Physics Enrichment Program HKUST |
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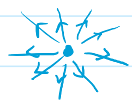
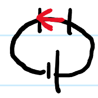
- E & M as a pair
- Basic : Finding $\vec{B}$ by integration of Biot - Savart Law.
- Divergent-less of $\vec{B}$
- Ampere's Law, Dot product line integral, Curl, Stoke's Theorem.
- Magnetic Vector Potential

E & M in pairs

(highlighted) = in HKDSE syllabus

	Electrostatics		Magnetostatics	
Fields	$\vec{E}$		$\vec{B}$	
Potential	V	Electric Potential	$\vec{A}$	Magnetic Vector Potential
"Point charge"	q		$I d\vec{l}$	current element 
Charge density	$\rho(x,y,z)$		$\vec{J}(x,y,z)$	current density 
Force	$q\vec{E}$		$I d\vec{l} \times \vec{B}$	Lorentz Force

From charge to field

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\, d^3\vec{r}\,'$$

Columb's Law

$$\hookrightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

for point charge

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\, d^3\vec{r}\,'$$

Biot-Savart's Law

$$\hookrightarrow \ulcorner\ \vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{L}}{r^2}\times\hat{r}\ \lrcorner$$

WRONG! We don't have point current!

From field to charge

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

Gauss Law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

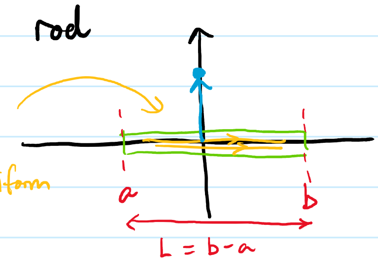
Ampere's Law

Potential definition	$\vec{E} = -\vec{\nabla} V$	$\vec{B} = \vec{\nabla} \times \vec{A}$
Energy	$qV = \iiint \rho(\vec{r}\,') V(\vec{r}\,')\, d^3\vec{r}\,'$ $= \iiint \frac{1}{2}\varepsilon_0 \lvert \vec{E}(\vec{r}\,')\rvert^2\, d^3\vec{r}\,'$	$I d\vec{l} \cdot \vec{A} = \iiint \vec{J}(\vec{r}\,') \cdot \vec{A}(\vec{r}\,')\, d^3\vec{r}\,'$ $= \iiint \frac{1}{2\mu_0} \lvert \vec{B}(\vec{r}\,')\rvert^2\, d^3\vec{r}\,'$

Basic skill : Find B field at certain point by integral

E.g. A finite rod

Current I flowing along, assume uniform

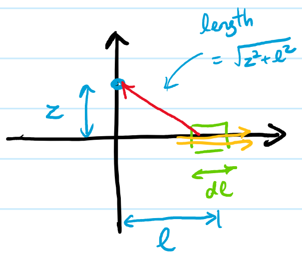


- What is the B field's magnitude & direction at a given point?

(Asking for potential is beyond your level)

Method : Integral as weighted sum

i.e. Every small element on the object contributes



its B field to the given point = weight

Current on each small element dl = $I d\vec{l}$

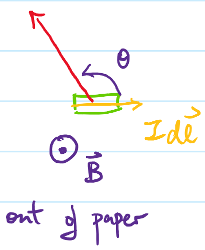
Distance from the target point = $\sqrt{z^2+l^2}$

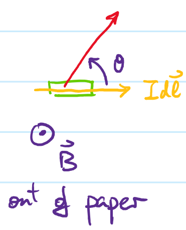
- By Biot-Savat Law, B field contribute by this element is

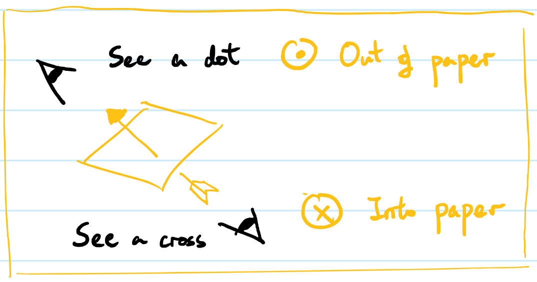
$$= \frac{\mu_0}{4\pi} \frac{I d\vec{l}}{(z^2+l^2)} \times \left(\text{direction of } \nwarrow\right)$$

cross product

☆ The direction of B field follow direction of the cross product







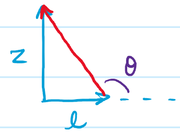
So direction of $\vec{B}$ by all small elements are always out of paper

- The magnitude of cross product between $\vec{dl}$ and (direction of $\nwarrow$)

which is a unit vector

$$= |d\vec{l}|\,|1|\,\sin\theta$$

And by the triangle



$$\sin\theta = \frac{z}{\sqrt{z^2+l^2}}$$

$\Rightarrow$ Total B field in out of paper direction is

$$\vec{B} = \int_a^b \frac{\mu_0}{4\pi} \frac{I d\vec{l}}{(z^2+l^2)} \times (\text{direction of } \nwarrow)$$

$$= \int_a^b \frac{\mu_0}{4\pi} \frac{I}{(z^2+l^2)} \cdot \underline{|d\vec{l}|} \cdot \frac{z}{\sqrt{z^2+l^2}}$$

Integration over l

Divergent-less of $\vec{B}$

$\boxed{\text{This is a pure observational statement about } \vec{B}}$

$\because$ Magnetic point charge (Magnetic monopole)

does not exist. At least no one has found it yet.

$\Rightarrow$ There is no sink / source of $\vec{B}$ field

$\Rightarrow$ B field can never form diverging / converging lines

i.e. <u>Flux through a close surface always = 0</u>

<u>B field is Divergent-less</u>

This property of B field can be formulate by so-called "Gauss's Law of magnetic field", i.e.

$$\begin{cases} \text{Integral form}: & \oiint \vec{B}\cdot d\vec{S} = 0 \\ \text{Differential form}: & \vec{\nabla}\cdot\vec{B} = 0 \end{cases}$$

This property is in fact important for making symmetry claims

as well as simplifying theoretical derivation.

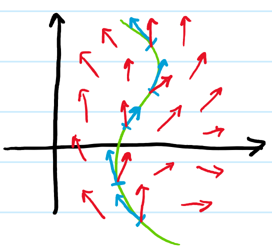
(We shall see later)

Line Integral along loops

The sign of dot product can be used to shown if 2 vectors are in the same or opposite direction

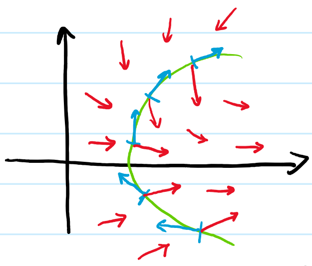
⇒ We can lay a path over a vector field, walk along the path, and record the dot product between the field vector and path direction

⇒ The sum of the dot products' value can tell us whether the field vectors are more of "along the path" or "opposite the path"



Most of the time angle between ↗ & ↗ are < 180°

⇒ more "along the path"



Most of the time angle between ↗ & ↗ are > 180°

⇒ more "opposite the path"

Extend the idea to infinitestimal scale, sum of dot product become dot product line integral on the vector field

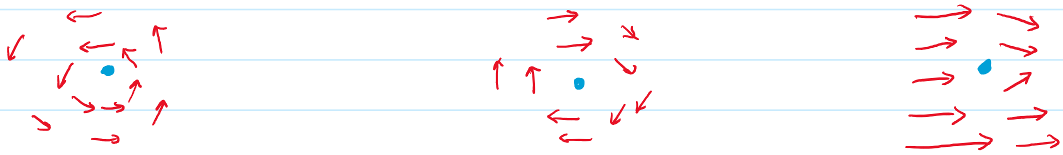
$$\int_{path} \vec{F}\cdot d\vec{l} > 0$$

Along the path

$$\int_{path} \vec{F}\cdot d\vec{l} < 0$$

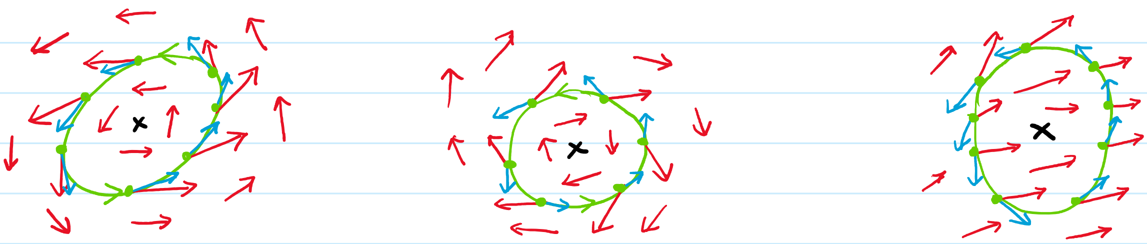
Opposite the path

Observation : Path integral is a good tool to measure rotation of fields



High "degree" of anti-clockwise rotation | High "degree" of clockwise rotation | Zero "degree" of rotation

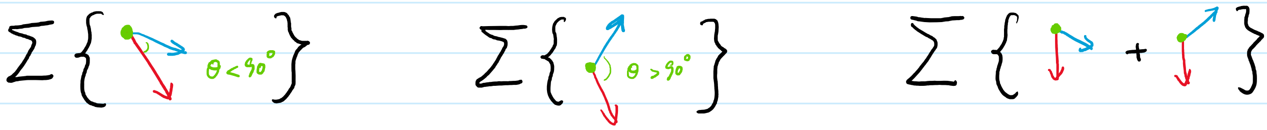
⇒ Do a path integral along a loop (anti clockwise by convention)



On this path, most of the dot product between ↗ & ↗ are +ve

On this path, most of the dot product between ↗ & ↗ are −ve

On this path, there are about the same amount of > 0 & < 0 dot product between ↗ & ↗



Expect integral result > 0 | Expect integral result < 0 | Expect integral result ≃ 0

The result of the path integral is a good indicator about the rotation of the vector field at a particular location :

- High +ve score = A strong trend of anti clockwise rotation
- High −ve score = A strong trend of clockwise rotation
- Close to 0 = Not really a center of rotation

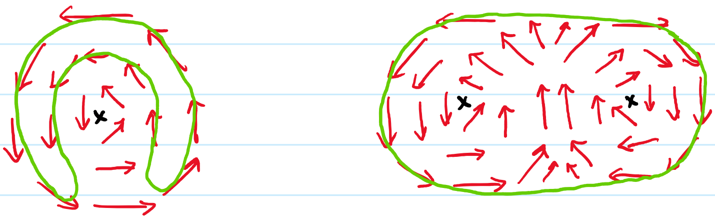
Curl

Problem : We have too much freedom to choose the loop

The loop can be irregular and of any size.

A bad loop cannot show the rotation characteristic in a meaningful way

E.g.



The path integral along these loop $\simeq 0$

$\Rightarrow$ Cannot show the rotation characteristics at all

Resolution : The loop should be infinitestimally small

$\Rightarrow$ Consequence : Become the curl operator

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

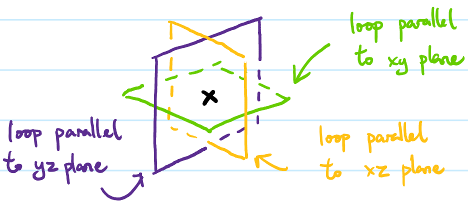
Recall $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y\right]\hat{x} + \left[\frac{\partial}{\partial z} F_x - \frac{\partial}{\partial x} F_z\right]\hat{y} + \left[\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x\right]\hat{z}$$

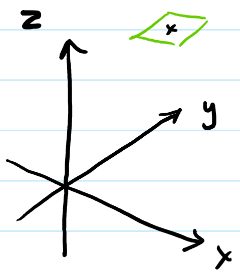
$[yz - zy]\,\hat{x}$ $\quad$ $[zx - xz]\,\hat{y}$ $\quad$ $[xy - yx]\,\hat{z}$

$=$ Some vector

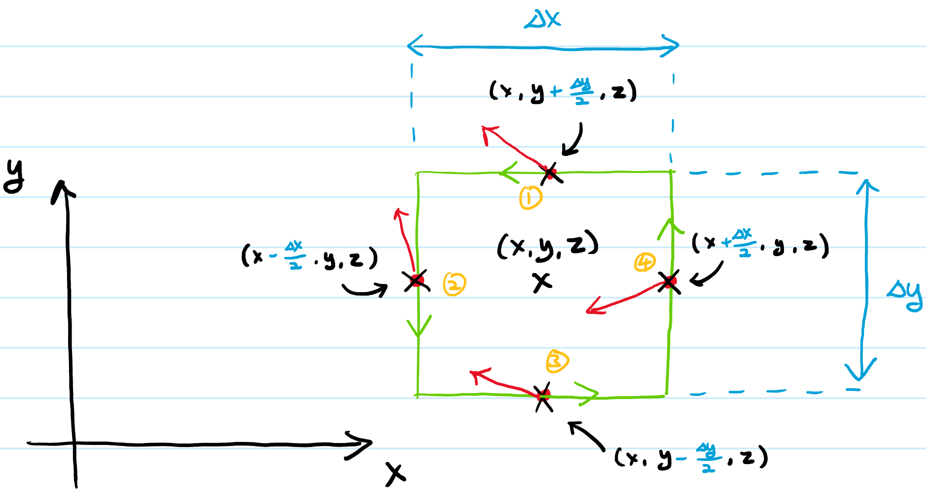
Geometrical interpretation



In 3D, we can draw 3 simple loops around any point. Each of the loops is parallel to 1 of the xy / yz / xz plane



For the simplicity of drawing, here only shows the interpretation on xy plane. Consider a very small loop of size $\Delta x \Delta y$ around a point at (x, y, z) and the field vectors on the loop.

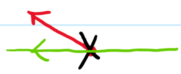


Because this loop is infinitesimally small, path integral along the loop

$\simeq$ Sum of dot product on each of the 4 edges

- Edge ① : Vector field on edge center $= \vec{F}(x, y+\frac{\Delta y}{2}, z)$

  Edge length $= \Delta x$, in -ve x direction



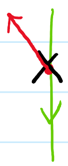
$\Rightarrow$ Dot product $= \vec{F}(x, y+\frac{\Delta y}{2}, z) \cdot \Delta x\,(-\hat{x})$

$$= -F_x\left(x, y+\frac{\Delta y}{2}, z\right) \Delta x$$

(only x component remains after dot product)

- Edge ② : Vector field on edge center $= \vec{F}(x-\frac{\Delta x}{2}, y, z)$

  Edge length $= \Delta y$, in -ve y direction



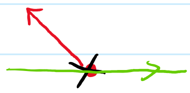
$\Rightarrow$ Dot product $= \vec{F}(x-\frac{\Delta x}{2}, y, z) \cdot \Delta y\,(-\hat{y})$

$$= -F_y\left(x-\frac{\Delta x}{2}, y, z\right) \Delta y$$

(only y component remains after dot product)

- Edge ③ : Vector field on edge center = $\vec{F}(x, y-\frac{\Delta y}{2}, z)$

Edge length = Δx , in +ve x direction



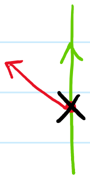
$$\Rightarrow \text{Dot product} = \vec{F}(x, y-\tfrac{\Delta y}{2}, z) \cdot \Delta x\,(+\hat{x})$$

$$= F_x(x, y-\tfrac{\Delta y}{2}, z)\,\Delta x$$

(only x component remains after dot product)

- Edge ④ : Vector field on edge center = $\vec{F}(x+\frac{\Delta x}{2}, y, z)$

Edge length = Δy , in +ve y direction



$$\Rightarrow \text{Dot product} = \vec{F}(x+\tfrac{\Delta x}{2}, y, z) \cdot \Delta y\,(+\hat{y})$$

$$= F_y(x+\tfrac{\Delta x}{2}, y, z)\,\Delta y$$

(only y component remains after dot product)

$\therefore$ The sum of the dot product of all 4 edges

$$= \underbrace{-F_x(x, y+\tfrac{\Delta y}{2}, z)\,\Delta x}_{\text{Edge ①}} + \underbrace{F_x(x, y-\tfrac{\Delta y}{2}, z)\,\Delta x}_{\text{Edge ③}}$$

$$+ \underbrace{F_y(x+\tfrac{\Delta x}{2}, y, z)\,\Delta y}_{\text{Edge ④}} - \underbrace{F_y(x-\tfrac{\Delta x}{2}, y, z)\,\Delta y}_{\text{Edge ②}}$$

$$= -\left[\frac{F_x(x, \boxed{y+\tfrac{\Delta y}{2}}, z) - F_x(x, \boxed{y-\tfrac{\Delta y}{2}}, z)}{\boxed{\Delta y}}\right]\Delta y\,\Delta x$$

$$+ \left[\frac{F_y(\boxed{x+\tfrac{\Delta x}{2}}, y, z) - F_y(\boxed{x-\tfrac{\Delta x}{2}}, y, z)}{\boxed{\Delta x}}\right]\Delta x\,\Delta y$$

Notice the definition of derivative

$$= -\boxed{\frac{\partial}{\partial y}} F_x(x,y,z)\,\Delta x\Delta y + \boxed{\frac{\partial}{\partial x}} F_y(x,y,z)\,\Delta x\Delta y$$

$$= \left[\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right](\Delta x\,\Delta y)$$

$$= (\text{Curl's } z \text{ component}) \cdot (\text{area of loop})$$

We can repeat for the loop of xz / yz plane to get another 2 similar terms

xy plane $\rightarrow$ $\left[\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right]\Delta x\,\Delta y = \begin{pmatrix}\text{curl's}\\ \text{z component}\end{pmatrix}\cdot\begin{pmatrix}\text{loop}\\ \text{area}\end{pmatrix}$

yz plane $\rightarrow$ $\left[\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right]\Delta y\,\Delta z = \begin{pmatrix}\text{curl's}\\ \text{x component}\end{pmatrix}\cdot\begin{pmatrix}\text{loop}\\ \text{area}\end{pmatrix}$

xz plane $\rightarrow$ $\left[\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right]\Delta z\,\Delta x = \begin{pmatrix}\text{curl's}\\ \text{y component}\end{pmatrix}\cdot\begin{pmatrix}\text{loop}\\ \text{area}\end{pmatrix}$

Together we can see that the geometrical meaning of curl as

$\{x/y/z\}$ component

$$\sim \frac{\text{Dot product loop integral on the plane normal to } \{x/y/z\}}{\text{Area of the loop}}$$

$$\sim \frac{\text{"Degree of rotation" about an axis in } \{x/y/z\} \text{ direction}}{\text{Area}}$$

And the curl vector is a compact form for storing the information of all 3 directions in a single expression

By this geometrical interpretation, we can directly go to a useful theorem

Stoke's Theorem (state without proof)

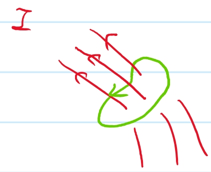
Must be close loop $\rightarrow$

$$\underbrace{\boxed{\oint}\vec{F}\cdot d\vec{l}}_{\text{total dot product line integral}} = \boxed{\iint}\underbrace{(\vec{\nabla}\times\vec{F})}_{\text{loop integral per area (in each direction)}}\boxed{\cdot\, d\vec{S}}$$

Total flux through some area $\iint(\ldots)\cdot d\vec{S}$

Ampere's Law

Ampere's Law is purely an observational statement relating B field & current



☆☆☆ Observe a line of current always generate B field in the shape of loops around it

$\Leftrightarrow$ If we find rotating B field, we are guarenteed to find a current source in the center of the rotation

The Ampere's Law can be written in 2 forms:

Integral form : $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$

(Dot product along loop) ~ (Integrate $\vec{B}$ along a loop) = (current contained in the loop) ~ (current)

Differential form : $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

(Dot product along loop per area) ~ (Curl of $\vec{B}$ over the loop) = (Current density in the loop) ~ (current density)

They can be inter-converted by Stoke's Theorem.

$$\text{LHS} = \oint \vec{B} \cdot d\vec{l} = \iint (\vec{\nabla} \times \vec{B}) \cdot d\vec{s}$$

$$\text{RHS} = \mu_0 I = \iint \mu_0 \vec{J} \cdot d\vec{s}$$

Example of using Ampere's Law (Integral form)

☆ Integral form is nice to use only in very symmetric scenerio

To simplify calculation, we should always choose a loop s.t.

① $\vec{B}$ has constant magnitude on the loop

② $\vec{B}$ forms the same angle with the loop's path

Only then the loop integral is easy to compute

Dot product $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\oint \vec{B}\cdot d\vec{\ell} = \oint |\vec{B}|\cos\theta \, d|\vec{\ell}|$$

$$= \underline{|\vec{B}|}\,\underline{\cos\theta} \oint d|\vec{\ell}|$$

Same magnitude & angle everywhere
So you can take them out of integral

$$= |\vec{B}|\cos\theta \cdot (\text{loop's perimeter})$$

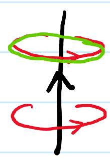
Only then we can use Ampere's Law integral form

to find the $\vec{B}$ field from current

$$|\vec{B}| = \mu_0 I \cdot \frac{1}{\text{loop's perimeter}} \cdot \frac{1}{\cos\theta}$$

E.g. 1 Infinite long wire with uniform current I.

This is a cylindrical symmetric case. Good loop = circle about the wire



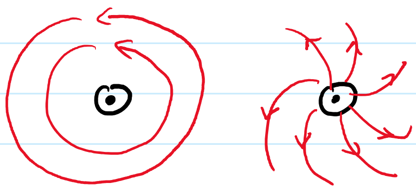
Ⅰ By cylindrical symmetry, we must have

- B field's magnitude only depends on radial distance from the wire. (const. of θ, z)

- B field's direction only depends on radial distance from the wire. (const. of θ, z)

☆ BUT: These symmetry conditions do not force B field to form loop.

For example, both of these configurations satisfy



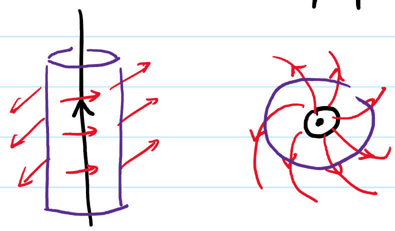
- Translational symmetry along z axis
- Rotational symmetry about z axis
- θ component of $B \neq 0$. So $\oint \vec{B} \cdot d\vec{l} \neq 0$

So how can we arrive the conclusion that B field must form a loop?

[2] By B field 's divergent-less

- Symmetry require B field 's radial compont = const. of θ, z.
- If the radial component $\neq 0$, the flux through a cylindrical surface $\neq 0$

  $\Rightarrow$ contradict to the property of divergent-less

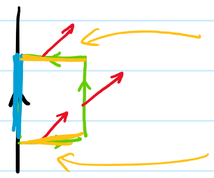


Look from above

$\therefore$ The B field must NOT carry any radial component

[3] Another "Ampere's loop"

We can also prove that the B field has no z component by considering this loop:



on axis $B = 0$

By translation symmetry $\vec{B}$ on these 2 edges are the same

But dot product with path give opposite sign

Because there is no current through this loop, by Ampere's Law the total dot product $= 0$

$\Rightarrow$ Dot product of $\vec{B}$ to the outer edge must be 0

This happens only if $\vec{B}$ has no z component

After these symmetry & divergentless claims, we can finally be certain that the B field only has angular components. ie. form circular loops

Only after then we can apply Ampere's Law:

$$\oint \vec{B}\cdot d\vec{\ell} = |\vec{B}|\cdot(\text{perimeter})\cdot\cos\theta = \mu_0 I$$

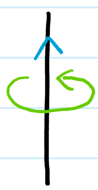
All we can get is the magnitude

$$|\vec{B}| = \mu_0 I \cdot \underbrace{\frac{1}{2\pi R}}_{\text{loop's perimeter}} \cdot \underbrace{\frac{1}{\cos 0^\circ}}_{\text{field lines tangent to the circular path}}$$

$$\Rightarrow \vec{B} \text{ field as a vector} = \frac{\mu_0 I}{2\pi R}\hat{\theta}$$

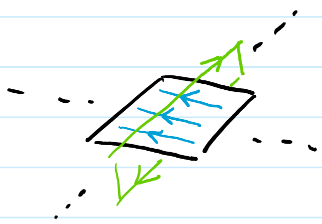
(This is in DSE formular sheet)

P.S. There are not many configurations that you can draw a good Ampere's loop. Even fewer than what you can do with Gauss's Law.



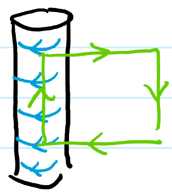
Infinitely long wire

→ circle



Infinitely large plane

→ rectangle

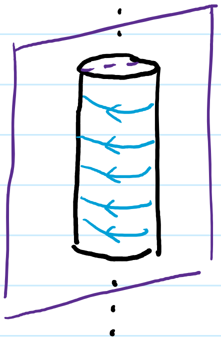


Infinitely long solenoid

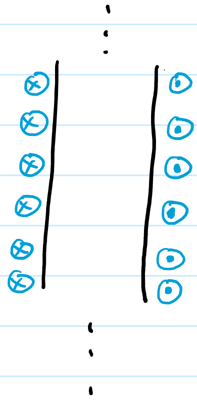
→ rectangle

(These are almost all you can find in textbook)

E.g. 2 Infinitely long solenoid with uniform density of coils.

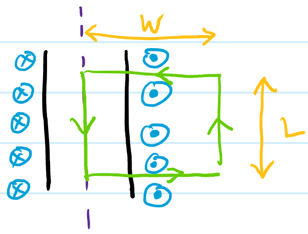


Cross section ⇒



To use Ampere's Law, the loop we choose should enclose some current

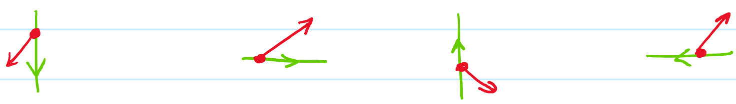
☆ We start with a loop that enclose 1 side of the current

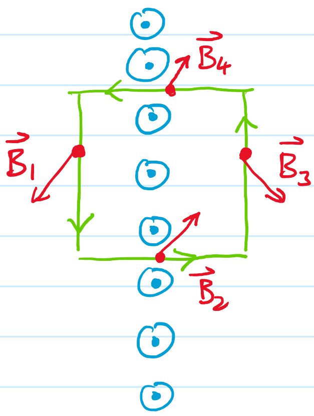


We can choose it to be a rectangle of size $W \times L$ for simplicity.

Then $\oint \vec{B} \cdot d\vec{l}$ = Sum of dot product on the 4 segments

$$= \vec{B}_1 \cdot (-\vec{L}) + \vec{B}_2 \cdot (\vec{W}) + \vec{B}_3 \cdot (\vec{L}) + \vec{B}_4 \cdot (-\vec{W})$$





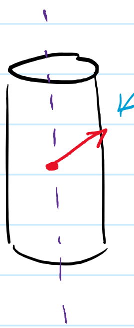
At the beginning, we cannot assume the magnitude and directions of $\vec{B}$ to be nice.

We must prove that they are nice by symmetry claims

[1] By rotational symmetry

The B field on the center axis must carry z component only (r, θ component $= 0$)

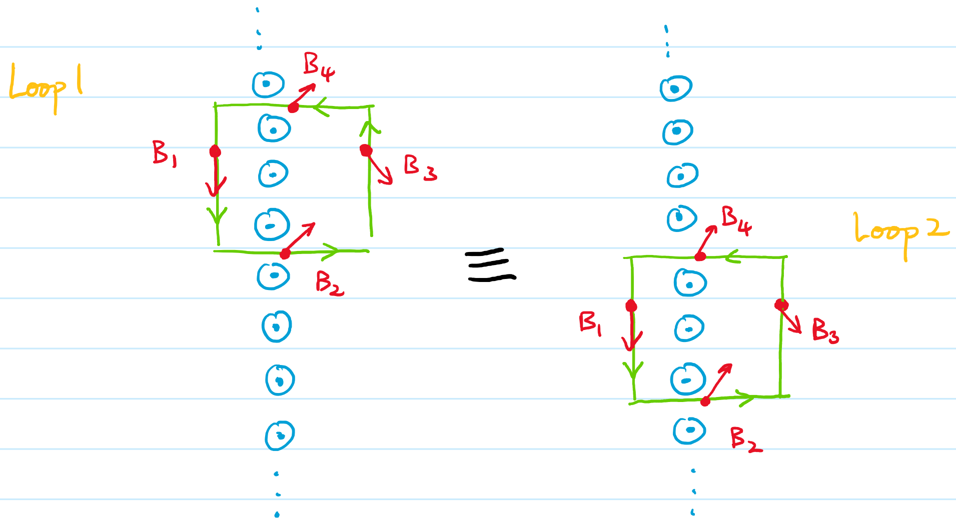
i.e. $\vec{B}_1$ must be along center axis



not rotational symmetric.

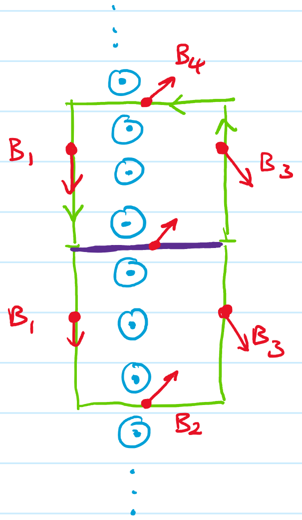
12 By translational symmetry

Note that the B field is magnitude unchange if we move the loop up/down. ($\because$ The solenoid is infinitely long)



Now consider joining both loop

since edge 2 on loop 1 & edge 4 on loop 2 are actually the same edge.



$\vec{B}_2$ & $\vec{B}_4$ must be of the same magnitude & direction

So sum of dot product now reads

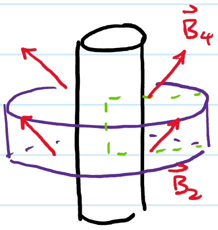
$$\vec{B}_1 \cdot (-\vec{L}) + \cancel{\vec{B}_2 \cdot (\vec{W})} + \vec{B}_3 \cdot (\vec{L}) + \cancel{\vec{B}_2 \cdot (-\vec{W})}$$

$$= (\vec{B}_3 - \vec{B}_1) \cdot (\vec{L})$$

which is independent of the size of W we choose

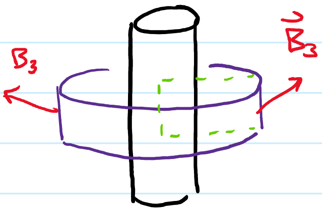
[3] Divergentless of $\vec{B}$

We can draw a cylindrical box around the solenoid



- Flux on top & bottom plane will be contributed by B_2 & B_4

$\Rightarrow \because \vec{B}_2 = \vec{B}_4$, total flux $= 0$



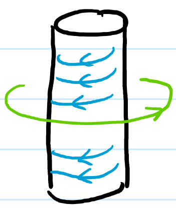
- Flux on the side surface is contributed by $\vec{B}_3$ only

☆ If $\vec{B}_3$ has radial component, flux through this surface will be non-zero $\Rightarrow$ divergence $\neq 0$

$\therefore$ $\vec{B}_3$ has no radial component

[4] Another "Ampere's loop"

We can also claim that $\vec{B}_3$ has no angular component by drawing such loop :



Because no current pass through this loop
Dot product integral should be 0

$\therefore$ $\vec{B}_3$ has no angular component

[4] Behavior at ∞

Since our choice of W does not affect the sum of dot product

Why not take $W = \infty$?

But if we are infinitely far from the solenoid, we should not feel any impact from it.

$\Rightarrow$ The only conclusion : $\vec{B}_3 = 0$

Finally, Ampere's Law write as

$$\oint \vec{B} \cdot d\vec{l} = \vec{B}_1 \cdot L$$

$$= \mu_0 (\text{no. of coil enclosed} \times \text{current per coil})$$

$$= \mu_0 N I$$

$$\Rightarrow \boxed{|\vec{B}| = \frac{\mu_0 N I}{L}}$$

(which is on the DSE formular sheet)

Magnetic Vector Potential

Recall that magnetic field is always divergentless. ($\vec{\nabla}\cdot\vec{B}=0$)

Mathematic fact : Any divergentless vector field can be expressed as the curl of some other vector field

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$$

Vector "Potential"

☆☆ It has the name "potential" just because it comes as a mathematical pair to electric potential. But it cannot be interpreted geometrically to be a potential surface

Furthermore if we substitute this into Ampere's Law differential form

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$$

$$-\vec{\nabla}^2\vec{A} = \mu_0\vec{J}$$

About this step:
Use vector identity $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$
Then chose "Columb Gauge" to make $\nabla\cdot A = 0$
But explaining what this means is beyond the level of this note

Or if we express it in matrix form :

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix} = \mu_0\begin{pmatrix}J_x\\J_y\\J_z\end{pmatrix}$$

which means each components of $\vec{A}$ and $\vec{J}$ give an independent PDE

$$\begin{cases}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A_x = -\mu_0 J_x\\[4pt] \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A_y = -\mu_0 J_y\\[4pt] \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A_z = -\mu_0 J_z\end{cases}$$

We have already learnt this type of PDE in electrostatics

The Poisson Equation : $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

And so the solution of $\vec{A}(\vec{r})$ depends on the boundary condition.

Only for the special case if given $\vec{A}(\infty) = 0$, we have

i for each of the $x/y/z$

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{J_i(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, d^3\vec{r}\,'$$

Combining the 3 independent components into 1 vector

$$\vec{A}(\vec{r}) = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

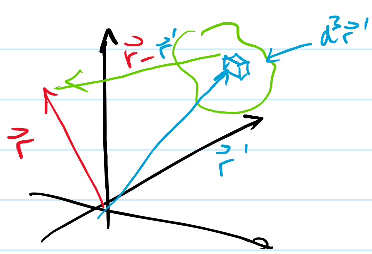
$$= \frac{\mu_0}{4\pi} \iiint \frac{J_x(\vec{r}\,')\hat{x} + J_y(\vec{r}\,')\hat{y} + J_z(\vec{r}\,')\hat{z}}{|\vec{r} - \vec{r}\,'|} \, d^3\vec{r}\,'$$

$$= \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, d^3\vec{r}\,'$$

$$\sim \frac{\mu_0}{4\pi} \sum \frac{\text{current}}{\text{distance}}$$

$\equiv$ Biot-Savart's Law of potential

We can then find $\vec{B}$ by



$$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$$

cross product

$$= \frac{\mu_0}{4\pi} \underbrace{\iiint}_{\substack{\text{everywhere} \\ \text{integrate to } \infty}} \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \times \underbrace{\left[\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|} \right]}_{\substack{\text{Just the unit vector} \\ \text{of } \vec{r} - \vec{r}\,'}} d^3\vec{r}\,'$$

$$\sim \frac{\mu_0}{4\pi} \sum \frac{\text{current}}{(\text{distance})^2}$$, point in the direction of (current $\times$ r)

$\equiv$ Just a fancier form of Biot Savart's Law for field

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I \, d\vec{l} \times \hat{r}}{r^2}$$

Remember there is no point current
We have to use differential to express
an infinitesimal current source

Why do we "invent" potential $\vec{A}$

The concept of magnetic potential is introduced not just for fun

We use $\vec{A}(\vec{r})$ because its PDE is easier to solve than solving for $\vec{B}(\vec{r})$'s PDE

- $\vec{\nabla}^2 \boxed{\vec{A}(\vec{r})} = -\mu_0 \vec{J}(\vec{r})$ — The 3 components of $\vec{A}(\vec{r})$ & $\vec{J}(\vec{r})$ pair up and separate into 3 independent single function PDE

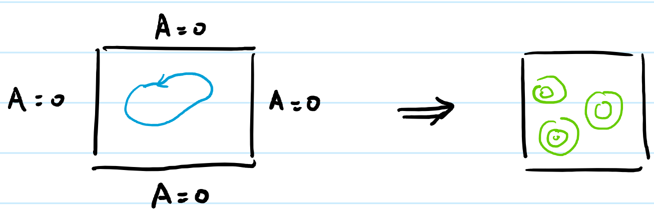
- $\vec{\nabla} \times \boxed{\vec{B}(\vec{r})} = \mu_0 \vec{J}(\vec{r})$ — PDE involving curl will mix up the pairing of the 3 components of $\vec{B}(\vec{r})$ & $\vec{J}(\vec{r})$

Therefore when given $\vec{J}(\vec{r})$ and asked for $\vec{B}(\vec{r})$, it is JUST BETTER to first solve $\nabla^2 \vec{A} = -\mu_0 \vec{J}$, and then take $\vec{B} = -\vec{\nabla} \times \vec{A}$

☆☆☆ In practice, the most common problems we need to solve look like this in general:

- Given
  - Distribution of Current — We can manually place them or model where they should be
  - Potential at the space boundary — But almost never see non-zero cases.

- Solve : Distribution of $\vec{B}$ field in the space



How does the B field look?

This is in fact the task of solving the PDE $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

This is an inhomogeneous PDE! Solving is awful!

$\Rightarrow$ Can we try to avoid that?

YES, but not always. And that is why you are taught these different methods:

- If the configuration is very symmetric

  $\Rightarrow$ Can use symmetry argument to convert Ampere's Law integral form into just multiplication: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I \Rightarrow B \cdot (\text{perimeter}) \cos\theta = \mu_0 I$

- If it is not symmetric enough, but the current only distribute in a finite size of space AND potential at ∞ is set to 0

  $\Rightarrow$ Can write Biot-Savart's Law for each current source, then add / integrate to compute the resultant B field

- If still no,

  $\Rightarrow$ Sorry, but please solve the PDE from scratch. ☹

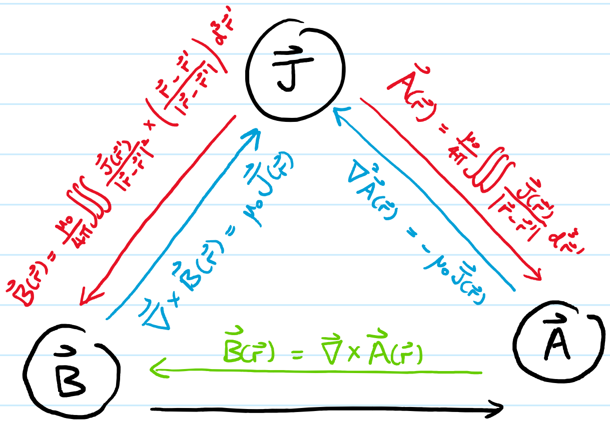
(And as mentioned, we prefer first solving $\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}$, then taking $\vec{B} = \vec{\nabla} \times \vec{A}$)

Summary

In magnetostatic, these 3 quantities are inter-convertible

$\vec{J}(\vec{r})$ $\vec{B}(\vec{r})$ $\vec{A}(\vec{r})$

i.e. If we know 1 of them, we can solve for other 2



Formular exists, but almost never used.

If you need to find $\vec{A}$ from $\vec{B}$, just go $\vec{B} \to \vec{J} \to \vec{A}$

— Ampere's Law

— Biot Savat's Law (☆ only if $\vec{B}/\vec{A} = 0$ at ∞)

— By definition