- Energy in electrostatics
- Energy in electrodynamics. Energy flux (Poynting Vector).
- Extra: Momentum in E&M. EM stress tensor.

Energy in Electrostatics

There are 3 frequently used formula to express the energy stored within / required to build an electrostatic configuration. equivalent

1. By potential / W.D. no one will use this

$$\frac{1}{2}\sum qV = -\frac{1}{2}\int q\vec{E}\cdot d\vec{l} \sim \frac{1}{2}\iiint \rho V\, d^3r = -\frac{1}{2}\int \left[\iiint \rho \vec{E}\, d^3r\right]\cdot d\vec{l}$$

2. By field

$$\text{Energy} = \frac{1}{2}\varepsilon_0 \int |\vec{E}|^2 d^3\vec{r} \Rightarrow \text{Energy Density} = \frac{1}{2}\varepsilon_0 |\vec{E}|^2$$

3. By capacitance

$$\frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \text{Energy stored in capacitor}$$

Derivation

The most fundamental expression of electrostatic is by potential

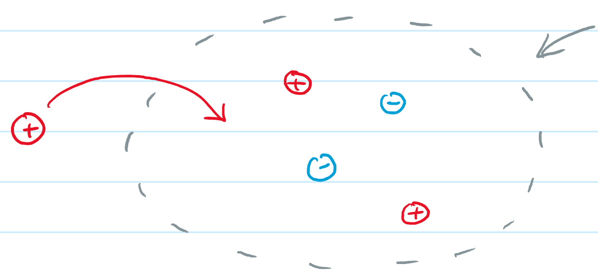
As $\vec{E}$ is a conservative field, moving a charge q from a position $\vec{r}_1$ to position $\vec{r}_2$ requires W.D.

$$W.D. = \int_{\vec{r}_1}^{\vec{r}_2} q\vec{E}\cdot d\vec{l} = q\left(V(\vec{r}_2) - V(\vec{r}_1)\right)$$

Imagine the process of building a configuration of charge

① . Charges are originally located "infinitely far" away from each other. Then we bring them close in one by one.

"Move" the charge from infinity to somewhere within finite distance



The "boundary" of infinity

② Moving in the 1^{st} charge to position $\vec{r}_1$ does not require any energy because the space is totally empty initially. But moving subsequent charges require W.D. against the E field from previous charges

2^{nd} charge to position $\vec{r}_2$: $$W.D. = -\int_{\infty}^{\vec{r}_2} q_2 \vec{E}_{(by\ q_1)} \cdot$$

$$= q_2\left(\frac{q_1}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|}\right)$$

Electrostatic $\vec{E}$ is always conservative
Can skip to potential directly

3^{rd} charge to position $\vec{r}_3$: $$W.D. = -\int_{\infty}^{\vec{r}_3} q_3 \vec{E}_{(by\ q_1\ \&\ q_2)} \cdot d\vec{r}$$

$$= q_3\left(\frac{q_1}{4\pi\epsilon_0 |\vec{r}_3 - \vec{r}_1|} + \frac{q_2}{4\pi\epsilon_0 |\vec{r}_3 - \vec{r}_2|}\right)$$

... And so on. Thus the W.D. for the n^{th} charge is

$$n^{th} \text{ charge} : \quad W.D. = q_n \sum_{i=1}^{n-1} \left(\frac{q_i}{4\pi\varepsilon_0 |\vec{r}_n - \vec{r}_i|} \right)$$

③ The total W.D. to move all n charge is therefore

$$W.D._{(total)} = \text{Sum of all contribution from previous charges}$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{i>j}^{n} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

Because exchanging i & j makes no difference in a term

$$\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2} \left[\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} + \frac{q_j q_i}{|\vec{r}_j - \vec{r}_i|} \right]$$

This allows the above sum to be written as

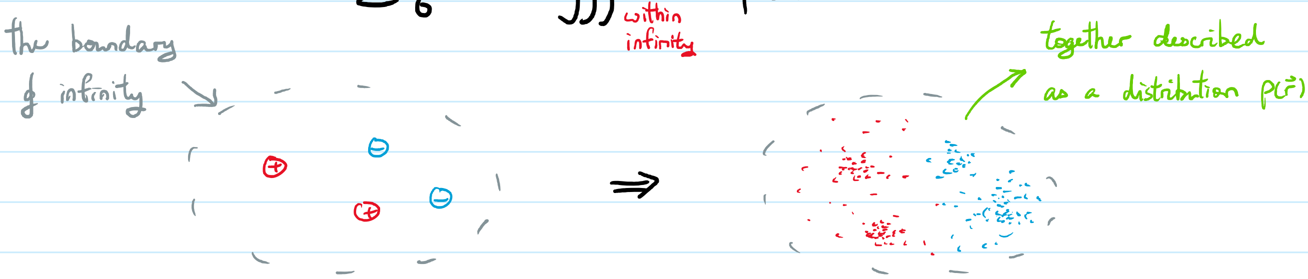
$$W.D._{(total)} = \text{Sum by all possible pair of charge}$$

$$\boxed{= \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \sum_{i=1}^{n} \sum_{i\neq j}^{n} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}$$

This is the formula for a configuration made of static point charges.

④ For continuous charge distribution, we can replace the sum over charges to integral over charge density :

$$\sum q \rightarrow \iiint_{\substack{\text{within}\\ \text{infinity}}} \rho(\vec{r})\, d^3\vec{r}$$



which convert the formula for energy into

$$\text{W.D.} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{2}\cdot\iiint \rho(\vec{r})\left[\underline{\iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3\vec{r}\,'}\right] d^3\vec{r}$$

$$= \boxed{\frac{1}{2}\iiint_{\substack{\text{within}\\ \text{infinity}}} \rho(\vec{r})\, V(\vec{r})\, d^3\vec{r}}$$

Colunb potential by all surrounding charges

This is the formula when given continuous charge distribution

Note : $\iiint_{\substack{\text{within}\\ \text{infinity}}} (\cdots)\, d^3\vec{r}$ simply means $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\cdots)\, dx\,dy\,dz$ or equivalent, as long as the integration range covers "everywhere".

⑤ The energy can be expressed by $\vec{E}$ by using Gauss Law

$$\rho(\vec{r}) = \varepsilon_0 \vec{\nabla}\cdot\vec{E}(\vec{r}) \quad \text{and} \quad \vec{E}(\vec{r}) = \vec{\nabla} V(\vec{r})$$

$$\Rightarrow \quad \frac{1}{2}\iiint_{\substack{\text{within}\\ \text{infinity}}} \rho(\vec{r})\, V(\vec{r})\, d^3\vec{r}$$

$$= \frac{1}{2}\iiint \varepsilon_0\left(\vec{\nabla}\cdot\vec{E}(\vec{r})\right) V(\vec{r})\, d^3\vec{r}$$

Chain rule
$\vec{\nabla}\cdot(a\vec{B}) = (\vec{\nabla}a)\cdot\vec{B} + a(\vec{\nabla}\cdot\vec{B})$

$$= \frac{\varepsilon_0}{2}\iiint \left[\underline{\nabla\cdot\left(E(\vec{r})V(\vec{r})\right)} - \vec{E}(\vec{r})\cdot\vec{\nabla}V(\vec{r})\right] d^3\vec{r}$$

↓ Divergence Theorem

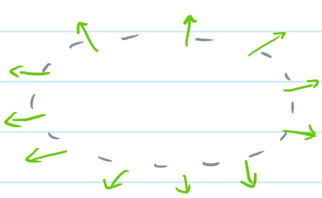
$$= \frac{\varepsilon_0}{2}\oiint \vec{E}(\vec{r})\, V(\vec{r})\, d^2\vec{r} - \frac{\varepsilon_0}{2}\iiint \vec{E}(\vec{r})\cdot\underline{\vec{\nabla}V(\vec{r})}\, d^3\vec{r}$$

$\vec{E} = -\vec{\nabla}V$

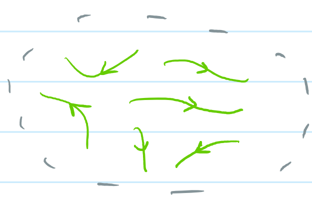
$$= \frac{\varepsilon_0}{2}\oiint \vec{E}(\vec{r})\, V(\vec{r})\, d^2\vec{r} + \frac{\varepsilon_0}{2}\iiint \underline{\vec{E}(\vec{r})\cdot\vec{E}(\vec{r})}\, d^3\vec{r}$$

$$= \underbrace{\frac{\varepsilon_0}{2}\oiint_{\substack{\text{on the}\\ \text{boundary of infinity}}} \vec{E}(\vec{r})\, V(\vec{r})\, d^2\vec{r}} + \underbrace{\frac{\varepsilon_0}{2}\iiint_{\substack{\text{within}\\ \text{infinity}}} |\vec{E}(\vec{r})|^2\, d^3\vec{r}}$$

Flux integral of something on region's boundary



Volume integral of another something inside the region



Claim : The flux integral = 0

Reasoning : On the boundary of infinity , $r \to \infty$

From Columb's Law , $\vec{E} \sim \frac{1}{r^2}$, $V \sim \frac{1}{r}$

So the flux integral is approximately equal to

$$\oiint \vec{E} V \, d^2\vec{r} \sim \iint \frac{1}{r^2} \cdot \frac{1}{r} \, d^2 r \sim \frac{1}{r} \to 0 \text{ when } r \to \infty$$

Therefore what is left is the term

$$\frac{1}{2} \iiint_{\substack{\text{within} \\ \text{infinity}}} \rho(\vec{r}) V(\vec{r}) \, d^3\vec{r} = \boxed{\frac{\varepsilon_0}{2} \iiint_{\substack{\text{within} \\ \text{infinity}}} |\vec{E}(\vec{r})|^2 \, d^3\vec{r}}$$

This is the formula when given E field.

The integral MUST be integrating to infinity, otherwise the flux integral $\neq 0$

⑥ In a capacitor , the charge stored inside must be under the same potential , or otherwise they will redistribute until they reach equi-potential . This simplifies the formula into

V is the same for all charge

$$\frac{1}{2} \iiint \rho(\vec{r}) V(\vec{r}) \, d^3\vec{r} = \frac{1}{2} V \iiint \rho(\vec{r}) \, d^3\vec{r}$$

$$= \frac{1}{2} QV$$

By definition $C = \frac{Q}{V}$

$$= \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2$$

Energy in Magnetostatic (?)

As a pairing to E field, we also have similar formula

for potential energy stored due to current interactions

$\boxed{1}$ By potential (Rarely used)

$$\frac{1}{2}\sum\left(\oint I\, d\vec{l}\right)\cdot\vec{A} \sim \frac{1}{2}\iiint \vec{J}\cdot\vec{A}\, d^3\vec{r}$$

$\boxed{2}$ By field

$$\text{Energy} = \frac{1}{2\mu_0}\int |\vec{B}|^2\, d^3\vec{r} \Rightarrow \text{Energy Density} = \frac{1}{2\mu_0}|\vec{B}|^2$$

$\boxed{3}$ By inductance

$$\frac{1}{2}LI^2 = \frac{1}{2}\frac{\Phi^2}{L}$$

Φ = Magnetic flux

☆ It is difficult to use the same derivation as in electrostatic because

- Current are in fact moving charge
- There does not exist point current sources.

If we want to move current while keeping everything charge neutral

we have to move them in form of loops.

But forces between current loop is awful to calculate.

The conventional way to derive them is via Poynting Theorem.

Poynting Theorem

We repeat the process like in electrostatic, but this time the charge is under non - static E field & B field.

Derivation

① The force on charge is now Lorentz force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

E & B are function of time and space $\Rightarrow \vec{E}(\vec{r},t)$ & $\vec{B}(\vec{r},t)$

which is not a conservative force. So the calculation of W.D. cannot be skipped by potential

$$W.D. = \int \vec{F} \cdot d\vec{l} = \int q\vec{E} \cdot d\vec{l} + \int q(\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$\vec{l}$ is the segment along a path

this is not $V(\vec{r})$ anymore
E field may form loops

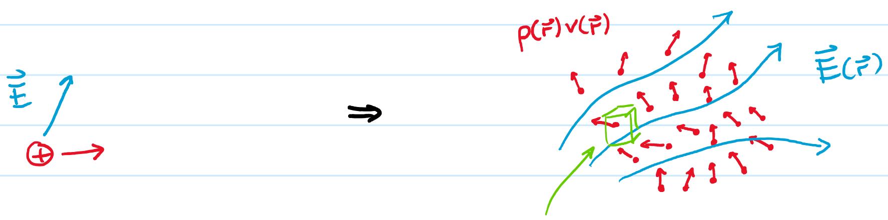
We have seen this in induction and it = 0

$$= \int q\vec{E} \cdot \frac{d\vec{l}}{dt} dt$$

$$= \int \vec{E} \cdot q\vec{v} \, dt$$

② In the microscopic view, we consider the W.D. per volume

$$\frac{W.D.}{Volume} \sim \int \vec{E} \cdot \rho\vec{v} \, dt$$



W.D. on a single charge
$$= \int (\vec{E} \cdot q\vec{v}) \, dt$$

W.D. on the charge in this volume
$$= \int \left(\iiint_{Volume} \vec{E} \cdot \rho\vec{v} \, d^3\vec{r} \right) dt$$

Then recall from the definition of current density $\vec{J} = \rho\vec{v}$

So $$\frac{W.D.}{\text{Volume}} = \int \vec{E}\cdot\vec{J}\, dt$$

or $$\frac{\partial}{\partial t}\left(\frac{W.D.}{\text{Volume}}\right) = \vec{E}\cdot\vec{J}$$

③ With this form, start substituting the Maxwell's equations

1. Subst. Ampere's Law to $\vec{J}$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \rightarrow \vec{J} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) - \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \varepsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$
$$= \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \frac{1}{2}\varepsilon_0\frac{\partial}{\partial t}|\vec{E}|^2$$

2. The first term can be rewritten with vector calculus identity and substitute Faraday's Law

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot(\vec{B}\times\vec{E}) + \vec{B}\cdot(\vec{\nabla}\times\vec{E})$$

$\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$

$$= \underbrace{-\vec{\nabla}\cdot(\vec{E}\times\vec{B})}_{\text{Switch order}} + \vec{B}\cdot\underbrace{\left(-\frac{\partial\vec{B}}{\partial t}\right)}_{\text{Faraday's Law}}$$

$$= -\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \frac{1}{2}\frac{\partial}{\partial t}|\vec{B}|^2$$

Finally we reach the equation

$$\frac{\partial}{\partial t}\left(\frac{W.D.}{\text{Volume}}\right) = -\frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \frac{1}{2\mu_0}\frac{\partial}{\partial t}|\vec{B}|^2 - \frac{1}{2}\varepsilon_0\frac{\partial}{\partial t}|\vec{E}|^2$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\left(\frac{1}{\mu_0}\vec{E}\times\vec{B}\right) = -\frac{\partial}{\partial t}\left[\frac{W.D.}{\text{Volume}} + \frac{1}{2}\varepsilon_0|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2\right]}$$

This is the Poynting Theorem

Interpretation

We can see the Poynting theorem is in the form

$$\vec{\nabla} \cdot (\text{Some vector}) = -\frac{\partial}{\partial t}(\text{Some scalar})$$

$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$

which is indicating some kind of conservation (like in charge conservation)

II The 3 terms on RHS

$\frac{\text{W.D.}}{\text{Volume}}$ = W.D on charge per volume due to surronding $\vec{E}/\vec{B}$

i.e. The amount that will become mechanical energy of the charge

$\frac{1}{2}\varepsilon_0|\vec{E}|^2$ = Energy density in E field. From previous derivation, this is exactly the PE stored in a charge config.

$\frac{1}{2\mu_0}|\vec{B}|^2$ = By symmetry, we can interpret this as the energy density in B field, or the PE stored in a config. of current / moving charge.

☆ If we only consider a close system, then energy (density) must conserve

i.e.
$$\frac{\partial}{\partial t}\left[\underbrace{\frac{\text{W.D}}{\text{Volume}}}_{\text{Mechanical energy of charge}} + \underbrace{\frac{1}{2}\varepsilon_0|\vec{E}|^2}_{\text{PE due to electric interaction}} + \underbrace{\frac{1}{2\mu_0}|\vec{B}|^2}_{\text{PE due to magnetic interaction}}\right] = 0$$

They are the only energy that are related to charges interaction.

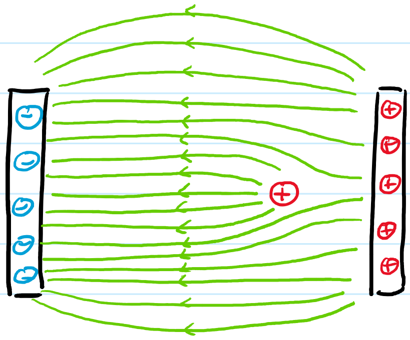
Other kinds of non-EM energy are ignored.

Illustration

If everything is static

total energy $= \iiint_{\text{whole space}} \frac{1}{2}\varepsilon_0 |\vec{E}|^2 \, d^3\vec{r}$

because there is only E field



Once the center charge is allowed to move

Its position changes after some time.

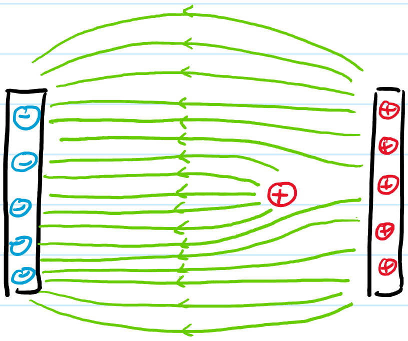
↳ W.D. on it / KE gain is calculated by the trajectory it moves

↳ The $\vec{E}$ field distribution change

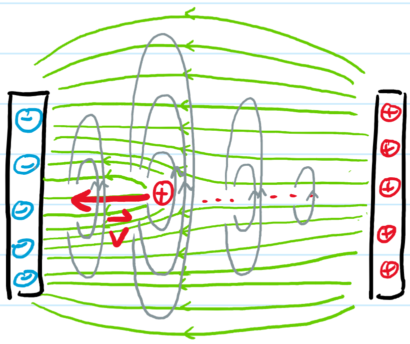
- It gains velocity

↳ B field is created

Total energy becomes $\frac{1}{2}mv^2 + \iiint_{\text{whole space}} \frac{1}{2}\varepsilon_0 |\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2 \, d^3\vec{r}$



After some time ⟶



Initially static

The charge becomes moving
- E field changes
- B field created

(The distortion due to accelerated charge are not drawn)

[2] Divergence on LHS : Poynting Vector (Field)

We define $\boxed{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \text{Poynting vector (field)}}$ to describe the flow of electromagnetic energy in a space

↳ i.e. Energy carried by charge & field

In the case of open system, the 3 kinds of energy do not have to be conserved. The Poynting vector field tells how much electromagnetic energy is entering / leaving the system.

By divergence theorem, we have the integral version

$$\iiint_{\text{a region}} \vec{\nabla}\cdot\vec{S}\, d^3\vec{r} \qquad \iiint_{\text{a region}} -\frac{\partial u}{\partial t}\, d^3\vec{r}$$

the sum of the 3 energy densities

$$u = \frac{\text{Mech. energy}}{\text{Volume}} + \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

$$\underbrace{\oiint_{\substack{\text{surface}\\\text{of the region}}} \vec{S}\cdot\hat{n}\, d^2\vec{r}}_{\text{Out flux of } \vec{S}} = \underbrace{\frac{d}{dt}U}_{\text{Loss of energy}}$$

Volume integral on energy density = total energy

Note 1 Flux integral on $\vec{S}$ gives rate of energy loss by time

We can literally interpret it as

$$\vec{S} \sim \frac{\boxed{\text{Power loss}}\text{ in a region}}{\text{surface area of the region}}$$

Note 2 : We can express his influx / outflux can be expressed purely by $\vec{E}/\vec{B}$ because

Mechanical energy can be converted to EM energy only if it involves an object with charge

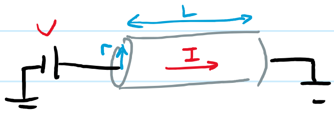
- Motion of charged object will create / distort its surrounding $\vec{E} / \vec{B}$

Note 3: We will see later that EM wave's propagation direction is also $\vec{E} \times \vec{B}$. The loss of energy as Poynting vector in fact indicates EM wave emission.

Example on Poynting vector

(The most common example in EM textbooks)

Consider a cylindrical wire

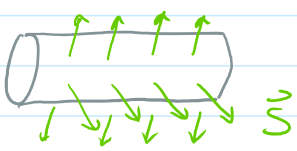


- Connected between voltage V, current passing through $= I$
- Radius r, length L

$\hookrightarrow$ E field in the wire $= \frac{V}{L}$ (along wire direction)

B field on the wire's surface $= \frac{\mu_0 I}{2\pi r}$ (in angular direction)

$\Rightarrow$ Poynting vector $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{VI}{2\pi r L}$ (radially outward)



$= \frac{\text{Power consumed by resistance}}{\text{Surface Area of the wire}}$

Extra : Momentum in E & M

If we put a charge in E field $\Rightarrow$ It accelerates

If we put a moving charge in B field $\Rightarrow$ It changes direction

$\Rightarrow$ There is a transfer of momentum to the charge

$\Rightarrow$ Where are the momentum from? From the field.

We can have a taste of what is the expression of conservation of momentum in E & M like.

Derivation

① Again start with Lorentz force, as it is the only way to transfer momentum to charge

$$\frac{d}{dt}\left(\begin{matrix}\text{momentum}\\ \text{of charge}\end{matrix}\right) = q\vec{E} + q\vec{v}\times\vec{B} \qquad \left(\text{Just } F = ma = \frac{dp}{dt}\right)$$

$$= \iiint_{\text{region}} \rho\vec{E} + \vec{J}\times\vec{B}\, d^3\vec{r}$$

$$= \iiint_{\text{region}} f\, d^3\vec{r}$$

Denote as "force per volume"

② Express ρ & $\vec{J}$ by $\vec{E}/\vec{B}$, and then symmetrize $\vec{E}$ & $\vec{B}$ by the 4 Maxwell's Equations

$$\rho\vec{E} + \vec{J}\times\vec{B} = \underbrace{\varepsilon_0(\vec{\nabla}\cdot\vec{E})}_{\text{Gauss's Law}}\vec{E} + \underbrace{\left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)}_{\text{Ampere's Law}}\times\vec{B}$$

$$= \varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

$\boxed{1}$ The last term can be expanded by Chain rule

$$\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \times \vec{B} = \varepsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \varepsilon_0 \vec{E} \times \underbrace{\frac{\partial \vec{B}}{\partial t}}_{\text{Faraday's Law}}$$

$$= \varepsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \underbrace{\varepsilon_0 \vec{E} \times (\vec{\nabla}\times\vec{E})}_{\text{Switch order of cross product}}$$

$$= \varepsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \varepsilon_0 (\vec{\nabla}\times\vec{E})\times\vec{E}$$

$\boxed{3}$ By divergence of $\vec{B} = 0$, adding this term changes nothing

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B} = 0$$

Now we arrive a symmetric expression between $\vec{E}$ & $\vec{B}$

$$\rho\vec{E} + \vec{J}\times\vec{B} = -\varepsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \quad \text{——— } \boxed{1}$$

$$\left.\begin{aligned} &+ \varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \varepsilon_0(\vec{\nabla}\times\vec{E})\times\vec{E} \\ &+ \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} \end{aligned}\right\} \boxed{2}$$

- $\boxed{1}$ ~ $\vec{E}\times\vec{B} \Rightarrow$ Can express by Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

- $\boxed{2}$ ~ Perhaps can be written into something understandable by applying some vector identities?

$$(\vec{\nabla}\cdot\vec{E})\vec{E} + \underbrace{(\vec{\nabla}\times\vec{E})\times\vec{E}}$$

Vector identity:
$(\nabla\times A)\times B = (B\cdot\nabla)A - B\cdot(\nabla A)$

$$= (\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \underbrace{\vec{E}\cdot(\vec{\nabla}\vec{E})}$$

Vector identity
$\nabla|A|^2 = 2A\cdot(\nabla A)$

$$= (\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}|\vec{E}|^2$$

(Do the same for $\vec{B}$)

③ Now come to the most annoying part — we need to write it as the divergence of something in order to construct a conservation equation $\left(-\frac{\partial(\cdots)}{\partial t} = \vec{\nabla}\cdot(\cdots)\right)$

But also we need the divergence of this thing to be vector

$\Rightarrow$ this thing must at least be expressed in matrix

$$\vec{\nabla}\cdot\vec{A} \sim \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \star$$

Divergence on vector gives a single number

$$\vec{\nabla}\cdot\mathbf{A} \sim \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)\begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix}$$

Divergence on matrix gives a vector

$$= \left(\left(\frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{yx}}{\partial x} + \frac{\partial A_{zx}}{\partial x}\right), \left(\frac{\partial A_{xy}}{\partial y} + \frac{\partial A_{yy}}{\partial y} + \frac{\partial A_{zy}}{\partial y}\right), \left(\frac{\partial A_{xz}}{\partial z} + \frac{\partial A_{yz}}{\partial z} + \frac{\partial A_{zz}}{\partial z}\right) \right)$$

$$= (\triangle\ \square\ \star) \leftarrow \text{a vector with 3 components}$$

Here we try to derive this matrix :

1. First 2 terms combine to be

$$(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}$$

$$= \left[\left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + (E_x\ E_y\ E_z)\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\right]\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)\left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}(E_x\ E_y\ E_z)\right]$$

$$= \left(\frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z} \right) \underline{\begin{pmatrix} E_xE_x & E_xE_y & E_xE_z \\ E_yE_x & E_yE_y & E_yE_z \\ E_zE_x & E_zE_y & E_zE_z \end{pmatrix}}$$

Part of the matrix we need

$\boxed{2}$ The last term can be expanded by

$$\frac{1}{2}\vec{\nabla}|\vec{E}|^2 = \frac{1}{2}\left(\frac{\partial|\vec{E}|^2}{\partial x} \; \frac{\partial|\vec{E}|^2}{\partial y} \; \frac{\partial|\vec{E}|^2}{\partial z} \right)$$

$$= \left(\frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z} \right) \underline{\frac{1}{2}\begin{pmatrix} |\vec{E}|^2 & 0 & 0 \\ 0 & |\vec{E}|^2 & 0 \\ 0 & 0 & |\vec{E}|^2 \end{pmatrix}}$$

Another part we need

(Also add the same expression for $\vec{B}$)

Finally we arrive at the expression of this matrix

$\boldsymbol{\sigma} = (\sigma_{ij})$ a 3×3 matrix with each component

$$\sigma_{ij} = \begin{cases} \varepsilon_0 E_iE_j + \frac{1}{\mu_0}B_iB_j & \text{if } i \neq j \\ \varepsilon_0\left(E_i^2 - \frac{1}{2}|\vec{E}|^2\right) + \frac{1}{\mu_0}\left(B_i^2 - \frac{1}{2}|\vec{B}|^2\right) & \text{if } i = j \end{cases}$$

This matrix $\boldsymbol{\sigma}$ is called Maxwell Stress Tensor

Together with Poynting vector and the Maxwell stress tensor the expression of momentum conservation in E&M is

$$\frac{\partial}{\partial t}\left(\frac{\text{momentum}}{\text{Volume}}\right) = \vec{f} = \rho\vec{E} + \vec{J}\times\vec{B} = \vec{\nabla}\cdot\boldsymbol{\sigma} - \mu_0\varepsilon_0\frac{\partial\vec{S}}{\partial t}$$

$$\boxed{\vec{\nabla}\cdot\boldsymbol{\sigma} = \frac{\partial}{\partial t}\left(\frac{\text{momentum}}{\text{Volume}} + \mu_0\varepsilon_0\vec{S}\right)}$$

Interpretation

Again we arrive in the form $\vec{\nabla} \cdot (\cdots) = - \frac{\partial (\cdots)}{\partial t}$

$$\vec{\nabla} \cdot (\text{Some matrix}) = - \frac{\partial}{\partial t} (\text{Some vector})$$

which is another formula of some conversion.

II The 2 terms on RHS

$\frac{\text{momentum}}{\text{Volume}}$ = Momentum on charge per volume

i.e. Just the $p = mv$ in Newton 2^{nd} Law

$\mu_0 \varepsilon_0 \vec{S}$ = Momentum density carried by the $\vec{E}/\vec{B}$ field

As EM wave travells in the same direction as $\vec{S}$

This is saying momentum transfer is by EM wave.

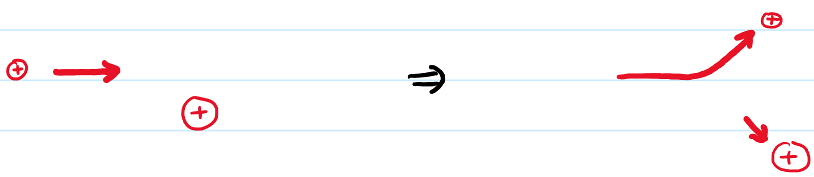
☆☆ If we only consider a close system,

momentum (density) must conserve

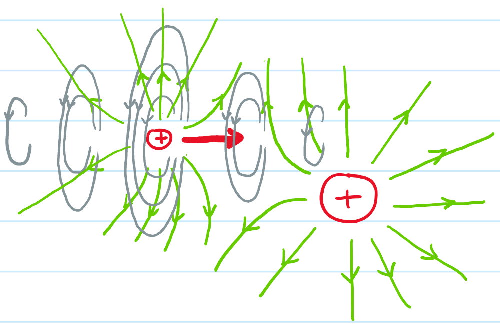
$$\text{i.e.} \quad \frac{\partial}{\partial t} \Bigg[\underbrace{\frac{\text{momentum}}{\text{Volume}}}_{\text{Momentum of charge}} + \underbrace{\mu_0 \varepsilon_0 \vec{S}}_{\text{Momentum in field}} \Bigg] = 0$$

Illustration

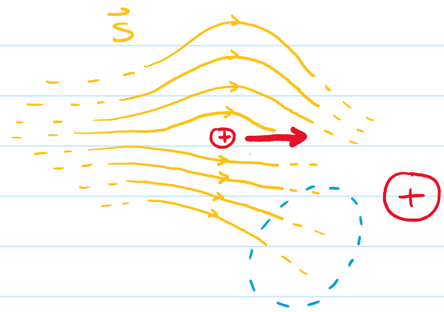
When 2 charges move closer to each other



$\vec{E}/\vec{B}$ field is approximately like $\qquad$ $\vec{S}$ field looks like this



$\Rightarrow$



This region stores a momentum in $\searrow$ direction

The $\vec{S}$ field is like a storage of momentum in the space. When the charge contacts with the $\vec{S}$ field there, it "consumes" the field to become its own momentum.

12 Divergence on LHS : Stress Tensor

In the case of open system, momentum do not have to be conserved. We use the "flux of stress tensor" to describe how much momentum is entering / leaving the system.

To understand what stress tensor is, again apply divergence theorem.

$$\underset{\text{a region}}{\iiint} \vec{\nabla} \cdot \boldsymbol{\sigma} \, d^3\vec{r} = \underset{\text{a region}}{\iiint} \vec{f} \, d^3\vec{r}$$

force per volume in the region

$$\underset{\text{Surface of the region}}{\oiint} \underbrace{\boldsymbol{\sigma} \cdot \hat{n}}_{\text{Matrix multiply on normal vector}} \, d^2\vec{r} = \vec{F}$$

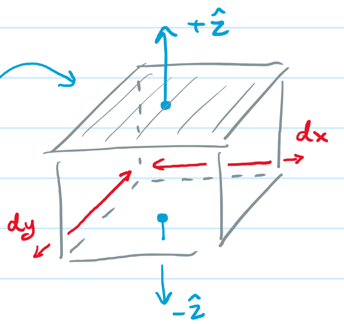
Net force on the region

Suppose the region is a cube,

and we only look at the

force on the top surface

(Then $\hat{n}\, d\vec{r}^{2} \Rightarrow +\hat{z}\, dx\, dy$)



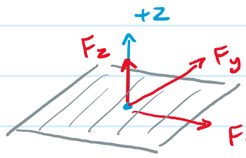
The surface integral becomes

$$\underset{\text{only top surface}}{\iint} \sigma \cdot \hat{n}\, d\vec{r}^{2} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\, dy = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

$$\Rightarrow \quad F_x = \sigma_{xz}\, dx\, dy \ , \quad F_y = \sigma_{yz}\, dx\, dy \ , \quad F_z = \sigma_{zz}\, dx\, dy$$

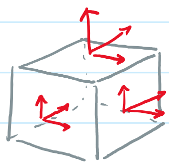
which means there are forces in

all 3 directions acting on the surface!



This happens on all 3 pairs of opposite surfaces

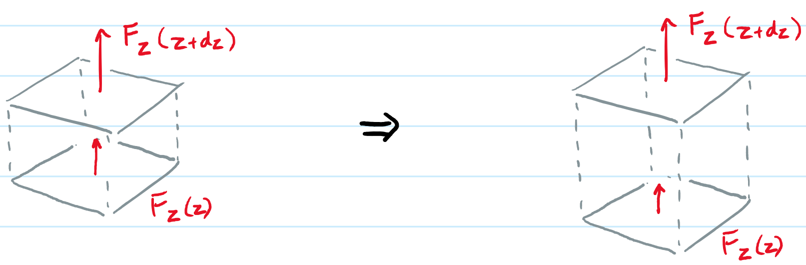
The pair gives physical meaning



Physical Correspondance

① Net forces perpendicular to the surfaces

$\hookrightarrow$ Tension / Compression (Stress force)



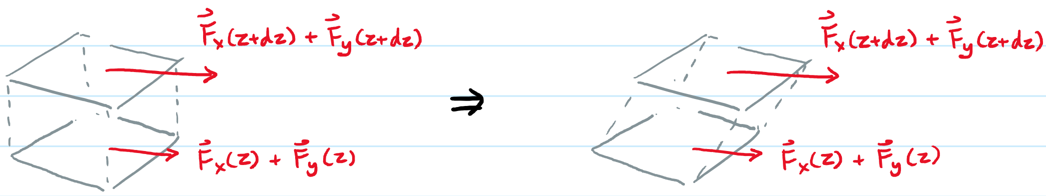
We can see that the diagonal elements in the σ matrix correspond to the stress forces on the 3 directions.

- Stress force on yz surfaces $= \sigma_{xx}\,\hat{x}\;dy\,dz$
- Stress force on xz surfaces $= \sigma_{yy}\,\hat{y}\;dx\,dz$
- Stress force on xy surfaces $= \sigma_{zz}\,\hat{z}\;dx\,dy$

Remark: In mechanics, stress is defined as $\sigma = \dfrac{\text{Force} \perp \text{surface}}{\text{Surface Area}}$ which is exactly the same thing here.

② Net forces parallel to the surfaces

↳ Shear forces (Strain force)



We can see that the off-diagonal elements in the σ matrix correspond to the strain forces on the 3 directions.

- Strain force on yz surfaces $= (\sigma_{yx}\,\hat{y} + \sigma_{zx}\,\hat{z})\;dy\,dz$
- Strain force on xz surfaces $= (\sigma_{xy}\,\hat{x} + \sigma_{zy}\,\hat{z})\;dx\,dz$
- Strain force on xy surfaces $= (\sigma_{xz}\,\hat{x} + \sigma_{yz}\,\hat{y})\;dx\,dy$

Remark: In mechanics, strain is defined as $\varepsilon = \dfrac{\text{Force} \parallel \text{surface}}{\text{Surface Area}}$ which is exactly the same thing here.

Short Summary

The term $\oiint \overleftrightarrow{\sigma} \cdot \hat{n}\, d^2 \vec{r} = F$ is simply saying

Net force on a region = Sum of all stress / strain forces on its surface

Finally in E&M, the conservation of momentum means

$$\underbrace{\vec{\nabla} \cdot \overleftrightarrow{\sigma}} = \underbrace{\frac{\partial}{\partial t}\left(\frac{\text{momentum}}{\text{Volume}} + \mu_0 \varepsilon_0 \vec{S} \right)}$$

The equivalent force onto the surface of a region due to external $\vec{E}/\vec{B}$

(σ_{ij} calculated by $\vec{E}/\vec{B}$ on the surface)

Rate of momentum increase in the region which can become $\begin{cases} \text{momentum on charge} \\ \text{momentum in field} \end{cases}$