}\, dx \longrightarrow$ Subst. $x = a\tan\theta \rightarrow$ $\sqrt{a^2 + x^2} = a\sec\theta$, $dx = a\sec^2\theta\, d\theta$

$\int \sqrt{x^2 - a^2}\, dx \longrightarrow$ subst. $x = a\sec\theta \rightarrow$ $\sqrt{x^2 - a^2} = a\tan\theta$, $dx = a\sec\theta\tan\theta\, d\theta$

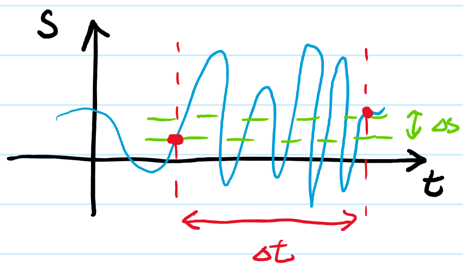
Calculus in Mechanics

Relations of s, v, a

In secondary school textbook we have

- Average velocity $v \sim \frac{\Delta s}{\Delta t}$
- Average acceleration $a \sim \frac{\Delta v}{\Delta t}$

But "Average" is never accurate to describe things.



Average velocity $\frac{\Delta s}{\Delta t}$ does not describe the drastic movement

For accurate description, we need to take limit $\Delta t \to 0$

↳ Instantaneous quantities = quantity at a "single" time point

- Instantaneous velocity $= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{d}{dt} s(t)$
- Instantaneous acceleration $= \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{d}{dt} v(t)$

$$\Rightarrow \quad s(t) \underset{\int dt}{\overset{\frac{d}{dt}}{\rightleftharpoons}} v(t) \underset{\int dt}{\overset{\frac{d}{dt}}{\rightleftharpoons}} a(t)$$

Example : Constant Acceleration Motion

$\Rightarrow$ $a(t) = a =$ some constant

↳ Integration to find $v(t)$ & $s(t)$

- Velocity : $v(t) = \int a \, dt = at + C$

At $t = 0$

$v(0) = C$ = initial velocity

Usually denote as "u" in textbook

↳ $v(t) = v(0) + at$

- Displacement : $s(t) = \int v(t) \, dt = v(0)t + \frac{1}{2}at^2 + C$

At $t = 0$

$s(0) = C$ = initial displacement

↳ $s(t) = s(0) + v(0)t + \frac{1}{2}at^2$

☆☆☆ In fact, only 2 of the 4 formulas of constant acceleration motions are independent

$$\begin{cases} v(t) = v(0) + at \\ s(t) = s(0) + v(0)t + \frac{1}{2}at^2 \\ s(t) - s(0) = \frac{v(t) + v(0)}{2} t \\ v(t)^2 - v(0)^2 = 2a \cdot s(0) \end{cases}$$

Definition nothing to explain

Can be derived by substitution of the above 2

This is also shown in fact that a "SUVAT" problem always provide you 3 of the 5 variables, then you can solve the remaining 2 with 2 equations.

Force & Energy

Fundamental relation between force & energy : Work Done

$$W.D. = F \cdot \Delta s = \left[\begin{matrix}\text{Average} \\ \text{Force}\end{matrix}\right] \cdot [\text{Displacement}]$$

↪ For a force $F(t)$ that is a function of time

Consider the W.D. in a very short time interval $[t, t+\Delta t]$

$$\Delta W.D. = \frac{F(t) + F(t+\Delta t)}{2} \cdot [s(t+\Delta t) - s(t)]$$

$$= \frac{F(t) + F(t+\Delta t)}{2} \left[\frac{s(t+\Delta t) - s(t)}{\Delta t}\right] \Delta t$$

⇒ Total W.D. in a long period time

= Sum of all W.D. of many small time interval

$$= \sum_i \left[\frac{F(t_i) + F(t_{i+1})}{2}\right] \cdot \left[\frac{s(t_{i+1}) - s(t_i)}{\Delta t}\right] \Delta t$$

↪ $\Delta t = t_{i+1} - t_i$

Take $\Delta t \to 0$, then $t_{i+1} \simeq t_i$ & $F(t_{i+1}) \simeq F(t_i)$

$$W.D. \to \int \frac{F(t) + F(t)}{2} \cdot \left[\frac{ds(t)}{dt}\right] dt$$

$$= \int F(t) \cdot \boxed{v(t)}\, dt$$

$$= \int \underline{F(t)} \cdot d(s(t))$$

$$= \int \underline{\underline{F^*[s(t)]}}\, d[s(t)]$$

More commonly, force is given as a function of displacement

E.g. Gravitational Force & PE

$$F(r) = -\frac{GMm}{r^2}$$

Force is a function of distance from the mass

Then the W.D. required to move from r_1 to r_2 is

$$W.D. = \int_{r_1}^{r_2} -\frac{GMm}{r^2}\,dr$$

Just $\int F(r)\,dr$

$$= -\frac{GMm}{r_2} - \frac{GMm}{r_1}$$

$$= \text{GPE at } r_2 - \text{GPE at } r_1$$

On the other hand, if given PE

$$PE = -\frac{GMm}{r}$$

Add -ve for convention

$$\text{Force by central mass} = -\frac{d}{dr}(PE)$$

$$= -\frac{GMm}{r^2}$$

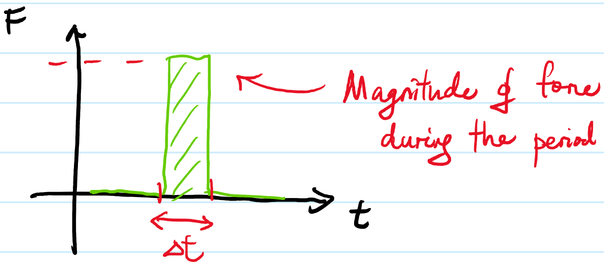
Momentum / Impulse vs Newton 2^{nd} Law

2 representations of Newton 2^{nd} Law

Force & Acceleration	Impulse & Momentum
$F = ma$	$I = \Delta(mv)$
Apply a force	Apply an impulse
⇒ Get acceleration	⇒ Change in momentum

Recap : Impulse

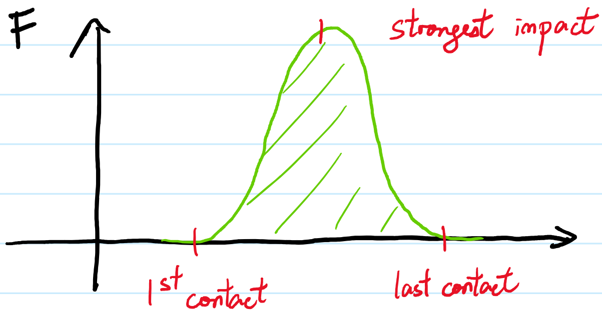
$$I \sim \underbrace{F}_{\text{Applied force}} \cdot \underbrace{\Delta t}_{\text{Duration}}$$

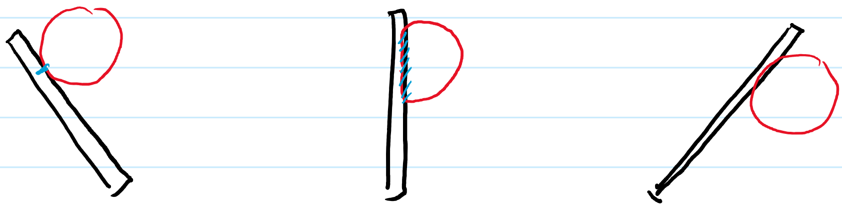


$= $ Area under curve in F-t graph

$= $ Change of momentum

☆☆☆ But in reality, when things are colliding, applied force is not constant during the whole contact time.





Contact area ~ 0	Contact area ~ max	Almost lose contact
$\Rightarrow F \sim 0$	$\Rightarrow F \sim max$	$\Rightarrow F \sim 0$

So in general, force should be a function of time

The total change of momentum during collision period :

$$\sum_i \Delta(mv)_i = \sum_i I_i = \sum_i \frac{F(t_i) + F(t_{i+1})}{2} \cdot \Delta t$$

$\hookrightarrow \Delta t = t_{i+1} - t_i$

Taking $\Delta t \to 0$

$$\int d(mv) = \int \frac{F(t) + F(t)}{2}\, dt = \int F(t)\, dt$$

This is the true Impulse - Momentum Relation

$$\int d(mv) = \int F(t)\, dt = I$$

Then differentiate both side by t

$$\frac{d}{dt}(mv) = F(t) \longrightarrow \boxed{\text{Rate of change of momentum} = \text{Force}}$$

☆ Normally we should not assume mass to be time independent

$$F(t) = \frac{d}{dt}\big(m(t)\, v(t)\big)$$

$$= m(t) \cdot \underbrace{\frac{dv(t)}{dt}}_{a(t)} + \frac{dm(t)}{dt} \cdot v(t)$$

$$\Rightarrow \boxed{F = ma + v\frac{dm}{dt}}$$

This is the more general Newton 2^{nd} Law.

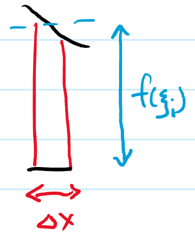
Only if we guarentee the mass to be constant

then we can write $F = ma$

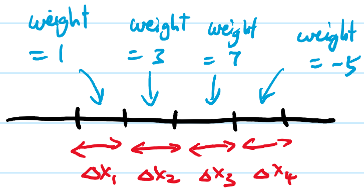
2 Interpretation of Integration

1st Interpretation : Area under curve

$$\int_a^b f(x)\,dx = \underbrace{\lim_{\Delta x_i \to 0} \sum_{i=0}^{n-1}}_{\text{Sum them all}} \underbrace{f(\xi_i)}_{\text{height of strip}} \underbrace{\Delta x_i}_{\text{width of strip}}$$



2nd Interpretation : Weighted sum



$$\int_a^b f(x)\,dx = \underbrace{\lim_{\Delta x_i \to 0} \sum_{i=0}^{n-1}}_{\text{Sum them all}} \underbrace{f(\xi_i)}_{\text{"weight" assigned to the interval } [x_i, x_{i+1}]} \underbrace{\Delta x_i}_{\text{length of the interval } [x_i, x_{i+1}]}$$

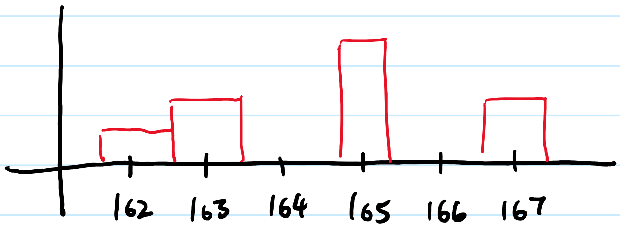
Example : Distribution of Students' height

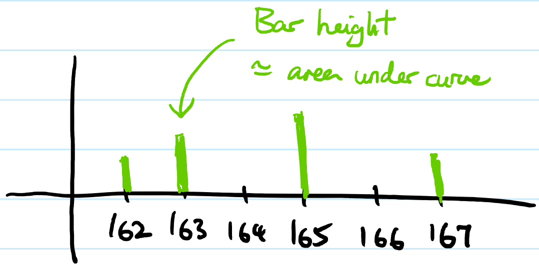
162	163	163	163	165	
165	165	165	167	167	167 ...

Average height = Weighted sum

$$= \sum_h \left(\begin{matrix} \text{Portion of students} \\ \text{with height } h \end{matrix} \right) \times (\text{height } h)$$

$$= \frac{1}{11} \times 162 + \frac{3}{11} \times 163 + \frac{4}{11} \times 165 + \frac{3}{11} \times 167$$





Total height of all students = sum of all bars' height

∼ Total area under each bar

Then transition from discrete to continuous distribution

- A single number → Interval between 2 number

↳ Range of students' height $[h_i, h_{i+1}]$

- Height of a bar → Area under the bar

↳ Portion of students whose height is within the range $[h_i, h_{i+1}]$

When the interval width → 0, this weighted sum arrives the formula the same as integration.