Calculate Resistance / Capacitance / Inductance of a config.

R → Ohm's Law

C → Gauss's Law

L → Faraday's Law & Ampere's Law

Resistance

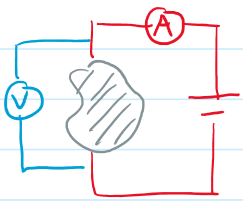
Definition : $R = \dfrac{\text{Potential applied}}{\text{Current created}} = \dfrac{dV}{dI}$ (Pre-calculus $R = \dfrac{V}{I}$)

Circuit symbol of resistor :  or 

(International) (US)

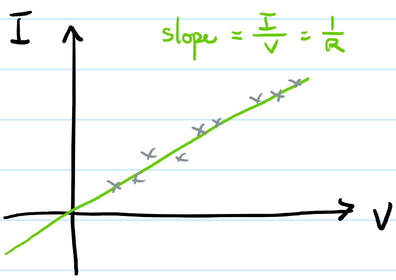
① Ohm's Law

The definition of resistance is essentially the Ohm's Law which was built purely from experimental results



Both voltage & current are measurables

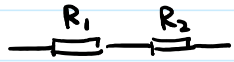
For most substances, the plot I vs V is a straight line through 0



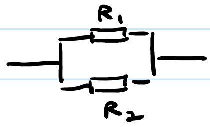
(Applied voltage is easier to change so it is almost always the x axis)

② Rules for resistance addition

Series : $R_{equiv} = R_1 + R_2$



Parallel : $\dfrac{1}{R_{equiv}} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$



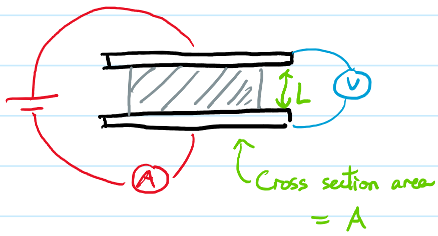
(Should have been covered in high school)

③ Conductivity

By definition
$$\begin{cases} V = -\int_{path} \vec{E} \cdot d\vec{l} \\ I = \iint_{surface} \vec{J} \cdot d\vec{s} \end{cases}$$

If we measure the resistance of a thin slice of object using the parallel plate set up, the definitions reduce to

$$\begin{cases} V = |\vec{E}| L \\ I = |\vec{J}| A \end{cases}$$



The Ohm's Law is now expressed by microscopic quantities :

$$R = \frac{dV}{dI} = \frac{L}{A} \frac{d|\vec{E}|}{d|\vec{J}|} \Rightarrow d|\vec{J}| = \frac{L}{RA} d|\vec{E}|$$

Naively define $\boxed{\text{conductivity} \quad \sigma = \frac{L}{RA}}$ which is a property related to resistance but independent of the object's size

④ Microscopic formula

Consider an infinitestimal object placed under a E field which cause a flow of charge / current $\vec{J}$ inside .

In general, there is no guarantee that the two are proportional

$$\vec{J} = f(\vec{E}) \quad \text{(like in dielectric/magnetic material)}$$

But in most daily life material, they are proportional, i.e.

$$|\vec{J}| \propto |\vec{E}|$$

they are called Ohmic materials.

Since
$$I = \iint \vec{J} \cdot d\vec{S} \sim |\vec{J}| A$$
$$V = -\int \vec{E} \cdot d\vec{l} \sim |\vec{E}| L$$
$$\Rightarrow R = \frac{V}{I} \sim \frac{|\vec{E}| L}{|\vec{J}| A} = \frac{L}{\sigma A}$$

We arrive at $\boxed{\vec{J} = \sigma \vec{E}}$, the microscopic Ohm's Law

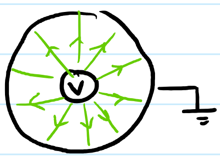
⑤ Finding resistance from conductivity

Even in Ohmic material, the conductivity may vary

by position, i.e. $\vec{J}(\vec{r}) = \sigma(\vec{r}) \vec{E}(\vec{r})$

And the object may not be of cylindrical shape

such that area A / length L are not well-defined

E.g.  Current flow like ray spreading

☆☆☆ The standard procedures of finding resistance

$\boxed{0}$ Begin with these 2 assumptions :

$$\begin{cases} \text{A constant potential } V \text{ set up at some boundary} \\ \text{A } \underline{\text{steady current}} \ I \text{ flowing through any cross-section} \end{cases}$$

↳ { Any charge distribution in the material must be constant
Also ensure no induced EMF

$\boxed{1}$ __Option 1__ : Solve $\vec{E} / \vec{J}$ by PDE

Require these 3 equations to hold at the same time

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} & \text{Gauss Law. Unknown charge distribution in material} \\ \vec{\nabla} \cdot \vec{J} = -\dfrac{\partial \rho}{\partial t} = 0 & \text{Continuity Eq. but with steady current} \\ \vec{J} = \sigma \vec{E} & \text{Ohm's Law} \end{cases}$$

One problem is that we may or may not know ρ

$\boxed{2}$ __Option 2__ : Symmetric claim

By definition $I = \iint_{\text{Any cross section}} \vec{J} \cdot d\vec{S}$

Note that "Any cross section" is a strong condition

↳ May be used as a symmetric claim.

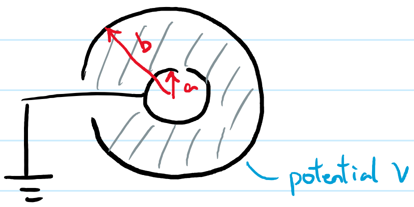
If we can find $\vec{J}$, we can then get $\vec{E} = \dfrac{\vec{J}}{\sigma}$

$\boxed{3}$ After we have either $\vec{J}$ or $\vec{E}$, we can compute

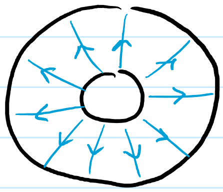
$I = \iint \vec{J} \cdot d\vec{S}$ and $V = \int \vec{E} \cdot d\vec{l}$ $\rightarrow$ $R = \dfrac{V}{I}$

Example : Material between 2 spherical shell (radius a & b)

conductivity $\sigma(r) = kr$ depends on radius

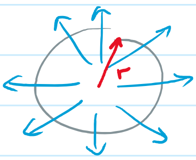


Current flow should look like $\Longrightarrow$



$\boxed{1}$ Begin from the steady current requirement

we can choose a spherical surface (like in Gauss Law)



$$I = \iint \vec{J} \cdot d\vec{s} = |\vec{J}| \cdot 4\pi r^2 = \text{constant}$$

$\Rightarrow$ The distribution of $\vec{J}$ must be $\propto \frac{1}{r^2}$

Let it writes as $|\vec{J}| = \frac{I}{4\pi r^2}$

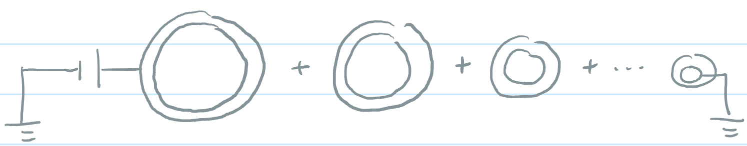
$\boxed{2}$ By Ohm's Law, $|\vec{E}| = \frac{1}{\sigma}|\vec{J}| = \frac{I}{k \cdot 4\pi r^3}$

Then the potential difference between 2 shells is

$$V = \int_b^a \frac{I}{k \cdot 4\pi r^3} dr = I \cdot \frac{1}{4\pi k}\left(\frac{-1}{2r^2}\right)\Big|_b^a = I \cdot \frac{1}{8\pi k}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)$$

And therefore $R = \frac{V}{I} = \frac{1}{8\pi k}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)$

☆ Note 1 : This example is equivalent to shells connected in series



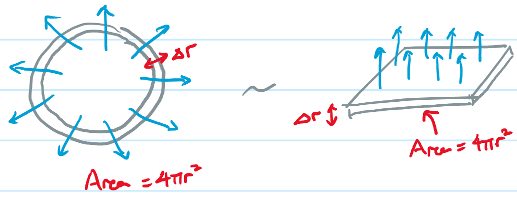
The shell with radius r has a resistance

$$\Delta R = \frac{1}{\sigma(r)} \cdot \frac{\Delta r}{4\pi r^2} \qquad \left(R = \frac{1}{\sigma} \cdot \frac{L}{A} \right)$$

In series = directly add

$$= \frac{1}{kr} \cdot \frac{\Delta r}{4\pi r^2}$$

$$\Rightarrow \int dR = \int_a^b \frac{1}{kr} \frac{dr}{4\pi r^2}$$



☆ **Note 2**: Taking Gauss Law over a spherical surface of radius r

$$\Rightarrow \iint_{\text{sphere}} \vec{E} \cdot d\vec{S} = |\vec{E}| \cdot 4\pi r^2 = \frac{I}{kr} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

$\Rightarrow$ Charge in the sphere depends on radius !

$\Rightarrow$ There are net charge distribution in the material.

There is no contradiction because these charge do not move

It is allowed to have regions with net charge

as long as the net flux of current is 0.

Recall in PDE approach, we have to write

$$\begin{cases} \nabla \cdot E = \dfrac{\rho}{\varepsilon_0} \\ \nabla \cdot J = \dfrac{\partial \rho}{\partial t} = 0 \end{cases}$$

ρ must no depend on time but it can be non zero and even depend on position

Capacitance

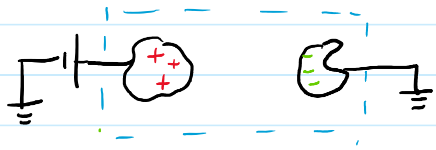
Definition : $C = \dfrac{\text{charge stored}}{\text{potential applied}} = \dfrac{dQ}{dV}$ $\left(\begin{array}{c}\text{Pre-calculus} \\ C = \frac{Q}{V}\end{array}\right)$

Circuit symbol of capacitor : —| |— (parallel plates)

① About Potential

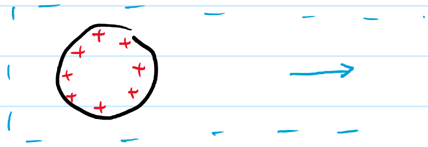
- If you are given 2 objects with opposite amount of charge

$\Rightarrow$ Take V be the potential difference between 2 objects.



- If you are given only 1 object

$\rightarrow$ Take V be the potential difference from infinity

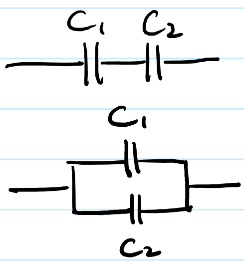


Capacitance = potential relative to ∞ per charge added

② Rules for capacitance addition

Series : $\dfrac{1}{C_{equiv}} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$

Parallel : $C_{equiv} = C_1 + C_2$



③ Finding capacitance

Capacitance is the relation between Q & V

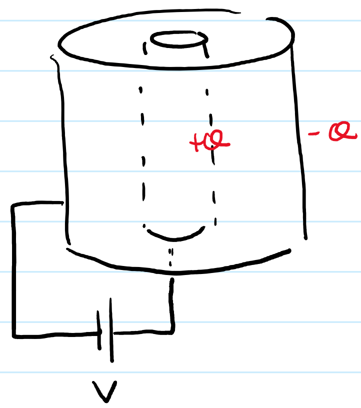
$\Rightarrow$ Assume we know one of them, then find the expression of the other by Gauss Law + symmetry / PDE / ... etc.

In most of the case, we start with Q

because V is usually only given the values at the 2 ends

(eg. V on the 2 parallel plates), not the whole distribution

Example: Cylindrical capacitor



$\boxed{1}$ Assume the charges on inner plate $= Q$
outer plate $= -Q$

$\boxed{2}$ Find $\vec{E}$ by Gauss Law

$$|\vec{E}| = \frac{Q}{2\pi\varepsilon_0 rL}, \text{ point outward}$$

$\boxed{3}$ Find V by $\int \vec{E} \cdot d\ell = \int_a^b |E| dr$

$$V = \frac{Q}{2\pi\varepsilon_0 L} \ln\left(\frac{b}{a}\right)$$

$\boxed{4}$ Capacitance $= \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$

④ Capacitance involving dielectric

only consider free charge as the charge stored.

$$C = \frac{Q_f}{V}$$

Because they are the only charge that can be released.

Then we follow the formula in dielectric to get $\vec{E}$, and finally V

$$Q_f \rightarrow \vec{D} \rightarrow \vec{E} \rightarrow V$$

Example : Cylindrical capacitor with layers of linear dielectric.

permitivity $\varepsilon(r) = kr$ depends on radius

[1] Assume the free charge on the cylinder be Q_f & $-Q_f$

We can find $\vec{D}$ by symmetry claims

$$|\vec{D}(r)| = \frac{Q_f}{2\pi r L}$$

[2] Linear dielectric guarentees that $\vec{D} = \varepsilon \vec{E}_{tot}$

So $$|\vec{E}(r)| = \frac{Q_f}{2\pi r \varepsilon L} = \frac{Q_f}{2\pi k r^2 L}$$

[3] Potential formed between metal plates is then

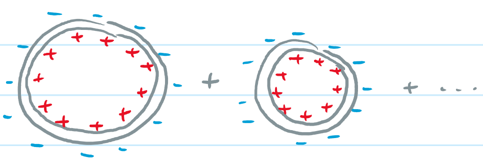
$$V = -\int_a^b \vec{E} \cdot d\vec{r} = \frac{Q_f}{2\pi k L}\left(\frac{1}{b} - \frac{1}{a}\right)$$

And therefore $$C = \frac{Q_f}{V} = \frac{2\pi k L}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

☆ **Note** : This example is equivalent to shells connected in series

The layer between radius r & $r+\Delta r$ has capacitance

$$\Delta C = \frac{Q_f}{\Delta V} = \frac{2\pi r L \cdot \varepsilon(r)}{\Delta r} = \frac{2\pi k r^2 L}{\Delta r}$$



In parallel
= add the recipical

$$\Rightarrow \frac{1}{C} = \int d\left(\frac{1}{C}\right) = \int_a^b \frac{dr}{2\pi k r^2 L}$$

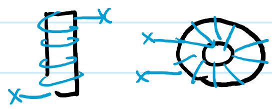
Inductance

Definition : $\frac{1}{L} = \frac{\text{rate of current change}}{\text{Induced emf}} = \frac{d}{d|\varepsilon|}\left(\frac{dI}{dt}\right)$ $\left(\text{Pre-calculus} \quad L = \frac{\frac{dI}{dt}}{|\varepsilon|} \right)$

Circuit symbol of inductor =  (solenoid)

There are not many shapes of inductor

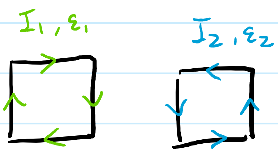
Basically you can only find solenoid or toroid



① 2 types of inductance

- Self inductance $\underline{\underline{L}}$ = Induced EMF acts on the original circuit
- Mutual inductance $\underline{\underline{M}}$ = Induced EMF is created on another circuit but has no effect on the original current

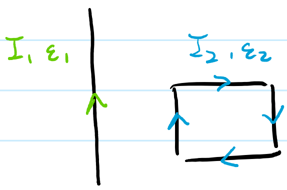
E.g. 1



2 objects induce emf on each other

$\varepsilon_1 = M_{12}\frac{dI_2}{dt}$, $\varepsilon_2 = M_{21}\frac{dI_1}{dt}$

E.g. 2



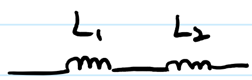
Changing I_1 induces ε_2 on RHS ring

(Is changing I_2 inducing ε_1 on wire ?)

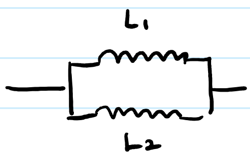
☆ It can be proved that $M_{12} = M_{21}$ always holds

② Rules for inductor addition

Series : $\frac{1}{L_{equiv}} = \frac{1}{L_1} + \frac{1}{L_2}$



Parallel : $L_{equiv} = L_1 + L_2$



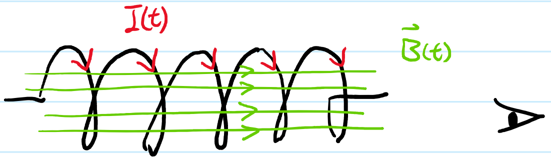
③ Finding Inductance

Inductance is the relation between $\frac{dI}{dt}$ & ε

$\Rightarrow$ Assume we know one of them, then find the expression of the other by Ampere + Faraday's Law + symmetry / PDE

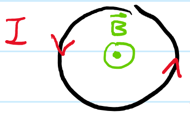
In most cases, we start with $\frac{dI}{dt}$. Start from ε is difficult

Example : Solenoid



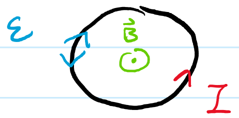
[1] By Ampere's Law, $B(t) = \frac{\mu_0 N I(t)}{l}$

[2] Flux by B field :



$$\Phi_B(t) = B(t) \cdot N \cdot 2\pi r^2 = \frac{\mu_0 N^2 \cdot \pi r^2}{l} I(t)$$

[3] Find emf by Faraday's Law (And direction by Lenz's Law)



$$|\varepsilon(t)| = \frac{d\Phi_B(t)}{dt} = \frac{\mu_0 N^2 \cdot 2\pi r^2}{l} \frac{dI(t)}{dt}$$

[4] (Self) Inductance $= L = \frac{|\varepsilon(t)|}{\frac{dI(t)}{dt}} = \frac{\mu_0 N^2 \cdot 2\pi r^2}{l}$

④ Inductance involving magnetic material

When it involves magnetic material, only consider free current as the source of the induced emf $\quad L = \frac{|\varepsilon|}{\frac{dI_f}{dt}}$

Because they are the real current that flows

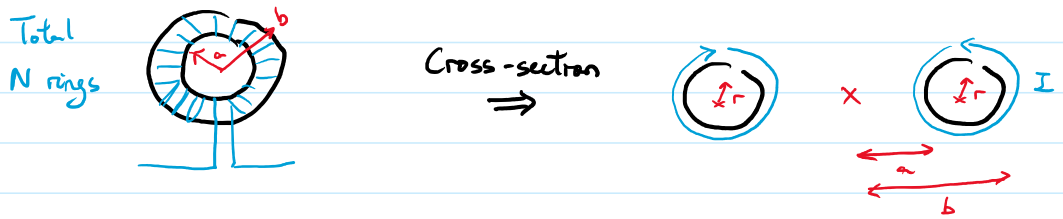
Then we follow the formula in magnetic material to get $\vec{B}$, and finally ε

$$I_f \to \vec{H} \to \vec{B} \to \varepsilon$$

(Examples are just similar to capacitors, with fewer varieties.)

Example : Toroid inductor with magnetic material in the middle

permeability $\mu(r) = kr$ depends on cross-section 's radius

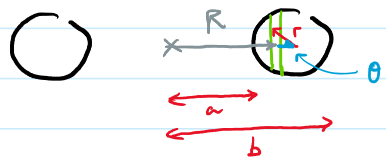


$\boxed{1}$ Assume the free current along the wire be I_f

We can find $\vec{H}$ by symmetry claims

$$|\vec{H}(R)| = \frac{NI}{2\pi R}$$

R = horizontal distance from the hole 's center



$\boxed{2}$ Linear magnetic material guarentees that $\vec{H} = \frac{1}{\mu}\vec{B}$

So $|B(r,R)| = \mu(r)\frac{NI_f}{2\pi R} = kr\cdot\frac{NI_f}{2\pi R}$

We can further express $R = \frac{b+a}{2} - r\cos\theta$

$\boxed{3}$ Magnetic flux in the cross-section is then

$$\Phi_B = \int_0^{\frac{b-a}{2}}\int_0^{2\pi} \frac{krNI_f}{2\pi\left(\frac{b+a}{2} - r\cos\theta\right)}\, r\,d\theta\,dr$$

And therefore $L = \frac{\varepsilon}{\frac{dI}{dt}} = \frac{\frac{d\Phi_B}{dt}}{\frac{dI}{dt}}$ (skipping the math)