- Vectors in Polar Coordinate → What do $\vec{s}/\vec{v}/\vec{a}$ become?
- Moment of Inertia → Why is it mr^2?
  → Can they be added like mass?
- Newton 2nd Law & Momentum in polar coordinate
- KE in polar coordinate

	Linear Motion Family	Rotational Motion Family	
Displacement	$d\vec{x}$, $d\vec{y}$	$d\vec{\theta}$	$d\vec{r}$
Velocity	$\vec{v}_x$, $\vec{v}_y$	$\vec{\omega}$	$\vec{v}_r$
Acceleration	$\vec{a}_x$, $\vec{a}_y$	$\vec{\alpha}$	$\vec{a}_r$
Inertia	m	I	m
Force	$\vec{F}_x$, $\vec{F}_y$	$\vec{\tau}$	$\vec{F}_r$

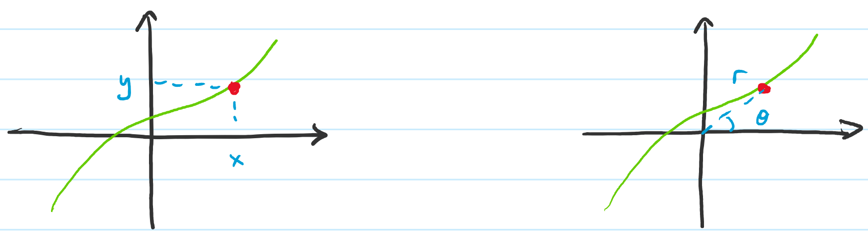
Newton 2nd Law	$\begin{cases} \vec{F}_x = m\vec{a}_x \\ \vec{F}_y = m\vec{a}_y \end{cases}$	$\begin{cases} \vec{\tau} = I\vec{\alpha} \\ \vec{F}_r = ma_r \end{cases}$
W.D.	$\vec{F}_x \cdot d\vec{x} + \vec{F}_y \cdot d\vec{y}$	$\vec{\tau} \cdot d\vec{\theta} + \vec{F}_r \cdot d\vec{r}$
KE	$\frac{1}{2}m(v_x^2 + v_y^2)$	$\frac{1}{2}I\omega^2 + \frac{1}{2}mv_r^2$
Momentum	$m\vec{v}_x + m\vec{v}_y$	$I\omega$, $m\vec{v}_r$

☆ Linear motion are described by 2 variables (x, y)

But in rotational motion we usually only see 1 variable.

This is because pure rotation is not a complete description to 2D motion

For a complete discription, you also need the radial component



Vector in polar coordinate

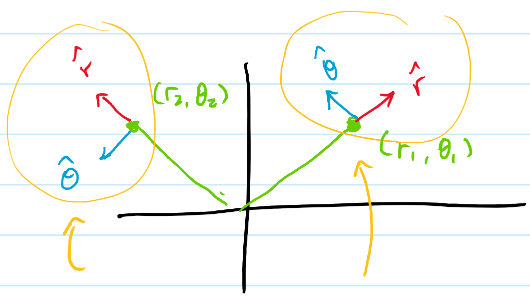
In x/y coordinate, every vector can be expressed using $\{\hat{x}, \hat{y}\}$

$$\vec{S} = S_x \hat{x} + S_y \hat{y}$$

In polar coordinate, we wish to express vectors using $\{\hat{r}, \hat{\theta}\}$

$$\vec{S} = S_r \hat{r} + S_\theta \hat{\theta}$$

☆ However the unit vector $\{\hat{r}, \hat{\theta}\}$ are not "constant"



- $\hat{r}$ should always be radially outward from origin
- $\hat{\theta}$ should always be $\perp$ to $\hat{r}$

The direction of the unit vector depends on position

⇒ They should be written as a function of (r, θ)

$$\{\hat{r}, \hat{\theta}\}_{\text{at } (r_1, \theta_1)} \neq \{\hat{r}, \hat{\theta}\}_{\text{at } (r_2, \theta_2)}$$

This makes a big difference in differentiation. Compare

① Vector in terms of $\{\hat{x}, \hat{y}\}$

$$\vec{s} = s_x \hat{x} + s_y \hat{y}$$

(☆ Product rule can apply to unit vector too)

$$\frac{d}{dt}\vec{s} = \frac{d}{dt}(s_x)\,\hat{x} + \cancelto{0}{s_x \frac{d}{dt}(\hat{x})} + \frac{d}{dt}(s_y)\,\hat{y} + \cancelto{0}{s_y \frac{d}{dt}(\hat{y})}$$

$\because$ $\hat{x}, \hat{y}$ are always pointing at the same direction

$\therefore$ Differentiation to them must be 0

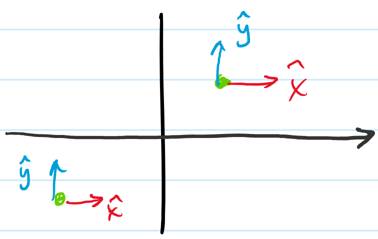
② Vectors in terms of $\{\hat{r}, \hat{\theta}\}$

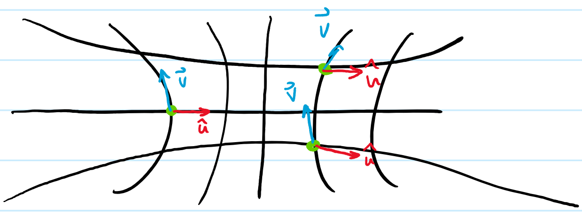
$$\vec{s} = s_r \hat{r} + s_\theta \hat{\theta}$$

$$\frac{d}{dt}\vec{s} = \frac{d}{dt}(s_r)\,\hat{r} + \underline{s_r \frac{d}{dt}(\hat{r})} + \frac{d}{dt}(s_\theta)\,\hat{\theta} + \underline{s_\theta \frac{d}{dt}(\hat{\theta})}$$

These 2 terms are non zero in general because they depend on position (r, θ), and $(r, \theta) = (r(t), \theta(t))$ can be function of time.

☆☆ This is also true for any non-rectangular coordinate



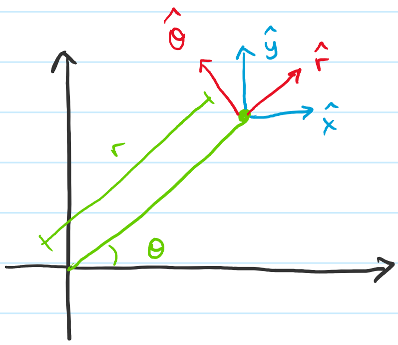


Differentiation to $\{\hat{r}, \hat{\theta}\}$

Because $\{\hat{x}, \hat{y}\}$ never change, we usually call it "ambient coordinate"

$\Rightarrow$ They can be used as reference for expressing any other coordinate

E.g. Expressing $\{\hat{r}, \hat{\theta}\}$ by $\{\hat{x}, \hat{y}\}$



The relations are simply trigolometric.

$$\begin{cases} \hat{r}_{at\,(r,\theta)} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \hat{\theta}_{at\,(r,\theta)} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \end{cases}$$

Obviously $\hat{r}$ & $\hat{\theta}$ are function to θ only. So

$$\boxed{\frac{d}{dr}(\hat{r}) = \frac{d}{dr}(\hat{\theta}) = 0}$$

$$\boxed{\begin{aligned} \frac{d}{d\theta}(\hat{r}) &= -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} = \hat{\theta}_{at\,(r,\theta)} \\ \frac{d}{d\theta}(\hat{\theta}) &= -\cos\theta\,\hat{x} - \sin\theta\,\hat{y} = -\hat{r}_{at\,(r,\theta)} \end{aligned}}$$

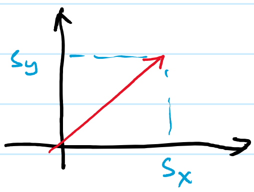
Their differentiation by θ yeilding each other is just a numerical coincidence.

Position Vector

Starting with a position vector in terms of $\{\hat{x}, \hat{y}\}$

$$\vec{s} = s_x\,\hat{x} + s_y\,\hat{y}$$

$$= \begin{pmatrix} s_x & s_y \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$



$$= \boxed{(s_x \;\; s_y)\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}}\boxed{\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x} \\ \hat{y}\end{pmatrix}}$$

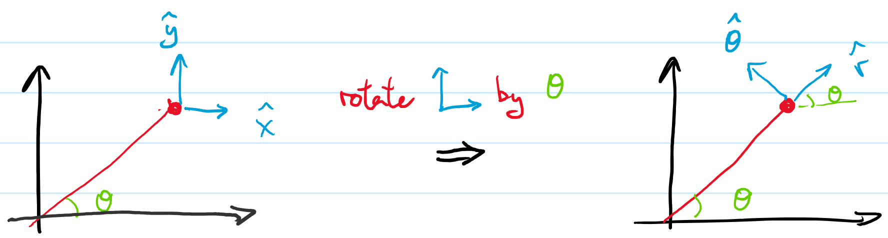
these 2 matrices are inverse of each other

i.e. their product $= \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$

they are called the "rotational matrix"

$$\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x} \\ \hat{y}\end{pmatrix} = \begin{pmatrix}\cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\end{pmatrix} = \begin{pmatrix}\hat{r}_{\text{at }(r,\theta)} \\ \hat{\theta}_{\text{at }(r,\theta)}\end{pmatrix}$$

i.e. $\begin{pmatrix}\text{Apply rotational} \\ \text{matrix of angle } \theta\end{pmatrix}\begin{pmatrix}\hat{x} \\ \hat{y}\end{pmatrix} \xrightarrow{\text{become}} \begin{pmatrix}\hat{r} \\ \hat{\theta}\end{pmatrix}_{\text{at }(r,\theta)}$



The remaining part can be thought as "inverse rotation" to the components

$$(s_x \;\; s_y)\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

$$= \left(|s|\cos\theta \;\; |s|\sin\theta\right)\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

$$= \left(|s|\cos^2\theta + |s|\sin^2\theta \quad -|s|\cos\theta\sin\theta + |s|\sin\theta\cos\theta\right)$$

$$= \left(|s| \quad 0\right)$$

⇒ The expression of a position vector in terms of $\{\hat{r}, \hat{\theta}\}$

$$\vec{s} = \begin{pmatrix} |s| & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix}_{\text{at } (r,\theta)} = |s| \; \hat{r}_{\text{at}(r,\theta)}$$

The radial component only tells the info about radial distance from origin

The info about the angle is hidden in the unit vector which is normally not written out

Comparing with

$$\vec{s} = s_x \hat{x} + s_y \hat{y}$$

x component tells the horizontal distance from origin

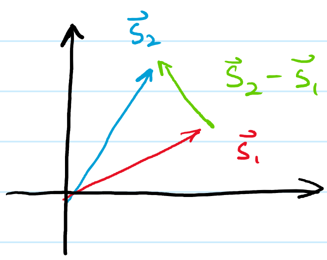
y component tells the vertical distance from origin

$\{\hat{x}, \hat{y}\}$ are just label

They tell us nothing about $\vec{s}$

Displacement Vector

Displacement = Subtraction between 2 position vectors.



In x/y coordinate, subtraction is done simply component by component

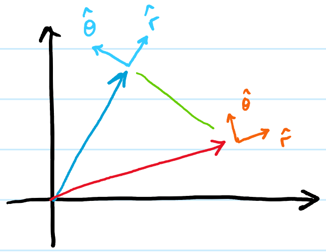
$$\vec{s}_2 - \vec{s}_1 = (s_{2x}\hat{x} + s_{2y}\hat{y}) - (s_{1x}\hat{x} + s_{1y}\hat{y})$$

$$= [s_{2x} - s_{1x}]\hat{x} + [s_{2y} - s_{1y}]\hat{y}$$

But in polar coordinate, you cannot

$$\vec{S}_2 - \vec{S}_1 = \left(S_{2r}\, \hat{r}_{at\,(r_2,\theta_2)} + S_{2\theta}\, \hat{\theta}_{at\,(r_2,\theta_2)} \right) - \left(S_{1r}\, \hat{r}_{at\,(r_1,\theta_1)} + S_{1\theta}\, \hat{\theta}_{at\,(r_1,\theta_1)} \right)$$

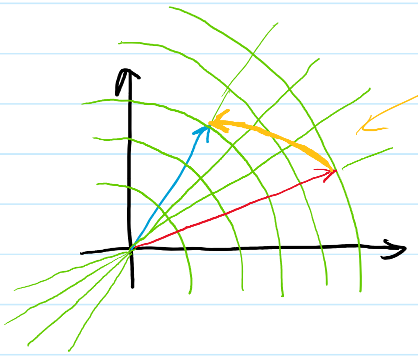
because $(\hat{r}, \hat{\theta})_{at\,(r_2,\theta_2)} \neq (\hat{r}, \hat{\theta})_{at\,(r_1,\theta_1)}$



You must go through a lot of sine/cosine to find the result vector

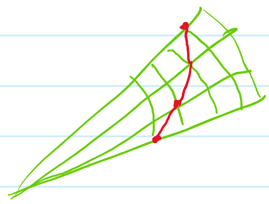
This happens in all non-rectangular coordinate. Subtraction directly by component does not give the correct vector.

Visually they are not even straight line



Not straight line

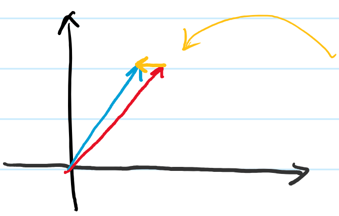
Not well defined by $\{\hat{r}, \hat{\theta}\}$



But we can still consider the infinitestimal displacement

i.e. The subtraction between 2 very close vector

So that the result vector is "approximately" a straight line



We can even take limit to length ~ 0

Suppose we have 2 position vector parametrized by variable t

When $\Delta t \to 0$, they become the same vector.

Their subtraction can be written as:

$$\vec{s}(t+\Delta t) - \vec{s}(t)$$

$$= |\vec{s}(t+\Delta t)|\, \hat{r}_{at\,[r(t+\Delta t),\, \theta(t+\Delta t)]} - |\vec{s}(t)|\, \hat{r}_{at\,[r(t),\, \theta(t)]}$$

$$= |\vec{s}(t+\Delta t)|\, \hat{r}_{at\,[r(t+\Delta t),\, \theta(t+\Delta t)]} \boxed{- |\vec{s}(t+\Delta t)|\, \hat{r}_{at\,[r(t),\, \theta(t)]}}$$

minus, then add back

$$\boxed{+ |\vec{s}(t+\Delta t)|\, \hat{r}_{at\,[r(t),\, \theta(t)]}} - |\vec{s}(t)|\, \hat{r}_{at\,[r(t),\, \theta(t)]}$$

$$= |\vec{s}(t+\Delta t)| \left[\hat{r}_{at\,[r(t+\Delta t),\, \theta(t+\Delta t)]} - \hat{r}_{at\,[r(t),\, \theta(t)]} \right]$$

Same component, different unit vector

$$+ \left[|\vec{s}(t+\Delta t)| - |\vec{s}(t)| \right] \hat{r}_{at\,[r(t),\, \theta(t)]}$$

different component, same unit vector

$$= |\vec{s}|\, \Delta(\hat{r}) + \Delta(|\vec{s}|)\, \hat{r}$$ Exactly the product rule

Take limit $\Delta t \to 0$. Then difference becomes differential.

$$d(\vec{s}(t)) = d\left[|\vec{s}(t)|\, \hat{r}_{at\,[r(t),\, \theta(t)]} \right]$$

$$= |\vec{s}(t)|\, d\left[\hat{r}_{at\,[r(t),\, \theta(t)]} \right] + d\left[|\vec{s}(t)| \right] \hat{r}_{at\,[r(t),\, \theta(t)]}$$

$$= |\vec{s}(t)|\, \hat{\theta}_{at\,[r(t),\, \theta(t)]}\, d\left[\theta(t)\right] + d\left[|\vec{s}(t)| \right] \hat{r}_{at\,[r(t),\, \theta(t)]}$$

By $\boxed{\frac{d\hat{r}}{dt}} = \frac{d\hat{r}}{d\theta} \cdot \frac{d\theta}{dt} = \boxed{\hat{\theta} \cdot \frac{d\theta}{dt}}$

$$\Rightarrow d\vec{s} = |\vec{s}|\,\hat{\theta}\,d\theta + d(|\vec{s}|)\,\hat{r} \quad \text{, if we try to omit all } t$$

☆☆ $d\vec{r} = r\,\hat{\theta}\,d\theta + \hat{r}\,dr$ is what usually written book

but such notation can be very confusing! Note that $d\vec{r} \neq dr$!

Infinitesimal Angular displacement

By cross product with $\hat{r}_{at\,[r,\theta]}$

$$\begin{cases} \hat{r} \times \hat{r} = \underline{0} \\ \hat{r} \times \hat{\theta} = \hat{z} \end{cases}$$

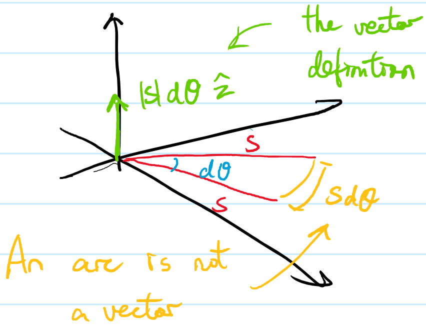
we can make the $\hat{r}$ term vanish and isolate out the info about $\hat{\theta}$ term.

$$\hat{r}_{at\,[r(t),\,\theta(t)]} \times d[\vec{s}(t)] = |\vec{s}(t)|\,d[\theta(t)] \cdot \hat{z} + d[|\vec{s}(t)|] \cdot 0$$

$$= |\vec{s}(t)|\,d[\theta(t)]\,\hat{z}$$

Re-arranging :

$$\boxed{d[\theta(t)] = \frac{\hat{r}_{at\,[r(t),\,\theta(t)]} \times d[\vec{s}(t)]}{|\vec{s}(t)|}}$$



This is the vector definition to angular displacement

(Its magnitude is just like $\theta = \frac{s}{r}$)

We can repeat differentiating to get the expression of velocity & acceleration vector on polar coordinate.

Velocity Vector

From infinitesimal displacement, we can get velocity as

$$\Rightarrow \text{Velocity} = \frac{\text{Infinitesimal displacement}}{\Delta t}$$

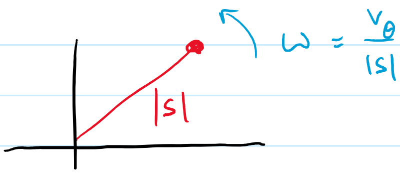
$$= \frac{d\vec{s}(t)}{dt}$$

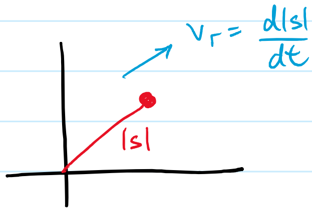
$$= |\vec{s}(t)| \cdot \hat{\theta}_{\text{at } [r(t),\, \theta(t)]} \cdot \boxed{\frac{d\theta(t)}{dt}} + \frac{d|\vec{s}(t)|}{dt} \cdot \hat{r}_{\text{at } [r(t),\, \theta(t)]}$$

Notation for angular velocity

$$= \underline{|\vec{s}(t)| \cdot \boxed{\omega(t)}\, \hat{\theta}_{\text{at } [r(t),\, \theta(t)]}} + \underline{\frac{d|\vec{s}(t)|}{dt} \cdot \hat{r}_{\text{at } [r(t),\, \theta(t)]}}$$

(Rate of change in angular direction = radius × angular velocity) (Rate of change in radial direction)





($\underline{\vec{V} = r\omega\hat{\theta} + \frac{dr}{dt}\hat{r}}$ is what usually appears in textbooks.)

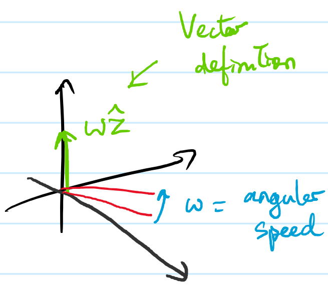
Again, we can take cross product with $\hat{r}_{\text{at}(r,\theta)}$ to isolate out the angular component.

$$\hat{r}_{\text{at}(r,\theta)} \times \vec{V}(t) = |\vec{s}(t)| \cdot \omega(t) \cdot \hat{z} + \frac{d|\vec{s}(t)|}{dt} \cdot 0$$

$$= |\vec{s}(t)| \cdot \omega(t) \cdot \hat{z}$$

Rearranging to have

$$\omega(t) \cdot \hat{z} = \frac{\hat{r}_{\text{at}(r,\theta)} \times \vec{V}(t)}{|\vec{s}(t)|}$$



This is the vector definition to angular velocity.
Its magnitude is just like $\omega = \frac{v}{r}$

Acceleration Vector

Differentiate the velocity vector

$$\frac{d}{dt}\vec{v}(t) = \frac{d}{dt}\left[|\vec{s}(t)|\,\omega(t)\cdot\hat{\theta}_{\text{at }[r(t),\theta(t)]}\right] + \frac{d}{dt}\left[\frac{d|\vec{s}(t)|}{dt}\,\hat{r}_{\text{at }[r(t),\theta(t)]}\right]$$

$\vdots$ (Skipping the steps. Just product rules)

$$\vec{a}(t) = \Bigg[\underbrace{\frac{d^2|s(t)|}{dt^2}}_{①} - \underbrace{|\vec{s}(t)|\cdot\omega(t)^2}_{②}\Bigg]\hat{r}_{\text{at }[r(t),\theta(t)]}$$

$$+\Bigg[\underbrace{2\cdot\frac{d|s(t)|}{dt}\cdot\omega(t)}_{③} + \underbrace{|s(t)|\cdot\boxed{\frac{d\omega(t)}{dt}}}_{④}\Bigg]\hat{\theta}_{\text{at }[r(t),\theta(t)]}$$

$\alpha(t)$ = rotation of angular acceleration

There are 4 terms in total

① $\frac{d^2}{dt^2}|s(t)|\,\hat{r}$ = Acceleration in radial direction.

② $-|\vec{s}(t)|\,[\omega(t)]^2\,\hat{r}$ = Centripetal force

radius $\times$ (angular velocity)^2 $\sim r\omega^2$

-ve for pointing to the origin.

③ $2\cdot\frac{d|\vec{s}(t)|}{dt}\cdot\omega(t)\,\hat{\theta}$ = Coriolis acceleration

This term appears only if radial distance is changing, i.e. $\frac{d|s|}{dt}\neq 0$

④ $|s(t)|\,\alpha(t)\,\hat{\theta}$ = Euler acceleration

This term appears only if there is angular acceleration, i.e. $\alpha\neq 0$

They correspond to the 4 kinds of pseudo force in a rotational reference frame

This time because there 2 terms in $\hat{\theta}$, taking cross product with $\hat{r}_{at\,[r(t),\theta(t)]}$ no longer gives trivial definition of $\vec{\alpha}(t)$

$$\hat{r}_{at\,[r(t),\theta(t)]} \times \vec{a}(t) = [\underbrace{\cdots}_{\text{the } \hat{r} \text{ component}}] \cdot 0 + \left[2 \cdot \frac{d|s(t)|}{dt} \omega(t) + |s(t)|\alpha(t)\right] \cdot \hat{z}$$

$$\alpha(t) \cdot \hat{z} = \frac{\hat{r}_{at\,[r(t),\theta(t)]} \times \vec{a}(t)}{|s(t)|} - 2 \frac{1}{|s(t)|} \boxed{\frac{d|s(t)|}{dt}} \omega(t)\, \hat{z}$$

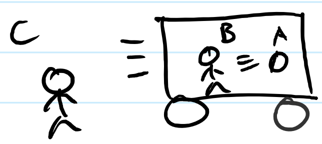
This is the vector definition for angular acceleration.

Note the $\alpha = \frac{a}{r}$ hold only if radial distance is a constant

---

Relative Angular velocity ? No

In linear motion, relative velocity can be added directly.



B throws ball A at $v = \vec{V}_{AB}$

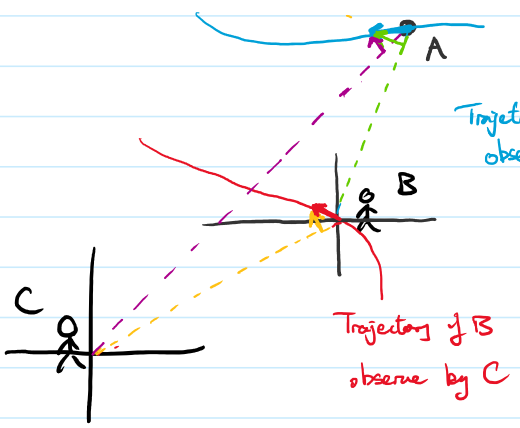
C sees the car moves at $v = \vec{V}_{BC}$

$\Rightarrow$ Relative velocity of A to C is

$$\vec{V}_{AC} = \vec{V}_{AB} + \vec{V}_{BC}$$

But for rotational motion

Note the angular velocity is just the $\hat{\theta}$ component of velocity



↖ = Angular compent of $\vec{V}_{BC}$

↖ = Angular component of $\vec{V}_{AB}$

↖ = Angular component of $\vec{V}_{AC}$

You MUST NOT add angular velocity directly

because you cannot add the angular component of a vector directly

Vectors in polar coordinate can be added only if you consider both its radial & angular components.

$$
\begin{aligned}
\vec{V}_{AC} &= \left(V_{AC,r} \boxed{\hat{r}_{AC}} + V_{AC,r} \cdot \boxed{\hat{\theta}_{AC}} \right) \\
&= \left(V_{AB,r} \boxed{\hat{r}_{AB}} + V_{AB,\theta} \cdot \boxed{\hat{\theta}_{AB}} \right) + \left(V_{BC,r} \boxed{\hat{r}_{BC}} + V_{BC,\theta} \boxed{\hat{\theta}_{BC}} \right) \\
&= \vec{V}_{AB} + \vec{V}_{BC}
\end{aligned}
$$

But still you cannot add the component directly

because $\{\hat{r}, \hat{\theta}\}_{AC} \neq \{\hat{r}, \hat{\theta}\}_{AB} \neq \{\hat{r}, \hat{\theta}\}_{BC}$

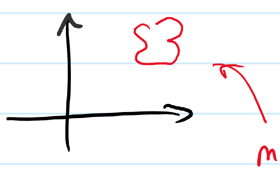
If you need to find the relative velocity, it is just more convenient by first converting back to x/y coordinate.

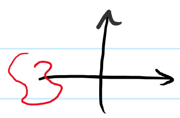
Moment of inertia

$$I = \int x^2 + y^2 \, dm = \sum m (x_i^2 + y_i^2)$$

☆☆☆ Moment of inertia depends on our choice of coordinate.

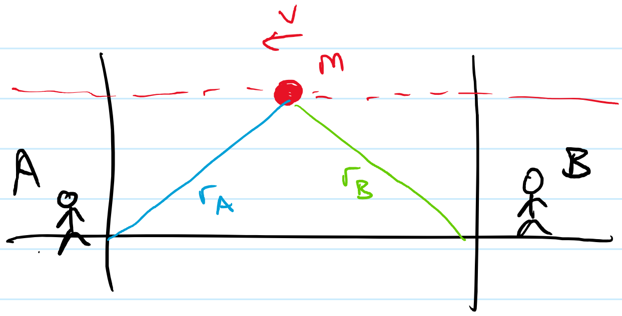
In comparison, mass is a constant quantity independent of coordinate





But this is not true for moment of inertia

Eg. A particle travelling on a straight line.



I observe by A $= m r_A^2$

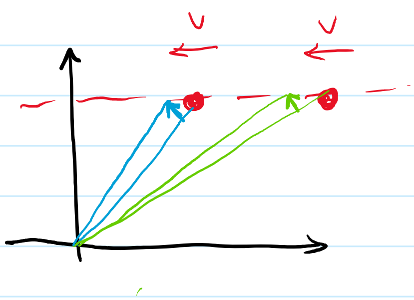
I observe by B $= m r_B^2$

Their observation to I are not equal

☆☆. Moment of inertia of objects can be added only if

The objects have the same angular velocity ω ①

measuring from the same origin ②



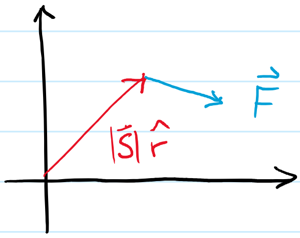
Eg. Two particles travelling at the same velocity v

But their angular components are different. So we cannot add their moment of inertia.

To explain these rules, we need to look into how moment of inertia is invented.

Torque & Angular momentum conservation.

Torque is defined by



$$\vec{\tau} = (\text{position vector}) \times (\text{Force on that position})$$

$$= |\vec{s}|\hat{r} \times \vec{F}$$

$$= |\vec{s}|\hat{r} \times m\vec{a}$$

$$= |\vec{s}|\hat{r} \times m\left[[a_r]\hat{r} + [a_\theta]\hat{\theta} \right]$$

0 $\hat{z}$

$$= |\vec{s}|\, m[a_\theta]\, \hat{z}$$

Same as how we isolate the angular component in previous derivation of $d\vec{\theta}$, $\vec{\omega}$, $\vec{\alpha}$

(While $m[a_r]\hat{r} = \vec{F}$
= Force in radial component)

Substitute the angular component of acceleration vector

$$\vec{\tau} = m|\vec{s}|\left[2 \cdot \frac{d|\vec{s}|}{dt}\omega + |\vec{s}|\frac{d\omega}{dt} \right]\hat{z}$$

$$= m\left[\boxed{2|s|}\frac{d|s|}{dt}\omega + |s|^2\frac{d\omega}{dt} \right]\hat{z} \qquad \text{Substitution}$$

$$= m\left[\frac{d}{dt}\left(\boxed{|s|^2}\right)\boxed{\omega} + \boxed{|s|^2}\frac{d\boxed{\omega}}{dt} \right]\hat{z} \qquad \text{Product rule!}$$

$$= m\left[\frac{d}{dt}\left(|s|^2\omega\right) \right]\hat{z}$$

$$= \frac{d}{dt}\left[m|s|^2\omega\,\hat{z} \right]$$

= Rate of change of "Some vector" $m|s|^2\omega\,\hat{z}$

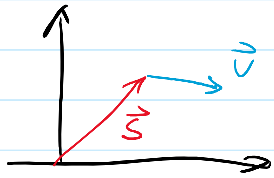
Compare with $\vec{F} = \frac{d}{dt}(mv)$ = Rate of change of "momentum"

Let's call this vector "Angular momentum" $\vec{L}$

$$\vec{L} = m|\vec{s}|^2 \boxed{\omega\hat{z}} = m|\vec{s}|^2 \boxed{\left(\frac{\hat{r}\times\vec{v}}{|\vec{s}|}\right)} = (m|\vec{s}|\hat{r})\times V = m\vec{s}\times\vec{v}$$

by vector definition

= (mass) (position vector) × (velocity at that position)



Also by comparing with linear momentum $\vec{p} = m\vec{v}$

$$\vec{p} = m\vec{v} = (\text{inertia}) \cdot (\text{velocity in linear family})$$

Why don't we make a new quantity of inertia as parity?

$$\vec{L} = (\text{sth. about inertia}) \cdot (\text{velocity in rotational family})$$

$$= \boxed{mr^2} \cdot \boxed{\omega\hat{z}}$$

So we invent a new quantity $I = mr^2$ = "moment of inertia"

Short Summary

① Define torque as a quantity relating to the angular component of force

↳ Find that it can be written as the rate of change of "some conserved quantity"

i.e. $\tau = \frac{d}{dt}\left(\text{some conserved quantity}\right)$

↳ Compare with $F = \frac{d}{dt}(\text{momentum})$

↳ Let's name this (some conserved quantity) as Angular momentum $\vec{L}$

② Find that Angular momentum can be written as

$$\vec{L} = \left(\text{some property about the object}\right) \times (\text{angular velocity})$$

↳ Compare with $\vec{p} = m\vec{v} = (\text{inertia}) \times (\text{velocity})$

↳ Let's name this $\left(\begin{array}{c}\text{some property} \\ \text{about the object}\end{array}\right)$ as moment of inertia I

which can be used like the "mass in rotational motion".

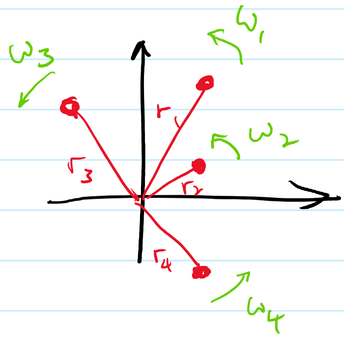
Now come back to the requirement for adding moment of inertia

By Newton 2nd Law in rotational motion :

$$\left(\begin{array}{c}\text{Total Torque to} \\ \text{the system}\end{array}\right) = \frac{d}{dt}\left(\begin{array}{c}\text{Total angular momentum} \\ \text{in the system.}\end{array}\right)$$

$$\sum \vec{\tau} = \frac{d}{dt}\left[\sum \vec{L}_i\right]$$

$$= \frac{d}{dt}\left[\sum m_i r_i \omega_i \hat{z}\right]$$



Only if $\omega_1 = \omega_2 = \omega_3 = \cdots$

Then ω_i can be taken out of the summation

$$\sum \tau = \frac{d}{dt}\left[\boxed{\sum m_i r_i^2}\right] \omega \hat{z}$$

Only then we can call this the moment of inertia of the system.

Otherwise you must write

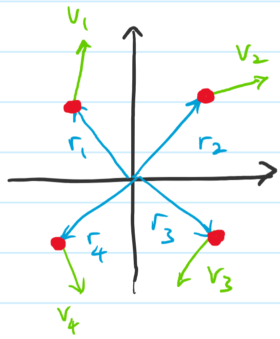
$$\vec{L} = \left[m_1 r_1 \omega_1 + m_2 r_2 \omega_2 + \cdots\right]\hat{z} \longleftarrow$$

meaningless to write $I = m_1 r_1^2 + m_2 r_2^2 + \cdots$ in this case

which has no expression to a single I

Angular momentum relative to CM

Again consider a many particle system.



Its total angular momentum

$$\vec{L} = \sum m_i r_i^2 \omega_i \hat{z} = \sum m_i \vec{r}_i \times \vec{v}_i$$

We can separate the motion of center of mass by taking

$$\begin{cases} \vec{r}_i = \vec{r}_{cm} + \vec{R}_{i,cm} \\ \vec{v}_i = \vec{v}_{cm} + \vec{V}_{i,cm} \end{cases}$$

Here : $\vec{r}_{cm}$, $\vec{v}_{cm}$ = Position / Velocity vector of the CM

$\vec{R}_{i,cm}$, $\vec{V}_{i,cm}$ = Relative position / velocity vector of each object to the CM.

$$\Rightarrow \vec{L} = \sum m_i (\vec{r}_{cm} + \vec{R}_{i,cm}) \times (\vec{v}_{cm} + \vec{V}_{i,cm})$$

break into 4 terms

$$= \sum m_i \left[\vec{r}_{cm} \times \vec{v}_{cm} + \vec{r}_{cm} \times \vec{V}_{i,cm} + \vec{R}_{i,cm} \times \vec{v}_{cm} + \vec{R}_{i,cm} \times \vec{V}_{i,cm} \right]$$

$$= \left(\sum m_i\right)(\vec{r}_{cm} \times \vec{v}_{cm}) + \boxed{\vec{r}_{cm} \times \sum \cancel{(m_i \vec{V}_{i,cm})}}^{\,0}$$

total velocity relative to CM = 0

$$+ \boxed{\sum \cancel{(m_i \vec{R}_{i,cm})} \times \vec{v}_{cm}}^{\,0} + \sum (m_i \vec{R}_{i,cm} \times \vec{V}_{i,cm})$$

total position different from CM = 0

These 2 terms vanish because of the definition of the CM.

$$\vec{r}_{cm} = \boxed{\frac{\sum m_i \vec{r}_i}{\sum m_i}}$$

this is in HKPhO syllabus

$$= \frac{\sum m_i (\vec{r}_{cm} + \vec{R}_{i,cm})}{\sum m_i}$$

$$= \frac{\sum m_i \boxed{\vec{r}_{cm}}}{\sum m_i} + \frac{\sum m_i \vec{R}_{i,cm}}{\sum m_i}$$

do not depend on i

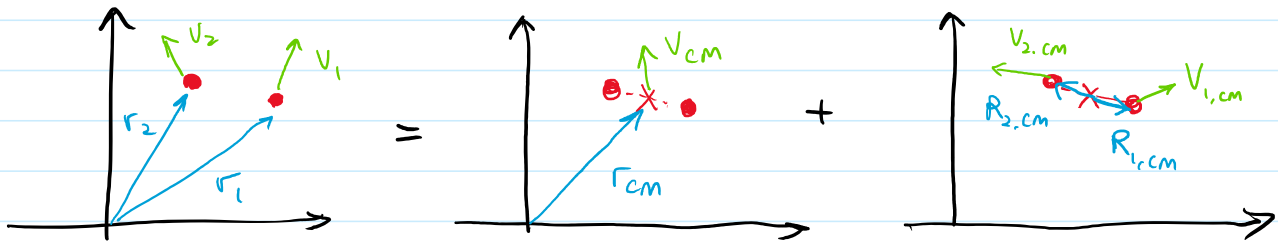
$$= \vec{r}_{cm} + \frac{\sum m_i \vec{R}_{i,cm}}{\sum m_i}$$

$$\therefore \sum m_i \vec{R}_{i,cm} = 0.$$

And also $\frac{d}{dt} \sum m_i \vec{R}_{i,cm} = \sum m_i \vec{V}_{i,cm} = 0$

$$\Rightarrow \vec{L} = \left(\sum m_i\right)\left(\vec{r}_{cm} \times \vec{v}_{cm}\right) + \sum \left(m_i \vec{R}_{i,cm} \times \vec{V}_{i,cm}\right)$$

$= \left(\text{Angular momentum of CM about origin}\right) + \left(\text{Angular momentum of each object using CM as origin}\right)$



So we can break down $\vec{L}$ of any object into 2 parts

[1] The angular momentum of CM relative to the origin

[2] The angular momentum of all individual object relative to CM

$$\vec{L} = \underbrace{M}_{\text{total mass}} \cdot \vec{r}_{cm} \times \vec{v}_{cm} + \sum \left(m_i \vec{R}_{i,cm} \times \vec{V}_{i,cm}\right)$$

$$= \boxed{M |\vec{r}_{cm}|^2} \omega_{cm} \hat{z} + \sum \left(\boxed{m_i |\vec{R}_{i,cm}|^2} \omega_{i,cm}\right) \hat{z}$$

Interpret as the I of a point object rotating about origin

Interpret as the I of each individual object rotating about CM

$$= \boxed{I_0} \omega_{cm} \hat{z} + \sum \left(\boxed{I_{i,cm}} \omega_{i,cm}\right) \hat{z}$$

☆☆☆ The "relative" angular momentum can only be applied using CM

$\vec{L} \neq$ (Angular momentum of a random point) + (Angular momentum of objects about a random point)

This point must be taken as the CM

Summary : Guide in finding $\vec{L}$ of a system

The universal approach:

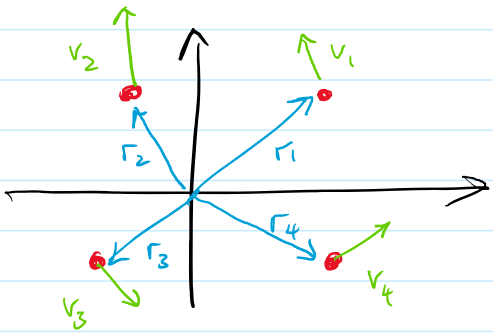
① Choose a "convenient point" as the origin. Never change it again

② Then $\vec{L}_i$ of each object in the system

is given by $\vec{L}_i = m_i \underbrace{\vec{r}_i}_{\text{distance of each object from the origin}} \times \underbrace{\vec{v}_i}_{\text{the object's velocity}}$

③ Total $\vec{L}$ of the system relative to the origin

$$= \sum m_i \vec{r}_i \times \vec{v}_i$$



☆ Note that $\vec{L}$ is well-defined even though the trajectory of the objects are not rotation.

It simply means $\vec{L} = \vec{L}(t)$ could be changing in time (and imply torque is present.)

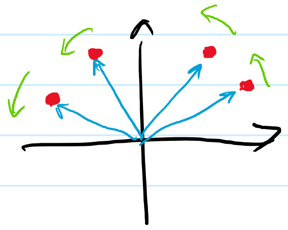
Special Scenrios:

> 90% of the practice problems of finding $\vec{L}$ can be classified into 2 cases, where the I of the system can be easily found and simplify the calculation :

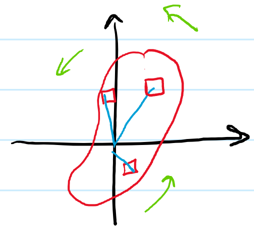
Case 1 : Objects are rotating around a common center at the same angular velocity ω.

E.g.,

Same ω



(Many discrete object)



(A solid continuous object)

Then

① Choosing the rotation center as the origin is obviously the most convenient.

② I of the system is easy to calculate as

$$I_O = \sum m_i r_i^2 = \int r^2 \, dm$$

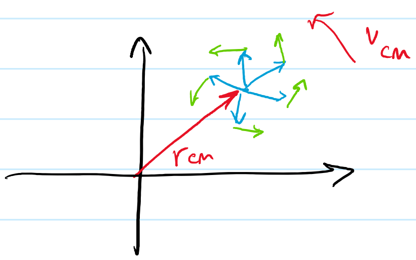
Relative to origin

And then we simply write $L = I_O \omega$

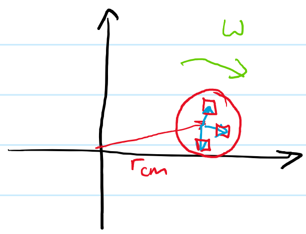
Case 2 : Objects are rotating around their CM.

(The CM may or may not be moving)

E.g.



(Satellite system)



(Rolling)

Then

① The origin may be chosen at any point you like.
However it may simplify more calculation if the position vector of CM $\perp$ velocity vector of CM. (Not always true)

② Depends on if the objects are moving about CM at

[1] Different ω (eg. satellite system)

$$\Rightarrow \vec{L} = M\vec{r}_{cm} \times \vec{v}_{cm} + \underbrace{\sum m_i \vec{r}_{i,cm} \times \vec{V}_{i,cm}}$$

simply by taking out the common ω

[2] Same ω (eg. rolling)

$$\Rightarrow \vec{L} = M\vec{r}_{cm} \times \vec{v}_{cm} + \underline{I_{cm}}\,\omega$$

I relative to CM.

KE in Rotational Motion

KE of 1 particle $= \frac{1}{2}m|\vec{v}_i|^2 = \frac{1}{2}m(v_x^2 + v_y^2)$

In polar coordinate, it becomes

$$\frac{1}{2}m|\vec{v}_i|^2 = \frac{1}{2}m\left[\left(\frac{d|\vec{s}|}{dt}\right)^2 + \left(|\vec{s}|\omega\right)^2\right]$$

$$\left(\text{By } \vec{v} = \left(\frac{d|\vec{s}|}{dt}\right)\hat{r} + \left(|\vec{s}|\frac{d\theta}{dt}\right)\hat{\theta}\right)$$

$$= \underbrace{\frac{1}{2}mv_r^2}_{\text{KE of radial component}} + \underbrace{\frac{1}{2}m|\vec{s}|^2\omega^2}_{\text{KE of angular component}}$$

☆ Only when the object is in pure rotation about the origin (ie. no radial movement)

Then we can use $KE = \frac{1}{2}m|\vec{s}|^2\omega^2 = \frac{1}{2}I_o\omega^2$

Rotational KE relative to CM

For a system with many particles.

The total KE is

$$KE = \sum \frac{1}{2} m_i |\vec{v}_i|^2$$

We can do similar things as in angular momentum

$$\begin{cases} \vec{r}_i = \vec{r}_{cm} + \vec{R}_{i,cm} \\ \vec{v}_i = \vec{v}_{cm} + \vec{V}_{i,cm} \end{cases}$$

$$\Rightarrow KE = \sum \frac{1}{2} m_i \left| \vec{v}_{cm} + \vec{V}_{i,cm} \right|^2$$

$$= \frac{1}{2} \sum m_i |\vec{v}_{cm}|^2 + \cancelto{0}{\left| \sum m_i \left(\vec{v}_{cm} \cdot \vec{V}_{i,cm} \right) \right|} + \frac{1}{2} \sum m_i |\vec{V}_{i,cm}|^2$$

$$= \vec{v}_{cm} \cdot \left(\sum m_i \vec{V}_{i,cm} \right) = 0$$

See the proof in angular momentum section

$$= \frac{1}{2} \left(\sum m_i \right) |\vec{v}_{cm}|^2 + \frac{1}{2} \sum \left(m_i |V_{i,cm}|^2 \right)$$

$$= \begin{pmatrix} \text{KE of CM} \\ \text{about the origin} \end{pmatrix} + \begin{pmatrix} \text{KE of each object} \\ \text{using CM as origin} \end{pmatrix}$$

So we can break down KE into 2 parts

1. The KE of CM
2. The KE of all individual object relative to CM.

$$KE = \underbrace{\frac{1}{2} M |\vec{v}_{cm}|^2}_{\substack{\text{KE of CM} \\ \text{moving as a point object}}} + \sum \left(\underbrace{\frac{1}{2} m_i \left| \vec{V}_{r,i,cm} \right|^2}_{\substack{\text{Radial KE of each} \\ \text{object, relative to CM.}}} + \underbrace{\frac{1}{2} I_{i,cm} W_{i,cm}^2}_{\substack{\text{Angular KE of each} \\ \text{object, relative to KE}}} \right)$$

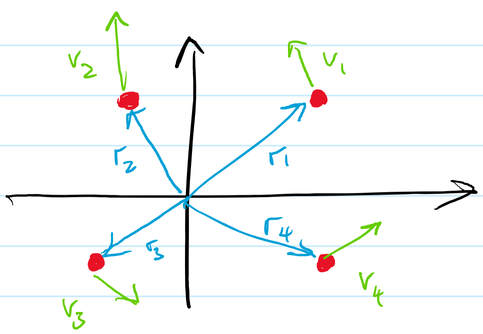
☆☆☆ Similarly, the "relative" KE can only be applied using CM..

$$KE \neq \left(\begin{array}{c}\text{KE of} \\ \underline{\text{a random point}}\end{array}\right) + \left(\begin{array}{c}\text{KE of objects moving} \\ \text{about } \underline{\text{a random point}}\end{array}\right)$$

The point must be taken as the CM

Summary: Guide in finding total/rotational KE of a system

We can write total KE in the form $\frac{1}{2} m_i |\vec{v}_i|^2$ in all problems until you are told to isolate the angular/radial parts.



$$KE = \sum \frac{1}{2} m_i (v_{i,x}^2 + v_{i,y}^2)$$

$$= \sum \frac{1}{2} m_i (v_{i,r}^2 + v_{i,\theta}^2)$$

$$= \sum \boxed{\frac{1}{2} m_i v_{i,r}^2} + \sum \boxed{\frac{1}{2} m_i |\vec{r}_i|^2 \omega_i^2}$$

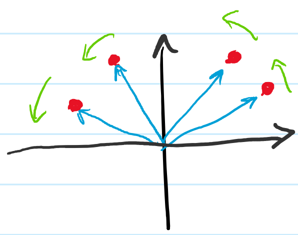
These formula are always true

Special Scenarios:

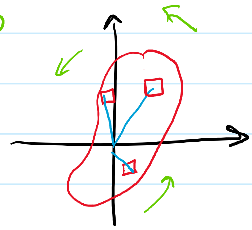
Similar to how we deal with $\vec{L}$, > 90% of the practice problems belong to these 2 cases:

Case 1: Objects are rotating around a common center at the same angular velocity ω.

E.g..



same ω



(Many discrete object) (A solid continuous object)

$\Rightarrow$ Pure rotation around the origin, so no radial KE

$$KE = \sum \frac{1}{2} m_i \cancel{v_{i,r}^2} \to 0 + \sum \frac{1}{2} m_i r_i^2 \omega_i^2$$

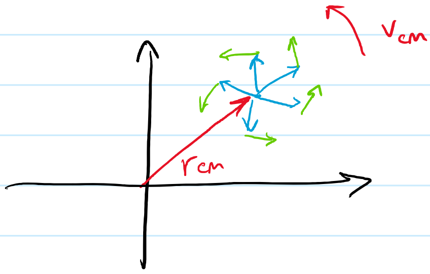
$$= \frac{1}{2}\left(\sum m_i r_i^2\right) \omega_i^2$$

$$= \frac{1}{2} I_0 \omega^2$$

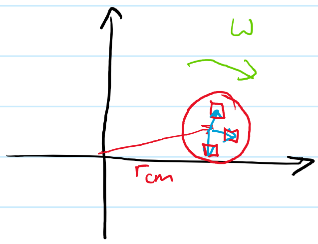
Case 2 : Objects are rotating around their CM.

(The CM may or may not be moving)

E.g.



(Satellite system)



(Rolling)

Depends on if the objects are moving about CM at

$\boxed{1}$ Different $\vec{v}$ (eg. satellite system)

$$\Rightarrow KE = \frac{1}{2} M |\vec{v}_{cm}|^2 + \sum \frac{1}{2} m_i |\vec{V}_{i,cm}|^2$$

$$= \frac{1}{2} M |\vec{v}_{cm}|^2 + \sum \frac{1}{2} m_i V_{i,r,cm}^2 + \sum \frac{1}{2} m_i |r_i|^2 \omega_i^2$$

$\boxed{2}$ Same ω, no radial movement relative to CM (eg. rolling)

$$\Rightarrow KE = \frac{1}{2} M |\vec{v}_{cm}|^2 + 0 + \frac{1}{2} I_{cm} \omega^2$$

take out the common ω

I relative to CM