

# Ordinary Differential Equation

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## Overview:

Newton's 2<sup>nd</sup> Law is a differential equation

$$F(t) = ma(t) = m \frac{d^2}{dt^2} x(t)$$

which is an equation of  $x(t)$ , but involves the 2<sup>nd</sup> derivative of  $x(t)$ .

In this note, I will introduce some basic techniques to solve differential equations that you will encounter in mechanics.

- Classification of differential equations
- 1<sup>st</sup> order linear ODE
- 2<sup>nd</sup> order linear ODE
- Linear ODE with non-homogeneous terms

The goal is to be familiar with solving the equation of motion for general harmonic oscillators, i.e.

$$-kx(t) + \gamma \frac{d}{dt} x(t) + F(t) = m \frac{d^2}{dt^2} x(t)$$

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## 1 Classification of Differential Equations

We can classify differential equations with these characteristics:

1. Number of variables in the wanted function
2. Order
3. Linearity
4. Type of Coefficients
5. Homogeneity

## 1.1 Number of variables in the wanted function

- If the function to be solved is a single variable function, the equation is called an **Ordinary Differential Equation (ODE)**.
- If the function to be solved is a multivariable function, the equation is called a **Partial Differential Equation (PDE)**.

It is easy to identify PDE from ODE because they must involve partial derivatives. Solving PDE can be a lot more complicated than ODE. We will not deal with PDE in this note at all.

## 1.2 Order

Order = Finding the highest derivative of the wanted function in the equation. E.g.

$$\underbrace{\frac{d}{dt}f(t)}_{1^{\text{st}}} + \underbrace{f(t)}_{0^{\text{th}}} = \ln t \quad \left( \begin{array}{l} \text{highest} = 1^{\text{st}} \text{ derivative} \\ \Rightarrow 1^{\text{st}} \text{ order} \end{array} \right)$$

$$\underbrace{\frac{d^2}{dt^2}f(t)}_{2^{\text{nd}}} + \underbrace{f(t)\frac{d}{dt}f(t)}_{0^{\text{th}} \times 1^{\text{st}}} = \sin t \quad \left( \begin{array}{l} \text{highest} = 2^{\text{nd}} \text{ derivative} \\ \Rightarrow 2^{\text{nd}} \text{ order} \end{array} \right)$$

$$\underbrace{\left(\frac{d}{dt}f(t)\right)^2}_{1^{\text{st}} \times 1^{\text{st}}} + \underbrace{(f(t))^2}_{0^{\text{th}} \times 0^{\text{th}}} = 1 \quad \left( \begin{array}{l} \text{highest} = 1^{\text{st}} \text{ derivative} \\ \Rightarrow 1^{\text{st}} \text{ order} \end{array} \right)$$

## 1.3 Linearity

Linear = Whether any terms contain multiplication between derivatives. E.g.

$$\underbrace{\frac{d^2}{dt^2}f(t)}_{\text{power } 1} - e^t \underbrace{\frac{d}{dt}f(t)}_{\text{power } 1} + \underbrace{f(t)}_{\text{power } 1} = 0 \quad (\text{Power} \leq 1 \Rightarrow \text{Linear})$$

$$\underbrace{\frac{d^2}{dt^2}f(t)}_{\text{power } 1} - \underbrace{\left(\frac{d}{dt}f(t)\right)^2}_{\text{power } 2} = \sin t \quad (\text{Power} > 1 \Rightarrow \text{Non-Linear})$$

$$\underbrace{\frac{d^2}{dt^2}f(t)}_{\text{power } 1} + \underbrace{f(t)\frac{d}{dt}f(t)}_{\text{power } 1 + \text{power } 1} = 5 \quad (\text{Power} > 1 \Rightarrow \text{Non-Linear})$$

## 1.4 Type of Coefficients

If all the coefficients of the derivatives are constant (i.e. not function of  $t$ ), The equation can be solved with much easier methods. E.g.

$$\underbrace{2}_{+2} \frac{d^2}{dt^2} f(t) - \underbrace{1}_{-1} \frac{d}{dt} f(t) + \underbrace{2}_{+2} f(t) = \cos t \quad (\text{All are constants})$$

$$\frac{(t^2 - 1)}{(t^2 - 1)} \frac{d^2}{dt^2} f(t) - \underbrace{t}_{-t} \frac{d}{dt} f(t) + \underbrace{2}_{+2} f(t) = 0 \quad (\text{Some are functions of } t)$$

## 1.5 Homogeneity

Homogeneity = Whether all terms contain the wanted functions or its derivatives. E.g.

$$\underbrace{\frac{d^2}{dt^2} f(t)}_{\text{Yes}} + 2 \cos(t) \underbrace{f(t)}_{\text{Yes}} = 0 \quad (\text{All yes} \Rightarrow \text{Homogeneous})$$

$$\underbrace{\frac{d}{dt} f(t)}_{\text{Yes}} + \underbrace{4f(t)}_{\text{Yes}} - \underbrace{\ln(t)}_{\text{No}} = 0 \quad (\text{Some terms not having } f(t) \Rightarrow \text{Non-homogeneous})$$

What ODEs can we solve analytically?

- Linear ODEs with any order
  - Constant coefficients
    - \* Homogeneous - **The  $e^{\lambda t}$  trick**
    - \* Non-homogeneous - **Method of undetermined coefficient**
  - Non-constant coefficients - **More complicated methods, E.g.**

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Integrating factor  
 Series expansion  
 Laplace/Fourier transform
- Non-linear ODEs - **No general methods. Only case by case.**

For a general harmonic oscillator problem, the Newton's 2<sup>nd</sup> Law writes

$$-kx(t) + \gamma \frac{d}{dt} x(t) + F(t) = m \frac{d^2}{dt^2} x(t)$$

where  $k$ ,  $\gamma$  and  $m$  are usually given as constants, and  $F(t)$  could be arbitrary function of  $t$ . So this is a **2<sup>nd</sup> order linear constant coefficient non-homogeneous ODE**.

## 2 1<sup>st</sup> Order Linear Constant Coefficient Homogeneous ODE

This is the simplest kind of ODE

$$\frac{d}{dt}f(t) + \lambda f(t) = 0$$

Here  $\lambda$  is a constant number. The solution is trivial by making use of the fact

$$\frac{d}{dt}e^{at} = a \cdot e^{at}$$

which is exactly saying  $f(t) = e^{at}$  is a solution to the equation  $\frac{d}{dt}f(t) - af(t) = 0$ . We can also observe that the relation still holds after multiplying any constant to  $f(t)$ . So we have

$$\text{General Solution : } f(t) = \underline{C}e^{-\lambda t}$$

$C$ =any constant number

In fact, we can show that this is the only solution to this ODE, by solving it with integration:

$$\frac{d}{dt}f(t) + \lambda f(t) = 0$$

$$\frac{1}{f(t)} \frac{df(t)}{dt} + \lambda = 0$$

$$\frac{d}{dt}[\ln f(t)] = -\lambda$$

$$\ln f(t) = \int -\lambda dt = -\lambda t + C$$

$$f(t) = e^{-\lambda t + C}$$

$$= \underline{C'}e^{-\lambda t}$$

Take  $C' = e^C$ , which is still a constant

**Example 2.1.** The **decay equation** is written as

$$\frac{d}{dt}N(t) = -kN(t)$$

where

–  $N(t)$  = Number of particles

–  $\frac{d}{dt}N(t)$  = Decay rate in number of particles

The equation theorizes phenomena where rate of decay is proportional to the number of particles present, i.e.  $\frac{dN}{dt} \propto N$ . From above, we can tell the general solution to be

$$N(t) = Ce^{-kt}$$

where  $C$  can be any constant. How do we tell what number we should substitute into  $C$  in a scenario? **By matching an initial condition.**

For example, given that at  $t = 0$ , we are told that there are  $N_0$  particles. Then by substitution,

$$N(0) = Ce^{-k \cdot 0} = C = N_0$$

which leads to a specific solution  $N(t) = N_0 e^{-kt}$ .

Side note:

The expression for *half life* comes from this solution. By definition, At half life  $t = \tau_{\frac{1}{2}}$ , number of particles remain =  $\frac{N_0}{2} = \frac{1}{2}$  the number at start ( $t = 0$ ). Thus

$$N\left(\tau_{\frac{1}{2}}\right) = N_0 e^{-k\tau_{\frac{1}{2}}} = \frac{N_0}{2}$$

$$\tau_{\frac{1}{2}} = \frac{\ln 2}{k}$$

### 3 2<sup>nd</sup> Order Linear Constant Coefficient Homogeneous ODE

The equation comes in the form

$$a \frac{d^2}{dt^2} f(t) + b \frac{d}{dt} f(t) + c f(t) = 0$$

where  $a, b, c$  are all constants. To solve it, we can apply the same trick as in 1<sup>st</sup> ODE - substitute  $f(t) = e^{\lambda t}$  :

$$a \frac{d^2}{dt^2} e^{\lambda t} + b \frac{d}{dt} e^{\lambda t} + c e^{\lambda t} = 0$$

$$a \lambda^2 e^{\lambda t} + b \lambda e^{\lambda t} + c e^{\lambda t} = 0$$

$$a \lambda^2 + b \lambda + c = 0 \quad \text{(a quadratic equation of } \lambda \text{)}$$

$$\Rightarrow \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can either take  $f_1(t) = Ce^{\frac{-b + \sqrt{b^2 - 4ac}}{2a}t}$  or  $f_2(t) = Ce^{\frac{-b - \sqrt{b^2 - 4ac}}{2a}t}$  as a solution.

#### 3.1 Superposition of Solutions

However,  $f_1(t)$  or  $f_2(t)$  alone is **NOT** the **general solution** because linear ODE allows superpositions (linear combination), i.e.

If  $f_1(t)$  and  $f_2(t)$  are solutions to a linear homogeneous ODE, any superposition  $C_1f_1(t) + C_2f_2(t)$  is also a solution to the ODE, for arbitrary constants  $C_1, C_2$ .

General Solution :  $f(t) = C_1e^{\frac{-b+\sqrt{b^2-4ac}}{2a}t} + C_2e^{\frac{-b-\sqrt{b^2-4ac}}{2a}t}$

### Proof

Given that  $f_1(t)$  and  $f_2(t)$  are solutions:

$$\begin{cases} a\frac{d^2}{dt^2}f_1(t) + b\frac{d}{dt}f_1(t) + cf_1(t) = 0 \\ a\frac{d^2}{dt^2}f_2(t) + b\frac{d}{dt}f_2(t) + cf_2(t) = 0 \end{cases}$$

To test whether  $C_1f_1(t) + C_2f_2(t)$  is a solution, we can do substitution:

$$\begin{aligned} \text{L.H.S.} &= a\frac{d^2}{dt^2}[C_1f_1(t) + C_2f_2(t)] + b\frac{d}{dt}[C_1f_1(t) + C_2f_2(t)] + c[C_1f_1(t) + C_2f_2(t)] \\ &= C_1 \cdot \left[ a\frac{d^2}{dt^2}f_1(t) + b\frac{d}{dt}f_1(t) + cf_1(t) \right] + C_2 \cdot \left[ a\frac{d^2}{dt^2}f_2(t) + b\frac{d}{dt}f_2(t) + cf_2(t) \right] \\ &= C_1 \cdot 0 + C_2 \cdot 0 \\ &= 0 \\ &= \text{R.H.S.} \end{aligned}$$

So  $C_1f_1(t) + C_2f_2(t)$  is also a solution. □

### Side note:

This superposition property can be easily extended to any  $N^{\text{th}}$  order linear ODE.

1. If a linear ODE is of  $N^{\text{th}}$  order, there must be  $N$  (linear) independent solution  
(Require rigorous proof from linear algebra):

$$f_1(t), f_2(t), \dots, f_N(t)$$

2. The general solution is then any superposition (linear combination) of these  $N$  solutions:

$$f(t) = C_1f_1(t) + C_2f_2(t) + \dots + C_Nf_N(t)$$

with  $C_1, C_2, \dots, C_N$  being some constants.

### 3.2 3 Sub-cases of the Solution

We may further derive the general solution according to the value of  $b^2 - 4ac$ .

#### 3.2.1 Case 1: $b^2 - 4ac > 0$

Both  $\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  are real number. Nothing can be further simplified. We would just keep the form

$$f(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t} \quad (\text{Both } \lambda \text{ real})$$

#### 3.2.2 Case 2: $b^2 - 4ac < 0$

Both  $\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  are complex number. We can separate their real and imaginary parts. Denote as

$$\text{Re}[\lambda_{\pm}] = -\frac{b}{2a} \stackrel{\text{def}}{=} p, \quad \text{Im}[\lambda_{\pm}] = \pm \frac{\sqrt{4ac - b^2}}{2a} \stackrel{\text{def}}{=} \pm q$$

which is just a re-labelling to  $\lambda_{\pm} \stackrel{\text{def}}{=} p \pm iq$ .

Then we can apply the **Euler formula**

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Rewriting  $f(t)$  as

$$\begin{aligned} f(t) &= C_1 e^{(p+iq)t} + C_2 e^{(p-iq)t} \\ &= e^{pt} [C_1 e^{iqt} + C_2 e^{-iqt}] \\ &= e^{pt} [C_1 (\cos qt + i \sin qt) + C_2 (\cos qt - i \sin qt)] \\ &= e^{pt} \left[ \underbrace{(C_1 + C_2)}_{\substack{\text{Both are constants.} \\ \text{Can combine.}}} \cos qt + i \underbrace{(C_1 - C_2)}_{\substack{\text{Both are constants.} \\ \text{Can combine.}}} \sin qt \right] \end{aligned}$$

$$f(t) = e^{pt} [C'_1 \cos qt + C'_2 \sin qt] \quad (\text{Both } \lambda \text{ complex})$$

which is an expression without the imaginary  $i$ , so that we can use it to describe physics.

We can also construct another convenient form for physics by trigonometry. Combine the  $\sin / \cos$  into 1 sinusoidal function by change of variables:

$$\begin{cases} C'_1 = A \cos \phi \\ C'_2 = -A \sin \phi \end{cases} \Leftrightarrow \begin{cases} A = \sqrt{C'^2_1 + C'^2_2} \\ \phi = \tan^{-1} \left( \frac{-C'_2}{C'_1} \right) \end{cases}$$

Such that

$$\begin{aligned} f(t) &= e^{pt} [C'_1 \cos qt + C'_2 \sin qt] \\ &= e^{pt} [(A \cos \phi) \cos qt + (-A \sin \phi) \sin qt] \end{aligned}$$

$$f(t) = e^{pt} \cdot A \cos (qt + \phi) \quad (\text{Both } \lambda \text{ complex})$$

Here we have used the cosine addition rule  $\cos(a + b) = \cos a \cos b - \sin a \sin b$ .

As a conclusion, we have reached 3 different forms of solution for the case  $b^2 - 4ac < 0$ , which all are convenient to use in some scenarios.

$$f(t) = \begin{cases} C_1 e^{(p+iq)t} + C_2 e^{(p-iq)t} & \text{(Complex form)} \\ e^{pt} [C'_1 \cos qt + C'_2 \sin qt] & \text{(CS form)} \\ e^{pt} \cdot A \cos(qt + \phi) & \text{(Amplitude form)} \end{cases} \quad (\text{Both } \lambda \text{ complex})$$

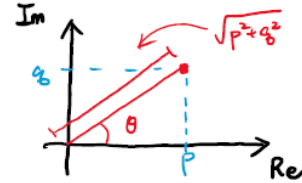
Side note 1:

The **Euler formula** is an extension to the definition of sin / cos function to complex number inputs. It can be proven by Taylor series expansion:

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4} \\ \sin x &= x - \frac{x^3}{3} + \frac{x^5}{5} \end{aligned}$$

It also allows any complex number  $p + iq$  to be expressed in polar form:

$$\begin{aligned} z &= p + iq \\ &= \sqrt{p^2 + q^2} \cos \theta + i \sqrt{p^2 + q^2} \sin \theta \\ &= \sqrt{p^2 + q^2} (\cos \theta + i \sin \theta) \\ &= \sqrt{p^2 + q^2} e^{i\theta} \end{aligned}$$



Side note 2:

The **Taylor series expansion** is a polynomial approximation to the any continuous functions  $f(x)$ , for finding the value of  $f(k + x)$  given the value of  $f(k)$ .

If the following is a "good" approximation (applicable when  $x$  is small enough)

$$f(k + x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

with  $k$  being a known value, then we can determine  $a_0, a_1, \dots, a_n$ :

$$a_0 = f(k), \quad a_1 = \left. \frac{df(t)}{dt} \right|_{t=k}, \quad a_2 = \frac{1}{2!} \left. \frac{d^2 f(t)}{dt^2} \right|_{t=k}, \quad a_3 = \frac{1}{3!} \left. \frac{d^3 f(t)}{dt^3} \right|_{t=k}, \dots$$

$$a_n = \left. \frac{1}{n!} \frac{d^n f(t)}{dt^n} \right|_{t=k}$$



Proof

Let  $t = k + x$ . Differentiate against  $t$  and substitute  $x = 0$  (so  $t$  becomes  $k + 0 = k$ ),

$$f(k+0) = a_0 + a_1(x) + a_2(x)^2 + a_3(x)^3 + a_4(x)^4 + \dots$$

$$\left. \frac{df(t)}{dt} \right|_{t=k} = +a_1 + 2a_2(x) + 3a_3(x)^2 + 4a_4(x)^3 + \dots$$

$$\left. \frac{d^2 f(t)}{dt^2} \right|_{t=k} = +2a_2 + (3 \cdot 2)a_3(x) + (4 \cdot 3)a_4(x)^2 + \dots$$

$$\left. \frac{d^3 f(t)}{dt^3} \right|_{t=k} = + (3 \cdot 2)a_3 + (4 \cdot 3 \cdot 2)a_4(x) + \dots$$

And so on. □

### 3.2.3 Case 3: $b^2 - 4ac = 0$

This case is problematic in that  $\lambda_+ = \lambda_- = \frac{b^2}{2a} \stackrel{\text{def}}{=} p$ . We can only get 1 independent solution  $Ce^{pt}$  by using the  $e^{\lambda t}$  trick.

But mathematicians say if the ODE is of  $N^{\text{th}}$  order, the general solution must be made of  $N$  independent functions.

How to find the remaining independent function in our 2<sup>nd</sup> order ODE? The method is called **reduction of order**.

1. Let the other independent function be  $v(t)e^{pt}$ . The goal is to find a suitable  $v(t)$  that help it form another solution. First substitute it into the original ODE.

$$a \frac{d^2}{dt^2} [v(t)e^{pt}] + b \frac{d}{dt} [v(t)e^{pt}] + c[v(t)e^{pt}] = 0$$

2. Do product rule for each term:

$$\begin{aligned} \frac{d^2}{dt^2} [v(t)e^{pt}] &= \frac{d^2}{dt^2} v(t)e^{pt} + 2 \frac{d}{dt} v(t) \frac{d}{dt} e^{pt} + v(t) \frac{d^2}{dt^2} e^{pt} \\ &= e^{pt} \left[ \frac{d^2}{dt^2} v(t) + 2p \frac{d}{dt} v(t) + p^2 v(t) \right] \\ \frac{d}{dt} [v(t)e^{pt}] &= \frac{d}{dt} e^{pt} + v(t) \frac{d}{dt} e^{pt} \\ &= e^{pt} \left[ \frac{d}{dt} v(t) + p v(t) \right] \end{aligned}$$

3. Group terms by derivatives of  $v(t)$  and solve it

$$\begin{aligned}
 0 &= ae^{pt} \left[ \frac{d^2}{dt^2} v(t) + 2p \frac{d}{dt} v(t) + p^2 v(t) \right] + be^{pt} \left[ \frac{d}{dt} v(t) + pv(t) \right] + c[v(t)e^{pt}] \\
 &= a \frac{d^2}{dt^2} v(t) - 2 \underbrace{(ap + b)}_{\substack{=0 \\ \text{because } p = -\frac{b}{2a}}} \frac{d}{dt} v(t) + \underbrace{(ap^2 + bp + c)}_{\substack{=0 \\ \text{because } p = -\frac{b}{2a} \text{ is the soln.} \\ \text{to } ax^2 + bx + c = 0}} \\
 &= a \frac{d^2}{dt^2} v(t)
 \end{aligned}$$

$$v(t) = C_1 t + C_2$$

where  $C_1, C_2$  are some constants. Therefore we find the other independent function to be  $(C_1 t + C_2)e^{pt}$ . Note that it already contains the first independent function  $Ce^{pt}$ . We may write

$$f(t) = C_1 e^{pt} + C_2 t e^{pt} \quad (\lambda \text{ equal})$$

*Side note 3:*

We can use the **Leibniz formula** to compute higher derivatives faster.

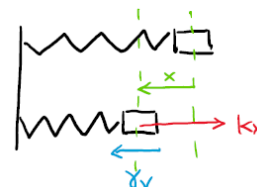
$$\begin{aligned}
 d(uv) &= \underbrace{du}_{1} \cdot \underbrace{v}_{1} + \underbrace{u}_{1} dv \\
 d^2(uv) &= \underbrace{d^2 u}_{1} \cdot \underbrace{v}_{1} + \underbrace{2 du}_{2} \cdot \underbrace{dv}_{1} + \underbrace{u}_{1} \cdot d^2 v \\
 d^3(uv) &= \underbrace{d^3 u}_{1} \cdot \underbrace{v}_{1} + \underbrace{3 d^2 u}_{3} \cdot \underbrace{dv}_{1} + \underbrace{3 d^2 u}_{3} \cdot \underbrace{d^2 v}_{1} + \underbrace{u}_{1} \cdot d^3 v \\
 &\vdots \\
 d^n(uv) &= \sum_{r=0}^n C_r^n (d^r u)(d^{n-r} v)
 \end{aligned}$$

The coefficients for each term are binomial coefficients  $C_r^n = \frac{n!}{r!(n-r)!}$ , which can be computed beforehand.

### 3.3 Application: Damped Harmonic Oscillator

Consider a spring-mass system with damping factor on the spring. When the spring is compressed by a displacement  $x$ , the forces on it are

- Spring's elastic force:  $-kx$  (Require  $k > 0$ )
- Damping force:  $-\gamma v$  (Require  $\gamma > 0$ )



In a naive model, the damping force is usually assumed proportional and opposite to the mass's velocity. Otherwise it will be a lot more difficult to calculate.

The Newton's 2<sup>nd</sup> Law writes:

$$\begin{aligned} (\text{total force}) &= -kx - \gamma v = ma \\ m \frac{d^2}{dt^2} x(t) + \gamma \frac{d}{dt} x(t) + kx(t) &= 0 \end{aligned}$$

Substitute  $x(t) = Ce^{\lambda t}$ , we can find  $\lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$ . Then further analyze by the 3

cases  $\gamma^2 - 4mk \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases}$ , which correspond to different physical behaviors.

### 3.3.1 Case 1: $\gamma^2 - 4mk > 0$ - Over-damped

Check the sign of  $\lambda_{\pm}$ . Since

$$\begin{aligned} \gamma^2 &> \gamma^2 - 4mk > 0 \\ \gamma &> \sqrt{\gamma^2 - 4mk} \\ 0 &> \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} = \lambda_+ \end{aligned}$$

Also  $\lambda_- = \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} < \lambda_+$ . So both  $\gamma$  are negative. We may write

$$\begin{aligned} x(t) &= C_1 e^{-|\lambda_+|t} + C_2 e^{-|\lambda_-|t} \\ &= \text{A sum of 2 exponentially decaying functions} \end{aligned}$$

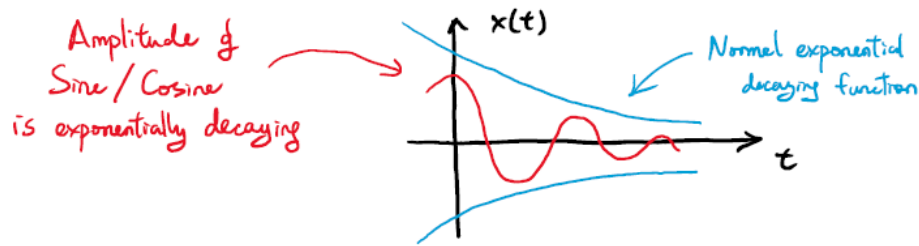


It is called "over-damped" as the damping force is too large, such that the mass can never return to its original position.

### 3.3.2 Case 2: $\gamma^2 - 4mk < 0$ - Under-damped

Write the solution in amplitude form:

$$\begin{aligned} x(t) &= e^{pt} \cdot A \cos(qt + \phi) \\ &= e^{\frac{\gamma}{2m}t} \cdot A \cos\left(\frac{\sqrt{4mk - \gamma^2}}{2m}t + \phi\right) \\ &= (\text{Exponentially Decay}) \times (\text{Sinusoidal}) \end{aligned}$$

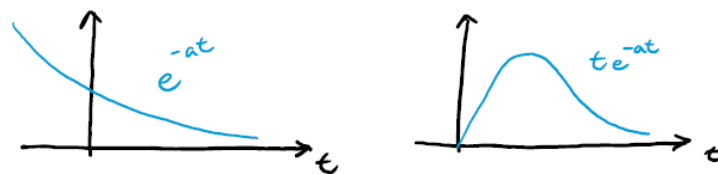


It is called "under-damped" as the damping force is not strong enough to stop the mass. The mass will oscillate forever although the amplitude will decrease with time.

### 3.3.3 Case 3: $\gamma^2 - 4mk = 0$ - Critical-damped

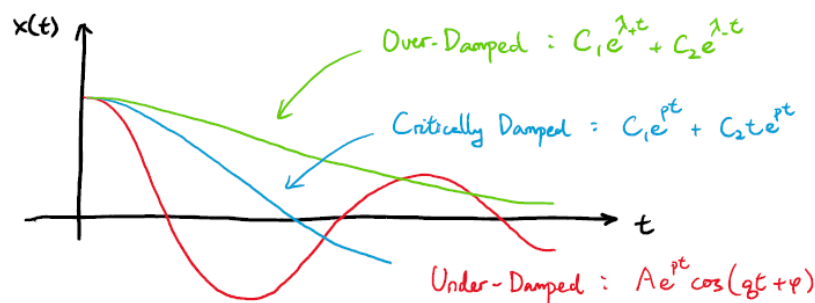
The general solution is

$$x(t) = C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t}$$



It is called "critical-damped" because it is in between the other 2 cases. It looks like over-damped case but the mass can return to the original position.

### Summary in 1 graph



## 4 Non-homogeneous Constant Coefficient Linear ODE

Now we consider non-homogeneous linear ODE. For example,

$$a \frac{d^2}{dt^2} f(t) + b \frac{d}{dt} f(t) + c f(t) = \underbrace{g(t)}_{\substack{\text{non-homogeneous term} \\ \text{i.e. not containing } f(t)}} \neq 0$$

The general solution to a non-homogeneous linear ODE is made of 2 parts:

$$f(t) = f_c(t) + f_p(t)$$

where

- $f_c(t)$  = **Complementary solution**. It is the general solution to the homogeneous counterpart of the ODE, i.e. the solution to

$$a \frac{d^2}{dt^2} f_c(t) + b \frac{d}{dt} f_c(t) + c f_c(t) = 0$$

- $f_p(t)$  = **Particular solution**. Its presence is to cancel the non-homogeneous term.

$$a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) = g(t)$$

Show by substitution to be clearer:

$$\begin{aligned} g(t) &= a \frac{d^2}{dt^2} [f_c(t) + f_p(t)] + b \frac{d}{dt} [f_c(t) + f_p(t)] + c [f_c(t) + f_p(t)] \\ &= \underbrace{\left[ a \frac{d^2}{dt^2} f_c(t) + b \frac{d}{dt} f_c(t) + c f_c(t) \right]}_{\substack{\text{Require the parts} \\ \text{constructed by } f_c(t) \\ \text{to become 0}}} + \underbrace{\left[ a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) \right]}_{\substack{\text{Require the parts} \\ \text{constructed by } f_p(t) \\ \text{to become } g(t)}} \\ &= 0 + g(t) \end{aligned}$$

## 4.1 An Example of Particular Solution

Consider a spring-mass system that is subject to gravity. The Newton's 2<sup>nd</sup> Law writes:

$$m \frac{d^2}{dt^2} x(t) = -kx(t) - mg$$



The simplest way to solve it is by grouping  $mg$  into  $x(t)$ .

$$m \frac{d^2}{dt^2} \left[ x(t) + \frac{mg}{k} \right] = -k \left[ x(t) + \frac{mg}{k} \right]$$

$\nwarrow \frac{mg}{k}$  is a constant. So  $\frac{d}{dt} \left( \frac{mg}{k} \right) = 0$

Then substitute  $y(t) = x(t) + \frac{mg}{k}$ . We can see that this ODE of  $y(t)$  is the same as equation of motion of a spring-mass system without gravity.

$$m \frac{d^2}{dt^2} y(t) = -ky(t)$$

which the solution is already known:  $y(t) = A \cos \left( \sqrt{\frac{k}{m}}t + \phi \right)$ . So we can solve  $x(t)$  and identify the particular solution.

$$\begin{aligned}
 x(t) &= y(t) - \frac{mg}{k} \\
 &= A \cos \left( \sqrt{\frac{k}{m}}t + \phi \right) - \frac{mg}{k}
 \end{aligned}$$

The complementary soln.  $f_c(t)$   
i.e. the soln. of the homogeneous ODE  
 $m \frac{d^2}{dt^2}y(t) = -ky(t)$ 
The particular soln.  $f_p(t)$   
i.e. for canceling the  
non-homogeneous term  $mg$

## 4.2 Method of Undetermined Coefficients

Finding a suitable  $f_p(t)$  for an arbitrary non-homogeneous term  $g(t)$  is hard. But in most applications,  $g(t)$  appears as a combination of common functions. In these cases, we can make smart guess of what functions  $f_p(t)$  is made of.

### 4.2.1 Families of common functions & their derivatives

Here we consider the function and the constituents of its derivatives as a family.

– Polynomial/Log: Derivatives of a polynomial must be made of polynomials of lower degree.

$$- t^n \rightarrow t^{n-1} \rightarrow \dots \rightarrow t^2 \rightarrow t \rightarrow 1 \quad (+ve \text{ integral power})$$

$$- \ln t \rightarrow t^{-1} \rightarrow t^{-2} \rightarrow \dots \quad (-ve \text{ integral power})$$

$$- t^{\frac{1}{2}} \rightarrow t^{-\frac{1}{2}} \rightarrow t^{-\frac{3}{2}} \rightarrow \dots \quad (\text{fractional power})$$

– Trigonometric: Derivatives of  $\sin(kt)/\cos(kt)$  cycle between themselves.

$$\sin(kt) \rightarrow \cos(kt) \rightarrow \sin(kt) \rightarrow \dots$$

– Exponential: Derivatives of  $e^{kt}$  always yield multiples of itself.

$$e^{kt} \rightarrow e^{kt} \rightarrow e^{kt} \rightarrow \dots$$

The product between different family yield a set of its own derivatives. For example,

$$t^2 \sin t \rightarrow \begin{cases} d(t^2 \sin t) = 2t \sin t + t^2 \cos t \\ d^2(t^2 \sin t) = 2 \sin t + 4t \cos t - t^2 \sin t \\ d^3(t^2 \sin t) = 6 \cos t - 6t \sin t - t^2 \cos t \\ \vdots \end{cases}$$

One can observe that all its derivatives are made up of 6 different functions, which form a family of function:

$$t^2 \sin t, t^2 \cos t, t \sin t, t \cos t, \sin t, \cos t$$

### 4.2.2 Method of Undetermined Coefficient

According to the ODE,

$$\begin{aligned} g(t) &= a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) \\ &= \sum (\text{constant}) \cdot (\text{Derivatives of } f_p(t)) \end{aligned}$$

Conversely, we can make a good guess by assuming  $f_p(t) = \frac{g(t)}{c} + (\text{Some Derivatives of } g(t))$ .  
Then

$$\begin{aligned} c \times [f_p(t)] &= c \times \left[ \frac{g(t)}{c} + (\text{Some Derivatives of } g(t)) \right] \\ b \times \left[ \frac{d}{dt} f_p(t) \right] &= b \times \left[ \frac{1}{c} \frac{d}{dt} g(t) + \left( \text{Some Derivatives of } \frac{d}{dt} g(t) \right) \right] \\ + ) \quad a \times \left[ \frac{d^2}{dt^2} f_p(t) \right] &= a \times \left[ \frac{1}{c} \frac{d^2}{dt^2} g(t) + \left( \text{Some Derivatives of } \frac{d^2}{dt^2} g(t) \right) \right] \\ \hline a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) &= g(t) + 0 \end{aligned}$$

If we can find a combination of  $g(t)$ 's derivatives that make up the "(Some Derivatives of  $g(t)$ )" terms, such that all terms on R.H.S except  $g(t)$  cancel one another, then we recover  $f_p(t)$ .

**Example 4.1.**

$$a \frac{d^2}{dt^2} f(t) + b \frac{d}{dt} f(t) + c f(t) = t^2 + 2t$$

- Since the ODE is of 2<sup>nd</sup> order, the highest derivative that can be found in "(Some Derivatives of  $g(t)$ )" is at most  $\frac{d^2}{dt^2} g(t)$ . Otherwise it cannot be canceled.
- Derivatives of  $t^2$  and  $t$  both belong to the "integral power polynomial family".

So we guess

$$\begin{aligned} f_p(t) &= (\text{Some combination of } t^2, t, 1) \\ &= At^2 + Bt + C \end{aligned}$$

for some constants  $A, B, C$  to be solved.

$$\begin{aligned} c \times [At^2 + Bt + C] &= c \times [At^2 + Bt + C] \\ b \times \left[ \frac{d}{dt} (At^2 + Bt + C) \right] &= b \times [2At + B] \\ + ) \quad a \times \left[ \frac{d^2}{dt^2} (At^2 + Bt + C) \right] &= a \times [2A] \\ \hline a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) &= (c \cdot A)t^2 + (c \cdot B + b \cdot 2A)t + (c \cdot C + b \cdot B + a \cdot A) \\ g(t) &= 1 \times t^2 + 2t + 0 \end{aligned}$$

By matching coefficients of  $t^2$ ,  $t$  and 1 respectively, we require

$$\begin{cases} c \cdot A = 1 \\ c \cdot B + b \cdot 2A = 2 \\ c \cdot C + b \cdot B + a \cdot A = 0 \end{cases}$$

which is a system of 3 equations with 3 unknowns  $A, B, C$ . (Leave the solving to you.)

**Example 4.2.**

$$a \frac{d^2}{dt^2} f(t) + b \frac{d}{dt} f(t) + c f(t) = \sin(2t)$$

- Since the ODE is of 2<sup>nd</sup> order, the highest derivative that can be found in "(Some Derivatives of  $g(t)$ )" is at most  $\frac{d^2}{dt^2} g(t)$ . Otherwise it cannot be canceled.
- Derivatives of  $\sin 2t$  will cycle between  $\sin 2t$  and  $\cos 2t$ .

So we guess

$$\begin{aligned} f_p(t) &= (\text{Some combination of } \sin 2t, \cos 2t) \\ &= A \sin 2t + B \cos 2t \end{aligned}$$

for some constants  $A, B$  to be solved.

$$\begin{aligned} c \times [A \sin 2t + B \cos 2t] &= c \times [A \sin 2t + B \cos 2t] \\ b \times \left[ \frac{d}{dt} (A \sin 2t + B \cos 2t) \right] &= b \times [2A \cos 2t - 2B \sin 2t] \\ +) \quad a \times \left[ \frac{d^2}{dt^2} (A \sin 2t + B \cos 2t) \right] &= a \times [-4A \sin 2t - 4B \cos 2t] \\ \hline a \frac{d^2}{dt^2} f_p(t) + b \frac{d}{dt} f_p(t) + c f_p(t) &= (c \cdot A - b \cdot 2B - a \cdot 4A) \sin 2t + (c \cdot B + b \cdot 2A - a \cdot 4B) \cos 2t \\ g(t) &= 1 \times \sin 2t + 0 \times \cos 2t \end{aligned}$$

By matching coefficients of  $\sin 2t$  and  $\cos 2t$  respectively, we require

$$\begin{cases} c \cdot A - b \cdot 2B - a \cdot 4A = 1 \\ c \cdot B + b \cdot 2A - a \cdot 4B = 0 \end{cases}$$

which is a system of 2 equations with 2 unknowns  $A, B$ . (Leave the solving to you.)

**Example 4.3.** What if particular solution has the same form as the complementary solution?

$$\frac{d^2}{dt^2} f(t) - k^2 f(t) = e^{kt}$$



Its homogeneous counterpart is  $\frac{d^2}{dt^2}f_c(t) - k^2 f_c(t) = 0$ . We can immediately write its general solution as

$$f_c(t) = C_1 e^{kt} + C_2 e^{-kt}$$

But the inhomogeneous term  $g(t) = e^{kt}$  is also in the same exponential family! How do we find a correct  $f_p(t)$ ? By **Reduction of order** again.

1. Let  $f_p(t) = v(t)e^{kt}$ . The goal is to find a suitable  $v(t)$  that helps it form a particular solution that does not belong to the same family as  $e^{kt}$ . First substitute it into the original ODE and break down by product rule.

$$\begin{aligned} \frac{d^2}{dt^2}[v(t)e^{kt}] - k^2 \frac{d}{dt}[v(t)e^{kt}] &= e^{kt} \\ \frac{d^2}{dt^2}v(t) \cdot \cancel{e^{kt}} + 2 \frac{d}{dt}v(t) \cdot \cancel{k e^{kt}} + v(t) \cdot \cancel{k^2 e^{kt}} - \cancel{k^2 v(t) e^{kt}} &= \cancel{e^{kt}} \\ \frac{d^2}{dt^2}v(t) + 2k \frac{d}{dt}v(t) &= 1 \end{aligned}$$

2. Because this equation has no 0<sup>th</sup> order of  $v(t)$ , we can integrate it once to reduce the total order. We arrive at a 1<sup>st</sup> order inhomogeneous ODE of  $v(t)$ .

$$\frac{d}{dt}v(t) + 2kv(t) = t + C_3$$

where  $C_3$  is an arbitrary integration constant. Now we have to carry out the standard procedure for solving non-homogeneous equation again:

$$v(t) = v_c(t) + v_p(t)$$

3. The complementary solution  $v_c(t)$  is trivial:

$$\begin{aligned} \frac{d}{dt}v_c(t) + 2kv_c(t) &= 0 \\ v_c(t) &= C_4 e^{-2kt} \end{aligned}$$

with  $C_4$  being some constant.

4. Use method of undetermined coefficients for  $v_p(t)$ . The non-homogeneous term  $t + C_3$  is a polynomial of degree 1. So its derivative can only be made of  $t$  and 1. Let  $v_p(t) = At + B$ .

$$\begin{aligned} 2k \times [At + B] &= 2k \times [At + B] \\ +) \quad 1 \times \left[ \frac{d}{dt}(At + B) \right] &= 1 \times [A] \\ \hline \frac{d}{dt}v_c(t) + 2kv_c(t) &= (2k \cdot A)t + (2k \cdot B + 1 \cdot A) \\ g(t) &= 1 \times t + C_3 \end{aligned}$$

By matching coefficients of  $t$  and 1 respectively, we require

$$\begin{cases} 2k \cdot A = 1 \\ 2k \cdot B + 1 \cdot A = C_3 \end{cases}$$

which gives  $A = \frac{1}{2k}$  and  $B = \frac{C_3}{2k} - \frac{1}{4k^2}$ . So,

$$v_p(t) = \frac{t}{2k} + \left( \frac{C_3}{2k} - \frac{1}{4k^2} \right)$$

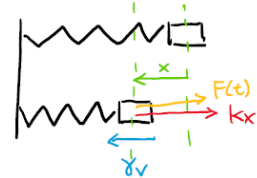
Finally we arrive at

$$\begin{aligned} v(t) &= v_c(t) + v_p(t) = C_4 e^{-2kt} + \frac{t}{2k} + \left( \frac{C_3}{2k} - \frac{1}{4k^2} \right) \\ \Rightarrow f_p(t) &= v(t) e^{kt} = C_4 e^{-kt} + \left( \frac{t}{2k} \right) e^{kt} + \underbrace{\left( \frac{C_3}{2k} - \frac{1}{4k^2} \right) e^{kt}}_{\substack{\text{All are constants.} \\ \text{Can combine.}}} \\ &= \underbrace{C_4 e^{-kt} + C'_3 e^{kt}}_{\substack{\text{Already in } f_c(t)}} + \underbrace{\left( \frac{t}{2k} \right) e^{kt}}_{\substack{\text{The true } f_p(t)}} \end{aligned}$$

### 4.3 Application: Forced Harmonic Oscillator

Consider a spring-mass system driven by an external force. When the spring is compressed by a displacement  $x$ , the forces on it are

- Spring's elastic force:  $-kx$  (Require  $k > 0$ )
- Damping force:  $-\gamma v$  (Require  $\gamma > 0$ )
- External force:  $F(t)$  (No restriction)



The Newton's 2<sup>nd</sup> Law writes:

$$(\text{total force}) = -kx - \gamma v + F(t) = ma$$

$$m \frac{d^2}{dt^2} x(t) + \gamma \frac{d}{dt} x(t) + kx(t) = F(t)$$

which is an inhomogeneous 2<sup>nd</sup> order ODE. We have already learnt the complementary solution:

$$x_c(t) = \begin{cases} C_1 e^{\frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} t} + C_2 e^{\frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} t} & (\gamma^2 - 4mk > 0) \\ e^{\frac{\gamma}{2m} t} \cdot A \cos \left( \frac{\sqrt{4mk - \gamma^2}}{2m} t + \phi \right) & (\gamma^2 - 4mk < 0) \\ C_1 e^{-\frac{\gamma}{2m} t} + C_2 t e^{-\frac{\gamma}{2m} t} & (\gamma^2 - 4mk = 0) \end{cases}$$

We cannot determine  $x_p(t)$  unless we are given the expression of  $F(t)$ . One common form of external force is vibration, which can be assume as sinusoidal:

$$F(t) = F_0 \cos \omega t$$

By method of undetermined coefficient, we can guess  $x_p(t) = A \cos(\omega t) + B \sin(\omega t)$ .

$$\begin{aligned}
 k \times [A \cos(\omega t) + B \sin(\omega t)] &= k \times [A \cos(\omega t) + B \sin(\omega t)] \\
 \gamma \times \left[ \frac{d}{dt}(A \cos(\omega t) + B \sin(\omega t)) \right] &= \gamma \times [-\omega A \sin(\omega t) + B \omega \cos(\omega t)] \\
 +) \quad m \times \left[ \frac{d^2}{dt^2}(-\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t)) \right] &= m \times [A \cos(\omega t) + B \sin(\omega t)]
 \end{aligned}$$


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$$\begin{aligned}
 m \frac{d^2}{dt^2} x_p(t) + \gamma \frac{d}{dt} x_p(t) + k x_p(t) &= (-\omega^2 A m + \omega B \gamma + A k) \cos \omega t \\
 &\quad + (-\omega^2 B m - \omega A \gamma + B k) \sin \omega t \\
 F(t) &= F_0 \cos 2t + 0
 \end{aligned}$$

By matching coefficients of  $\cos \omega t$  and  $\sin \omega t$  respectively, we require

$$\begin{cases} -\omega^2 A m + \omega B \gamma + A k = F_0 \\ -\omega^2 B m - \omega A \gamma + B k = 0 \end{cases}$$

Solving, yield

$$A = \frac{F_0(m\omega^2 - k)}{(\gamma\omega)^2 + (m\omega^2 - k)^2}, \quad B = \frac{F_0\gamma\omega}{(\gamma\omega)^2 + (m\omega^2 - k)^2}$$

By combining  $\sin / \cos$  into 1 sinusoidal,  $x_p(t)$  becomes

$$\begin{aligned}
 x_p(t) &= A \cos \omega t + B \sin \omega t \\
 &= \sqrt{A^2 + B^2} \cos \left[ \omega t + \tan^{-1} \left( \frac{B}{A} \right) \right] \\
 &= \frac{F_0}{\sqrt{(\gamma\omega)^2 + (m\omega^2 - k)^2}} \cos \left[ \omega t + \tan^{-1} \left( \frac{\gamma\omega}{m\omega^2 - k} \right) \right]
 \end{aligned}$$

### Special Case: Resonance

When the frequency of the external force equals to the natural frequency of harmonic oscillator  $\sqrt{\frac{k}{m}}$ , or  $m\omega^2 - k = 0$ . The displacement  $x_p(t)$  becomes

$$\begin{aligned}
 x_p(t) &= \frac{F_0}{\gamma\omega} \cos \left[ \omega t - \frac{\pi}{2} \right] \\
 &= \frac{F_0}{\gamma} \sqrt{\frac{m}{k}} \sin \left( \sqrt{\frac{k}{m}} t \right)
 \end{aligned}$$

In particular, if the damping force also diminish (i.e.  $\gamma \rightarrow 0$ ), The amplitude of  $x_p(t)$  will explode to infinity.

— The End —