- Deriving Wave Equation $\begin{cases} \text{Transverse Wave} & \text{(already in lecture note)} \\ \text{Longitudinal Wave} \end{cases}$

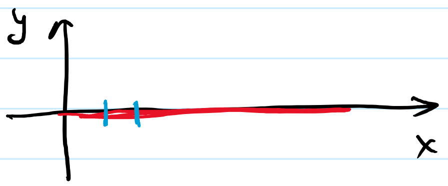
- Initial Value Problem

- Boundary Value Problem

$$\boxed{\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} y(x,t)}$$

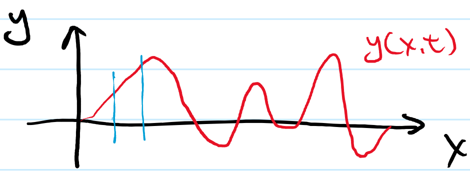
Model of transverse wave. = Elastic string

- When the string lies flat



Each string segment has width Δx

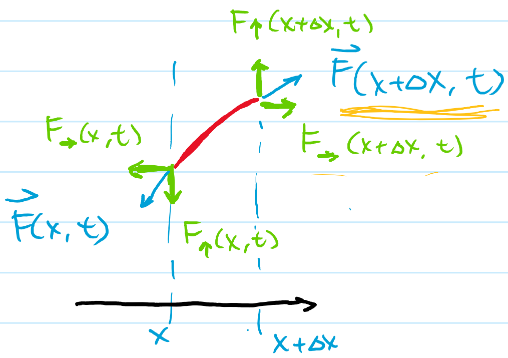
- When the string shakes



String segment jump up & down and become curved.

Transverse wave = Height of string at different position/time is described by the function $y(x,t)$

① Analysis by Newton 2nd Law.



Tension $\vec{F}$ must be a function of x because it must be different everywhere along the string

We can separate the tension into horizontal & vertical component, to write 2 Newton 2^{nd} Law.

$$\begin{cases} \rightarrow : \; F_{\rightarrow}(x+\Delta x, t) - F_{\rightarrow}(x,t) = 0 \\ \uparrow : \; F_{\uparrow}(x+\Delta x, t) - F_{\uparrow}(x,t) = (\mu \Delta x) \cdot a_{\uparrow} \end{cases}$$

horizontal acceleration = 0 because the string segment only jump up & down

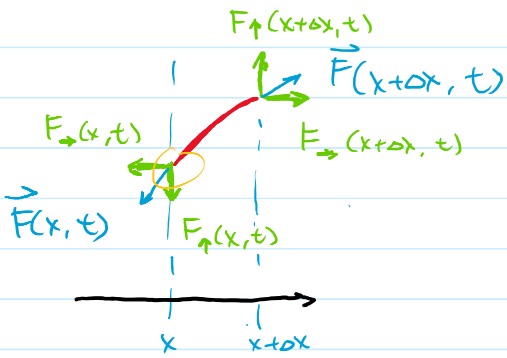
μ = density per unit length.

$\Rightarrow$ $\mu \Delta x$ = mass of the string segment

Note: There must not be gravity or the 2^{nd} Law become.

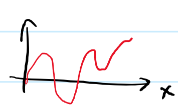
$$F_{\uparrow}(x+\Delta x, t) - F_{\uparrow}(x,t) - \underbrace{(\mu \Delta x) g}_{\text{additional } mg \text{ term.}} = (\mu \Delta x)\, a_{\uparrow}$$

② Analysis by the string 's geometry



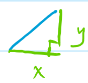
The tension $\vec{F}$ must be parallel to the slope at the 2 end points

- Because the string 's shape = the graph of $y(x,t)$ at some fix t



$\Rightarrow$ slope $= \dfrac{\partial}{\partial x} y(x,t)$

- And slope of the tension $\vec{F}$ can be calculated as $\dfrac{F_{\uparrow}}{F_{\rightarrow}}$



$\Rightarrow$ Relation at the end points

$$\begin{cases} \dfrac{F_{\uparrow}(x+\Delta x, t)}{F_{\rightarrow}(x+\Delta x, t)} = \dfrac{\partial}{\partial x} y(x,t) \Big|_{\text{at } x+\Delta x} \\ \dfrac{F_{\uparrow}(x,t)}{F_{\rightarrow}(x,t)} = \dfrac{\partial}{\partial x} y(x,t) \Big|_{\text{at } x} \end{cases}$$

③ Substitute into Newton 2nd Law.

$$(\mu \Delta x) a_{\uparrow} = F_{\uparrow}(x+\Delta x, t) - F_{\uparrow}(x, t)$$

$$= F_{\rightarrow}(x+\Delta x, t)\left[\frac{\partial}{\partial x} y(x,t)\right]_{\text{at } x+\Delta x} - F_{\rightarrow}(x,t)\left[\frac{\partial}{\partial x} y(x,t)\right]_{\text{at } x}$$

By Newton 2nd Law in horizontal direction, they are equal

so they can be taken out as a constant

$$= \boxed{F_{\leftrightarrow}}\left[\frac{\partial}{\partial x} y(x,t)\Big|_{\text{at } x+\Delta x} - \frac{\partial}{\partial x} y(x,t)\Big|_{\text{at } x}\right]$$

$$\mu a_{\uparrow} = F_{\leftrightarrow}\left[\frac{\frac{\partial}{\partial x} y(x,t)\Big|_{\boxed{\text{at } x+\Delta x}} - \frac{\partial}{\partial x} y(x,t)\Big|_{\boxed{\text{at } x}}}{\boxed{\Delta x}}\right]$$

vertical acceleration

= 2nd derivative over t

This is just like derivative $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$\mu \boxed{\frac{\partial^2}{\partial t^2} y(x,t)} = F_{\leftrightarrow} \cdot \boxed{\frac{\partial^2}{\partial x^2} y(x,t)}$$

become 2nd derivative over x

$$\frac{\mu}{F_{\leftrightarrow}} \frac{\partial^2}{\partial t^2} y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t)$$

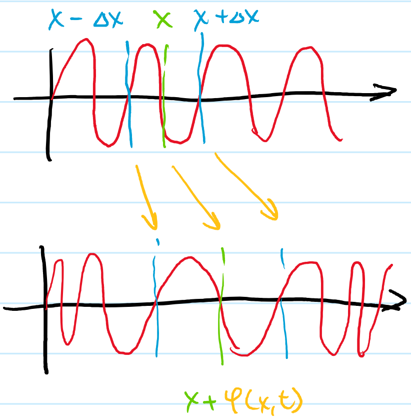
This is the wave Equation

By comparing with the general form, we can identify the wave speed as $v = \sqrt{\frac{F_{\leftrightarrow}}{\mu}} = \sqrt{\frac{\text{horizontal tension}}{\text{mass per length}}}$

Model of Longitudinal wave = slinky

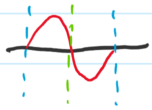
- When the slinky is static
  peaks are of equal spacing

- When the slinky shakes
  peaks become unevenly distributed



Longitudinal wave = Displacement of slinky at different position / time

is described by the function $\varphi(x,t)$

① Take out a small segment  for analysis

Node	Original Position	(segment length)	New Position	(segment length)
Left	$x-\Delta x$		$x-\Delta x+\varphi(x-\Delta x,t)$	
		Δx		$\Delta x+\varphi(x,t)-\varphi(x-\Delta x,t)$
Center	x		$x+\varphi(x,t)$	
		Δx		$\Delta x+\varphi(x+\Delta x,t)-\varphi(x,t)$
Right	$x+\Delta x$		$x+\Delta x+\varphi(x+\Delta x,t)$	

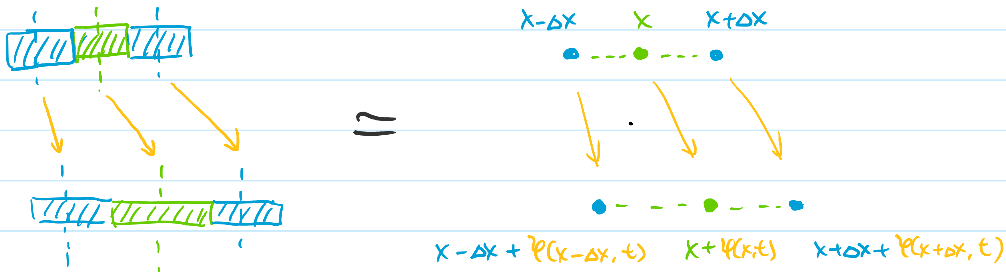
② Consider the segment to be very small

We can make these approximations :

[1] The mass concentrates on the nodes

$\Rightarrow$ Mass within $\pm\frac{\Delta x}{2}$ from the nodes approximately move together with the node

i.e. Displacement of node $=\varphi(x,t)$ $\Rightarrow$ Acceleration of node $=\frac{\partial^2}{\partial t^2}\varphi(x,t)$



$\boxed{2}$ Elastic force is only between nodes.

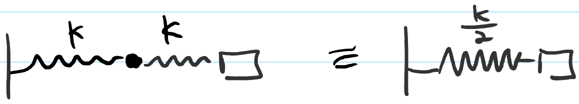
We can write the elastic force as

$$F_L = -k\cdot\begin{pmatrix}\text{change in length}\\\text{of left segment}\end{pmatrix} = -k\left[\varphi(x,t)-\varphi(x-\Delta x,t)\right]$$

$$F_R = -k\cdot\begin{pmatrix}\text{change in length}\\\text{of right segment}\end{pmatrix} = -k\left[\varphi(x+\Delta x,t)-\varphi(x,t)\right]$$

Note: k is the spring constant (just like $\vec{F}=-k\vec{x}$)

However spring constant depends on length of spring

Eg. 

$(k \propto \frac{1}{L})$

So we want to remove the length dependancy by taking

$$k \propto \frac{1}{L} \Rightarrow \boxed{k = \frac{Y}{L} = \frac{\text{Young Modulus}}{\text{Total relaxed length}}}$$

where Young modulus only depends on the material

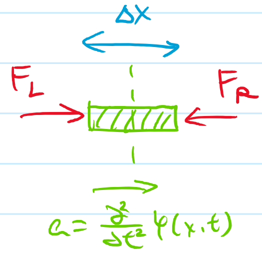
So the elastic force is written as

$$F_L = -\frac{Y}{\Delta x}\left[\varphi(x,t)-\varphi(x-\Delta x,t)\right]$$

$$F_R = -\frac{Y}{\Delta x}\left[\varphi(x+\Delta x,t)-\varphi(x,t)\right]$$

relaxed length
= original separation between nodes
= Δx

③ Newton 2^{nd} Law to the nodes



$$ma = \sum F$$

$$\underbrace{(\mu\Delta x)}_{m} \cdot \underbrace{\frac{\partial^2}{\partial t^2}\varphi(x,t)}_{a} = F_L - F_R$$

$$= \frac{Y}{\Delta x}\Big[\varphi(x+\Delta x,t) - \varphi(x,t) - \varphi(x,t) + \varphi(x-\Delta x,t)\Big]$$

$$= Y\left[\boxed{\frac{\varphi(x+\Delta x,t) - \varphi(x,t)}{\Delta x}} - \boxed{\frac{\varphi(x,t) - \varphi(x-\Delta x,t)}{\Delta x}}\right]$$

become derivative

$$= Y\left[\frac{\partial}{\partial x}\varphi(x,t)\Big|_{\text{at } x+\Delta x} - \frac{\partial}{\partial x}\varphi(x,t)\Big|_{\text{at } x}\right]$$

$$\mu\frac{\partial^2}{\partial t^2}\varphi(x,t) = Y\left[\boxed{\frac{\frac{\partial}{\partial x}\varphi(x,t)\Big|_{\text{at } x+\Delta x} - \frac{\partial}{\partial x}\varphi(x,t)\Big|_{\text{at } x}}{\Delta x}}\right]$$

become 2^{nd} derivative

$$= Y\frac{\partial^2}{\partial x^2}\varphi(x,t)$$

$$\frac{\mu}{Y}\frac{\partial^2}{\partial t^2}\varphi(x,t) = \frac{\partial^2}{\partial x^2}\varphi(x,t)$$

is a wave equation

By comparing with the general form, we can identify the

wave speed as $v = \sqrt{\frac{Y}{\mu}} = \sqrt{\frac{\text{Young modulus}}{\text{mass per segment}}}$

General Solution to Wave Equation

$$\frac{\partial^2}{\partial x^2}\psi(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t)$$

Wave Equation is a Partial Differential Equation (PDE)

The general solution looks like :

$$\boxed{\psi(x,t) = f(x+vt) + g(x-vt)}$$

where $f(\ldots)$, $g(\ldots)$ can be any single variable function

E.g. $f(u) = \sin u + u^2$ (subst. $u \to x+vt$)

$$\Rightarrow f(x+vt) = \sin(x+vt) + (x+vt)^2$$

Proof : Simply by substitution

E.g. $\frac{\partial}{\partial t} f(x+vt) = \frac{\partial f(x+vt)}{\partial(x+vt)} \cdot \frac{\partial(x+vt)}{\partial t}$

$= f'(x+vt) \cdot v$

$\frac{\partial^2}{\partial t^2} f(x+vt) = \frac{\partial f'(x+vt)}{\partial(x+vt)} \cdot \frac{\partial(x+vt)}{\partial t} \cdot v$

$= f''(x+vt) \cdot v^2$

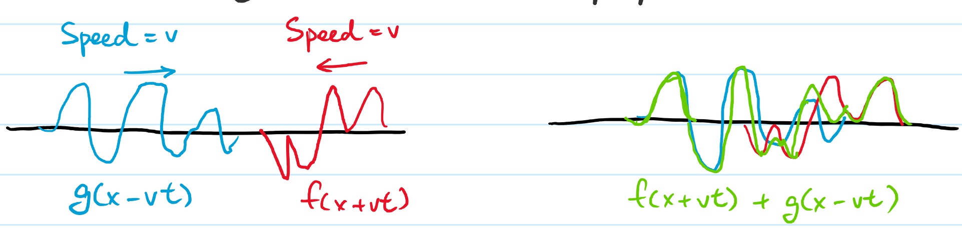
Do similar step to find $\frac{\partial^2}{\partial x^2} f(x+vt)$. One should find them equal.

Physical Interpretation

$f(x+vt)$ = A waveform travelling to the left

$g(x-vt)$ = A waveform travelling to the right.

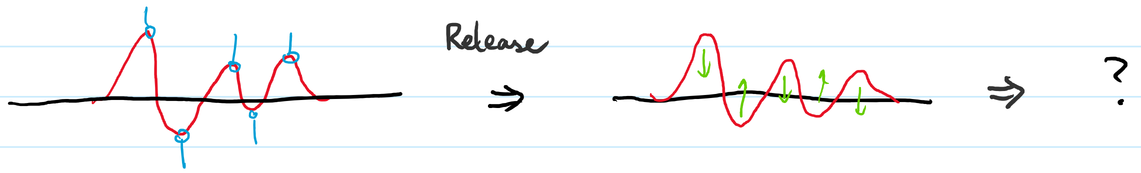
Adding together = Superposition.



Initial Value Problem

Given a string being stretch to the shape φ at time $t=0$

After the string is released, how will the waveform evolve?



i.e. Solve for $\varphi(x,t)$ by given $\underline{\varphi(x,0)}$ & $\underline{\frac{\partial}{\partial t}\varphi(x,t)\Big|_{t=0}}$

initial waveform. — initial velocity at every point

Derivation

$\boxed{1}$ Break down into general form $\varphi = f + g$

$$\varphi(x,0) = f(x+0) + g(x+0) \quad \text{from the initial position} \quad —\ (1)$$

$$\frac{\partial}{\partial t}\varphi(x,t)\Big|_{t=0} = vf'(x+0) - vg'(x+0) \quad \text{from the initial velocity.} \quad —\ (2)$$

$\boxed{2}$ Differentiate (1)

$$\frac{d}{dx}\varphi(x,0) = f'(x+0) + g'(x+0) \quad —\ (3)$$

$\boxed{3}$ Isolate $f'(x+0)$ & $g'(x+0)$ by (2) & (3)

(3) $+ \frac{1}{v}$ (2) :

$$\frac{d}{dx}\varphi(x,0) + \frac{1}{v}\frac{\partial}{\partial t}\varphi(x,t)\Big|_{t=0} = 2\cdot f'(x+0) \quad —\ (4)$$

(3) $- \frac{1}{v}$ (2) :

$$\frac{d}{dx}\varphi(x,0) - \frac{1}{v}\frac{\partial}{\partial t}\varphi(x,t)\Big|_{t=0} = 2\cdot g'(x+0) \quad —\ (5)$$

4 Integrate ④, ⑤ about x

④ $$2 \cdot \int f'(x)\, dx = 2f(x) = \int \left[\frac{d}{dx}\varphi(x,0) + \frac{1}{v}\frac{\partial}{\partial t}\varphi(x,t)\Big|_{t=0} \right] dx$$

C_1 = integration constant

$$= \underline{\varphi(x,0) + C_1} + \int \left[\frac{1}{v}\frac{\partial}{\partial t}\varphi(x,t)\Big|_{t=0} \right] dx$$

$$= \varphi(x,0) + C_1 + \int_{s=0}^{s=x} \frac{1}{v}\frac{\partial}{\partial t}\varphi(s,t)\Big|_{t=0} ds$$

s = dummy variable to replace x. Just to bring convenience to later steps

⑤ $$2 \cdot \int g'(x)\, dx = 2g(x) = \int \left[\frac{d}{dx}\varphi(x,0) - \frac{1}{v}\frac{\partial}{\partial t}\varphi(x,t)\Big|_{t=0} \right] dx$$

C_2 = another integration constant

$$= \varphi(x,0) + C_2 - \int_{s=0}^{s=x} \frac{1}{v}\frac{\partial}{\partial t}\varphi(s,t)\Big|_{t=0} ds$$

5 Replace the "x" in f(x) by "x+vt"
"x" in g(x) by "x−vt"

$$\Rightarrow \begin{cases} 2 \cdot f(x+vt) = \varphi(x+vt, 0) + C_1 + \displaystyle\int_{s=0}^{s=x+vt} \frac{1}{v}\frac{\partial}{\partial t}\varphi(s,t)\Big|_{t=0} ds \\ 2 \cdot g(x-vt) = \varphi(x-vt, 0) + C_2 + \displaystyle\int_{s=x-vt}^{s=0} \frac{1}{v}\frac{\partial}{\partial t}\varphi(s,t)\Big|_{t=0} ds \end{cases}$$

Switch upper/lower bound = change sign.

6 Add the 2 expressions together = The general solution

$$\varphi(x,t) = f(x+vt) + g(x-vt)$$

$$= \frac{1}{2}\left[\varphi(x+vt, 0) + \varphi(x-vt, 0) \right] + C_1 + C_2$$

$$+ \frac{1}{2}\int_{s=x-vt}^{s=x+vt} \frac{1}{v}\frac{\partial}{\partial t}\varphi(s,t)\Big|_{t=0} ds$$

[7] Subst. $t=0$ into the expression to find what is $C_1 + C_2$

$$\psi(x,t=0) = \frac{1}{2}\left[\psi(x+0,0) + \psi(x-0,0)\right] + C_1 + C_2$$

$$+ \boxed{\frac{1}{2}\int_{s=x-0}^{s=x+0} \frac{1}{v}\frac{\partial}{\partial t}\psi(s,t)\Big|_{t=0} ds}$$

$= 0$ because upper bound $=$ lower bound

$$= \frac{1}{2}\cdot 2\,\psi(x,0) + C_1 + C_2$$

$$\psi(x,0) = \psi(x,0) + C_1 + C_2$$

$$\therefore\ C_1 + C_2 = 0$$

So finally we arrive the general soln. to the initial value problem.

$$\boxed{\underbrace{\psi(x,t)}_{\text{later waveform}} = \frac{1}{2}\left[\underline{\psi(x+vt,0)} + \underline{\psi(x-vt,0)}\right] + \frac{1}{2v}\int_{s=x-vt}^{s=x+vt} \underbrace{\frac{\partial}{\partial t}\psi(x,t)\Big|_{t=0}}_{\text{info of the initial velocity}} ds}$$

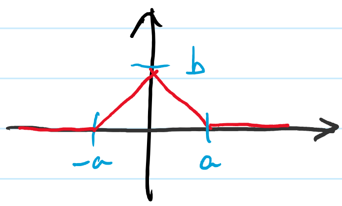
terms that use info of the initial waveform

later waveform.

info of the initial velocity

E.g. Given the initial waveform

$$\psi(x,0) = \begin{cases} b - \dfrac{b|x|}{a} & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$$



And is static at the beginning (i.e. velocity $= 0$ everywhere)

$\Rightarrow$ We can find how the wave evolve by substitute into the general soln.

$$\psi(x,t) = \frac{1}{2}\left[\psi(x+vt) + \psi(x-vt)\right] + \int_{x-vt}^{x+vt} \cancelto{0}{\frac{\partial}{\partial t}\psi(s,t)\Big|_{t=0}}\, ds$$

Because no initial velocity

$$= \underbrace{\frac{1}{2}\left(b - \frac{b|x+vt|}{a}\right)} + \underbrace{\frac{1}{2}\left(b - \frac{b|x-vt|}{a}\right)}$$

This is a function of $x+vt$

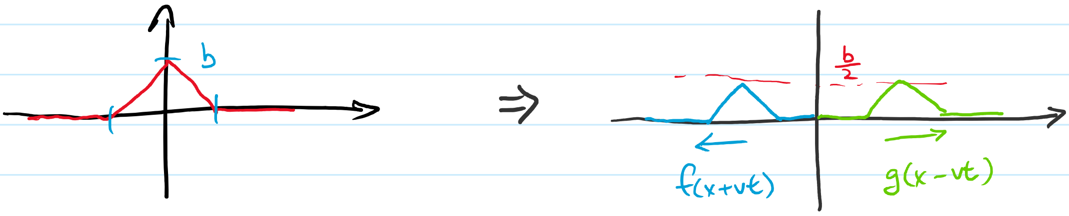
$\therefore$ This is $f(x+vt)$

(The left travelling wave)

This is a function of $x-vt$

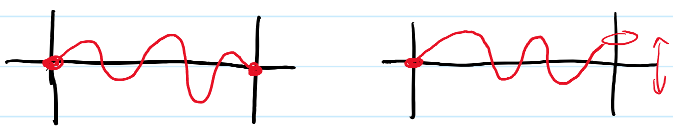
$\therefore$ This is $g(x-vt)$

(The right travelling wave)



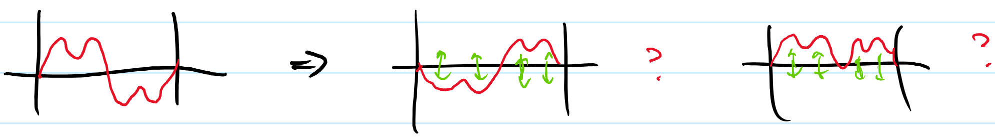
Boundary Value Problem ~ standing wave

Eg.



You are given the info. at the end points of the string and the initial waveform. How will the wave evolve?

Eg.



Method of Separation of Variable.

Assume the solution can be expressed as a product

of 2 single variable function of x or t only

$$\psi(x,t) = \underbrace{X(x)}_{\substack{\text{Single Variable function} \\ \text{only depends on } x}} \underbrace{T(t)}_{\substack{\text{Single Variable function} \\ \text{only depends on } t}}$$

Substitute into the Wave Eq.

$$\frac{\partial^2}{\partial x^2}\left[X(x)\,T(t)\right] = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[X(x)\,T(t)\right]$$

$$T(t)\frac{\partial^2}{\partial x^2}\left[X(x)\right] = \frac{1}{v^2}X(x)\frac{\partial^2}{\partial t^2}\left[T(t)\right]$$

$T(t)$: Not depends on x, can be taken out of $\frac{\partial^2}{\partial x^2}$

$X(x)$: Not depends on t, can be taken out of $\frac{\partial^2}{\partial t^2}$

$$\frac{1}{X(x)}\frac{\partial^2}{\partial x^2}\left[X(x)\right] = \frac{1}{v^2}\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}\left[T(t)\right] = \text{Some const.} = -k^2$$

writing like this for convenience in later steps

LHS is a function of x only

RHS is a function of t only

The only possibility for them to be equal = Both equal to a constant

$\Rightarrow$ Separate into 2 independent ODEs.

$$\begin{cases} \dfrac{1}{X(x)}\dfrac{d^2}{dx^2}X(x) = -k^2 & \Rightarrow \quad \dfrac{d^2}{dx^2}X(x) + k^2 X(x) = 0 \\[2ex] \dfrac{1}{v^2}\dfrac{1}{T(t)}\dfrac{d^2}{dt^2}T(t) = -k^2 & \Rightarrow \quad \dfrac{d^2}{dt^2}T(t) + k^2 v^2 T(t) = 0 \end{cases}$$

Both are 2nd order linear ODE

Remember in SHM, solution of $\frac{d^2}{dt^2}f(t) + k^2 f(t) = 0$ is $C_1 \cos kt + C_2 \sin kt$

$$\therefore \text{Soln} \begin{cases} X(x) = C \cos kx + D \sin kx \\ T(t) = A \cos kvt + B \sin kvt \end{cases}$$

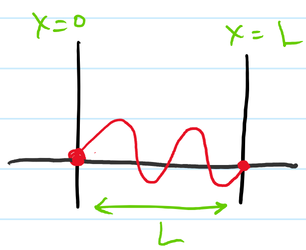
A, B, C, D are constants depending on the boundary condition.

2 Important boundary conditions

– Dirichlet Condition

= 2 fixed ends

$$\begin{cases} \psi(0, t) = 0 \\ \psi(L, t) = 0 \end{cases}$$



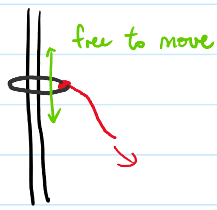
– Neumann Condition

= 2 free ends (free to move up & down)

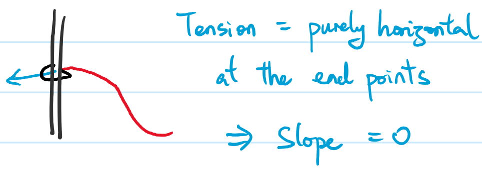
⇒ No vertical force / acceleration

⇒ Tension in vertical direction = 0

⇒ Slope at the endpoints = 0



$$\begin{cases} \frac{\partial}{\partial x}\psi(x,t)\Big|_{x=0} = 0 \\ \frac{\partial}{\partial x}\psi(x,t)\Big|_{x=L} = 0 \end{cases}$$



① Solution under Dirichlet condition

By $\psi(0,t) = \psi(L,t) = 0$

⇒ Must have $X(0) = X(L) = 0$

From $X(x) = C\cos kx + D\sin kx$

- $X(0) = C \cdot 1 + D \cdot 0 = 0$

$\therefore C = 0$

- $X(L) = 0 + D\sin kL = 0$

This is true only if $kL = n\pi$ $\quad (n = 0, 1, 2, 3 \ldots)$

but taking $n = 0$ gives $X(x) = \sin 0 = 0$ which is not interesting

$\therefore$ k can only take values of $\frac{n\pi}{L}$

$$\therefore X(x) = D \sin \frac{n\pi}{L} x$$

$$\therefore \psi(x,t) = T(t)\, X(x)$$

$$= \left[A\cos\left(\frac{n\pi}{L} vt\right) + B\sin\left(\frac{n\pi}{L} vt\right)\right] \cdot \left[\sin \frac{n\pi}{L} x\right]$$

for any $n = 1, 2, 3, \ldots$

const. D can merge with const. A & B

By Superposition property of linear differential equation

Because [$\psi(x,t)$ with $n=1$] is a soln.

[$\psi(x,t)$ with $n=2$] is a soln.

$\vdots$

$\Rightarrow$ The general soln is the sum of all possible solutions

i.e. Sum [$\psi(x,t)$ with $n = 1, 2, 3 \ldots \infty$]

$$\boxed{\therefore \psi(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{L} vt\right) + B_n \sin\left(\frac{n\pi}{L} vt\right)\right]\left[\sin \frac{n\pi}{L} x\right]}$$

② Solution under Neumann Condition

By $\frac{\partial}{\partial x}\psi(x,t)\Big|_{x=0} = 0$ & $\frac{\partial}{\partial x}\psi(x,t)\Big|_{x=L} = 0$

$\Rightarrow$ Must have $\frac{d}{dx}X(x)\Big|_{x=0} = 0$ & $\frac{d}{dx}X(x)\Big|_{x=L} = 0$

From $X(x) = C\cos kx + D\sin kx$

$$\frac{d}{dx}X(x)\Big|_{x=0} = -kC\cdot 0 + kD\cdot 1 = 0 \qquad \therefore D = 0$$

$$\frac{d}{dx}X(x)\Big|_{x=L} = -kC\cdot \sin kL = 0$$

This is true only if $kL = n\pi$ $(n = 0, 1, 2, \dots)$

$\therefore$ k can only take values of $\frac{n\pi}{L}$

$\therefore$ $X(x) = C\cos\frac{n\pi}{L}x$

This time we keep $n=0$ because it gives $X(x) = C\cos 0 = C$

Then subst. $k = \frac{n\pi}{L}$ to $T(t)$

$\boxed{1}$ If $k = 0$

ODE of $T(t)$ becomes $\frac{d^2}{dt^2}T(t) = 0\cdot T(t)$

Solution is a linear function $T(t) = A + Bt$

$\boxed{2}$ If $k \neq 0$

Simply keep the form $T(t) = A\cos\frac{n\pi}{L}vt + B\sin\frac{n\pi}{L}vt$

$$\therefore \psi(x,t) = \begin{cases} [A+Bt][C] = A + Bt \\ \left[A\cos\left(\frac{n\pi}{L}vt\right) + B\sin\left(\frac{n\pi}{L}vt\right)\right]\left[\cos\frac{n\pi}{L}x\right] \end{cases}$$

const. C can merge with A & B

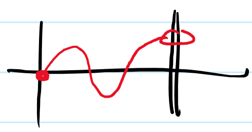
Again, the general soln. is the sum of all possible n

$$\boxed{\psi(x,t) = \left[A_0 + B_0 t\right] + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi}{L}vt\right) + B_n \sin\left(\frac{n\pi}{L}vt\right)\right]\left[\cos\frac{n\pi}{L}x\right]}$$

<u>P.S.</u> We can also solve for other boundary condition following these steps :

1. Use $x = 0$'s condition to eliminate 1 of C or D
2. Use $x = L$'s condition to determine what value k can be
3. Subst back into $\psi(x,t) = T(t)\,X(x)$ for each possible k
4. The general soln = sum of $\psi(x,t)$ of all possible k

<u>EX</u>: 1 fixed end + 1 open end



You should get

$$\psi(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2n-1}{2}\frac{\pi}{L}vt\right) + B_n \sin\left(\frac{2n-1}{2}\frac{\pi}{L}vt\right)\right]\left[\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x\right)\right]$$

<u>Interpreting the solutions</u>

The general solution is a superposition of all simplier solutions of different n. Each n has its corresponding $X_n(x)$ and $T_n(t)$

$$\psi(x,t) = \sum_n \boxed{\psi_n(x,t)} = \sum \left[X_n(x)\,T_n(t)\right]$$

- $X_n(x)$ only depends on x $\Rightarrow$ Carry info about <u>waveform</u>
- $T_n(t)$ only depends on t $\Rightarrow$ Carry info about <u>frequency</u>

☆☆☆ Note that every $\varphi_n(x,t)$ are <u>independent</u> of the others.

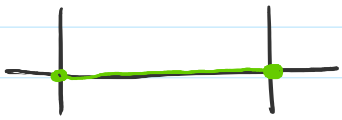
They are called the <u>mode</u> of a standing wave

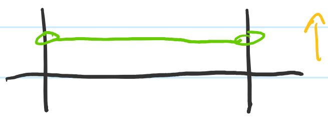
i.e. $\begin{pmatrix} n^{th} \text{ mode} \\ \varphi_n(x,t) \end{pmatrix} = \begin{pmatrix} \text{Its own waveform} \\ X_n(x) \end{pmatrix} + \begin{pmatrix} \text{Its own frequency} \\ T_n(t) \end{pmatrix}$

We can draw the graph for each $\varphi_n(x,t)$:

	Dirichlet	Neumann	1 side open + 1 side fix

$n=0$



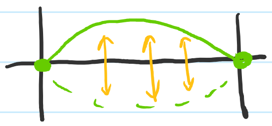


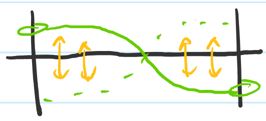
(Same as Dirichlet)

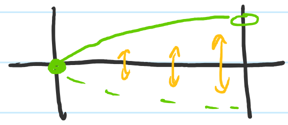
$X(x) = 0$
so $\varphi(x,t) = 0$
nothing interesting

$X(x) = \text{const.} = \text{flat line}$
$T(t) = A + Bt$
$= \text{increasing / decreasing at constant speed}$
$\Rightarrow \varphi(x,t) = \text{a flat line moving at constant speed}$

$n=1$

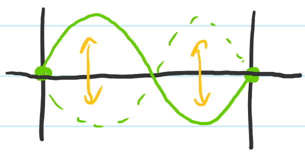


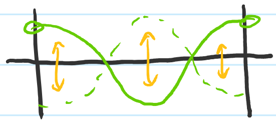


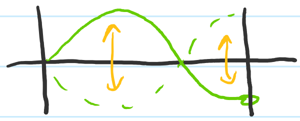


$\text{frequency} = \frac{\pi v}{L}$ $\qquad$ $\text{frequency} = \frac{\pi v}{L}$ $\qquad$ $\text{frequency} = \frac{1}{2}\frac{\pi v}{L}$

$n=2$







$\text{frequency} = \frac{2\pi v}{L}$ $\qquad$ $\text{frequency} = \frac{2\pi v}{L}$ $\qquad$ $\text{frequency} = \frac{3}{2}\frac{\pi v}{L}$

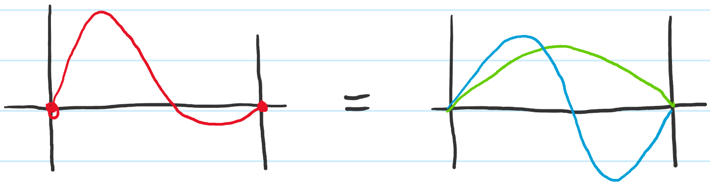
⋮ and so on

If the standing wave is vibrating purely as one of its mode

its [waveform + frequency] will never change.

★★★ By the superposition of solutions, any possible vibration on the string can be written as a sum of many modes

E.g.



this wave (at $t=0$) $= \sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L}$

$=$ 1^{st} mode (at $t=0$) $+$ 2^{nd} mode (at $t=0$)

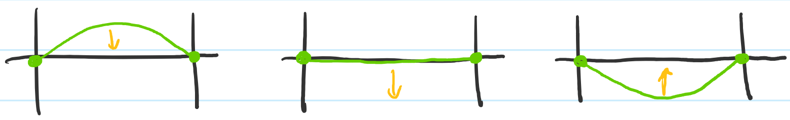
However each mode has its own frequency of vibration

So the shape of the combined wave will not maintain

$t=0$ $\qquad t=\frac{L}{2v}$ $\qquad t=\frac{L}{v}$

1^{st} mode

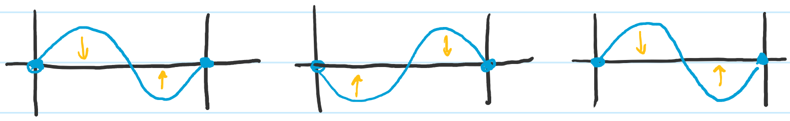
period $= \frac{2\pi}{\left(\frac{\pi}{L}v\right)} = \frac{2L}{v}$



(only complete $\frac{1}{2}$ period)

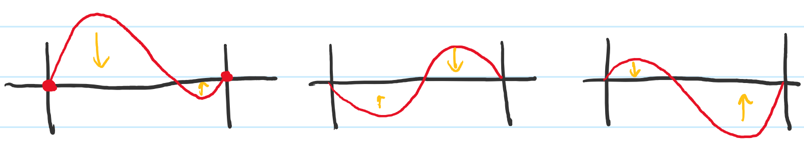
2^{nd} mode

period $= \frac{2\pi}{\left(\frac{2\pi}{L}v\right)} = \frac{L}{v}$



(already complete 1 period)

Sum



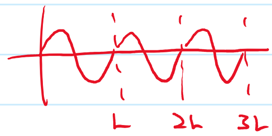
The waveform becomes sort of irregular.

Fourier Series

= Tool for breaking down the waveform into modes.

For a periodic function $f(x) = f(x+L)$

Period $= L$



Can be expanded by a series

$$f(x) = \frac{\boxed{a_0}}{2} + \sum_{n=1}^{\infty} \left[\boxed{a_n} \cos\left(\frac{2\pi n}{L} x\right) + \boxed{b_n} \sin\left(\frac{2\pi n}{L} x\right) \right]$$

The constants a_n, b_n can be calculated by

$$\left.\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cdot \cos\left(\frac{2\pi n}{L} x\right) dx \\ b_n &= \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{2\pi n}{L} x\right) dx \end{aligned}\right\} \text{Called Fourier Coefficient}$$

The derivations come from these integrals :

for any integer m, n

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m = n = 0 \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_0^{2\pi} \sin mx \cos nx \, dx = 0$$

(These integral are easy using product to sum formula of sin / cos)

E.g. $\displaystyle\int_0^L f(x) \cos\left(\frac{2\pi n}{L} x\right) dx$

$$= \int_0^L \left[\frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos\left(\frac{2\pi m}{L} x\right) + b_m \sin\left(\frac{2\pi m}{L} x\right) \right] \right] \cos\left(\frac{2\pi n}{L} x\right) dx$$

$$= \int_0^L \left[\frac{a_0}{2}\cos\left(\frac{2\pi n}{L}x\right) + a_1\cos\left(\frac{2\pi\cdot 1}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right) + \cdots \right.$$

$m \neq 0$. so integration $= 0$

Area under a cosine curve of full period $= 0$

$$\left. + \, b_1 \sin\left(\frac{2\pi\cdot 1}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right) + \cdots \right] dx$$

integration of any $\sin mx \cos nx = 0$

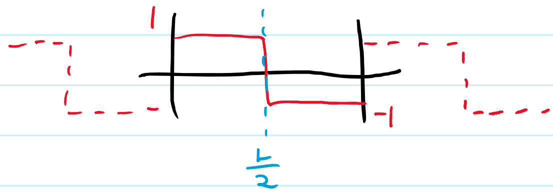
$$= \text{All terms vanish except } \int_0^L \cos\left(\frac{2\pi n}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right) dx$$

$$= a_n \cdot \frac{L}{2}$$

$$\therefore \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right) dx$$

E.g. Square Wave

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{L}{2} \\ -1 & \text{for } \frac{L}{2} < x < L \end{cases}$$



Directly compute the Fourier Coefficient :

$$a_n = \frac{2}{L}\int_0^{\frac{L}{2}} 1 \cdot \cos\left(\frac{2\pi n}{L}x\right) dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} -1 \cdot \cos\left(\frac{2\pi n}{L}x\right) dx = 0$$

$$b_n = \frac{2}{L}\int_0^{\frac{L}{2}} 1 \cdot \sin\left(\frac{2\pi n}{L}x\right) dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} -1 \cdot \sin\left(\frac{2\pi n}{L}x\right) dx$$

$$= \frac{2}{n\pi}\left[1 - (-1)^n\right] = \begin{cases} 0 & \text{for } n = \text{even} \\ \frac{4}{n\pi} & \text{for } n = \text{odd} \end{cases}$$

$\Rightarrow$ The Fourier Series of square wave :

$$f(x) = \frac{4}{\pi}\left[\sin\left(\frac{2\pi}{L}x\right) + \frac{1}{3}\sin\left(\frac{2\pi}{L}\cdot 3x\right) + \frac{1}{5}\sin\left(\frac{2\pi}{L}\cdot 5x\right) + \cdots \right]$$

☆☆ By Fourier Series, we can break down any waveform into sum of $\boxed{X(x)}$ of different mode n.

To find the evolution of the wave with t simply add back the $\boxed{T(t)}$ part for each mode

$$f(x) = \frac{4}{\pi}\left[\underbrace{\sin\left(\frac{2\pi}{L}x\right)}_{\text{evolve with frequency } \propto \frac{2\pi}{L}} + \underbrace{\frac{1}{3}\sin\left(\frac{2\pi}{L}\cdot 3x\right)}_{\text{evolve with frequency } \propto \frac{2\pi}{L}\cdot 3} + \frac{1}{5}\sin\left(\frac{2\pi}{L}\cdot 5x\right) + \cdots \right]$$

$$\boxed{\varphi_1(x,t) = \frac{4}{\pi}\boxed{\sin\left(\frac{2\pi}{L}x\right)}\boxed{A_1\cos\left(\frac{2\pi}{L}vt\right) + B_1\sin\left(\frac{2\pi}{L}vt\right)}}$$

$$\boxed{\varphi_3(x,t) = \frac{4}{3\pi}\boxed{\sin\left(\frac{2\pi\cdot 3}{L}x\right)}\boxed{A_3\cos\left(\frac{2\pi}{L}\cdot 3vt\right) + B_3\sin\left(\frac{2\pi}{L}\cdot 3vt\right)}}$$

So the whole evolution of wave follows:

$$\varphi(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n\pi}\left[\sin\left(\frac{2\pi n}{L}x\right)\right]\left[A_n\cos\left(\frac{2\pi}{L}\cdot nvt\right) + B_n\sin\left(\frac{2\pi}{L}\cdot nvt\right)\right]$$

where the constants A_n, B_n need to be determined by the initial condition.