

- Matrix operations . Determinants .
- Theory of systems of Linear Equations
- Eigenvalue & Eigenvector
- Solving System of ODE

E.g.  $\begin{vmatrix} k_1 & m_1 & m_1 \\ m_1 & k_2 & m_2 \\ m_1 & m_2 & k_3 \end{vmatrix}$

## Matrix

Dimension of matrix = (No. of rows)  $\times$  (No. of column)

Use bold font to represent  $\rightarrow$  a matrix  $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

Each entry is called an element

dimension =  $2 \times 3$

Column matrix  
(Column vector)

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_3 \end{pmatrix}$$

Row matrix  
(Row vector)

$$(a_1, a_2, \dots, a_n)$$

Square matrix  
No. of rows = No. of columns

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

## Matrix Arithmetics

Calculation of matrix is like calculating numbers , but with new definitions

### ① Multiplication with constants

Multiply a constant = Multiply each element with the constant

$$kA = k \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{pmatrix}$$

## ② Addition / Subtraction

Elements at the same position add/subtract independently.

Only matrices of the same dimension can be added/subtracted

$$A \pm B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \end{pmatrix}$$

## ③ Transpose

= Flip the matrix along the diagonal line

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

## ④ Matrix multiplication

Given 2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$$

(dimension :  $m \times n$ )

(dimension :  $n \times p$ )

Their product

$$C = AB = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{pmatrix}$$

↑  
we don't write  $\times$  sign in matrix multiplication

is a  $m \times p$  matrix

$$\text{Calculation of each element} = c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

$$\left( \begin{array}{c|ccccc} & & \vdots & & \\ \cdots & \cdots & c_{ij} & \cdots & \cdots & \vdots \\ & \overbrace{\hspace{1cm}}^{\text{jth column}} & \vdots & & & \end{array} \right) = \left( \begin{array}{c|ccccc} & & \vdots & & \\ & & \downarrow & & \\ & & \text{i}^{\text{th}} \text{ row} & & \\ \hline a_{i1} & a_{i2} & \cdots & a_{in} & & \\ \hline & & \overbrace{\hspace{1cm}}^{\text{every element}} & & & \end{array} \right) \left( \begin{array}{c|ccccc} & & & b_{1j} & & \\ & & & \overbrace{\hspace{1cm}}^{\text{jth column}} & b_{2j} & \cdots \\ & & & \vdots & \vdots & \\ & & & b_{nj} & & \end{array} \right)$$

Visualizing the multiplication rule :

$$\begin{pmatrix} 0 & 0 & 0 \\ \cdot & \cdot & \cdot \end{pmatrix} = \left( \begin{array}{c|c} \cdots & \cdots \end{array} \right) \times \left( \begin{array}{c|c} & \\ \downarrow & \downarrow \\ & \end{array} \right)$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{pmatrix} = \left( \begin{array}{c|c} & \\ \downarrow & \downarrow \\ & \end{array} \right) \times \left( \begin{array}{c|c} & \\ \downarrow & \downarrow \\ & \end{array} \right)$$

★ Matrices can be multiplied only if

$$\text{No. of columns in } 1^{\text{st}} \text{ matrix} = \text{No. of rows in } 2^{\text{nd}} \text{ matrix}$$

So in general  $AB \neq BA$

E.g.

$$A = \left( \begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right) \quad \begin{matrix} \uparrow 3 \\ \uparrow 2 \end{matrix} \quad B = \left( \begin{array}{c|c} 7 & \\ 8 & \\ 9 & \end{array} \right) \quad \begin{matrix} \uparrow 1 \\ \uparrow 3 \end{matrix}$$

$$AB = \left( \begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right) \left( \begin{array}{c|c} 7 & \\ 8 & \\ 9 & \end{array} \right) = \left( \begin{array}{c|c} 1 \times 7 + 2 \times 8 + 3 \times 9 & \\ 4 \times 7 + 5 \times 8 + 6 \times 9 & \end{array} \right) = \left( \begin{array}{c|c} 50 & \\ 122 & \end{array} \right)$$

$2 \times 3 \rightarrow 3 \times 1$

But  $BA$  does not exist

$$\left( \begin{array}{c|c} 7 & \\ 8 & \\ 9 & \end{array} \right) \left( \begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right)$$

$3 \times 1 \xrightarrow{*} 2 \times 3$

The closest we have is  $(AB)^T = B^T A^T$  (always true)

$$\left( \begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right) \left( \begin{array}{c|c} 7 & \\ 8 & \\ 9 & \end{array} \right) \xleftarrow{\text{transpose}} \left( \begin{array}{c|cc} 7 & 8 & 9 \\ 1 & 2 & 3 \end{array} \right) \left( \begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right)$$

### ⑤ Identity Matrix

= Any square matrix with diagonal elements = 1 and else = 0

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Property :  $IA = AI = A$  if  $A$  is a square matrix

It is just like multiplying 1 to numbers

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times 0 & 1 \times 0 + 2 \times 1 \\ 3 \times 1 + 4 \times 0 & 3 \times 0 + 4 \times 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

### ⑥ Zero matrix

Matrices with all elements = 0 (just like the 0 in numbers)

$$O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad O_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

### ⑦ Inverse of matrices ( $\sim$ Division of matrices)

If  $AB = I = BA$  (Require both to be square matrices)

Then  $A$  is the inverse of  $B$ , i.e.  $A = B^{-1}$

$B$  is the inverse of  $A$ , i.e.  $B = A^{-1}$

Faster method to compute inverse : Row operations

- Exchange any 2 rows  $(R_i \leftrightarrow R_j)$
- Multiply a row with constant  $(R_i \leftarrow kR_i)$
- Add a multiple of a row to another  $(R_i \leftarrow R_i + kR_j)$

E.g. Find inverse of  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

① Joining with identity matrix on RHS (Use  $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , because  $2 \times 2$ )

$$\left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right)$$

② Use row operations to turn LHS to be identity matrix

$$\left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right) \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3-3 \times 1 & 4-3 \times 2 & 0-3 \times 1 & 1-3 \times 0 \end{array} \right)$$

Turn this to 0

$$\left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right) \xrightarrow{R_1 \leftarrow R_1 + R_2} \left( \begin{array}{cc|cc} 1+0 & 2+(-2) & 1+(-3) & 0+1 \\ 0 & -2 & -3 & 1 \end{array} \right)$$

Turn this to 0

$$\left( \begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array} \right) \xrightarrow{R_2 \leftarrow \frac{-1}{2}R_2} \left( \begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 \times \frac{-1}{2} & -2 \times \frac{-1}{2} & -3 \times \frac{-1}{2} & 1 \times \frac{-1}{2} \end{array} \right)$$

Turn this to 1

$$\left( \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \mid \left[ \begin{array}{cc} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{array} \right] \right)$$

identity

This is the inverse

Checking

$$\left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right) \left( \begin{array}{cc} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{array} \right) = \left( \begin{array}{cc} 1 \times -2 + 2 \times \frac{3}{2} & 1 \times 1 + 2 \times \frac{-1}{2} \\ 3 \times -2 + 4 \times \frac{3}{2} & 3 \times 1 + 4 \times \frac{-1}{2} \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

\* Not all matrices have inverse

$$\text{E.g. } \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

it is impossible to turn these entries to (0 1)

Matrices without inverse are called singular

## Determinant

Notation :  $\det(A) =$

Use straight lines →  
instead of parenthesis

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

\* We can only calculate determinant for square matrices

It is a special operation (~function) on matrices that returns a number

(Rule of Sarrus)

① For  $2 \times 2$  &  $3 \times 3$  matrices, the computations have a convenient trick

$$2 \times 2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = + (a_{11} \cdot a_{22}) - (a_{21} \cdot a_{12})$$

$$3 \times 3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = + (a_{11} \cdot a_{22} \cdot a_{33}) + (a_{12} \cdot a_{23} \cdot a_{31}) + (a_{13} \cdot a_{21} \cdot a_{32}) - (a_{31} \cdot a_{22} \cdot a_{13}) - (a_{32} \cdot a_{23} \cdot a_{11}) - (a_{33} \cdot a_{21} \cdot a_{12})$$

copy

② The above trick does not work with  $4 \times 4$  or above

One must compute  $\det(A)$  by the general way : minor & cofactor

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$$

Intermediate steps required :

Minor of  $a_{ij}$  - Remove the  $i^{\text{th}}$  row &  $j^{\text{th}}$  column

then take the determinant of the remaining matrix

$$\underline{\text{Cofactor of } a_{ij}} = (\text{Minor of } a_{ij}) \times (-1)^{i+j} = \text{cof}(a_{ij})$$

$$\text{cof}(a_{ij}) = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{(j+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ \hline a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix}$$

Remove  $j^{\text{th}}$  column

Remove  $i^{\text{th}}$  row

$\star\star$  Need to  
find determinant  
of this smaller matrix

Then,  $\det(A)$  can be computed by either

$$\sum a_{ij} \text{cof}(a_{ij}) \quad \text{or} \quad \sum a_{ij} \text{cof}(a_{ij})$$

along any row  
sum all j

along any column  
sum all i

(These formulas work for  $2 \times 2$  &  $3 \times 3$  too, but much more annoying)

E.g.

$$\begin{pmatrix} |1 & 2 & 3| \leftarrow \\ |4 & 5 & 6| \\ |7 & 8 & 9| \end{pmatrix}$$

We can choose any row/column to start with  
(E.g. we can choose the row/column with the most 0)  
In this example. I pick the 1<sup>st</sup> row

For  $a_{11} = \boxed{1}$ ,  $i=1$   $j=1$

$$\begin{pmatrix} |1 & 2 & 3| \\ |4 & 5 & 6| \\ |7 & 8 & 9| \end{pmatrix} \rightarrow \text{Cofactor} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3$$

For  $a_{12} = \boxed{2}$ ,  $i = 1, j = 2$

$$\begin{pmatrix} 1 & \boxed{2} & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow \text{Cofactor} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 6$$

For  $a_{13} = \boxed{3}$ ,  $i = 1, j = 3$

$$\begin{pmatrix} 1 & 2 & \boxed{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow \text{Cofactor} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$$

$$\Rightarrow \text{Determinant} = \boxed{1} \times -3 + \boxed{2} \times 6 + \boxed{3} \times -3 = 0$$


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Supplementary : Some useful properties/formulae

[1] If any 2 rows / any 2 columns of a matrix are equal

its determinant = 0

[2] If we change any column  $j$  from  $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$  to  $\begin{pmatrix} k a_{1j} + b_{1j} \\ k a_{2j} + b_{2j} \\ \vdots \\ k a_{nj} + b_{nj} \end{pmatrix}$

determinant becomes  $k \cdot \left| \begin{array}{c|cc|c} a_{1j} & \dots & a_{nj} & b_{1j} \\ \dots & \dots & \dots & \dots \\ a_{2j} & \dots & a_{nj} & b_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{nj} & \dots & a_{nj} & b_{nj} \end{array} \right|$

with all other columns the same as in the original matrix

This also applies to any row.

[3] Interchange any 2 rows / any 2 column  $\Rightarrow \det(A) \rightarrow -\det(A)$

[4]  $\det(A^T) = \det(A)$

[5]  $\det(AB) = \det(A)\det(B)$ . And so  $\det(A^{-1}) = \frac{1}{\det(A)}$

## Theory of System of Linear Equation

We can always rewrite a system of linear equation into matrix form

$$\left\{ \begin{array}{l} 2y - z = 1 \\ x - y + z = 0 \\ 2x + y - z = -2 \end{array} \right. \Leftrightarrow \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$A \quad x = b$$

Note that if  $A$  has inverse  $A^{-1}$ , then we can do

$$\underbrace{A^{-1}(Ax)}_{= \text{identity}} = A^{-1}b \quad \begin{matrix} (\text{matrix}) \times (\text{its inverse}) \\ = \text{identity} \end{matrix}$$

$$x = A^{-1}b \quad \text{Anything} \times (\text{identity}) = \text{no change}$$

So if  $A$  has inverse, the system of equation can be solved

just by finding  $A^{-1}$  and then multiply it to  $b$

Problem : Some matrices do not have inverse

We first need to check if  $A^{-1}$  exists. How to check?

Theorem without proof [1]

From the inverse formula that nobody uses :

$$A^{-1} = \frac{[\text{cof}(a_{ij})]^T}{\det(A)}$$

$\Rightarrow$  If  $\det(A) = 0$ ,  $A^{-1}$  will not exist

Theorem without proof [2]

If  $A^{-1}$  exists, the system of linear equation  $Ax = b$

one & only one  
has an unique solution  $x = A^{-1}b$

$$\underline{\text{E.S.}} \quad \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$\mathbf{A} \quad \mathbf{x} \quad \mathbf{b}$

Compute  $\det(\mathbf{A})$  :

$$\left| \begin{array}{ccc|c} 0 & 2 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & 1 & -1 & -2 \end{array} \right| = +0 + (2 \times 1 \times 2) + (-1 \times 1 \times 1) - (2 \times -1 \times -1) - 0 - (-1 \times 1 \times 2)$$

$$= 3 \neq 0$$

From the above theorems we can tell that the soln. is unique

$$\text{Then we can find } \mathbf{A}^{-1} = \frac{1}{3} \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 4 & -2 \end{pmatrix} \quad (\text{skipping the steps})$$

And so the unique solution is

$$\begin{aligned} \mathbf{A}^{-1} \mathbf{b} &= \frac{1}{3} \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -2 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}$$

$$\therefore \left\{ x = \frac{-2}{3}, y = \frac{5}{3}, z = \frac{7}{3} \right\}$$

However this is never the recommended method in solving linear equations. You can solve it much faster by row operations / substitution.

## Homogeneity of Linear Equation

Similar to ODE, if all terms in the equation contain the unknown, the equation is homogeneous. Otherwise non-homogeneous

- Homogeneous  $A \underline{x} = \underline{0}$ 
  - the unknown  $\xrightarrow{\text{No other term other than } Ax}$
- Non-homogeneous  $A \underline{x} = \underline{b}$   $\xleftarrow{\text{A term that does not contain } x}$

### Case ① : Homogeneous

Obviously  $\underline{x} = \underline{0}$  is a solution  $\Rightarrow$  the "trivial solution"

(Anything multiply 0 = 0)

① If  $\det(A) \neq 0 \Rightarrow A^{-1}$  exists

$\Rightarrow$  Solution must be unique

$\Rightarrow \underline{x} = \underline{0}$  is the only solution

② If  $\det(A) = 0 \Rightarrow A^{-1}$  does not exist

$\Rightarrow \underline{x}$  has multiple solutions

$$\text{E.g. } \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x+y=0 \\ 0+0=0 \end{cases}$$

then we can choose  $x$  to be anything (and  $y = -x$ )

$\Rightarrow$  Infinitely many solutions

## Case ② : Non-homogeneous

Just like in non-homogeneous ODE, the solution

can be separated into 2 parts:

$$x = \frac{x_c}{\text{complementary}} + \frac{x_p}{\text{particular}}$$

Complementary soln.  $\mathbf{x}_c$  = Soln. to its homogeneous counterpart  
 $=$  Soln. when  $\mathbf{A}\mathbf{x} = \mathbf{0}$

Particular soln.  $x_p$  = A special soln. for cancelling the non-homogeneous term  $b$

□ If  $\det(A) \neq 0$

- The only soln. to  $\mathbf{A}\mathbf{x}_c = \mathbf{0}$  is  $\underline{\mathbf{x}_c = \mathbf{0}}$

- If  $\underline{x}_p = \underline{A}^{-1}\underline{b}$  then  $\underline{A}\underline{x}_p = \underline{b}$

$\Rightarrow$  The unique solution is  $\underline{x} = \underline{x}_c + \underline{x}_p = \mathbf{A}^{-1}\mathbf{b}$

② If  $\det(A) = 0$

- $x_c$  has infinitely many solutions

-  $x_p$  may or may not exist. Requires actual solving.

$$\Rightarrow \left\{ \begin{array}{l} x_p \text{ exist } \Rightarrow x = x_c \text{ (infinitely many)} + x_p \text{ (exist)} \\ \qquad \qquad \qquad = \text{Infinitely many} \\ x_p \text{ not exist } \Rightarrow x = x_c \text{ (infinitely many)} + x_p \text{ (not exist)} \\ \qquad \qquad \qquad = \text{Not exist.} \end{array} \right.$$

E.g. 1

$$\left\{ \begin{array}{l} x + 2y + 3z = 0 \\ 2x + 3y + 8z = 0 \\ 3x + 2y + 17z = 0 \end{array} \right. \Leftrightarrow \text{homogeneous} \quad \left( \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 8 & 0 \\ 3 & 2 & 17 & 0 \end{array} \right) \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right)$$

⋮

Check if  $\det(A) = 0 \rightarrow$  Yes. (skip the steps here)

∴ Soln. must be infinitely many

Then we shall solve it exactly by row operations

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 8 & 0 \\ 3 & 2 & 17 & 0 \end{array} \right) \xrightarrow{\begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array}} \left( \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -4 & 8 & 0 \end{array} \right) \xrightarrow{\begin{array}{l} R_3 \leftarrow R_3 - 4R_2 \\ R_2 \leftarrow -R_2 \end{array}} \left( \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Transforming back to  $x/y/z$ , reads as

$$\left\{ \begin{array}{l} x + 2y + 3z = 0 \\ y - 2z = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = -2y - 3z = -7z \\ y = 2z \end{array} \right.$$

Therefore the soln. is  $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7z \\ 2z \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} z$  where  $z$  can be any constant

E.g. 2

$$\left\{ \begin{array}{l} x + 2y + 3z = 3 \\ 2x + 3y + 8z = 4 \\ 3x + 2y + 17z = 1 \end{array} \right. \Leftrightarrow \text{non-homogeneous} \quad \left( \begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 3 & 2 & 17 & 1 \end{array} \right) \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{c} 3 \\ 4 \\ 1 \end{array} \right)$$

A is the same as in last example, but b is non-zero.

∴ Soln. must be either infinitely many or not exist

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 3 & 2 & 17 & 1 \end{array} \right) \xrightarrow{\begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array}} \left( \begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -4 & 8 & -8 \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} R_3 \leftarrow R_3 - 4R_2 \\ R_2 \leftarrow -R_2 \end{array}} \left( \begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Transforming back to  $x/y/z$ , reads as

$$\left\{ \begin{array}{l} x + 2y + 3z = 3 \\ y - 2z = 2 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = -2y - 3z + 3 = -7z - 1 \\ y = 2z + 2 \end{array} \right.$$

Therefore the soln. is  $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7z - 1 \\ 2z + 2 \\ z \end{pmatrix}$

$z$  can be  
any constant

$$= \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} \underline{z} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

solution when  $b = 0$  particular  
i.e. the complementary soln. soln.

E.g. 3

$$\left\{ \begin{array}{l} x + 2y + 3z = 3 \\ 2x + 3y + 8z = 4 \\ 3x + 2y + 17z = 2 \end{array} \right. \Leftrightarrow \left( \begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 3 & 2 & 17 & 2 \end{array} \right) \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{c} 3 \\ 4 \\ 2 \end{array} \right)$$

non-homogeneous      A    x     $\nearrow b$

A is the same as in last example, but different b

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 3 & 2 & 17 & 2 \end{array} \right) \xrightarrow{\begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array}} \left( \begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -4 & 8 & -7 \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} R_3 \leftarrow R_3 - 4R_2 \\ R_2 \leftarrow -R_2 \end{array}} \left( \begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

The last row reads as  $0x + 0y + 0z = 1$ . So no solution.

## Eigenvalue Problem

Given a matrix  $A$ . Find any non-zero column matrix  $x$

and number  $\lambda$  satisfying  $Ax = \lambda x$

Then  $\begin{cases} x = \text{Eigenvector of } A \\ \lambda = \text{Eigenvalue of } A \text{ corresponding to } x \end{cases}$

Theorem :  $\lambda$  is an eigenvector of  $A \Leftrightarrow \det(A - \lambda I) = 0$

Proof :  $Ax = \lambda x = \lambda Ix$

$$Ax - \lambda Ix = 0$$

$$(A - \lambda I)x = 0$$

Note that this is a homogeneous system of linear equations

① If  $\det(A - \lambda I) \neq 0$

$\Leftrightarrow$  The only soln. is  $x = 0$  (reject)

② If  $\det(A - \lambda I) = 0$

$\Leftrightarrow x$  has infinitely many soln.

There will always be infinitely many  $x$  satisfying  $Ax = \lambda x$

because if  $x$  is an eigenvector,  $kx$  is also an eigenvector, for any constant  $k$ .

$$A(\underline{kx}) = k(Ax) = k(\lambda x) = \lambda(\underline{kx})$$

## Steps for solving Eigenvalue Problem

① Solve the equation  $\det(A - \lambda I) = 0$  to get some  $\lambda$

② For each  $\lambda$ , subst. back into  $(A - \lambda I)x = 0$  to solve  $x$

E.g.  $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

① Solve  $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \det \left[ \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

$$= \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix}$$

$$= + (1-\lambda)(1-\lambda) - (4)(1)$$

$$= \lambda^2 - 2\lambda - 3 \quad \text{require} = 0$$

$$\Rightarrow \lambda = 3 \text{ or } -1 \quad (2 \text{ eigenvalues})$$

② Subst. back each  $\lambda$  into  $(A - \lambda I)x = 0$  to solve  $x$

II For  $\lambda = 3$

$$(A - \lambda I)x = \left[ \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Read as  $\begin{cases} -2x_1 + x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{cases}$  2^{nd} \text{ Eq. is simply } (1^{st} \text{ Eq.}) \times (-2)

$$\rightarrow x_2 = 2x_1$$

$x_1$  can be any  
constant

$$\therefore \text{General soln. } x = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \underline{x_1}$$

② For  $\lambda = -1$

$$(A - \lambda I)x = \left[ \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - (-1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Read as  $\begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{cases}$  2<sup>nd</sup> Eq. is simply  
(1<sup>st</sup> Eq.)  $\times$  (2)

$$\Rightarrow x_2 = -2x_1$$

$\therefore$  General soln :  $x = \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \underline{x_1}$

$x_1$  can be any constant

Conclusion :  $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$  has 2 eigenvectors

Any multiples of these 2 column matrices are also acceptable answers  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  with eigenvalue = 3  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$  with eigenvalue = -1

Note 1 : Eigenvalues & Eigenvector can be complex no.

E.g.  $\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} \lambda = -1 \quad 1-2i \quad 1+2i \\ x: \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1-i \\ -2i \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 2i \\ 1 \end{pmatrix} \end{cases}$

Note 2 : An  $n \times n$  matrix has at most  $n$  eigenvalues.

Because some matrices have repeated eigenvalues.

E.g.  $\begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \Rightarrow \lambda = 3$  only and  $x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} s + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t$

s & t can be any constants

## System of ODEs

Only deal with the case : Linear + Homogeneous + Constant Coefficients

Otherwise too difficult to solve by hand !

### 1<sup>st</sup> order Linear (Homogeneous) Constant Coefficient

E.g.

$$\begin{cases} \frac{d}{dt} x_1(t) = a x_1(t) + b x_2(t) \\ \frac{d}{dt} x_2(t) = c x_1(t) + d x_2(t) \end{cases}$$

① Rewrite into matrix form

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \frac{d}{dt} \mathbf{x} = \mathbf{A}\mathbf{x}$$

② Use the trick  $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$

$k_1, k_2$  are some constants

But here we let  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 e^{\lambda t} \\ k_2 e^{\lambda t} \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$

Then  $\frac{d}{dt} \mathbf{x} = \lambda \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$

③ Subst. back. Note that it becomes an eigenvalue problem.

$$\frac{d}{dt} \mathbf{x} = \lambda \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t} = \mathbf{A}\mathbf{x}$$

$$\lambda \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$\Rightarrow$  To the matrix  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

-  $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$  needs to be its eigenvector

-  $\lambda$  needs to be its eigenvalue

④ After solving for  $\lambda$  &  $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ , the general soln. is

the superposition of all its eigenvector

$$x = \underbrace{C_1}_{\text{constant}} \underbrace{\begin{pmatrix} k_{11} \\ k_{12} \end{pmatrix}}_{\substack{\uparrow \\ \text{eigenvector 1}}} e^{\lambda_1 t} + \underbrace{C_2}_{\text{constant}} \underbrace{\begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix}}_{\substack{\uparrow \\ \text{eigenvector 2}}} e^{\lambda_2 t}$$

$\uparrow \text{eigenvalue 1}$        $\uparrow \text{eigenvalue 2}$

For example,  $2 \times 2$  matrix  $\rightarrow$  at most 2 eigenvector  
 $\rightarrow$  at most 2 terms in the soln.

E.g.  $\begin{cases} \frac{d}{dt} x_1 = x_1 + x_2 \\ \frac{d}{dt} x_2 = 4x_1 + x_2 \end{cases} \Rightarrow \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

We have used this matrix in the last example and found

$$\text{Eigenvalue} = 3 \quad -1$$

$$\text{Eigenvector} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

So the general soln to this system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

$C_1, C_2$  are constants to be matched when given initial condition

Ex

$$\begin{cases} \frac{d}{dt} x_1 = x_1 + x_2 - 2x_3 \\ \frac{d}{dt} x_2 = x_2 - 4x_3 \\ \frac{d}{dt} x_3 = 2x_1 - x_3 \end{cases}$$

Write down its general soln.

## 2<sup>nd</sup> order Linear Homogeneous Constant Coefficient

E.g.  $\begin{cases} \frac{d^2}{dt^2}x_1(t) = a \frac{d}{dt}x_1(t) + b \frac{d}{dt}x_2(t) + cx_1(t) + dx_2(t) \\ \frac{d^2}{dt^2}x_2(t) = p \frac{d}{dt}x_1(t) + q \frac{d}{dt}x_2(t) + rx_1(t) + sx_2(t) \end{cases}$

We can convert it to a 1<sup>st</sup> order system by letting

$$\frac{d}{dt}x_1(t) = u_1(t), \quad \frac{d}{dt}x_2(t) = u_2(t)$$

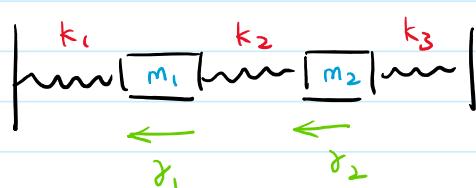
which then the system becomes

$$\begin{aligned} \frac{d^2}{dt^2}x_1 &= \frac{d}{dt}u_1 & \frac{d}{dt}u_1(t) &= au_1(t) + bu_2(t) + cx_1(t) + dx_2(t) \\ \frac{d^2}{dt^2}x_2 &= \frac{d}{dt}u_2 & \frac{d}{dt}u_2(t) &= pu_1(t) + qu_2(t) + rx_1(t) + sx_2(t) \\ \text{Definitions of } u_1, u_2 & \left[ \begin{array}{l} \frac{d}{dt}x_1(t) = u_1(t) \\ \frac{d}{dt}x_2(t) = u_2(t) \end{array} \right] \end{aligned}$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ p & q & r & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix}$$

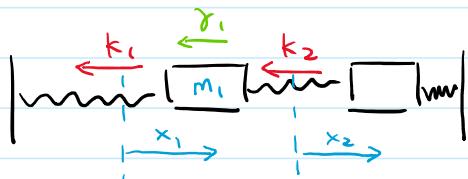
(Hand solving this is a nightmare)

## Application : Coupled spring-mass system with damping



Let displacement of  $m_1 = x_1(t)$ , displacement of  $m_2 = x_2(t)$

## Newton 2<sup>nd</sup> Law for $m_1$ :

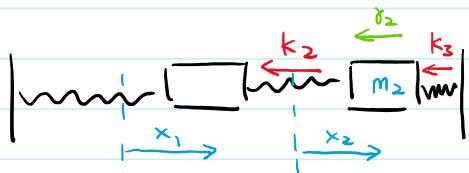


Left spring ( $k_1$ ) change in length =  $|x_1|$

Right spring ( $k_2$ ) change in length =  $|x_2 - x_1|$

$$\therefore F = m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 - k_2(x_2 - x_1) - \gamma_1 \frac{dx_1}{dt}$$

## Newton 2<sup>nd</sup> Law for $m_2$ :



Left spring ( $k_2$ ) change in length =  $|x_2 - x_1|$

Right spring ( $k_3$ ) change in length =  $|x_2|$

$$\therefore F = m_2 \frac{d^2 x_2}{dt^2} = -k_3 x_2 - k_2(x_2 - x_1) - \gamma_2 \frac{dx_2}{dt}$$

Combining the 2 equations and transform into a 1<sup>st</sup> order system:

$$\left\{ \begin{array}{l} \frac{du_1}{dt} = -\frac{\gamma_1}{m_1} u_1 - \frac{k_1+k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2 \\ \frac{du_2}{dt} = -\frac{\gamma_2}{m_2} u_2 + \frac{k_2}{m_2} x_1 + \frac{k_2+k_3}{m_2} x_2 \\ \frac{dx_1}{dt} = u_1 \\ \frac{dx_2}{dt} = u_2 \end{array} \right.$$

(Too annoying to solve)

Instead, here only demonstrates the results of a simplified case:

Suppose  $k_1 = k_3$ ,  $\gamma_1 = \gamma_2 = 0$ ,  $m_1 = m_2 = m$

Then the Newton 2<sup>nd</sup> Laws are just

$$\begin{aligned}\frac{d^2}{dt^2}x_1 &= -\frac{k_1+k_2}{m}x_1 + \frac{k_2}{m}x_2 \\ \frac{d}{dt^2}x_2 &= \frac{k_2}{m}x_1 - \frac{k_1+k_2}{m}x_2 \\ \Rightarrow \frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -\frac{k_1+k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_1+k_2}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\end{aligned}$$

We don't have to break it to 4 equations because

we can substitute  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{wt}$  and get

$$\underbrace{\omega^2 \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{wt}}_{\lambda \quad x} = \underbrace{\begin{pmatrix} -\frac{k_1+k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_1+k_2}{m} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{wt}}_x$$

is an eigenvalue problem, but with eigenvalue =  $\omega^2$

On solving can find

$$\text{Eigenvalue} = \omega^2 = -\frac{k_1}{m}, -\frac{k_1+2k_2}{m}$$

$$\text{Eigenvector} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So the general soln is

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k_1}{m}}t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\sqrt{\frac{k_1}{m}}t} \\ &\quad + C_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{k_1+2k_2}{m}}t} + C_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{\frac{k_1+2k_2}{m}}t}\end{aligned}$$

## Physical Interpretation

Note that the general soln. is a superposition of

2 kinds of vibration of different frequencies  
(mode of vibration)

### Motion ①

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k_1}{m}}t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{\frac{k_1}{m}}t}$$

$$\sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} A \cos\left(\sqrt{\frac{k_1}{m}}t + \phi\right)$$

$\Rightarrow$  In this pattern of vibration,  $x_1 = x_2 = A \cos(\sqrt{\frac{k_1}{m}}t + \phi)$

$$\text{[m}_1\overset{x}{\underset{x}{\text{--}}} \text{m}_2\text{]} \equiv \text{[m}_1\overset{x}{\underset{x}{\text{--}}} \text{m}_2\text{]}$$

middle spring is not compressed/extended at all

$m_1, m_2$  always move together  $\Rightarrow$  Call this the "in phase" mode

### Motion ②

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{k_1+2k_2}{m}}t} + C_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\sqrt{\frac{k_1+2k_2}{m}}t}$$

$$\sim \begin{pmatrix} 1 \\ -1 \end{pmatrix} A \cos\left(\sqrt{\frac{k_1+2k_2}{m}}t + \phi\right)$$

$\Rightarrow$  In this pattern of vibration,  $x_1 = -x_2 = A \cos(\sqrt{\frac{k_1+2k_2}{m}}t + \phi)$

$$\text{[m}_1\overset{-}{\underset{+}{\text{--}}} \text{m}_2\text{]} \leftrightarrow \text{[m}_1\overset{+}{\underset{-}{\text{--}}} \text{m}_2\text{]}$$

$m_1, m_2$  are always moving in opposite direction

$\Rightarrow$  Call this the "out of phase" mode