- Deriving Lorentz Transformation matrix
- Explaining relativistic phenomenon: Time dilation, Length contraction, Velocity Addition
- Energy - momentum 4 vector
- Spacetime interval

Einstein's 2 Postulate

① Principle of relativity

- All physics must be the same in any inertial frame (observer is not in acceleration)
- There is no absolute velocity

② Light speed is invariant, i.e. constant in all reference frame

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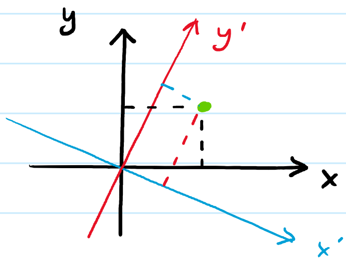
Matrix as Linear Transformation

We can use matrix to calculate linear coordinate transformation

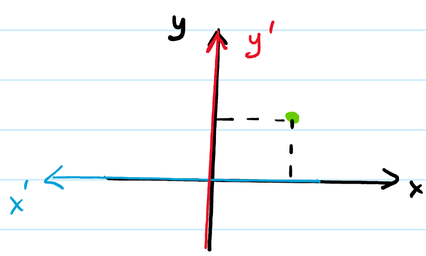
E.g. 1 Rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



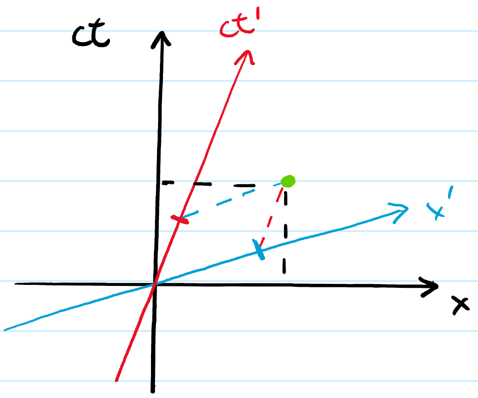
E.g. 2 Reflection

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



In special relativity, Lorentz transformation can be expressed in a matrix form

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma_\beta \\ -\gamma_\beta & \gamma \end{pmatrix}}_{\text{The Lorentz matrix}} \underbrace{\begin{pmatrix} ct \\ x \end{pmatrix}}_{\text{Spacetime Coordinate}} \quad \text{with} \quad \begin{cases} \gamma = \dfrac{1}{\sqrt{1-\beta^2}} \\ \beta = \dfrac{v}{c} \end{cases}$$



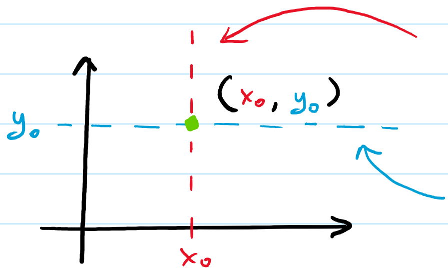
The coordinate transform can be graphically presented as the

Minkwoski Diagram

Note: To read the coordinate on a particular axis

We draw lines/surfaces that is "normal" to that axis

E.g.1 Rectangular Coordinate



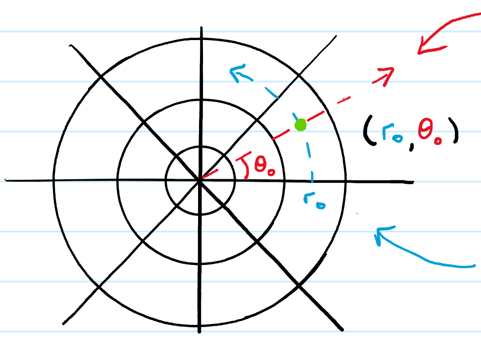
To find x coordinate, draw a line normal to x axis

$\Rightarrow$ i.e. a line parallel to y axis

To find y coordinate, draw a line normal to y axis

$\Rightarrow$ i.e. a line parallel to x axis

E.g.2 Polar Coordinate



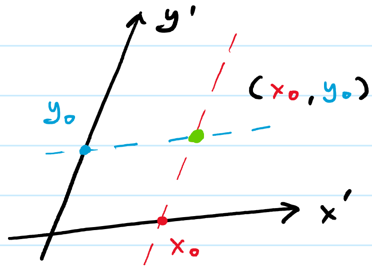
To find θ coordinate, draw a radial line which is always geometrically "normal" to $\hat{\theta}$ direction

$\Rightarrow$ i.e. a line "parallel" to r axis

To find r coordinate, draw an arc which is always geometrically "normal" to $\hat{r}$ direction

$\Rightarrow$ i.e. a line "parallel" to θ axis

Reading the coordinate from Minkwoski diagram is just similar



Deriving Lorentz Transformation Matrix

Definitions :

① Event ⇔ 4-vector

Every "event" is denoted by a 4 vector

$$\vec{X} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

← Time when it happens

position when it happens

② Observer ⇔ Coordinate System

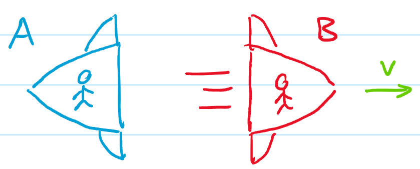
The same event is described by different 4-vector in the eyes of different observers

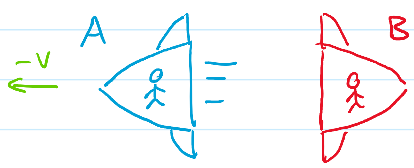
$$\vec{X}_A = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad \vec{X}_B = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

Both refer to the same point. But written in different coordinate system

Situation : 2 observers A, B on spaceship. Both are in inertial frames

no acceleration





A thinks he is stationary and B is moving relative to him

B thinks he is stationary and A is moving relative to him

Step 0.

Since x/y/z coordinate are symmetric, we can generalize

s.t. the spaceships are only moving in x directions.

$\Rightarrow$ y/z coordinates are the same in the eyes of both observer

$\Rightarrow$ Only need to write the transformation as

B's coordinate

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} p(v) & q(v) \\ r(v) & s(v) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \Lambda_v \begin{pmatrix} ct \\ x \end{pmatrix}$$

A's coordinate

p, q, r, s are some functions depending on v, velocity of B relative to A

Step 1.

Inverse of Lorentz matrix, Λ_v^{-1}, must exist and equal to Λ_{-v}, because

In A's viewpoint:

The coordinate read by B

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda_v \begin{pmatrix} ct \\ x \end{pmatrix}$$

An event in A's eyes

A sees B moving at velocity v, so he can substitute v in Λ

But in B's viewpoint:

The coordinate read by A

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \Lambda_{-v} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

An event in B's eyes

B sees A moving at velocity $-v$, so he can substitute $-v$ in Λ

So we get $\Lambda_{-v} \Lambda_v = I$

i.e. $$\begin{pmatrix} p(-v) & q(-v) \\ r(-v) & s(-v) \end{pmatrix} \begin{pmatrix} p(v) & q(v) \\ r(v) & s(v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Step 2.

Suppose A & B start moving at $(ct, x) = (ct', x') = (0,0)$

We can use a table to list out what they see after some time

In A's viewpoint

	Coordinate of A	Coordinate of B
Time $= 0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Time $= t$ according to A's clock	$\begin{pmatrix} ct \\ 0 \end{pmatrix}$ see himself not moving	$\begin{pmatrix} ct \\ vt \end{pmatrix}$ see B moving

In B's viewpoint

– Can be found by transforming by Λ_v

	Coordinate of A	Coordinate of B
Time $= 0$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_v \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_v \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Time $= t$ according to A's clock	$\begin{pmatrix} ? \\ ?? \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_v \begin{pmatrix} ct \\ 0 \end{pmatrix}$	$\begin{pmatrix} ??? \\ 0 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_v \begin{pmatrix} ct \\ vt \end{pmatrix}$

At the time when A look at B, he does not know what is his coordinate in B's eyes. But he can be sure that at any time B looks at himself, his position must be 0 because he always sees himself not moving

$\Rightarrow$ This gives a relation : $r(v)\, ct + s(v)\, vt = 0$

or $r(v) = -\frac{v}{c} s(v)$

Step 3.

Repeat step 2 but this time starts with B.

In B 's viewpoint

	Coordinate of A	Coordinate of B
Time = 0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Time = t' according to B 's clock	$\begin{pmatrix} ct' \\ \underline{-vt'} \end{pmatrix}$ see A moving	$\begin{pmatrix} ct' \\ \underline{0} \end{pmatrix}$ see himself not moving

In A 's viewpoint

	Coordinate of A	Coordinate of B
Time = 0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_{-v} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_{-v} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Time = t' according to B 's clock	$\begin{pmatrix} ??? \\ 0 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_{-v} \begin{pmatrix} ct' \\ -vt' \end{pmatrix}$	$\begin{pmatrix} ? \\ ?? \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_{-v} \begin{pmatrix} ct' \\ 0 \end{pmatrix}$

$\Rightarrow$ Get a similar relation $r(-v) = \frac{v}{c} s(-v)$

Step 4

Combine the results of steps 1, 2, 3

$$\begin{pmatrix} p(-v) & q(-v) \\ \frac{v}{c} s(-v) & s(-v) \end{pmatrix} \begin{pmatrix} p(v) & q(v) \\ -\frac{v}{c} s(v) & s(v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives total 4 equations. 2 of them give us info about p & q

From $\left(\boxed{}\right)\left(\boxed{}\right) = \left(\boxed{}\right)$

$$\frac{v}{c} s(-v) \cdot p(v) + s(-v) \cdot \frac{-v}{c} s(v) = 0$$

$$\underline{p(v) = s(v)}$$

From $\left(\ \square\ \right)\left(\ \square\ \right) = \left(\ \square\ \right)$

$$\frac{v}{c} s(-v) \cdot q(v) + s(v)\, s(-v) = 1$$

$$q(v) = \frac{(1 - s(v)s(-v))\, c}{s(-v) \cdot v}$$

Step 5.

Apply principle of constant light speed

Both A & B look at a light beam travelling at light speed c

In A's viewpoint : $\begin{pmatrix} ct \\ ct \end{pmatrix}$ In B's viewpoint : $\begin{pmatrix} ct' \\ ct' \end{pmatrix}$

Their coordinates can be transformed to each other by

$$\begin{pmatrix} ct' \\ ct' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}_v \begin{pmatrix} ct \\ ct \end{pmatrix}$$

$$\Rightarrow \begin{cases} ct' = (p(v) + q(v)) \cdot ct \\ ct' = (r(v) + s(v)) \cdot ct \end{cases}$$

Since $p(v) = s(v)$ (from step 4)

We must have $q(v) = r(v) = -\frac{v}{c} s(v)$

Step 6.

So far we reach the following

$$\Lambda_v = \begin{pmatrix} s(v) & -\frac{v}{c} s(v) \\ -\frac{v}{c} s(v) & s(v) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} s(v)$$

In fact there are infinitely many choices of $s(v)$ s.t. $\Lambda_{-v}\Lambda_v = 1$

The conventional choice is to make $\det(\Lambda_v) = \det(\Lambda_{-v}) = 1$

i.e. $\det(\Lambda) = s^2(v) \cdot \left(1 - \frac{v^2}{c^2}\right) = 1$

$$s(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

The reason why this is the choice, will be left until we introduced the concept of "spacetime interval"

As a conclusion, we derived the Lorentz transform matrix as

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} = \gamma\begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} = \underline{\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}}$$

The conventional notation with

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Explaining Phenomenon with Lorentz Matrix

① Time Dilation

Situation : 2 events ① & ②

2 observers static & moving

"Static" observer = see 2 events happen at the same position

"Moving" observer = Moving at velocity v relative to static observer

	In "static"'s coordinate	In "moving"'s coordinate
Event ①	$\begin{pmatrix} ct_1 \\ x \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} ct_1 \\ x \end{pmatrix}$
Event ②	$\begin{pmatrix} ct_2 \\ x \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} ct_2 \\ x \end{pmatrix}$
Difference	$\begin{pmatrix} c(t_2-t_1) \\ 0 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} c(t_2-t_1) \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma c(t_2-t_1) \\ \text{something} \neq 0 \end{pmatrix}$

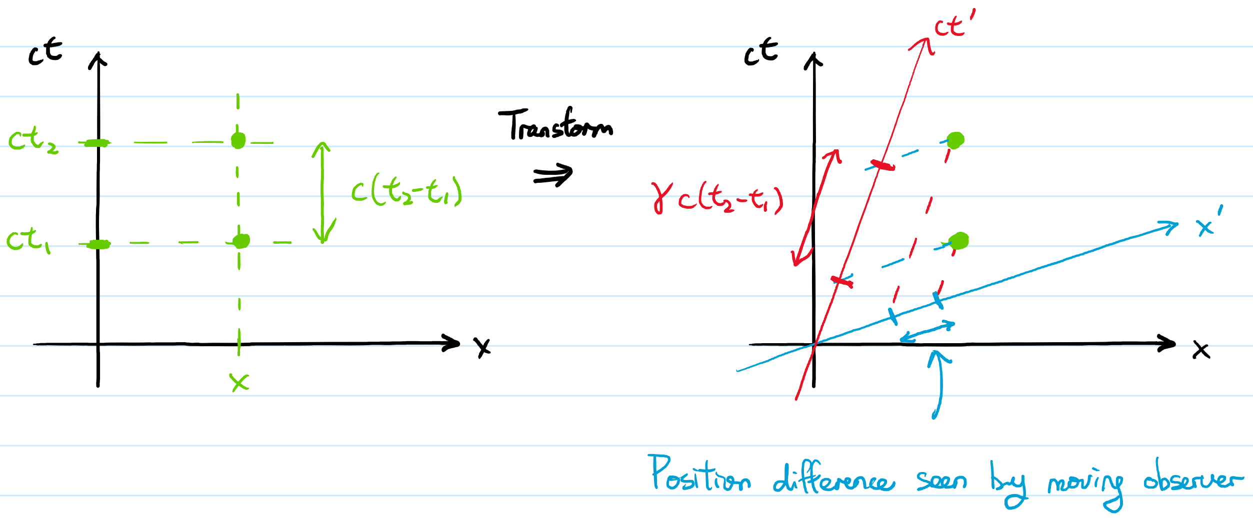
⇒ Description of time dilation :

If an observer sees 2 events $\begin{cases} \text{separate with duration } T \\ \text{happen at the same position} \end{cases}$

$\gamma \geq 1$

A moving observer sees them $\begin{cases} \text{separate with duration } \gamma T \\ \text{happen at different positions} \end{cases}$

Visualizing by Minkowski Diagram :



② Length Contraction

Situation : The 2 ends of a rod ① & ②

2 observers Co-moving & Moving

"Co-moving" observer = Move together with the rod

⇒ The rod is not moving in his eyes

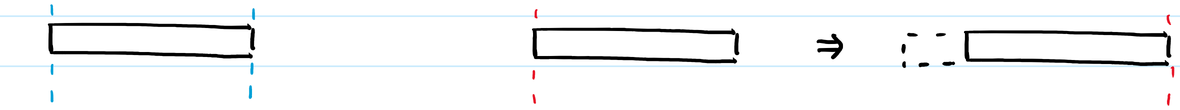
"Moving" observer = Moving at velocity v relative to co-moving observer

	In "Co-moving"'s coordinate	In "moving"'s coordinate
"Co-moving" Read position of End ①	$\begin{pmatrix} ct \\ x_1 \end{pmatrix}$ Read at the same time	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} ct \\ x_1 \end{pmatrix}$
"Co-moving" Read position of End ②	$\begin{pmatrix} ct \\ x_2 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} ct \\ x_2 \end{pmatrix}$
Difference	$\begin{pmatrix} 0 \\ x_2 - x_1 \end{pmatrix}$	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} \text{something} \neq 0 \\ \gamma (x_2 - x_1) \end{pmatrix}$

Problem: When measuring length, we need to read the positions of the 2 ends at the same time

Co-moving Observer

Moving Observer



Positions of the 2 ends are marked at the same time

Position of 1 end is maked first

Position of another end is marked after the rod has moved

How to fix: Require time difference for moving to be 0

	In "Co-moving"'s coordinate	In "moving"'s coordinate
"Moving" Read position of End ①	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} ct' \\ x_1' \end{pmatrix} = \begin{pmatrix} \text{some } t_1 \\ x_1 \end{pmatrix}$	$\begin{pmatrix} ct' \\ x_1' \end{pmatrix}$
"Moving" Read position of End ②	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} ct' \\ x_2' \end{pmatrix} = \begin{pmatrix} \text{some } t_2 \\ x_2 \end{pmatrix}$	$\begin{pmatrix} ct' \\ x_2' \end{pmatrix}$
Difference	$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} 0 \\ x_2' - x_1' \end{pmatrix} = \begin{pmatrix} \text{something} \neq 0 \\ x_2 - x_1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ x_2' - x_1' \end{pmatrix}$

The rod "never moves" in the eye of co-moving observer

$\Rightarrow$ Position of the 2 ends are always & x_1 & x_2

This gives the relation $\gamma_{-v} \cdot (x_2' - x_1') = x_2 - x_1$

$\hookrightarrow$ But $\gamma_v = \gamma_{-v}$

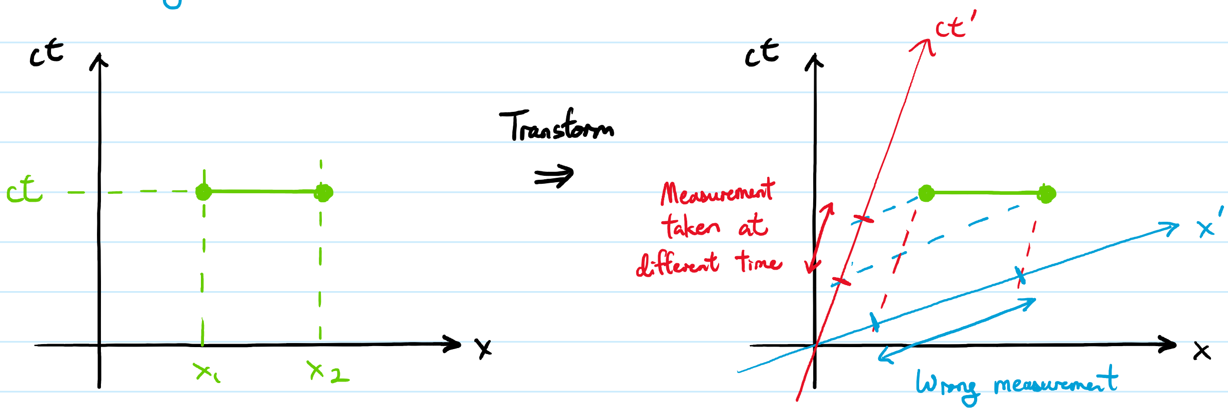
$\Rightarrow$ Description of length contraction :

If a moving observer $\begin{cases} \text{measure length of object as } \frac{L}{\gamma} \leftarrow \gamma \geq 1 \\ \text{take measurement } \underline{\text{at the same time}} \end{cases}$

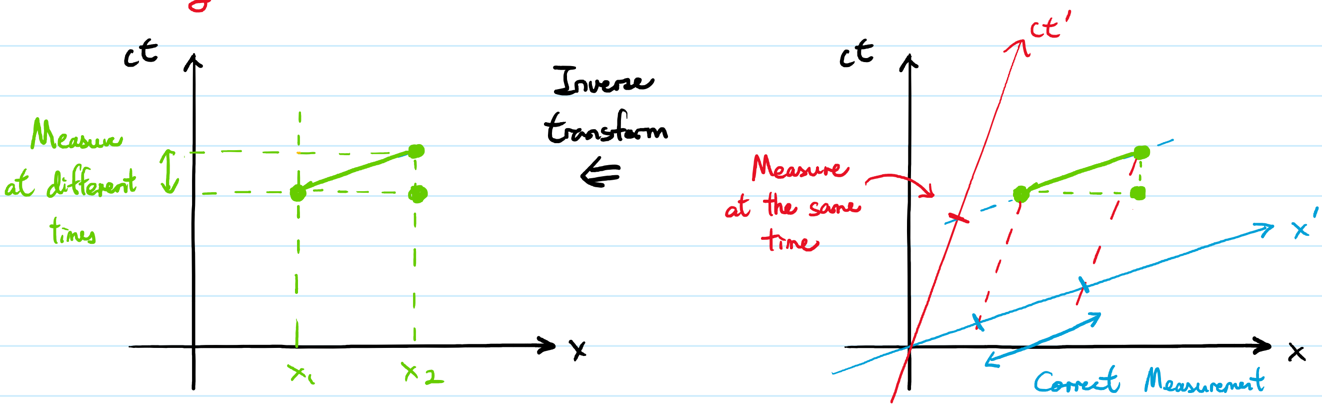
The co-moving observer to the object $\begin{cases} \text{measure length of object as } L \\ \text{see moving observer taking measurement } \underline{\text{at different time}} \end{cases}$

Visualizing by Minkwoski Diagram

Co-moving observer measure at the same time :



Moving observer measure at the same time :



Important note :

[1] In most textbooks , this "co-moving" observer is also called "static" because he is static relative to the rod. However this may be confusing when dealing with multiple objects. (e.g. "static" to which object ?)

[2] The more common name of "co-moving" observer is in fact, "proper" observer, or "proper" frame. But "co-moving" is kind of more accurate since it emphasizes the observer is moving together with the object.

③ Velocity Addition

Situation : Let A moving at velocity v relative to B

Let B moving at velocity u relative to C

What is the velocity of A relative to C ?

Suppose there happens an event, which A/B/C read the coordinate of this event as

$$A : \begin{pmatrix} ct_A \\ x_A \end{pmatrix} , \quad B : \begin{pmatrix} ct_B \\ x_B \end{pmatrix} , \quad C : \begin{pmatrix} ct_C \\ x_C \end{pmatrix}$$

These coordinates can be transformed to one and other through

- A moving at v relative to B

$$\Rightarrow \begin{pmatrix} ct_A \\ ct_B \end{pmatrix} = \Lambda_v \begin{pmatrix} ct_B \\ x_B \end{pmatrix}$$

- B moving at u relative to C

$$\Rightarrow \begin{pmatrix} ct_B \\ x_B \end{pmatrix} = \Lambda_u \begin{pmatrix} ct_C \\ x_C \end{pmatrix}$$

- A moving at ω relative to C (the unknown to be solved)

$$\Rightarrow \begin{pmatrix} ct_A \\ x_A \end{pmatrix} = \Lambda_\omega \begin{pmatrix} ct_C \\ x_C \end{pmatrix}$$

We can clearly see that

$$\begin{pmatrix} ct_A \\ x_A \end{pmatrix} = \Lambda_\omega \begin{pmatrix} ct_C \\ x_C \end{pmatrix} = \Lambda_v \Lambda_u \begin{pmatrix} ct_C \\ x_C \end{pmatrix}$$

So we must have $\Lambda_\omega = \Lambda_v \Lambda_u$. i.e.

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_\omega = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_u$$

We can use any entry to get the ans.

E.g. $\begin{pmatrix} \square & \\ & \end{pmatrix} = \begin{pmatrix} \boxed{\quad\quad} \\ \; \end{pmatrix}\begin{pmatrix} \boxed{\,} & \\ & \end{pmatrix}$

$$\Rightarrow \quad \gamma_\omega = \gamma_v \gamma_u + \gamma_v \beta_v \cdot \gamma_u \beta_u$$

$$\frac{1}{\sqrt{1-\frac{\omega^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \left(1 + \frac{uv}{c^2}\right)$$

$$\Rightarrow \quad \omega = \frac{u+v}{1+\frac{uv}{c^2}}$$

The 4-vector Family

We have used 4-vector to record the coordinate of events in space-time. They are actually called "4-position" vector $\vec{X} = \begin{pmatrix} ct \\ x \end{pmatrix}$

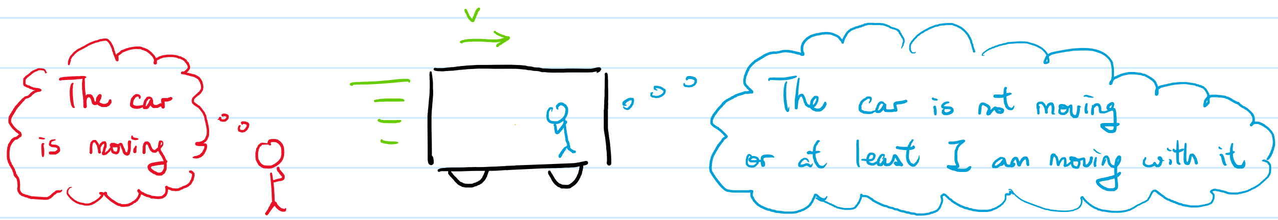
As a "position" vector, we can differentiate it w.r.t. time to get a "velocity" vector.

Question : W.r.t whose timeline?

Answer : In theory we can use any observer's timeline

But every object has a unique reference frame :

Its "Co-moving" (i.e. "proper") frame



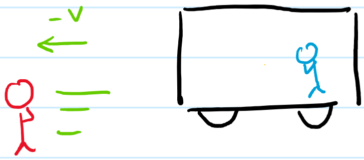
For convenience, we can always first differentiate w.r.t. the "proper time" τ, i.e. timeline of the co-moving observer, then use change of variable if we want to differentiate w.r.t other timeline

$$\frac{dA}{d\tau} = \text{rate of something w.r.t proper time}$$

$$\Rightarrow \frac{dA}{dt} = \frac{dA}{d\tau}\frac{d\tau}{dt} = \text{rate of something w.r.t. to other time}$$

And because any observer moving with velocity $-v$ relative to this object will read the coordinate of this object as

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} c\tau \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma c\tau \\ \gamma v\tau \end{pmatrix} = \begin{pmatrix} ct \\ vt \end{pmatrix}$$



The distance the object has moved, according to his time scale

Therefore $t = \gamma_v \tau$, and so $\dfrac{d\tau}{dt} = \dfrac{1}{\gamma_v}$

4-velocity

The 4-velocity of an object observed by its co-moving observer is

$$\vec{U} = \frac{d}{d\tau}\vec{X} = \frac{d}{d\tau}\begin{pmatrix} c\tau \\ \text{some const.} \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

Must be its proper time

In the eye of its co-moving observer
- Time scale unit $= \tau$
- The object never moves.

Then for an observer who sees the object moving at velocity v the object's 4-velocity:

$$\vec{U}' = \frac{d}{d\tau}\vec{X}' = \frac{d}{d\tau}\Lambda_{-v}\vec{X} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_{-v} \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v c \\ \gamma_v v \end{pmatrix}$$

Must be the object's proper time

Note: Differentiating w.r.t. to other timeline other than proper time are NOT 4-velocity. Even if the timeline is from the same moving observer.

Quantities like this do not have significant meaning.

$$\frac{d}{dt}\vec{X}' = \frac{d\tau}{dt}\cdot\frac{d\vec{X}}{d\tau} = \frac{1}{\gamma_v}\begin{pmatrix}\gamma_u c\\ \gamma_u v\end{pmatrix}$$

Timeline of 1 random observer
See object moving at velocity u

Position coordinate by a different observer
See object moving at velocity v

And if this timeline & position coordinate are from the same observer, we get trivial result

$$\vec{X}' = \begin{pmatrix}ct'\\ vt'\end{pmatrix} \Rightarrow \frac{d}{dt'}\vec{X}' = \begin{pmatrix}c\\ v\end{pmatrix} = \frac{1}{\gamma_v}\begin{pmatrix}\gamma_v c\\ \gamma_v v\end{pmatrix}$$

Still not 4-vector!

4-Momentum

With 4-velocity, we can define 4-momentum as

$$\vec{P} = m\vec{U} = m\frac{d}{d\tau}\vec{X}$$

m is called the "rest mass" of the object

We can directly see that

- For co-moving observer : $\vec{P} = \begin{pmatrix}mc\\ 0\end{pmatrix}$
- For another observer moving with relative velocity $-v$: $\vec{P} = \begin{pmatrix}\gamma_v mc\\ \gamma_v mv\end{pmatrix}$

Interpretation

① $p = \gamma_v mv$ = Relativistic momentum of the object

$$= \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \simeq mv\left(\underline{1} + \underline{\frac{1}{2}\frac{v^2}{c^2} + \cdots}\right)$$

Measurement is accurate up to this in Newtonian mechanics

Relativistic corrections

Note : Some people combine γ_v with m to become the "observed mass" $\gamma_v m$, and claim mass to be changing when velocity change,

And that's why m is specifically called "rest mass"

② $\frac{E}{c} = \gamma mc$ = Relativistic energy of the object

$$= \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} = mc\left(1+\frac{1}{2}\frac{v^2}{c^2}+\cdots\right)$$

$$= \frac{1}{c}\left(\underbrace{mc^2}_{\text{Some energy more fundamental than KE}} + \underbrace{\frac{1}{2}mv^2}_{\text{Newtonian KE}} + \underbrace{\cdots}_{\text{Relativistic correction}}\right)$$

↳ Identify mc^2 = "Rest energy"

i.e. Some intrinsic energy carried by objects even if the object is not moving

Note : Different textbook may have different convention on what "KE" they refer to, e.g.

- KE $= \frac{1}{2}mv^2$? (Newtonian KE)
- KE $= (\gamma-1)mc^2$? (Relativistic KE)

It may be confusing if one switches between textbooks.

Extra : 4 - Acceleration

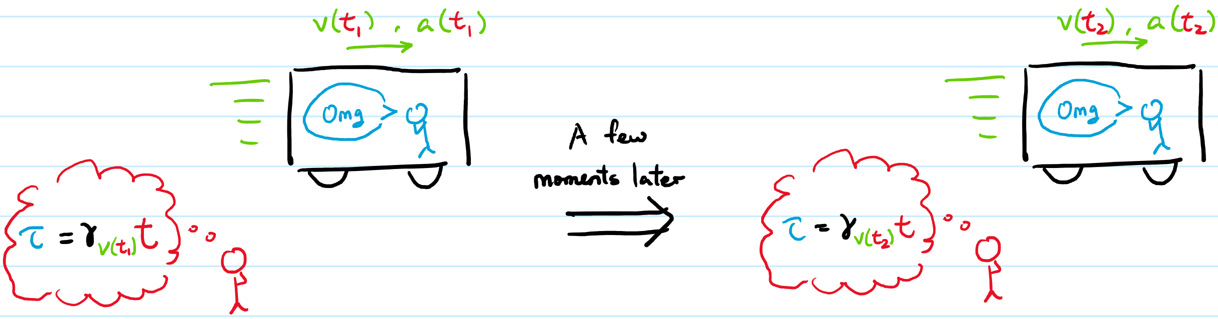
The 4-acceleration of an object is defined like in Newtonian

$$\vec{A} = \frac{d^2}{d\tau^2}\vec{X}$$

Problem 1 : The co-moving observer is in a non-inertial frame

His time scale is no longer maintaining at a constant

ratio with other inertial frame observer.



Problem 2 : The "co-moving" observer always read his

position as constant. So by differenting :

$$\frac{d^2}{d\tau^2}\vec{X} = \frac{d^2}{d\tau^2}\begin{pmatrix} c\tau \\ \text{some const.} \end{pmatrix} = \frac{d}{d\tau}\begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Apply Lorentz transform always give $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Contradiction?

Solution : This time we start with the coordinate read by

some random inertial frame observer. He reads

the object's movement using his own timeline t

$$\left.\begin{array}{l} - \text{Trajectory} = x(t) \\ - \text{Velocity} = v(t) \\ - \text{Acceleration} = a(t) \end{array}\right\} \text{Just the same as in Newtonian mechanics}$$

[1] Express $\vec{A}$ as a function of t

$$\vec{A} = \frac{d^2}{d\tau}\vec{X} = \frac{d}{dt}\left(\frac{d}{dt}\vec{X}\cdot\frac{dt}{d\tau}\right)\cdot\frac{dt}{d\tau}$$

$$= \frac{d}{dt}\left(\frac{d}{dt}\vec{X}\cdot\gamma_v\right)\cdot\gamma_v$$

$\because$ γ_v is a function of t (i.e. $\gamma_{v(t)}$)

$$\Rightarrow \vec{A} = \left(\frac{d^2}{dt^2}\vec{X}\right)\gamma^2_{v(t)} + \left(\frac{d}{dt}\vec{X}\right)\cdot\left(\frac{d}{dt}\gamma_{v(t)}\right)\cdot\gamma_{v(t)}$$

Then by $\frac{d}{dt}\gamma = \frac{d}{dt}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{v}{c^2}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}\frac{dv}{dt} = \gamma^3\cdot\frac{v}{c^2}\frac{dv}{dt}$

$$\Rightarrow \vec{A} = \gamma^2_{v(t)}\left(\frac{d^2}{dt^2}\vec{X}\right) + \gamma^4_{v(t)}\cdot\frac{v}{c^2}\frac{dv}{dt}\left(\frac{d}{dt}\vec{X}\right)$$

Finally if we substitute $\vec{X} = \begin{pmatrix} ct \\ \underline{x(t)} \end{pmatrix}$ ← He reads the object's trajectory as some $x(t)$

$$\vec{A} = \gamma^2_{v(t)}\begin{pmatrix} 0 \\ a(t) \end{pmatrix} + \gamma^4_{v(t)}\cdot\frac{v(t)\cdot a(t)}{c^2}\begin{pmatrix} c \\ v(t) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^4_{v(t)}\frac{v(t)\cdot a(t)}{c} \\ \gamma^2_{v(t)}a(t) + \gamma^4_{v(t)}\left(\frac{v(t)\cdot a(t)}{c^2}\right)v(t) \end{pmatrix}$$

This is the 4-acceleration vector for any object

That is observed moving velocity $v(t)$ and acceleration $a(t)$

[2] We can transform 4-acceleration vector to another inertial observer's by Lorentz transform matrix, and derive what the acceleration of the object $a(t')$ in another observer's eye:

$$\begin{pmatrix} \gamma^4_{v(t')} \dfrac{v(t')\cdot a(t')}{c} \\ \gamma^2_{v(t')} a(t') + \gamma^4_{v(t')} \left(\dfrac{v(t')\cdot a(t')}{c^2}\right) v(t') \end{pmatrix} = \Lambda_u \begin{pmatrix} \gamma^4_{v(t)} \dfrac{v(t)\cdot a(t)}{c} \\ \gamma^2_{v(t)} a(t) + \gamma^4_{v(t)} \left(\dfrac{v(t)\cdot a(t)}{c^2}\right) v(t) \end{pmatrix}$$

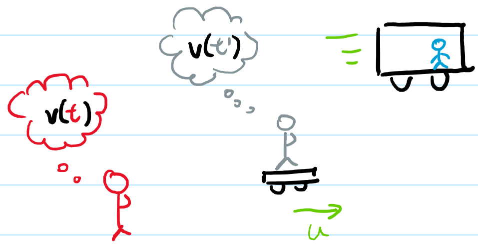
If their relative velocity $= u$

But this is obviously terrible for finding the expression of $a(t')$ in terms of $a(t)$, so no textbook will even try it

A more reasonable derivation is starting from velocity addition:

In Galilean we have

$$\underbrace{V_{\bullet\bullet}}_{v(t')} + \underbrace{V_{\bullet\bullet}}_{u} = \underbrace{V_{\bullet\bullet}}_{v(t)}$$



$$\Rightarrow \quad v(t') = v(t) - u$$

So in relativity it becomes

$$v(t') = \frac{v(t) - u}{1 - \frac{v(t)\cdot u}{c^2}}$$

Finally by differentiation, we can get $a(t')$

$$a(t') = \frac{d}{dt'} v(t')$$

$$= \frac{dt}{dt'} \cdot \frac{d}{dt}\left(\frac{v(t) - u}{1 - \frac{v(t)\cdot u}{c^2}}\right)$$

$$\frac{dt}{dt'} = \frac{\frac{dt}{d\tau}}{\frac{dt'}{d\tau}} = \frac{\gamma_{v(t)}}{\gamma_{v(t')}}$$

$$= \frac{\gamma_{v(t)}}{\gamma_{v(t')}} \cdot \left[\frac{1}{1 - \frac{v(t)\cdot u}{c^2}} + \frac{v(t) - u}{\left(1 - \frac{v(t)\cdot u}{c^2}\right)^2} \cdot \frac{u}{c^2}\right] \frac{d v(t)}{dt}$$

$$\gamma_{v(t')} = \gamma_{v(t)}\gamma_u\left(1 - \frac{uv(t)}{c^2}\right)$$

See velocity addition

$$= \frac{1}{\gamma_u\left(1 - \frac{uv(t)}{c^2}\right)} \left[\frac{\overset{\frac{1}{\gamma_u^2}}{\left(1 - \frac{u^2}{c^2}\right)}}{\left(1 - \frac{v(t)\cdot u}{c^2}\right)}\right] a(t)$$

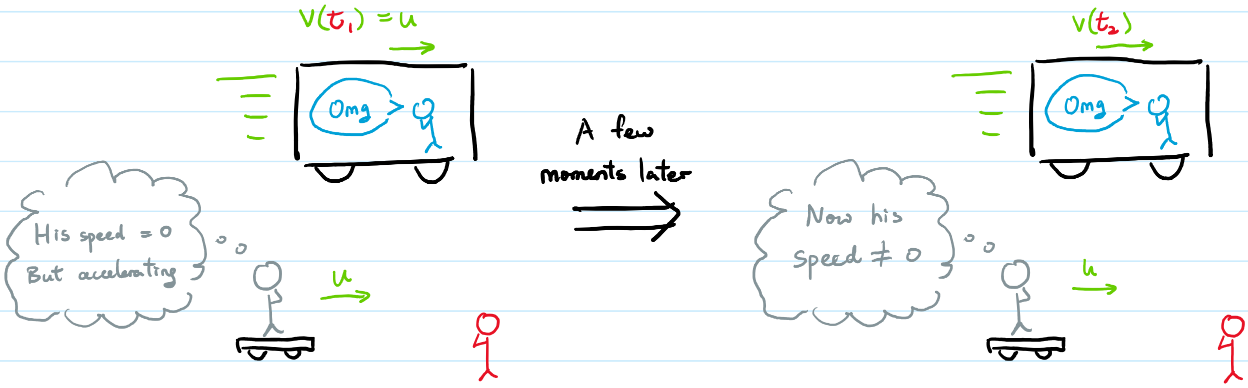
$$= \frac{a(t)}{\gamma_u^3\left(1 - \frac{uv(t)}{c^2}\right)}$$

[3] It is not possible to use Lorentz transform to convert from the coordinate of an inertial observer to that of a non-inertial observer (the co-moving observer)

But we can still define an "instantaneous co-moving observer"

i.e. He is moving at constant speed u. But in a tiny moment the object is also moving at u

$\Rightarrow$ They appear static to each other in that tiny moment



From the above derivation, his observed 4-acceleration at the moment of $v(t) = u$ is exactly when $v(t') = 0$

$$\Rightarrow \vec{A}' = \begin{pmatrix} 0 \\ a(t') \end{pmatrix}$$

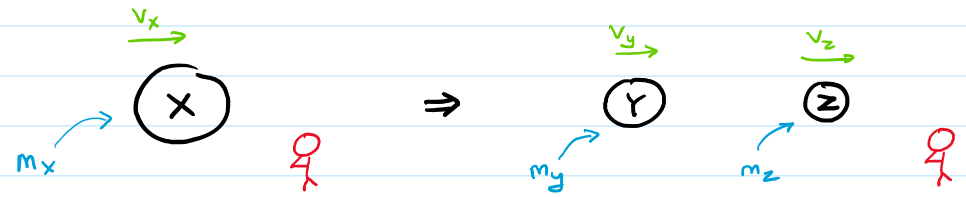
[4] There is no contradiction that the non-inertial frame co-moving observer observe his 4-acceleration $= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This is because

Inertial frame observer	$\Leftrightarrow$	Non-inertial frame observer
sees acceleration		sees Pseudo-force

Applications of 4-momentum

4-momentum is the truely conservative quantity since it can be Lorentz transformed, i.e. it conserves in all inertial frames. It is a replacement of energy & momentum conservation in Newtonian.

Example 1: Decay of particle (1D "explosion")



Before decay: Particle X observed to be at v_x

After decay: Particle Y observed to be at v_y

Particle Z observed to be at v_z

To find a relation between v_x, v_y, v_z, we can naively write out their 4-momentum:

$$\vec{P}_x = \gamma_{v_x} m_x \begin{pmatrix} c \\ v_x \end{pmatrix}, \quad \vec{P}_y = \gamma_{v_y} m_y \begin{pmatrix} c \\ v_y \end{pmatrix}, \quad \vec{P}_z = \gamma_{v_z} m_z \begin{pmatrix} c \\ v_z \end{pmatrix}$$

The conservation is simply $\vec{P}_x = \vec{P}_y + \vec{P}_z$, i.e.

$$\gamma_{v_x} m_x \begin{pmatrix} c \\ v_x \end{pmatrix} = \gamma_{v_y} m_y \begin{pmatrix} c \\ v_y \end{pmatrix} + \gamma_{v_z} m_z \begin{pmatrix} c \\ v_z \end{pmatrix}$$

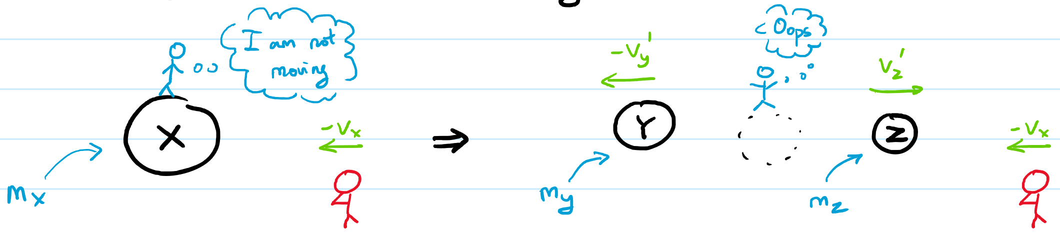
or

$$\begin{cases} \gamma_{v_x} m_x c = \gamma_{v_y} m_y c + \gamma_{v_z} m_z c \\ \gamma_{v_x} m_x v_x = \gamma_{v_y} m_y v_y + \gamma_{v_z} m_z v_z \end{cases}$$

Easy to write out, but annoying to solve!

<u>Useful trick</u> : Change to the co-moving frame of 1 particle

E.g. In X's co-moving frame, then $\vec{P}_x = m_x \begin{pmatrix} c \\ 0 \end{pmatrix}$



We don't have to explicitly calculate v_y', v_z' from matrix

because we already have the velocity addition formula

$\because$ Relative velocity : $\quad v_{\bullet\bullet} + v_{\bullet\bullet} = v_{\bullet\bullet}$

$\uparrow$ v_y' or v_z' $\quad \uparrow$ v_x $\quad \uparrow$ v_y or v_z

$$\Rightarrow \quad v_y' = \frac{v_y - v_x}{1 - \frac{v_x v_y}{c^2}} \quad , \quad v_z' = \frac{v_z - v_x}{1 - \frac{v_z v_x}{c^2}}$$

Then write again the 4 momentum conservation

$$m_x \begin{pmatrix} c \\ \underline{0} \end{pmatrix} = \gamma_{v_y'} m_y \begin{pmatrix} c \\ v_y' \end{pmatrix} + \gamma_{v_z'} m_z \begin{pmatrix} c \\ v_z' \end{pmatrix}$$

Having a 0 make this "relatively easier" to solve

<u>Example 2</u> : Relativistic Doppler Effect

Two observers of different speed see a photon (light particle)

From quantum mechanics.

- Energy of photon $= h\underline{f}$ ← frequency
- Momentum of photon $= \frac{h}{\underline{\lambda}}$ ← wavelength

where h = planck constant $\simeq 6.63 \times 10^{-34}$ $m^2 kg\, s^{-1}$

We can interestingly derive the relativistic doppler effect formula by

frequency / wavelength seen by moving observer $\begin{pmatrix} \frac{hf'}{c} \\ \frac{h}{\lambda'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma_\beta \\ -\gamma_\beta & \gamma \end{pmatrix}_v \begin{pmatrix} \frac{hf}{c} \\ \frac{h}{\lambda} \end{pmatrix}$ frequency / wavelength seen by static observer

And for light in vacuum it must have $c = f' \cdot \lambda' = f \cdot \lambda$

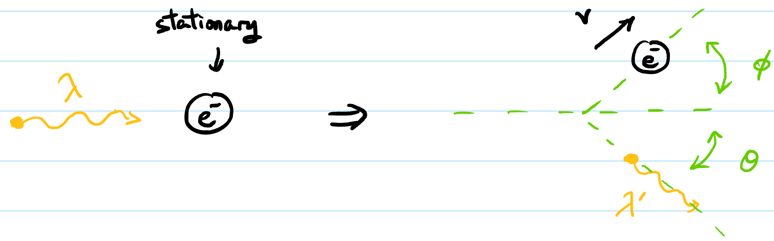
The matrix equation tells

$$\begin{aligned} \frac{f'}{c} &= \gamma \cdot \frac{f}{c} - \gamma_\beta \cdot \frac{1}{\lambda} \\ &= \gamma \cdot \frac{f}{c} - \gamma_\beta \cdot \frac{f}{c} \\ &= \frac{1}{\sqrt{1-\beta^2}} (1-\beta) \cdot \frac{f}{c} \\ &= \sqrt{\frac{1-\beta}{1+\beta}} \frac{f}{c} \end{aligned}$$

$$\begin{aligned} \frac{1}{\lambda'} &= -\gamma_\beta \frac{f}{c} + \gamma \cdot \frac{1}{\lambda} \\ &= -\gamma_\beta \frac{1}{\lambda} + \gamma \cdot \frac{1}{\lambda} \\ &= \frac{1}{\sqrt{1-\beta^2}} \cdot (1-\beta) \frac{1}{\lambda} \\ &= \sqrt{\frac{1-\beta}{1+\beta}} \frac{1}{\lambda} \end{aligned}$$

They are essentially the same formula.

Example 3 : Compton Scattering (4-momentum in 2D)

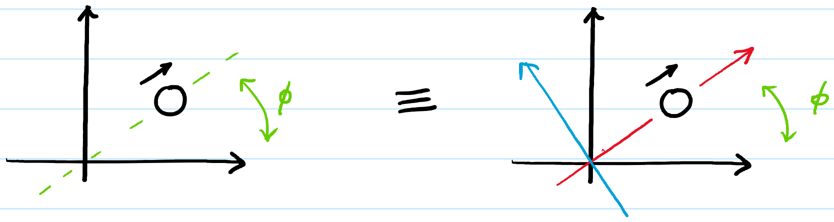


① How to write 4-momentum in 2D ?

In 1D, it is $\vec{P} = \gamma_v m \begin{pmatrix} c \\ v \end{pmatrix} \begin{matrix} t \\ x \end{matrix}$

↪ Give it 1 more dimension : $\vec{P} = \gamma_v m \begin{pmatrix} c \\ v \\ 0 \end{pmatrix} \begin{matrix} t \\ x \\ y \end{matrix}$

↪ Note that "horizontally travelling" vs "diagonally travelling" are in fact different by the choice of coordinate system



So the x/y coordinate are related by rotational transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \cdot \gamma_v m \begin{pmatrix} c \\ v \\ 0 \end{pmatrix} = \gamma_v m \begin{pmatrix} c \\ v\cos\phi \\ v\sin\phi \end{pmatrix}$$

② Back to Compton Scattering. 4 Momentum writes as

$\frac{hf}{c} = \frac{h}{\lambda} \rightarrow$
$$\begin{pmatrix} \frac{h}{\lambda} \\ \frac{h}{\lambda} \\ 0 \end{pmatrix} + \begin{pmatrix} mc \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_v mc \\ \gamma_v mv\cos\phi \\ \gamma_v mv\sin\phi \end{pmatrix} + \begin{pmatrix} \frac{h}{\lambda'} \\ \frac{h}{\lambda'}\cos\theta \\ \frac{h}{\lambda'}\sin\theta \end{pmatrix}$$

Rewrite row by row as

$$\begin{cases} \boxed{1} \quad \left(\frac{h}{\lambda} + mc - \frac{h}{\lambda'}\right)^2 = \gamma_v^2 m^2 c^2 \\ \boxed{2} \quad \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta\right)^2 = \gamma_v^2 m^2 v^2 \cos^2\phi \\ \boxed{3} \quad \left(\frac{h}{\lambda'}\sin\theta\right)^2 = \gamma_v^2 m^2 v^2 \sin^2\phi \end{cases}$$

Taking square because it is easier for $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

- Add $\boxed{2}$ & $\boxed{3}$ to remove ϕ

$$\Rightarrow \left(\frac{h}{\lambda}\right)^2 - 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right)\cos\theta + \left(\frac{h}{\lambda'}\right)^2 = \gamma_v^2 m^2 v^2$$

- Use $\boxed{1}$ minus the above,

$$\Rightarrow m^2c^2 + 2mc\cdot\frac{h}{\lambda} - 2mc\cdot\frac{h}{\lambda'} - 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right)(1-\cos\theta) = \gamma_v^2 m^2 (c^2 - v^2)$$

- By $\gamma_v^2(c^2-v^2) = \gamma_v^2 c^2\left(1-\frac{v^2}{c^2}\right) = c^2$

$$\Rightarrow \quad 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right)(1-\cos\theta) = 2mc\cdot\left(\frac{h}{\lambda}-\frac{h}{\lambda'}\right)$$

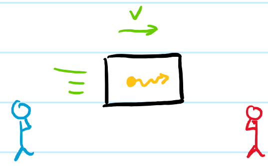
$$\boxed{\frac{h}{mc}(1-\cos\theta) = \lambda' - \lambda}$$

which is the usual formula in textbook

Example 4 : Refractive index of moving object

Consider a co-moving observer and a moving observer

watching a light particle in a material



Co-moving observer sees :

- Material with refractive index n
- Frequency of light never changes, independent of material

  Wavelength of light changes from λ to $\frac{\lambda}{n}$, compare to in vacuum

$\Rightarrow$ He writes the 4-momentum as $\begin{pmatrix} \frac{hf}{c} \\ \frac{nh}{\lambda} \end{pmatrix} = \begin{pmatrix} \frac{h}{\lambda} \\ \frac{nh}{\lambda} \end{pmatrix}$

Then transform to the moving observer's frame

$$\begin{pmatrix} \frac{h}{\lambda'} \\ \frac{n'h}{\lambda'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} \frac{h}{\lambda} \\ \frac{nh}{\lambda} \end{pmatrix}$$

1^{st} row gives : $\lambda' = \dfrac{1}{\gamma(1-n\beta)}\cdot\lambda$

2^{nd} row gives : $n' = \dfrac{\lambda'}{\lambda}\gamma(n-\beta)$

$$= \frac{n-\beta}{1-n\beta} \simeq n + \underbrace{(1-n^2)\beta - n^2\beta + \cdots}_{\text{relativistic correction}}$$

Spacetime Interval

In mathematics, the formula of metric (distance) can be defined separately from the vectors's definition.

In fact, any functions $d(\vec{x}, \vec{y})$ that satisfy the followings can be used as a metric function:

① $d(\vec{x}, \vec{y}) \geq 0$ (non-negativity)

② $d(\vec{x}, \vec{y}) = 0 \Leftrightarrow \vec{x} = \vec{y}$ (identity of indiscernibles)

③ $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$ (symmetry)

④ $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$ (triangle inequality)

You can easily verify that our "common sense" of distance

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$ satisfies the above.

But then mathematicians take out these properties so as to "make up" different definitions of distance in other weird space.

The distance in spacetime

Definition: The "natural choice of distance" between 4-vectors is

For $\vec{X}_1 = \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $\vec{X}_2 = \begin{pmatrix} ct_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$$\Rightarrow \|\vec{X}_2 - \vec{X}_1\| = \sqrt{-c^2(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is usually called the Minkwoski metric

Idea: Consider a light beam travelling between 2 points

$$\underbrace{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}_{\text{Distance it travels}} = \underbrace{c(t_2-t_1)}_{\text{Duration it takes}}$$

$$\Rightarrow \sqrt{-c^2(t_2-t_1)^2+(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = 0$$

which is supposed to be a phenomenon independent of observer. In fact, we can show that

$$\text{If} \quad \vec{X}_2' - \vec{X}_1' = \Lambda_v (\vec{X}_2 - \vec{X}_1)$$

$$\begin{pmatrix} c(t_2'-t_1') \\ x_2'-x_1' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}_v \begin{pmatrix} c(t_2-t_1) \\ x_2-x_1 \end{pmatrix} = \begin{pmatrix} \gamma c(t_2-t_1) - \gamma\beta(x_2-x_1) \\ -\gamma\beta c(t_2-t_1) + \gamma(x_2-x_1) \end{pmatrix}$$

$$\Rightarrow \|\vec{X}_2 - \vec{X}_1\| = -(\gamma c(t_2-t_1) - \gamma\beta(x_2-x_1))^2 + (-\gamma\beta c(t_2-t_1) + \gamma(x_2-x_1))^2$$

$$= \gamma^2 c^2 (t_2-t_1)^2(\beta^2-1) + \gamma^2(x_2-x_1)^2(1-\beta^2)$$

$$= -c^2(t_2-t_1)^2 + (x_2-x_1)^2$$

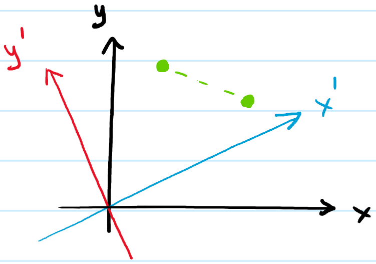
$$= \|\vec{X}_2 - \vec{X}_1\|$$

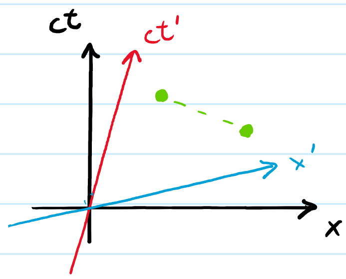
This kind of "distance" is independent of observer. An "invariant".

Analogy

In rotation of coordinate system the Pythagoras distance between 2 points never change

In Lorentz transform of inertial frame the Minkowski metric between 2 events never change





Note 1 : Minkowski metric is in fact a "pseudo-metric" because it does not satisfy property ① & ③ of metric.

So it is rather called "interval" but not distance.

Note 2 : Recall that we choose $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ because we want $\det(\Lambda_v) = 1$, is in fact a consequence from wanting this kind of distance being invariant.

$$\begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \end{pmatrix} = \begin{pmatrix} pc\Delta t + q\Delta x \\ qc\Delta t + p\Delta x \end{pmatrix}$$

If require $-(c\Delta t)^2 + (\Delta x)^2 = -(pc\Delta t + q\Delta x)^2 + (qc\Delta t + p\Delta x)^2$

$$= (q^2 - p^2)(c\Delta t)^2 + (p^2 - q^2)(\Delta x)^2$$

it must have $\det \begin{pmatrix} p & q \\ q & p \end{pmatrix} = p^2 - q^2 = 1$

Interval & Causality

Traditionally, the spacetime interval between 2 events can be used to determine causality between the events.

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

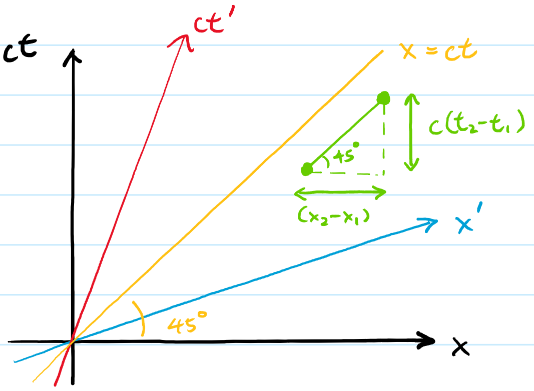
$$= \begin{cases} > 0 & \text{Space-like} \\ = 0 & \text{Light-like} \\ < 0 & \text{Time-like} \end{cases}$$

① Light - like Interval

$(\Delta s)^2 = 0$ implies $(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 = c^2 (t_2-t_1)^2$

i.e. To travel from (x_1, y_1, z_1) to (x_2, y_2, z_2) exactly in the duration of $t_2 - t_1$, you must travel at light speed.

$\Rightarrow$ An event occuring at (ct_1, x_1, y_1, z_1) can only influence another event at (ct_2, x_2, y_2, z_2) with light signal



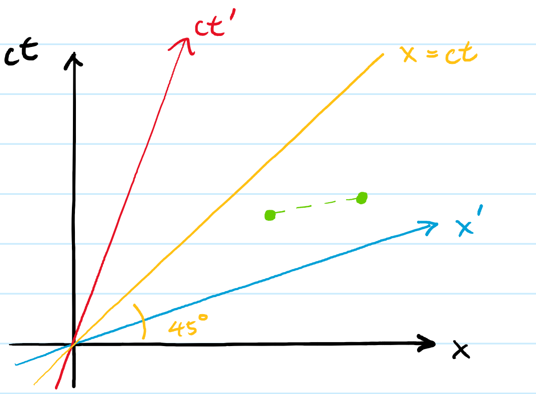
Any 2 events connecting with a 45° line are casually related by light signal only

② Space - like Interval

$(\Delta s)^2 > 0$ implies $(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 > c^2 (t_2-t_1)^2$

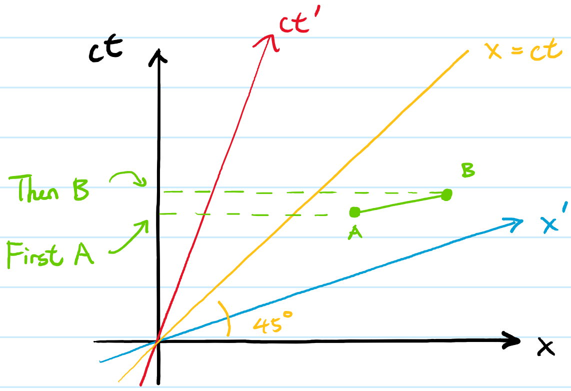
i.e. A communication between the 2 events require a signal / medium that travels faster than light

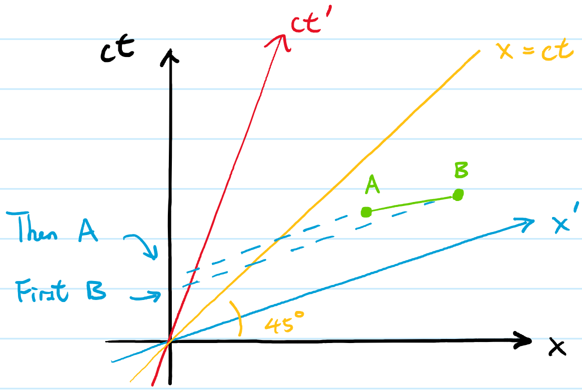
$\Rightarrow$ Impossible to influence each other (No causality)



Line with slope < 45° are trajectories of objects travelling at speed $\frac{x}{t} > c$, i.e. not physical

The occurance of 2 space-like separated events can be reversed according to different observers.





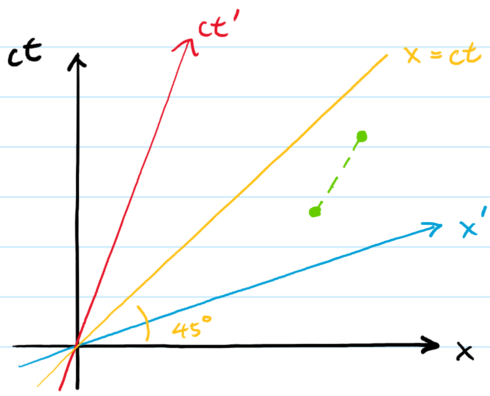
But this will not lead to logical contradiction since there is no physical ways for one event happens to cause another.

③ Time-like Interval

$(\Delta s)^2 < 0$ implies $(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 < c^2 (t_2-t_1)^2$

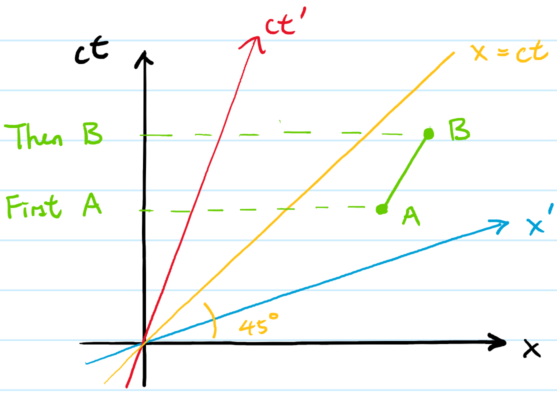
i.e. A communication between the 2 events only require a signal / medium that travels slower than light

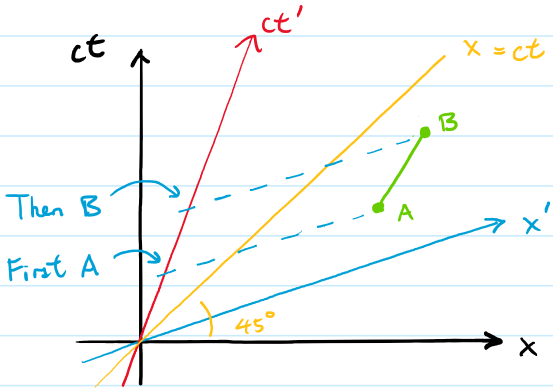
$\Rightarrow$ Possible for one event affecting the other.



Line with slope > 45° are trajectories of objects travelling at speed $\frac{x}{t} < c$

The 2 events will always occur in the same time order, independent of observer. So they may be casually related while logic will never break.





Norm of 4-vectors

Since the spacetime interval is invariant to Lorentz transform we can use the "length" of other 4-vectors to get some identities that are useful for calculation.

4-velocity $\begin{pmatrix} \gamma c \\ \gamma v \end{pmatrix}$: $-(\gamma c)^2 + (\gamma v)^2 = -\gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = \underline{-c^2}$

is a constant

4-momentum $\begin{pmatrix} \frac{E}{c} \\ p \end{pmatrix} = \begin{pmatrix} \gamma m c \\ \gamma m v \end{pmatrix}$: $-\left(\frac{E}{c}\right)^2 + p^2 = -m^2 c^2$

$\Leftrightarrow \quad E^2 = m^2 c^4 + p^2 c^2$

The energy mass relation

4-acceleration $\begin{pmatrix} \gamma^4_{v(t)} \dfrac{v(t) \cdot a(t)}{c} \\ \gamma^2_{v(t)} a(t) + \gamma^4_{v(t)} \left(\dfrac{v(t) \cdot a(t)}{c^2} \right) v(t) \end{pmatrix}$

is a constant

$$\Rightarrow \; -\gamma^8 \left(\frac{v \cdot a}{c}\right)^2 + \gamma^4 a^2 + 2\gamma^6 \left(\frac{v \cdot a}{c}\right)^2 + \gamma^8 \left(\frac{v \cdot a}{c}\right)^2 \left(\frac{v}{c}\right)^2 = \underline{\gamma^4 \left(a^2 + \gamma^2 \left(\frac{v \cdot a}{c}\right)^2\right)}$$