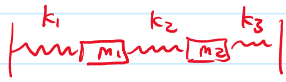
- Matrix operations. Determinants.
- Theory of systems of Linear Equations
- Eigenvalue & Eigenvector
- Solving System of ODE E.g. 

Matrix

Dimension of matrix = (No. of row) × (No. of column)

Use bold font to represent a matrix →

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \updownarrow 2 \qquad \Rightarrow \text{dimension} = 2\times 3$$

$$\xleftrightarrow{\quad 3 \quad}$$

Each entry is called an element

Column matrix (Column vector)	Row matrix (Row vector)	Square matrix No. of row = No. of column
$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_3 \end{pmatrix}$	$(a_1 \ a_2 \ \dots \ a_n)$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Matrix Arithmetics

Calculation of matrix is like calculating numbers, but with new definitions

① Multiplication with constants

Multiply a constant = Multiply each element with the constant

$$k\mathbf{A} = k \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{pmatrix}$$

② Addition / Subtraction

Elements at the same position add/subtract independently.

Only matrices of the same dimension can be added/subtracted

$$A \pm B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \end{pmatrix}$$

③ Transpose

= Flip the matrix along the diagonal line

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

④ Matrix multiplication

Given 2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

(dimension : $m \times n$) (dimension : $n \times p$)

Their product

$$C = AB = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$ is a $m \times p$ matrix

we don't write
$\times$ sign in matrix multiplication

Calculation of each element : $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

ith row, jth column, every element

$$\begin{pmatrix} & \vdots & \\ \cdots & c_{ij} & \cdots \\ & \vdots & \end{pmatrix} = \begin{pmatrix} \vdots \\ a_{i1} \; a_{i2} \cdots a_{in} \\ \vdots \end{pmatrix} \begin{pmatrix} \cdots & b_{1j} & \cdots \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{pmatrix}$$

Visualizing the multiplication rule :

☆ Matrices can be multiplied only if

No. of column in 1^{st} matrix = No. of row in 2^{nd} matrix

So in general $AB \neq BA$

E.g.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

(A: 3 columns, 2 rows; B: 1 column, 3 rows)

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1\times 7 + 2\times 8 + 3\times 9 \\ 4\times 7 + 5\times 8 + 6\times 9 \end{pmatrix} = \begin{pmatrix} 50 \\ 122 \end{pmatrix}$$

$2\times 3 \rightarrow 3\times 1$

But BA does not exist $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$3\times 1 \not\rightarrow 2\times 3$

The closest we have is $(AB)^T = B^T A^T$ (always true)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \xleftrightarrow{\text{transpose}} \begin{pmatrix} 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

⑤ Identity Matrix

= Any square matrix with diagonal elements = 1 and else = 0

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Property : $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$ if $\mathbf{A}$ is a square matrix

It is just like multiplying 1 to numbers

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\times1+2\times0 & 1\times0+2\times1 \\ 3\times1+4\times0 & 3\times0+4\times1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

⑥ Zero matrix

Matrices with all elements = 0 (just like the 0 in numbers)

$$\mathbf{0}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{0}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

⑦ Inverse of matrices (~ Division of matrices)

If $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$ (Require both to be square matrices)

Then $\mathbf{A}$ is the inverse of $\mathbf{B}$, i.e. $\mathbf{A} = \mathbf{B}^{-1}$

$\mathbf{B}$ is the inverse of $\mathbf{A}$, i.e. $\mathbf{B} = \mathbf{A}^{-1}$

Faster method to compute inverse : Row operations

- Exchange any 2 rows ($R_i \leftrightarrow R_j$)
- Multiply a row with constant ($R_i \leftarrow kR_i$)
- Add a multiple of a row to another ($R_i \leftarrow R_i + kR_j$)

E.g. Find inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

① Joining with identity matrix on RHS (Use $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, because 2x2)

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

② Use row operations to turn LHS to be identity matrix

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ \underline{\underline{3}} & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3-3\times 1 & 4-3\times 2 & 0-3\times 1 & 1-3\times 0 \end{array}\right)$$

turn this to 0

$$\left(\begin{array}{cc|cc} 1 & \underline{\underline{2}} & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|cc} 1+0 & 2+(-2) & 1+(-3) & 0+1 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

turn this to 0

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & \underline{\underline{-2}} & -3 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow \frac{-1}{2} R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 \times \frac{-1}{2} & -2 \times \frac{-1}{2} & -3 \times \frac{-1}{2} & 1 \times \frac{-1}{2} \end{array}\right)$$

turn this to 1

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right)$$

identity

<u>This is the inverse</u>

Checking

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} 1\times -2 + 2\times \frac{3}{2} & 1\times 1 + 2\times \frac{-1}{2} \\ 3\times -2 + 4\times \frac{3}{2} & 3\times 1 + 4\times \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

☆ Not all matrices have inverse

E.g. $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ \underline{\underline{0}} & \underline{\underline{0}} & 0 & 1 \end{array}\right)$

it is impossible to turn these entries to (0 1)

Matrices without inverse are called <u>singular</u>

Determinant

Notation : $\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

Use straight lines instead of parenthesis

☆ We can only calculate determinant for square matrices

It is a special operation ($\sim$ function) on matrices that returns a number

① For 2×2 & 3×3 matrices, the computations have a convenient trick (Rule of Sarrus)

$$2\times 2 : \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +(a_{11}\cdot a_{22}) - (a_{21}\ a_{12})$$

$$3\times 3 : \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

copy

$$= +(a_{11}\cdot a_{22}\cdot a_{33}) - (a_{12}\cdot a_{23}\cdot a_{31}) + (a_{13}\cdot a_{2}\cdot a_{32})$$

$$-(a_{31}\cdot a_{22}\cdot a_{13}) - (a_{32}\cdot a_{23}\cdot a_{11}) \quad (a_{33}\cdot a_{21}\cdot a_{12})$$

② The above trick does not work with 4x4 or above

One must compute $\det(\mathbf{A})$ by the general way : minor & cofactor

$$\text{Let } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

Intermediate steps required :

Minor of a_{ij} - Remove the i^{th} row & j^{th} column

then take the determinant of the remaining matrix

Cofactor of a_{ij} $=$ (Minor of a_{ij}) $\times (-1)^{i+j}$ $=$ $cof(a_{ij})$

Remove j^{th} column

$$cof(a_{ij}) = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

Remove i^{th} row

☆☆ Need to find determinant of this smaller matrix

Then, det (**A**) can be computed by either

$$\sum a_{ij} \, cof(a_{ij}) \quad \text{or} \quad \sum a_{ij} \, cof(a_{ij})$$

along any row, sum all j — along any column, sum all i

(These formulae work for 2x2 & 3x3 too, but much more annoying)

E.g

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We can choose any row/column to start with

(E.g we can choose the row/column with the most 0)

In this example. I pick the 1^{st} row

For $a_{11} = \boxed{1}$, $i=1$ $j=1$

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightarrow \text{Cofactor} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3$$

For $a_{12} = \boxed{2}$, $i = 1, j = 2$

$$\begin{pmatrix} 1 & \boxed{2} & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow \text{Cofactor} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 6$$

For $a_{13} = \boxed{3}$, $i = 1, j = 3$

$$\begin{pmatrix} 1 & 2 & \boxed{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow \text{Cofactor} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$$

$$\Rightarrow \text{Determinant} = \boxed{1} \times -3 + \boxed{2} \times 6 + \boxed{3} \times -3 = 0$$

Supplementary : Some useful properties/formula

1. If any 2 rows / any 2 columns of a matrix are equal

   its determinant $= 0$

2. If we change any colum j from $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$ to $\begin{pmatrix} k\, a_{1j} + b_{1j} \\ k\, a_{2j} + b_{2j} \\ \vdots \\ k\, a_{nj} + b_{nj} \end{pmatrix}$

   determinant becomes $k \cdot \begin{vmatrix} & a_{1j} & \\ \cdots & a_{2j} & \cdots \\ & \vdots & \\ & a_{nj} & \end{vmatrix} + \begin{vmatrix} & b_{1j} & \\ \cdots & b_{2j} & \cdots \\ & \vdots & \\ & b_{nj} & \end{vmatrix}$

   with all other columns the same as in the original matrix

   This also applies to any row.

3. Interchange any 2 rows / any 2 column $\Rightarrow$ $\det(\mathbf{A}) \to -\det(\mathbf{A})$

4. $\det(\mathbf{A}^T) = \det(\mathbf{A})$

5. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$. And so $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$

Theory of System of Linear Equation

We can always rewrite a system of linear equation into matrix form

$$\begin{cases} \quad\;\; 2y - z = 1 \\ x - y + z = 0 \\ 2x + y - z = -2 \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}_{\mathbf{b}}$$

Note that if $\mathbf{A}$ has inverse $\mathbf{A}^{-1}$, then we can do

$$\underbrace{\mathbf{A}^{-1}(\mathbf{A}}_{=\text{ identity}}\mathbf{x}) = \mathbf{A}^{-1}\mathbf{b}$$

(matrix) × (its inverse) = identity

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Anything × (identity) = no change

So if $\mathbf{A}$ has inverse, the system of equation can be solved just by finding $\mathbf{A}^{-1}$ and then multiply it to $\mathbf{b}$

Problem : Some matrices do not have inverse

We first need to check if $\mathbf{A}^{-1}$ exists. How to check?

Theorem without proof $\boxed{1}$

From the inverse formula that nobody uses :

$$\mathbf{A}^{-1} = \frac{[\operatorname{cof}(a_{ij})]^T}{\det(\mathbf{A})}$$

$$\Rightarrow \text{If } \det(\mathbf{A}) = 0, \; \mathbf{A}^{-1} \text{ will not exist}$$

Theorem without proof $\boxed{2}$

If $\mathbf{A}^{-1}$ exists, the system of linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has an $\underbrace{\text{unique}}_{\text{one \& only one}}$ solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

E.g. $\underbrace{\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}_{\mathbf{b}}$

Compute $\det(\mathbf{A})$:

$$\begin{vmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & -1 \\ 2 & 1 \end{matrix} = +0 + (2\times1\times2) + (-1\times1\times1) - (2\times-1\times-1) - 0 - (-1\times1\times2)$$

$$= 3 \neq 0$$

From the above theorems we can tell that the soln. is unique

Then we can find $\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 4 & -2 \end{pmatrix}$ (skipping the steps)

And so the unique solution is

$$\mathbf{A}^{-1}\mathbf{b} = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -2 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore \quad \left\{ x = \frac{-2}{3},\ y = \frac{5}{3},\ z = \frac{7}{3} \right\}$$

☆☆☆ However this is never the recommended method in solving linear equations. You can solve it much faster by row operations / substitution.

Homogeneity of Linear Equation

Similar to ODE, if all terms in the equation contain the unknown, the equation is homogeneous. Otherwise non-homogeneous

- Homogeneous $\mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{0}}$ — No other term other than Ax

  the unknown

- Non-homogeneous $\mathbf{A}\mathbf{x} = \underline{\mathbf{b}}$ — A term that does not contain x

Case ① : Homogeneous

Obviously $\mathbf{x} = \mathbf{0}$ is a solution $\Rightarrow$ the "trivial solution"

(Anything multiply $0 = 0$)

$\boxed{1}$ If $\det(\mathbf{A}) \neq 0 \Rightarrow \mathbf{A}^{-1}$ exists

$\Rightarrow$ Solution must be unique

$\Rightarrow$ $\mathbf{x} = \mathbf{0}$ is the only solution

$\boxed{2}$ If $\det(\mathbf{A}) = 0 \Rightarrow \mathbf{A}^{-1}$ does not exist

$\Rightarrow$ $\mathbf{x}$ has multiple solutions

E.g. $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x + y = 0 \\ 0 + 0 = 0 \end{cases}$

then we can choose x to be anything (and $y = -x$)

$\Rightarrow$ Infinitely many solutions

Case ② : Non-homogeneous

Just like in non-homogeneous ODE, the solution can be separated into 2 parts :

$$\mathbf{x} = \underbrace{\mathbf{x}_c}_{\text{complementary}} + \underbrace{\mathbf{x}_p}_{\text{particular}}$$

Complementary soln. $\mathbf{x}_c$ = Soln. to its homogeneous counterpart

= Soln. when $A\mathbf{x} = \mathbf{0}$

Particular soln. $\mathbf{x}_p$ = A special soln. for cancelling the non-homogeneous term $\mathbf{b}$

1 If $\det(A) \neq 0$

- The only soln. to $A\mathbf{x}_c = \mathbf{0}$ is $\mathbf{x}_c = \mathbf{0}$
- If $\mathbf{x}_p = A^{-1}\mathbf{b}$ then $A\mathbf{x}_p = \mathbf{b}$

$\Rightarrow$ The unique solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = A^{-1}\mathbf{b}$

2 If $\det(A) = 0$

- $\mathbf{x}_c$ has infinitely many solutions
- $\mathbf{x}_p$ may or may not exist. Require actual solving.

$\Rightarrow$
- $\mathbf{x}_p$ exist $\Rightarrow$ $\mathbf{x} = \mathbf{x}_c$ (infinitely many) + $\mathbf{x}_p$ (exist) = Infinitely many
- $\mathbf{x}_p$ not exist $\Rightarrow$ $\mathbf{x} = \mathbf{x}_c$ (infinitely many) + $\mathbf{x}_p$ (not exist) = Not exist

E.g. 1

$$\begin{cases} x + 2y + 3z = 0 \\ 2x + 3y + 8z = 0 \\ 3x + 2y + 17z = \underline{0} \end{cases} \iff \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 3 & 2 & 17 \end{pmatrix}} \underset{\mathbf{x}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{\mathbf{b}}{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

homogeneous

$\vdots$

Check if $\det(\mathbf{A}) = 0 \rightarrow$ Yes. (skip the steps here)

$\therefore$ Soln. must be infinitely many

Then we shall solve it exactly by row operations

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 8 & 0 \\ 3 & 2 & 17 & 0 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -4 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow -R_2]{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Transforming back to $x/y/z$, reads as

$$\begin{cases} x + 2y + 3z = 0 \\ y - 2z = 0 \end{cases} \Rightarrow \begin{cases} x = -2y - 3z = -7z \\ y = 2z \end{cases}$$

Therefore the soln. is $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7z \\ 2z \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} \underline{\underline{z}}$

where z can be any constant

E.g. 2

$$\begin{cases} x + 2y + 3z = 3 \\ 2x + 3y + 8z = 4 \\ 3x + 2y + 17z = \underline{1} \end{cases} \iff \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 3 & 2 & 17 \end{pmatrix}} \underset{\mathbf{x}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{\mathbf{b}}{\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}}$$

non-homogeneous

$\mathbf{A}$ is the same as in last example, but $\mathbf{b}$ is non-zero.

$\therefore$ Soln. must be either infinitely many or not exist

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 3 & 2 & 17 & 1 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -4 & 8 & -8 \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow -R_2]{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Transforming back to $x/y/z$, reads as

$$\begin{cases} x + 2y + 3z = 3 \\ \quad\; y - 2z = 2 \end{cases} \Rightarrow \begin{cases} x = -2y - 3z + 3 = -7z - 1 \\ y = 2z + 2 \end{cases}$$

Therefore the soln. is $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7z - 1 \\ 2z + 2 \\ z \end{pmatrix}$

z can be any constant

$$= \underbrace{\begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} z}_{\text{solution when } b = 0 \text{, i.e. the complementary soln.}} + \underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{\text{particular soln.}}$$

E.g.3

$$\begin{cases} x + 2y + 3z = 3 \\ 2x + 3y + 8z = 4 \\ 3x + 2y + 17z = 2 \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 3 & 2 & 17 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}}_{\mathbf{b}}$$

non-homogeneous

$\mathbf{A}$ is the same as in last example, but different $\mathbf{b}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 3 & 2 & 17 & 2 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -4 & 8 & -7 \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow -R_2]{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The last row reads as $0x + 0y + 0z = 1$. So no solution.

Eigenvalue Problem

Given a matrix $\mathbf{A}$. Find any non-zero column matrix $\mathbf{x}$

and number λ satisfying $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$

Then $\begin{cases} \mathbf{x} = \text{Eigenvector of } \mathbf{A} \\ \lambda = \text{Eigenvalue of } \mathbf{A} \text{ corresponding to } \mathbf{x} \end{cases}$

Theorem: λ is an eigenvector of $\mathbf{A} \Leftrightarrow \det(\mathbf{A} - \lambda \mathbf{I}) = 0$

Proof:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} = \lambda \mathbf{I}\mathbf{x}$$

$$\mathbf{A}\mathbf{x} - \lambda \mathbf{I}\mathbf{x} = 0$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

Note that this is a homogeneous system of linear equations

$\boxed{1}$ If $\det(\mathbf{A} - \lambda \mathbf{I}) \neq 0$

$\Leftrightarrow$ The only soln. is $\mathbf{x} = \mathbf{0}$ (reject)

$\boxed{2}$ If $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$\Leftrightarrow$ $\mathbf{x}$ has infinitely many soln.

There will always be infinitely many $\mathbf{x}$ satisfying $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$

because if $\mathbf{x}$ is an eigenvector, $k\mathbf{x}$ is also

an eigenvector, for any constant k.

$$\mathbf{A}(k\mathbf{x}) = k(\mathbf{A}\mathbf{x}) = k(\lambda \mathbf{x}) = \lambda(k\mathbf{x})$$

Steps for solving Eigenvalue Problem

① Solve the equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$ to get some λ

② For each λ, subst. back into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ to solve $\mathbf{x}$

E.g. $\mathbf{A}=\begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}$

① Solve $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\left[\begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]$$

$$= \begin{vmatrix}1-\lambda & 1\\ 4 & 1-\lambda\end{vmatrix}$$

$$= +(1-\lambda)(1-\lambda) - (4)(1)$$

$$= \lambda^2 - 2\lambda - 3 \quad \text{require} = 0$$

$$\Rightarrow \lambda = 3 \text{ or } -1 \quad (2 \text{ eigenvalues})$$

② Subst. back each λ into $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ to solve $\mathbf{x}$

$\boxed{\text{I}}$ For $\lambda = 3$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \left[\begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix} - 3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$= \begin{pmatrix}-2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Read as $\begin{cases}-2x_1 + x_2 = 0\\ 4x_1 - 2x_2 = 0\end{cases}$

2^{nd} Eq. is simply (1^{st} Eq.) $\times (-2)$

$$\rightarrow x_2 = 2x_1$$

x_1 can be any constant

$$\therefore \text{General soln.} \quad \mathbf{x} = \begin{pmatrix}x_1\\ 2x_1\end{pmatrix} = \begin{pmatrix}1\\ 2\end{pmatrix}x_1$$

$\boxed{2}$ For $\lambda = -1$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Read as $\begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{cases}$ 2^{nd} Eq. is simply $(1^{st}$ Eq.$) \times (2)$

$$\Rightarrow \quad x_2 = -2x_1$$

x_1 can be any constant

$\therefore$ General soln : $\mathbf{x} = \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \underline{\underline{x_1}}$

Conclusion : $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ has 2 eigenvectors

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with eigenvalue $= 3$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ with eigenvalue $= -1$

Any multiples of these 2 column matrices are also acceptable answers

<u>Note 1</u> : Eigenvalues & Eigenvector can be complex no.

E.g. $\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} \lambda = \quad -1 \qquad 1-2i \qquad 1+2i \\ \mathbf{x} : \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1-i \\ -2i \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 2i \\ 1 \end{pmatrix} \end{cases}$

<u>Note 2</u> : An $n \times n$ matrix has at most n eigenvalues.

Because some matrices have repeated eigenvalues.

s & t can be any constants

E.g. $\begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \Rightarrow$ $\lambda = 3$ only and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} s + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t$

System of ODEs

Only deal with the case : Linear + Homogeneous + Constant Coefficients

Otherwise too difficult to solve by hand!

1st order Linear Homogeneous Constant Coefficient

E.g.
$$\begin{cases} \frac{d}{dt} x_1(t) = a\, x_1(t) + b\, x_2(t) \\ \frac{d}{dt} x_2(t) = c\, x_1(t) + d\, x_2(t) \end{cases}$$

① Rewrite into matrix form

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$$

② Use the trick $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$

k_1, k_2 are some constants

But here we let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 e^{\lambda t} \\ k_2 e^{\lambda t} \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$

Then $\frac{d}{dt}\mathbf{x} = \lambda \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$

③ Subst. back. Note that it becomes an eigenvalue problem.

$$\frac{d}{dt}\mathbf{x} = \lambda \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t} = \mathbf{A}\mathbf{x}$$

$$\lambda \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$\Rightarrow$ To the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ needs to be its eigenvector
- λ needs to be its eigenvalue

④ After solving for λ & $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, the general soln. is the superposition of all its eigenvector

$$\mathbf{x} = \underbrace{C_1}_{\text{constant}} \underbrace{\begin{pmatrix} k_{11} \\ k_{12} \end{pmatrix}}_{\text{eigenvector 1}} e^{\overbrace{\lambda_1}^{\text{eigenvalue 1}} t} + \underbrace{C_2}_{\text{constant}} \underbrace{\begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix}}_{\text{eigenvector 2}} e^{\overbrace{\lambda_2}^{\text{eigenvalue 2}} t}$$

For example, 2x2 matrix → at most 2 eigenvector
→ at most 2 terms in the soln.

E.g.

$$\begin{cases} \frac{d}{dt} x_1 = x_1 + x_2 \\ \frac{d}{dt} x_2 = 4x_1 + x_2 \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We have used this matrix in the last example and found

Eigenvalue : 3 $\quad -1$

Eigen vector : $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

So the general soln to this system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

C_1, C_2 are constants to be matched when given initial condition

EX

$$\begin{cases} \frac{d}{dt} x_1 = x_1 + x_2 - 2x_3 \\ \frac{d}{dt} x_2 = x_2 - 4x_3 \\ \frac{d}{dt} x_3 = 2x_1 - x_3 \end{cases}$$

Write down its general soln.

2^{nd} order Linear Homogeneous Constant Coefficient

E.g.
$$\begin{cases} \frac{d^2}{dt^2} x_1(t) = a \frac{d}{dt} x_1(t) + b \frac{d}{dt} x_2(t) + c x_1(t) + d x_2(t) \\ \frac{d^2}{dt^2} x_2(t) = p \frac{d}{dt} x_1(t) + q \frac{d}{dt} x_2(t) + r x_1(t) + s x_2(t) \end{cases}$$

We can convert it to a 1^{st} order system by letting

$$\frac{d}{dt} x_1(t) = u_1(t) \quad , \quad \frac{d}{dt} x_2(t) = u_2(t)$$

which then the system becomes

$\frac{d^2}{dt^2} x_1 = \frac{d}{dt} u_1$

$\frac{d^2}{dt^2} x_2 = \frac{d}{dt} u_2$

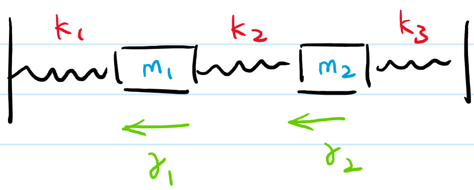
Definitions of u_1, u_2

$$\begin{cases} \frac{d}{dt} u_1(t) = a u_1(t) + b u_2(t) + c x_1(t) + d x_2(t) \\ \frac{d}{dt} u_2(t) = p u_1(t) + q u_2(t) + r x_1(t) + s x_2(t) \\ \frac{d}{dt} x_1(t) = u_1(t) \\ \frac{d}{dt} x_2(t) = u_2(t) \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ p & q & r & s \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix}$$

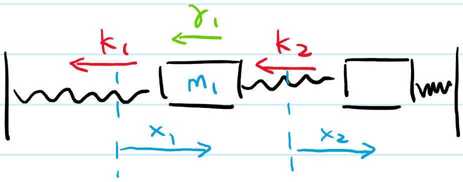
(Hand solving this is a nightmare)

Application : Coupled spring-mass system with damping



Let displacement of $m_1 = x_1(t)$, displacement of $m_2 = x_2(t)$

Newton 2^{nd} Law for m_1 :

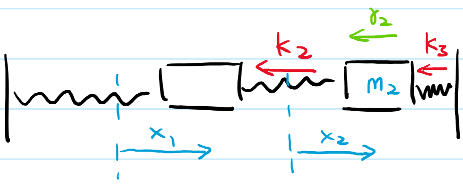


Left spring (k_1) change in length $= |x_1|$

Right spring (k_2) change in length $= |x_2 - x_1|$

$$\therefore \quad F = m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 - k_2(x_1 - x_2) - \gamma_1 \frac{d}{dt} x_1$$

Newton 2^{nd} Law for m_2 :



Left spring (k_2) change in length $= |x_2 - x_1|$

Right spring (k_3) change in length $= |x_2|$

$$\therefore \quad F = m_2 \frac{d^2 x_2}{dt^2} = -k_3 x_2 - k_2(x_2 - x_1) - \gamma_2 \frac{d}{dt} x_2$$

Combing the 2 equations and transform into a 1^{st} order system :

$$\begin{cases} \frac{d}{dt} u_1 = -\frac{\gamma_1}{m_1} u_1 - \frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2 \\ \frac{d}{dt} u_2 = -\frac{\gamma_2}{m_2} u_2 + \frac{k_2}{m_2} x_1 + \frac{k_2 + k_3}{m_2} x_2 \\ \frac{d}{dt} x_1 = u_1 \\ \frac{d}{dt} x_2 = u_2 \end{cases}$$

(Too annoying to solve)

Instead, here only demonstrates the results of a simplified case:

Suppose $k_1 = k_3$, $\gamma_1 = \gamma_2 = 0$, $m_1 = m_2 = m$

Then the Newton 2^{nd} Laws are just

$$\frac{d^2}{dt^2} x_1 = -\frac{k_1+k_2}{m} x_1 + \frac{k_2}{m} x_2$$

$$\frac{d}{dt^2} x_2 = \frac{k_2}{m} x_1 - \frac{k_1+k_2}{m} x_2$$

$$\Rightarrow \frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{k_1+k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_1+k_2}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We don't have to break it to 4 equations because

we can substitute $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\omega t}$ and get

$$\underbrace{\omega^2}_{\lambda} \underbrace{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}}_{x} \cancel{e^{\omega t}} = \underbrace{\begin{pmatrix} -\frac{k_1+k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_1+k_2}{m} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}}_{x} \cancel{e^{\omega t}}$$

is an eigenvalue problem, but with eigenvalue $= \omega^2$

On solving can find

$$\text{Eigenvalue} : \omega^2 \qquad -\frac{k_1}{m} \qquad -\frac{k_1+2k_2}{m}$$

$$\text{Eigenvector} : \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So the general soln is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k_1}{m}}t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\sqrt{\frac{k_1}{m}}t}$$

$$+ C_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{k_1+2k_2}{m}}t} + C_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{\frac{k_1+2k_2}{m}}t}$$

Physical Interpretation

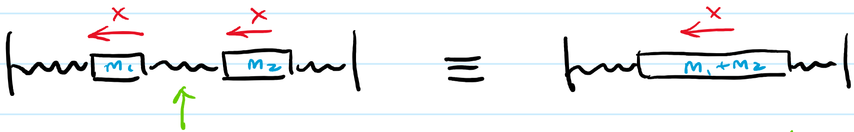
Note that the general soln. is a superposition of 2 kinds of vibration of different frequencies
(mode of vibration)

Motion ①

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k_1}{m}}t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\sqrt{\frac{k_1}{m}}t}$$

$$\sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} A\cos\left(\sqrt{\frac{k_1}{m}}t + \phi\right)$$

$\Rightarrow$ In this pattern of vibration, $x_1 = x_2 = A\cos\left(\sqrt{\frac{k}{m}}t + \phi\right)$



middle spring is not compressed/extended at all

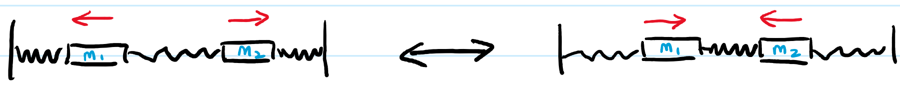
m_1, m_2 always move together $\Rightarrow$ Call this the "in phase" mode

Motion ②

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{k_1+2k_2}{m}}t} + C_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{\frac{k_1+2k_2}{m}}t}$$

$$\sim \begin{pmatrix} 1 \\ -1 \end{pmatrix} A\cos\left(\sqrt{\frac{k_1+2k_2}{m}}t + \phi\right)$$

$\Rightarrow$ In this pattern of vibration, $x_1 = -x_2 = A\cos\left(\sqrt{\frac{k_1+2k_2}{m}}t + \phi\right)$



m_1 m_2 are always moving in opposite direction

$\Rightarrow$ Call this the "out of phase" mode