

Single Variable calculus review

Limit - Naive Definition + How to calculate

Differentiation - Geometrical meaning + How to calculate

Integration - Geometrical meaning + How to calculate

Use of calculus in basic mechanics

Limit :

$$(\text{Naive}) \text{ Definition} : \lim_{x \rightarrow a} f(x) = L \quad (\neq \pm\infty)$$

means $f(x)$ is getting closer to L when x is getting closer to a

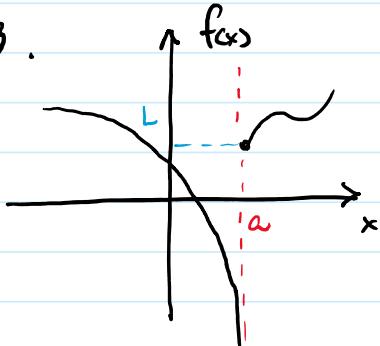
Two sides limit

$$\text{Right hand limit} : \lim_{x \rightarrow a^+} f(x) = L \quad (\neq \pm\infty)$$

$$\text{Left hand limit} : \lim_{x \rightarrow a^-} f(x) = L \quad (\neq \pm\infty)$$

Existence of limit : A limit "exist" if both sides limit exist & equal

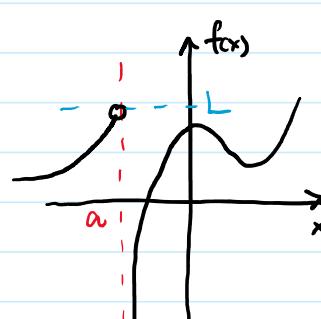
E.g.



RH limit exists

LH limit not exist

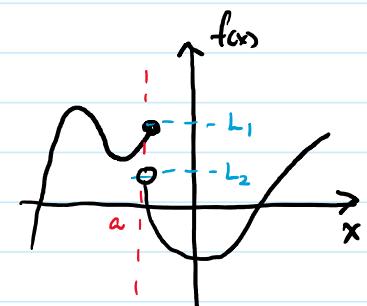
\therefore limit not exist



RH limit not exist

LH limit exist

\therefore limit not exist



Both sides limit exist

but they are not equal

\therefore limit not exist

Evaluation of limit (Assume both $f(x)$ & $g(x)$'s limits exist)

Addition / Subtraction $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$

Product $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = (\lim_{x \rightarrow a} f(x)) \cdot (\lim_{x \rightarrow a} g(x))$

Quotient $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ (only if $\lim_{x \rightarrow a} g(x) \neq 0$)

Trigonometric Relation

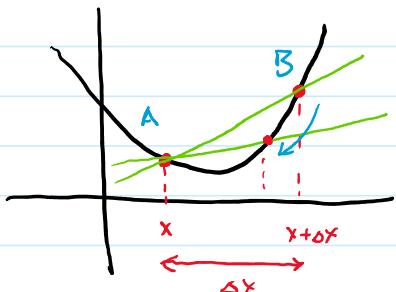
$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \cos x = 1$$

Differentiation

Def : $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{d}{dx} f(x) = f'(x)$ (If this limit exists
f(x) is "differentiable" at the given x)

Geometrical Meaning : Slope of tangent line of function



Slope of line AB = $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

when $\Delta x \rightarrow 0$, AB become tangent

Evaluation

Addition / Subtraction $\frac{d}{dx} [f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

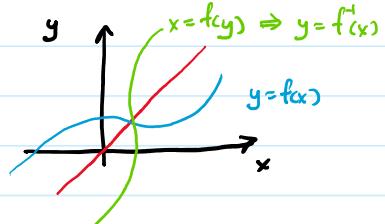
Product Rule $\frac{d}{dx} [f(x) \cdot g(x)] = \left[\frac{d}{dx} f(x) \right] \cdot g(x) + f(x) \cdot \left[\frac{d}{dx} g(x) \right]$

$$\text{Quotient Rule } \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{1}{[g(x)]^2} \left[g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x) \right]$$

(This is just product rule of $\frac{d}{dx} [f(x) \cdot \frac{1}{g(x)}]$)

$$\text{Chain rule } \frac{d}{dx} [g(f(x))] = \frac{d}{d[f(x)]} g(f(x)) \cdot \frac{d}{dx} f(x)$$

$$\text{Inverse function } \frac{df^{-1}(x)}{dx} = \frac{1}{\frac{dy}{dx}|_{y=f(x)}}$$



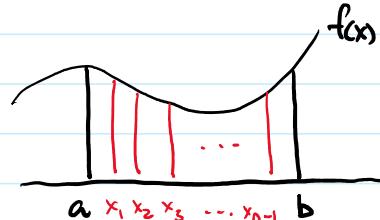
$$\text{Polynomial } \frac{d}{dx} x^n = nx^{n-1}$$

$$\text{Trigonometric } \frac{d}{dx} \begin{cases} \sin x \\ \cos x \\ \tan x \end{cases} = \begin{cases} \cos x \\ -\sin x \\ \sec^2 x \end{cases}$$

$$\text{Exponential } \frac{d}{dx} \begin{cases} e^x \\ \ln x \end{cases} = \begin{cases} e^x \\ \frac{1}{x} \end{cases}$$

Integration

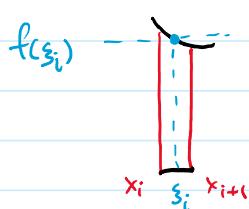
Def : Definite Integral



① Choose $n-1$ points between $[a, b]$

$$a < x_1 < x_2 < \dots < x_{n-1} < b = \begin{matrix} \text{separate into} \\ \text{total } n \text{ strips} \end{matrix}$$

② In each strip, pick an arbitrary point $x = \xi_i$



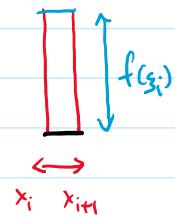
Approximate the strip's height
 $\approx \simeq f(\xi_i)$

③ Approximate area of each strip $\approx f(\xi_i) \cdot [x_{i+1} - x_i]$

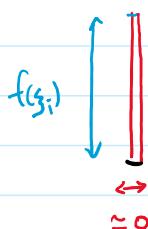
\Rightarrow Total area under curve = Summing all strips

$$= \sum_{i=0}^{n-1} f(\xi_i) \cdot [x_{i+1} - x_i]$$

with $x_0 = a$, $x_n = b$



④ And then take each of $[x_{i+1} - x_i]$ limit to 0



No matter how "arbitrary" we choose ξ_i in $[x_i, x_{i+1}]$

the height of the strip is basically equal to $f(\xi_i)$

\hookrightarrow Area under curve \approx Summing the area of infinitely many strips that have ≈ 0 width

$$\text{Def : } \int_a^b f(x) dx = \lim_{\Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$$

Def : Indefinite integral

Given a function $f(x)$. If there exists another function

$F(x)$ such that $\frac{d}{dx} F(x) = f(x)$

Then $F(x)$ is called antiderivative / primitive of $f(x)$

Indefinite integral = Set of all antiderivatives of $f(x)$

= Set of all $F(x)$ s.t. $\frac{d}{dx} F(x) = f(x)$

$$\text{E.g. } f(x) = x^2 \Rightarrow \int f(x) dx = \frac{x^3}{3} + C$$

C
arbitrary constant

\Rightarrow Set of antiderivative of x^2

$$= \left\{ \frac{x^3}{3} + 1, \frac{x^3}{3} - 2, \frac{x^3}{3} + \pi, \dots \right\}$$

Fundamental Theorem of Calculus

\Rightarrow Differentiation & Integration are opposite operations

\therefore Definite integration of $f(x)$ over $[a, b]$

$$= \int_a^b f(x) dx = F(b) - F(a)$$

Then differentiate both sides

Treat b be a free variable, b can be any real no.

$$\begin{aligned} \frac{d}{db} \int_a^b f(x) dx &= \frac{d}{db} F(b) - 0 \\ a &= \text{some constant} \quad \uparrow \\ &= \frac{d}{dx} F(x) \quad \text{b is just a symbol representing a free variable. You can replace it with other letters} \\ &= f(x) \end{aligned}$$

Rule of Integration

Addition / Subtraction $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

Integration by part (correspondant of product rule)

$$\int f(x) \cdot \frac{d}{dx} g(x) \cdot dx = f(x)g(x) - \int \frac{d}{dx} f(x) \cdot g(x) \cdot dx$$

$$\int f dg = f \cdot g - \int g \cdot df$$

$$\frac{d}{dx} \left(\int f dg + \int g df \right) = \frac{d}{dx} (f \cdot g)$$

Integration by change of variable (correspondant of chain rule)

$$\begin{aligned} \int g[f(x)] \cdot \frac{d}{dx} f(x) dx &= \int g[f(x)] \cdot d[f(x)] \\ &= \int g[u] du \Big|_{u=f(x)} \end{aligned}$$

Polynomial $\int x^n dx = \frac{x^{n+1}}{n+1}$

Trigonometric $\int \left\{ \begin{array}{l} \sin x \\ \cos x \\ \sec^2 x \end{array} \right\} dx = \left\{ \begin{array}{l} -\cos x \\ \sin x \\ \tan x \end{array} \right\}$

Exponential $\int \left\{ \begin{array}{l} e^x \\ \frac{1}{x} \end{array} \right\} dx = \left\{ \begin{array}{l} e^x \\ \ln|x| \end{array} \right\}$

Substitution of trigonometric function

$$\int \sqrt{a^2 - x^2} dx \rightarrow \text{Subst. } x = a \sin \theta \rightarrow \begin{aligned} \sqrt{a^2 - x^2} &= a \cos \theta \\ dx &= a \cos \theta d\theta \end{aligned}$$

$$\int \sqrt{a^2 + x^2} dx \rightarrow \text{Subst. } x = a \tan \theta \rightarrow \begin{aligned} \sqrt{a^2 + x^2} &= a \sec \theta \\ dx &= a \sec^2 \theta d\theta \end{aligned}$$

$$\int \sqrt{x^2 - a^2} dx \rightarrow \text{Subst. } x = a \sec \theta \rightarrow \begin{aligned} \sqrt{x^2 - a^2} &= a \tan \theta \\ dx &= a \sec \theta \tan \theta d\theta \end{aligned}$$

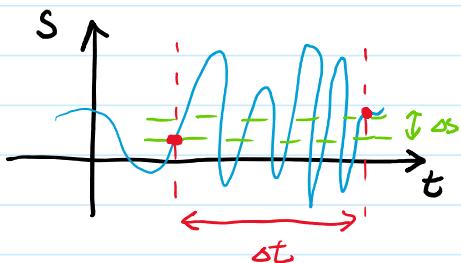
Calculus in Mechanics

Relations of s, v, a

In secondary school textbook we have

- Average velocity $v \sim \frac{\Delta s}{\Delta t}$
- Average acceleration $a \sim \frac{\Delta v}{\Delta t}$

But "Average" is never accurate to describe things.



Average velocity $\frac{\Delta s}{\Delta t}$ does not describe the drastic movement

For accurate description, we need to take limit $\Delta t \rightarrow 0$

↪ Instantaneous quantities = quantity at a "single" time point

$$- \text{Instantaneous velocity} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} s(t)$$

$$- \text{Instantaneous acceleration} = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} v(t)$$

$$\Rightarrow s(t) \xrightarrow[\int dt]{\frac{ds}{dt}} v(t) \xrightarrow[\int dt]{\frac{dv}{dt}} a(t)$$

Example : Constant Acceleration Motion

$$\Rightarrow a(t) = a = \text{some constant}$$

↪ Integration to find $v(t)$ & $s(t)$

$$- \text{Velocity} = v(t) = \int a dt = at + C$$

At $t = 0$

$$v(0) = C = \text{initial velocity}$$

Usually denote as "u" in textbook

$$\hookrightarrow v(t) = v(0) + at$$

$$- \text{Displacement} : s(t) = \int v(t) dt = v(0)t + \frac{1}{2}at^2 + C$$

At $t = 0$

$$s(0) = C = \text{initial displacement}$$

$$\hookrightarrow s(t) = s(0) + v(0)t + \frac{1}{2}at^2$$

★ ★ ★ In fact, only 2 of the 4 formulas of constant acceleration motions are independent

$$\left. \begin{array}{l} v(t) = v(0) + at \\ s(t) = s(0) + v(0)t + \frac{1}{2}at^2 \\ s(t) - s(0) = \frac{v(t) + v(0)}{2} t \\ v(t)^2 - v(0)^2 = 2a \cdot [s(t) - s(0)] \end{array} \right] \begin{array}{l} \text{Definition} \\ \text{nothing to explain} \\ \text{Can be derived by} \\ \text{substitution of the above 2} \end{array}$$

This is also shown in fact that a "SUVAT" problem always provide you 3 of the 5 variables, then you can solve the remaining 2 with 2 equations.

Force & Energy

Fundamental relation between force & energy : Work Done

$$W.D. = F \cdot ds = \left[\frac{\text{Average Force}}{\text{Time}} \right] \cdot [\text{Displacement}]$$

For a force $F(t)$ that is a function of time

Consider the W.D. in a very short time interval $[t, t+\Delta t]$

$$\begin{aligned} \Delta W.D. &= \frac{F(t) + F(t+\Delta t)}{2} \cdot [s(t+\Delta t) - s(t)] \\ &= \frac{F(t) + F(t+\Delta t)}{2} \left[\frac{s(t+\Delta t) - s(t)}{\Delta t} \right] \Delta t \end{aligned}$$

⇒ Total W.D. in a long period time

= Sum of all W.D. of many small time interval

$$= \sum_i \left[\frac{F(t_i) + F(t_{i+1})}{2} \right] \cdot \left[\frac{s(t_{i+1}) - s(t_i)}{\Delta t} \right] \Delta t$$

$$\hookrightarrow \Delta t = t_{i+1} - t_i$$

Take $\Delta t \rightarrow 0$, then $t_{i+1} \approx t_i$ & $F(t_{i+1}) \approx F(t_i)$

$$W.D. \rightarrow \int \frac{F(t) + F(t)}{2} \cdot \left[\frac{ds(t)}{dt} \right] dt$$

$$= \int F(t) \cdot \boxed{v(t)} dt$$

$$= \int \underline{F(t)} \cdot d(s(t))$$

$$= \int \underline{F^*[s(t)]} d[s(t)]$$

More commonly, force
is given as a function
of displacement

E.g. Gravitational Force & PE

$$F(r) = -\frac{GMm}{r^2}$$

↑

Force is a function of distance from the mass

Then the W.D. by gravitational force from r_1 to r_2 is

$$W.D. = \int_{r_1}^{r_2} -\frac{GMm}{r^2} dr \quad \text{Just } \int F(r) dr$$

$$\left\{ \begin{array}{l} \text{W.D. by gravitation} = \left(\frac{GMm}{r_2} \right) - \left(\frac{GMm}{r_1} \right) \\ \Rightarrow \text{loss in PE} \\ \\ \text{W.D. against gravitation} = - (GPE \text{ at } r_2 - GPE \text{ at } r_1) \\ \Rightarrow \text{gain in PE} \end{array} \right.$$

On the other hand, if given PE

$$PE = -\frac{GMm}{r} \quad \text{Add -ve for convention}$$

$$\text{Force by central mass} = -\frac{d}{dr}(PE)$$

$$= -\frac{GMm}{r^2}$$

Momentum / Impulse vs Newton 2nd Law

2 representations of Newton 2nd Law

Force & Acceleration

$$F = ma$$

Apply a force

⇒ Get acceleration

Impulse & Momentum

$$I = \Delta(mv)$$

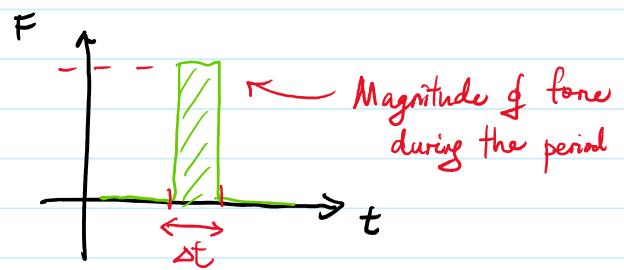
Apply an impulse

⇒ Change in momentum

Recap : Impulse

$$I \sim F \cdot \Delta t$$

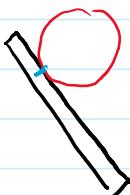
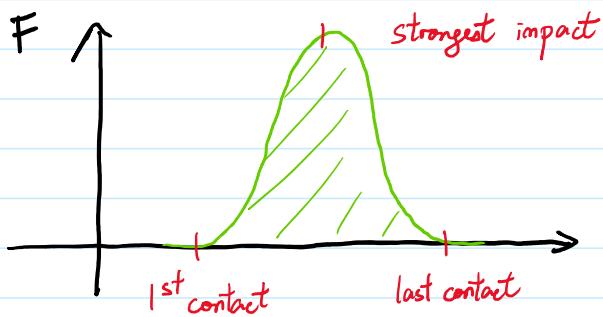
Applied force Duration



= Area under curve in F-t graph

= Change of momentum

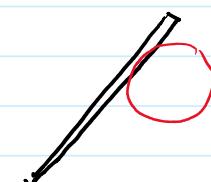
☆☆☆ But in reality, when things are colliding, applied force is not constant during the whole contact time.



Contact area ~ 0
 $\Rightarrow F \sim 0$



Contact area $\sim \text{max}$
 $\Rightarrow F \sim \text{max}$



Almost lose contact
 $\Rightarrow F \sim 0$

So in general, force should be a function of time

The total change of momentum during collision period :

$$\sum_i \Delta(mv)_i = \sum_i I_i = \sum_i \frac{F(t_i) + F(t_{i+1})}{2} \cdot \Delta t$$

$\hookrightarrow \Delta t = t_{i+1} - t_i$

Taking $\Delta t \rightarrow 0$

$$\int d(mv) = \int \frac{F(t) + F(t)}{2} dt = \int F(t) dt$$

This is the true Impulse - Momentum Relation

$$\int d(mv) = \int F(t) dt = I$$

Then differentiate both sides by t

$$\frac{d}{dt}(mv) = F(t) \rightarrow \boxed{\text{Rate of change of momentum} = \text{Force}}$$

★ Normally we should not assume mass to be time independent

$$\begin{aligned} F(t) &= \frac{d}{dt}(m(t)v(t)) \\ &= m(t) \cdot \frac{dv(t)}{dt} + \frac{dm(t)}{dt} \cdot v(t) \\ &\quad \xrightarrow{\text{C}} a(t) \\ \Rightarrow \quad &\boxed{F = ma + v \frac{dm}{dt}} \end{aligned}$$

This is the more general Newton 2nd Law.

Only if we guarantee the mass to be constant

then we can write $F=ma$

2 Interpretation of Integration

1st Interpretation : Area under curve

$$\int_a^b f(x) dx = \lim_{\Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$$

Sum them all height of strip width of strip

2nd Interpretation : Weighted sum

$$\int_a^b f(x) dx = \lim_{\Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$$

Sum them all length of the
"weight" assigned to the interval $[x_i, x_{i+1}]$

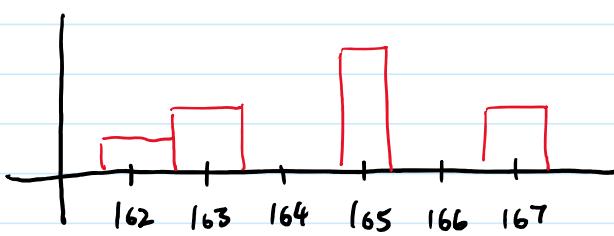
Example : Distribution of Students' height

	162	163	163	163	165	
	165	165	165	167	167	...

Average height = Weighted sum

$$= \sum_h \left(\begin{array}{l} \text{Portion of students} \\ \text{with height } h \end{array} \right) \times (\text{height } h)$$

$$= \frac{1}{11} \times 162 + \frac{3}{11} \times 163 + \frac{4}{11} \times 165 + \frac{3}{11} \times 167$$



Total height of all students = sum of all bars' height

~ Total area under each bar

Then transition from discrete to continuous distribution

- A single number → Interval between 2 numbers

↪ Range of students' height $[h_i, h_{i+1}]$

- Height of a bar → Area under the bar

↪ Portion of students whose height is within the range $[h_i, h_{i+1}]$

When the interval width $\rightarrow 0$, this weighted sum

arrives the formula the same as integration.