All you need to care = Compute path difference (p.d.)

- Common cases that involve 2 rays only. E.g. $\begin{cases} \text{2 slit} \\ \text{Newton ring} \end{cases}$

  (Also other variants.)

- N slit case

- Fraunhofer diffraction.

Interference & Path difference

(Light) Path Difference (p.d.) = Difference in distance travelled by 2 light wave (in unit of wavelength)

Specific name for 2 special cases:

- Constructive interference : If p.d. $= m\lambda$

- Destructive interference : If p.d. $= (m+\frac{1}{2})\lambda$

2 special case in the change of p.d.

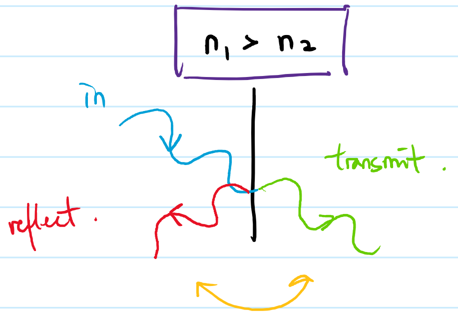
① When travelling in material with refractive index n

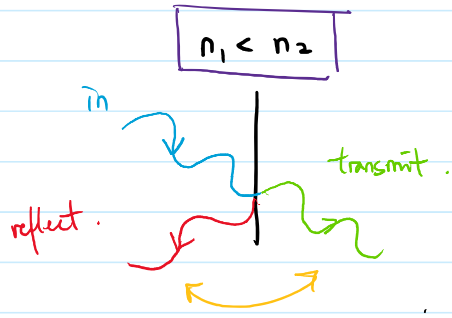
Wavelength change from $\lambda \to \frac{\lambda}{n}$ (i.e. p.d. $\times n$)

($\because$ light speed $\to \frac{c}{n}$, but frequency = constant)

② When hitting and reflected by material of higher refractive index

Gain an extra phase π (i.e. p.d. $+\frac{\lambda}{2}$)





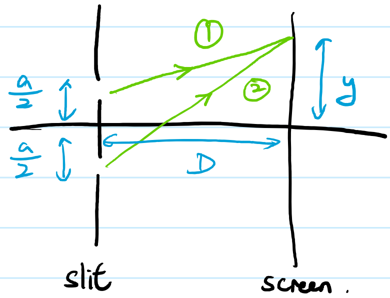
Reflected wave = Symmetric to Transmitted wave

Refracted wave = Flipped

Review 1 : Double Slit interference

Assumption : $D \gg a$ & $D \gg y$

So we can use Taylor approx.



① Distance travelled by the 2 rays

Ray ① : $\sqrt{(y-\frac{a}{2})^2+D^2}$

Ray ② : $\sqrt{(y+\frac{a}{2})^2+D^2}$

$$\Rightarrow \text{p.d.} = \sqrt{(y+\tfrac{a}{2})^2+D^2} - \sqrt{(y-\tfrac{a}{2})^2+D^2}$$

$$= D\sqrt{1+\left(\tfrac{y}{D}+\tfrac{a}{2D}\right)^2} - D\sqrt{1+\left(\tfrac{y}{D}-\tfrac{a}{2D}\right)^2}$$

$$\simeq D\left[1+\tfrac{1}{2}\left(\tfrac{y}{D}+\tfrac{a}{2D}\right)^2 - 1 - \tfrac{1}{2}\left(\tfrac{y}{D}-\tfrac{a}{2D}\right)^2\right]$$

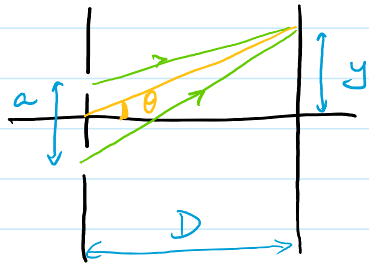
$(1+x)^n \simeq 1+nx$ if $x \ll 1$

$$= D\cdot\tfrac{1}{2}\cdot 4\left(\tfrac{y}{D}\right)\left(\tfrac{a}{2D}\right)$$

$$= \frac{ya}{D}$$

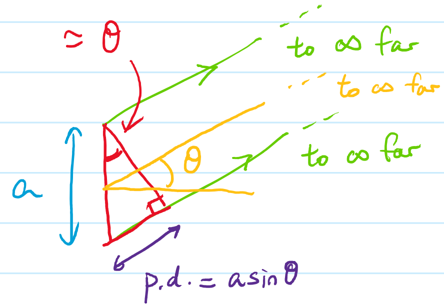
② Expressing screen position by elevation angle.

[1] From geometry



$$\sin\theta \simeq \tan\theta = \frac{y}{D}$$

[2] By the assumption $D \gg y$



The rays are so long that

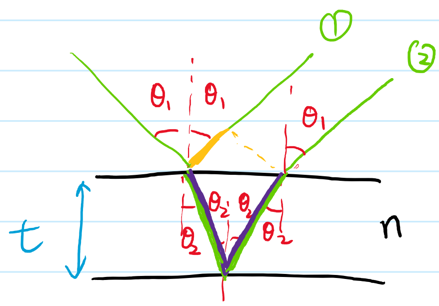
they look parallel

$$\Rightarrow \quad p.d. = \frac{ya}{D} = a\sin\theta$$

③ Finally arrive the equations in textbooks.

$$p.d. = \frac{ya}{D} = a\sin\theta = \begin{cases} m\lambda & \rightarrow \text{ constructive interference} \\ (m+\frac{1}{2})\lambda & \rightarrow \text{ destructive interference} \end{cases}$$

Review 2 : Thin Flim



Ray ② travels longer distance than

Ray ① by $2\frac{t}{\cos\theta_2} - 2t\tan\theta_2\sin\theta_1$

But be caution :

[1] Ray ② is travelling in a material of refractive index n

↪ p.d. correct to $2n\frac{t}{\cos\theta_2} - 2t\tan\theta_2\sin\theta_1$

[2] Ray ① hit and reflected by the material of higher n

↳ Ray ① gets extra p.d. $\frac{\lambda}{2}$

↳ p.d. correct to $2n\frac{t}{\cos\theta_2} - \frac{\lambda}{2} - 2t\tan\theta_2\sin\theta_1$

⇒ Interference condition

$$\text{p.d.} = 2n\frac{t}{\cos\theta_2} - \frac{\lambda}{2} - 2t\tan\theta_2\sin\theta_1 = \begin{cases} m\lambda \rightarrow \text{constructive} \\ (m+\frac{1}{2})\lambda \rightarrow \text{destructive} \end{cases}$$

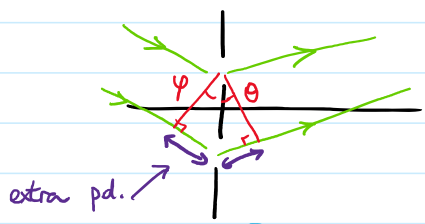
Thin flim interference is a simple way to measure the thickness of a very thin "transparent" material

$$\Rightarrow t = \frac{m\lambda\cos\theta_2}{2n - 2\sin\theta_1\sin\theta_2}$$

By locating the angular position of destructive interference (finding those θ_1 which this happens)

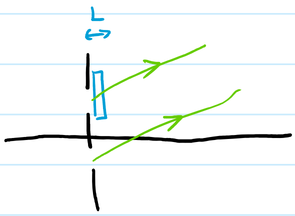
Common Variations to related problems

E.g. 1 Double slit but light enter at an angle.



$\Rightarrow$ Total p.d. $= a\sin\theta + a\sin\varphi$

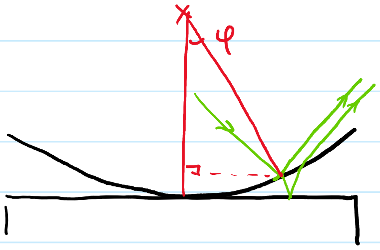
E.g. 2 Double slit but 1 slit covered by a thin material



$\Rightarrow$ Total p.d $= a\sin\theta - \frac{(n-1)L}{\cos\theta}$

$n-1 = \begin{pmatrix}\text{distance} \\ \text{in material}\end{pmatrix} - \begin{pmatrix}\text{distance} \\ \text{in air}\end{pmatrix}$

E.g. 3 Newton's Ring



= Thin flim but with varying thickness

$\Rightarrow$ Change to $t = R(1-\cos\varphi)$

Interference by complex form

Mathematically we can express a wave in the form.

$$\vec{E}(\vec{r},t) = |\vec{E}_0|\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

some amplitude

time travelled

angular frequency

distance travelled

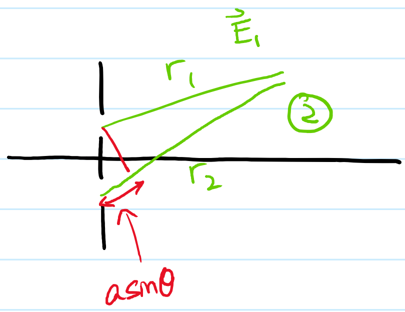
$|\vec{k}| = \frac{2\pi}{\lambda}$, along propagation direction

For 2 wave to interfere, they must have

- Same frequency $\rightarrow$ Same ω
- Same wavelength $\rightarrow$ Same k
- Arrive at the same position at the same time $\rightarrow$ Same t

The only thing difference in the $e^{i(\dots)}$ factor = distance travelled.

E.g. Double Slit again



$$E_1(r,t) = \vec{E}_0\, e^{i(kr_1 - \omega t)}$$

$$= \vec{E}_0\, e^{i(k\sqrt{D^2 + (y-\frac{a}{2})^2} - \omega t)}$$

$$\vec{E}_2(r,t) = \vec{E}_0\, e^{i(kr_2 - \omega t)}$$

$$= \vec{E}_0\, e^{i\left(k\sqrt{D^2 + (y+\frac{a}{2})^2} - \omega t\right)}$$

$$\simeq \vec{E}_0\, e^{i(k(r_1 + a\sin\theta) - \omega t)}$$

$\Rightarrow$ Interference = Superposition of 2 waves

$$\vec{E}_{tot}(r,t) = \vec{E}_1(r,t) + \vec{E}_2(r,t)$$

$$= \vec{E}_0\, e^{i(kr_1 - \omega t)} + \vec{E}_0\, e^{i(k(r_1 + a\sin\theta) - \omega t)}$$

$$= \vec{E}_0\, e^{i(kr_1 - \omega t)} \cdot \left[1 + e^{ika\sin\theta}\right]$$

$$= E_0\, e^{i(kr_1 - \omega t)} \cdot e^{ik\frac{a\sin\theta}{2}} \cdot \left[e^{-ik\frac{a\sin\theta}{2}} + e^{ik\frac{a\sin\theta}{2}}\right]$$

$$= \underline{E_0\, e^{i(kr_1 - \omega t)}} \cdot \underline{e^{ik\frac{a\sin\theta}{2}}} \cdot 2\cos\left[\frac{ka\sin\theta}{2}\right]$$

Original wave

Additional phase (something like e^{ix})

Amplitude variation (a real function of θ)

☆ $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

Computing the intensity

$$I \propto |\vec{E}_{tot}|^2$$

$|e^{i\theta}| = e^{i\theta} \times e^{-i\theta} = 1$

these phase factor has nothing to with amplitude

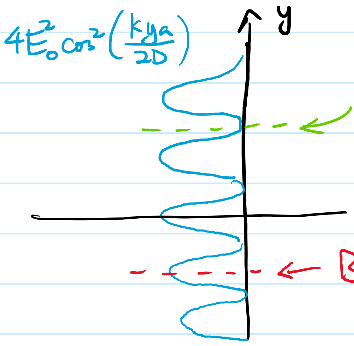
$$= E_0^2 \left|e^{i(kr_1 - \omega t)}\right|^2 \cdot \left|e^{ik\frac{a\sin\theta}{2}}\right|^2 \cdot \underline{4\cos^2\left[\frac{ka\sin\theta}{2}\right]}$$

the only factor related

$$= 4E_0^2 \cos^2\left[\frac{ka\sin\theta}{2}\right]$$

$$= 4E_0^2 \cos^2\left[\frac{kya}{2D}\right] \sim \cos^2(??\, y)$$

Plotting the intensity distribution and identify the fringes' positions



Dark fringes : Intensity = 0 when $\frac{kya}{2D} = \left(m+\frac{1}{2}\right)\pi$

$\Rightarrow \frac{ya}{D} = \left(m+\frac{1}{2}\right)\lambda$

Bright fringes : Intensity = $4E_0^2$ when $\frac{kya}{2D} = m\pi$

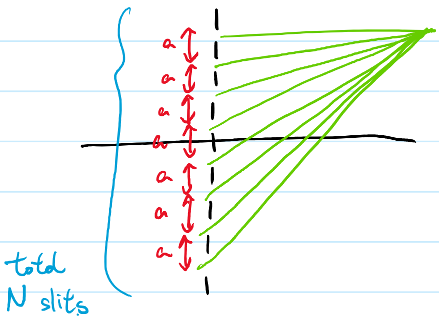
$\Rightarrow \frac{ya}{D} = m\lambda$

N slits Interference

Similar to double slit, but superposition by way more wave.

Note that path difference between 2 waves from adjacent slits

$= a\sin\theta$



① We can express the path difference of each ray relative to center one.

- If N = odd, take

Ray that comes out from center

$-\left(\frac{N-1}{2}\right)a\sin\theta, \ldots, -a\sin\theta, 0, a\sin\theta, \ldots \left(\frac{N-1}{2}\right)a\sin\theta$

- If N = even, take

center is between middle 2 slits

$-\left(\frac{N-1}{2}\right)a\sin\theta, \ldots, -\frac{a}{2}\sin\theta, \frac{a}{2}\sin\theta, \ldots, \left(\frac{N-1}{2}\right)a\sin\theta$

In any case, p.d. is summed from $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

② Then write the result wave as superposition

$$\vec{E}_{tot} = \vec{E}_{-\left(\frac{N-1}{2}\right)} + \vec{E}_{-\left(\frac{N-3}{2}\right)} + \dots + \vec{E}_{\left(\frac{N-3}{2}\right)} + \vec{E}_{\left(\frac{N-1}{2}\right)}$$

$$= \vec{E}_0 e^{i(kr_0-\omega t)} \cdot \Big[e^{-ik\left(\frac{N-1}{2}\right)a\sin\theta} + e^{-ik\left(\frac{N-3}{2}\right)a\sin\theta} + \dots$$

$$+ \dots + e^{ik\left(\frac{N-3}{2}\right)a\sin\theta} + e^{ik\left(\frac{N-1}{2}\right)a\sin\theta} \Big]$$

$$= \vec{E}_0 e^{i(kr_0-\omega t)} \cdot \sum_{p=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{ikpa\sin\theta}$$

③ Note that this is just a geometric sum.

$$u + ur + ur^2 + \dots + ur^{n-1} = \frac{u(r^n-1)}{r-1}$$

u = first term, r = ratio between terms., n = no. of terms

In the above, we have $\begin{cases} u = e^{-ik\left(\frac{N-1}{2}\right)a\sin\theta} \\ r = e^{ika\sin\theta} \end{cases}$

So the sum becomes

$$\vec{E}_{tot}(r,t) = \vec{E}_0 e^{i(kr_0-\omega t)} \cdot \left[e^{ik\left(\frac{N-1}{2}\right)a\sin\theta} \frac{\left[e^{ika\sin\theta}\right]^N - 1}{e^{ika\sin\theta} - 1} \right]$$

$$= \vec{E}_0 e^{i(kr_0-\omega t)} \cdot \frac{e^{\frac{-ikNa\sin\theta}{2}}}{e^{\frac{-ika\sin\theta}{2}}} \left[\frac{e^{ikNa\sin\theta} - 1}{e^{ika\sin\theta} - 1} \right]$$

$$= \vec{E}_0 e^{i(kr_0-\omega t)} \cdot \frac{\left[e^{\frac{ikNa\sin\theta}{2}} - e^{\frac{-ikNa\sin\theta}{2}} \right]}{\left[e^{\frac{ika\sin\theta}{2}} - e^{\frac{-ika\sin\theta}{2}} \right]}$$

$$= \vec{E}_0 e^{i(kr_0 - \omega t)} \cdot \frac{\sin\left[\frac{kNa\sin\theta}{2}\right]}{\sin\left[\frac{ka\sin\theta}{2}\right]}$$

By $e^{i\theta} = \cos\theta + i\sin\theta$

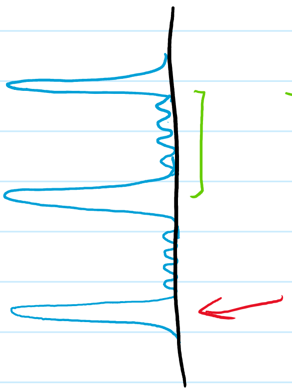
$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$= \vec{E}_0 e^{i(kr_0 - \omega t)} \cdot \frac{\sin\left[\frac{kNya}{2D}\right]}{\sin\left[\frac{kya}{2D}\right]}$$

④ Compute the intensity

$$I \propto |E_{tot}|^2$$

$$= |E_0|^2 \left[\frac{\sin\left[\frac{kNya}{2D}\right]}{\sin\left[\frac{kya}{2D}\right]}\right]^2$$



Total N-1 destructive interference

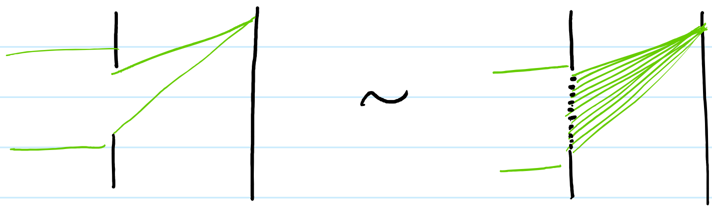
N-2 weak constructive interference

Very bright constructive interference

Fraunhofer Diffraction

Diffraction = Interference after through a wide aperture

Model: Aperture ~ ∞ Slits with 0 separation



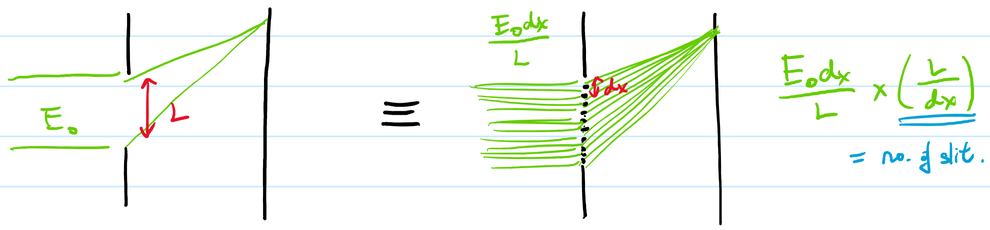
⇒ Turn summation to integral.

① Let aperture width $= L$

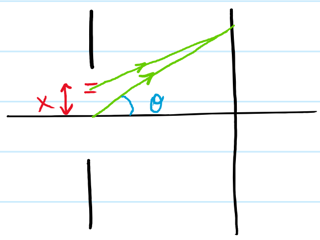
Divide the aperture into many "slit" with slit width dx

$\Rightarrow$ Amplitude of each ray that pass through a slit

$= E_0 \cdot \frac{dx}{L}$ (So all together $\equiv$ 1 beam of E_0)



② For each slit at height x from the center



$\Rightarrow$ P.d. relative to center ray $= x\sin\theta$

$\Rightarrow$ Express superposition as integral

$$\vec{E}_{tot}(r,t) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \vec{E}_0 \frac{dx}{L} \cdot e^{i(k(r_0 + x\sin\theta) - \omega t)}$$

$$= \frac{\vec{E}_0}{L} e^{i(kr_0 - \omega t)} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{ikx\sin\theta} dx$$

$$= \frac{\vec{E}_0}{L} e^{i(kr_0 - \omega t)} \frac{1}{ik\sin\theta} \left[e^{ikx\sin\theta} \right]_{x=-\frac{L}{2}}^{x=\frac{L}{2}}$$

$$= \frac{\vec{E}_0}{L} e^{i(kr_0 - \omega t)} \frac{1}{ik\sin\theta} \left[e^{ik\frac{L\sin\theta}{2}} - e^{-ik\frac{L\sin\theta}{2}} \right]$$

$$= \frac{\vec{E}_0}{L} e^{i(kr_0 - \omega t)} \frac{2\cancel{i} \sin\left[\frac{kL\sin\theta}{2}\right]}{\cancel{i}k\sin\theta}$$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$= \vec{E}_0 e^{i(kr_0 - \omega t)} \cdot \frac{\sin\left[\frac{kL\sin\theta}{2}\right]}{\frac{kL\sin\theta}{2}}$$

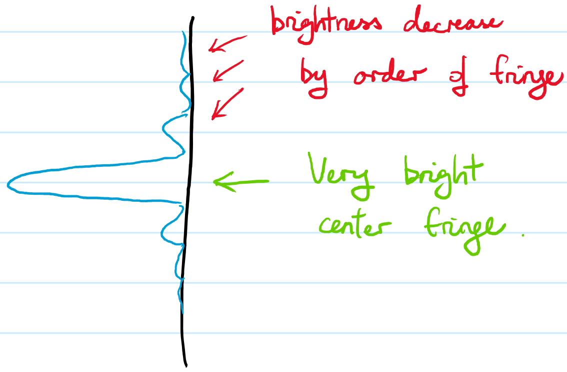
$$= \vec{E}_0 \, e^{i(kr_0 - \omega t)} \cdot \text{sinc}\left[\frac{kL\sin\theta}{2}\right]$$

Define new function $\text{sinc}(x) = \frac{\sin x}{x}$

Because it is very common to appear

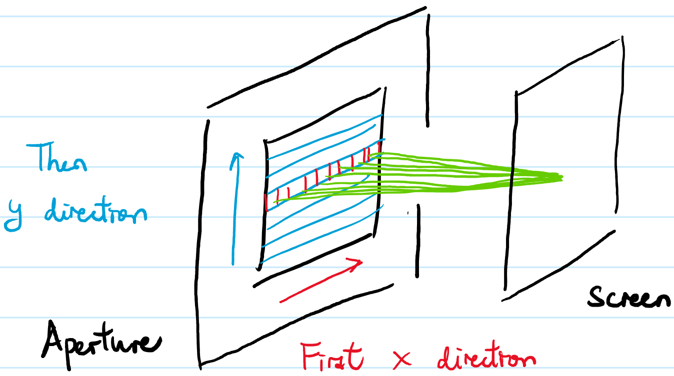
Plotting the intensity would look like :

$$I \propto |\vec{E}_0|^2 \, \text{sinc}^2\left(\frac{kL\sin\theta}{2}\right)$$



Diffraction through 2D aperture

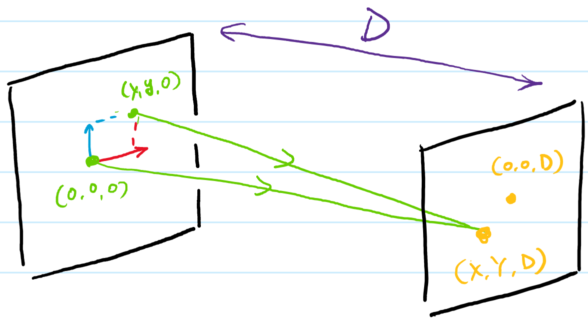
= Superposition of many 1D diffraction



$\Rightarrow$ 2D integral

Find the p.d. between a wave from (x, y) v.s a wave from $(0, 0)$

when both wave hit the same position (X, Y) on screen



Wave from $(0,0,0)$: $\vec{E}_1 = \vec{E}_0 e^{i(\vec{k}\cdot[X\hat{x} + Y\hat{y} + D\hat{z}] - \omega t)}$

$= \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}_0 - \omega t)}$

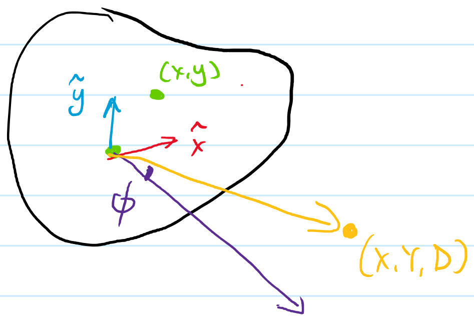
Wave from $(x,y,0)$: $\vec{E}_2 = \vec{E}_0 e^{i(\vec{k}\cdot[(X-x)\hat{x} + (Y-y)\hat{y} + D\hat{z}] - \omega t)}$

$= \vec{E}_0 e^{i(k\cdot\vec{r}_0 - \omega t - \underbrace{\vec{k}\cdot(x\hat{x}+y\hat{y})}_{\text{this is the p.d.}})}$

Further breaking down the p.d. by geometry :

$$\vec{k}\cdot(x\hat{x}+y\hat{y}) = |\vec{k}| \frac{X\hat{x} + Y\hat{y} + D\hat{z}}{r_0}\cdot(x\hat{x}+y\hat{y})$$

$$= |\vec{k}| \frac{xX + yY}{r_0}$$



$\vec{k}$ pointing to the point (X,Y,D) on screen

can be written as $\vec{k} = |\vec{k}| \frac{X\hat{x} + Y\hat{y} + D\hat{z}}{r_0}$

$\left(r_0 = \sqrt{X^2 + Y^2 + D^2}\right)$

So the superposition is expressed as the integral

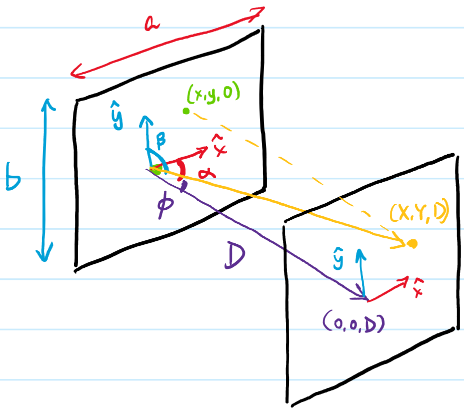
$$\vec{E}_{tot}(r,t) = \iint_{\text{bound of aperture}} \underbrace{\vec{E}_0 \frac{dx\,dy}{A}} e^{i(kr_0 - \omega t - |k|\frac{xX+yY}{r_0})}$$

The amplitude of each wave that passes through an infinitestimal area

A = Total area of the aperture

We shall look at 2 common apertures :

① <u>Rectangular Aperture</u> , width = a , height = b .



We shall keep the p.d. in this form :

$$|\vec{k}| \cdot \frac{xX + yY}{r_0}$$

$$= |\vec{k}| \left(x\sin\alpha + y\sin\beta \right)$$

So the integral becomes

$$\vec{E}_{tot}(r,t) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \vec{E}_0 \frac{dx\,dy}{a \cdot b} e^{i(kr_0 - \omega t - kx\sin\alpha - ky\sin\beta)}$$

$$= \frac{\vec{E}_0}{a \cdot b} e^{i(kr_0 - \omega t)} \cdot \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-ikx\sin\alpha} \cdot e^{-iky\sin\beta} \, dx\,dy$$

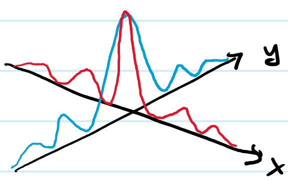
x, y are independent
can integrate separately

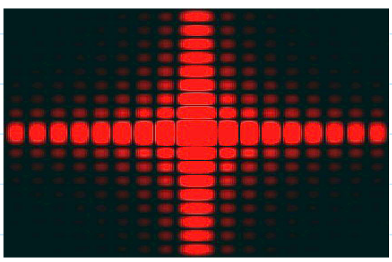
$$= \frac{\vec{E}_0}{ab} e^{i(kr_0 - \omega t)} \left[\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-ikx\sin\alpha} dx \right] \cdot \left[\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-iky\sin\beta} dy \right]$$

$$= \frac{\vec{E}_0}{ab} e^{i(kr_0 - \omega t)} \left[\frac{e^{-ik\frac{a\sin\alpha}{2}} - e^{ik\frac{a\sin\alpha}{2}}}{2 \cdot \left(-ik\frac{\sin\alpha}{2} \right)} \right] \left[\frac{e^{-ik\frac{b\sin\beta}{2}} - e^{ik\frac{b\sin\beta}{2}}}{2 \cdot \left(-ik\frac{\sin\beta}{2} \right)} \right]$$

$$= \vec{E}_0 e^{i(kr_0 - \omega t)} \cdot \text{sinc}\left[\frac{ka\sin\alpha}{2} \right] \text{sinc}\left[\frac{kb\sin\beta}{2} \right]$$

Plotting the intensity looks like :





② Circular aperture, radius R

First change to polar coordinate

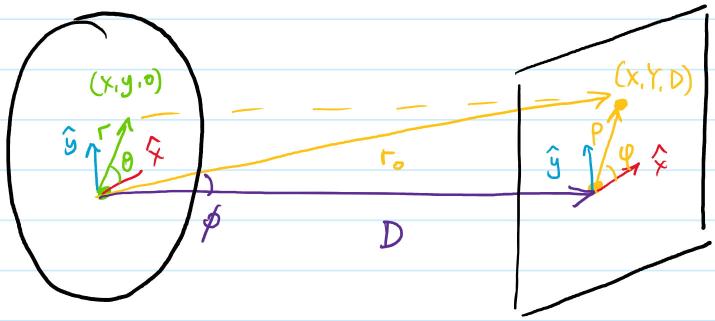
- Points on aperture : $(x, y, 0) \rightarrow (r\cos\theta, r\sin\theta, 0)$
- Points on screen : $(X, Y, D) \rightarrow (\rho\cos\varphi, \rho\sin\varphi, 0)$

The p.d. becomes

$$|\vec{k}| \cdot \frac{xX + yY}{r_0} = \frac{|\vec{k}|}{r_0} \cdot (r\cos\theta\, \rho\cos\varphi + r\sin\theta\, \rho\sin\varphi)$$

$$= \frac{|\vec{k}|}{r_0} r\rho \cos(\theta - \varphi)$$

$$= |\vec{k}|\, r\cos(\theta - \varphi) \sin\phi$$



The integral becomes

$$\vec{E}_{tot} = \int_0^R \int_0^{2\pi} \vec{E}_0 \frac{r\,d\theta\,dr}{\pi R^2}\, e^{-i(kr_0 - \omega t - kr\cos(\theta - \varphi)\sin\phi)}$$

$$= \frac{\vec{E}_0}{\pi R^2} e^{-i(kr_0 - \omega t)} \int_0^R r \underbrace{\int_0^{2\pi} e^{ikr\cos(\theta-\varphi)\sin\phi}\, d\theta}\, dr$$

Unfortunately, the integral to θ has no analytical form

The best we can do is represent it as an infinite series

In fact, it belongs to a group of polynomial called Bessel functions

(which are the general soln to a specific kind of PDE
but we will not discuss here)

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\cos\theta - n\theta)}\, d\theta$$

$$= \frac{x^n}{2^n n!}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} + \cdots\right]$$

Identify that $\int_0^{2\pi} e^{ikr\sin\phi\cos(\theta-\varphi)}\, d\theta = \int_0^{2\pi} e^{ix\cos\theta}\, d\theta = 2\pi J_0(x)$

take $n=0$

if we take $x = kr\sin\phi$. Therefore

$$\vec{E}_{tot} = \frac{1}{\pi R^2}\vec{E}_0 e^{i(kr_0-\omega t)} \cdot 2\pi\int_0^R r\, J_0(kr\sin\phi)\, dr$$

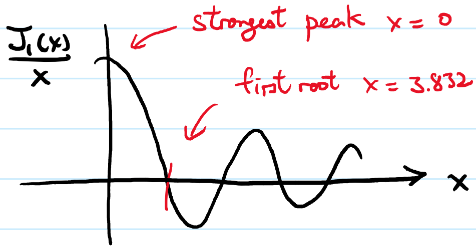
$$= \frac{2}{R^2}\vec{E}_0\, e^{i(kr_0-\omega t)} \cdot \frac{1}{(k\sin\phi)^2}\int_0^{r=R} kr\sin\phi \cdot J_0(kr\sin\phi)\, d(kr\sin\phi)$$

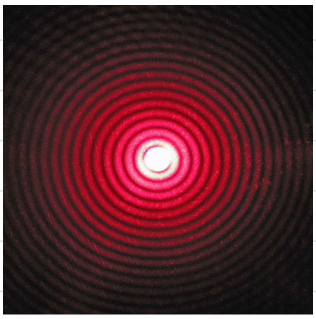
Property of Bessel function $\frac{d}{dx}\left[x^{n+1}J_{n+1}(x)\right] = x^{n+1}J_n(x)$

$$= \frac{2}{R^2}\vec{E}_0\, e^{i(kr_0-\omega t)}\, \frac{1}{(k\sin\phi)^2}\Big[kr\sin\phi\, J_1(kr\sin\phi)\Big]_0^{r=R}$$

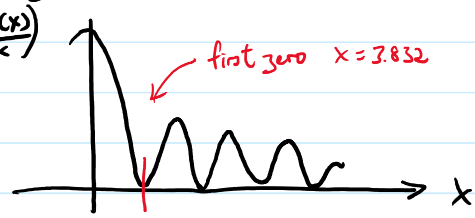
$$= 2\vec{E}_0\, e^{i(kr_0-\omega t)} \cdot \left[\frac{J_1(kR\sin\phi)}{kR\sin\phi}\right]$$

The plot of $\frac{J_1(x)}{x}$ looks like this:



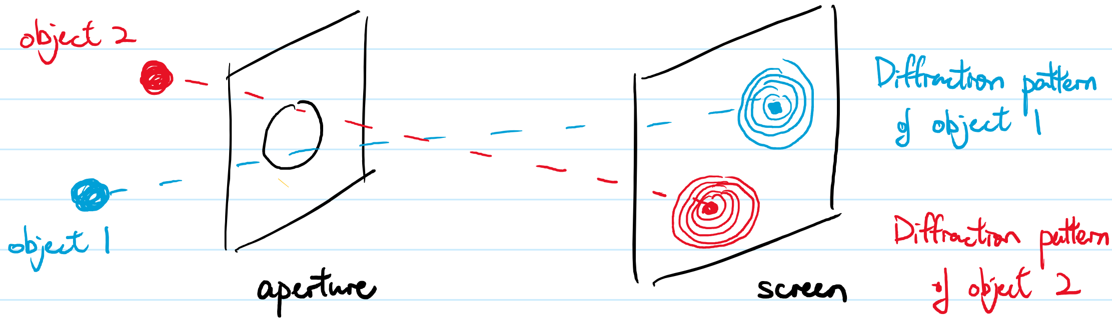


Intensity $\sim \left(\frac{J_1(x)}{x}\right)^2$

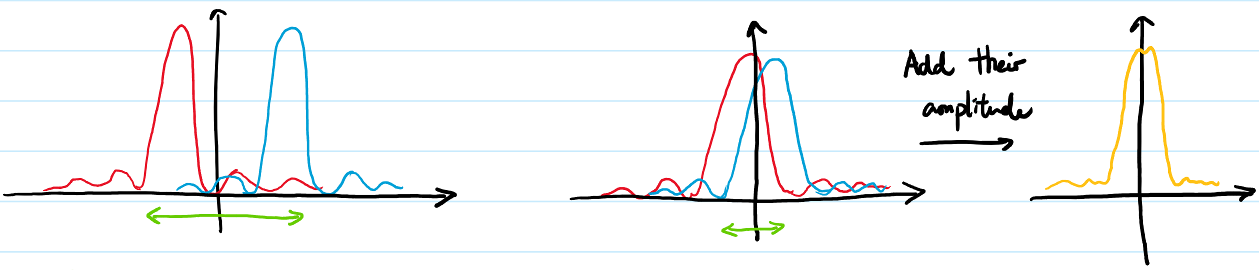


Rayleigh Criterion

Situation : 2 objects are "resolvable" if their images observed through an aperture are distant enough.



Look at the intensity plot along a line that through the peaks

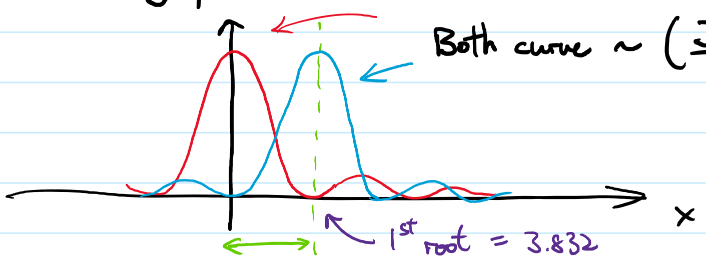


The peaks are far enough
Can confidently tell there are 2 objects.

The peaks are too close together
We cannot tell if there are really 2 peaks

Quantitatively, we define 2 objects to be resolvable if the peak of 1 image is located at least on the the 1st dark fringe or further from the peak of another image

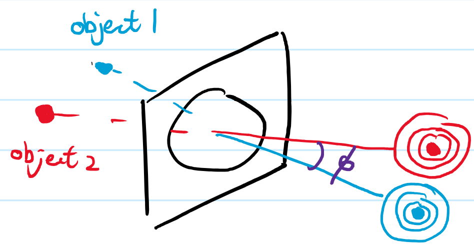
i.e. In the intensity plot



Both curve $\sim \left(\frac{J(x)}{x}\right)^2$

resolvable if peaks' separation > 3.832

Therefore we require $kR\sin\phi > 3.832$



Also by $k = \frac{2\pi}{\lambda}$,

$$\frac{2\pi R}{\lambda}\sin\phi > 3.832$$

$$\sin\phi > 0.6099\frac{\lambda}{R}$$

Or more commonly we use diameter of the aperture $D = 2R$

$$\Rightarrow \sin\phi > 1.219\frac{\lambda}{D}$$

This where you get the mysterious number 1.22.