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Non-Cartesian Coordinate

by Tony Shing

Overview:

- 2D: Polar coordinate
- 3D: Cylindrical coordinate & Spherical coordinate

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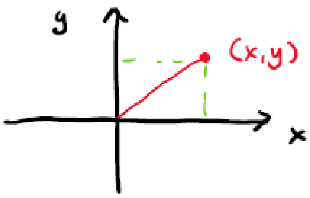
1 Polar Coordinate

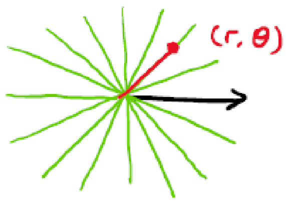
You should have already learnt what polar coordinate is from high school, and its conversion with rectangular coordinate.

From (x, y) to (r, θ)



$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

From (r, θ) to (x, y)



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

What about double integral over an area that is represented by polar coordinate?

$$\boxed{\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \Leftrightarrow \quad \iint f(r,\theta)\,\underline{r}\,\mathrm{d}r\,\mathrm{d}\theta}$$

Caution: An extra r

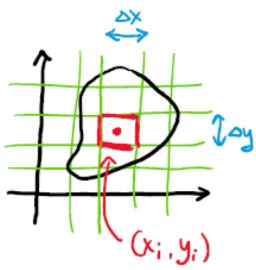
The reason for this extra r comes from the dimension of the area element.

- In x/y coordinate, area of each grid is fixed.

$$I = \sum_i \underline{f(x_i, y_i)} \cdot \underline{\Delta x \Delta y} \sim \iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

"Weight" assigned to the point (x_i, y_i)

Each grid are of area $\Delta x \Delta y$

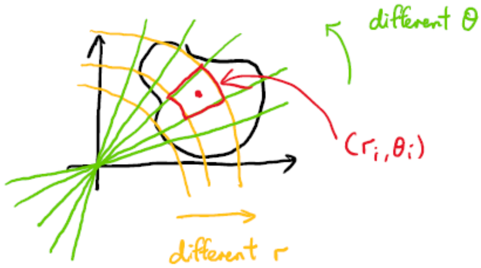


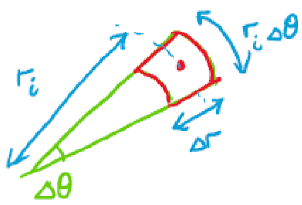
– In polar coordinate, area of the grids depends on its r coordinate. When $\Delta\theta$ is very small, the grid's area $\approx$ (height)$\times$(width) $\approx \Delta r \times r_i\Delta\theta$. So

$$I = \sum_i \underline{f(r_i,\theta_i)} \cdot \underline{r_i \cdot \Delta r\Delta\theta} \sim \iint f(r,\theta) r \, \mathrm{d}r \, \mathrm{d}\theta$$

"Weight" assigned to the point (r_i,θ_i)

The grid's area depends on r coordinate $= \Delta r \times r_i\Delta\theta$





Each grid's area depends on its r coordinate.

When $\Delta\theta$ is very small,
grid's area $\approx$ (height)$\times$(width)
$\approx (\Delta r) \times (r\Delta\theta)$

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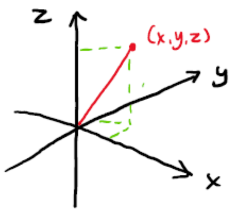
2 Cylindrical Coordinate

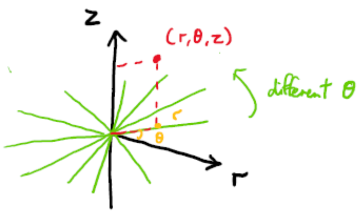
Cylindrical coordinate is essentially polar coordinate + z-axis.

From (x,y,z) to (r,θ,z)



$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

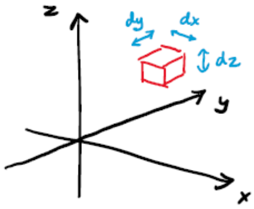
From (r,θ,z) to (x,y,z)



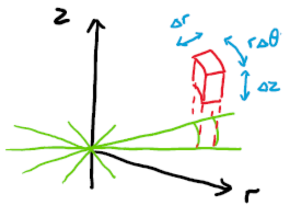
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Therefore the triple integral expression is very similar to the double integral in polar coordinate, with an extra r present in the volume element.

$$\boxed{\iiint f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \quad \Leftrightarrow \quad \iiint f(r,\theta,z) \, \underline{r} \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z}$$

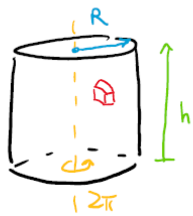


$\Leftrightarrow$



Example 2.1. Given a solid cylinder with mass density distribution $\rho(r, \theta, z) = r^2$, and dimension: radius $= R$, height $= H$. Making use of cylindrical coordinate,

- Mass of each volume element $=$ (density)$\times$(volume) $= \rho(r, \theta, z) r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z$
- Total mass $= \iiint \rho(r, \theta, z) r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z$
- Upper/Lower bound for each dimension are:
  - Range of r: From $r = 0$ to $r = R$
  - Range of θ: From $\theta = 0$ to $\theta = 2\pi$ (whole circle)
  - Range of z: From $z = 0$ to $z = H$



The calculaton of the total mass is then

density function $\rho(r, \theta, z) = r^2$

$$\int_{z=0}^{z=H} \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \underline{r^2} \cdot r \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z$$

$$= \int_{z=0}^{z=H} \int_{r=0}^{r=R} \underbrace{\left(\int_{\theta=0}^{\theta=2\pi} r^3 \, \mathrm{d}\theta \right)}_{\text{First integrate } \theta} \mathrm{d}r \, \mathrm{d}z$$

$$= \int_{z=0}^{z=H} \int_{r=0}^{r=R} \left[r^3 \theta \right] \Big|_{\theta=0}^{\theta=2\pi} \mathrm{d}r \, \mathrm{d}z$$

$$= \int_{z=0}^{z=H} \underbrace{\left(\int_{r=0}^{r=R} 2\pi r^3 \, \mathrm{d}r \right)}_{\text{Then integrate } r} \mathrm{d}z$$

$$= \int_{z=0}^{z=H} \left[\frac{2\pi r^4}{4} \right] \Big|_{r=0}^{r=R} \mathrm{d}z$$

$$= \underbrace{\int_{z=0}^{z=H} \frac{\pi R^4}{2} \, \mathrm{d}z}_{\text{Finally integrate } z}$$

$$= \frac{\pi R^4 H}{2}$$

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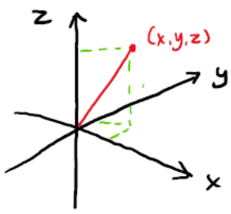
3 Spherical Coordinate

Spherical coordinate is a description of position on a sphere by 3 parameters:

- r = Radius, $\sim$ Altitute
- θ = Polar angle, $\sim$ Latitude
- ϕ = Azimuthal angle, $\sim$ Longitude

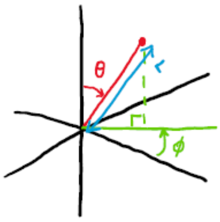
The conversion to rectangular coordinate is as follow:

From (x, y, z) to (r, θ, ϕ)



$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

From (r, θ, ϕ) to (x, y, z)

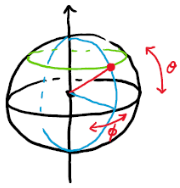


$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

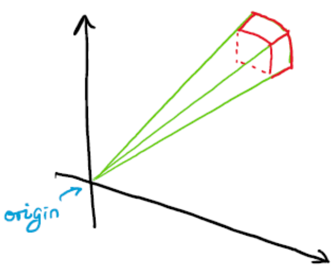
Caution 1: The notations here are adopting the **physics convention**, where ϕ is the angle on the x-y plane and θ is the inclination to the z-axis. This is the convention found in modern Physics textbooks. However in many mathematics textbook and older physics books, you may find the **mathematics convention** where the meaning of ϕ and θ are swapped.

Caution 2: The choice of the angles are not the same as we use geography. Note that

- Range of $\theta = [0, \pi]$. But in geography, latitude angle is $90° - \theta$, which ranged between $[-90°, 90°]$.
- Range of $\phi = [0, 2\pi)$. But in geography, longitutude is ranged between $(-180°, 180°]$.

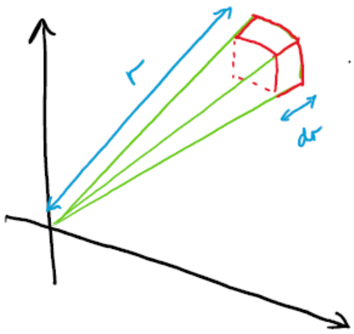


When dealing with triple integral, the unit volume in spherical coordinate is $\sim (\Delta r) \times (r\Delta\theta) \times (r\sin\theta\Delta\phi)$. We may visualize as follows:

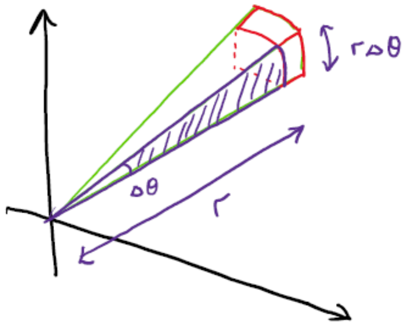


Volume element
= Skin of a watermelon cone

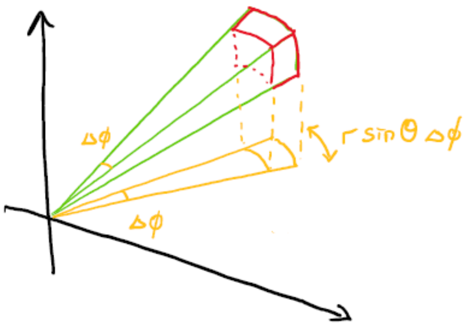
The side along r direction: $\mathrm{d}r$

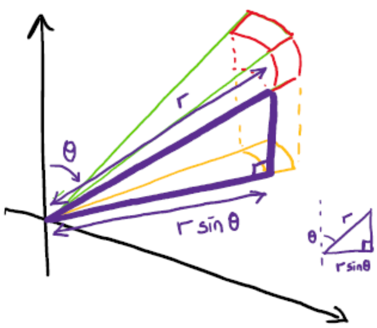


The side along θ direction: $r\,\mathrm{d}\theta$



The side along ϕ direction: $r\sin\theta\,\mathrm{d}\phi$





The radius is $r\sin\theta$, not r. Because it is the radius's projection

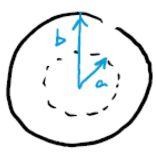
As a result, the triple integral has to be written as

$$\boxed{\iiint f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad \Leftrightarrow \quad \iiint f(r,\theta,\phi)\,\underline{r^2\sin\theta}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi}$$

Caution: An extra $r^2\sin\theta$

Example 3.1. Given a hollow but thick sphere with radius range from $r = a$ to $r = b$, and with mass density distribution $\rho(r,\theta,\phi) = r^4$. Making use of spherical coordinate,

- Mass of each volume element = (density)×(volume) = $\rho(r,\theta,\phi)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- Total mass = $\iiint \rho(r,\theta,\phi)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
- Upper/Lower bound for each dimension are:
  - Range of r: From $r = a$ to $r = b$
  - Range of θ: From $\theta = 0$ to $\theta = \pi$
  - Range of ϕ: From $\phi = 0$ to $\phi = 2\pi$



The calculaton of the total mass is then

$$\int_{r=a}^{r=b}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\underbrace{r^4}_{\text{density function } \rho(r,\theta,\phi)=r^4}\cdot r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r$$

$$=\int_{r=a}^{r=b}\int_{\phi=0}^{\phi=2\pi}\underbrace{\left(\int_{\theta=0}^{\theta=\pi}r^6\sin\theta\,\mathrm{d}\theta\right)}_{\text{First integrate }\theta}\mathrm{d}\phi\,\mathrm{d}r$$

$$=\int_{r=a}^{r=b}\int_{\phi=0}^{\phi=2\pi}\left[-r^6\cos\theta\right]\Big|_{\theta=0}^{\theta=\pi}\mathrm{d}\phi\,\mathrm{d}r$$

$$=\int_{r=a}^{r=b}\underbrace{\left(\int_{\phi=0}^{\phi=2\pi}2r^6\,\mathrm{d}\phi\right)}_{\text{Then integrate }\phi}\mathrm{d}r$$

$$=\int_{r=a}^{r=b}\left[2r^6\phi\right]\Big|_{\phi=0}^{\phi=2\pi}\mathrm{d}r$$

$$=\underbrace{\int_{r=a}^{r=b}4\pi r^6\,\mathrm{d}r}_{\text{Finally integrate }r}$$

$$=\frac{4\pi}{7}(b^7-a^7)$$

— The End —