

- Solve a circuit
 - { - Write Kirchoff's Law
 - Solve by ODE / phasor / impedance
 - Advanced circuit techniques
-

Circuit Analysis

Step 1: Simplify circuit, eg. Series / Parallel combination

Step 2: Write Kirchoff's Law

Step 3: Solve by ODE / phasor / impedance method

what is taught to
physics students
(very slow)

what is taught to
people who does not
know complex no.
(geometrical method)

most commonly used
in practice
(fastest)

Kirchoff's Law

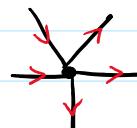
① Kirchoff Current Law (KCL)

② Kirchoff Voltage Law (KVL)

- KCL \equiv conservation of charge at a junction

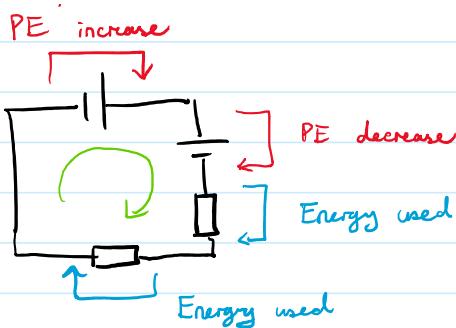
$$\sum I_{\text{in}} = \sum I_{\text{out}}$$

rate of charge
flow in rate of charge
 flow out



- KVL \equiv conservation of energy over a loop

$$\sum \Delta V = 0 \text{ for any loop}$$



All PE gained should be used up after the charge travel back to its starting point. Or else energy accumulate

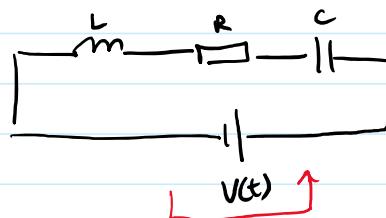
The 3 methods to solve circuit

(1) ODE

By definition, $I = \frac{dQ}{dt} = \text{rate of charge flow through}$

$$\Rightarrow \left\{ \begin{array}{l} C = \frac{Q}{V} \Rightarrow V_C = \frac{Q}{C} \\ R = \frac{V}{I} \Rightarrow V_R = IR = R \frac{dQ}{dt} \\ L = \frac{V}{\frac{dI}{dt}} \Rightarrow V_L = L \frac{dI}{dt} = L \frac{d^2Q}{dt^2} \end{array} \right.$$

E.g. The easiest circuit = LRC in series



By KVL:

$$V(t) - \frac{Q}{C} - R \frac{dQ}{dt} - L \frac{d^2Q}{dt^2} = 0$$

This is a 2nd order linear inhomogeneous ODE

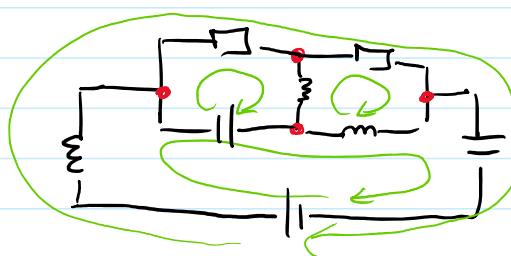
$$Q(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t} + (\text{particular soln})$$

$$\text{when } \lambda_{\pm} = -\frac{R}{2L} \pm \frac{1}{2} \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

But if the circuit gets more complicated

You get a lot more ODEs

E.g.



4 junctions = Can write at most 4 KCL equations

4 loops = Can write at most 4 KVL equations

\Rightarrow A system of ODE with 8 equalities.

(Although they are not all independent of each other)

* When the circuit gets more complicated, solving ODE system

is too annoying. So no one does it in practice.

② Phasor = Geometrical method to avoid complex numbers.

* Only applicable for sinusoidal voltage input of pure frequency.

i.e. $V(t) = V_0 e^{i\omega t} = V_1 \cos(\omega t) + V_2 \sin(\omega t)$

ω must be taken as some constant

Given the voltage term being sinusoidal, the ODE can be simplified

$$V_0 e^{i\omega t} = L \frac{d^2}{dt^2} \phi + R \frac{d}{dt} \phi + \frac{\phi}{C}$$

① let the soln of $\varphi(t)$ be in the form;

$$\varphi(t) = \varphi_0 e^{i\omega t} \quad (\text{Note that } \varphi_0 \text{ can be a complex number})$$

② By substitution, all $e^{i\omega t}$ are cancelled

$$V_0 e^{i\omega t} = L \cdot \varphi_0 (-\omega^2) e^{i\omega t} + R \cdot \varphi_0 (i\omega) e^{i\omega t} + \frac{\varphi_0}{C} e^{i\omega t}$$

③ What remains is a pure algebraic equation of φ_0 .

$$\frac{V_0}{\varphi_0} = -\omega^2 L + i\omega R + \frac{1}{C}$$

~~Method of phasor~~ Method of phasor = Draw this relation on the complex plane

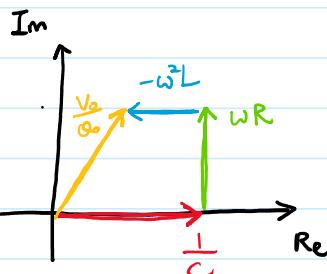
① Draw an arrow for each term



$\frac{1}{C}$ = some +ve real no.

$i\omega R$ = some +ve complex no.

$-\omega^2 L$ = some -ve real no.



② By the equality, the vector sum

equals $\frac{V_0}{\varphi_0}$

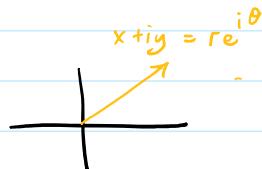
From the diagram, we can measure

$\left\{ \begin{array}{l} \text{Magnitude of } \frac{V_0}{\varphi_0} \\ \text{Angle of } \frac{V_0}{\varphi_0} \text{ (i.e. phase)} \end{array} \right.$

But of course we can also solve by algebra.

$$\frac{V_0}{\varphi_0} = \left(\frac{1}{C} - \omega^2 L \right) + i\omega R$$

$$= \sqrt{\left(\frac{1}{C} - \omega^2 L \right)^2 + (\omega R)^2} e^{i \tan^{-1} \left(\frac{\omega R}{\frac{1}{C} - \omega^2 L} \right)}$$



$$\therefore Q(t) = Q_0 e^{i\omega t} = \frac{V_0}{\sqrt{\left(\frac{1}{C} - \omega^2 L\right)^2 + (\omega R)^2}} e^{i[\omega t - \tan^{-1}\left(\frac{\omega R}{\frac{1}{C} - \omega^2 L}\right)]}$$

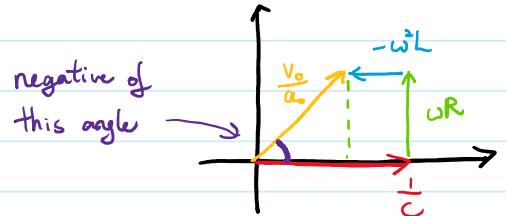
Terminology of lead / lag in phase angle

Compare with $V(t) = V_0 e^{i\omega t}$

Q_0 has an extra phase angle $-\tan^{-1}\left(\frac{\omega R}{\frac{1}{C} - \omega^2 L}\right)$

So we say $Q(t)$ is lagging behind $V(t)$

by a phase of $\tan^{-1}\left(\frac{\omega R}{\frac{1}{C} - \omega^2 L}\right)$



Can also find the phase difference for voltage across each component

- Voltage through C

$$V_C = \frac{Q(t)}{C} = \frac{V_0}{C} \cdot \frac{1}{\sqrt{\dots}} e^{i[\omega t - \tan^{-1}(\dots)]}$$

- Voltage through R

$$i = e^{i\frac{\pi}{2}}$$

$$V_R = R \frac{dQ(t)}{dt} = \frac{V_0 R}{\sqrt{\dots}} (i\omega) e^{i[\omega t - \tan^{-1}(\dots)]}$$

$$= \frac{V_0 R \omega}{\sqrt{\dots}} e^{i[\omega t - \tan^{-1}(\dots) + \frac{\pi}{2}]}$$

Lead V_C by $\frac{\pi}{2}$

- Voltage through L

$$V_L = L \frac{d^2Q}{dt^2} = \frac{V_0 L}{\sqrt{\dots}} (-\omega^2) e^{i[\omega t - \tan^{-1}(\dots)]}$$

$$= \frac{V_0 L \omega^2}{\sqrt{\dots}} e^{i[\omega t - \tan^{-1}(\dots) + \pi]}$$

Lead V_C by π

③ Impedance Method

* Only applicable for sinusoidal voltage input of pure frequency.

i.e. $V(t) = V_0 e^{i\omega t} = V_1 \cos(\omega t) + V_2 \sin(\omega t)$
 ω must be taken as some constant

Actually the algebraic version of phasor method

Starting from the previous ODE

$$V_0 e^{i\omega t} = L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} \frac{dQ}{dt}$$

① By letting $Q(t) = Q_0 e^{i\omega t}$, current is

$$I(t) = \frac{dQ}{dt} = i\omega Q_0 e^{i\omega t}$$

② Rewrite the ODE in terms of $I(t)$

$$\begin{aligned} V_0 e^{i\omega t} &= -\tilde{\omega} L Q_0 e^{i\omega t} + i\omega R Q_0 e^{i\omega t} + \frac{Q_0}{C} e^{i\omega t} \\ &= i\omega L I(t) + R I(t) + \frac{1}{i\omega C} I(t) \\ V(t) &= \left[i\omega L + R + \frac{1}{i\omega C} \right] I(t) \end{aligned}$$

(why express by $I(t)$ but not $Q(t)$? Because current can be measured directly)

③ This becomes something looks like Ohm's Law

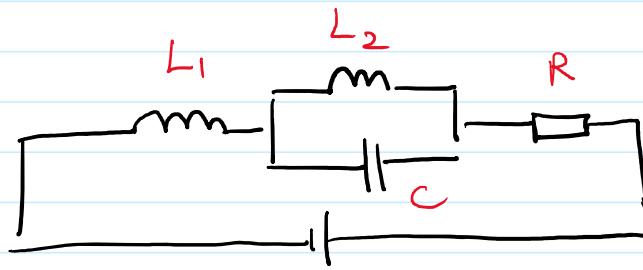
$$V(t) = [Z_L + Z_R + Z_C] I(t)$$

By defining the impedance of each components:

$$Z_L = i\omega L \quad Z_R = R \quad Z_C = \frac{1}{i\omega C}$$

And the series / parallel addition are the same as resistor.

E.g.



$$V(t) = V \cos(\omega t)$$

$$\frac{1}{Z_L} + \frac{1}{Z_C}$$

① $\frac{1}{Z_1} = \frac{1}{L_2} + i\omega C$ with $\frac{1}{Z_1} = \frac{1}{i\omega L_2} + i\omega C$

② $\frac{1}{Z_2} = \frac{1}{L_1} + Z_1 + R$ with $Z_2 = i\omega L_1 + Z_1 + R$

After that we can find the current by $V(t) = Z_2 \cdot I(t)$

Question: What if $V(t)$ is not sinusoidal? (not sine/cosine)

② Break $V(t)$ into $\sum V(\omega) e^{i\omega t}$ by Fourier Transform.

In fact an integral, as frequency
can be a continuous spectrum

③ Each $V(\omega)$ will result in a different current response

i.e. Current will be a function of ω ,

$$V(\omega) = Z(\omega) \overset{\curvearrowleft}{I(\omega)} \quad Z_C, Z_L \text{ are always function of } \omega$$

④ The overall current is the superposition of all $I(\omega)$

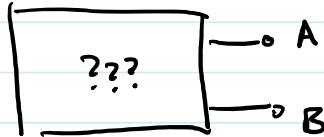
$$I(t) = \sum I(\omega) e^{i\omega t}$$

(This is actually the topic of frequency analysis in circuit)
usually appear in Yr 3/4 of electrical engineering

Linear circuit blackbox

Situation: Suppose we have a blackbox of circuit

The only things we have access are its 2 terminals



And it only contains linear components.

(E.g. L, R, C, voltage/current source)

What properties can we analyse from it?

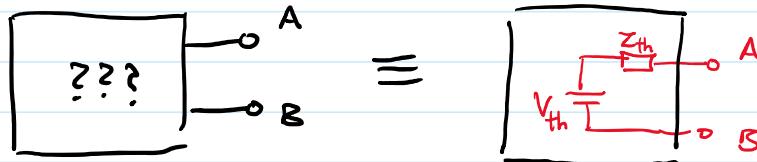
In circuit theory, there are 2 theorems we may apply:

① Thevenin's theorem

Any blackbox circuit made of only linear components

can be simplified into some voltage source V_{th} in

series connection with some impedance Z_{th} .

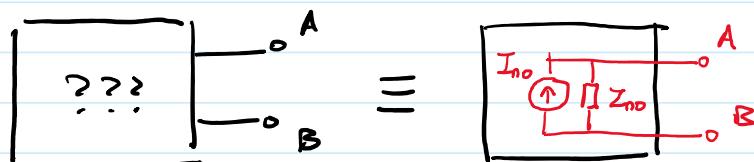


② Norton's theorem

Any blackbox circuit made of only linear components

can be simplified into some current source I_{no} in

parallel connection with some impedance Z_{no}



That means without further information, we can only classify these blackbox by 2 parameters. (Either V_{th}/Z_{th} or I_{no}/Z_{no})

- Want to understand what is in the box \Rightarrow Impossible
- Just use the box as a component in circuit \Rightarrow Simple

Measure V_{th}/Z_{th} or I_{no}/Z_{no}

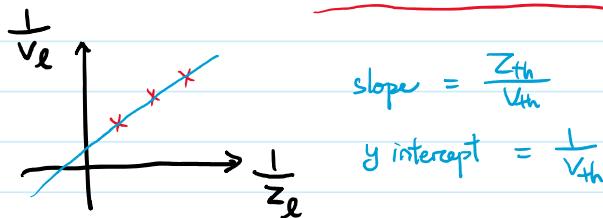
Because the equivalent Thvenin / Norton circuits are so simple, we simply need a voltmeter (oscilloscope) and a few different impedance component - E.g.



From above we have $V_L = \frac{V_{th}}{Z_L + Z_{th}} \cdot Z_L$

or

$$\frac{1}{V_L} = \frac{Z_{th}}{V_{th}} \cdot \frac{1}{Z_L} + \frac{1}{V_{th}}$$



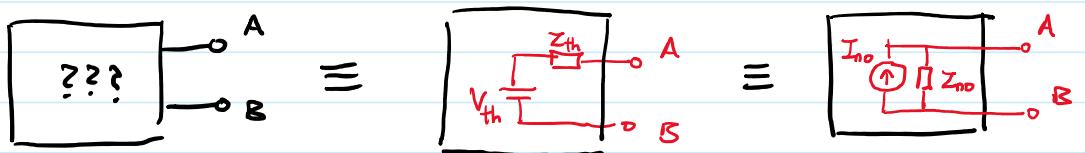
Repeat many times with different Z_L , one should be able to find V_{th} and Z_{th} numerically.

Advanced circuit simplifying techniques

2 very useful tricks so that you do not have to write out Kirchoff's Law for every node / loop.

① Thevenin - Norton Equivalence

From the equivalent circuit in Thevenin / Norton's Theorem

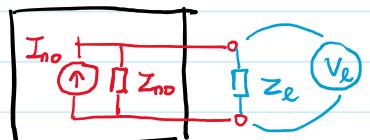
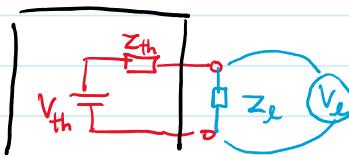


where $Z_{th} = Z_{no}$ and $V_{th} = I_{no} Z_{no} = I_{no} Z_{th}$

Proof :

Let the box being connected to an impedance Z_L

and the voltage across is measured to be V_L



$$\frac{V_L}{Z_L} = \frac{V_{th}}{Z_{th} + Z_L}$$

$$\Rightarrow V_L = \frac{V_{th} Z_L}{Z_{th} + Z_L}$$

$$\frac{V_L}{Z_L} + \frac{V_L}{Z_{no}} = I_{no}$$

$$\Rightarrow V_L = \frac{I_{no} Z_{no} Z_L}{Z_{no} + Z_L}$$

The relationships between V_L and Z_L should be the same

since they are equivalent circuit. Also they should be

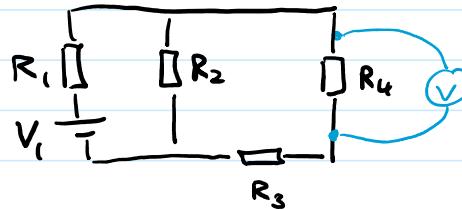
independent to the choice of Z_L . So we must have

$$\frac{V_{th}}{Z_{th} + Z_e} = \frac{I_{no} Z_{no}}{Z_{no} + Z_e}$$

$$\Rightarrow Z_e(V_{th} - I_{no} Z_{no}) + (V_{th} Z_{no} - I_{no} Z_{no} Z_{th}) = 0$$

To be independent of Z_e , we must have the coefficients = 0

E.g. 1

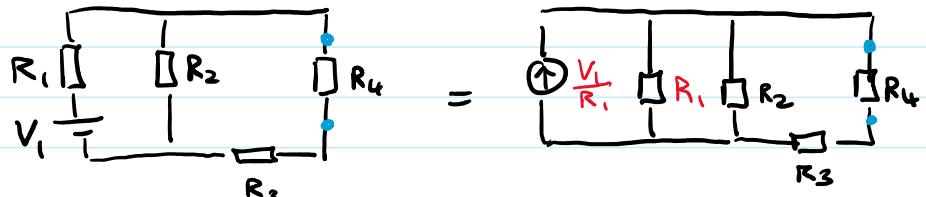


Final voltage across R_4

By Thvenin - Norton Equivalence, these patterns are replaceable

$$V_{th} \frac{R}{Z} = I_{no} \left(\frac{1}{R} \right) R \quad \text{with } V_{th} = I_{no} R$$

So



$$= \frac{V_i}{R_1} \left(\frac{R_1 R_2}{R_1 + R_2} \right) R_4$$

$$= \frac{\frac{V_i}{R_1} \cdot \frac{R_1 R_2}{R_1 + R_2}}{R_3 + R_4} R_4$$

At this step we can already tell the voltage across R_4 is

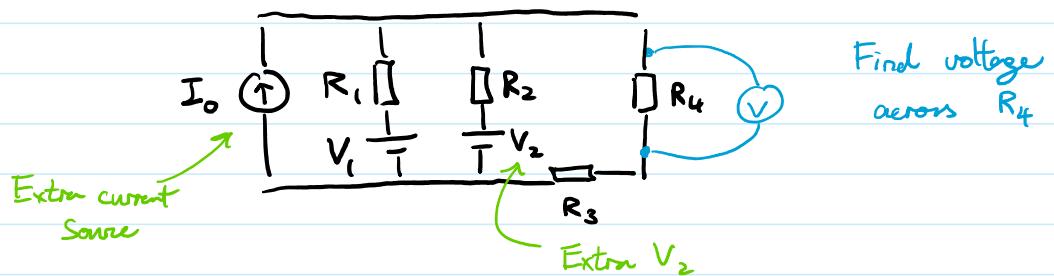
$$V_i \cdot \frac{R_2}{R_1 + R_2} \cdot \frac{R_4}{\frac{R_1 R_2}{R_1 + R_2} + R_3 + R_4}$$

② Superposition Theorem

In a linear circuit with multiple independent voltage / current source, the voltage / current across any components equal to the sum of the contribution of each source after removing the others by

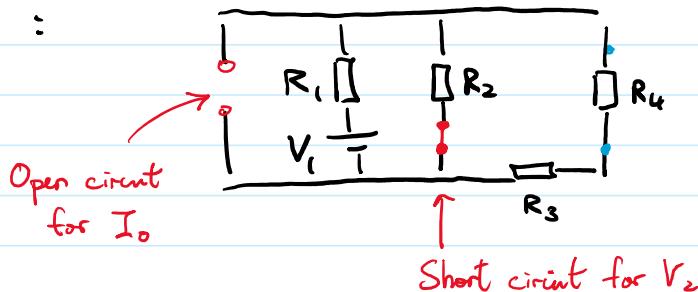
- If it is a voltage source, replace with short circuit
- If it is a current source, replace with open circuit

E.g. 2



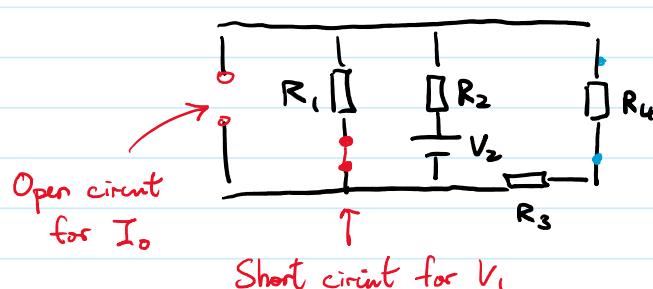
Calculate contribution from each source, after removing others :

By V_1 :



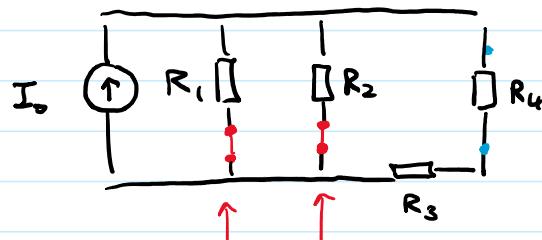
$$\Rightarrow \text{Voltage across } R_4 = V_1 \cdot \frac{R_2}{R_1 + R_2} \cdot \frac{R_4}{\frac{R_1 R_2}{R_1 + R_2} + R_3 + R_4}$$

By V_2 :



$$\Rightarrow \text{Voltage across } R_4 = V_2 \cdot \frac{R_1}{R_1+R_2} \cdot \frac{R_4}{\frac{R_1R_2}{R_1+R_2} + R_3 + R_4}$$

By I_o :



Short circuit for V_1 and V_2

$$\Rightarrow \text{Voltage across } R_4 = I_o \cdot \frac{R_1R_2}{R_1+R_2} \cdot \frac{R_4}{\frac{R_1R_2}{R_1+R_2} + R_3 + R_4}$$

Finally, total voltage across R_4 sum of all 3 cases

$$= (V_1R_2 + V_2R_1 + I_oR_1R_2) \cdot \frac{1}{R_1+R_2} \cdot \frac{R_4}{\frac{R_1R_2}{R_1+R_2} + R_3 + R_4}$$