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Matrix Method for Special Relativity

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Overview:

- Lorentz transformation matrix
- Explaining relativistic phenomena with the matrix method

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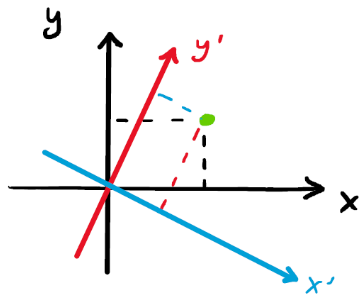
1 Lorentz Transform

1.1 Matrix as Linear Transformation

First we shall re-visit matrix as a tool of coordinate transformation - By applying a matrix onto a position vector, we can change the expression of a position in one coordinate system into the expression in another coordinate system. Here are some very common transformations that you need to remember:

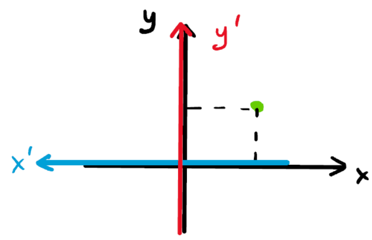
- **Rotation matrix**

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



- **Reflection matrix**

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Remember the very important fact about coordinate transform:

The point is always the same one, but the point's coordinate can change because we are free to choose the coordinate system.

> *Side Note:*
>
> In fact, the coordinate transformation by a matrix $\mathbf{A}$ will map the coordinate expression onto the coordinate system spanned by the vectors $\left\{\frac{1}{\lambda_1}\vec{\boldsymbol{v}}_1, \frac{1}{\lambda_1}\vec{\boldsymbol{v}}_1, \ldots, \frac{1}{\lambda_n}\vec{\boldsymbol{v}}_n\right\}$, where λ_i are the eigenvalues of $\mathbf{A}$ and $\vec{\boldsymbol{v}}_i$ are the corresponding eigenvectors.

1.2 Lorentz Transformation Matrix

1.2.1 Spacetime Coordinate

The topic of relativity is to study the transform between **spacetime coordinate system**:

- Every "**event**" in the spacetime can be labelled with a coordinate:
  - An event happens at time t at position (x, y, z) is given the coordinate (ct, x, y, z).
  - The time t is multiplied by speed of light c such that all 4 coordinates have the unit of positions.
- Different **observers** can describe the same event using their own coordinate systems, leading to different expressions of the same event. E.g. For the same event,
  - Observer A may describe it as (ct, x, y, z), while
  - Observer B describe it as (ct', x', y', z').

In special relativity, we only deals with observers in different **inertial frames**, i.e. they do not experience accelerations. For simplicity, we can choose

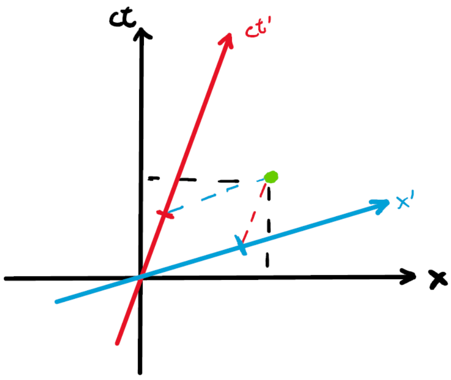
- The relative velocity between observers is along the x-axis.
- The origins of the observers' coordinate "coincide", i.e. $(ct, x, y, z) = (0, 0, 0, 0)$ is the same point as $(ct', x', y', z') = (0, 0, 0, 0)$.

Then the y/z coordinate of an event will be the same when described by both observers. We can focus on the transformation on t and x coordinate only.

In the following section, we will derive the coordinate transform matrix - the Lorentz matrix, which relates the observed ct and x coordinate between observers.

$$= \underbrace{\begin{pmatrix} ct' \\ x' \end{pmatrix}}_{\text{An event's coordinate observed by B}} = \underbrace{\begin{pmatrix} \mathbf{\Lambda} \end{pmatrix}}_{\text{The Lorentz Transformation A 2x2 matrix}} \underbrace{\begin{pmatrix} ct \\ x \end{pmatrix}}_{\text{An event's coordinate observed by A}}$$

Geometrically, this transformation can be visualized as a change in the coordinate on a $t - x$ plane called **Minkowski diagram**:



1.2.2 2 Einstein's Posulates

Special relativity is proposed based on only two principles. They are essential to derive the expression of Lorentz transformation.

1. **Principle of Relativity**
   All physics must be the same to any inertial observers. i.e. All the formula yield the correct results, although values to be substituted are different for different observers.

2. **Principle of invariant light speed**
   Speed of light is the same for all observers. (This includes non-inertial frame observers.)

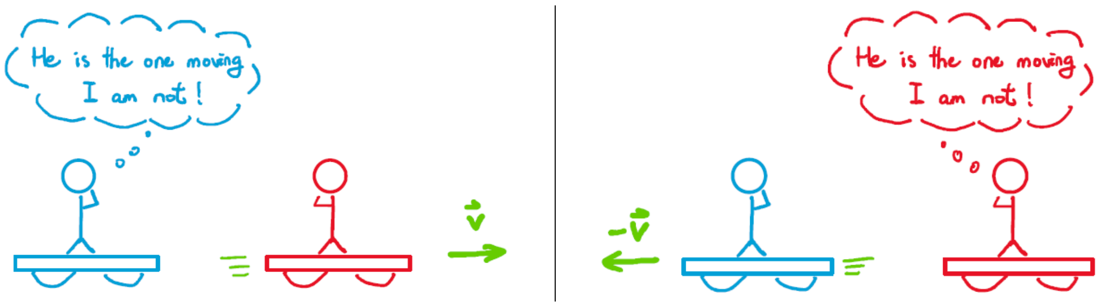
1.2.3 Deriving the Matrix

Let the two observers A, B differ in relative velocity v. Bear in mind that when there is no acceleration, observers cannot distinguish if it is the object moving relative to him or him moving relative to the object.

– A always thinks that B is the one moving, while A himself never moves.

– B always thinks that A is the one moving, while B himself never moves.

– If A sees B moving with velocity v, B sees A moving with velocity $-v$.



We are going to show that:

When B moves at velocity v relative to A

$$\underbrace{\begin{pmatrix} ct' \\ x' \end{pmatrix}}_{\text{An event's coordinate observed by B}} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}}_{\text{The Lorentz transformation matrix } \Lambda} \underbrace{\begin{pmatrix} ct \\ x \end{pmatrix}}_{\text{An event's coordinate observed by A}}$$

with $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\beta = \frac{v}{c}$.

Proof

In general, A and B are no different other than they move with a speed v relative to each other. So the Lorentz transform between them should only be related to v. We can write

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \mathbf{\Lambda}(v) \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

The subscripts on p, q, r, s indicate that they are functions of v.

1. *The inverse of* $\mathbf{\Lambda}$ *must exist*
   We should always be able to transfrom from B's coordinates back to A's coordinates. Since B sees A moving at velocity $-v$, the inverse of $\mathbf{\Lambda}(v)$ should have the same expression but with all v changed to $-v$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \mathbf{\Lambda}^{-1}(v) \begin{pmatrix} ct' \\ x' \end{pmatrix} = \mathbf{\Lambda}(-v) \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} p_{-v} & q_{-v} \\ r_{-v} & s_{-v} \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

$$\boxed{\mathbf{\Lambda}^{-1}(v) = \mathbf{\Lambda}(-v)}$$

2. *From A transform to B*
   We have previously chosen their coordinate systems to coincide, i.e. $(ct, x) = (0, 0)$ is the same point as $(ct', x') = (0, 0)$. When some time T advanced in A's clock, A will see

   - A himself has not moved.
   - B's position changed to vT because A sees B moving with velocity v.

	A seen by A	B seen by A
When A's clock shows $t = 0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
When A's clock shows $t = T$	$\begin{pmatrix} cT \\ 0 \end{pmatrix}$	$\begin{pmatrix} cT \\ vT \end{pmatrix}$

   Multiplying Lorentz matrix to these coordinate will transform to what is seen by B:

	A seen by B	B seen by B
When A's clock shows $t = 0$	$\begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
When A's clock shows $t = T$	$\begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} \begin{pmatrix} cT \\ 0 \end{pmatrix}$	$\begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} \begin{pmatrix} cT \\ vT \end{pmatrix}$

Notice the bottom right entry - when B looks at himself, he should always see himself not moving, i.e. always at his origin ($x' = 0$).

$$\begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} \begin{pmatrix} cT \\ vT \end{pmatrix} = \begin{pmatrix} \textit{don't care} \\ \underline{\underline{0}} \end{pmatrix} \sim \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

This gives us the first relations between the Lorentz matrix's elements.

$$\begin{pmatrix} \cdots & \cdots \\ r_v & s_v \end{pmatrix} \begin{pmatrix} cT \\ vT \end{pmatrix} = \begin{pmatrix} \cdots \\ r_v cT + s_v vT \end{pmatrix} = \begin{pmatrix} \cdots \\ 0 \end{pmatrix}$$

$$\boxed{r_v = -s_v \left(\frac{v}{c}\right)}$$

3. *From B transform to A*

We can interchange the A, B's roles by switching $x' \leftrightarrow x$, $t' \leftrightarrow t$, and $v \leftrightarrow -v$ to repeat the previous step. When some time T' advanced in B's clock, B will see

- B himself has not moved.
- A's position changed by $-vT'$ because B sees A moving with velocity $-v$.

	A seen by B	B seen by B
When B's clock shows $t' = 0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
When B's clock shows $t' = T'$	$\begin{pmatrix} cT' \\ -vT' \end{pmatrix}$	$\begin{pmatrix} cT' \\ 0 \end{pmatrix}$

Multiplying these coordinate with the *inverse* Lorentz matrix, i.e. use $-v$ instead of v, will transform to what is seen by A:

	A seen by A	B seen by A
When B's clock shows $t' = 0$	$\begin{pmatrix} p_{-v} & q_{-v} \\ r_{-v} & s_{-v} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} p_{-v} & q_{-v} \\ r_{-v} & s_{-v} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
When B's clock shows $t' = T'$	$\underline{\underline{\begin{pmatrix} p_{-v} & q_{-v} \\ r_{-v} & s_{-v} \end{pmatrix} \begin{pmatrix} cT' \\ -vT' \end{pmatrix}}}$	$\begin{pmatrix} p_{-v} & q_{-v} \\ r_{-v} & s_{-v} \end{pmatrix} \begin{pmatrix} cT' \\ 0 \end{pmatrix}$

Notice the bottom left entry - when A looks at himself, he should always see himself not moving, i.e. always at his origin ($x = 0$).

$$\begin{pmatrix} p_{-v} & q_{-v} \\ r_{-v} & s_{-v} \end{pmatrix} \begin{pmatrix} cT' \\ -vT' \end{pmatrix} = \begin{pmatrix} \textit{don't care} \\ \underline{\underline{0}} \end{pmatrix} \sim \begin{pmatrix} ct \\ x \end{pmatrix}$$

This gives us the second relations between the Lorentz matrix's elements.

$$\begin{pmatrix} \cdots & \cdots \\ r_{-v} & s_{-v} \end{pmatrix} \begin{pmatrix} cT' \\ vT' \end{pmatrix} = \begin{pmatrix} \cdots \\ r_{-v}cT' - s_{-v}vT' \end{pmatrix} = \begin{pmatrix} \cdots \\ 0 \end{pmatrix}$$

$$\boxed{r_{-v} = s_{-v}\left(\frac{v}{c}\right)}$$

4. <u>*(Matrix)* $\times$ *(Its inverse)* $= \mathbf{I}$</u>
Substitute the results from step 2 and 3 to $\mathbf{\Lambda}$ and $\mathbf{\Lambda}^{-1}$, then multiply them:

$$\mathbf{\Lambda}^{-1}\mathbf{\Lambda} = \begin{pmatrix} p_{-v} & q_{-v} \\ s_{-v}\left(\frac{v}{c}\right) & s_{-v} \end{pmatrix} \begin{pmatrix} p_v & q_v \\ s_v\left(-\frac{v}{c}\right) & s_v \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

The bottom left entry gives us a relation between p and s.

$$\begin{pmatrix} \cdots & \cdots \\ s_{-v}\left(\frac{v}{c}\right) & s_{-v} \end{pmatrix} \begin{pmatrix} p_v & \cdots \\ s_v\left(-\frac{v}{c}\right) & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots \\ 0 & \cdots \end{pmatrix}$$

$$s_{-v}\left(\frac{v}{c}\right)p_v + s_{-v}s_v\left(-\frac{v}{c}\right) = 0$$

$$\boxed{p_v = s_v}$$

5. <u>*Principle of constant light speed*</u>
Both A and B should see a light beam travelling at speed $= c$.

	Light beam seen by A			Light beam seen by B
When A's clock shows $t = 0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	AND	When B's clock shows $t' = 0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
When A's clock shows $t = T$	$\begin{pmatrix} cT \\ cT \end{pmatrix}$		When B's clock shows $t' = T'$	$\begin{pmatrix} cT' \\ cT' \end{pmatrix}$

These tables should be true for ANY value of T and T'. We can choose the value of T' such that the light beam being observed by both A and B as the same event.

	Light beam seen by A	Light beam seen by B
When A's clock shows $t = 0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
When A's clock shows $t = T$	$\begin{pmatrix} cT \\ cT \end{pmatrix}$	$\begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} \begin{pmatrix} cT \\ cT \end{pmatrix} = \begin{pmatrix} cT' \\ cT' \end{pmatrix}$

This gives us a relation between q and s:

$$\begin{cases} (p_v + q_v)cT = cT' \\ (r_v + s_v)cT = cT' \end{cases}$$

In the previous steps, we have already found $p_v = s_v$ and $r_v = -s_v\left(\frac{v}{c}\right)$ so it remains

$$\boxed{q_v = r_v = -s_v\left(\frac{v}{c}\right)}$$

6. <u>*Choose* $\det(\mathbf{\Lambda}) = 1$</u>
   So far we have found each entry in the Lorentz transformation matrix in terms of s_v.

$$\mathbf{\Lambda} = \begin{pmatrix} p_v & q_v \\ r_v & s_v \end{pmatrix} = \begin{pmatrix} s_v & -s_v\left(\frac{v}{c}\right) \\ -s_v\left(\frac{v}{c}\right) & s_v \end{pmatrix} = \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} s_v$$

   All we are left is the form of s_v. From its inverse property $\mathbf{\Lambda}^{\text{-}1}(v) = \mathbf{\Lambda}(-v)$ and $\mathbf{\Lambda}^{\text{-}1}\mathbf{\Lambda} = \mathbf{I}$,

$$\mathbf{\Lambda}^{\text{-}1}\mathbf{\Lambda} = \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} s_{-v} \cdot \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} s_v = s_v s_{-v} \begin{pmatrix} 1 - \frac{v^2}{c^2} & 0 \\ 0 & 1 - \frac{v^2}{c^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

   s_v can be ANY functional form as long as $\boxed{s_v s_{-v} = 1 - \frac{v^2}{c^2}}$. For example,

   - $s_v = s_{-v} = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$
   - $s_v = \dfrac{1}{1 - \frac{v}{c}}$ and $s_{-v} = \dfrac{1}{1 + \frac{v}{c}}$
   - ...

   Out of all the choice, we are choosing the s_v which makes $\det(\mathbf{\Lambda}) = 1$:

$$1 = \det(\mathbf{\Lambda}) = s_v^2 \cdot \left(1 - \frac{v^2}{c^2}\right)$$

$$\boxed{s_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

   The true reason behind this choice is because we want to construct certain "spacetime" invariants that carry physical meaning. We shall explain more in later sections.

As a conclusion, we have derived the Lorentz transformation matrix as the form:

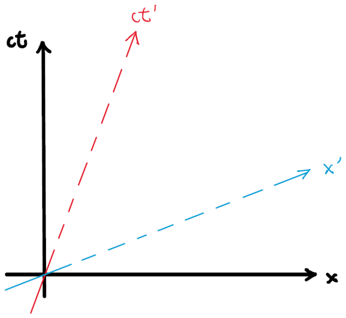
$$\boxed{\mathbf{\Lambda}(v) \overset{\text{def}}{=} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{pmatrix} 1 & -\dfrac{v}{c} \\ -\dfrac{v}{c} & 1 \end{pmatrix} \overset{\text{def}}{=} \gamma_v \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}_v = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v}$$

The conventional form is denoted by these letters:

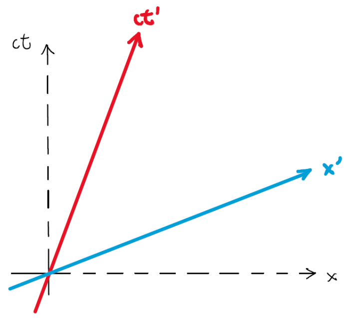
$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \text{and} \quad \beta = \frac{v}{c}$$

1.2.4 Reading Minkowski Diagram

The Minkowski diagram can be used to read the coordinate's value of the same point (event) according to different observer.

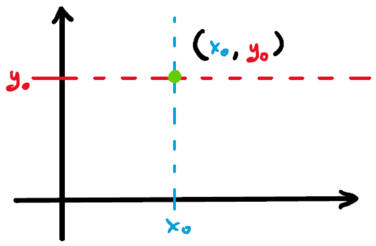


Coordinates seen by one observer uses the perpendicular axis

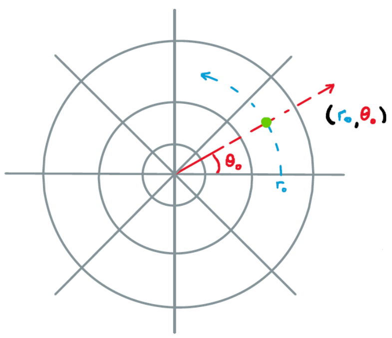


Coordinates seen by another observer uses the sloped axis

To read the value on the sloped axis, we can draw a line that is "normal" to that axis and tells by the intercept. Just like how we read the coordinates in rectangular and polar coordinate.

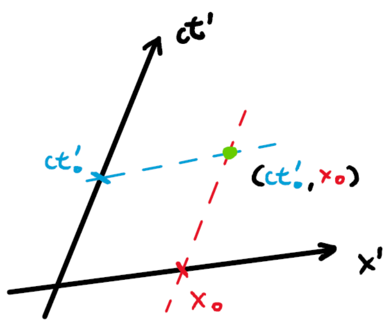


- Find x coordinate: Draw a line normal to x-axis
  $\Rightarrow$ The line needs to be parallel to y-axis
- Find y coordinate: Draw a line normal to y-axis
  $\Rightarrow$ The line needs to be parallel to x-axis



- Find r coordinate: Draw a line "normal" to r-axis
  $\Rightarrow$ The line is a circular arc, "parallel" to θ-axis
- Find θ coordinate: Draw a line "normal" to θ-axis
  $\Rightarrow$ The line is a radial line, "parallel" to r-axis

It is similar for Minkowski diagram, to read the coordinate on the sloped axis:



- Find t' coordinate: Draw a line "normal" to θ-axis
  $\Rightarrow$ The line is a radial line, "parallel" to r-axis
- Find x' coordinate: Draw a line "normal" to x'-axis
  $\Rightarrow$ The line has the same slope as the t'-axis

We can also determine that the angle between t, t' axes (or between x, x' axes) to be

$$\tan\theta = \frac{vt}{ct} \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{v}{c}\right) = \tan^{-1}\beta$$

2 Relativistic Phenomena

In the following section, we are going to examine these 4 phenomena with the matrix method:

- Time dilation
- Length contraction
- Relative velocity addition under relativity
- Relativistic Doppler effect

2.1 Time Dilation

The standard setup is to have 2 events: "1" and "2" described by two observers A, B:

- A is the **"co-moving"** observer - he sees the two events happen at the same position x, but different time $t = t_1$ and $t = t_2$.
- B is the **"moving"** observer - he moves at velocity v relative to the co-moving observer.

We can tabulate the spacetime coordinate of the two events as

	Seen by A	Seen by B
Event 1	$\begin{pmatrix} ct_1 \\ x \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} ct_1 \\ x \end{pmatrix}$
Event 2	$\begin{pmatrix} ct_2 \\ x \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} ct_2 \\ x \end{pmatrix}$
Difference in coordinates	$\begin{pmatrix} c(t_2 - t_1) \\ \underline{\underline{0}} \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} c(t_2 - t_1) \\ 0 \end{pmatrix}$

This explains what actually happens when we describe time dilation:

- If the "co-moving" observer sees two events happen with a time difference $t_2 - t_1$ in between,

$$\begin{pmatrix} c \cdot \Delta t \\ \Delta x \end{pmatrix}_A = \begin{pmatrix} c \cdot \underline{(t_2 - t_1)} \\ 0 \end{pmatrix}$$

- Any other moving observer, with a speed v relative to the co-moving observer, will see a time difference $\gamma_v(t_2 - t_1)$.

$$\begin{aligned} \begin{pmatrix} c \cdot \Delta t \\ \Delta x \end{pmatrix}_B &= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} c(t_2 - t_1) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_v c(t_2 - t_1) \\ -\gamma_v v(t_2 - t_1) \end{pmatrix} \\ &= \begin{pmatrix} c \cdot \gamma_v \cdot \underline{\text{(Time diff. seen by A)}} \\ \textit{Something } \neq 0 \end{pmatrix} \end{aligned}$$

Because $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \geq 1$, this effect is described as **time dilation**:

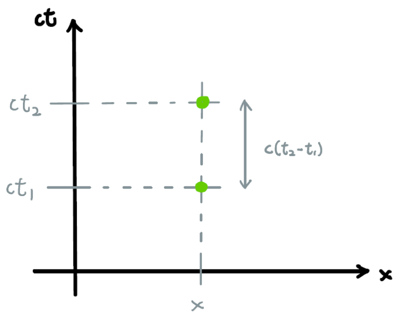
Time difference Δt measured by a co-moving observer is always the shortest, while other observer will measure a "longer" time difference $\gamma \Delta t$.

You should remember that although the time scale is changed, it comes with a side effect:

- The events happens at the same position in according to the co-moving observer,
- But the positions are different according to other observers.

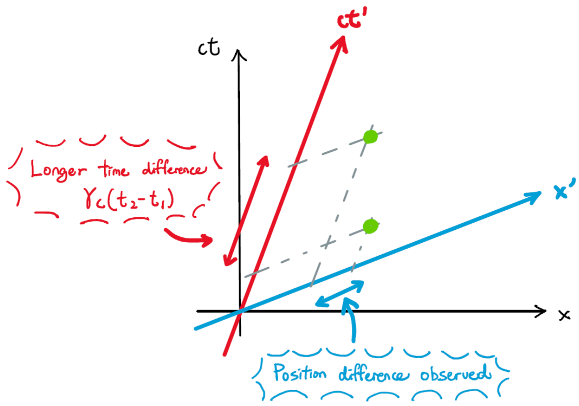
We can visualize this effect on the Minkowski diagram:

Co-moving observer sees:



Two events happen at the same location

Moving observer sees:



Side note:

In most introductory textbooks, "proper" is the word to describe what means by "co-moving" in this note.

- **Proper observer** = The co-moving observer.
- **Proper time** = The time difference / time scale of the co-moving observer.
- **Proper length** = The position difference / length scale of the co-moving observer, i.e. "rest" length.

However, I personally prefer saying "co-moving" because it is the most accurate - the observer and the object are literally "moving together".

- "Proper" is not a good adjective about motions (What is a "proper" motion?).
- "Rest" is just wrong, because the objects are not really at rest.

2.2 Length Contraction

The standard setup is by observing the two endings of a rod, labeled as "1" and "2", by two observers A, B:

- A is the **"co-moving"** observer - he moves together with the rod - the two ends of the rod are always at the same position $x = x_1$ and $x = x_2$, at any time t.
- B is the **"moving"** observer - he moves at velocity v relative to the co-moving observer.

We can tabulate the spacetime coordinate of the two endings of the rod as

	Seen by A	Seen by B
A reads position of Ending 1	$\begin{pmatrix} ct \\ x_1 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} ct \\ x_1 \end{pmatrix}$
A reads position of Ending 2	$\begin{pmatrix} ct \\ x_2 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} ct \\ x_2 \end{pmatrix}$
Difference in coordinates	$\begin{pmatrix} 0 \\ x_2 - x_1 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} 0 \\ x_2 - x_1 \end{pmatrix}$

Here analyze what are observed:

- If the "co-moving" observer checks the position of the two endings at the same time, getting a length measurement to the rod as $x_2 - x_1$:

$$\begin{pmatrix} c \cdot \Delta t \\ \Delta x \end{pmatrix}_A = \begin{pmatrix} 0 \\ \underline{x_2 - x_1} \end{pmatrix}$$

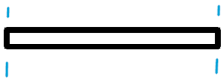
- Then according to the other observer, the co-moving observers checks the positions of the two endings at different positions and different time:

$$\begin{aligned}
\begin{pmatrix} c \cdot \Delta t \\ \Delta x \end{pmatrix}_B &= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} 0 \\ x_2 - x_1 \end{pmatrix} \\
&= \begin{pmatrix} -\gamma_v\beta(x_2 - x_1) \\ \gamma_v(x_2 - x_1) \end{pmatrix} \\
&= \begin{pmatrix} \textit{Something } \neq 0 \\ \underline{\gamma_v \cdot (\text{Length measured by A})} \end{pmatrix}
\end{aligned}$$

Obviously, the moving observer should not simply subtract his recorded positions to claim it as the measured length of the rod, because the records are taken at different time!

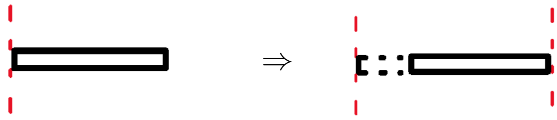
Co-moving observer sees:

Left/right record at the same time



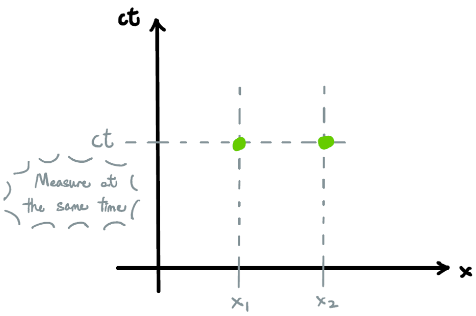
Moving observer sees:

Record left $\rightarrow$ rod moves $\rightarrow$ record right



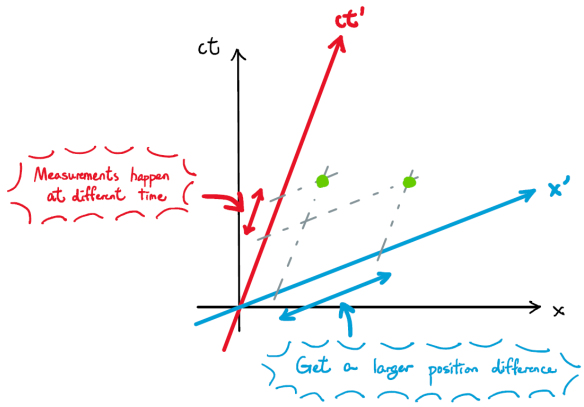
We may see it clearer with Minkowski diagram:

Co-moving observer sees:



Two measurements happen at the same time

Moving observer sees:



For B to take correct measurement, we require his measurements to be taken at the same time. Then we can use the inverse Lorentz transform to tell what is observed by A:

	Seen by A	Seen by B
B reads position of Ending (1)	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} ct' \\ x_1' \end{pmatrix}$	$\begin{pmatrix} ct' \\ x_1' \end{pmatrix}$
B reads position of Ending (2)	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} ct' \\ x_2' \end{pmatrix}$	$\begin{pmatrix} ct' \\ x_2' \end{pmatrix}$
Difference in coordinates	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} 0 \\ x_2' - x_1' \end{pmatrix}$	$\begin{pmatrix} \underline{\underline{0}} \\ x_2' - x_1' \end{pmatrix}$

But note that A is the co-moving observer - he will always find the position of Ending (1) at $x = x_1$ and Ending (2) at $x = x_2$, at ANY time! So we must have

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} ct' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cdots \\ \underline{\underline{x_1}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} ct' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cdots \\ \underline{\underline{x_2}} \end{pmatrix}$$

The difference in coordinates give

$$\begin{pmatrix} c \cdot \Delta t \\ \Delta x \end{pmatrix}_A = \begin{pmatrix} \cdots \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} 0 \\ x_2' - x_1' \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma_{-v}\beta(x_2' - x_1') \\ \gamma_{-v}(x_2' - x_1') \end{pmatrix}$$

$$= \begin{pmatrix} \textit{Something} \ \neq 0 \\ \underline{\gamma_{-v} \cdot (\text{Length measured by B})} \end{pmatrix}$$

i.e. If the co-moving observer measure a length $x_2 - x_1$, any moving observer will measure a length $\frac{1}{\gamma_{-v}}(x_2 - x_1)$. Because $\frac{1}{\gamma_v} = \sqrt{1 - \frac{v^2}{c^2}} \leq 1$, this effect is described as **length contraction**:

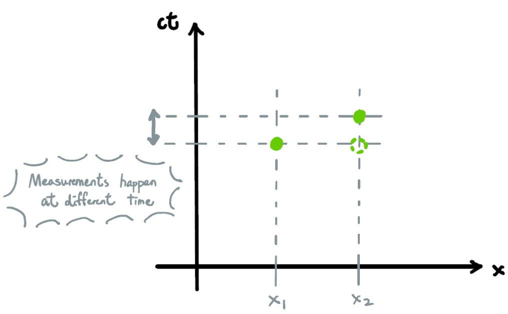
Position difference Δx measured by a co-moving observer is always the longest, while other observer will measure a "shorter" time difference $\frac{1}{\gamma}\Delta x$.

You should remember that although the length scale is changed, it comes with a side effect:

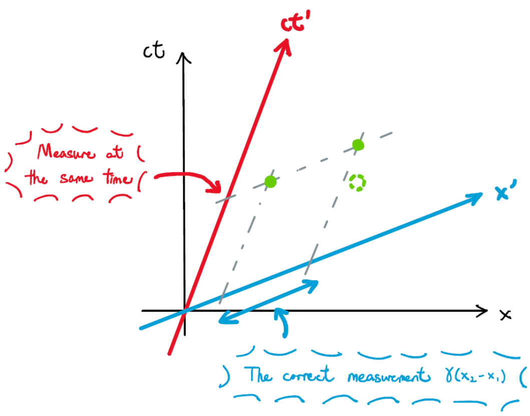
- The positions of endings are recorded at the same time in according to the moving observer,
- But the record time are different according to co-moving observer.

We can visualize this effect using the Minkowski diagram:

Co-moving observer sees: | Moving observer sees:



Positions of the 2 ends never change



2.3 Velocity Addition

Given 3 observers who are moving relative to each other:

- B is moving at velocity v relative to A
- C is moving at velocity u relative to B
- C is moving at velocity w relative to A

What are the relations between v, u and w? We can express the coordinate of C in terms of B's axes (ct', x'), and then transform it through $\mathbf{\Lambda}_{-v}$ to what is observed by A:

	C seen by B	C seen by A
When B's clock shows $t' = 0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
When B's clock shows $t' = T'$	$\begin{pmatrix} cT \\ uT \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} cT \\ uT \end{pmatrix} = \begin{pmatrix} \gamma_v cT + \gamma_v \frac{vu}{c} T \\ \gamma_v vT + \gamma_v uT \end{pmatrix}$

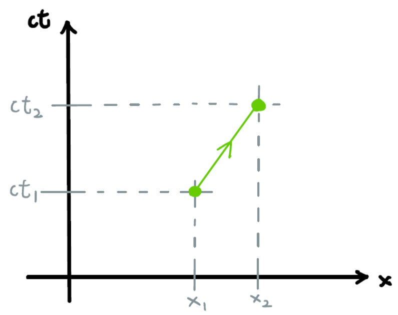
Since C is moving at velocity w relative to A,

$$
\begin{aligned}
w &= \frac{\text{(Change in position seen by A)}}{\text{(Change in time seen by A)}} \\
&= \frac{\gamma_v vT + \gamma_v uT}{\gamma_v cT + \gamma_v \frac{vu}{c^2} T}
\end{aligned}
$$

$$
\boxed{w = \frac{v+u}{1+\frac{vu}{c^2}}}
$$

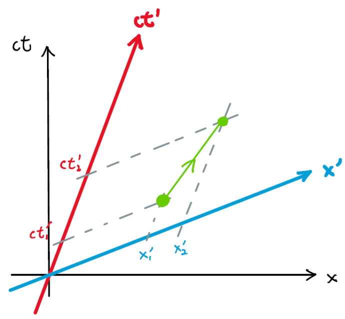
This is the relative velocity addition formula in relativity.

Trajectory of C seen by A



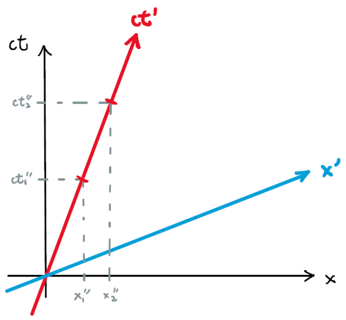
Distance travel $= x_2 - x_1$
Time spent $= t_2 - t_1$

Trajectory of C seen by B



Distance travel $= x'_2 - x'_1$
Time spent $= t'_2 - t'_1$

Trajectory of B seen by A



Distance travel $= x''_2 - x''_1$
Time spent $= t''_2 - t''_1$

In Minkowski diagram,
speed is calculated as $\frac{\Delta x}{\Delta t} = \frac{c}{\text{Slope}}$ of a line.

Geometrically, velocity addition formula
is the relation between the slopes
relative to different axes.

Side note:

Alternatively, we can show the velocity addition formula by using Lorentz matrix as a tool to switch frame of reference.

– For any coordinate observed by A, we can multiply $\mathbf{\Lambda}_v$ to change it into what is observed by B.

$$\begin{pmatrix} ct_B \\ x_B \end{pmatrix} = \mathbf{\Lambda}_v \begin{pmatrix} ct_A \\ x_A \end{pmatrix}$$

– For any coordinate observed by B, we can multiply $\mathbf{\Lambda}_u$ to change it into what is observed by C.

$$\begin{pmatrix} ct_C \\ x_C \end{pmatrix} = \mathbf{\Lambda}_u \begin{pmatrix} ct_B \\ x_B \end{pmatrix}$$

– For any coordinate observed by A, we can multiply $\mathbf{\Lambda}_w$ to change it into what is observed by C.

$$\begin{pmatrix} ct_C \\ x_C \end{pmatrix} = \mathbf{\Lambda}_w \begin{pmatrix} ct_A \\ x_A \end{pmatrix}$$

This gives a relation between different $\mathbf{\Lambda}$:

$$\begin{pmatrix} ct_C \\ x_C \end{pmatrix} = \mathbf{\Lambda}_w \begin{pmatrix} ct_A \\ x_A \end{pmatrix} = \mathbf{\Lambda}_u \mathbf{\Lambda}_v \begin{pmatrix} ct_A \\ x_A \end{pmatrix}$$

$$\mathbf{\Lambda}_w = \mathbf{\Lambda}_u \mathbf{\Lambda}_v$$

We can use any of the entry to reach the velocity addition formula, for example,

$$\begin{pmatrix} \gamma_w & \cdots \\ \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \gamma_u & -\gamma_u \beta_u \\ \cdots & \cdots \end{pmatrix} \begin{pmatrix} \gamma_v & \cdots \\ -\gamma_v \beta_v & \cdots \end{pmatrix}$$

$$\begin{aligned} \gamma_w &= \gamma_u \gamma_v + \gamma_u \beta_u \gamma_v \beta_v \\ \frac{1}{\sqrt{1-\frac{w^2}{c^2}}} &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \left(1 + \frac{uv}{c^2}\right) \\ \Rightarrow \quad w &= \frac{u+v}{1+\frac{uv}{c^2}} \end{aligned}$$

2.4 Relativistic Doppler Effect

Relativistic Doppler effect is only applicable to light - explaining what will be observed by different observers for an object that *by definition* travels at the same speed to all observer. For objects that travels slower than light, you should use the classical Doppler effect formula.

Recall the terminologies in wave:

- Period T = Time separation between each pulse emission.
- Wavelength λ = Distance travelled of the pulse within 1 period of time.

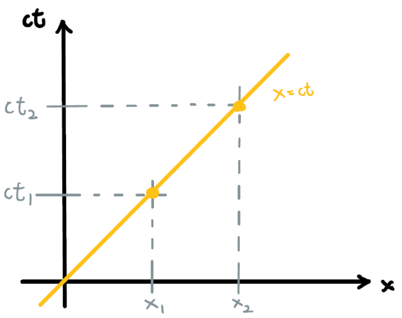
For light, its travelling speed c is a constant to all observers. This makes $\lambda = cT$ and $\lambda' = cT'$. With a table we can see
Therefore the observed wavelength is

$$\begin{aligned}\lambda' &= \gamma(1-\beta)cT = \gamma(1-\beta)\lambda \\ &= \frac{1-\beta}{\sqrt{1-\beta^2}}\lambda \\ &= \sqrt{\frac{1-\beta}{1+\beta}}\lambda\end{aligned}$$

and period is

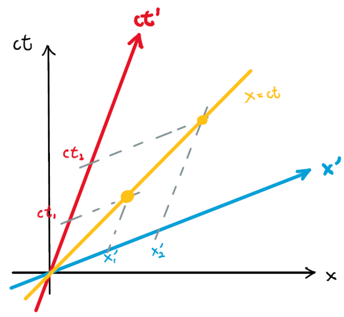
$$\begin{aligned}T' &= \frac{\lambda'}{c} \\ &= \sqrt{\frac{1-\beta}{1+\beta}}\frac{\lambda}{c} \\ &= \sqrt{\frac{1-\beta}{1+\beta}}T\end{aligned}$$

Trajectory of light seen by A



Distance travel = $x_2 - x_1$
Time spent = $t_2 - t_1$

Trajectory of light seen by B



Distance travel = $x_2' - x_1'$
Time spent = $t_2' - t_1'$

— The End —