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Magnetostatics

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Overview:

- Basic problems: Find $\vec{\boldsymbol{B}}$ by Biot-Savat law with integration
- Divergent-less of B-field
- Ampere's law, line integral, curl & Stokes' theorem
- Magnetic vector potential & Poisson equation

In electromagnetism, theoretically every problem can be solved through a set of PDEs called the **Maxwell Equations**.

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0} \qquad \longrightarrow \quad \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} = 0$$

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} = -\frac{\partial\vec{\boldsymbol{B}}}{\partial t} \qquad \longrightarrow \quad \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} = \mu_0\vec{\boldsymbol{J}} + \mu_0\epsilon_0\frac{\partial\vec{\boldsymbol{E}}}{\partial t}$$

However, a *system of PDEs* is just too complicate to be solved. So we need to learn different "tricks" to avoid them, which are enough for some simple scenarios.

Magnetostatics only concerns the 2^{nd} and 4^{th} equation of the set - Gauss's law on B-field and Ampere's law.

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1 Basic Skill: Biot-Savat Law with Integration

Physically, currents are just moving charges. There is no such thing called "point current". However we can imagine a line of current being divided into many infinitestimally small segments such that each current segment "looks like" a vector point source.

$$\begin{pmatrix}\text{Current}\\\text{line}\end{pmatrix} = I\vec{\boldsymbol{L}} \quad\Longrightarrow\quad \int I\,\mathrm{d}\vec{\boldsymbol{l}} \sim \sum(\text{Current})\begin{pmatrix}\text{Unit}\\\text{length}\end{pmatrix}$$

(add figure here: current segment source)

This is why you will never find Biot-Savat law in your high school textbook.

$$\underbrace{\vec{\boldsymbol{B}} = \frac{1}{4\pi\epsilon_0}\frac{I\vec{\boldsymbol{L}}}{r^2}\times\hat{\boldsymbol{r}}}_{\substack{\text{Always remember that}\\\text{You should not write this}}} \quad\Longrightarrow\quad \underbrace{\int \mathrm{d}\vec{\boldsymbol{B}} = \int_{\substack{\text{whole}\\\text{line}}}\frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{\boldsymbol{l}}}{r^2}\times\hat{\boldsymbol{r}}}_{\substack{\text{Biot-Savat law can only be}\\\text{written as an integral}}}$$

Furthermore, because we are living in a 3D world, current does not always travel along a line segment, but may flow on a surface or through an object such that the current is position dependent. In these cases, we should describe current as a distribution of flow (i.e. vector field).

(add figure here: current volume density + surface density)

When current is travelling on a surface (so that each wire is like a ribbon strip), we can describe current as **surface current density $\vec{\boldsymbol{K}}$**.

$$I\,\mathrm{d}\vec{\boldsymbol{l}} \qquad \Longrightarrow \qquad \vec{\boldsymbol{K}}\,\mathrm{d}s$$

When current is travelling in an object (so that each wire is like a continuous volume), we can describe current as **volume current density $\vec{\boldsymbol{J}}$**.

$$I\,\mathrm{d}\vec{\boldsymbol{l}} \qquad \Longrightarrow \qquad \vec{\boldsymbol{J}}\,\mathrm{d}\tau$$

Caution:

In magnetostatics problem, we require the current to be a constant flow. Although you may have learnt that a moving point charge q traveling at velocity v acts like a point current source qv, this current is only temporary.

(add figure here: point charge not equal continuous flow)

The true formula of B-field by point charge needs to consider the travelling time of B-field. This is completely out of our scope.

Example 1.1. Suppose there is a wire lying on the x-axis, with its ends at $x = a$ and $x = b$. Let there be uniform current I flowing along it. What is the B-field on an arbituary point on the z axis?

(add figure here: rod)

We can analyze by dividing the rod into infinitestimal pieces:

- Each segment has a length $\mathrm{d}x$
- Each unit of current segment is thus $I\,\mathrm{d}\vec{\boldsymbol{x}} = I\,\mathrm{d}x\,\hat{\boldsymbol{x}}$.
- For the segment at position x, its distance from the targeted point is $\sqrt{z^2 + x^2}$.

(add figure here: infinte element)

Thus we can calculate $\vec{\boldsymbol{B}}$ by Biot-Savat law.

$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi} \int_a^b \frac{I\,\mathrm{d}x\,\hat{\boldsymbol{x}}}{z^2 + x^2} \times \left(\substack{\text{directon to} \\ \text{target point}}\right)$$

To do the cross product, we need to resolve the direction's component from the segment to the target point. By the triangle:

(add figure here: component)

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \frac{x}{\sqrt{z^2+x^2}}\hat{\boldsymbol{x}} + \frac{z}{\sqrt{z^2+x^2}}\hat{\boldsymbol{z}} \\
\Rightarrow \quad \hat{\boldsymbol{x}} \times \hat{\boldsymbol{r}} &= \frac{z}{\sqrt{z^2+x^2}}(-\hat{\boldsymbol{y}})
\end{aligned}$$

In this situation, $-\hat{\boldsymbol{y}}$ is the out of paper direction.

(add figure here: directions notation)

The B-field should be integrated by

$$B_y = -\frac{\mu_0}{4\pi}\int_a^b \frac{I\,\mathrm{d}x}{z^2+x^2}\frac{z}{\sqrt{z^2+x^2}}$$

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2 Divergent-less of B-field

Similar to E-field, there is the **Gauss's law for B-field**, which has two different expressions:

$$\oiint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = 0 \qquad \text{(Integral form)}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = 0 \qquad \text{(Differential form)}$$

This law is purely an observation to B-field, claiming that

Magnetic point source does not exist	$\Leftrightarrow$	Total flux of B-field on a closed surface always $= 0$

Because so far no one has found any magnetic monopoles, we determine that B-field lines must exist as closed loops, and can never form diverging/converging patterns like E-field does.

This law is not as important as the other 3 in the Maxwell equation because it does not involve any source terms. It is only sometimes useful when we need to make symmetry claims or simplify derivations.

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3 Ampere's Law

The Ampere's Law (in magnetostatics) has two different expressions:

$$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I \qquad \text{(Integral form)}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}} \qquad \text{(Differential form)}$$

It is easier to study the physical meaning and visualize by the integral form. After that we can generalize to the differential form by introducing an operator called **curl**.

3.1 Revisit: Dot Product Line Integral

The literal description in Ampere's law integral form is

$$\begin{pmatrix}\text{Dot product line integral}\\ \text{of magnetic field along a loop}\end{pmatrix} = \oint \vec{\boldsymbol{B}}\cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I = (\text{Constant})(\text{Current enclosed})$$

Recall that we can use the sign of a dot product between 2 vectors to determine if the vectors are in similar / opposite directions.

(add figure here: dot product sign)

Now consider that we are travelling in a vector field along some path. At each step, we can take note of field vector there and our travelling direction, then compute their dot product.

If the sum of all the dot products > 0, we are travelling more or less the same direction relative to the field's flow.

If the sum of all the dot products < 0, we are travelling more or less the opposite direction relative to the field's flow.

(add figure here: flow along + flow opposite)

If we divide our path into infinitestimal small segments, then the sum become line integral.

$$\int_{\text{path}} \vec{\boldsymbol{F}}\cdot \mathrm{d}l \quad \begin{cases} > 0 & \Rightarrow \quad \sim \text{Our path is along the flow} \\ < 0 & \Rightarrow \quad \sim \text{Our path is opposite to the flow} \end{cases}$$

3.2 Detection of Rotating Flow

The situation becomes interesting if we choose our path to be a closed loop - this loop integral becomes an indicator whether there are rotation trends around the loop.

By convention, we always use an anti-clockwise loop (Just like right hand rule).

- If $\oint_{\substack{\text{our}\\ \text{loop}}} \vec{\boldsymbol{F}}\cdot \mathrm{d}l > 0$, our loop may have enclosed some rotation centers of anti-clockwise flow.

  (add figure here: High degree of anti clock flow)

  (add figure here: loop overlay -> +ve)

- If $\oint_{\substack{\text{our}\\ \text{loop}}} \vec{\boldsymbol{F}}\cdot \mathrm{d}l < 0$, our loop may have enclosed some rotation centers of clockwise flow.

  (add figure here: High degree of clock flow)

  (add figure here: loop overlay -> -ve)

- If $\oint_{\substack{\text{our}\\ \text{loop}}} \vec{\boldsymbol{F}}\cdot \mathrm{d}l \approx 0$, our loop probably does not enclose any rotation centers.

  (add figure here: No rotation flow)

  (add figure here: loop overlay -> 0)

3.3 Curl

However there is a problem in using loop integral of dot product - if we choose the loop too arbituarily, the calculated dot product is ambiguous to tell where the rotation centers are.

(add figure here: arbituary surface)

E.g. If we choose an irregular loop, it may not catch the rotation centers.

To tackle this problem, we need to introduce the **curl** operator:

$$\underset{\substack{\uparrow\\ \text{Like gradient operator}\\ \text{but with a cross}}}{\underline{\vec{\boldsymbol{\nabla}}\times}}\ \bullet \overset{\text{def}}{=} \left(\frac{\partial \bullet_z}{\partial y}-\frac{\partial \bullet_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial \bullet_x}{\partial z}-\frac{\partial \bullet_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial \bullet_y}{\partial x}-\frac{\partial \bullet_x}{\partial y}\right)\hat{\boldsymbol{z}} \overset{\text{def}}{=} \underset{\substack{\uparrow\\ \text{Sometimes we}\\ \text{just write "curl"}}}{\underline{\text{curl}}}\ \bullet$$

The curl operator can be applied on a vector function, and will return another vector function.

$$\begin{aligned}
\vec{\boldsymbol{\nabla}}\times F &= \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\times\begin{pmatrix}F_x & F_y & F_z\end{pmatrix}\\
&= \begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}\\
&= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}}\\
&= (\text{A vector})
\end{aligned}$$

Each component of the curl of a vector field is related to its **loop integral along an infinitestimal small loop in each direction.**

3.3.1 Geometrical Interpretation

To visualize, we can draw 3 infinitestimal small loop around a point (x, y, z). Because of symmetry in all 3 directions, it suffices to just analyze the loop that is parallel to the x-y plane.

(add figure here: infinitestmal loops + floating loop)

The dot product on each side of the loop are calculated as follow:

(add figure here: magnify loop)

- Edge 1 :
  - Vector field on the edge center $= \vec{\boldsymbol{F}}\left(x, y+\frac{\Delta y}{2}, z\right)$
  - Edge length $= \Delta x$, in $-x$ direction

$\Rightarrow$ Dot product $= \vec{\boldsymbol{F}}\left(x, y+\frac{\Delta y}{2}, z\right)\cdot(\Delta x)(-\hat{\boldsymbol{x}}) = -F_{\underline{\underline{x}}}\left(x, y+\frac{\Delta y}{2}, z\right)\Delta x$

(add figure here: edge 1)

- Edge 2 :
  - Vector field on the edge center $= \vec{\boldsymbol{F}}\left(x - \frac{\Delta x}{2}, y, z\right)$
  - Edge length $= \Delta y$, in $-y$ direction

$\Rightarrow$ Dot product $= \vec{\boldsymbol{F}}\left(x - \frac{\Delta x}{2}, y, z\right) \cdot (\Delta y)(-\hat{\boldsymbol{y}}) = -F_{\underline{y}}\left(x - \frac{\Delta x}{2}, y, z\right)\Delta y$

(add figure here: edge 2)

- Edge 3 :
  - Vector field on the edge center $= \vec{\boldsymbol{F}}\left(x, y - \frac{\Delta y}{2}, z\right)$
  - Edge length $= \Delta x$, in $+x$ direction

$\Rightarrow$ Dot product $= \vec{\boldsymbol{F}}\left(x, y - \frac{\Delta}{2}, z\right) \cdot (\Delta x)(+\hat{\boldsymbol{x}}) = F_{\underline{x}}\left(x, y - \frac{\Delta y}{2}, z\right)\Delta x$

(add figure here: edge 3)

- Edge 4 :
  - Vector field on the edge center $= \vec{\boldsymbol{F}}\left(x + \frac{\Delta x}{2}, y, z\right)$
  - Edge length $= \Delta y$, in $+y$ direction

$\Rightarrow$ Dot product $= \vec{\boldsymbol{F}}\left(x + \frac{\Delta x}{2}, y, z\right) \cdot (\Delta y)(+\hat{\boldsymbol{y}}) = F_{\underline{y}}\left(x + \frac{\Delta x}{2}, y, z\right)\Delta y$

(add figure here: edge 4)

Therefore the total dot product along the loop is

$$\underbrace{F_y\left(x + \tfrac{\Delta x}{2}, y, z\right)\Delta y}_{\text{Edge 4}} - \underbrace{F_y\left(x - \tfrac{\Delta x}{2}, y, z\right)\Delta y}_{\text{Edge 2}} + \underbrace{-F_x\left(x, y + \tfrac{\Delta y}{2}, z\right)\Delta x}_{\text{Edge 1}} + \underbrace{F_x\left(x, y - \tfrac{\Delta y}{2}, z\right)\Delta x}_{\text{Edge 3}}$$

$$= \left(\frac{F_x\left(x + \frac{\Delta x}{2}, y, z\right) - F_x\left(x - \frac{\Delta x}{2}, y, z\right)}{\Delta x} - \frac{F_x\left(x, y + \frac{\Delta y}{2}, z\right) - F_x\left(x, y - \frac{\Delta y}{2}, z\right)}{\Delta y} \right)(\Delta x \Delta y)$$

$$= \left(\underset{\text{This is exactly partial x}}{\boxed{\frac{F_x\left(\boxed{x + \frac{\Delta x}{2}}, y, z\right) - F_x\left(\boxed{x - \frac{\Delta x}{2}}, y, z\right)}{\boxed{\Delta x}}}} - \underset{\text{This is exactly partial y}}{\boxed{\frac{F_x\left(x, \boxed{y + \frac{\Delta y}{2}}, z\right) - F_x\left(x, \boxed{y - \frac{\Delta y}{2}}, z\right)}{\boxed{\Delta y}}}} \right)(\Delta x \Delta y)$$

$$= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)(\Delta x \Delta y)$$

$$= \left(\begin{matrix} \text{Curl's} \\ z \text{ component} \end{matrix} \right) \left(\begin{matrix} \text{Unit area} \\ \text{parallel to xy plane} \end{matrix} \right)$$

$$= \left(\begin{matrix} \text{Curl's} \\ z \text{ component} \end{matrix} \right) \left(\begin{matrix} \text{Unit area} \\ \text{normal to z direction} \end{matrix} \right)$$

We can expect the similar results in the other 2 directions. Gather them together:

$$\begin{cases} \begin{pmatrix}\text{Loop integral}\\ \text{normal to}\\ x \text{ direction}\end{pmatrix} = \left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right)(\mathrm{d}y\,\mathrm{d}z) = (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}})_x(\mathrm{d}y\,\mathrm{d}z) = \begin{pmatrix}\text{Curl's}\\ x \text{ component}\end{pmatrix}\begin{pmatrix}\text{Unit area}\\ \text{normal to}\\ x \text{ direction}\end{pmatrix} \\ \begin{pmatrix}\text{Loop integral}\\ \text{normal to}\\ y \text{ direction}\end{pmatrix} = \left(\dfrac{\partial F_z}{\partial x} - \dfrac{\partial F_x}{\partial z}\right)(\mathrm{d}z\,\mathrm{d}x) = (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}})_y(\mathrm{d}z\,\mathrm{d}x) = \begin{pmatrix}\text{Curl's}\\ y \text{ component}\end{pmatrix}\begin{pmatrix}\text{Unit area}\\ \text{normal to}\\ y \text{ direction}\end{pmatrix} \\ \begin{pmatrix}\text{Loop integral}\\ \text{normal to}\\ z \text{ direction}\end{pmatrix} = \left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right)(\mathrm{d}x\,\mathrm{d}y) = (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}})_z(\mathrm{d}x\,\mathrm{d}y) = \begin{pmatrix}\text{Curl's}\\ z \text{ component}\end{pmatrix}\begin{pmatrix}\text{Unit area}\\ \text{normal to}\\ z \text{ direction}\end{pmatrix} \end{cases}$$

Therefore we can geometrically interpret curl as

$$\boxed{\begin{pmatrix}\textbf{Curl's}\\ \textbf{i}^{\textbf{th}} \textbf{ component}\end{pmatrix} = (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}})_i = \frac{(\text{Loop integral normal to i}^{\text{th}} \text{ direction})}{(\text{Area enclosed by the loop})} \sim \begin{pmatrix}\textbf{loop integral}\\ \underbrace{\textbf{density}}_{\text{This density is by area}}\end{pmatrix}}$$

3.3.2 Stokes' Theorem

With the geometrical interpretation, we can directly state (without proof) a convenient formula related to divergence - the **Stokes' theorem**:

$$\boxed{\oint \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \iint (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}) \cdot \mathrm{d}\vec{\boldsymbol{s}}}$$

which is basically

$$\begin{pmatrix}\text{Total}\\ \text{Loop integral}\end{pmatrix} \sim \sum_{\text{All area}} \begin{pmatrix}\text{Loop integral}\\ \text{per area}\end{pmatrix} \times (\text{Area})$$

3.4 Ampere's Law - Explanation

The Ampere's law is purely an observation about the relation between B-field and currents:

Total line integral of B-field along a closed loop $\neq 0$	$\Leftrightarrow$	There are currents circled by the loop

The two forms of Ampere's law are describing this same observation:

– Integral form:

$$(\vec{\boldsymbol{E}}\text{'s flux}) \sim \oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \frac{Q}{\epsilon_0} \sim (\text{Charge})$$

– Differential form:

$$\begin{pmatrix}\vec{\boldsymbol{E}}\text{'s flux}\\ \text{density}\end{pmatrix} \sim \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0} \sim \begin{pmatrix}\text{Charge}\\ \text{density}\end{pmatrix}$$

And the two form can be inter-converted by divergence theorem.

$$\oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \frac{Q}{\epsilon_0}$$

Divergence Theorem

Charge to charge density

$$\iiint (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}})\, \mathrm{d}\tau = \frac{1}{\epsilon_0} \iiint \rho\, \mathrm{d}\tau$$

3.5 Applying Gauss's Law Integral Form

In beginner electromagnetism, there is only one type of Gauss's law related problems:

Given the charge distribution, find the E-field everywhere by Gauss's law integral form in some very symmetrical scenarios.

which is basically asking you to *revert* the flux calculation:

$$\oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \frac{Q}{\epsilon_0} \qquad \Rightarrow \qquad \vec{\boldsymbol{E}} = \text{ ??? in terms of } Q$$

If Q has a very ugly distribution, there is nothing we can do except solving some partial differential equations. But if Q distributes very symmetrically, $\vec{\boldsymbol{E}}$ should also be symmetrical, such that the flux integral can be broken into multiplications.

In these cases, we can choose a "Gaussian" surface to to be integrated where

1. $\vec{\boldsymbol{E}}$ has constant magnitude everywhere on the surface.
2. $\vec{\boldsymbol{E}}$ forms the same angle with the surface normal vector everywhere on the surface

Only then, the flux integral can be broken down as

$$\oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \oiint \underline{|\vec{\boldsymbol{E}}||\,\mathrm{d}\vec{\boldsymbol{s}}\,|\cos\theta} \longleftarrow \begin{array}{c}\text{Just dot product}\\ \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta\end{array}$$

$$= \underline{|\vec{\boldsymbol{E}}|}\ \underline{\cos\theta} \oiint |\,\mathrm{d}\vec{\boldsymbol{s}}\,|$$

Same magnitude everywhere
Can move out of integral!

Form same angle everywhere
Can move out of integral!

$$= |\vec{\boldsymbol{E}}|\ \cos\theta\ (\text{Total surface area})$$

such that we can find the magnitude of $\vec{\boldsymbol{E}}$ with simple division

$$\boxed{|\vec{\boldsymbol{E}}| = \frac{(\text{Total flux})}{(\text{Total surface area})\cos\theta} = \frac{Q/\epsilon_0}{(\text{Total surface area})\cos\theta}}$$

In fact, there are not many of these "very symmetrical" cases. These examples below, with their respective Gaussian surface, are basically all the variations you can find in textbooks.

(add figure here: different gaussian surface)

Example 3.1. Given a solid sphere with uniform charge density ρ and radius R. By rotational symmetry, the E-field must satisfy:

- Only point in radial direction.
- Magnitude does not depend on angular directions θ, ϕ.

(add figure here: spherical symm)

Therefore we can choose the Gaussian surface to be a sphere of radius r to find the magnitude of E-field at distance r from the sphere center.

$$\begin{aligned} |\vec{\boldsymbol{E}}| &= \frac{Q/\epsilon_0}{(\text{Total surface area})\cos\theta} \\ &= \frac{Q}{\epsilon_0} \cdot \frac{1}{(4\pi r^2)} \cdot \frac{1}{\cos \underline{0^\circ}} \longleftarrow \begin{array}{l}\text{E-field = radial}\\ \therefore \text{Normal to surface}\end{array} \\ &= \frac{Q}{4\pi\epsilon_0 r^2} \\ \Rightarrow \quad \vec{\boldsymbol{E}} &= \frac{Q}{4\pi\epsilon_0 r^2}\underline{\hat{\boldsymbol{r}}} \longleftarrow \text{You have to manually add the unit vector} \end{aligned}$$

1. For radial distance $r < R$, the total charge enclosed in the gaussian surface is only the core of the sphere, up to radius r. So we should take $Q = \frac{4}{3}\pi r^3 \rho$.

$$\vec{\boldsymbol{E}} = \frac{\frac{4}{3}\pi r^3 \rho}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} = \frac{\rho r}{3\epsilon_0}\hat{\boldsymbol{r}}$$

(add figure here: inside)

2. For radial distance $r > R$ the total charge enclosed in the gaussian surface is the whole sphere, So we should take $Q = \frac{4}{3}\pi R^3 \rho$.

$$\vec{\boldsymbol{E}} = \frac{\frac{4}{3}\pi R^3 \rho}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{\boldsymbol{r}}$$

(add figure here: outside)

Example 3.2. Given an infinitely long hollow cylinder with inner radius $= a$ and outer radius $= b$, and its charge density is proportional to distance from center r, i.e. $\rho(\vec{\boldsymbol{r}}) = kr$
For cylinder, we can claim by rotational symmetry around the axis and translation symmetry along the axis, that the E-field must satisfy:

- Only point in r direction.
- Magnitude does not depend on θ or z.

(add figure here: cylinder symm)

Therefore we can choose the Gaussian surface to be a cylindrical sheet radius r and an arbituary length L (which will be cancelled later) to find the magnitude of E-field at distance r from the rotation axis.

$$\begin{aligned}
|\vec{\boldsymbol{E}}| &= \frac{Q/\epsilon_0}{(\text{Total surface area})\cos\theta} \\
&= \frac{Q}{\epsilon_0}\cdot\frac{1}{\underline{(2\pi rL)}}\cdot\frac{1}{\cos\underline{0^\circ}} \longleftarrow \begin{array}{c}\text{E-field = radial}\\ \therefore\text{ Normal to curved surface}\end{array} \\
& \qquad\qquad \begin{array}{c}\text{E-field = radial}\\ \therefore\text{ Only go through the curved surface}\\ \text{Top/bottom surface has no flux}\end{array} \\
&= \frac{Q}{2\pi\epsilon_0 rL} \\
\Rightarrow \quad \vec{\boldsymbol{E}} &= \frac{Q}{2\pi\epsilon_0 rL}\underline{\hat{\boldsymbol{r}}} \longleftarrow \text{You have to manually add the unit vector}
\end{aligned}$$

(add figure here: not side surface cuz flux = 0)

This time the charge density depends on position, so the total charge enclosed by the surface needs to be computed by integration.

1. For radial distance $r < a$, there is no charge enclosed because the cylinder is hollow. So $Q = 0$ implying $\vec{\boldsymbol{E}} = 0$.

   (add figure here: inner)

2. For radial distance $a < r < b$, total enclosed charge are distributed from radius $= a$ to radius $= r$, which calculates as

$$Q = \int_a^r \rho\cdot 2\pi rL\,\mathrm{d}r = 2\pi kL\int_a^r r^2\,\mathrm{d}r = \frac{2\pi kL}{3}(r^3 - a^3)$$

   So the E-field is

$$\vec{\boldsymbol{E}} = \frac{Q}{2\pi\epsilon_0 rL}\hat{\boldsymbol{r}} = \frac{k}{3\epsilon_0}\left(r^2 - \frac{a^3}{r}\right)\hat{\boldsymbol{r}}$$

   (add figure here: middle)

3. For radial distance $r > b$, total enclosed charge are distributed from radius $= a$ to radius $= b$, which calculates as

$$Q = \int_a^b \rho\cdot 2\pi rL\,\mathrm{d}r = 2\pi kL\int_a^r r^2\,\mathrm{d}r = \frac{2\pi kL}{3}(b^3 - a^3)$$

   So the E-field is

$$\vec{\boldsymbol{E}} = \frac{Q}{2\pi\epsilon_0 rL}\hat{\boldsymbol{r}} = \frac{k}{3\epsilon_0 r}(b^3 - a^3)\hat{\boldsymbol{r}}$$

   (add figure here: outer)

4 Magnetic Potential

4.1 Mathematical Origin

The reason to create a electrical potential function $V(\vec{r})$ is rather mathematical:

- Observation: E-field by static charge never forms loops. $\Rightarrow$ Static E-field is conservative.
- Mathematical fact: Any conservative field can be expressed as the gradient of some scalar function (i.e. potential).

Therefore we can define a scalar function $V(\vec{r})$ such that

$$\boxed{\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})}$$

And the reverse can be calculated by

$$\boxed{V(\vec{r_0}) = -\int_{\substack{\text{any path}\\ \text{from } \infty \text{ to } \vec{r_0}}} \vec{E}(\vec{r}) \cdot \mathrm{d}\vec{r} = -\int_{\infty}^{\vec{r_0}} \vec{E}(\vec{r})\, \mathrm{d}\vec{r}}$$

4.2 Poisson Equation

If we substitute $\vec{E} = -\vec{\nabla} V$ into the Gauss's law, we arrive at a new equation:

$$\begin{aligned}
\frac{\rho}{\epsilon_0} &= \vec{\nabla} \cdot \vec{E} \\
&= \vec{\nabla} \cdot \left(-\vec{\nabla} V\right) \\
&= -\vec{\nabla} \cdot \left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right) \\
&= -\left(\underbrace{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}\right) \stackrel{\text{def}}{=} -\underline{\nabla^2} V
\end{aligned}$$

This is called **Laplacian Operator**

$$\boxed{\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}}$$

This equation belongs to a class of PDE called **Poisson equation**, which is one of the earliest studied PDEs in history. It is the general method to relate potential between charge distribution. that works for any configurations of potential or charge distribution.

(add figure here: find one and other, add Q = lap V and V = ??? Q with arrow)

However, being general does not mean it is always easy to solve:

- $V(\vec{r})$ to $\rho(\vec{r})$: The Laplacian operator is just a sum of 2^{nd} order derivatives. Easy!
- $\rho(\vec{r})$ to $V(\vec{r})$: Need to solve the Laplacian equation, which is a 2^{nd} order non-homogeneous linear PDE. Awful!

Unfortunately in realistic problems, it is more frequent to ask for $V(\vec{r})$ from $\rho(\vec{r})$, because we can usually confine the charge distribution in a small region by using a very small test object; but for potential, it is always everywhere.

(add figure here: charge in small object -> can treat like point charge. but V spread everywhere)

We are not going to discuss its general solution here - it can take several book chapters to derive and analyze the solution forms at different boundary conditions.

In 1D wave equation, the boundary conditions are just about the 2 end points. Either fixed or free.

(add figure here: wave eq vs possion bc)

In Poisson equation, the boundary conditions are about the line/face boundary of the region of interest. Too many variations.

(add figure here: wave eq vs possion bc)

Although the general case is terribly complicated, we have already learnt the solution in one very special case - When the region of interest is infinitely large + potential vanishes at the boundary, i.e. $V(\vec{r}=\infty)=0$, the solution is exactly the Coulomb's law.

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \underset{\substack{\text{infinitely large} \\ \text{space}}}{\iiint} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \, \mathrm{d}^3\vec{r}'}$$

$$\sim \frac{1}{4\pi\epsilon_0} \sum_{\text{everywhere}} \frac{(\text{charge})}{(\text{distance})}$$

$$\equiv \underset{\text{(But written in a fancier vector form)}}{\text{Coulomb's law for electric potential}}$$

(add figure here: infinitely large region + potential element plot)

4.3 Finding $\vec{E}$ from Q

On the other hand, Poisson equation provides an alternative to calculate E-field distribution from charge distribution. If we compare the Gauss's law and Poisson equation:

- Poisson equation $\left(\nabla^2 V = -\frac{\rho}{\epsilon_0}\right)$: $V(\vec{r})$ is a scalar function. Only 1 function $V(\vec{r})$ to be solved.
- Gauss's law $\left(\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}\right)$: $\vec{E}(\vec{r})$ is a vector function with 3 components $E_x(\vec{r})$, $E_y(\vec{r})$, $E_z(\vec{r})$, which are 3 inter-depending functions to be solved.

Obviously, there is no reason to try to solve the more difficult PDE of $\vec{E}$ if alternatively we can solve the easier PDE of V, and then take gradient to get $\vec{E}$ (i.e. via $\vec{E} = -\vec{\nabla}V$).

(add figure here: triangle)

Similar to potential, if the given boundary condition is $V(\vec{\boldsymbol{r}} = \infty) = 0$, the solution for the Gauss's law as a PDE should also be the Coulomb's law.

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}} V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \iiint\limits_{\substack{\text{infinitely large}\\ \text{space}}} \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^2} \left[\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right] \mathrm{d}^3\vec{\boldsymbol{r}}'}$$

$$\sim \frac{1}{4\pi\epsilon_0} \sum_{\text{everywhere}} \frac{(\text{charge})}{(\text{distance})^2} \left(\substack{\text{unit}\\ \text{vector}}\right)$$

$$\equiv \underset{\text{(But written in a fancier vector form)}}{\text{Coulomb's law for electric field}}$$

Here we shall summarize the methods of solving magnetostatics problems:

1. Very symmetric configurations $\Rightarrow$ Ampere's law integral form. No calculus required.
2. Not so symmetric but satisfies $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}} = \infty) = 0 \Rightarrow$ Multiple integral with Biot-Savat law.
3. All the above do not apply $\Rightarrow$ Solve Poisson equation explicitly. PDE hell

— The End —