Displacement Current

Physics Enrichment Program HKUST |
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- Definition of current & Charge conservation
- Fixing Ampere's Law – Displacement current
- Electromagnetic Induction as a complicated relation

Charge – Current Relation

In electromagnetism, charge is the real source of everything while current is simply moving charge.

Sometimes it is convenient to express everything only in terms of charge rather than mixing charge / current together. E.g. when solving dynamics of moving charged object

They are both charge (q) & current (I) source

But in the perspective of mechanics,

it is more intuitive to think as moving (q, v) charges.

We need the definition of current for inter-converting the current & moving charge perspective. The literal definition is

$$\boxed{\text{Current} = \text{Rate of charge flow through a surface}}$$

But across textbooks, you may have a lot of such formulae :

"Line" Definition :

$$\boxed{I = \frac{q}{t} \text{ (pre-calculus) } \quad \text{or} \quad I = \frac{dq}{dt}}$$

Consider the wire to be a line.

Does not even involve a "surface"

"Cylinder" Definition :

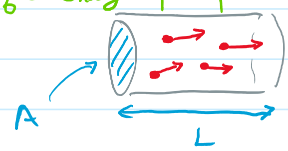
$$\boxed{I = nAqv = nAq \cdot \frac{dL}{dt}}$$

Consider the wire to be a cylinder

n = number of charge per volume

q = charge per particle

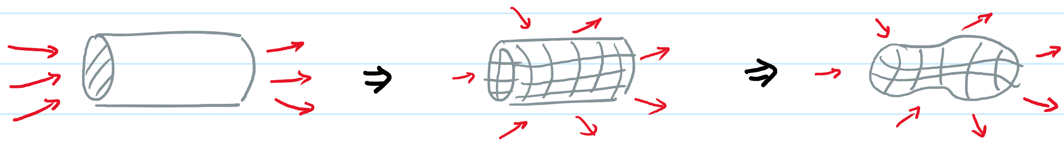
So that the "surface" come in consideration



Microscopic Definition :

$$\boxed{I = \iint n q \vec{v} \cdot d\vec{S}}$$

In general the "wire" can be of any shape, not just cylinder



The "through a surface" condition becomes a flux integral

And we can define current density through :

$$I = \iint n q \vec{v} \cdot d\vec{S} = \iint \underline{\underline{\rho}} \vec{v} \cdot d\vec{S}$$

$$\equiv \iint \underbrace{\vec{J}}_{\text{Current density}} \cdot d\vec{S}$$

nq = (number per volume) × (charge per particle)

~ charge per volume

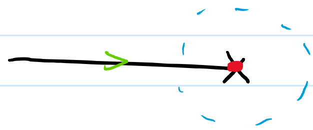
Note that $\boxed{\vec{J}(\vec{r}) = \rho(\vec{r})\, \vec{v}(\vec{r})}$ is a field distribution.

Charge conservation

The simplest expression of charge conservation is simply

$$I_{in} = \frac{d}{dt} Q_{enclosed}$$

current flow into the region — charge in the region



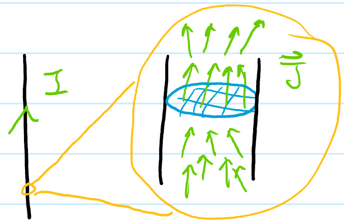
If we express this by the fancier microscopic definitions :

① Charges $\longrightarrow$ Charge density

$$Q = \iiint_V \rho(r)\, d\tau$$

② Current $\longrightarrow$ Current density

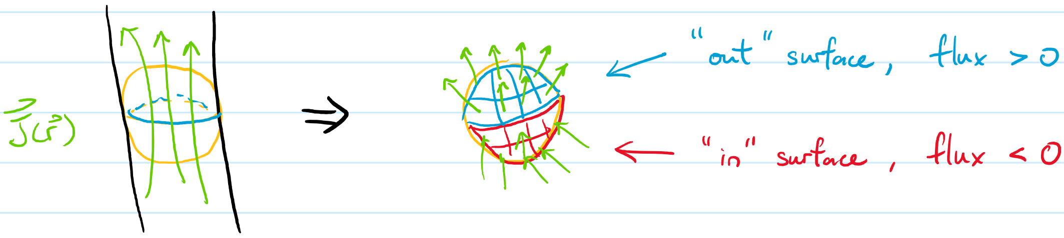
$$I = \iint_S \vec{J} \cdot d\vec{s}$$



Think of there is some current flowing through a region

We can divide the region's surface into 2 parts

by the sign of $\vec{J}$ flux.



By charge conservation, we require

$$[\text{Current in}] - [\text{Current out}] = \left[\begin{array}{c}\text{Rate of charge}\\ \text{Accumulation}\end{array}\right]$$

$$\iint_{\text{in surface}} \vec{J} \cdot d\vec{s} - \iint_{\text{out surface}} \vec{J} \cdot d\vec{s} = \frac{\partial}{\partial t} \iiint_{\text{The volume}} \rho \, d\tau$$

combine

$$-\oint\!\!\oint \vec{J}\cdot d\vec{S} = \frac{\partial}{\partial t}\iiint \rho\, d\tau$$

Surface of the volume

The volume

By convention, outward flux = +ve
but having outward current flux = lost of charge
So need to have this additional minus sign

☆☆☆ Note that this ρ - $\vec{J}$ relation is no more than the definition of current. It is just a fancier form and has taken the convention of outward current

$$-\oint\!\!\oint \vec{J}\cdot d\vec{S} = \frac{\partial}{\partial t}\iiint \rho\, d\tau \iff -\underline{\underline{I_{out}}} = \frac{d}{dt}Q_{enclosed}$$

enclosed charge in the region

We can also derive its differential form by Divergent theorem.

This is known as the continuity equation of charges

$$-\boxed{\oint\!\!\oint} \vec{J}\cdot d\vec{S} = -\boxed{\iiint \vec{\nabla}}\cdot\vec{J}\, d\tau = \frac{\partial}{\partial t}\iiint \rho\, d\tau$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\rho}$$

which is a PDE that one may use to find $\rho(\vec{r},t)$ / $\vec{J}(\vec{r},t)$
when one of them is given and you need to find another

Displacement Current

Ampere's Law is ambiguous in defining what means by "through a loop"

Intuitively, we may fill the loop with a surface

and claim : $\begin{cases} \text{Poke through the surface} = \checkmark \\ \text{Not Poke through the surface} = \times \end{cases}$

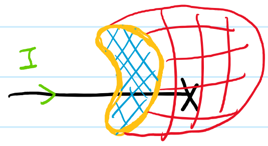
However, the choice of surface can be arbitrary

and create ambiguity at the endpoints of wire.

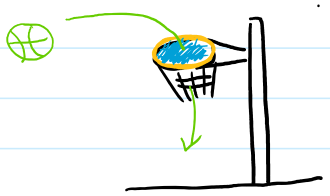
E.g. The wire poke through the blue surface but not the red surface.

Should we count this current as enclosed by the loop?

$$\oint_{loop} \vec{B} \cdot d\vec{\ell} = \mu_0 I \leftarrow ???$$



Simple analogy : Basketball through the ring

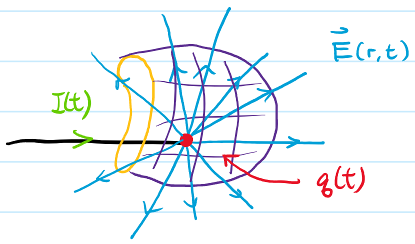


Choose this blue surface
⇒ Your throw counts

Choose this red surface
⇒ Your throw does not count

Solution :

- By charge conservation, a termination of current $I(t)$ will result in accumulating charges $q(t)$
- Charges emit E field $\vec{E}(r,t)$, and E field produce flux $\Phi_E(t)$ on the chosen surface



$\Rightarrow$ Add a term I_d that depends on $\Phi_E(t)$ in Ampere's Law

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 (I + I_d)$$

Such that

- When the wire poke through the chosen surface

$$I \neq 0 , \quad I_d = 0$$

- When the wire does not poke through the chosen surface

$$I = 0 , \quad I_d \neq 0$$

This term I_d is called <u>displacement current</u>, found to be

$$I_d = \varepsilon_0 \frac{\partial}{\partial t} \iint_S \vec{E}\cdot d\vec{s}$$

Integrate on the chosen surface

or in differential form

$$\vec{\nabla}\times\vec{B} = \mu_0 (\vec{J} + \vec{J}_d) , \quad \text{with} \quad \vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Derivation

The phenomenon is in fact a consequence of charge conservation

By substituting Gauss Law into continuity equation :

$$\begin{aligned} 0 &= \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\rho \qquad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \\ &= \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\varepsilon_0 \vec{\nabla}\cdot\vec{E}) \\ &= \vec{\nabla}\cdot\left(\vec{J} + \varepsilon_0 \frac{\partial}{\partial t}\vec{E} \right) \end{aligned}$$

Recall that if a vector field has zero divergence,

it can be expressed as the curl of another vector field

(Just like B field, $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\times\vec{A} = \vec{B}$)

This suggests us to modify the original Ampere's Law into

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\Big[\vec{J} + \underbrace{\varepsilon_0 \frac{\partial}{\partial t}\vec{E}}_{J_d}\Big]$$

or in integral form

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I \quad \Rightarrow \quad \oint \vec{B}\cdot d\vec{l} = \mu_0\Big[I + \underbrace{\varepsilon_0 \frac{\partial}{\partial t}\iint \vec{E}\cdot d\vec{S}}_{I_d}\Big]$$

Complication of Electro-magnetic induction

From Faraday's Law & Ampere's Law, we can see that

Time varying $\vec{E} / \vec{B}$ will induce each other

$$\begin{cases} \text{Faraday}: & \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \text{Time changing } \vec{B} \text{ create } \vec{E} \\ \text{Ampere}: & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} & \text{Time changing } \vec{E} \text{ create } \vec{B} \end{cases}$$

if we treat them as individual equation, we can end up in such loop:

Some initial $\vec{B}(t)$ $\longrightarrow$ Solve for an induced $\vec{E}(t)$ (Faraday)

$\swarrow$

Solve for an induced $\Delta\vec{B}(t)$ (Ampere) $\longrightarrow$ Solve for its induced $\Delta\vec{E}(t)$ (Faraday)

$\swarrow$

Solve for its induced $\Delta(\Delta\vec{B}(t))$ (Ampere) $\rightarrow$ ...

What can we do about this ?

Soln. 1 : Take approximation $\Delta B(t) \simeq 0$

i.e. Assume the induced $\vec{E}(t)$ does not induce additional B field

This approximation is OK because $\mu_0 \varepsilon_0 = \frac{1}{c^2} = \frac{1}{(3\times 10^8)^2}$

The size of displacement current $\mu_0 \varepsilon_0 \frac{\partial \vec{E}(t)}{\partial t}$ is usually too small

compare with the original current, and thus $\Delta B(t) \sim 0$

Not ideal. But good enough in many cases.

In fact most textbook problems are taking this approximation.

Soln 2 : Solve both of the equations together

i.e. Solving the system of Maxwell's Equation

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} & \text{(Gauss's Law)} \\ \vec{\nabla}\cdot\vec{B} = 0 & \text{(Gauss's Law of B)} \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} & \text{(Faraday's Law)} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} & \text{(Ampere's Law)} \end{cases}$$

A system of PDE ! Terrible !

But there are situations that you must do so. E.g.

- Plasma physics
- Neutron star / black hole

(i.e. When E/B too strong to be ignored)