Physics Enrichment Program HKUST |
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- All you need to remember : { Snell 's Law, Mirror Formula, Lens Formula }

- Extra : Matrix method for paraxial optics

- Extra - Extra : Abberration (intro only)

The formula

- Reflection on spherical surface $\frac{1}{s} + \frac{1}{s'} = \frac{-2}{R}$

- Refraction on spherical surface $\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$

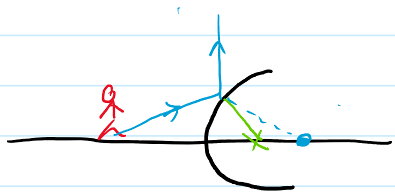
Sign convention

Object / Image distance (s / s')
+ve = anything real
−ve = anything virtual

Radius of curvature (R) :
+ve = convex towards real object
−ve = concave towards real object
opposite sign for virtual object

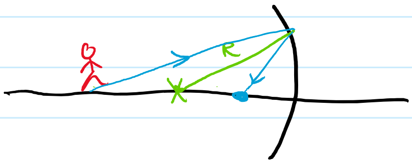
☆ This convention is subjectively better because
{ convex / concave } , { real / virtual } is very commonly asked

Eg. 1 Mirrors



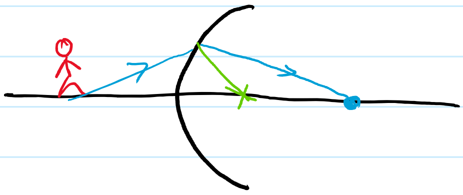
- see convex surface
⇒ R = +ve

- Image form behind mirror
⇒ Cannot be projected ≡ virtual
⇒ s' = −ve

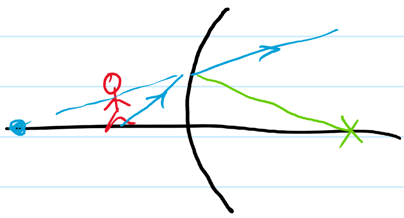


- see concave surface
  $\Rightarrow R = -ve$
- Image form in front of mirror
  $\Rightarrow$ Can be projected $\equiv$ Real
  $\Rightarrow s' = +ve$

Eg. 2 Lens



- see convex surface
  $\Rightarrow R = +ve$
- Image form behind lens
  $\Rightarrow$ Can be projected $\equiv$ Real
  $\Rightarrow s' = +ve$



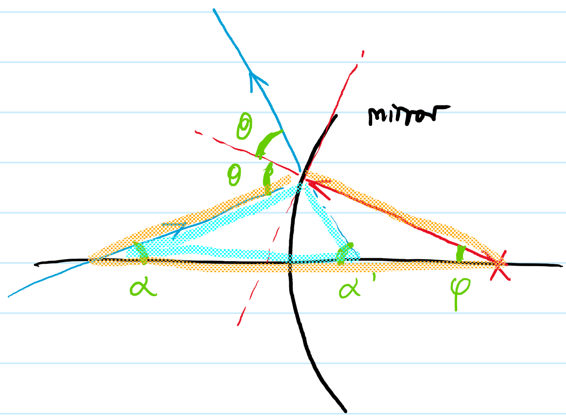
- see convex surface
  $\Rightarrow R = +ve$
- Image from in front of lens
  $\Rightarrow$ Cannot be projected $\equiv$ Virtual
  $\Rightarrow s' = -ve$

Derivation

Reflection on spherical mirror

$\boxed{1}$ By exterior $\angle$ of $\triangle$

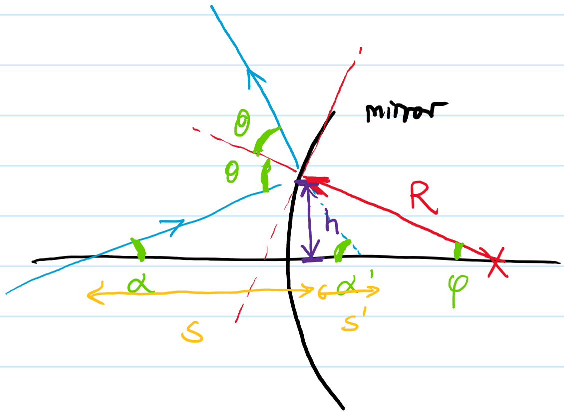
$$\begin{cases} \alpha + \varphi = \theta \\ \alpha + \alpha' = 2\theta \end{cases}$$



$\Rightarrow \alpha - \alpha' = -2\varphi$ = Change in angle of incident ray

[2] Apply small angle approximation. Assume α very small.

$$\begin{cases} \tan\alpha \simeq \alpha \simeq \dfrac{h}{s} \\ \tan\alpha' \simeq \alpha' \simeq \dfrac{h}{s'} \\ \sin\varphi \simeq \varphi \simeq \dfrac{h}{R} \end{cases}$$



$$\Rightarrow \frac{h}{s} - \frac{h}{s'} = \frac{-2h}{R}$$

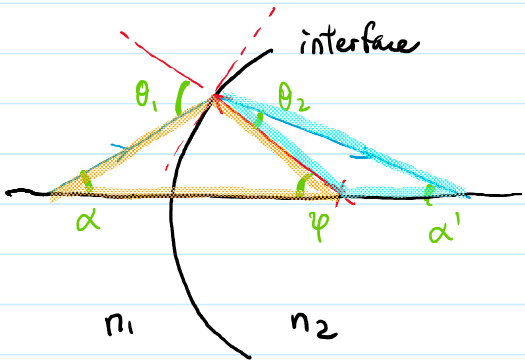
$$\boxed{\frac{1}{s} + \frac{1}{s'} = \frac{-2}{R}}$$

Adopt convention : $s' = -ve$ if virtual (behind mirror)

Refraction on spherical lens

[1] By Snell's Law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

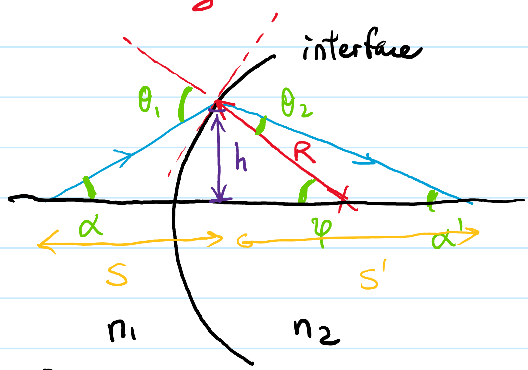


[2] By exterior ∠ of △

$$\begin{cases} \theta_1 = \alpha + \varphi \\ \varphi = \theta_2 + \alpha' \end{cases}$$

[3] Apply small angle approximation, Assume α very small

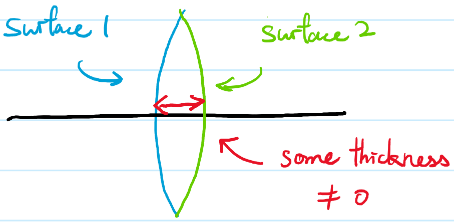
$$\begin{cases} \tan\alpha \simeq \alpha \simeq \dfrac{h}{s} \\ \tan\alpha' \simeq \alpha' \simeq \dfrac{h}{s'} \\ \sin\varphi \simeq \varphi \simeq \dfrac{h}{R} \\ n_1\sin\theta_1 \simeq n_1\theta_1 \simeq n_2\theta_2 \simeq n_2\sin\theta_2 \end{cases}$$



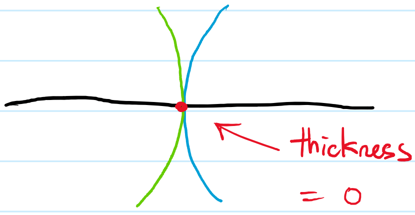
$$\Rightarrow \boxed{\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}}$$

The lens maker's formula (Thin lens formula)

Approximation : Sepration between the spherical surface = 0



In reality



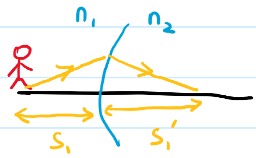
The ideal case in the formula

Let surrounding refractive index $= n_1$

Lens material's refractive index $= n_2$

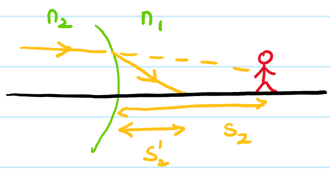
$\boxed{1}$ Through the 1^{st} interface (Radius R_1)

$$\frac{n_1}{s_1} + \frac{n_2}{s_1'} = \frac{n_2 - n_1}{R_1}$$



$\boxed{2}$ Through the 2^{nd} interface (Radius R_2)

$$\frac{n_2}{s_2} + \frac{n_1}{s_2'} = \frac{n_1 - n_2}{R_2}$$



Note that object of 2^{nd} interface = image of 1^{st} interface

- This image is behind 2^{nd} interface $\equiv$ virtual object

$\Rightarrow$ Object distance get $-ve$ sign

i.e. $s_2 = -s_1'$

- Virtual object see 2^{nd} interface as convex

$\Rightarrow$ Radius of curvature get -ve sign

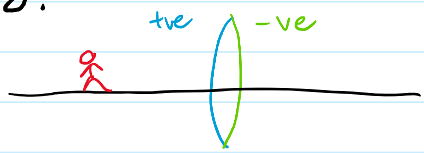
i.e. Change to $-R_2$

$\boxed{3}$ Substitute to remove s_1', get

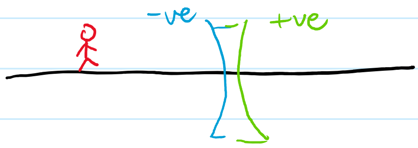
$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{f}$$

Easy remembering : Sign of R = what is seen by original object

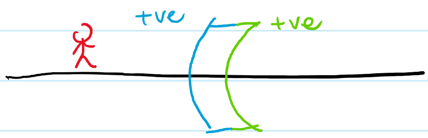
Eg.



$$\Rightarrow \quad \frac{1}{+R_1} - \frac{1}{-R_2} = \text{very +ve}$$



$$\Rightarrow \quad \frac{1}{-R_1} - \frac{1}{+R_2} = \text{very -ve}$$



$$\Rightarrow \quad \frac{1}{+R_1} - \frac{1}{+R_2} \simeq 0$$

Extra : Matrix method for paraxial optics

Paraxial = Light rays have very small angle of elevation ($< 5°$)

s.t. almost parallel to principle axis

(Also called Gaussian optics because proposed by Gauss)

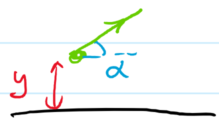
☆ Consequence : Can use small angle approximation

- All the previous formula belong to this domain
- Optical system can be approximated by linear transform.

The framework

Represent distance from principle axis & elevation angle of light ray as a column vector

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} \begin{matrix} \leftarrow \text{Vertical distance} \\ \leftarrow \text{Elevation angle} \end{matrix}$$

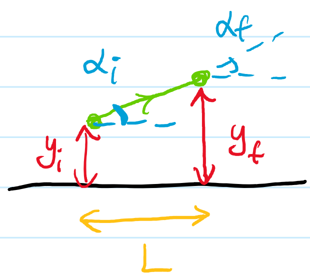


Transition = Matrix multiplication

① Translation

After travelling for a horizontal distance L

$$\begin{cases} y_f = y_i + L \tan \alpha_i \approx y_i + L\alpha_i \\ \alpha_f = \alpha_i \end{cases}$$



$$\Rightarrow \begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}}_{\text{Matrix for translation}} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$$

② Reflection

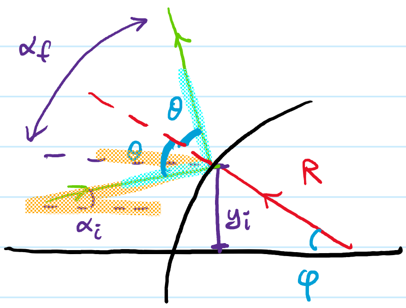
Consider the light ray right before reflect vs. right after reflect

- Change in vertical distance $= 0$ (No translation yet)

$$\Rightarrow y_f = y_i$$

- Change in elevation angle depends on where it is reflected

By Geometry,



$$\begin{cases} \theta = \alpha_i + \varphi \\ 2\theta = \alpha_i + \alpha_f \end{cases}$$

And small angle approximation

$$\sin\varphi \simeq \varphi \simeq \frac{y_i}{R}$$

$$\Rightarrow \quad \alpha_f = \alpha_i + \frac{2y_i}{R}$$

$\therefore$ Express reflection by the matrix

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}}_{\text{Matrix of reflection}} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$$

③ Refraction

Similar to reflection

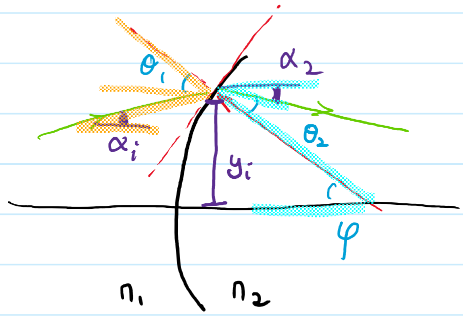
- Change in vertical distance $= 0$ (No translation yet)

$$\Rightarrow y_f = y_i$$

- Change in elevation angle depends on where it is refracted
- Snell's Law : $n_1 \sin\theta_1 = n_2 \sin\theta_2$

By geometry

$$\begin{cases} \theta_1 = \alpha_1 + \varphi \\ \varphi = \alpha_2 + \theta_2 \end{cases}$$



And small angle approximation

$$\sin\varphi \simeq \varphi \simeq \frac{y_i}{R}$$

After substitution

$$\Rightarrow \alpha_f = \left[\frac{1}{R}\left(\frac{n_1}{n_2}-1\right)\right] y_i + \left[\frac{n_1}{n_2}\right] \alpha_i$$

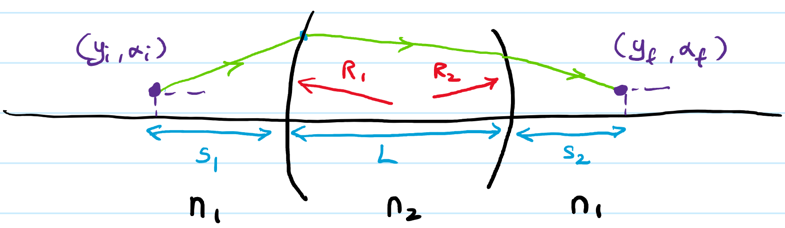
$\therefore$ Express refraction by matrix

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2}-1\right) & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$$

Application

Matrix method simplify design process & computer programmable

E.g. Design a thick lens with a desired thickness



(y_f, α_f) is related (y_i, α_i) simply by a series of matrix

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & s_2 \\ 0 & 1 \end{bmatrix}}_{\text{translate by } s_2} \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{-1}{R_2}\left(\frac{n_2}{n_1}-1\right) & \frac{n_2}{n_1} \end{bmatrix}}_{\text{refract at 2nd interface}} \underbrace{\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}}_{\text{translate by } L} \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n_1}{n_2}-1\right) & \frac{n_1}{n_2} \end{bmatrix}}_{\text{refract at 1st interface}} \underbrace{\begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix}}_{\text{translate by } s_1} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$$

- Easy to obtain object - image distance relation, instead of applying lens formula many times
- Easy to tune parameters for desired purposes

E.g. $\begin{cases} \text{Can vary} = R_1, R_2, L, \frac{n_2}{n_1} \\ \text{Require equivalent focal length} = \frac{1}{s_1 + \frac{L}{2}} + \frac{1}{s_2 + \frac{L}{2}} = \frac{1}{f} \end{cases}$

Extra - Extra : Aberration Theory

Problem: Perfect imaging only happen at paraxial condition ($< 5^\circ$)

Aberration = Imaging error between object and the image in reality

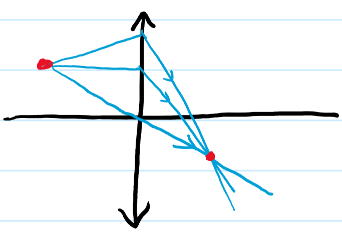
2 Main groups of aberration

① Achromatic aberration

= Error unrelated to colour of light.

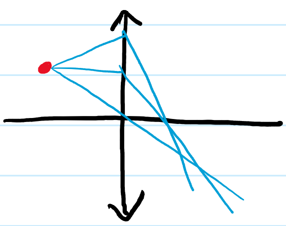
= Error from small angle approximation.

E.g.



Perfect focus

All light from same source go to same spot



Reality

Geometrically they cannot arrive at the same spot

How to mathematically describe this error?

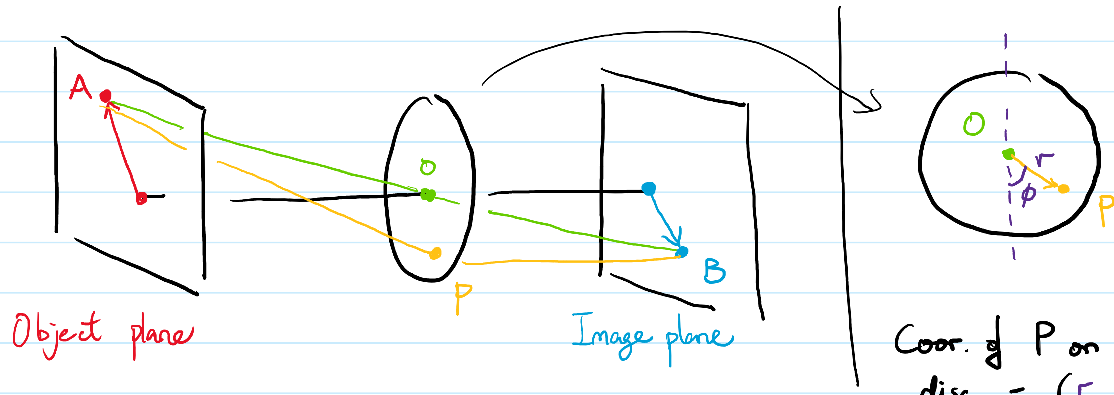
By the light path length difference between a light ray through the image device's center vs a light ray pass through else where.

= The distance light travelled in Fermat's light principle. Not just geometrical distance.

☆☆ If the image is focused, any rays should have the same light path distance, no matter where it passes through.

For the error due to spherical device, we can write it as

$$a(r,\phi) = (\text{length of light path through device center})$$
$$- (\text{length of light path through anywhere else})$$
$$= \overset{\text{light}}{\overline{AOB}} - \overset{\text{light}}{\overline{APB}}$$



Coor. of P on this disc $= (r, \phi)$

This difference in light path length can be expressed as an infinite sum if we keep all the terms in small angle approximation

$$a(r,\phi) = \sum_m \sum_n C_{mn} \left(\frac{r}{R}\right)^m \cos^n \phi$$

R = radius of the disc so $0 \le \frac{r}{R} \le 1$

where C_{mn} are the constants that determine the size of the aberration type $\left(\frac{r}{R}\right)^m \cos^n\phi$

Note that when m, n go bigger, $\left(\frac{r}{R}\right)^m \cos^n\phi$ becomes smaller

i.e. the higher order abberration types are more neligible.

Siedel Aberration

The first 5 terms in the sum contribute to the major imaging error to most optical devices.

They are grouped to called Siedal Aberration or 3rd order aberration ($\sin\theta \simeq \theta - \frac{\theta^3}{3!} + \cdots$ error begin with θ^3)

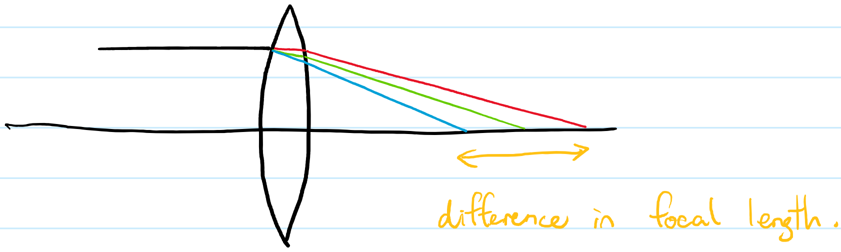
They are given individual name mostly by their geometric effect.

1. Spherical Aberration $\propto \left(\frac{r}{R}\right)^4$
2. Coma $\propto \left(\frac{r}{R}\right)^3 \cos\phi$
3. Astigmatism $\propto \left(\frac{r}{R}\right)^2 \cos^2\phi$
4. Curvature of field $\propto \left(\frac{r}{R}\right)^2$
5. Distortion $\propto \left(\frac{r}{R}\right)\cos\phi$

② Chrometic Aberration

= Error by refractive index dependency on wavelength.

Usually quantified by the difference in focal length between serval standard wave length.



Idea of optical device = Meet design requirement + minimize aberration.

- Most calculation handle by computer software nowadays.
- Most aberrations have standard "solution"

  E.g. Chromatic aberration → flint glass + crown glass combination
- But different types of aberration are mathematically related

  Reducing one type may enlarge another type.

  ⇒ Need to sacrifice certain reduction depending on needs.

  E.g. Telescope targets = paraxial objects.

  ↳ Can safely ignore most achromatic aberration when choosing the right lens.
- Also need to care other engineering requirement / production cost.

  E.g. Weight / Size of device? Adjustable focal length?