- Motion under Lorentz Force
- Magnetic Induction
  - Motional / Transformer EMF
  - Faraday's Law
  - Lenz's Law
- "Fixing" Ampere's Law - Displacement Current

Brief History of Electrodynamics

Ancient - 1500s Different electrostatic phenomenon are known. But not unified and no explaination yet.

William Gibert (1600) First person to use the term "electrical" to describe those phenomenon. First proposal that electrical effect is due to flows of particles

Benjamin Franklin (1750) Developed a one "fluid" theory of electricity and called this fluid "charge"

Columb (1784) Experimentally prove that force between charged object $\propto \frac{1}{r^2}$

$\left(F = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{r^2} \right)$

Volta (1800) First battery from electro-chemistry.

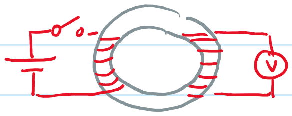
(Finally can study steady current)

Ampere (1820) Derive formula of the force between current wire

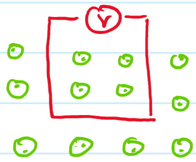
$\left(F = I\vec{l}_1 \times \left(\frac{\mu_0 I}{2\pi r} \hat{l}_2 \right) \right)$

Faraday (1831)

Discover & experiment on magnetic induction



Reading appears at the instant switch is on / off



Reading appears when the wireframe move, change shape, or when magnetic field change

Lenz (1834)

Explain direction of induced current by energy conservation

Maxwell (1860)

- Unify many past discoveries into 20 equations
- Fix Ampere's Law with displacement current
- Introduce the vector field description $\vec{E} / \vec{B}$

(Before then, everything was described by force)

Oliver Heaviside (1893)

Reformulate the 20 Maxwell equations into the 4 by vector calculus

Lorentz (1895)

Derive the correct formula of force on charge under both E & B field

☆ Historically, Lorentz force was formulated much later than anything. But in modern textbooks, we treat Lorentz force as fundamental and use it to explain the phenomenon of magnetic induction.

Motion under Lorentz force

The Newton 2^{nd} Law is written as

$$m\vec{a} = m\frac{d\vec{v}}{dt} = q\vec{E}(\vec{r},t) + q\vec{v}\times\vec{B}(\vec{r},t)$$

In general, the $\vec{E}$ & $\vec{B}$ field may vary with time & position

Here demonstrates the special case when $\vec{E}$ & $\vec{B}$ are constant, and

in particular $\vec{B}$ is in z-direction.

$$m\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ \cancel{B_x}^{0} & \cancel{B_y}^{0} & B_z \end{vmatrix}$$

for $\vec{B} = (0,0,B_z)$

$$\hookrightarrow \begin{cases} m\dfrac{dv_x}{dt} = qE_x + qv_yB_z \\ m\dfrac{dv_y}{dt} = qE_y - qv_xB_z \\ m\dfrac{dv_z}{dt} = qE_z \end{cases}$$

① Note that motion in z direction is independent to x/y

$$m\frac{dv_z}{dt} = qE_z \Rightarrow v_z(t) = \frac{qE_z}{m}t + v_0$$

$$\Rightarrow \boxed{z(t) = \frac{1}{2}\frac{qE_z}{m}t + \underbrace{v_0}_{}t + \underbrace{z_0}_{\text{some const.}}}$$

which is a constant acceleration motion

② The x/y motion are coupled. But this system is easy to solve by differentiating once more and substituting

$$\frac{d^2}{dt^2}v_x = \frac{qB_z}{m}\frac{dv_y}{dt} = \frac{qB_z}{m}\left(\frac{qE_y}{m} - \frac{qB_z}{m}v_x\right)$$

$$= -\frac{q^2B_z^2}{m^2}\left(v_x - \frac{E_y}{B_z}\right)$$

which is an SHM equation

$$\Rightarrow \quad v_x(t) = -\underline{C}\sin\left(\frac{qB_z}{m}t+\underline{\varphi}\right)+\frac{E_y}{B_z}$$

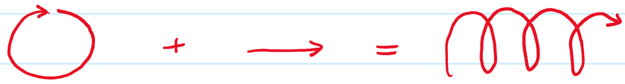
some constant

$$\Rightarrow \quad \boxed{x(t) = \underline{C'}\cos\left(\frac{qB_z}{m}t+\underline{\varphi}\right)+\frac{E_y}{B_z}t+\underline{x_0}}$$

And then
$$v_y(t) = \frac{m}{qB_z}\frac{dv_x}{dt}-\frac{E_x}{B_z}$$
$$= -C\cos\left(\frac{qB_z}{m}t+\varphi\right)-\frac{E_x}{B_z}$$

$$\Rightarrow \quad \boxed{y(t) = -\underline{C'}\sin\left(\frac{qB_z}{m}t+\underline{\varphi}\right)-\frac{E_x}{B_z}t+\underline{y_0}}$$

The result motion is a circular motion superposition to a drifting



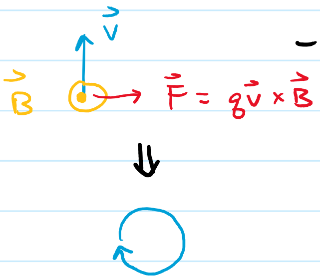
[1] The drifting direction is not intuitive

- With E_x only, drifting direction $= -y$
- With E_y only, drifting direction $= x$

(In general, the drifting direction is in $\vec{E}\times\vec{B}$ and velocity $= \frac{|\vec{E}\times\vec{B}|}{|\vec{B}|^2}$)

[2] The circular motion has the properties :

- Angular velocity is constant $= \frac{qB_z}{m}$



- Radius & initial velocity are depending on each other

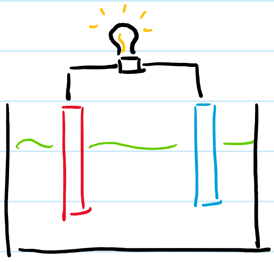
$$x(t) = R\cos\left(\frac{qB_z}{m}t\right) \Rightarrow v_x(t) = -\frac{qB_z}{m}R\sin\left(\frac{qB_z}{m}t\right)$$

(Can also arrive the same result by $\frac{mv^2}{r} = qvB$)

- Rotation is clockwise if B field is out of paper

Electromotive Force (EMF)

EMF is a term originally invented from chemisty to explain the observation in electrochemistry (Volta, 1801)



Metal electrodes in electrolyte

⇒ Current generate spontaneously

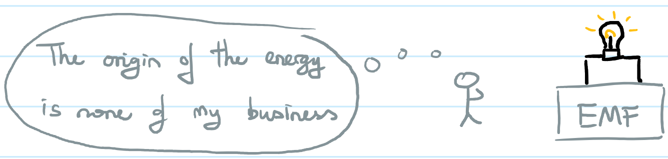
⇒ Some kind of "force" pushing the current ?

EMF took the unit of voltage, because people at the time tend to describe things like mechanical system, e.g. some source of force that drives the motion of charges. But voltage is the only thing they could measure out of the battery.

Magnetic Induced EMF

Later when magnetic induction was discovered by Faraday. He borrowed the same term to explain the observed current.

Today we are still keeping the term EMF, even though we know much better how the current is induced. Maybe because it is just simplier to use a general term for all kinds of voltage supply, rather than distinguishing them by the energy's origin.



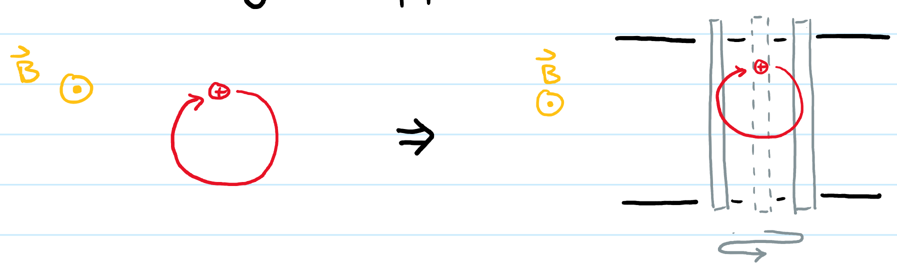
The methods to create magnetic induced EMF can be classified into 2 types :

- Motional EMF – EMF due to movement of charge "container"
  – Can be explained via Lorentz Force
- Transformer EMF – EMF due to magnetic field change
  – Explanation require relativity

Motional EMF

We already know that by Lorentz force, charges travels in a circle under constant $\vec{B}$ field.

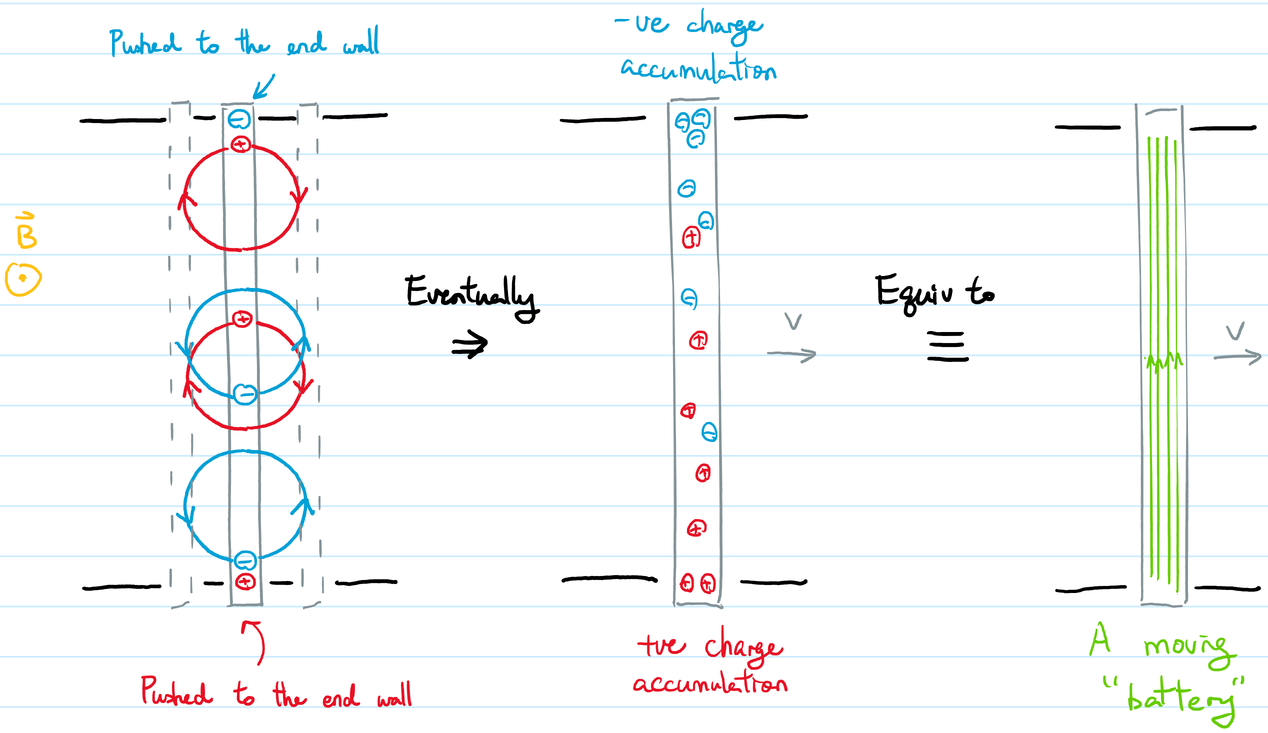
① We can further imagine if the charge is contained inside a massless long-thin pipe



The massless pipe will be dragged by the charge.

If the pipe is restricted to move horizontally, its motion must be SHM.

② Consider the pipe to be closed ends and contain both types of charge (so electrically neutral overall). When the pipe moves, charges at the 2 ends cannot move in a full circle :



The charges at the 2 ends will be pushed against the end walls by Lorentz force and cannot move in circle

Thus after some time, there will be charge accumulation at the 2 ends and potential energy will be stored, and the energy will not run out as long as it keeps moving

If it can run out, it is a capacitor

↪ The moving pipe acts like a battery!

↪ It can be used to drive circuit! i.e. EMF

③ The electric potential in the pipe is built up due to Lorentz force dragging the charge to the end walls.

Thus the total energy built up can be computed by the W.D. of the Lorentz force :

$$q \cdot d\mathcal{E} = \vec{F}_{Lorentz} \cdot d\vec{y}$$ ← $F \cdot ds$ = W.D. for infinitesimal distance

take y = vertical direction

$\mathcal{E}$ = Emf ~ Voltage

$$= q(\vec{v} \times \vec{B}) \cdot d\vec{y}$$

$$= q(d\vec{y} \times \vec{v}) \cdot \vec{B}$$

Vector Identity

$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

$$= -q(\vec{v} \times d\vec{y}) \cdot \vec{B}$$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(Rewrite into this form for later use)

Note 1: It is still true that W.D. by Lorentz Force = 0

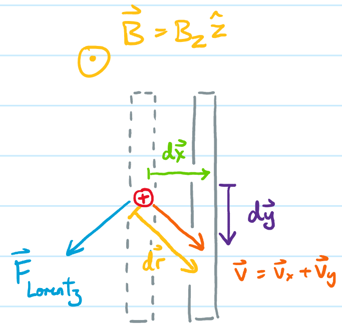
On above, we only considered work done by vertical component.

However its horizontal component does negative work,

making the net change in energy of the charge = 0

The total W.D. by horizontal component is

$$W.D. = \vec{F}_{lorentz} \cdot d\vec{r} = 0 \quad (\because \vec{F}_{lorentz} \perp d\vec{r})$$



$$= q[(\vec{v}_x + \vec{v}_y) \times \vec{B}] \cdot (d\vec{x} + d\vec{y})$$

$$= q\,\underbrace{(\vec{v}_x \times \vec{B})}_{\substack{\text{pointing downward} \\ (+y \text{ direction})}} \cdot d\vec{y} + q\,\underbrace{(\vec{v}_y \times \vec{B})}_{\substack{\text{pointing leftward} \\ (-x \text{ direction})}} \cdot d\vec{x}$$

W.D. that pushes the charge to move in the pipe's direction

W.D. that stops the charge from moving horizontally

Note 2: Because Lorentz force will slow down the pipe's motion, while the pipe can be used as a EMF (battery) only if the pipe keeps moving.

⇒ It requires an external driving force to forever keep the pipe acting as a battery. And so energy conserves.

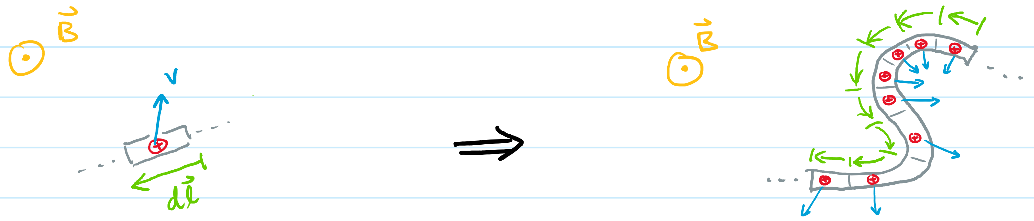
④ <u>Generalization</u>: Take the pipe's orientation as arbitrary. So that

$$q\,d\varepsilon = -q\left[\vec{v}\times \underline{d\vec{\ell}}\right]\cdot\vec{B}$$

not restricted to y only

$\vec{\ell} = \ell_x\hat{x} + \ell_y\hat{y} + \ell_z\hat{z}$

Then we can consider many of such pipe connecting together and possibly forming an arbitrary shape (and even a loop)



An infinitesimal small pipe can release energy per charge:

$$d\varepsilon = -(\vec{v}\times d\vec{\ell})\cdot\vec{B}$$

Many pipes connect together
Sum them up by line integral

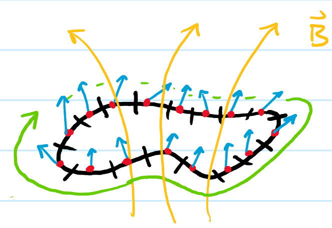
$$\int d\varepsilon = -\int (\vec{v}\times d\vec{\ell})\cdot\vec{B}$$

☆ Note that $\vec{v}$ for each segment can be different

⇒ The loop's shape may change after some time.

By then the EMF output may change time to time.

If the pipes <u>do not have resistance</u> and they form a loop, energy will accumulate forever as KE of the charges

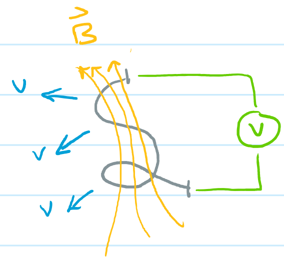


Short Summary

With a pipe of charge in an arbitrary shape, by keeping it moving inside magnetic field, it can be used like a battery.

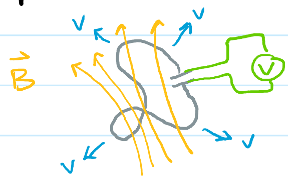
The EMF it can supply can be calculated as

$$\mathcal{E} = -\int_{\text{Along the pipe}} (\vec{v} \times d\vec{\ell}) \cdot \vec{B}$$



In the special case if the pipe forms a loop, it becomes

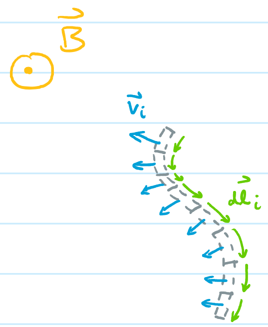
$$\mathcal{E} = -\oint_{\text{Along the loop}} (\vec{v} \times d\vec{\ell}) \cdot \vec{B}$$



(However in real life no one will refer this moving wire as a battery. We more often call this a generator ☺

Then the term "EMF source" includes both battery and generator.)

Geometric relation to flux



Consider a wire with changing shape in a static B field. Each segment $d\vec{\ell}_i$ moves according to its velocity $\vec{v}_i$

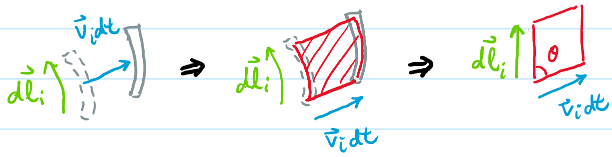
In a small time interval dt, the area swept by each segment can be approximated as

$$(\text{Swept area})_i \simeq \left(\begin{matrix}\text{Parallelogram by}\\ \vec{v}_i dt \ \& \ d\vec{l}_i\end{matrix}\right)$$

$$\simeq |\vec{v}_i dt|\,|d\vec{l}_i| \cdot \sin(\text{Angle between } \vec{v}_i dt \ \& \ d\vec{l}_i)$$

$$= (\vec{v}_i dt) \times (d\vec{l}_i)$$

$$\Rightarrow \frac{d}{dt}(\text{Swept Area})_i = \vec{v}_i \times d\vec{l}_i$$



Compare with the expression of W.D. by Lorentz force

$$\mathcal{E} = -\sum_{\text{all segment}} \left[(\vec{v}_i \times d\vec{l}_i)\cdot\vec{B}\right] = -\sum_{\text{all segment}} \left[\frac{d(\text{Swept area})_i}{dt}\cdot\vec{B}\right]$$

B is independent of t
⇒ can put into $\frac{d}{dt}$

$$= -\sum_{\text{all segment}} \left[\frac{d}{dt}\left[(\text{Swept Area})_i \cdot \vec{B}\right]\right]$$

$$= -\frac{d}{dt}\left[\sum_{\text{all segment}} \left[(\text{Swept Area})_i \cdot \vec{B}\right]\right]$$

$$\Rightarrow -\int (\vec{v} \times d\vec{l})\cdot\vec{B} = -\frac{d}{dt}\left[\iint_{\substack{\text{Swept}\\ \text{Regions}}} d\vec{S}\cdot\vec{B}\right]$$

$$= -\frac{d}{dt}\left(\begin{matrix}\text{Magnetic Flux through}\\ \text{the swept area}\end{matrix}\right)$$

☆☆ Note that since

- $\vec{B}$ is static. Not depending on t. (Even though it may vary by position)
- $d\vec{S}$ is just a notation saying that the integral is about summing many small areas. Not related to t at all.

The only thing that vary with t = Shape of the wire

= Integration range of the integral

So to be more accurate, we can write the equation as

$$\varepsilon = -\int_{\text{along wire}} (\vec{v} \times d\vec{\ell}) \cdot \vec{B} = -\frac{d}{dt} \iint_{\substack{\text{Area}(t) \\ \text{Swept by the wire}}} d\vec{S} \cdot \vec{B}$$

Emphasize the part depending on t is in the integration range

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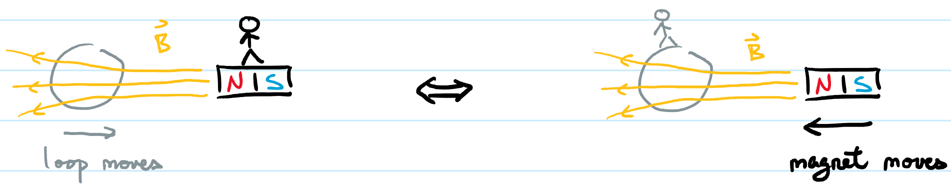
Transformer EMF

Motional EMF $\Rightarrow$ Generated from loop's motion

Transformer EMF $\Rightarrow$ Generated from B field change

The modern explanation is by Relativity, that they are simply observation to the same thing under different reference frames.



But historically, they were unified by Maxwell before Einstein's relativity

By vector calculus, the EMF-flux relation can be understand like product rule

$d(u \cdot v) = du \cdot v + u \cdot dv$

$$\varepsilon = -\frac{d}{dt}\left[\iint_{\text{Swept Area}(t)} d\vec{S} \cdot \vec{B}(t)\right]$$

$$\sim -\underbrace{\frac{d}{dt}\left[\iint_{\text{Swept Area}(t)} d\vec{S} \cdot \vec{B}(t_0)\right]}_{\frac{d}{dt}\text{ on integration range only}} - \iint_{\text{Swept Area}(t_0)} d\vec{S} \cdot \underbrace{\frac{d}{dt}\vec{B}(t)}_{\frac{d}{dt}\text{ on } \vec{B} \text{ only}}$$

Keep as constant

$$= \text{Motional EMF} + \text{Transformer EMF}$$

Maxwell - Faraday Equation

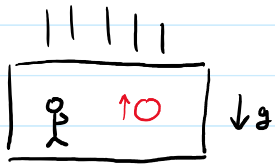
In a frame of reference which there are solely transformer EMF, the magnetic force $q\vec{v} \times \vec{B} = 0$ but we still observe the charges moving and building up potential.

How to fix the theory so that Newton's Law is not broken?

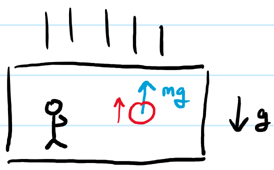
$\Rightarrow$ Add a E field and claim it to be induced by changing B field

2 Similar Resolutions

Observer in accelerating frame



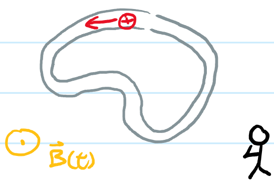
$\Rightarrow$



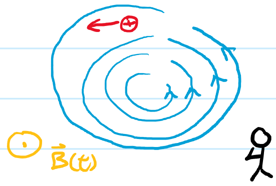
The ball flies up without force
Newton 2^{nd} Law is broken!

Add ficticious force
Newton 2^{nd} Law all good

Observer who only sees transformer EMF



$\Rightarrow$



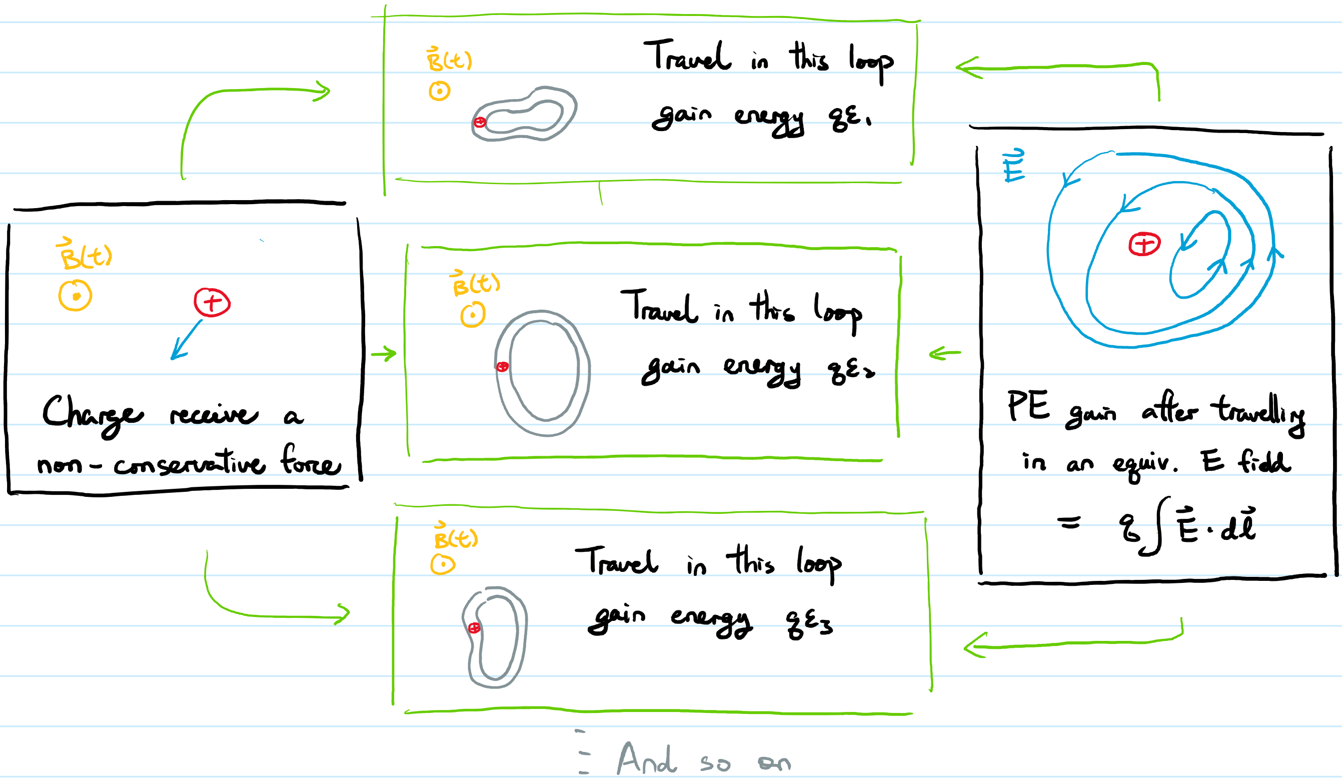
The charge moves without forces
Newton 2^{nd} Law is broken!

Add induced E field
Newton 2^{nd} Law all good

Then the question becomes finding the equivalent induced E field

- This E field must be non conservative, because after travelling along a loop, the charge will gain energy
- Energy by the induced field = W.D. by the field

$$= \int q\vec{E} \cdot d\vec{l}$$



Therefore, mathematically we want

$$\varepsilon_{\text{transformer}} = -\iint_{\text{Loop's Area}} d\vec{S}\cdot\frac{d\vec{B}(t)}{dt} = \oint_{\text{Loop}} \vec{E}_{\text{induced}}\cdot d\vec{l}$$

Moreover, we know that E field comes from charges are always conservative. i.e. $\oint \vec{E}_{\text{charge}}\cdot d\vec{l} = 0$

So we can simply take $\vec{E}_{\text{tot}} = \vec{E}_{\text{charge}} + \vec{E}_{\text{induced}}$ to have

$$\boxed{-\iint_{\text{Loop's Area}} \left(\frac{d\vec{B}}{dt}\right)\cdot d\vec{S} = \oint_{\text{Loop}} \vec{E}_{\text{tot}}\cdot d\vec{l}}$$

This is the integral form of Maxwell - Faraday Equation

We can also further write it into the differential form by Stoke's Theorem :

$$\oint_{\text{Loop}} \vec{E}\cdot d\vec{l} = \iint_{\substack{\text{Area in}\\ \text{Loop}}} (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\iint_{\substack{\text{Area in}\\ \text{Loop}}} \left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{s}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E}_{tot} = -\frac{\partial \vec{B}}{\partial t}}$$

which is a PDE relating changing B field and induced E field

(because $\vec{E}_{charge}$ is conservative and thus $\vec{\nabla}\times\vec{E}_{charge} = 0$)

Note : Maxwell - Faraday's Equation is only relevant to transformer EMF. In motional EMF there is no induced E field, but only magnetic force.

A more complete description to contain both EMF should be resorted back to Lorentz force

$$\text{W.D. on charge} = \int_{\text{path}} (q\vec{E}_{tot} + q\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$= \underbrace{\int_{\text{path}} q\vec{E}_{charge}\cdot d\vec{l}}_{\text{PE gain by travelling in potential due to charge}} + \underbrace{\int_{\text{path}} q\vec{E}_{induced}\cdot d\vec{l}}_{\text{Transformer EMF}} + \underbrace{\int_{\text{path}} q(\vec{v}\times\vec{B})\cdot d\vec{l}}_{\text{Motional EMF}}$$

(Finding $\vec{E}_{induced}$ requires solving $\nabla\times\vec{E} = \frac{\partial B}{\partial t}$)

$$\left(\begin{array}{c}\text{If the path}\\ \text{is a loop}\end{array}\right) \quad 0 + \iint_{\substack{\text{Loop's}\\ \text{area}}} \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{s} + \frac{\partial}{\partial t}\iint_{\substack{\text{Loop's}\\ \text{area}(t)}} \vec{B}\cdot d\vec{s}$$

(time independent) (time independent)

Lenz's Law

Faraday's Law, in particular the integral form, only talks about the magnitude of the emf, but not its direction.

$$\mathcal{E} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{S} \sim -\frac{d}{dt}\left[B(t)\cdot S(t)\right]$$

The minus is not useful for determining the direction of emf

We can use Lenz Law as a shortcut to determine the direction:

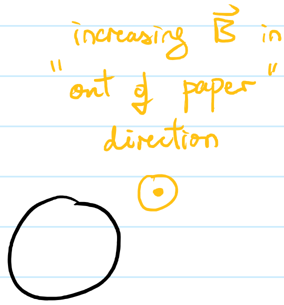
$\boxed{\text{Principle: Nature hates change of magnetic flux}}$

And you only need your right hand

E.g. 1

[0] B field increasing in magnitude

↳ B flux is "more out of paper"

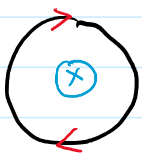


[1] Nature "hates" magnetic flux changing

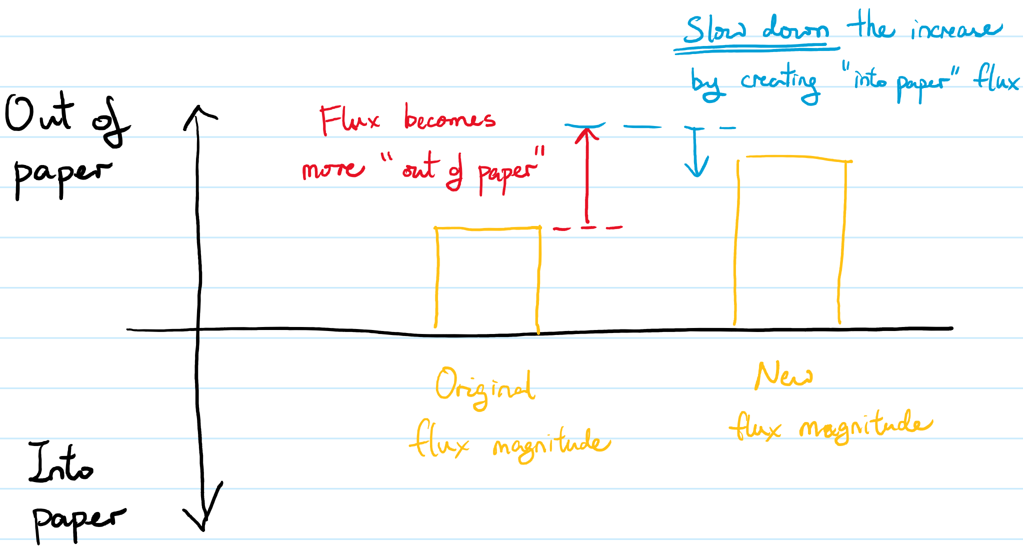
[2] To oppose the "more out of paper" flux

Nature needs to create an "into paper" B field to compensate the increase

[3] By your right hand , "into paper" B field can be generated if current flow clockwise



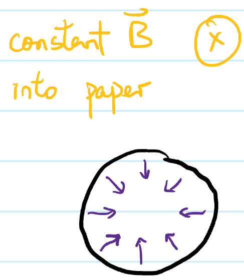
Clockwise current generate into paper B field, to counter the B flux change



E.g.2

$\boxed{0}$ Loop 's area decrease under const. B field

↪ flux is " less into paper"

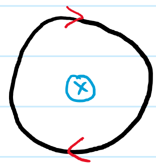


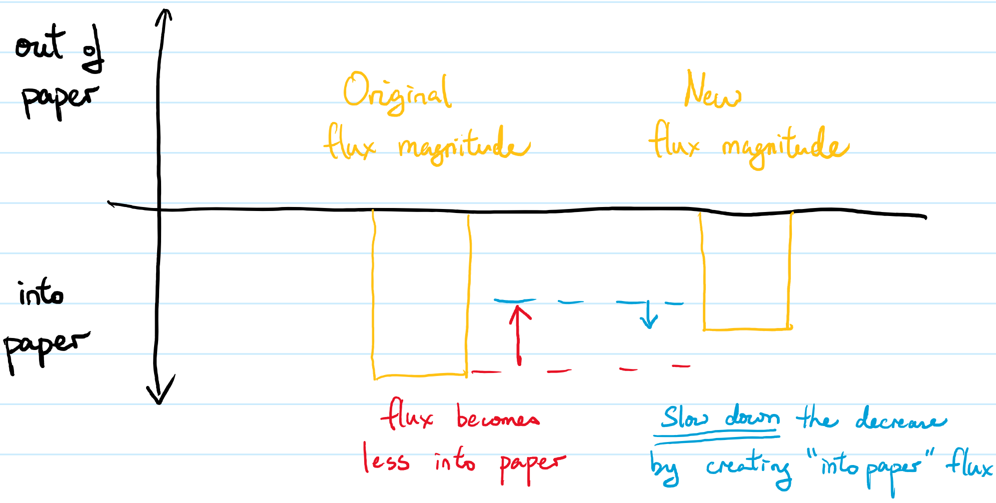
$\boxed{1}$ Nature "hates" magnetic flux changing

$\boxed{2}$ To counter the " less into paper " flux

Nature needs to create an " into paper" B field to compensate the decrease

$\boxed{3}$ By your right hand, " into paper" B field can be generated if current flow clockwise





Standard problems related to Faraday's Law

① Finding emf with either $\vec{B}$ or area change

⇒ Direct calculation of the surface integral

$$\varepsilon = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{S}$$

Use Lenz Law to find direction

$$\sim -\frac{d}{dt}\left[B(t)\cdot S(t)\right]$$

if the B field is uniform over the area, this just reduces to multiplying the surface area

② Finding the induced $\vec{E}$ field in the space, given how $\vec{B}$ changes

⇒ Equivalently asking to solve the PDE $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Can we avoid that? Yes, but only in some symmetric cases

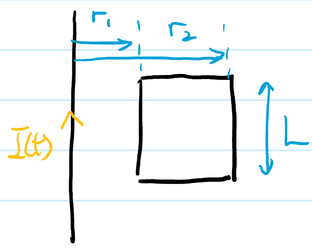
$\boxed{1}$ $\vec{E}$ is of the same magnitude on the chosen loop

$\boxed{2}$ $\vec{E}$ make the same angle with the line segment of the loop

Then we can avoid solving the PDE by solving the integral form

$$-\frac{d}{dt}\iint \vec{B}\cdot d\vec{S} = \oint \vec{E}\cdot d\vec{l} = |\vec{E}|\cdot\cos\theta\cdot(\text{loop's perimeter})$$

E.g. A loop next to a infinitely long wire, with time varying current $I(t)$



Step 1: Find $\vec{B}(t)$ from Ampere's Law

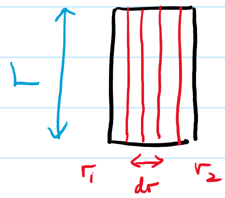
$$\vec{B}(r,t) = \frac{\mu_0}{2\pi}\frac{I(t)}{r} \quad \text{(into paper)}$$

Step 2: Find magnetic flux through loop

B depends on r only
so it is constant on each strip

$$\Phi_B = \iint \vec{B} \cdot d\vec{s}$$

$$= \int_{r_1}^{r_2} \frac{\mu_0 I(t)}{2\pi r} \cdot (L\,dr)$$



$$= \frac{\mu_0 I(t) L}{2\pi} \left[\ln(r_2) - \ln(r_1) \right]$$

Step 3 :

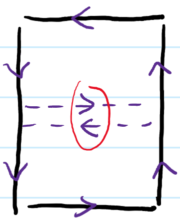
$\boxed{1}$ If only ask for emf → Just do differentiation.

$$\varepsilon = \frac{\mu_0 L}{2\pi r} \left(\frac{dI(t)}{dt} \right) \left[\ln(r_2) - \ln(r_1) \right]$$

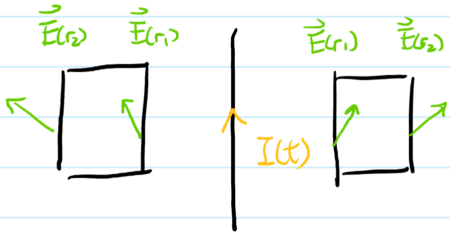
In this problem, only I depends on t

$\boxed{2}$ If further ask for E field → Need symmetry claims

Cancelled
in line integral



$\boxed{1}$ Top / bottom wire's $\vec{E}$ should be of same magnitude & direction
⇒ Contribution cancel in line integral



$\boxed{2}$ Left / right wire's $\vec{E}$ should depends on r only, due to cylindrical symmetry

So we have

$$\varepsilon = \frac{\mu_0 L}{2\pi r} \left(\frac{dI(t)}{dt} \right) \left[\ln(r_2) - \ln(r_1) \right]$$

$$\oint \vec{E} \cdot d\vec{l} = |E''(r_2)|\,L - |E''(r_1)|\,L + \left(\begin{array}{l} \text{contribution of} \\ \text{top/ bottom edge} \end{array} = 0 \right)$$

where E'' = component of $\vec{E}$ parallel to the left/right edges

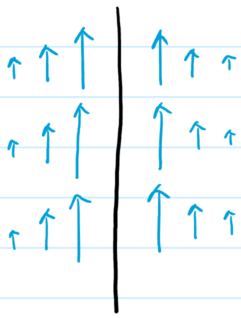
$\Rightarrow$ By comparing terms, we can claim

$$\left|E^{\parallel}(r,t)\right| = \frac{\mu_0}{2\pi}\left(\frac{dI(t)}{dt}\right)\cdot\frac{\ln(r)}{r}$$

The direction (up/down) can be determined by Lenz Law

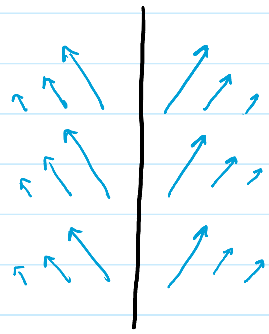
Follow up : Does $\vec{E}$ only contain components parallel to left/right edges?

With only $E^{\parallel}$



Magnitude of $\vec{E} \propto \frac{\ln r}{r}$

If $E^{\perp}$ exist



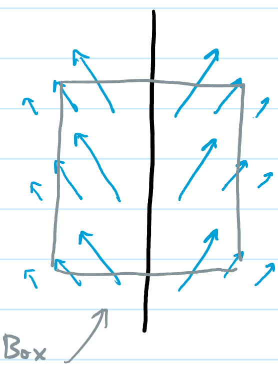
$\vec{E}$: component // wire $\propto \frac{\ln r}{r}$

Can there be component $\perp$ wire ?

ANS : For induced E field $\rightarrow$ Only has // component

For total E field $\rightarrow$ Can have $\perp$ component

☆ If the wire is not charge neutral, the E field can be diverging / converging. The $\perp$ component can be computed by Gauss Law.



Gaussian Box

Divergence > 0

$\Rightarrow$ Wire has +ve charge

Displacement Current

Ampere's Law is ambiguous in defining what means by "through a loop"

Intuitively, we may fill the loop with a surface

and claim : $\begin{cases} \text{Poke through the surface} = \checkmark \\ \text{Not Poke through the surface} = \times \end{cases}$

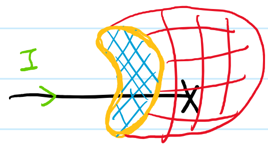
However, the choice of surface can be arbitrary

and create ambiguity at the endpoints of wire.

E.g. The wire poke through the blue surface but not the red surface.

Should we count this current as enclosed by the loop?

$$\oint_{loop} \vec{B} \cdot d\vec{\ell} = \mu_0 I \leftarrow ???$$



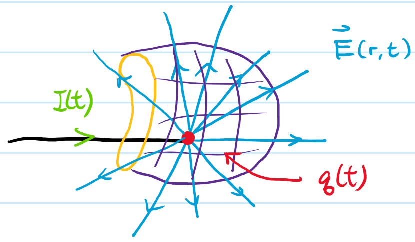
Simple analogy : Basketball through the ring

Choose this blue surface
⇒ Your throw counts

Choose this red surface
⇒ Your throw does not count

Solution :

- By charge conservation, a termination of current $I(t)$ will result in accumulating charges $q(t)$
- Charges emit E field $\vec{E}(r,t)$, and E field produce flux $\Phi_E(t)$ on the chosen surface



$\Rightarrow$ Add a term I_d that depends on $\Phi_E(t)$ in Ampere's Law

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 (I + I_d)$$

Such that

- When the wire poke through the chosen surface
  $I \neq 0 \, , \quad I_d = 0$
- When the wire does not poke through the chosen surface
  $I = 0 \, , \quad I_d \neq 0$

This term I_d is called displacement current, found to be

$$I_d = \varepsilon_0 \frac{\partial}{\partial t} \iint_S \vec{E}\cdot d\vec{s}$$

Integrate on the chosen surface

or in differential form

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_d) \, , \quad \text{with} \quad \vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The proof is in principle, charge conservation.

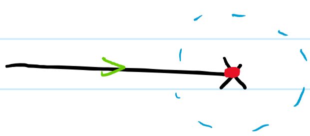
Charge conservation

The simplest expression of charge conservation is simply

$$I_{in} = \frac{d}{dt} Q_{enclosed}$$

current flow into the region

charge in the region



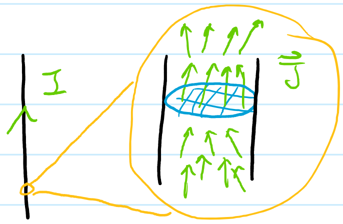
We can express this by the density terms.

(1) Charges $\rightarrow$ Charge density

$$Q = \iiint_V \rho(r)\, d\tau$$

(2) Current $\rightarrow$ Current density.

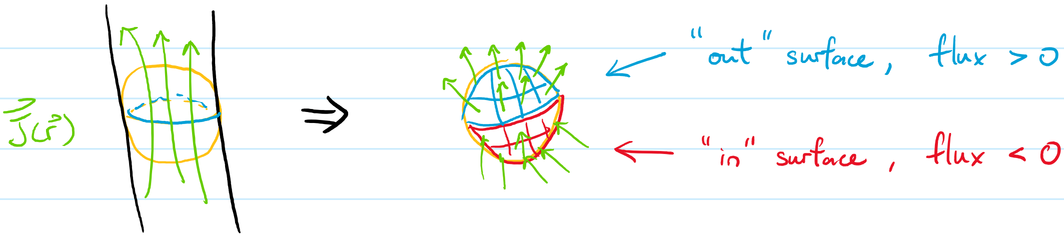
$$I = \iint_S \vec{J} \cdot d\vec{s}$$



Think of there is some current flowing through a region

We can divide the region's surface into 2 parts

by the sign of $\vec{J}$ flux.



By charge conservation, we require

[Current in] - [Current out] = [Rate of charge Accumulation]

$$\iint_{\text{in surface}} \vec{J} \cdot d\vec{s} - \iint_{\text{out surface}} \vec{J} \cdot d\vec{s} = \frac{\partial}{\partial t} \iiint_{\text{The volume}} \rho\, d\tau$$

combine

$$-\oiint_{\text{Surface of the volume}} \vec{J} \cdot d\vec{s} = \frac{\partial}{\partial t} \iiint_{\text{The volume}} \rho\, d\tau$$

By convention, outward flux = +ve
but having outward current flux = lost of charge
So need to have this additional minus sign

This is known as the continuity equation of charges

We can also derive its differential form by Divergent theorem.

$$-\oiint \vec{J}\cdot d\vec{s} = -\iiint \vec{\nabla}\cdot\vec{J}\, d\tau = \frac{\partial}{\partial t}\iiint \rho\, d\tau$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\rho}$$

which is a PDE that one may use to find $\rho(\vec{r},t)$ / $\vec{J}(\vec{r},t)$ when one of them is given and you need to find another

Deriving Displacement Current

Subst. the above by Gauss Law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

$$\begin{aligned} 0 &= \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\rho \\ &= \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\varepsilon_0 \vec{\nabla}\cdot\vec{E}) \\ &= \vec{\nabla}\cdot\left(\vec{J} + \varepsilon_0 \frac{\partial}{\partial t}\vec{E}\right) \end{aligned}$$

Recall that if a vector field has zero divergence,

it can be expressed as the curl of another vector field

(Just like B field, $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\times\vec{A} = \vec{B}$)

This suggests us to modify the original Ampere's Law into

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad\Rightarrow\quad \vec{\nabla}\times\vec{B} = \mu_0\Big[\vec{J} + \underbrace{\varepsilon_0\frac{\partial}{\partial t}\vec{E}}_{J_d}\Big]$$

or in integral form

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I \quad\Rightarrow\quad \oint \vec{B}\cdot d\vec{l} = \mu_0\Big[I + \underbrace{\varepsilon_0\frac{\partial}{\partial t}\iint \vec{E}\cdot d\vec{s}}_{I_d}\Big]$$

Complication of Electro-magnetic induction

From Faraday's Law & Ampere's Law, we can see that

Time varying $\vec{E}$ / $\vec{B}$ will induce each other

$$\begin{cases} \text{Faraday} : & \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \text{Time changing } \vec{B} \text{ create } \vec{E} \\ \text{Ampere} : & \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} & \text{Time changing } \vec{E} \text{ create } \vec{B} \end{cases}$$

So if we try to treat them as individual equation, we can end up in such loop:

Some initial $\vec{B}(t)$ $\longrightarrow$ Solve for an induced $\vec{E}(t)$ (Faraday)

$\swarrow$

Solve for an induced $\Delta\vec{B}(t)$ (Ampere) $\longrightarrow$ Solve for its induced $\Delta\vec{E}(t)$ (Faraday)

$\swarrow$

Solve for its induced $\Delta(\Delta\vec{B}(t))$ (Ampere) $\longrightarrow$ ...

What can we do about this?

Soln. 1: Take approximation $\Delta B(t) \simeq 0$

i.e. Assume the induced $\vec{E}(t)$ does not induce additional B field

This approximation is OK because $\mu_0 \varepsilon_0 = \frac{1}{c^2} = \frac{1}{(3\times10^8)^2}$

The size of displacement current $\mu_0 \varepsilon_0 \frac{\partial \vec{E}(t)}{\partial t}$ is usually too small compare with the original current, and thus $\Delta B(t) \sim 0$

Not ideal. But good enough in many cases.

In fact most textbook problems are taking this approximation.

Soln 2 : Solve both of the equations together

i.e. Solving the system of Maxwell's Equation

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0} & \text{(Gauss's Law)} \\ \vec{\nabla}\cdot\vec{B} = 0 & \text{(Gauss's Law of B)} \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \text{(Faraday's Law)} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} & \text{(Ampere's Law)} \end{cases}$$

A system of PDE ! Terrible !

But there are situations that you must do so . E.g.

- Plasma physics
- Neutron star / black hole