

Newton 2nd Law is a differential equation

$$F(t) = m a(t) = m \frac{d^2}{dt^2} x(t)$$

Need to solve this equation of $x(t)$, which involves $x(t)$'s derivatives

E.g. SHM (Simple harmonic motion)

$$-kx = ma = m \frac{d^2}{dt^2} x \quad \text{has both } x \text{ and } \frac{d^2}{dt^2} x$$

How to solve this kind of equations? What other variants can we solve?

Classification of Differential Equations

① No. of Variables in the wanted function

- Single Variable \Rightarrow Ordinary Differential Equation (ODE)
- Multi-variable \Rightarrow Partial Differential Equation (PDE)
(not going to discuss here)

② Order = highest derivative in the equation

E.g. $\underline{\frac{df(t)}{dt}} + f(t) = \ln t$ highest = 1st derivative
 \Rightarrow 1st order

$$\underline{\frac{d^2}{dt^2} f(t) + f(t) \frac{df(t)}{dt}} = \sin t$$
 highest = 2nd derivative
 \Rightarrow 2nd order

$$\underline{\left(\frac{df(t)}{dt} \right)^2} + \left(f(t) \right)^2 = 1$$
 highest = 1st derivative
 \Rightarrow 1st order

③ Linearity = whether any terms contain multiplication between derivatives

$$\text{E.g. } \frac{d^2}{dt^2} f(t) - c^t \frac{d}{dt} f(t) + f(t) = 0 \Rightarrow \text{linear}$$

power 1 power 1 power 1

$$\frac{d^2}{dt^2} f(t) + \left(\frac{d}{dt} f(t) \right)^2 = \sin t \Rightarrow \text{non-linear}$$

power 1 power 2

$$\frac{f(t)}{\frac{d}{dt} f(t)} = 5 \Rightarrow \text{non linear}$$

power 1 power 1

④ Constant Coefficients = whether derivatives are multiplying with constants or function of t

$$\text{E.g. } \underline{2} \frac{d^2}{dt^2} f(t) - \underline{-1} \frac{d}{dt} f(t) + \underline{2} f(t) = \cos t$$

2 -1 2 are all constants

$$\underline{(t^2-1)} \frac{d^2}{dt^2} f(t) - \underline{-t} \frac{d}{dt} f(t) + \underline{2} f(t) = 0$$

t²-1 -t 2 some are functions of t

⑤ Homogeneity = whether all terms contain the function or its derivatives

$$\text{E.g. } \frac{d^2}{dt^2} f(t) + 2 \cos t f(t) = 0 \Rightarrow \text{Homogeneous}$$

Yes Yes

$$\frac{d}{dt} f(t) + 4 f(t) = \ln t \Rightarrow \text{Non-homogeneous}$$

Yes Yes No!

For SHM - problem, the Newton 2nd Law writes

$$F = ma = m \frac{d^2}{dt^2} x(t) = -k x(t)$$

is a 2nd Order Linear Constant Coefficient Homogeneous ODE

1st Order Linear Constant Coefficient Homogeneous ODE

This is the simplest kind of ODE

$$\frac{d}{dt} f(t) + \lambda f(t) = 0 \quad \lambda = \text{some constant}$$

Making use of the fact $\frac{d}{dt} e^{at} = a \cdot e^{at}$

which is exactly $\frac{d}{dt} f(t) - af(t) = 0$ with $f(t) = e^{at}$

$$\Rightarrow \text{Solution of ODE: } f(t) = C e^{-\lambda t}$$

$C = \text{any constant number}$

This is the only solution

On the other hand we can also solve it by integration:

$$\frac{1}{f(t)} \frac{df(t)}{dt} + \lambda = 0$$

$$\frac{d}{dt} [\ln f(t)] = -\lambda$$

$$\ln f(t) = \int -\lambda dt = -\lambda t + C$$

$$\begin{aligned} f(t) &= e^{-\lambda t + C} \\ &= C' e^{-\lambda t} \end{aligned} \quad \text{take } C' = e^C$$

(However basically no other ODE can be solved as simple as this)

E.g. Decay Equation

$$\frac{d}{dt} N(t) = -k N(t) \quad \text{some constant}$$

$N(t)$ = no. of particles

$\frac{d}{dt} N(t)$ = rate of decay in no. of particles \propto no. of particle remains

$$\Rightarrow \text{General soln: } N(t) = C e^{-kt}$$

We only know C is some constant. What value should we choose?

⇒ By matching a given initial condition

E.g. Given at $t = 0$, no. of particles = N_0

Then by substitution $N(0) = Ce^{-k \cdot 0} = C = N_0$

⇒ We arrive a specific soln $N(t) = N_0 e^{-kt}$

Side note: We can derive half life from this soln. form

i.e. At half life $t = \tau_{\frac{1}{2}}$, no. of particles remain

$$= \frac{N_0}{2} = \frac{1}{2} \text{ of original at } t = 0$$

$$\Rightarrow N(\tau_{\frac{1}{2}}) = N_0 e^{-k\tau_{\frac{1}{2}}} = \frac{N_0}{2}$$

$$\tau_{\frac{1}{2}} = \frac{\ln 2}{k}$$

What other ODEs can we solve analytically?

- Any linear ODEs

↳ Constant Coefficient

↳ Homogeneous → Just use e^{xt} trick

] we will
only discuss
these

↳ Non-homogeneous → Undetermined Coefficients

↳ Non-constant Coefficient → More complicate tricks

E.g. Integrating factor

Series expansion

Laplace/Fourier transform

- Non-linear ODE - No general method. Only case by case.

2nd Order Linear Constant Coefficient Homogeneous ODE

$$a \frac{d^2}{dt^2} f(t) + b \frac{d}{dt} f(t) + c f(t) = 0 \quad a, b, c \text{ are some constants}$$

E.g. Basic SHM $m \frac{d^2}{dt^2} x(t) + kx(t) = 0 \Rightarrow \text{identify } \begin{cases} a = m \\ b = 0 \\ c = k \end{cases}$

To solve this, we can use the $e^{\lambda t}$ trick

$$\Rightarrow a \frac{d^2}{dt^2} e^{\lambda t} + b \frac{d}{dt} e^{\lambda t} + c e^{\lambda t} = 0$$

$$a \lambda^2 e^{\lambda t} + b \lambda e^{\lambda t} + c e^{\lambda t} = 0$$

$a\lambda^2 + b\lambda + c = 0$ is a quadratic equation of λ

$$\Rightarrow \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{So we can take } f(t) = C_1 e^{\frac{-b + \sqrt{b^2 - 4ac}}{2a} t} \text{ or } C_2 e^{\frac{-b - \sqrt{b^2 - 4ac}}{2a} t}$$

However this is incomplete since for linear ODE

superpositions of any of its solutions are also its solution

i.e. If $f_1(t), f_2(t)$ are soln. $C_1 f_1(t) + C_2 f_2(t)$ is also

a soln., for any arbitrary constants C_1, C_2

Proof by substitution

Knowing $\begin{cases} a(t) \frac{d^2}{dt^2} f_1(t) + b(t) \frac{d}{dt} f_1(t) + c(t) f_1(t) = 0 \\ a(t) \frac{d^2}{dt^2} f_2(t) + b(t) \frac{d}{dt} f_2(t) + c(t) f_2(t) = 0 \end{cases}$

because $f_1(t), f_2(t)$ are soln. to the ODE. Then

$$a(t) \frac{d^2}{dt^2} [C_1 f_1(t) + C_2 f_2(t)] + b(t) \frac{d}{dt} [C_1 f_1(t) + C_2 f_2(t)] + c(t) [C_1 f_1(t) + C_2 f_2(t)]$$

$$= C_1 \cdot \left[a(t) \frac{d^2}{dt^2} f_1(t) + b(t) \frac{d}{dt} f_1(t) + c(t) f_1(t) \right] \\ + C_2 \cdot \left[a(t) \frac{d^2}{dt^2} f_2(t) + b(t) \frac{d}{dt} f_2(t) + c(t) f_2(t) \right]$$

$$= C_1 \cdot 0 + C_2 \cdot 0$$

$\therefore C_1 f_1(t) + C_2 f_2(t)$ is also a solution

This property can be easily extended to any linear N^{th} order ODE

① If a linear ODE is of N^{th} , there will be N independent

solutions : $f_1(t), f_2(t), \dots, f_N(t)$ (require rigorous
math proof)

② The general soln is any linear combination of these N solutions

$$f(t) = C_1 f_1(t) + C_2 f_2(t) + \dots + C_N f_N(t)$$

with C_1, C_2, \dots, C_N being some constant number

3 Sub-cases of the General Solution

We can derive the general solution further by the value of $b^2 - 4ac$

Case 1 : $b^2 - 4ac > 0$

Both $\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are real no.

Nothing can be done to further simplify

$$f(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t}$$

Case 2 : $b^2 - 4ac < 0$

Both $\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are complex no.

\Rightarrow We can separate the λ into their real/imaginary parts

$$\operatorname{Re}[\lambda_{\pm}] = -\frac{b}{2a} = p, \quad \operatorname{Im}[\lambda_{\pm}] = \pm \frac{\sqrt{4ac-b^2}}{2a} = \pm q$$

$$\Rightarrow \text{Re-label as } \lambda_{\pm} = p \pm iq$$

Then we can make use of the Euler formula

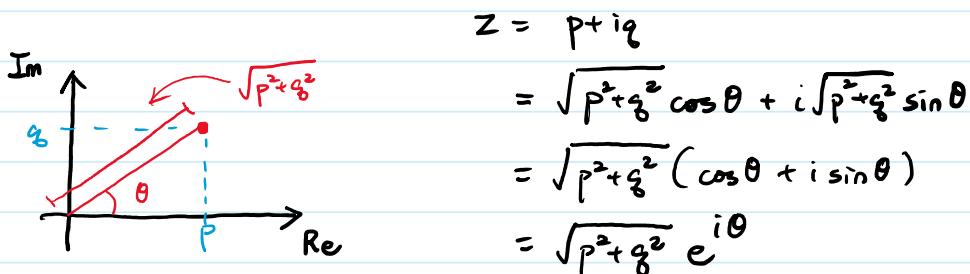
$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is just an extension of sin/cos function to complex inputs

It can be shown by Taylor expansion

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \end{aligned}$$

This allows any complex no. $p+iq$ can be expressed in polar form :



Rewriting $f(t)$ as

$$\begin{aligned} f(t) &= C_1 e^{(p+iq)t} + C_2 e^{(p-iq)t} \\ &= e^{pt} [C_1 e^{iqt} + C_2 e^{-iqt}] \\ &= e^{pt} [C_1 (\cos qt + i\sin qt) + C_2 (\cos qt - i\sin qt)] \\ &= e^{pt} [(C_1 + C_2) \cos qt + (C_1 - C_2) \sin qt] \\ &= e^{pt} [C'_1 \cos qt + C'_2 \sin qt] \end{aligned}$$

Now we return to an expression without the complex i

It can be further combined into 1 sinusoidal function by taking

$$\begin{cases} C_1' = A \cos \varphi \\ C_2' = -A \sin \varphi \end{cases} \Rightarrow \begin{cases} A = \sqrt{C_1'^2 + C_2'^2} \\ \varphi = \tan^{-1}\left(-\frac{C_2'}{C_1'}\right) \end{cases}$$

Such that

$$\begin{aligned} f(t) &= e^{pt} (A \cos \varphi \cos gt - A \sin \varphi \sin gt) \\ &= e^{pt} \cdot A \cos(gt + \varphi) \end{aligned}$$

Therefore it is frequent to see 3 kinds of expression

to the solution in the $b^2 - 4ac < 0$ case

$$f(t) = \begin{cases} C_1 e^{(p+ig)t} + C_2 e^{(p-ig)t} \\ e^{pt} [C_1' \cos gt + C_2' \sin gt] \\ e^{pt} \cdot A \cos(gt + \varphi) \end{cases}$$

Case 3: $b^2 - 4ac = 0$

This case is problematic because $\lambda_+ = \lambda_- = \frac{-b^2}{2a} = p$

\Rightarrow The general soln. only has 1 independent function $C e^{pt}$

But mathematician says if the ODE is of 2nd order,

there must be 2 independent functions in the general soln.

How to find the other function?

We use the method of Reduction of Order

i.e. let the other function be $v(t) e^{pt}$ and try to find $v(t)$

1 Subst. into original ODE

$$a \frac{d^2}{dt^2} [v(t) e^{pt}] + b \frac{d}{dt} [v(t) e^{pt}] + c [v(t) e^{pt}] = 0$$

2 Do product rule for each term

$$\frac{d^2}{dt^2} [v(t) e^{pt}] = \frac{d^2}{dt^2} v(t) e^{pt} + 2 \frac{d}{dt} v(t) \frac{d}{dt} e^{pt} + v(t) \frac{d^2}{dt^2} e^{pt}$$

$$= e^{pt} \left[\frac{d^2}{dt^2} v(t) + 2p \frac{d}{dt} v(t) + p^2 v(t) \right]$$

$$\frac{d}{dt} [v(t) e^{pt}] = \frac{d}{dt} v(t) e^{pt} + v(t) \frac{d}{dt} e^{pt}$$

$$= e^{pt} \left[\frac{d}{dt} v(t) + p v(t) \right]$$

Fast trick to do n^{th} derivative : Leibniz Formula

$$d(uv) = du \overset{1}{\cdot} v + u dv \overset{1}{\cdot}$$

$$d^2(uv) = d^2u \overset{1}{\cdot} v + 2dudv \overset{2}{\cdot} + udv \overset{1}{\cdot}$$

$$d^3(uv) = d^3u \overset{1}{\cdot} v + 3d^2u dv \overset{3}{\cdot} + 3du dv^2 \overset{3}{\cdot} + udv^3 \overset{1}{\cdot}$$

⋮

The coefficients follows binomial theorem.

$$d^n(uv) = \sum_{r=0}^n C_r^n (d^r u) (d^{n-r} v)$$

3 Group terms by derivatives of $v(t)$ and solve

$$a \frac{d^2}{dt^2} v(t) - \underline{(2ap+b)} \frac{d}{dt} v(t) + \underline{(ap^2+bp+c)} = 0$$

$\underset{=0}{}$

because $p = \frac{-b}{2a}$

$\underset{=0}{}$

because p is a soln to $ax^2+bx+c=0$

$$\Rightarrow \frac{d^2}{dt^2} v(t) = 0$$

$$\Rightarrow v(t) = C_1 t + C_2$$

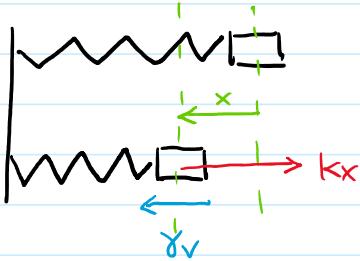
Therefore the other independent function is $(C_1 t + C_2) e^{pt}$

Note that it already contain the first function $C e^{pt}$

$\therefore \boxed{f(t) = C_1 e^{pt} + C_2 t e^{pt}}$

Application : SHM with damping

When a spring mass is compressed by



displacement x , it experiences :

- Spring's force : $-kx$ ($k > 0$)
- Damping force : $-\gamma v$ ($\gamma > 0$)

(Assume proportional to velocity
Otherwise difficult to calculate)

Newton 2nd Law writes as :

$$ma = -kx - \gamma v$$

$$\Rightarrow m \frac{d^2}{dt^2} x(t) + \gamma \frac{dx}{dt} x(t) + k x(t) = 0$$

$$x(t) \sim C e^{\lambda t} \text{ with } \lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

The 3 cases of $\gamma^2 - 4mk$ $\left\{ \begin{array}{l} > 0 \\ < 0 \\ = 0 \end{array} \right.$ correspond

to different physical behaviors

Case 1 : $\gamma^2 - 4mk > 0 \Leftrightarrow \lambda_{\pm} > 0 \Leftrightarrow$ Over-Damped

Check the sign of λ_{\pm} . Since

$$\gamma^2 > \gamma^2 - 4mk > 0 \quad (\text{m, k are +ve numbers})$$

$$\gamma > \sqrt{\gamma^2 - 4mk}$$

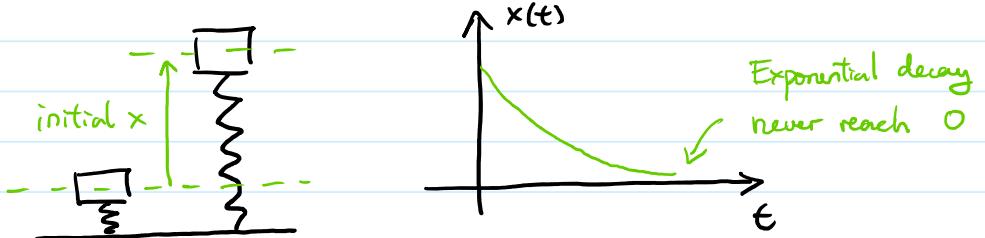
$$0 > \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} = \lambda_+$$

$$\text{Also } \lambda_- = \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} < \lambda_+$$

\therefore Both λ_+ & λ_- are negative no.

$$\Rightarrow x(t) = C_1 e^{-\lambda_+ t} + C_2 e^{-\lambda_- t}$$

= a sum of 2 exponentially decaying function



\Rightarrow "Over-damped" by a too large damping force

s.t. it cannot return to original position

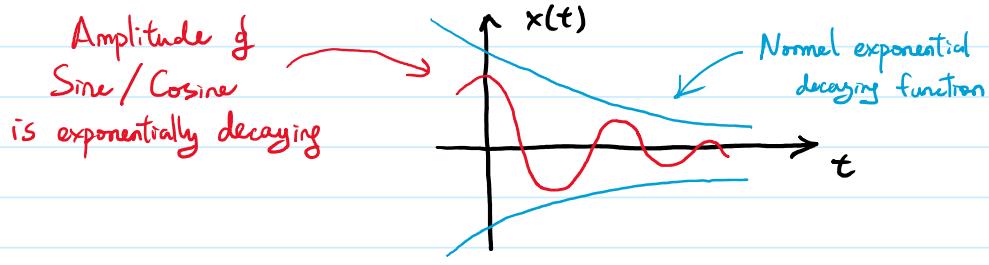
Case 2 : $\gamma^2 - 4mk < 0 \Leftrightarrow \lambda_{\pm}$ complex \Leftrightarrow Under-Damped

Write the soln. in the form

$$x(t) = e^{pt} \cdot A \cos(gt + \varphi)$$

$$= e^{-\frac{\gamma}{2m}t} \cdot A \cos\left(\frac{\sqrt{4mk - \gamma^2}}{2m}t + \varphi\right)$$

$$= (\text{Exponential Decay}) \times (\text{Sinusoidal})$$

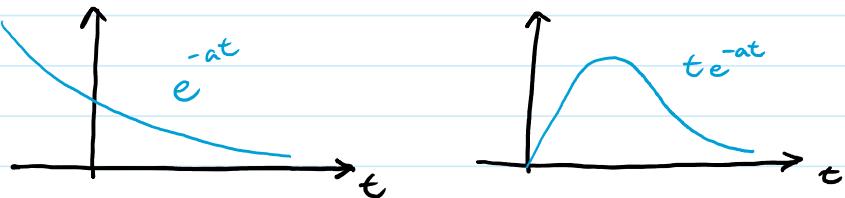


- "Under-damped" because the damping force is not strong enough to stop the vibration.
- In the special case $\gamma = 0$, returns to normal SHM

$$x(t) = A \cos(\sqrt{\frac{k}{m}}t + \varphi)$$

Case 3: $\gamma^2 - 4mk = 0$ $\Leftrightarrow \lambda_+ = \lambda_- \Leftrightarrow$ Critically-Damped

$$x(t) = C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t}$$

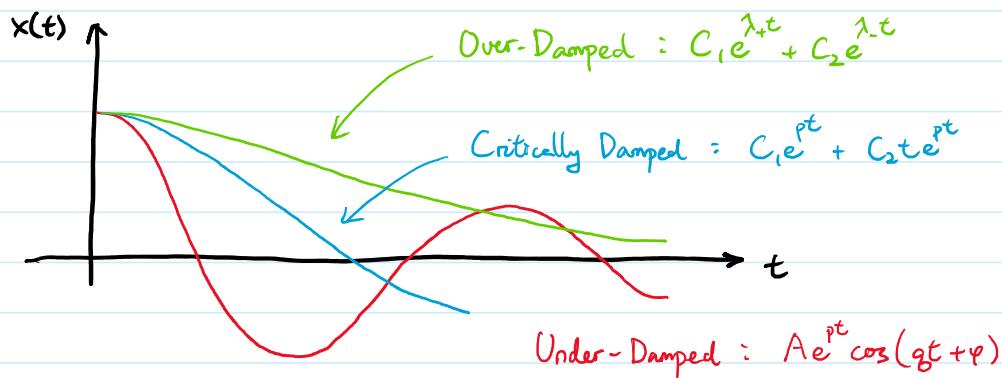


"Critical" because it is in between the other 2 cases

It looks like over-damped case but it can return

to original position at when $C_1 + C_2 t = 0$

Summary in 1 graph



2nd Order Linear Constant Coefficient Non-homogeneous ODE

$$a \frac{d^2}{dt^2} f(t) + b \frac{d}{dt} f(t) + c f(t) = \underline{g(t)}$$

non - homogeneous term

The general soln. to a non-homogeneous equation can be

broken into 2 parts : $f(t) = f_c(t) + f_p(t)$

- $f_p(t) = \underline{\text{Particular Solution}}$

= The term to cancel the non-homogeneous term

- $f_c(t) = \underline{\text{Complementary Solution}}$

= General soln to its homogeneous counterpart

$$\text{i.e. } a \frac{d^2}{dt^2} f_c(t) + b \frac{d}{dt} f_c(t) + c f_c(t) = 0$$

This can be shown simply by substitution :

$$\begin{aligned} & a \frac{d^2}{dt^2} [f_c(t) + f_p(t)] + b \frac{d}{dt} [f_c(t) + f_p(t)] + c [f_c(t) + f_p(t)] \\ &= \left[a \frac{d^2}{dt^2} f_c(t) + b \frac{d}{dt} f_c(t) + c f_c(t) \right] + \left[a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) \right] \end{aligned}$$

$f_c(t)$ is a solution when this = 0

We already learnt how to solve it

We need a "particular" $f_p(t)$ s.t.

this part = $g(t)$

An example of particular solution

A simpler mass-spring system under gravity can be described as

$$m \frac{d^2}{dt^2} x(t) = -k x(t) - mg$$



The simplest way to solve it is by grouping mg into $x(t)$

$$m \frac{d^2}{dt^2} \left[x(t) + \frac{mg}{k} \right] = -k \left[x(t) + \frac{mg}{k} \right]$$

$\uparrow \frac{mg}{k}$ is a constant. So $\frac{d}{dt} \left(\frac{mg}{k} \right) = 0$

Let $y(t) = x(t) + mg$. Then we can see that this ODE of $y(t)$ is the same as a spring-mass system without gravity

$$m \frac{d^2}{dt^2} y(t) = -ky(t)$$

We already know the soln is $y(t) = A \cos(\sqrt{\frac{k}{m}}t + \varphi)$. So

$$\begin{aligned} x(t) &= y(t) - \frac{mg}{k} \\ &= \underline{A \cos(\sqrt{\frac{k}{m}}t + \varphi)} - \underline{\frac{mg}{k}} \end{aligned}$$

The complementary soln $f_c(t)$
i.e. soln when without the term $-mg$

The particular soln $f_p(t)$
i.e. the term for canceling $-mg$ in the ODE

Method of Undetermined Coefficients

Finding $f_p(t)$ for *any* $g(t)$ is hard. But in applications $g(t)$ are usually common functions. In these cases we can smartly guess what $f_p(t)$ is made of.

Common functions & their derivatives

- Polynomial / Log :
$$\left\{ \begin{array}{ll} t^n \rightarrow t^{n-1} \rightarrow \dots \rightarrow t^2 \rightarrow t \rightarrow 1 & \text{+ve integer power} \\ \ln t \rightarrow t^{-1} \rightarrow t^{-2} \rightarrow \dots & \text{-ve integer power} \\ t^{\frac{1}{2}} \rightarrow t^{\frac{1}{2}} \rightarrow t^{-\frac{1}{2}} \rightarrow \dots & \text{non integer power} \end{array} \right.$$

- Trigonometric $\sin t \leftrightarrow \cos t, \sin 2t \leftrightarrow \cos 2t, \dots$
(especially \sin/\cos)
- Exponential $e^t, e^{2t}, e^{-t}, \dots$

Note that a combination of these functions will yield a set of its own derivatives.

$$\text{E.g. } t^2 \sin t \rightarrow \begin{cases} t \sin t & t^2 \cos t \\ \sin t & t \cos t \\ t^3 \sin t & \cos t \end{cases}$$

Idea of the method:

$g(t) = \text{Combination of } f_p(t) \text{ & its derivatives}$

$$\Rightarrow \left\{ \begin{array}{l} f_p(t) \text{ may contain } g(t) + \text{some other terms} \\ \frac{d}{dt} f_p(t) \text{ may contain } \frac{d}{dt} g(t) + \text{some other terms} \\ \frac{d^2}{dt^2} f_p(t) \text{ may contain } \frac{d^2}{dt^2} g(t) + \text{some other terms} \end{array} \right. \quad \vdots \quad \uparrow$$

\Rightarrow Try to find a combination that all these terms cancel out

\Rightarrow What are the "some other terms"? Obviously they

should be in terms of $\frac{d}{dt} g(t), \frac{d^2}{dt^2} g(t), \dots$

so that we can be sure that they will be cancelled.

Calculation Examples

($a, b, c = \text{some constants}$)

Let the ODE be $a \frac{d^2}{dt^2} f(t) + b \frac{d}{dt} f(t) + c f(t) = g(t)$

Here demonstrates how to find $f_p(t)$ for different $g(t)$

$$\underline{\text{E.g. 1}} \quad g(t) = t^2 + 2t$$

- ODE is of 2nd order, so $f_p(t)$ at most contains $\frac{d^2}{dt^2}g(t) \propto 1$

- Derivatives of t^2 already contain derivatives of t

\Rightarrow Guess $f_p(t) = \text{Some combination of } t^2, t, 1$

$$= At^2 + Bt + C$$

and then solve for A, B, C

$$a \times \left[\frac{d^2}{dt^2} [At^2 + Bt + C] \right] = [2A] \times a$$

$$b \times \left[\frac{d}{dt} [At^2 + Bt + C] \right] = [2At + B] \times b$$

$$+) c \times [At^2 + Bt + C] = [At^2 + Bt + C] \times c$$

$$a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) = (c \cdot A)t^2 + (c \cdot B + b \cdot 2A)t + (c \cdot C + b \cdot B + a \cdot 2A)$$

$$g(t) = t^2 + 2t + 0$$

By comparing coefficient, we can solve A, B, C by

$$\begin{cases} c \cdot A = 1 \\ c \cdot B + b \cdot 2A = 2 \\ c \cdot C + b \cdot B + a \cdot 2A = 0 \end{cases} \quad \text{3 Equations, 3 unknowns}$$

$$\underline{\text{E.g. 2}} \quad g(t) = \sin 2t$$

- Derivatives of $\sin 2t$ cycle between $\sin 2t$ & $\cos 2t$

\Rightarrow Guess $f_p(t) = \text{Some combinations of } \sin 2t \text{ & } \cos 2t$

$$= A \sin 2t + B \cos 2t$$

and then solve for A, B

$$a \times \left[\frac{d^2}{dt^2} [A \sin 2t + B \cos 2t] \right] = [-4A \sin 2t - 4B \cos 2t] \times a$$

$$b \times \left[\frac{d}{dt} [A \sin 2t + B \cos 2t] \right] = [2A \cos 2t - 2B \sin 2t] \times b$$

+) $c \times [A \sin 2t + B \cos 2t] = [A \sin 2t + B \cos 2t] \times c$

$$a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) = (-a \cdot 4A - b \cdot 2B + c \cdot A) \sin 2t + (-a \cdot 4B + b \cdot 2A + c \cdot B) \cos 2t$$

$$g(t) = \sin 2t + 0 \cdot \cos 2t$$

By comparing coefficients, we can solve A, B by

$$\begin{cases} -a \cdot 4A - b \cdot 2B + c \cdot A = 1 \\ -a \cdot 4B + b \cdot 2A + c \cdot B = 0 \end{cases} \quad 2 \text{ Equations } 2 \text{ unknowns.}$$

E.g. 3 Suppose the ODE is $\frac{d^2}{dt^2} f(t) - \lambda^2 f(t) = e^{\lambda t} = g(t)$

From the homogeneous part $\frac{d^2}{dt^2} f(t) - \lambda^2 f(t) = 0$

$$\text{we can tell } f_c(t) = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$$

So $g(t) = e^{\lambda t}$ is now contained in $f_c(t)$. How to find $f_p(t)$?

\Rightarrow Reduction of order again. Let $f_p(t) = v(t) e^{\lambda t}$

$$\Rightarrow \frac{d^2}{dt^2} [v(t) e^{\lambda t}] - \lambda^2 [v(t) e^{\lambda t}] = e^{\lambda t}$$

$$\frac{d^2}{dt^2} v(t) \cdot e^{\lambda t} + 2 \frac{d}{dt} v(t) \cdot \lambda e^{\lambda t} + v(t) \cdot \lambda^2 e^{\lambda t} - \lambda^2 v(t) e^{\lambda t} = e^{\lambda t}$$

$$\frac{d^2}{dt^2} v(t) + 2\lambda \frac{d}{dt} v(t) - 1 = 0$$

$$(\text{integrate 1 time}) \quad \frac{d}{dt} v(t) + 2\lambda v(t) = t + C$$

some constant

arriving at another inhomogeneous 1st order ODE. solving to get

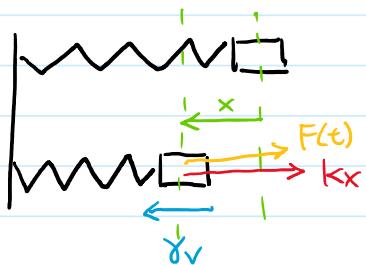
$$v(t) = \frac{t}{2\lambda} + \left(\frac{C}{2\lambda} - \frac{1}{4\lambda^2} \right) + D e^{-2\lambda t}$$

the true $f_p(t)$

these 2 are already in $f_c(t)$

$$\Rightarrow f_p(t) = v(t) e^{\lambda t} = \frac{t}{2\lambda} e^{\lambda t} + \left(\frac{C}{\lambda} - \frac{1}{\lambda^2} \right) e^{\lambda t} + D e^{-\lambda t}$$

Application : Forced SHM



Apply an external force to the spring

mass system, s.t. it experiences

- Spring's force : $-kx$ ($k > 0$)

- Damping force : $-\gamma v$ ($\gamma > 0$)

- External force : $F(t)$ (assume time dependent)

Newton 2nd Law written as :

$$ma = -kx - \gamma v + F(t)$$

$$\Rightarrow m \frac{d^2}{dt^2}x(t) + \gamma \frac{dx}{dt}x(t) + kx(t) = F(t)$$

is a non-homogeneous 2nd order ODE

We already know $f_c(t) = \begin{cases} C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ C_3 e^{pt} + C_4 t e^{pt} \\ A e^{pt} \cos(qt + \phi) \end{cases}$

And $f_p(t)$ depends on the form of $F(t)$. For example, let

$$F(t) = F_0 \cos \omega t$$

We can guess $f_p(t) = \text{Combination of } \sin \omega t / \cos \omega t$

$$= A \cos \omega t + B \sin \omega t$$

$$m \times \left[\frac{d^2}{dt^2} [A \cos \omega t + B \sin \omega t] \right] = [-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t] \times m$$

$$\gamma \times \left[\frac{d}{dt} [A \cos \omega t + B \sin \omega t] \right] = [-A\omega \sin \omega t + B\omega \cos \omega t] \times \gamma$$

$$+ k \times [A \cos \omega t + B \sin \omega t] = [A \cos \omega t + B \sin \omega t] \times k$$

... (skip the steps)

$$\Rightarrow \begin{cases} -A\omega^2m + B\gamma\omega + kA = F_0 \\ -B\omega^2m - A\gamma\omega + kB = 0 \end{cases}$$

Solving by substitution to get

$$A = \frac{F_0(m\omega^2 - k)}{(\gamma\omega)^2 + (m\omega^2 - k)^2}, \quad B = \frac{F_0\gamma\omega}{(\gamma\omega)^2 + (m\omega^2 - k)^2}$$

$$\begin{aligned} \therefore x_p(t) &= A \cos \omega t + B \sin \omega t \\ &= \sqrt{A^2 + B^2} \cos \left[\omega t + \tan^{-1} \left(\frac{B}{A} \right) \right] \\ &= \frac{F_0}{\sqrt{(\gamma\omega)^2 + (m\omega^2 - k)^2}} \cos \left[\omega t + \tan^{-1} \left(\frac{-\gamma\omega}{m\omega^2 - k} \right) \right] \end{aligned}$$

Special Case : When $\omega = \sqrt{\frac{k}{m}}$ = the "natural frequency"

$$\Rightarrow m\omega^2 - k = 0$$

$$\begin{aligned} \Rightarrow x_p &= \frac{F_0}{\gamma\omega} \cos \left[\omega t - \frac{\pi}{2} \right] \\ &= \frac{F_0}{\gamma} \sqrt{\frac{m}{k}} \sin \sqrt{\frac{k}{m}} t \end{aligned}$$

In particular if $\gamma \rightarrow 0$ (no damping force)

$$\text{Amplitude} = \frac{F_0}{\gamma} \sqrt{\frac{m}{k}} \rightarrow \infty$$

Amplitude will grow to infinitely large = Resonance