Physics Enrichment Program HKUST |
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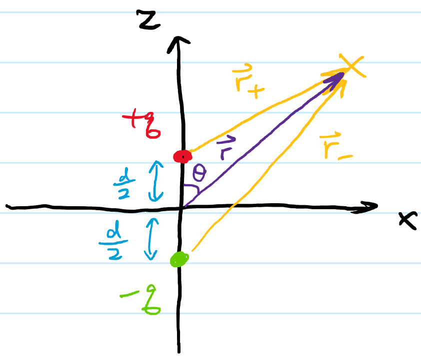
- Standard derivation of electric dipole's potential / field
- Dielectric material
  - Polarization, bound charge
  - Electric displacement, free charge
  - Special case: Linear dielectric

Electric Dipole's field / potential

Recall that potential is easier to calculate than field because field is a vector and is annoying to solve each component

$\Rightarrow$ Usual approach: Find $V / \vec{A} \xrightarrow{\text{div / curl}} \vec{E} / \vec{B}$

Simple electric dipole



Model:

- 2 opposite point charge $+q/-q$
- Separation between charges $= d$
- Only consider the potential very far away, $|\vec{r}| \gg d$

(So the 2 charges look like locating at the same position when zoom out)

Standard derivation

Just cosine law

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{r}_+|} + \frac{-q}{|\vec{r}_-|}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2 + \frac{d^2}{4} - 2r\left(\frac{d}{2}\right)\cos\theta}} - \frac{1}{\sqrt{r^2 + \frac{d^2}{4} + 2r\left(\frac{d}{2}\right)\cos\theta}}\right]$$

By Taylor Expansion to 1st order $(1+x)^n \simeq 1+nx$ when $x \ll 1$

In the model we have $r \gg d \Rightarrow \frac{d}{r} \ll 1$

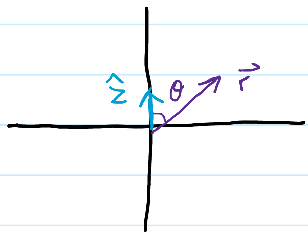
and $\left(\frac{d}{r}\right)^2$ is small enough to be ignored

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r}\left[\left(1+\frac{d}{2r}\cos\theta\right) - \left(1-\frac{d}{2r}\cos\theta\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r} \cdot \frac{d\boxed{\cos\theta}}{r}$$

$$= \frac{qd}{4\pi\varepsilon_0 r^2}\boxed{(\hat{z}\cdot\hat{r})}$$

from geometry



By defining electric dipole moment as $\vec{p} = q\vec{d} = q|d|\hat{z}$

$$\Rightarrow \boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$$

Important : $\vec{p}$ points from -ve to +ve charges. $\vec{p}$

From then we can find $\vec{E}$ by $\vec{E} = -\vec{\nabla}V$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3(\vec{p}\cdot\hat{r}) - \vec{p}}{r^3}}$$

(Skip the steps here)

Electric field in matter

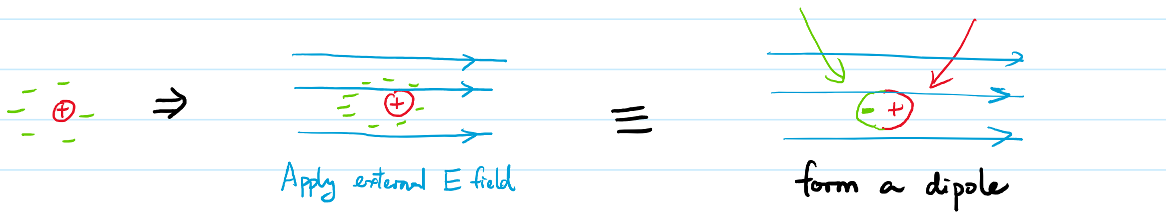
Classification of matter by response to electric field

① Conductor

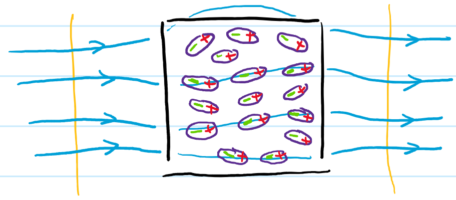
- Charges only distribute on material surface and can free flow
- Charge distribution always results in a constant potential on the conductor surface, under whatever external E field

② Dielectric (Insulator)

- Charges are bounded within atom



- Every atom in the material with both
    - The external E field
    - E fields from other dipoles



☆☆ So even if the applied E field is regular, the field distribution in the material can be a mess.

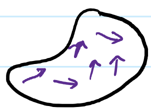
(How messy is depending on the material structure)

Polarization (field) / Dipole density

Consider an object made of many dipoles. The potential distribution is

$$V_{dip}(\vec{r}) = \sum \text{(each dipole's contribution)}$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_i \frac{\vec{p}_i \cdot \hat{r}_i}{|\vec{r}_i|^2}$$



Extending to continuous distribution, like in Columb's Law

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\underline{\vec{P}}(\vec{r})}{|\vec{r}-\vec{r}'|^2} \cdot \left[\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \right] d^3\vec{r}'$$

$$\boxed{\text{Define : } \quad \underline{\vec{P}}(\vec{r}) \equiv \text{Dipole density at position } \vec{r}}$$

Symbol = capital P. Give an underline for clarity reason

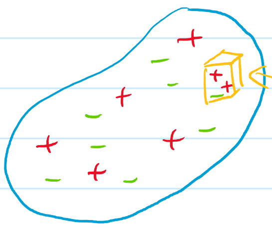
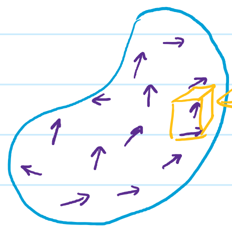
This formular is very similar to Columb's Law except it is $\sim \frac{1}{r^2}$

$$\frac{1}{4\pi\varepsilon_0}\iiint \frac{\boxed{\rho(\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|} \cdot \left[\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\right] d^3\vec{r}\,' \quad \text{V.S.} \quad \frac{1}{4\pi\varepsilon_0}\iiint \frac{\boxed{\vec{P}(\vec{r})}}{|\vec{r}-\vec{r}\,'|^2} \cdot \left[\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\right] d^3\vec{r}\,'$$

Source = charge $\qquad$ Source = dipole

$V \sim \frac{1}{r}$ $\qquad$ $V \sim \frac{1}{r^2}$

☆ You may understand dipole density by comparing with charge density

Every position $\vec{r}$ in a material carry a "weight" of charge $\rho(\vec{r})$	Every position $\vec{r}$ in a material carry a "weight" of dipole $\vec{P}(\vec{r})$
So the total charge is by integration	So the "total dipole" is by integration
$Q_{tot} = \iiint \rho(\vec{r}\,')\, d^3\vec{r}$	$P_{tot} = \iiint \vec{P}(\vec{r})\, d^3\vec{r}$
$\rho \sim \frac{Q_{tot}}{V}$	$\underline{P} \sim \frac{P_{tot}}{V}$
 density of charge in this small volume $= \rho(\vec{r})$	 density of dipole in this small volume $= \vec{P}(\vec{r})$

☆☆ $\vec{P}(\vec{r})$ has several names. eg.

- Dipole density ← I think this is the most accurate
- Polarization Field
- Polarization ← The most common name in book

Bound charges

Because charges distribute as dipoles in materials, it is more natural to describe the charge distribution in terms of dipole density $\vec{P}$ rather than charge density ρ.

But $\vec{P}$ is a vector field so integration is annoying.

$\Rightarrow$ Rewrite the formula into a more convenient form

(After skipping a lot of annoying vector calculus simplification)

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \boxed{\oiint} \frac{\vec{P}(\vec{r}') \boxed{\cdot d\vec{r}'}}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\varepsilon_0} \boxed{\iiint} \frac{-\vec{\nabla}\cdot\vec{P}(\vec{r}') \boxed{d\vec{r}'}}{|\vec{r}-\vec{r}'|}$$

Integral of total flux on the material surface

Volume integral in the whole material

How to give these 2 terms physical meaning?

Recall Columb's Law of potential

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \sim \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho}{r} d^3r \sim \frac{\text{charge}}{\text{distance}}$$

The numerator in the above 2 terms can be interpreted as somekind of charge distribution

$$\frac{1}{4\pi\varepsilon_0}\oiint \frac{\boxed{\vec{P}(\vec{r}')\cdot d^2\vec{r}'}}{\boxed{|\vec{r}-\vec{r}'|}} \sim \frac{\text{charge}}{\text{distance}} \sim \frac{1}{4\pi\varepsilon_0}\iiint \frac{\boxed{-\nabla\cdot\vec{P}(\vec{r}')}\, d^3\vec{r}'}{\boxed{|\vec{r}-\vec{r}'|}}$$

Defining the bound charge distribution

normal vector on the surface

$$\sigma_b(\vec{r}') \equiv \vec{P}(\vec{r}')\cdot\hat{n} = \text{Surface bound charge density}$$

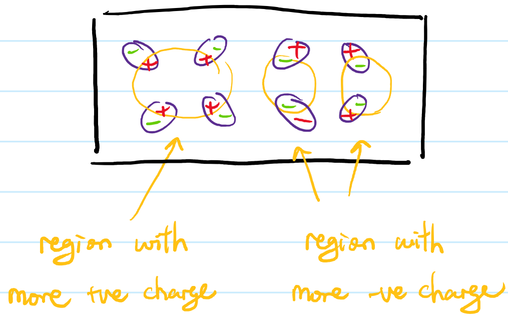
$$\rho_b(\vec{r}) \equiv -\vec{\nabla}\cdot\vec{P}(\vec{r}') = \text{Volume bound charge density}$$

So we can interprete the dipole potential as a result of contributions of 2 kinds of source

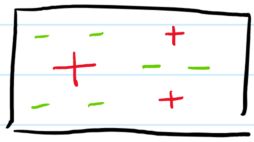
$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oint\!\!\oint \frac{\sigma_b(\vec{r}')\, d\vec{a}'}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho_b(\vec{r}')\, d^3\vec{r}'}{|\vec{r}-\vec{r}'|}$$

$$= \left[\begin{array}{l}\text{Contribution by the}\\ \text{charges that are}\\ \text{on the material surface}\end{array}\right] + \left[\begin{array}{l}\text{Contribution by the}\\ \text{charges that are}\\ \text{inside the material}\end{array}\right]$$

Geometrical interpretation of bound charges

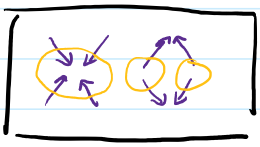


View as region with charges →



These charges cannot move freely as they are bounded to the atom and thus the material structure

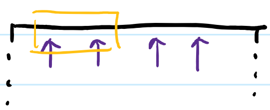
↓ View as $\vec{P}$ field



- Region with converging $\vec{P}$ field ($\vec{\nabla}\cdot\vec{P} < 0$)

  = contain more +ve charge

  $\Rightarrow \rho_b \sim -\vec{\nabla}\cdot\vec{P}$



- Outward flux of $\vec{P}$ field ($\vec{P}\cdot\hat{n} > 0$)

  = surface with +ve charge accumulate

  $\Rightarrow \sigma_b \sim \vec{P}\cdot\hat{n}$

Short Summary ①

Given a polarized object, if we know its dipole density distribution $\vec{P}$, we can compute the following directly:

— The (bound) charge distribution on the object

$$\sigma_b = \vec{P} \cdot \hat{n} \quad , \quad \rho_b = -\vec{\nabla} \cdot \vec{P}$$

— The electric potential & E field it creates

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{P}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \cdot \left[\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|} \right] d^3\vec{r}\,'$$

$$= \frac{1}{4\pi\varepsilon_0} \oint\!\!\!\oint \frac{\sigma_b(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^2\vec{r}\,' + \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho_b(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3\vec{r}\,'$$

and then $\vec{E} = -\nabla V$

(However these calculation are very annoying)

☆☆ Note that $\vec{P}$ here is the dipole density naturally carried by the object. To find $\vec{P}$, we need to derive from the material 's property (i.e. the atomic arrangement)

Electric Displacement / D field

Situation : Most objects do not carry polarization. Unless external E field is applied to them.

Problem : External E field affects dipole arrangement in the material

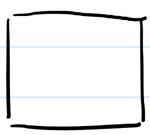
↪ Rearrangement of dipoles form bound charge distributions

↪ Bound charge distributions emit their own E field

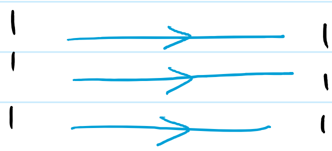
⇒ Total E field = Superposition of the 2

= (Contribution by external E field) + (Contribution by bound charges i.e. dipoles in the material)

Depending on the material structure, the result E field can be very irregular.

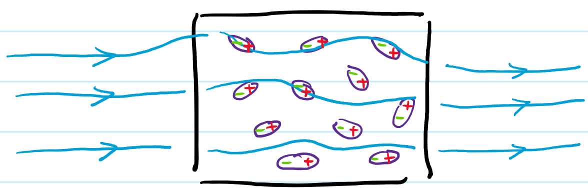


No external E field
material shows no polarization



External E field is originally very regular

But when put together, the E field distribution may be distorted



How to find the total E field when external E field is present ?

Free charge distribution

Note that to create the external E field, we also need certain charge distribution on our equipment.

$\Rightarrow$ These charge distribution is (somewhat) controllable.

$\hookrightarrow$ the thing we have 100% control is voltage

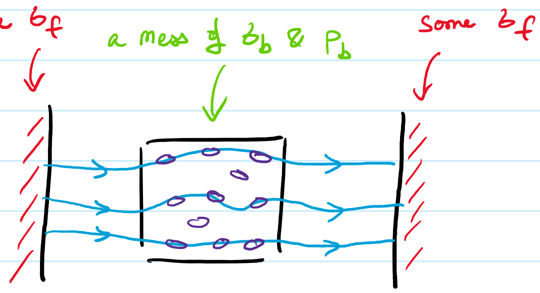
$\Rightarrow$ Call these charges "free charges"

$$\begin{cases} \sigma_f(\vec{r}) = \text{Surface free charge density} \\ \rho_f(\vec{r}) = \text{Volume free charge density} \end{cases}$$

But since we usually create the external E field by parallel plate

σ_f is almost all you see.

ρ_f is very rare.



Then by Gauss's Law

$$\varepsilon_0 \vec{\nabla} \cdot \vec{E}_{tot}(\vec{r}) = (\text{all charge distribution})$$

$$= \begin{pmatrix} \text{bound charge} \\ \text{distribute on} \\ \text{material} \end{pmatrix} + \begin{pmatrix} \text{free charge} \\ \text{distribute on} \\ \text{set up} \end{pmatrix}$$

$$= \rho_b(\vec{r}) + \rho_f(\vec{r})$$

By definition, $\rho_b(\vec{r}) = -\vec{\nabla} \cdot \vec{P}(\vec{r})$, so

$$\rho_f(\vec{r}) = \varepsilon_0 \vec{\nabla} \cdot \vec{E}_{tot}(\vec{r}) + \vec{\nabla} \cdot \vec{P}(\vec{r})$$

$$= \vec{\nabla} \cdot \left(\varepsilon_0 \vec{E}_{tot}(\vec{r}) + \vec{P}(\vec{r}) \right)$$

We can define a new vector field $\vec{D}$

$$\boxed{\vec{D}(\vec{r}) = \varepsilon_0 \vec{E}_{tot}(\vec{r}) + \vec{P}(\vec{r})}$$

So that $\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D}(\vec{r}) &= \rho_f(\vec{r}) \\ \vec{D}(\vec{r})\cdot\hat{n} &= \sigma_f(\vec{r}) \end{aligned}}$ looks like Gauss Law.

Finding $\vec{E}_{tot}$

So far we have 2 approaches to find $\vec{E}_{tot}$

[1] If we know both ρ_b/σ_b and ρ_f/σ_f, we can solve the PDE $\vec{\nabla}\cdot\vec{E}_{tot} = -\nabla^2 V = \frac{\rho_b+\rho_f}{\varepsilon_0}$

[2] If we know both $\vec{P}$ and $\vec{D}$, simply use $\vec{E}_{tot} = \frac{1}{\varepsilon_0}(\vec{D}-\vec{P})$

Method [1] requires solving PDE but [2] does not.

So we would always want to use [2] if possible.

- $\vec{P}$ can be derived by knowing the material's atomic arrangement and bonding strength, and thus telling how many dipoles are generated under E field.
- $\vec{D}$ may be derived from ρ_f/σ_f, while ρ_f/σ_f can be told from our experimental set up.

Bad news: We cannot fully determine $\vec{D}$ by knowing where the free charges are if the config. is not symmetric

Although we have $\vec{\nabla}\cdot\vec{D} = \rho_f$ which is similar to $\vec{\nabla}\cdot\vec{E}_{tot} = \frac{\rho_{tot}}{\varepsilon_0}$

We cannot solve for $\vec{D}$ by this because $\vec{D}$ may not be conservative (curl-less). Recall the PDE way to find $\vec{E}_{tot}$:

Given $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

and $\vec{E}$ must be conservative

$(\nabla\times\vec{E} = 0)$ in electrostatic

We can define $\vec{E} = -\vec{\nabla}V$

and solve $-\vec{\nabla}^2 V = \frac{\rho}{\varepsilon_0}$

The $\vec{E}$ we get is unique

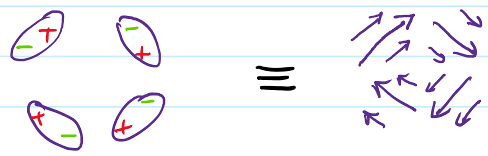
Given $\vec{\nabla}\cdot D = \rho_f$

but $\vec{D}$ may not be conservative

$(\vec{\nabla}\times\vec{D} =$ sth. we don't know$)$

$\Rightarrow$ Cannot continue.

Example of $\vec{D}$ having curl $\neq 0$:



Such dipole configuration gives rotating $\vec{P}$ field $(\vec{\nabla}\times\vec{P} \neq 0)$

In order to achieve $\vec{\nabla}\times\vec{E}_{tot} = 0$, we require

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\varepsilon_0\vec{E} + \vec{P})$$

$$= \vec{\nabla}\times\vec{P} \neq 0$$

Good news : As free charge comes from our set up,

We can choose a good layout to make $\vec{\nabla}\times\vec{D} = 0$

In symmetric config. we can even solve by Gauss Law integral form

i.e. $\oiint \vec{D}\cdot d\vec{S} = Q_f \Rightarrow |\vec{D}| = \dfrac{(\text{total free charge})}{(\text{total area})\cdot\cos\theta}$

Short Summary ②

When applying external E field on a material,

the overall $\vec{E}_{tot}$ can be found by $\varepsilon_0 E_{tot} = \vec{D} - \vec{P}$

- Finding $\vec{P}$ require knowing the material's property

  i.e. how the dipoles form under a E field

  ↪ Need a model / description of $\vec{P}$ vs $\vec{E}$

- Finding $\vec{D}$ is possible only if the free charges are distributed in a symmetric layout (so that we have 100% certainty that $\vec{\nabla} \times \vec{D} = 0$), then we can solve for $\vec{\nabla} \cdot \vec{D} = \rho_f$ just like normal Gauss Law

Special Model : Linear dielectric

There exists material with strange polarization response.

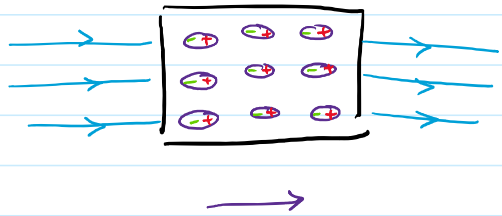
In general, the dipole density is some function of overall $\vec{E}$

$$\vec{P} = f(\vec{E})$$

But most daily life material are linear dielectric, i.e.

$$|\vec{P}| \propto |\vec{E}|$$

i.e. the polarization is scaled linearly to the total E field



Mathematically we can write

$$\vec{P}(\vec{r}) = \varepsilon_0 \chi_e \vec{E}_{tot}(\vec{r})$$

Defining the proportionality "constant" $\boxed{\chi_e = \text{electric susceptibility}}$

In fact, χ_e can be a matrix (i.e. polarization is directional) but for simplicity, you will not see it in textbook question.

Also by definition of $\vec{D}$

$$\vec{D}(\vec{r}) = \varepsilon_0 \vec{E}_{tot}(\vec{r}) + \vec{P}(\vec{r})$$

$$= \varepsilon_0 \boxed{(1+\chi_e)} \vec{E}_{tot}(\vec{r})$$

$$= \varepsilon_0 \boxed{\varepsilon_r} \vec{E}_{tot}(\vec{r})$$

ε_r = relativity permitivity (no unit)

$$= \boxed{\varepsilon} \vec{E}_{tot}(\vec{r})$$

$\varepsilon = \varepsilon_0 \varepsilon_r$ = permitivity (same unit as ε_0)

Short Summary ③

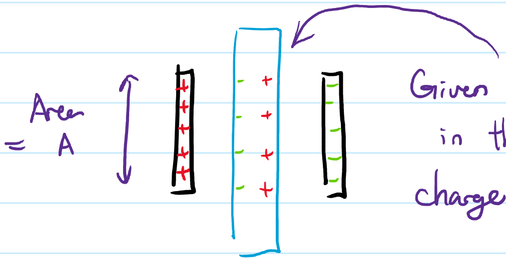
Knowing the material being linear makes calculation easy

[1] Given we know the free charges distribution is symmetric in our setup, we can find $\vec{D}$ from ρ_f

[2] Knowing the material is linear, $\vec{E}_{tot} = \frac{1}{\varepsilon}\vec{D}$ inside the material, and $\vec{E}_{tot} = \frac{1}{\varepsilon_0}\vec{D}$ in vacuum.

[3] Dipole density can also be found by $\vec{P} = \varepsilon_0 \chi_e \vec{E}_{tot}$ and thus the bound charges.

Example 1 : Dielectric between parallel plates with const. charges.



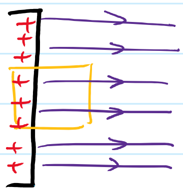
Given relative permittivity ε_r in the electric

charge density on parallel plate $= \sigma$

Step 1 : The only free charges are the charges on the metal plate and we know the density is uniform

$\Rightarrow$ This is a symmetric case (when the plates are large)

$\Rightarrow$ Can find $\vec{D}$ like using Gauss Law integral form



$$\Rightarrow |\vec{D}| = \frac{(\text{total free charge in box})}{(\text{area})}$$

$$= \sigma$$

☆☆☆ $\vec{D}$ field is the same no matter in what material.

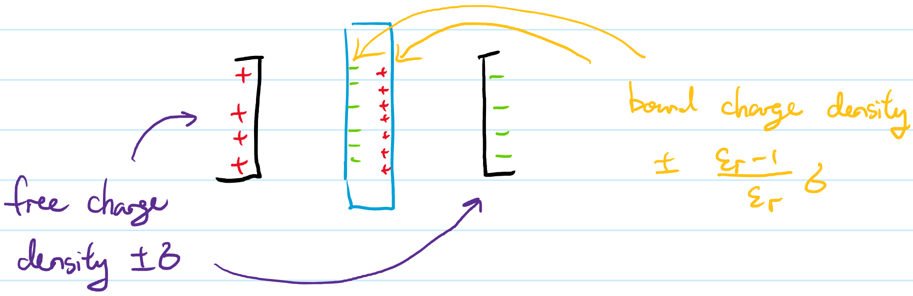
Step 2 : $|\vec{E}_{tot}|$ depends on the material

$$|\vec{E}_{tot}| = \frac{1}{\varepsilon_0}|\vec{D}| = \frac{\sigma}{\varepsilon_0} \quad \text{in vacuum.}$$

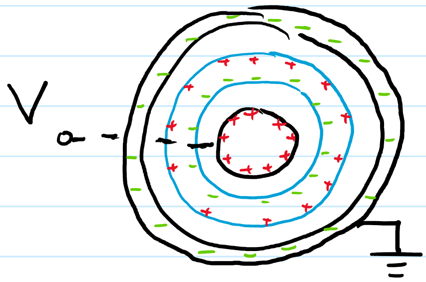
$$|\vec{E}_{tot}| = \frac{1}{\varepsilon_0\varepsilon_r}|\vec{D}| = \frac{\sigma}{\varepsilon_0\varepsilon_r} \quad \text{in the dielectric}$$

Step 3 : $|\vec{P}| = \varepsilon_0 \chi_e \underbrace{|\vec{E}_{tot}|}_{\text{take the E from dielectric}} = \frac{\varepsilon_r - 1}{\varepsilon_r} \cdot \sigma$ on the dielectric

which is the bound charge distribution on dielectric.



Example 2 : Dielectric between spherical shells with constant voltage



Given relative permitivity ε_r in the dielectric

inner metal sphere radius $= R_1$
outer metal sphere radius $= R_2$
dielectric shell radius : r_1, r_2 with $r_1 < r_2$

Step 1 : We don't know the size of free charge distribution.

But we always have to start from $\vec{D}$, so first let it as an unknown and derive its true value later.

$\Rightarrow$ Let charge on metal sphere $= \pm Q$

$\Rightarrow$ This is a symmetric case

$\Rightarrow$ Can find $\vec{D}$ by Gauss Law integral form

$$|\vec{D}| = \frac{Q}{4\pi r^2}, \quad r = \text{distance from center}$$

☆☆☆ $\vec{D}$ field is the same no matter in what material

Step 2 : $|\vec{E}_{tot}|$ depends on material

$$|\vec{E}_{tot}| = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{in air} \quad (R_1 < r < r_1,\ r_2 < r < R_2)$$

$$|\vec{E}_{tot}| = \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2} \quad \text{in dielectric} \quad (r_1 < r < r_2)$$

$$\text{Also } |\vec{P}| = \varepsilon_0 \chi_e |\vec{E}_{tot}| = \frac{\varepsilon_r - 1}{\varepsilon_r} \cdot \frac{Q}{4\pi r^2}$$

Note that $\vec{P}$ varies with radius $\Rightarrow$ bound charge are not only on dielectric's surface, but also inside the dielectric

Step 3 : From $|\vec{E}_{tot}|$ we can get $V = -\int_{R_1}^{R_2} \vec{E} \cdot d\vec{r}$

$$V = -\int_{R_1}^{r_1} \frac{Q}{4\pi\varepsilon_0 r^2}\,dr - \int_{r_1}^{r_2} \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}\,dr - \int_{r_2}^{R_2} \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{R_1} + \frac{1}{\varepsilon_r}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) + \frac{1}{R_2} - \frac{1}{r_2}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_2} - \frac{1}{R_1} + \left(\frac{1}{\varepsilon_r} - 1\right)\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right]$$

Reversing this, we find $Q = \frac{4\pi\varepsilon_0 V}{(\cdots)}$. Remainings are substitution of Q.