

- Limit of Vector functions

- Differentiation
 - { On vectors
 - By vectors

- Parametrization & Line Integral

Review : Vector Geometry

Vectors ~ Objects with Magnitude & Direction

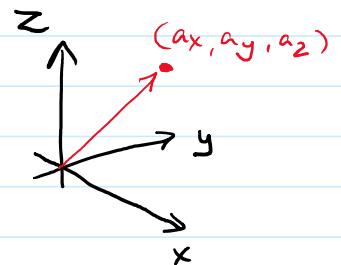
- Usually visualized as an arrow

pointing from origin to some coordinate

- Can be expressed as row/column matrices

$$(a_x \ a_y \ a_z)$$

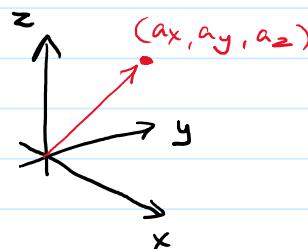
$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$



① Norm (Magnitude)

Notation : $|\vec{a}|$

Geometrically the length of the vector. Follows Pythagoras Theorem.



$$\Rightarrow |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

(2) Unit Vectors

Denote by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ = A vector pointing in the same direction
 "hat" symbol but with length (norm) = 1

The 3 "basis" vectors are chosen to be unit vectors

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ pointing in } x \text{ direction}$$

$$\hat{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ pointing in } y \text{ direction}$$

$$\hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ pointing in } z \text{ direction}$$

So any vectors can be written as

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = a_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$$

$a_x, a_y, a_z \rightarrow$ Components of the vector

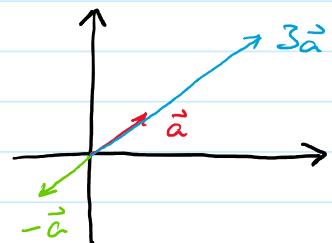
$\hat{x}, \hat{y}, \hat{z} \rightarrow$ Directions of the vector

(3) Multiplication by Constant

In matrix form : Multiply each element by the constant

$$k\vec{a} = k \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix}$$

Geometrically as extending / contracting the vector in the same / opposite direction

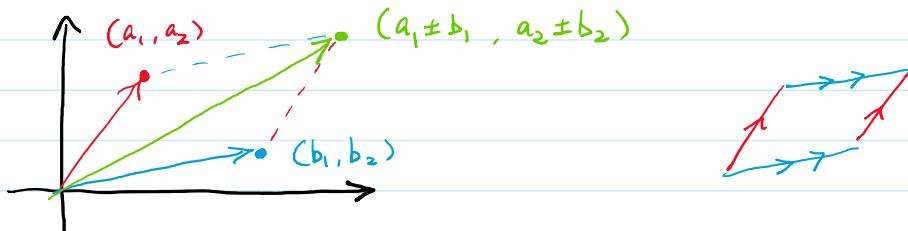


(4) Addition / Subtraction

In matrix form : Add/ Subtract element by element

$$\vec{a} \pm \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \end{pmatrix}$$

Geometrically, addition can be depicted as parallelogram rule



And subtraction is simply adding a negative vector

(5) Dot product (Scalar product)

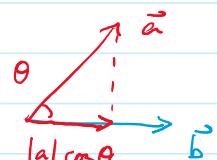
A multiplication between vectors that gives a scalar

$$\begin{aligned} \text{Def: } \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos \theta \quad \text{geometrical angle between } \vec{a}, \vec{b} \\ &= a_x b_x + a_y b_y + a_z b_z \\ &= (a_x \ a_y \ a_z) \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \end{aligned}$$

Dot product can also be visualized as length of projection

when dotting with a unit vector

$$\vec{a} \cdot \hat{b} = |\vec{a}| |\hat{b}| \cos \theta = a \cos \theta$$



In particular if it is dotted with a basis unit vector we get its component at that direction

$$\text{E.g. } (a_x \ a_y \ a_z) \cdot \hat{x}$$

$$= (a_x \ a_y \ a_z) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= a_x = x \text{ component}$$

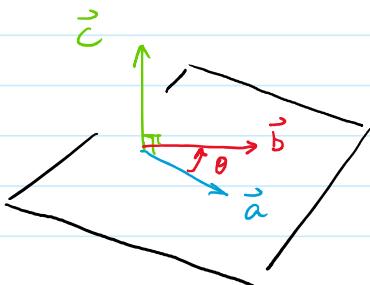
⑥ Cross Product (Vector Product)

A multiplication between vectors that gives a vector

$$\underline{\text{Def}}: \vec{a} \times \vec{b} = \vec{c} = -\vec{b} \times \vec{a} \quad \begin{pmatrix} \text{Anti-symmetric} \\ \text{get minus sign when} \\ \text{switching order} \end{pmatrix}$$

$$\text{with } |\vec{c}| = |\vec{a}| |\vec{b}| \sin \theta$$

and direction of $\vec{c} \perp$ plane formed by \vec{a} & \vec{b}



The exact vector can also be computed by determinant

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

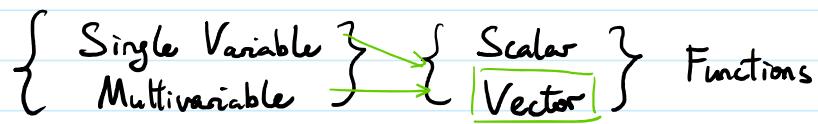
$$\text{E.g. } \vec{a} = \hat{x} + 2\hat{y}, \quad \vec{b} = 3\hat{x} - \hat{y} \quad (\text{Both are on } x-y \text{ plane})$$

$$\Rightarrow \vec{c} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} = -7\hat{z} \quad (\perp \text{ to } x-y \text{ plane})$$

★ Note that cross product is only for 3D vectors

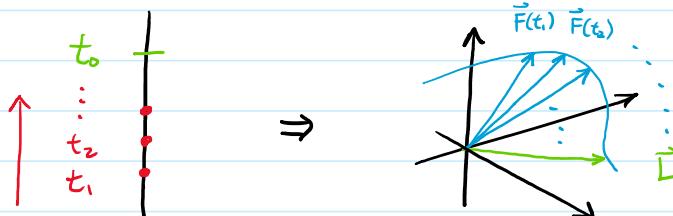
Limit of Vector Functions

Recall that we classify the functions by their no. of inputs/outputs



Single Variable Vector Functions

$$\lim_{t \rightarrow t_0} \vec{F}(t) = \vec{L} \iff \text{When } t \text{ approaches } t_0, \vec{F}(t) \text{ approaches } \vec{L}$$



Equivalent to having the distance between $\vec{F}(t)$ & \vec{L} diminishing

$$\lim_{t \rightarrow t_0} |\vec{F}(t) - \vec{L}| = 0$$

$$\iff \lim_{t \rightarrow t_0} \sqrt{(F_x(t) - L_x)^2 + (F_y(t) - L_y)^2 + (F_z(t) - L_z)^2}$$

$$\iff \begin{cases} \lim_{t \rightarrow t_0} |F_x(t) - L_x| = 0 \\ \lim_{t \rightarrow t_0} |F_y(t) - L_y| = 0 \\ \lim_{t \rightarrow t_0} |F_z(t) - L_z| = 0 \end{cases}$$

All 3 components approaching their own limits

Multivariable Vector Function (similar concept, but hard to draw)

$$\lim_{(x_1, x_2, \dots, x_n) \rightarrow (a_1, a_2, \dots, a_n)}$$

When every input approaches their target value

$$\vec{F}(x_1, x_2, \dots, x_n) = \vec{L}$$

The output approaches the target vector

Differentiation on Vector Functions

On Single Variable Vector Function

$$\frac{d}{dt} \vec{F}(t) = \lim_{\Delta t \rightarrow 0} \left[\frac{\vec{F}(t + \Delta t) - \vec{F}(t)}{\Delta t} \right]$$

$$= \left[\lim_{\Delta t \rightarrow 0} \frac{F_1(t + \Delta t) - F_1(t)}{\Delta t} \right] \hat{u}_1 + \left[\lim_{\Delta t \rightarrow 0} \frac{F_2(t + \Delta t) - F_2(t)}{\Delta t} \right] \hat{u}_2 + \dots$$

$$= \left[\frac{d}{dt} F_1(t) \right] \hat{u}_1 + \left[\frac{d}{dt} F_2(t) \right] \hat{u}_2 + \dots + \left[\frac{d}{dt} F_n(t) \right] \hat{u}_n$$

$$= \begin{pmatrix} \frac{d}{dt} F_1(t) \\ \frac{d}{dt} F_2(t) \\ \vdots \\ \frac{d}{dt} F_n(t) \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix} = \text{Differentiating each components individually}$$

Arithmetics are similar as in single variables

- Addition / Subtraction

$$\frac{d}{dt} [\vec{F}(t) \pm \vec{G}(t)] = \frac{d}{dt} \vec{F}(t) \pm \frac{d}{dt} \vec{G}(t)$$

- Product Rule

* Vectors can multiply each other in 2 different ways

$$\frac{d}{dt} [\vec{F}(t) \cdot \vec{G}(t)] = \left[\frac{d}{dt} F(t) \right] \cdot \vec{G}(t) + \vec{F}(t) \cdot \left[\frac{d}{dt} \vec{G}(t) \right]$$

$$\frac{d}{dt} [\vec{F}(t) \times \vec{G}(t)] = \left[\frac{d}{dt} F(t) \right] \times \vec{G}(t) + \vec{F}(t) \times \left[\frac{d}{dt} \vec{G}(t) \right]$$

Order of "F x G" must be the same
because order matters in cross product

(Cannot have quotient rule, obviously)

On Multivariable Vector Function

The major difference is that there is 1 partial D for each input

$$\left. \begin{array}{l} \text{Functions with } m \text{ inputs have } m \text{ partial D} \\ \frac{\partial}{\partial x_1} \vec{F}(x_1, x_2, \dots, x_m) = \frac{\partial}{\partial x_1} F_1(x_1, x_2, \dots, x_m) \hat{u}_1 + \frac{\partial}{\partial x_1} F_2(x_1, x_2, \dots, x_m) \hat{u}_2 \\ \quad \vdots \\ \quad \vdots \\ \quad \vdots \\ \quad \vdots \\ \frac{\partial}{\partial x_m} \vec{F}(x_1, x_2, \dots, x_m) = \frac{\partial}{\partial x_m} F_1(x_1, x_2, \dots, x_m) \hat{u}_1 + \frac{\partial}{\partial x_m} F_2(x_1, x_2, \dots, x_m) \hat{u}_2 \\ \quad \vdots \\ \quad \vdots \\ \quad \vdots \end{array} \right\}$$

To make the expression easier to read, we can write them as matrices

$$\frac{\partial}{\partial x_1} \vec{F}(x_1, x_2, \dots, x_m) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} \\ \vdots \\ \frac{\partial F_n}{\partial x_1} \end{pmatrix} \quad \dots \quad \frac{\partial}{\partial x_m} \vec{F}(x_1, x_2, \dots, x_m) = \begin{pmatrix} \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_m} \\ \vdots \\ \frac{\partial F_n}{\partial x_m} \end{pmatrix}$$

Then compress into 1 big matrix :

$$\frac{\partial}{\partial \vec{x}} \vec{F}(\vec{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}$$

n rows for outputting an n components vector

$$\vec{F}(\dots) = (F_1, F_2, \dots, \underline{F_n}) \quad \xleftarrow{\text{m columns for function with m inputs}}$$

$$\vec{x} = (x_1, x_2, \dots, \underline{x_m})$$

(This matrix is called Jacobian matrix)

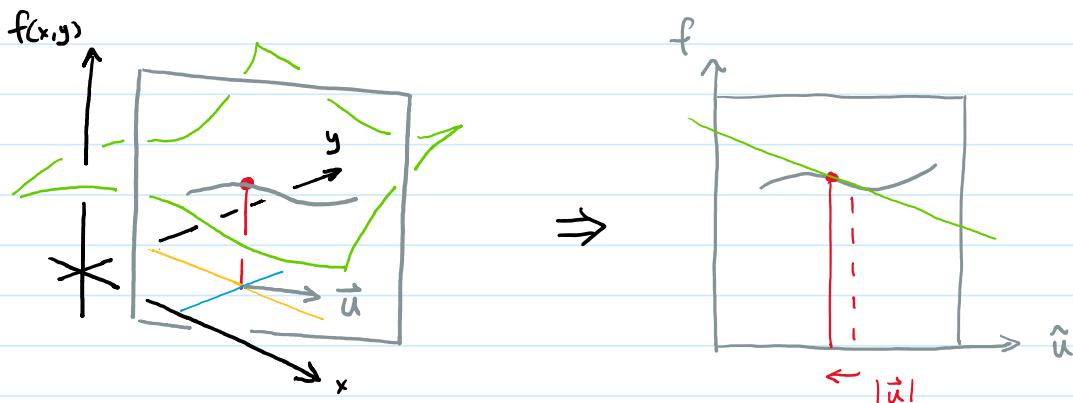
Differentiation by Vector - Gradient

Recall definition of partial differentiation :

$$\frac{\partial}{\partial x} f(x, y) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \text{Slope in } x \text{ direction}$$

$$\frac{\partial}{\partial y} f(x, y) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \text{Slope in } y \text{ direction}$$

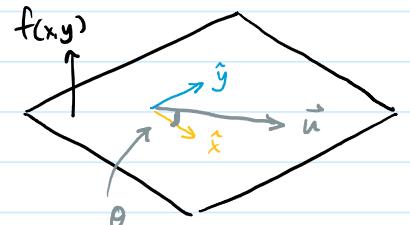
What if we want the slope in arbitrary directions?



Suppose we want to find the slope along the direction of \vec{u}

We know \vec{u} can be decomposed into component form of \hat{x}/\hat{y}

$$\begin{aligned}\vec{u} &= u_x \hat{x} + u_y \hat{y} \\ &= |\vec{u}| \cos \theta \hat{x} + |\vec{u}| \sin \theta \hat{y}\end{aligned}$$



And the unit vector of \vec{u} can be expressed as

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \cos \theta \hat{x} + \sin \theta \hat{y} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

To find the slope along \vec{u} , first vary $f(x,y)$ to $f(x+u_x, y+u_y)$

then divide by $|\vec{u}|$. Finally by limiting $|\vec{u}| \rightarrow 0$,

it becomes differentiation.

$$D_{\hat{u}} f(x,y) = \lim_{|\vec{u}| \rightarrow 0} \frac{f(x+u_x, y+u_y) - f(x,y)}{|\vec{u}|}$$

Just a notation

of differentiating in
 \vec{u} 's direction

$$\begin{aligned}
 &= \lim_{|\vec{u}| \rightarrow 0} \frac{f(x+|\vec{u}|\cos\theta, y+|\vec{u}|\sin\theta) - f(x,y+|\vec{u}|\sin\theta)}{|\vec{u}|} \\
 &\quad + \lim_{|\vec{u}| \rightarrow 0} \frac{f(x, y+|\vec{u}|\sin\theta) - f(x,y)}{|\vec{u}|} \\
 &= \lim_{|\vec{u}| \rightarrow 0} \frac{f(x+|\vec{u}|\cos\theta, y+|\vec{u}|\sin\theta) - f(x,y+|\vec{u}|\sin\theta)}{|\vec{u}|\cos\theta} \\
 &\quad + \lim_{|\vec{u}| \rightarrow 0} \frac{f(x, y+|\vec{u}|\sin\theta) - f(x,y)}{|\vec{u}|\sin\theta} \\
 &= \left(\frac{\partial}{\partial x} f(x,y) \right) \cos\theta + \left(\frac{\partial}{\partial y} f(x,y) \right) \sin\theta \\
 &= \left(\frac{\partial}{\partial x} f(x,y) \quad \frac{\partial}{\partial y} f(x,y) \right) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}
 \end{aligned}$$

This is independent of \vec{u} This is $\hat{u} = \cos\theta \hat{x} + \sin\theta \hat{y}$

$$\Rightarrow \text{Slope in } \vec{u} \text{ direction} = \left(\text{something independent of direction} \right) \cdot \hat{u}$$

\Rightarrow Give a special name to this quantity : Gradient

Notation : $\vec{\nabla} f$, grad f

$\vec{\nabla}$ symbol = pronounce as "Del",
formal name = "nabla"

Note 1 : Gradient itself is NOT the slope. It is just a "property" of the function while we can use it to obtain slopes along any direction \vec{u} .

Note 2 : Because slope in all direction = $\vec{\nabla}f \cdot \hat{u}$

$$= |\vec{\nabla}f| |\hat{u}| \cos \theta \leq |\vec{\nabla}f|$$

The max slope at any point is $|\vec{\nabla}f|$ and this happens

when $\cos \theta = 1$, i.e. \hat{u} is chosen parallel to $\vec{\nabla}f$

$\Rightarrow \vec{\nabla}f$ is a vector function which shows the direction with max slope.

Gradient in matrix form

Gradient is conventionally written as a row vector

$$\vec{\nabla}f(\vec{x}) = \left(\frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \dots \frac{\partial f}{\partial x_m} \right)$$

For vector function, we can take gradient to each component

$$\left. \begin{array}{l} \text{If } \vec{F}(\dots) \\ \text{outputs } n \\ \text{values} \end{array} \right\} \quad \begin{aligned} \vec{\nabla} F_1(\vec{x}) &= \left(\frac{\partial F_1}{\partial x_1} \frac{\partial F_1}{\partial x_2} \dots \frac{\partial F_1}{\partial x_m} \right) \\ \vec{\nabla} F_2(\vec{x}) &= \left(\frac{\partial F_2}{\partial x_1} \frac{\partial F_2}{\partial x_2} \dots \frac{\partial F_2}{\partial x_m} \right) \\ &\vdots && \vdots \\ \vec{\nabla} F_n(\vec{x}) &= \left(\frac{\partial F_n}{\partial x_1} \frac{\partial F_n}{\partial x_2} \dots \frac{\partial F_n}{\partial x_m} \right) \end{aligned}$$

Then compress all rows into 1 matrix

$$\vec{\nabla} \vec{F}(\vec{x}) = \left(\begin{array}{cccc} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_m} \end{array} \right)$$

↑
n rows for
outputting an
n components
vector

← m columns for
function with m inputs

(which is again the Jacobian matrix)

Short Summary

	Input	Single Var.	Multi Var.
Output			
Scalar		$\frac{df}{dt}$	$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_m} \right)$
Vector		$\begin{pmatrix} \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial t} \\ \vdots \\ \frac{\partial F_n}{\partial t} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial F_1}{\partial x_1}, \frac{\partial F_1}{\partial x_2}, \dots, \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1}, \frac{\partial F_2}{\partial x_2}, \dots, \frac{\partial F_2}{\partial x_m} \\ \vdots \\ \vdots \\ \frac{\partial F_n}{\partial x_1}, \frac{\partial F_n}{\partial x_2}, \dots, \frac{\partial F_n}{\partial x_m} \end{pmatrix}$

Chain Rule in matrix form

Writing chain rule of multivariable function by matrix is much cleaner

E.g. $\vec{g}(\vec{f}(\vec{x}))$ is a function composition of no. of variables ($n \rightarrow m \rightarrow p$)

$$\begin{aligned}\vec{x} &= (x_1, x_2, \dots, x_n) \\ \vec{f}(\dots) &= (f_1(\dots), f_2(\dots), \dots, f_m(\dots)) \\ \vec{g}(\dots) &= (g_1(\dots), g_2(\dots), \dots, g_p(\dots))\end{aligned}$$

Then the chain rule can be written as

$$\frac{\partial \vec{g}}{\partial \vec{x}} = \frac{\partial \vec{g}}{\partial \vec{f}} \cdot \frac{\partial \vec{f}}{\partial \vec{x}}$$

$$\begin{pmatrix} p \times n \\ \text{matrix} \end{pmatrix} \begin{pmatrix} p \times m \\ \text{matrix} \end{pmatrix} \begin{pmatrix} m \times n \\ \text{matrix} \end{pmatrix}$$

$$\begin{array}{c} \text{i}^{\text{th}} \text{ row} \\ \downarrow \end{array} \left(\begin{array}{ccc} & \vdots & \\ \dots & \frac{\partial g_i}{\partial x_j} & \dots \\ \text{j}^{\text{th}} \text{ column} & \vdots & \end{array} \right) = \left(\begin{array}{c} \vdots \\ \frac{\partial g_i}{\partial f_1} \quad \frac{\partial g_i}{\partial f_2} \quad \dots \quad \frac{\partial g_i}{\partial f_m} \\ \vdots \\ \text{every element} \end{array} \right) \left(\begin{array}{c} \dots \quad \frac{\partial f_1}{\partial x_j} \quad \dots \\ \text{i}^{\text{th}} \text{ row} \\ \text{j}^{\text{th}} \text{ column} \quad \frac{\partial f_2}{\partial x_j} \\ \dots \quad \frac{\partial f_m}{\partial x_j} \\ \text{every element} \end{array} \right)$$

Or as individual terms

$$\begin{aligned} \frac{\partial g_i}{\partial x_j} &= \sum_{k=1}^m \frac{\partial g_i}{\partial f_k} \cdot \frac{\partial f_k}{\partial x_j} \\ &= \frac{\partial g_i}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_j} + \frac{\partial g_i}{\partial f_2} \cdot \frac{\partial f_2}{\partial x_j} + \dots + \frac{\partial g_i}{\partial f_m} \cdot \frac{\partial f_m}{\partial x_j} \end{aligned}$$

Example

$$\begin{cases} f(p, q) = \sqrt{p+q} \\ \vec{g}(t) = (t-1, t^2) \end{cases} \Rightarrow f(\vec{g}(t)) = \sqrt{t^2+t-1}$$

- Differentiate on t directly : $\frac{d}{dt} (\sqrt{t^2+t-1}) = \frac{1}{2} \frac{2t+1}{\sqrt{t^2+t-1}}$

- Differentiate by chain rule :

$$\left(\frac{\partial f}{\partial p} \quad \frac{\partial f}{\partial q} \right) = \left(\frac{1}{2} \frac{1}{\sqrt{p+q}} \quad \frac{1}{2} \frac{1}{\sqrt{p+q}} \right) \quad \left| \begin{array}{l} p = t-1 \\ q = t^2 \end{array} \right.$$

$$\left(\begin{array}{c} \frac{\partial g_1}{\partial t} \\ \frac{\partial g_2}{\partial t} \end{array} \right) = \left(\begin{array}{c} 1 \\ 2t \end{array} \right)$$

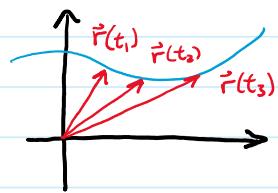
$$\Rightarrow \frac{d}{dt} [f(\vec{g}(t))] = \left(\frac{\partial f}{\partial p} \quad \frac{\partial f}{\partial q} \right) \left(\begin{array}{c} \frac{\partial g_1}{\partial t} \\ \frac{\partial g_2}{\partial t} \end{array} \right)$$

$$= \left(\frac{1}{2} \frac{1}{\sqrt{p+q}} \quad \frac{1}{2} \frac{1}{\sqrt{p+q}} \right) \left(\begin{array}{c} 1 \\ 2t \end{array} \right) = \frac{1}{2} \frac{2t+1}{\sqrt{t^2+t-1}}$$

Curve Parametrization

Recall single var. vector function

e.g. $\vec{r}(t) = [x(t), y(t)]$

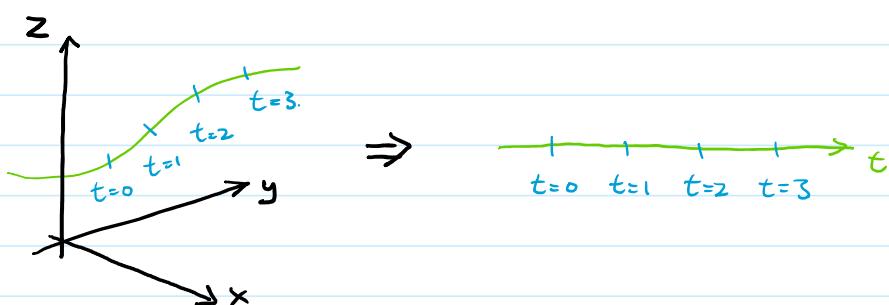


This is a curve on a plane. Any point along the curve only needs 1 input (t) to fully locate it

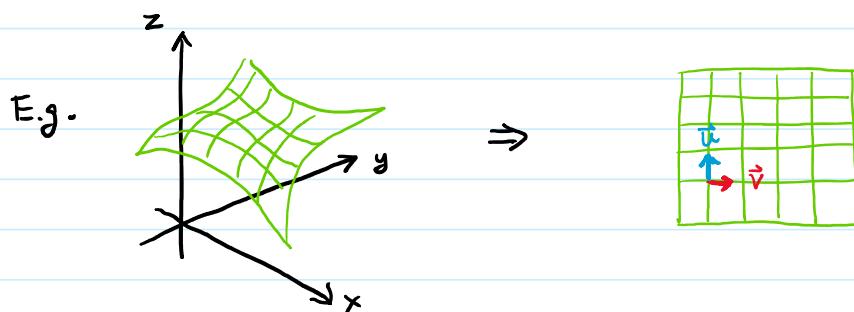
Intuitively : Curve = 1D object = Only 1 free variable

Parametrization = Choose a coordinate system on the object to describe every points , rather than using the environment coordinate ($x/y/z$)

(Note : Parametrization is never unique , because there are infinitely many ways of choosing a coordinate system)



(We can do similar thing to higher dimensional objects)
but the maths are way more complicated



E.g. Parametrize the curve $y = 3x^{\frac{3}{2}}$

Choice 1: Let $x = t^2$, then $y = 3(t^2)^{\frac{3}{2}} = 3t^3$

\Rightarrow Parametrize as $[t^2, 3t^3]$

Choice 2: Let $x = t$, then $y = 3t^{\frac{3}{2}}$

\Rightarrow Parametrize as $[t, 3t^{\frac{3}{2}}]$

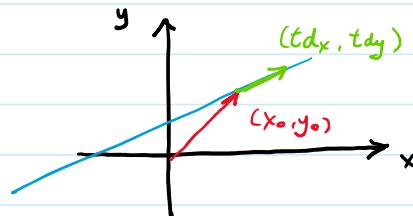
\vdots

Obviously there are infinitely more possibilities

Some common parametrization

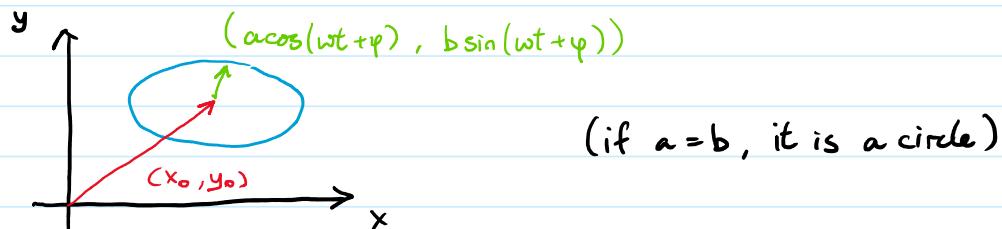
1 Straight line

$$[x(t), y(t)] = [x_0 + tdx, y_0 + tdy]$$



2 Ellipse / Circle

$$[x(t), y(t)] = [x_0 + a \cos(\omega t + \varphi), y_0 + b \sin(\omega t + \varphi)]$$

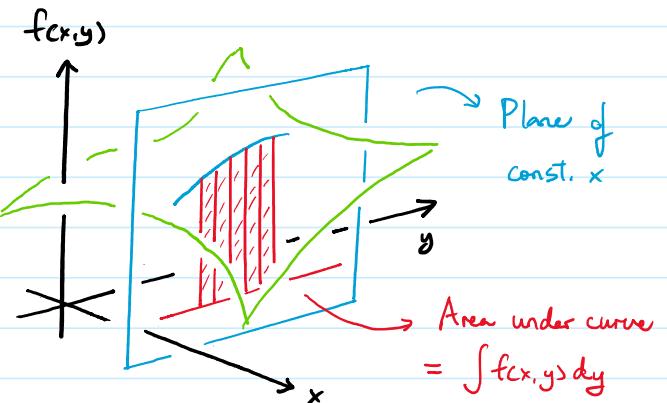
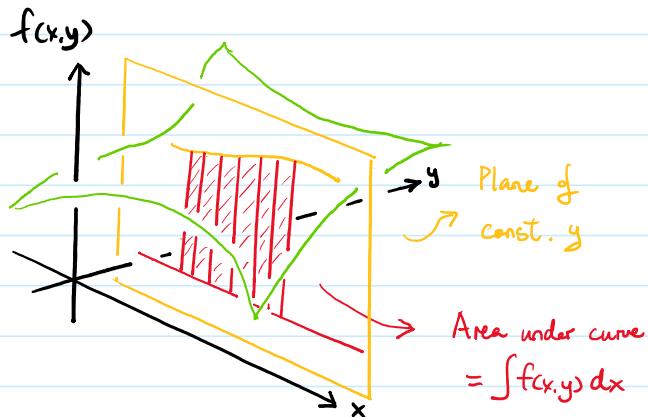


Line Integral on Scalar Function

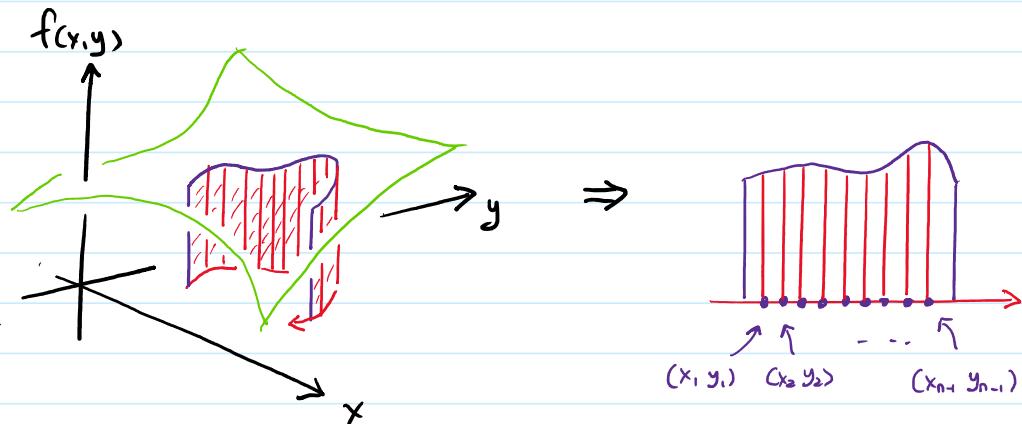
Recall

$\int f(x,y) dx$ = Integrate along x -axis, at constant y

$\int f(x,y) dy$ = Integrate along y -axis, at constant x



What about integrating along an arbitrary curve?



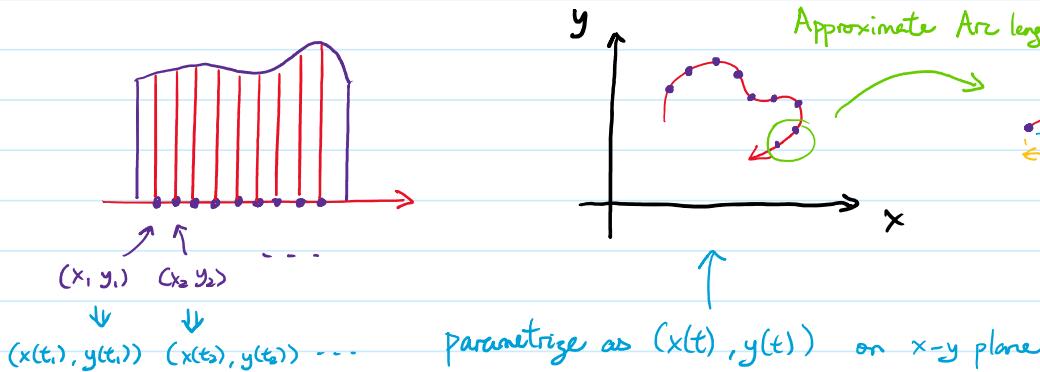
Recall integration on single variable function

$$\int f(x) dx = \lim_{\Delta x_i \rightarrow 0} \sum_{i=1}^n \underbrace{f(\xi_i)}_{\text{height of slice}} \underbrace{\Delta x_i}_{\text{width of slice}}$$

We can write something similar for line integral

$$\int f(x,y) d\vec{r} = \lim_{|\Delta \vec{r}_i| \rightarrow 0} \sum_{i=1}^n \underbrace{f(\xi_{i,x}, \xi_{i,y})}_{\text{height of slice}} \underbrace{|\Delta \vec{r}_i|}_{\text{width of slice}}$$

The width of each slice can be estimated by Pythagoras Thm.



Width of interval $|\Delta \vec{r}_i|$

$$= \sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2}$$

$$= |t_{i+1} - t_i| \sqrt{\left[\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i} \right]^2 + \left[\frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} \right]^2}$$

$$= |\Delta t_i| \sqrt{\left[\frac{\Delta x_i}{\Delta t_i} \right]^2 + \left[\frac{\Delta y_i}{\Delta t_i} \right]^2}$$

$$\sim dt \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \quad \text{when } \Delta t_i \rightarrow 0$$

So in the line integral definition :

$$\left| \int f(x, y) d\vec{r} \right| = \lim_{|\Delta \vec{r}_i| \rightarrow 0} \sum_{i=1}^n f(\xi_{i,x}, \xi_{i,y}) |\Delta \vec{r}_i|$$

Notation you
can find in
textbook

$$= \lim_{|\Delta t_i| \rightarrow 0} \sum_{i=1}^n f(\xi_{i,x}, \xi_{i,y}) |\Delta t_i| \sqrt{\left(\frac{\Delta x_i}{\Delta t_i} \right)^2 + \left(\frac{\Delta y_i}{\Delta t_i} \right)^2}$$

$$= \boxed{\int f(x, y) \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt}$$

This is how you really calculate line integral
eg. along a line on x-y plane

You have to decide how to parametrize the curve

Weighted Sum Interpretation

This is just like the interpretation in 1D integral

$$\int f(x) dx = \lim_{\Delta x_i \rightarrow 0} \sum_{i=1}^n f(\xi_{ix}, \xi_{iy}) |\Delta \vec{r}_i|$$

Sum them all "weight" assigned
to the interval length of the
interval $[(x_i, y_i), (x_{i+1}, y_{i+1})]$

More on notations

1 Integration range

Single var. integration only requires knowing the upper/lower bounds.

However in line integral we need to describe the whole curve

⇒ Too many words to write under the integral sign

⇒ denote the curve as "C" and define C in text

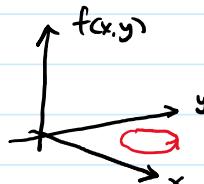
$$\int_C f(x, y) d\vec{r} \text{ with } C = \text{Curve of } \dots$$

2 Loop Integral

If the curve to be integrated along is a closed loop

We draw an extra circle on the integral sign

$$\oint_C f(x, y) d\vec{r}$$



This is because integrals over closed loop carry nice

properties for a lot of functions and are used in theorems

⇒ Create a new symbol for these specific uses.

Line Integration on Vector Functions

Note that $f(x,y) d\vec{r}$ is like a product between number \times vector

If $f(x,y)$ is replaced by a vector function, we have 2 possibilities:

Dot Product Line Integral

$$\int_C \vec{F}(x,y) \cdot d\vec{r} = \lim_{|\Delta\vec{r}_i| \rightarrow 0} \sum_{i=1}^n \vec{F}(\xi_{ix}, \xi_{iy}) \cdot \Delta\vec{r}_i$$

Cross Product Line Integral

$$\int_C \vec{F}(x,y) \times d\vec{r} = \lim_{|\Delta\vec{r}_i| \rightarrow 0} \sum_{i=1}^n \vec{F}(\xi_{ix}, \xi_{iy}) \times \Delta\vec{r}_i$$

★ Because order matters in cross product

$$\int_C \vec{F} \times d\vec{r} = - \int_C d\vec{r} \times \vec{F}$$

← this form is very common

Calculation

Demonstrate with dot product line integral (Similar for cross product)

$$\int_C \vec{F}(x,y) \cdot d\vec{r} = \lim_{|\Delta\vec{r}_i| \rightarrow 0} \sum_{i=1}^n \boxed{\vec{F}(\xi_{ix}, \xi_{iy}) \cdot \Delta\vec{r}_i}$$

Notice that this is a summation of all dot product along a curve

Express each dot product by the curve parameter t :

$$\begin{aligned}
 \vec{F}(x(t_i), y(t_i)) \cdot \Delta\vec{r}_i &= \vec{F}(x(t_i), y(t_i)) \cdot [\vec{r}(t_{i+1}) - \vec{r}(t_i)] \\
 &= \vec{F}(x(t_i), y(t_i)) \cdot \frac{\vec{r}(t_{i+1}) - \vec{r}(t_i)}{t_{i+1} - t_i} (t_{i+1} - t_i) \\
 &= \vec{F}(x(t_i), y(t_i)) \cdot \left(\frac{\Delta\vec{r}_i}{\Delta t_i} \right) \Delta t_i \\
 &\sim \left(\vec{F}(x, y) \cdot \frac{d\vec{r}}{dt} \right) dt
 \end{aligned}$$

\therefore Dot product line integral compute as :

$$\boxed{\int_C \vec{F}(x,y) \cdot d\vec{r}} = \boxed{\int_C \vec{F}(x(t),y(t)) \cdot \frac{d\vec{r}(t)}{dt} dt}$$

Notation in textbook

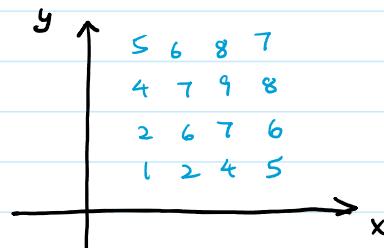
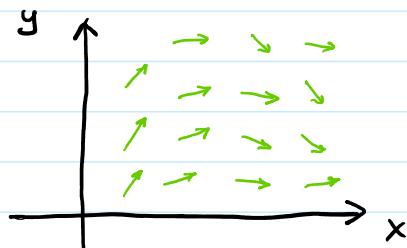
This is how you really calculate
You have to decide how to
parametrize the curve

Illustration

The function $\vec{F}(x,y)$ can be plotted as a "field of vectors"

i.e. At each point (x,y) , there is a vector (F_x, F_y)

(In scalar function, at each point (x,y) there is a number $f(x,y)$)



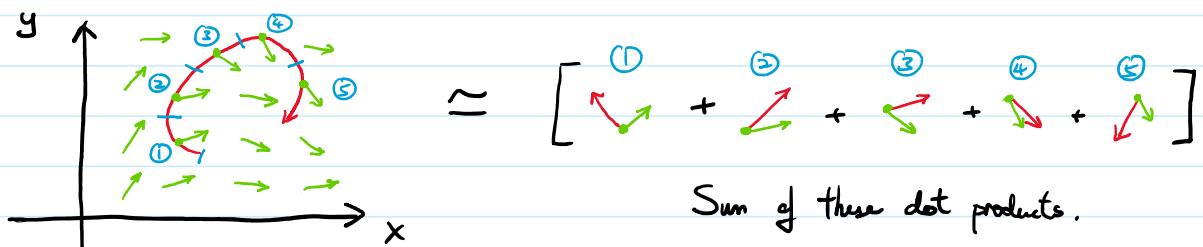
Vector function : Every pt. gives 1 vector

Scalar function : Every pt. gives 1 number

The quantity $\vec{F} \cdot d\vec{r}$ is like ,

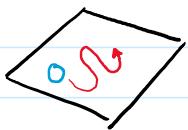
- Find the vector field on each segment
- Find the vector interval of each segment

Do dot product for each segment , then sum them all



Sum of these dot products.

Example An object moving on a surface with a positional



$$\text{dependent friction force} = \vec{F}(x,y) = (x^2y, xy+1)$$

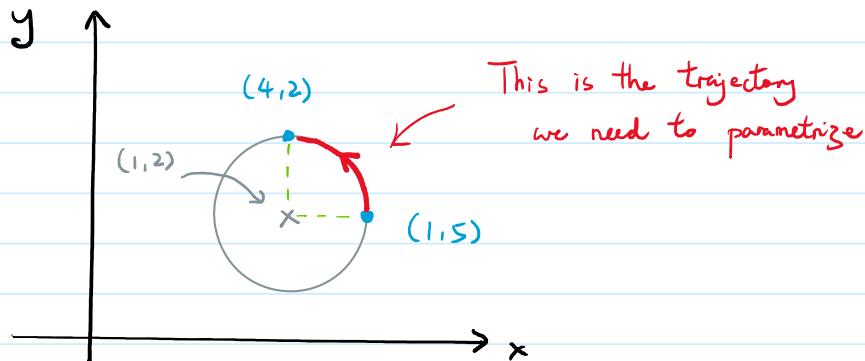
Suppose the object is moving in a circular trajectory

that is

- center at $(1, 2)$, radius = 3
- starts at $(1, 5)$, ends at $(4, 2)$
- travelling in anticlockwise direction

Find the W.D. along this trajectory.

① Begins with parametrization to the trajectory.



Recall that parametrization of circle is usually in the form

$$\begin{cases} x(\theta) = x_0 + R \cos(\theta + \varphi) \\ y(\theta) = y_0 + R \sin(\theta + \varphi) \end{cases}$$

From the given information, we can identify

$$(x_0, y_0) = (1, 2), R = 3, \varphi = 0$$

$$\Rightarrow \begin{cases} x(\theta) = 1 + 3 \cos \theta \\ y(\theta) = 2 + 3 \sin \theta \end{cases}$$

And so $\begin{cases} \theta = 0 \Leftrightarrow (x, y) = (1, 5) \\ \theta = \frac{\pi}{2} \Leftrightarrow (x, y) = (2, 4) \end{cases}$

(3) Substitute into dot product line integral

$$W.D. = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \vec{F}(x, y) \cdot \frac{d\vec{r}(\theta)}{d\theta} d\theta$$

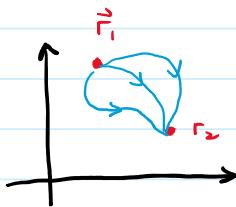
$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} (x(\theta)y(\theta) - x(\theta)y(\theta) + 1) \frac{d}{d\theta} \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} (1+3\cos\theta)(2+3\sin\theta)(-3\sin\theta) \frac{dx}{d\theta} d\theta$$
$$+ \left[(1+3\cos\theta)(2+3\sin\theta) + 1 \right] (3\cos\theta) \frac{dy}{d\theta} d\theta$$

The remaining steps is just an annoying single variable integration.

Gradient Theorem

Theorem : If $f(\vec{r})$ is a continuous scalar function



Integration always gives the same value

$\vec{\nabla}f(\vec{r})$ is its gradient vector (field)

Then $\int_{\text{start from } \vec{r}_1}^{\text{end at } \vec{r}_2} \vec{\nabla}f(\vec{r}) \cdot d\vec{r} = f(\vec{r}_2) - f(\vec{r}_1)$

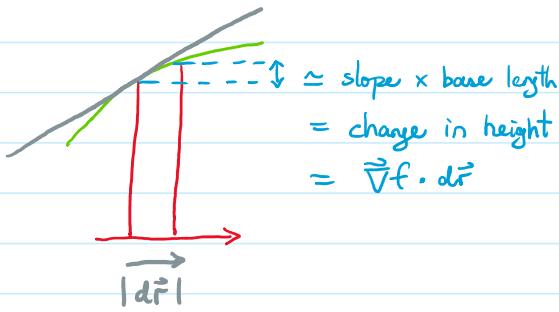
is independent of the curve to be integrated

Illustration

Recall $\vec{\nabla}f \cdot \hat{u} = \text{slope in } \hat{u} \text{ direction}$

$$\text{So } \vec{\nabla}f \cdot d\vec{r} = \vec{\nabla}f \cdot \left| \frac{d\vec{r}}{|d\vec{r}|} \right| |d\vec{r}|$$

unit vector



$$= (\text{slope in } d\vec{r} \text{ direction}) \times (\text{Base length})$$

$$= (\text{Change in height along } d\vec{r})$$

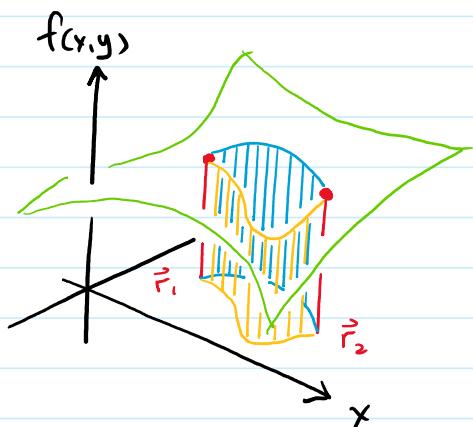
\Rightarrow Sum of all $\vec{\nabla}f \cdot d\vec{r}$ along a curve from \vec{r}_1 to \vec{r}_2

= Net height change by travelling from \vec{r}_1 to \vec{r}_2

When the landscape is continuous,

the net height change should be

independent of the path travelled



Application : Conservative Force & Potential

Any vector function $\vec{F}(\vec{r})$ is conservative if it equals to

the gradient of some scalar function $U(\vec{r})$

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r})$$

↑
Have a minus sign
in physics convention

★ The nice property when a force vector field is conservative:

Total W.D. along any path between \vec{r}_1 , \vec{r}_2

$$= \int_{\vec{r}_1 \rightarrow \vec{r}_2} \vec{F} \cdot d\vec{r} \quad (\text{Force} \cdot \text{displacement in each segment})$$

then sum all of them

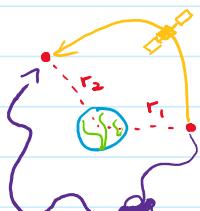
$$= \int_{\vec{r}_1 \rightarrow \vec{r}_2} -\vec{\nabla} U \cdot d\vec{r}$$

Gradient Theorem

$$= - (U_{(\vec{r}_2)} - U_{(\vec{r}_1)})$$

Only depends on the 2 end points. Independent to the path.

$$\text{E.g. } \vec{F}(\vec{r}) = \frac{GMm}{|\vec{r}|^2} \hat{r} \quad \text{is conservative}$$



That's why we can define Gravitational potential Energy

$$\Delta U(r) = - \int_{\vec{r}_1 \rightarrow \vec{r}_2} \vec{F} \cdot d\vec{r} = - \left(\frac{GMm}{|\vec{r}_2|} - \frac{GMm}{|\vec{r}_1|} \right)$$

And we never need to calculate line integral even in 3D

P.S. To prove that a force field is conservative requires us to show that its curl = 0

$$\text{i.e., } \vec{\nabla} \times \vec{F} = 0$$

However we will not touch curl until E&M