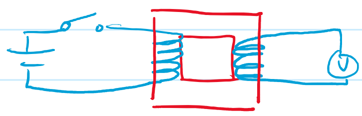
- Faraday's Law
  - Motional Emf
  - Transformer Emf
- "Fixing" Ampere's Law - Displacement Current

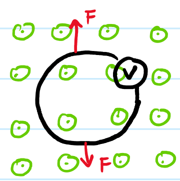
History of Faraday's Law

- **Faraday** : First induction Experiment (1831)

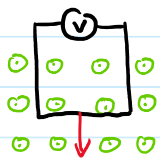


Reading appears at the instant the switch is closed / opened

Later in 1800s, discover more configurations that can create voltage reading



- Change shape of loop in B field



- Move a loop into space with B field

- **Lenz** : Lenz's Law (1834)

  Explain the direction of induced current by energy conservation

- **Maxwell** : Maxwell - Faraday's Equation

  Formulate discoveries of Faraday / Lenz into maths.

Methods of induced current / emf can be classified into 2 types

- Motional Emf – Can be explained by Lorentz force
- Transformer Emf – Can only be explained by relativity

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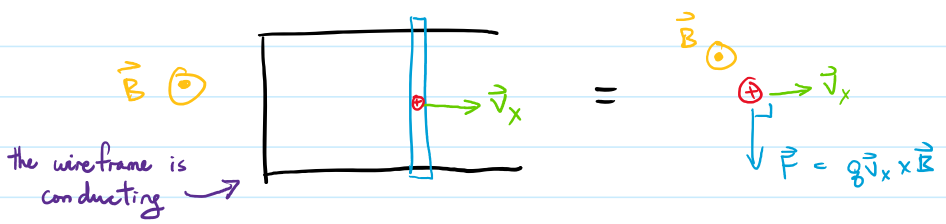
① Motional Emf

Model : A massless rod that carries a charge moving on B field

[1] Initially the rod move with horizontal velocity $\vec{v}_x$

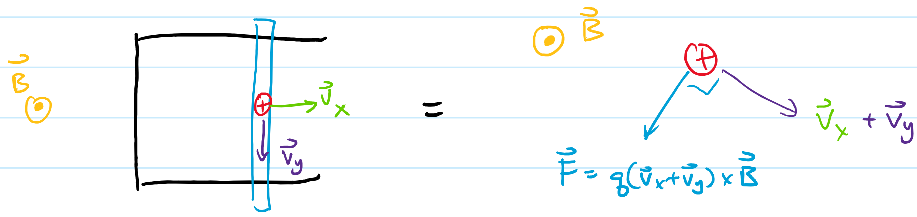


[2] Charge accelerate by $\vec{F}$ in the rod's direction

⇒ Get an additional vertical velocity $\vec{v}_y$ gradually

⇒ New velocity = $\vec{v}_x + \vec{v}_y$

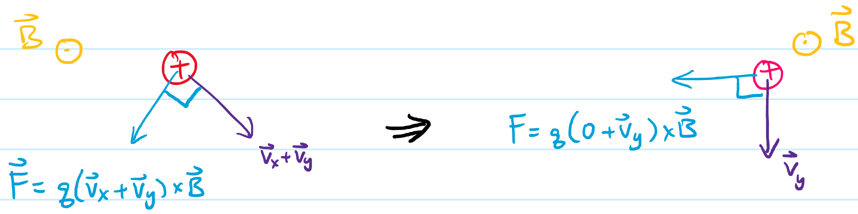
⇒ Net force change gradually to $q(\vec{v}_x + \vec{v}_y) \times \vec{B}$



[3] Observe the new $\vec{F}$

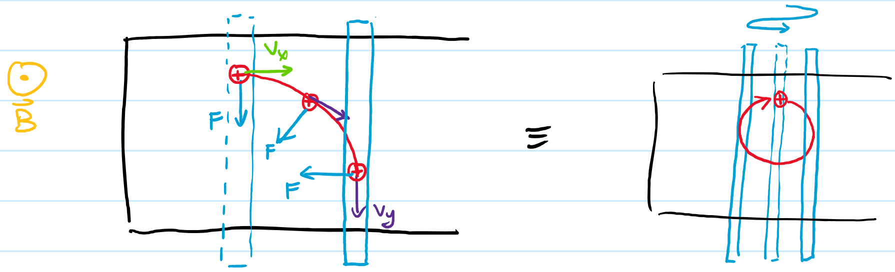
- Horizontal component is opposite to $\vec{v}_x$ ⇒ $\vec{v}_x \downarrow$
- Vertical component is along $\vec{v}_y$ ⇒ $\vec{v}_y \uparrow$

These changes continue until $\vec{v}_x$ disappear



☆ If there is only a single charge in this rod, it is the same as a free charge in B field

- Charge → Circular motion
- Rod → SHM (if restricted to move horizontally)



☆☆ But in reality, the rod would carry some resistence

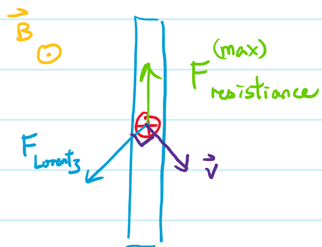
⇒ Velocity decreases by dissipating into heat

⇒ Motion stops when both v_x & v_y becomes 0

Question : How much energy can be dissipated ?

ANS : W.D. against resistive force from the rod

= W.D. by Lorentz force 's vertical component



From force diagram we can see that y component of Lorentz force will be cancelled by the resistive force and x component will keep decreasing $\vec{v}$

We can write this mathematically as :

$$\underline{q\varepsilon} = \underline{\vec{F}_{Lorentz} \cdot \vec{y}}$$

$F \cdot s$ = work done
take y for vertical direction

ε = Emf ~ voltage

$$= q(\vec{v} \times \vec{B}) \cdot \vec{y}$$

Vector identity
$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

$\Rightarrow q\varepsilon$ = energy stored by the charge

$$= q(\vec{y} \times \vec{v}) \cdot \vec{B}$$

= energy that can dissipated in a circuit

$$= -q(\vec{v} \times \vec{y}) \cdot \vec{B}$$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Note that in this expression, $\vec{v}$ & $\vec{B}$'s direction are not specified.

Only the rod is chosen in y direction.

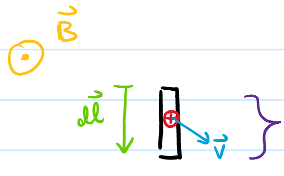
$\Rightarrow$ We can further the rod to be in arbituary direction :

$$q\varepsilon = -q\left[\vec{v} \times \underline{\vec{\ell}}\right] \cdot \vec{B}$$

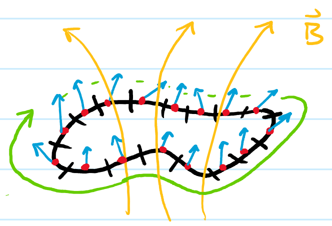
not restricted to y only
$\vec{\ell} = \ell_x \hat{x} + \ell_y \hat{y} + \ell_z \hat{z}$

This is the amount of energy the charge can dissipated if the charge has been "dragged" by Lorentz force to travel along the rod for distance $|\vec{\ell}|$.

Now imagine this "rod" is infinitestimal small of length $|d\vec{\ell}|$ and assemble many many of this rod into a loop :



$\Rightarrow$



An infinitestimal small rod dissipated energy per charge :

$$d\varepsilon = -(\vec{v} \times d\vec{\ell}) \cdot \vec{B}$$

Travel along a loop
$\Rightarrow$ Sum them up by loop integral

$$\oint d\varepsilon = -\oint (\vec{v} \times d\vec{\ell}) \cdot \vec{B}$$

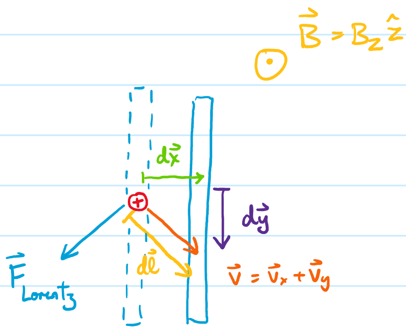
Note : It is still true that W.D. by Lorentz Force = 0

On above, we only considered work done by vertical component

However its horizontal component does negative work,

making the net charge in energy of the charge = 0

The total W.D. by horizontal component is



$$W.D. = \vec{F}_{lorentz} \cdot d\vec{\ell} = 0 \quad (\because \vec{F}_{lorentz} \perp d\vec{\ell})$$

$$= q\left[(\vec{v}_x + \vec{v}_y) \times \vec{B}\right] \cdot (d\vec{x} + d\vec{y})$$

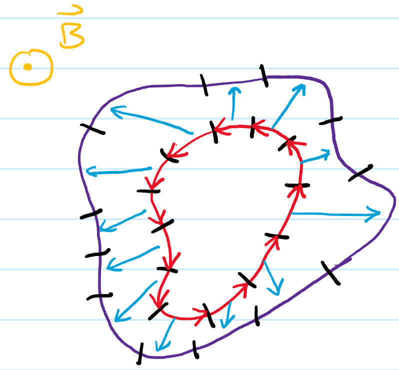
$$= q\,(\underbrace{\vec{v}_x \times \vec{B}}_{\substack{\text{pointing downward} \\ (+y \text{ direction})}}) \cdot d\vec{y} + q\,(\underbrace{\vec{v}_y \times \vec{B}}_{\substack{\text{pointing leftward} \\ (-x \text{ direction})}}) \cdot d\vec{x}$$

(W.D. by Lorentz force Move the charge along the rod to become current) + (W.D. against charge's rightward motion, slow it down in x directions)

And they are equal in magnitude so that energy is conserved

Geometric relation to flux

Consider a loop with changing shape in a static B field. Each segment $d\ell_i$ will move according to its velocity $\vec{v}_i$



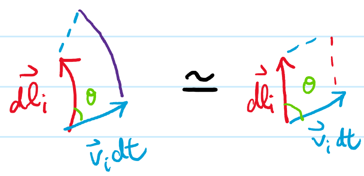
In a small time interval dt, the area swept by each segment can be approximated as

$$(\text{swept area})_i = d\begin{pmatrix}\text{area in}\\ \text{the loop}\end{pmatrix} \quad \text{change due to segment } i$$

$$\simeq |d\vec{l}_i| \cdot |\vec{v}_i \Delta t| \cdot \sin(\text{angle between } d\vec{l}_i \text{ \& } \vec{v}_i)$$

$$= (\vec{v}_i dt) \times d\vec{l}_i$$

$$\frac{d}{dt}\begin{pmatrix}\text{Area in}\\ \text{the loop}\end{pmatrix}_i = \vec{v}_i \times d\vec{l}_i$$



Compare with the expression of W.D. by Lorentz force

$$\mathcal{E} = -\sum_{\text{all segment}} \left[(\vec{v}_i \times d\vec{l}_i) \cdot \vec{B}\right] = -\sum_{\text{all segment}} \left[\frac{d(\text{Loop area})_i}{dt} \cdot \vec{B}\right]$$

$\vec{B}$ is independent of t
$\Rightarrow$ can put into $\frac{d}{dt}$

$$= -\sum_{\text{all segment}} \left[\frac{d}{dt}\left[(\text{Loop area})_i \cdot \vec{B}\right]\right]$$

$$= -\frac{d}{dt}\left[\sum_{\text{all segment}} \left[(\text{Loop area})_i \cdot \vec{B}\right]\right]$$

$$\Rightarrow \quad -\oint (\vec{v} \times d\vec{l}) \cdot \vec{B} = -\frac{d}{dt}\left[\iint_{\text{Loop area}} d\vec{S} \cdot \vec{B}\right]$$

$$= -\frac{d}{dt}\begin{pmatrix}\text{Magnetic flux}\\ \text{through the loop}\end{pmatrix}$$

☆☆ Note that since

- $\vec{B}$ is static. Not depending on t.
- $d\vec{S}$ is just a notation of saying this integral is about summing a lot of small area. Not depending on t.

The only thing that depends on t = Shape of the loop

= Range of the area integral

So to be more accurate, we can write the equation as

$$\mathcal{E} = -\oint_{Loop} (\vec{v} \times d\vec{l}) \cdot \vec{B} = -\frac{d}{dt} \iint_{\text{Loop area}(t)} d\vec{S} \cdot \vec{B}$$

Emphasize the time varying part is the integration range

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② Transformer Emf

Not explainable until birth of relativity

Maxwell's contribution = Unified Faraday's discoveries

Motional Emf ⇒ Generated when loop's shape change

Transformer Emf ⇒ Generated when B field change

⇒ Can expand the motional Emf formula like a product rule

$d(u \cdot v) = du \cdot v + u \cdot dv$

$$\mathcal{E} = -\frac{d}{dt}\left[\iint_{\text{Loop Area}(t)} d\vec{S} \cdot \vec{B}(t)\right]$$

$$\sim -\underbrace{\frac{d}{dt}\left[\iint_{\text{Loop Area}(t)}\right.}_{\frac{d}{dt}\text{ on integration range only}} \left. d\vec{S} \cdot \vec{B}(t_0)\right] - \iint_{\text{Loop Area}(t_0)} d\vec{S} \cdot \underbrace{\frac{d}{dt} \vec{B}(t)}_{\frac{d}{dt}\text{ on }\vec{B}\text{ only}}$$

keep the original form

= Motional Emf + Transformer Emf

③ Maxwell - Faraday Equation

By taking $\varepsilon = \oint \vec{E} \cdot d\vec{l}$ = Potential gain after travelling along the loop once

$$\Rightarrow \boxed{\oint_{\text{Loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{\substack{\text{Area in} \\ \text{the loop}}} \vec{B} \cdot d\vec{S}}$$

We can further write into differential form by Stoke's Theorem

$$\oint_L \vec{E} \cdot d\vec{l} = \iint_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S}$$

We can move the differentiation into the integral, then

then remove the integral sign

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{d}{dt} \vec{B}}$$

Lenz's Law

Faraday's Law, in particular the integral form, only talks about the magnitude of the emf, but not its direction.

$$\varepsilon = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S} \sim -\frac{d}{dt} [B(t) \cdot S(t)]$$

The minus is not useful for determining the direction of emf

We can use Lenz Law as a shortcut to determine the direction:

$\boxed{\text{Principle : Nature hates change of magnetic flux}}$

And you only need your right hand

Demonstration by examples

E.g. 1

increasing $\vec{B}$ in "out of paper" direction

$\boxed{0}$ B field increasing in magnitude

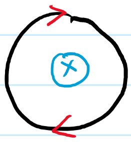
↳ B flux is "more out of paper"

$\boxed{1}$ Nature "hates" magnetic flux changing

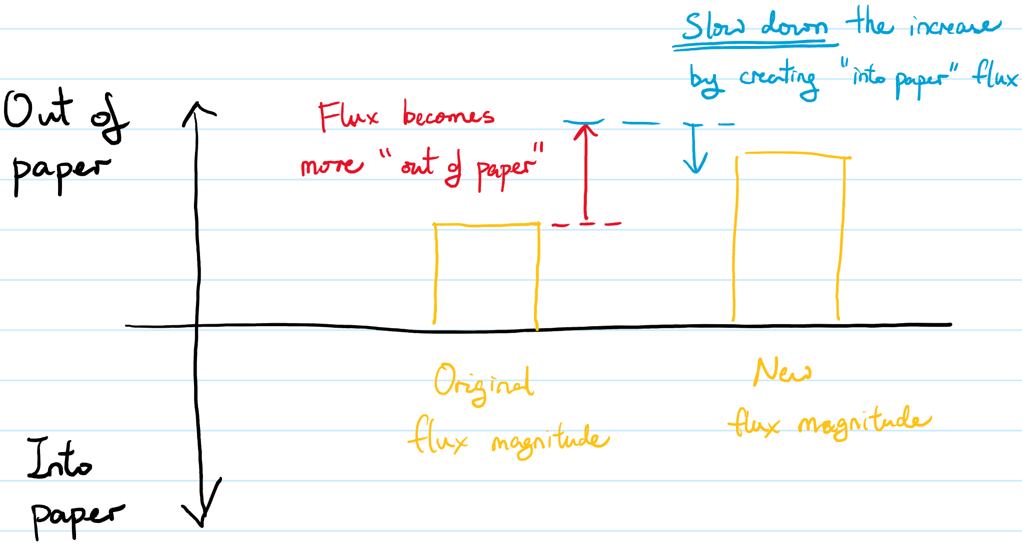
$\boxed{2}$ To oppose the "more out of paper" flux

Nature needs to create an "into paper" B field to compensate the increase

$\boxed{3}$ By your right hand, "into paper" B field can be generated if current flow clockwise



Clockwise current generate into paper B field, to counter the B flux change



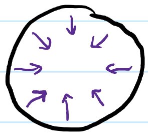
E.g.2

constant $\vec{B}$ into paper ⓧ

$\boxed{0}$ Loop 's area decrease under const. B field

↳ flux is " less into paper "

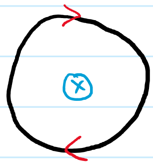
$\boxed{1}$ Nature " hates " magnetic flux changing

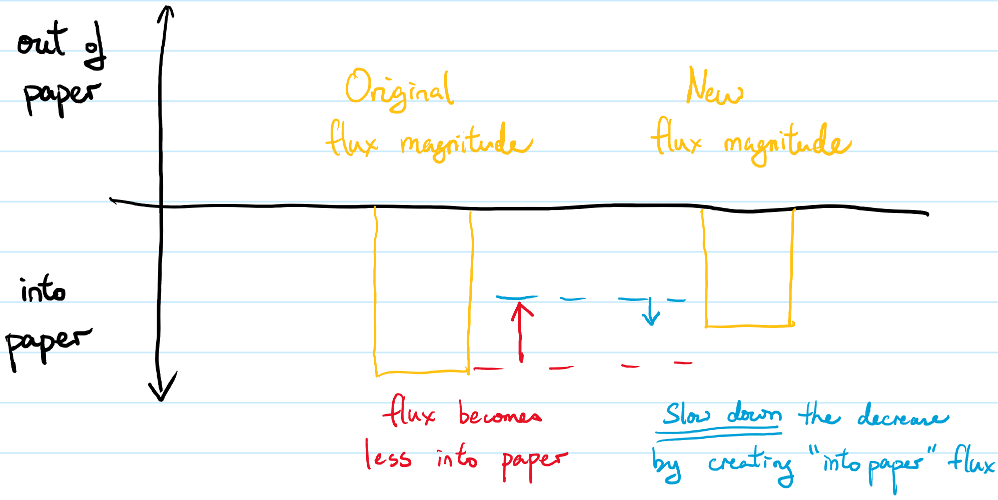


$\boxed{2}$ To counter the " less into paper " flux

Nature needs to create an " into paper " B field
to compensate the decrease

$\boxed{3}$ By your right hand , " into paper " B field
can be generated if current flow clockwise





Standard problems related to Faraday 's Law

① Finding emf with either $\vec{B}$ or area change

⇒ Direct calculation of the surface integral

$$\varepsilon = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{S}$$

if the B field is uniform over the area , this just reduces to multiplying the surface area

Use Lenz Law to find direction

$$\sim -\frac{d}{dt}\left[B(t)\cdot S(t)\right]$$

② Finding the induced $\vec{E}$ field in the space, given how $\vec{B}$ changes

⇒ Equivalently asking to solve the PDE $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Can we avoid that? Yes, but only in some symmetric cases

$\boxed{1}$ $\vec{E}$ is of the same magnitude on the chosen loop

$\boxed{2}$ $\vec{E}$ make the same angle with the line segment of the loop

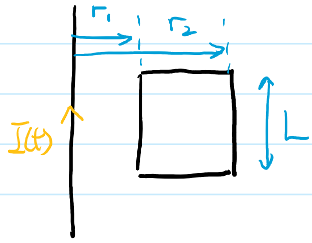
Then we can avoid solving the PDE by solving the integral form

$$-\frac{d}{dt}\iint \vec{B}\cdot d\vec{s} = \oint \vec{E}\cdot d\vec{l} = |\vec{E}|\cdot \cos\theta \cdot (\text{loop's perimeter})$$

E.g. A loop next to a infinitely long wire, with time varying current $I(t)$

Step 1: Find $\vec{B}(t)$ from Ampere's Law

$$\vec{B}(r,t) = \frac{\mu_0}{2\pi}\frac{I(t)}{r} \quad (\text{into paper})$$



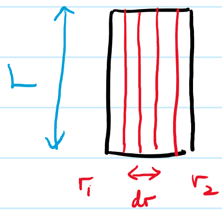
Step 2: Find magnetic flux through loop

$$\Phi_B = \iint \vec{B}\cdot d\vec{s}$$

$$= \int_{r_1}^{r_2} \frac{\mu_0 I(t)}{2\pi r}\cdot (L\,dr)$$

$$= \frac{\mu_0 I(t) L}{2\pi}\left[\ln(r_2) - \ln(r_1)\right]$$

B depends on r only
so it is constant on each strip



Step 3:

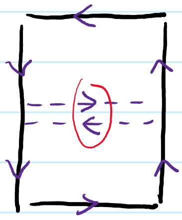
$\boxed{1}$ If only ask for emf → Just do differentiation.

$$\varepsilon = \frac{\mu_0 L}{2\pi r}\left(\frac{dI(t)}{dt}\right)\left[\ln(r_2) - \ln(r_1)\right]$$

In this problem, only I depends on t

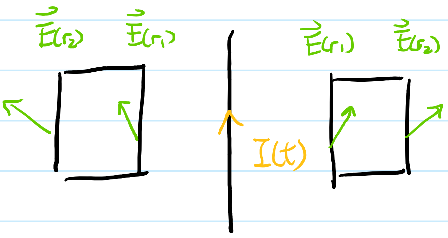
[2] If further ask for E field → Need symmetry claims

Cancelled in line integral



[1] Top / bottom wire 's $\vec{E}$ should be of same magnitude & direction

$\Rightarrow$ Contribution cancel in line integral



[2] Left / right wire 's $\vec{E}$ should depends on r only, due to cylindrical symmetry

So we have

$$\varepsilon = \frac{\mu_0}{2\pi r}\left(\frac{dI(t)}{dt}\right)\left[\ln(r_2) - \ln(r_1)\right]$$

$$= \oint \vec{E} \cdot d\vec{\ell}$$

$$= |E^{\parallel}(r_2)|\,L - |E^{\parallel}(r_1)|\,L + \left(\text{contribution of top/bottom edge} = 0\right)$$

where $E^{\parallel}$ = component of $\vec{E}$ parallel to the left/right edges

$\Rightarrow$ By comparing terms, we can claim

$$|E^{\parallel}(r,t)| = \frac{\mu_0}{2\pi}\left(\frac{dI(t)}{dt}\right) \cdot \frac{\ln(r)}{r}$$

The direction (up/down) can be determined by Lenz Law

Follow up : Does $\vec{E}$ only contain components parallel to left/right edges ?

Displacement Current

Ampere's Law is ambiguous in defining what means by "through a loop"

Intuitively, we may fill the loop with a surface

and claim : $\begin{cases} \text{Poke through the surface} = \checkmark \\ \text{Not Poke through the surface} = \times \end{cases}$

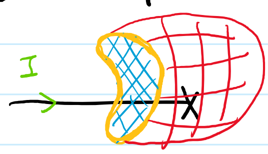
However, the choice of surface can be arbitrary

and create ambiguity at the endpoints of wire.

E.g. The wire poke through the blue surface but not the red surface.

Should we count this current as enclosed by the loop?

$$\oint_{loop} \vec{B}\cdot d\vec{\ell} = \mu_0 I \leftarrow ???$$



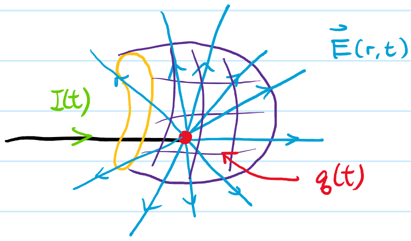
Simple analogy : Basketball through the ring

Choose this blue surface
⇒ Your throw counts

Choose this red surface
⇒ Your throw does not count

Solution :

- By charge conservation, a termination of current $I(t)$ will result in accumulating charges $q(t)$
- Charges emit E field $\vec{E}(r,t)$, and E field produce flux $\Phi_E(t)$ on the chosen surface



$\Rightarrow$ Add a term I_d that depends on $\Phi_E(t)$ in Ampere's Law

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 (I + I_d)$$

Such that

- When the wire poke through the chosen surface

$$I \neq 0 , \quad I_d = 0$$

- When the wire does not poke through the chosen surface

$$I = 0 , \quad I_d \neq 0$$

This term I_d is called <u>displacement current</u>, found to be

$$I_d = \varepsilon_0 \frac{\partial}{\partial t} \iint_S \vec{E}\cdot d\vec{s}$$

Integrate on the chosen surface

or in differential form

$$\vec{\nabla}\times\vec{B} = \mu_0 (\vec{J} + \vec{J}_d) , \quad \text{with } \vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The proof is in principle, charge conservation.

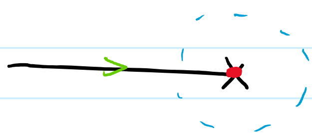
Charge conservation

The simplest expression of charge conservation is simply

$$I_{in} = \frac{d}{dt} Q_{enclosed}$$

current flow into the region

charge in the region



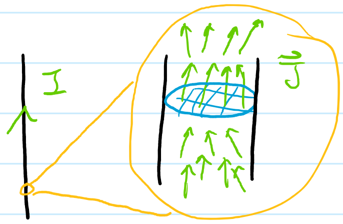
We can express this by the density terms.

(1) Charges $\rightarrow$ Charge density

$$Q = \iiint_V \rho(r)\, d\tau$$

(2) Current $\rightarrow$ Current density

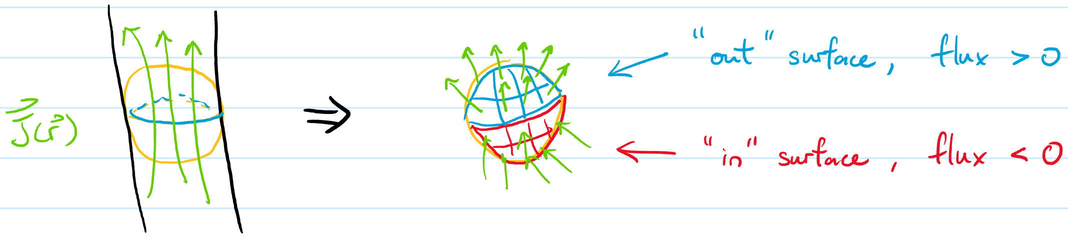
$$I = \iint_S \vec{J} \cdot d\vec{S}$$



Think of there is some current flowing through a region

We can divide the region's surface into 2 parts

by the sign of $\vec{J}$ flux.



By charge conservation, we require

[Current in] $-$ [Current out] $=$ [Rate of charge Accumulation]

$$\iint_{\text{in surface}} \vec{J} \cdot d\vec{S} - \iint_{\text{out surface}} \vec{J} \cdot d\vec{S} = \frac{\partial}{\partial t} \iiint_{\text{The volume}} \rho \, d\tau$$

combine

$$- \oint\!\!\oint_{\text{Surface of the volume}} \vec{J} \cdot d\vec{S} = \frac{\partial}{\partial t} \iiint_{\text{The volume}} \rho \, d\tau$$

By convention, outward flux = +ve
but having outward current flux = lost of charge
So need to have this additional minus sign

This is known as the continuity equation of charges

We can also derive its differential form by Divergent theorem.

$$-\boxed{\oint\!\!\oint} \vec{J}\cdot d\vec{s} = -\boxed{\iiint \vec{\nabla}}\cdot\vec{J}\, d\tau = \frac{\partial}{\partial t}\iiint \rho\, d\tau$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\rho}$$

which is a PDE that one may use to find $\rho(\vec{r},t)$ / $\vec{J}(\vec{r},t)$ when one of them is given and you need to find another

Deriving Displacement Current

Subst. the above by Gauss Law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

$$0 = \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\rho$$

$$= \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\varepsilon_0 \vec{\nabla}\cdot\vec{E})$$

$$= \vec{\nabla}\cdot(\vec{J} + \varepsilon_0 \frac{\partial}{\partial t}\vec{E})$$

Recall that if a vector field has zero divergence, it can be expressed as the curl of another vector field

(Just like B field, $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\times\vec{A} = \vec{B}$)

This suggests us to modify the original Ampere's Law into

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \underbrace{\varepsilon_0 \frac{\partial}{\partial t}\vec{E}}_{J_d}\right]$$

or in integral form

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I \Rightarrow \oint \vec{B}\cdot d\vec{l} = \mu_0\left[I + \underbrace{\varepsilon_0 \frac{\partial}{\partial t}\iint \vec{E}\cdot d\vec{s}}_{I_d}\right]$$

Complication of Electro-magnetic induction

From Faraday's Law & Ampere's Law, we can see that

Time varying $\vec{E}$ / $\vec{B}$ will induce each other

$$\begin{cases} \text{Faraday}: & \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \text{Time changing } \vec{B} \text{ create } \vec{E} \\ \text{Ampere}: & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} & \text{Time changing } \vec{E} \text{ create } \vec{B} \end{cases}$$

So if we try to treat them as individual equation, we

can end up in such loop:

Some initial $\vec{B}(t)$ $\longrightarrow$ Solve for an induced $\vec{E}(t)$ (Faraday)

Solve for an induced $\Delta\vec{B}(t)$ (Ampere) $\longrightarrow$ Solve for its induced $\Delta\vec{E}(t)$ (Faraday)

Solve for its induced $\Delta(\Delta\vec{B}(t))$ (Ampere) $\longrightarrow$ . . .

What can we do about this?

Soln. 1: Take approximation $\Delta B(t) \simeq 0$

i.e. Assume the induced $\vec{E}(t)$ does not induce additional B field

This approximation is OK because $\mu_0\varepsilon_0 = \dfrac{1}{c^2} = \dfrac{1}{(3\times10^8)^2}$

The size of displacement current $\mu_0\varepsilon_0\dfrac{\partial\vec{E}(t)}{\partial t}$ is usually too small

compare with the original current, and thus $\Delta B(t) \sim 0$

Not ideal. But good enough in many cases.

In fact most textbook problems are taking this approximation.

__Soln 2__ : Solve both of the equations together

i.e. Solving the system of Maxwell's Equation

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} & \text{(Gauss's Law)} \\ \vec{\nabla}\cdot\vec{B} = 0 & \text{(Gauss's Law of B)} \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} & \text{(Faraday's Law)} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} & \text{(Ampere's Law)} \end{cases}$$

A system of PDE ! Terrible !

But there are situations that you must do so . E.g.

- Plasma physics
- Neutron star / black hole