Edited 2025

Non-Cartesian Coordinate

by Tony Shing

Overview:

- 2D: Polar coordinate
- 3D: Cylindrical coordinate & Spherical coordinate

1 Polar Coordinate

You should have already learnt what polar coordinate is from high school, and its conversion with rectangular coordinate.

From (x, y) to (r, θ)

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

From (r, θ) to (x, y)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

(add figure here: polar conv)

What about double integral over an area that is represented by polar coordinate?

$$\boxed{\iint f(x, y)\, dx\, dy \quad \Leftrightarrow \quad \iint f(r, \theta)\, \underset{\substack{\uparrow \\ \text{Caution: An extra } r}}{\underline{r}}\, dr\, d\theta}$$

The reason for this extra r comes from the dimension of the area element.

- In x/y coordinate, area of each grid is fixed.

$$I = \sum_i \underbrace{f(x_i, y_i)}_{\substack{\text{"Weight" assigned} \\ \text{to the point } (x_i, y_i)}} \cdot \underbrace{\Delta x \Delta y}_{\substack{\text{Each grid are} \\ \text{of area } \Delta x \Delta y}} \sim \iint f(x, y)\, dx\, dy$$

(add figure here: rect element)

- In polar coordinate, area of the grids depends on its r coordinate. When $\Delta\theta$ is very small, the grid's area $\approx$ (height)$\times$(width) $\approx \Delta r \times r_i\Delta\theta$. So

$$I = \sum_i \underbrace{f(r_i,\theta_i)}_{\substack{\text{"Weight" assigned} \\ \text{to the point } (r_i,\theta_i)}} \cdot \underbrace{r_i \cdot \Delta r\Delta\theta}_{\substack{\text{The grid's area} \\ \text{depends on } r \text{ coordinate} \\ = \Delta r \times r_i\Delta\theta}} \sim \iint f(r,\theta) r \,\mathrm{d}r \,\mathrm{d}\theta$$

(add figure here: polar element)

---

2 Cylindrical Coordinate

Cylindrical coordinate is essentially polar coordinate + z-axis.

From (x,y,z) to (r,θ,z)

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

From (r,θ) to (x,y)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

(add figure here: cyl conv)

Therefore the triple integral expression is very similar to the double integral in polar coordinate, with an extra r present in the volume element.

$$\boxed{\iiint f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad \Leftrightarrow \quad \iiint f(r,\theta,z)\,\underline{r}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z}$$

(add figure here: cyl vol element)

Example 2.1. Given a solid cylinder with mass density distribution $\rho(r,\theta,z) = r^2$, and dimension: radius $= R$, height $= H$. Making use of cylindrical coordinate,

- Mass of each volume element $=$ (density)$\times$(volume) $= \rho(r,\theta,z)r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$
- Total mass $= \iiint \rho(r,\theta,z)r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$
- Upper/Lower bound for each dimension are:
  - Range of r: From $r=0$ to $r=R$

- Range of θ: From $\theta = 0$ to $\theta = 2\pi$ (whole circle)
- Range of z: From $z = 0$ to $z = H$

(add figure here: dim of cyl)

The calculaton of the total mass is then

density function $\rho(r, \theta, z) = r^2$

$$
\begin{aligned}
&\int_{z=0}^{z=H} \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \underline{r^2} \cdot r \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z \\
=& \int_{z=0}^{z=H} \int_{r=0}^{r=R} \underbrace{\left(\int_{\theta=0}^{\theta=2\pi} r^3 \, \mathrm{d}\theta \right)}_{\text{First integrate } \theta} \mathrm{d}r \, \mathrm{d}z \\
=& \int_{z=0}^{z=H} \int_{r=0}^{r=R} \left[r^3 \theta \right] \Big|_{\theta=0}^{\theta=2\pi} \mathrm{d}r \, \mathrm{d}z \\
=& \int_{z=0}^{z=H} \underbrace{\left(\int_{r=0}^{r=R} 2\pi r^3 \, \mathrm{d}r \right)}_{\text{Then integrate } r} \mathrm{d}z \\
=& \int_{z=0}^{z=H} \left[\frac{2\pi r^4}{4} \right] \Big|_{r=0}^{r=R} \mathrm{d}z \\
=& \underbrace{\int_{z=0}^{z=H} \frac{\pi R^4}{2} \, \mathrm{d}z}_{\text{Finally integrate } z} \\
=& \frac{\pi R^4 H}{2}
\end{aligned}
$$

---

3 Spherical Coordinate

Spherical coordinate is a description of position on a sphere by 3 parameters:

- r = Radius, $\sim$ Altitute
- ϕ = Azimuthal angle, $\sim$ Longitude
- θ = Polar angle, $\sim$ Latitude

The conversion to rectangular coordinate is as follow:

From (x, y, z) to (r, θ, ϕ)

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

From (r, θ, ϕ) to (x, y, z)

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

(add figure here: sph conv)

Caution 1: The notations here are adopting the **physics convention**, where ϕ is the angle on the x-y plane and θ is the inclination to the z-axis. This is the convention found in modern Physics textbooks. However in many mathematics textbook and older physics books, you may find the **mathematics convention** where the meaning of ϕ and θ are swapped.

Caution 2: The choice of the angles are not the same as we use geography. Note that

- Range of $\phi = [0, 2\pi)$. But in geography, longitutude is ranged between $(-180°, 180°]$.
- Range of $\theta = [0, \pi]$. But in geography, latitude angle is $90° - \theta$, which ranged between $[-90°, 90°]$.

When dealing with triple integral, the unit volume in spherical coordinate is $\sim (\Delta r) \times (r\Delta\theta) \times (r\sin\theta\Delta\phi)$. We may analyze its dimension with the following pictures.

(add figure here: sph unit vol dim)

As a result, the triple integral has to be written as

$$\boxed{\iiint f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad \Leftrightarrow \quad \iiint f(r,\theta,\phi)\,\underline{r^2\sin\theta}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi}$$

Caution: An extra $r^2\sin\theta$

Example 3.1. Given a hollow but thick sphere with radius range from $r = a$ to $r = b$, and with mass density distribution $\rho(r, \theta, \phi) = r^4$. Making use of spherical coordinate,

- Mass of each volume element = (density)×(volume) = $\rho(r,\theta,\phi)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- Total mass = $\iiint \rho(r,\theta,\phi)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- Upper/Lower bound for each dimension are:

- Range of r: From $r = a$ to $r = b$
- Range of θ: From $\theta = 0$ to $\theta = \pi$
- Range of ϕ: From $\phi = 0$ to $\phi = 2\pi$

(add figure here: dim of sph)

The calculaton of the total mass is then

density function $\rho(r, \theta, \phi) = r^4$

$$\int_{r=a}^{r=b} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \underline{r^4} \cdot r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}r$$

$$= \int_{r=a}^{r=b} \int_{\phi=0}^{\phi=2\pi} \underbrace{\left(\int_{\theta=0}^{\theta=\pi} r^6 \sin\theta \, \mathrm{d}\theta \right)}_{\text{First integrate } \theta} \mathrm{d}\phi \, \mathrm{d}r$$

$$= \int_{r=a}^{r=b} \int_{\phi=0}^{\phi=2\pi} \left[-r^6 \cos\theta \right] \Big|_{\theta=0}^{\theta=\pi} \mathrm{d}\phi \, \mathrm{d}r$$

$$= \int_{r=a}^{r=b} \underbrace{\left(\int_{\phi=0}^{\phi=2\pi} 2r^6 \, \mathrm{d}\phi \right)}_{\text{Then integrate } \phi} \mathrm{d}r$$

$$= \int_{r=a}^{r=b} \left[2r^6 \phi \right] \Big|_{\phi=0}^{\phi=2\pi} \mathrm{d}r$$

$$= \underbrace{\int_{r=a}^{r=b} 4\pi r^6 \, \mathrm{d}r}_{\text{Finally integrate } r}$$

$$= \frac{4\pi}{7} (b^7 - a^7)$$

— The End —