

Single Variable Calculus on Mechanics

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Overview:

- Review on limits, differentiation and integration formulas.
- Calculus relations between quantities in mechanics.

1 Review on Calculus

1.1 Limits

The naive definition of limit writes as:

$$\lim_{x \rightarrow a} f(x) = L \quad (\text{where } L \neq \pm\infty)$$

which implies $f(x)$ is getting closer to L when x is getting closer to a .

1.1.1 Two Sides Limits

x can approach a from either the $+x$ or $-x$ direction. Therefore there can be two different limits.

- Right hand limit : $\lim_{x \rightarrow a^+} f(x) = L$
- Left hand limit : $\lim_{x \rightarrow a^-} f(x) = L$

By definition, we say $f(x)$ has a limit at a only if **both side limits at a exist**.

(add figure here: examples of limit not exist)

1.1.2 Evaluation on Limits

These formula are useful for resolving the limits of more complicated functions. **They work only if both the limits of $f(x)$ and $g(x)$ exist.**

- Addition/Subtraction : $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- Product : $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = (\lim_{x \rightarrow a} f(x)) \cdot (\lim_{x \rightarrow a} g(x))$
- Quotient : $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad (\text{require } \lim_{x \rightarrow a} g(x) \neq 0)$

- Trigonometric relation : $\begin{cases} \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \\ \lim_{x \rightarrow 0} \cos x = 1 \end{cases}$
- Definition of natural exponential function : $\lim_{n \rightarrow +\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

Side note: The formulas may look trivial, but they do require rigorous proofs to show that they work on all kinds of functions, which is a topic belonging to *mathematical analysis*.

1.2 Differentiation

The definition of differentiation on a function $f(x)$ is exactly evaluating the following limit:

$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \stackrel{\text{def}}{=} \frac{d}{dx} f(x) \text{ or } f'(x)$$

Both notation $\frac{d}{dx}$ and $f'(x)$ are commonly used. I will stick to $\frac{d}{dx}$ because it is more obvious to read.

Geometrically, differentiation means evaluating the slope of (tangent lines of) the function.

(add figure here: from secant to tangent)

1.2.1 Evaluation of Differentiation

These formula are important for evaluating differentiation of more complicated functions. [They work only if both the differentiation of \$f\(x\)\$ and \$g\(x\)\$ exist.](#)

- Addition/Subtraction : $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$
- Scaling : $\frac{d}{dx}[kf(x)] = k \frac{d}{dx}f(x)$ (given $k = \text{constant}$)
- Product rule : $\frac{d}{dx}[f(x) \cdot g(x)] = \left[\frac{d}{dx}f(x)\right] \cdot g(x) + f(x) \cdot \left[\frac{d}{dx}g(x)\right]$
- Quotient rule : $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{1}{[g(x)]^2} \left[g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x) \right]$

(Note that quotient rule is the same as applying product rule on $\frac{d}{dx}\left[f(x) \cdot \frac{1}{g(x)}\right]$)

- Chain rule : $\frac{d}{dx}[g(f(x))] = \frac{d}{d[f(x)]}g(f(x)) \cdot \frac{d}{dx}f(x)$
- Inverse function : $\frac{d}{dx}f^{-1}(x) = \frac{1}{\left.\frac{d}{dy}f(y)\right|_{y=f^{-1}(x)}}$

(add figure here: inverse function)

- Polynomial function : $\frac{d}{dx}x^n = nx^{n-1}$
- Trigonometric function : $\frac{d}{dx} \begin{cases} \sin x \\ \cos x \\ \tan x \end{cases} = \begin{cases} \cos x \\ -\sin x \\ \sec^2 x \end{cases}$
- Exponential function : $\frac{d}{dx} \begin{cases} e^x \\ \ln x \end{cases} = \begin{cases} e^x \\ \frac{1}{x} \end{cases}$

1.3 Integration

1.3.1 Definite Integral (Riemann Integral)

The geometrical meaning to definite integral is to compute the area under curve. The construction is through the following steps:

(add figure here: area under curve divide into strips)

1. Choose $n - 1$ points between $[a, b]$:

$$a < x_1 < x_2 < \dots < x_{n-1} < b = \begin{matrix} \text{separate into} \\ \text{total } n \text{ steps} \end{matrix}$$

2. Within each strip, pick an arbitrary point $x = \xi_i$ at its base. The strip's height can be approximated as $\approx f(\xi_i)$.

(add figure here: stripe height)

3. The approximated area of each strip is then (height) \times (base) $\approx f(\xi_i) \times [x_{i+1} - x_i]$.
 \Rightarrow Total area under curve = Summing the areas of all strips

$$= \sum_{i=0}^{n-1} f(\xi_i) \cdot [x_{i+1} - x_i] \quad (\text{with } x_0 = a, x_n = b)$$

4. Finally take the width of each strip $[x_{i+1} - x_i] = \Delta x_i$ limit to 0.
 No matter how "arbitrary" we chose ξ_i within $[x_i, x_{i+1}]$, the width become so small so that the height of the strip is limit to $f(\xi_i)$.

\Rightarrow Area under curve \approx Summing the area of infinitely many strips that have widths ≈ 0 . Here comes the formal definition:

$$\int_b^a f(x) dx \stackrel{\text{def}}{=} \lim_{\Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$$

1.3.2 An alternative interpretation of definite integral

Other than the area under curve, we can also interpret a definite integral as a weighted sum. This is in fact more useful in physics' applications.

- 1st interpretation - Area under curve

$$\int_b^a f(x) dx \stackrel{\text{def}}{=} \lim_{\Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} \underbrace{f(\xi_i)}_{\substack{\uparrow \\ \text{height of} \\ \text{strips}}} \cdot \underbrace{\Delta x_i}_{\substack{\uparrow \\ \text{width of} \\ \text{strips}}}$$

\uparrow
 Sum them all

- 2nd interpretation - Weighted sum

$$\int_b^a f(x) dx \stackrel{\text{def}}{=} \lim_{\Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} \underbrace{f(\xi_i)}_{\substack{\uparrow \\ \text{"weight" assigned to the} \\ \text{interval } [x_i, x_{i+1}]}} \cdot \underbrace{\Delta x_i}_{\substack{\nwarrow \\ \text{length of interval} \\ [x_i, x_{i+1}]}}$$

\uparrow
 Sum them all

(add figure here: graphical diff)

Example 1.1. Distribution graph of students' heights

Suppose a class of 18 students have the following heights (in cm):

162 163 163 163 165 165 165 165 167 167 167 170 170 172 172 172 174 175

Their total height is the the sum of each possible heights but weighted by number of people of that height:

$$\begin{aligned} \text{Total height} &= \sum \text{"weight"} \times \text{height} \\ &= \sum_h \left(\text{Number of students with height } h \right) \times (\text{height } h) \\ &= 1 \times 162 + 3 \times 163 + 4 \times 165 + 3 \times 167 + 2 \times 170 + 3 \times 172 + 1 \times 174 + 1 \times 175 \end{aligned}$$

If we plot out the bar chart, we can see the similarity to area under curve - Total height of students = Sum of all bars' height Total area under every bar.

(add figure here: bar chart)

Then we can transition from discrete to continuous distribution:

- **A standalone number** → **An interval between two numbers.**
E.g. A height of 163 cm → A range of height between [162.5, 163.5]cm
- **Height of a bar** → **The area under the bar**
E.g. Portion of students with height between the range [162.5, 163.5]cm

When the interval width is limit to 0, the weighted sum arrives at the same definition as a definite integral.

1.3.3 Indefinite Integral

In contrast to definite integral, the definition of indefinite integral is purely algebraic. Given a function $f(x)$,

- If there exists another function $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$, then $F(x)$ is called the **antiderivative** or **primitive** of $f(x)$.
- The **indefinite integral** of $f(x)$ is the **set of all antiderivative** of $f(x)$.

The emphasis is on the term "set" because $f(x)$ always has infinitely many antiderivatives. This is also the reason why a constant C should always be added after computing indefinite integral.

$$\begin{aligned} f(x) &= x^2 \\ \Rightarrow \int f(x) dx &= \frac{x^3}{3} + C \\ &= \left\{ \text{Set of all functions in the form } \frac{x^3}{3} + \text{real number} \right\} \end{aligned}$$

1.3.4 Evaluation of Indefinite Integral

These formula are important for evaluating integration of more complicated functions. (Yet not every function can be integrated analytically.) [The proofs for these formula can be reached directly by reversing the differentiation operation.](#)

- Addition/Subtraction : $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int_a^b g(x) dx$
- Scaling : $\int kf(x) dx = k \int f(x) dx$ (given $k = \text{constant}$)
- Integration by Part (**Corresponding to product rule**) :

$$\begin{aligned} \int f(x) \cdot \frac{d}{dx}g(x) dx &= f(x) \cdot g(x) - \int \frac{d}{dx}f(x) \cdot g(x) dx \\ f(x) \cdot g(x) &= \int f(x) \cdot \frac{d}{dx}g(x) dx + \int \frac{d}{dx}f(x) \cdot g(x) dx \\ \Rightarrow \frac{d}{dx}(f(x) \cdot g(x)) &= f(x) \cdot \frac{d}{dx}g(x) + \frac{d}{dx}f(x) \cdot g(x) \end{aligned}$$

- Integration by Change of Variable (**Corresponding to chain rule**) :

$$\begin{aligned} \int g[f(x)] \cdot \frac{d}{dx}f(x) dx &= \int g[f(x)] \cdot d[f(x)] \\ &= \int g(u) du \Big|_{u=f(x)} \end{aligned}$$

- Polynomial function : $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$$- \text{Trigonometric function} : \int \begin{cases} \sin x \\ \cos x \\ \sec^2 x \end{cases} dx = \begin{cases} -\cos x \\ \sin x \\ \tan x \end{cases} + C$$

$$- \text{Exponential function} : \int \begin{cases} e^x \\ \frac{1}{x} \end{cases} dx = \begin{cases} e^x \\ \ln |x| \end{cases} + C$$

- Substitution of trigonometric function

$$\begin{aligned} \int \sqrt{r^2 - x^2} dx &\Rightarrow \text{Subst. } x = r \sin \theta \Rightarrow \begin{cases} \sqrt{r^2 - x^2} = r \cos \theta \\ dx = r \cos \theta d\theta \end{cases} \\ \int \sqrt{r^2 + x^2} dx &\Rightarrow \text{Subst. } x = r \tan \theta \Rightarrow \begin{cases} \sqrt{r^2 + x^2} = r \sec \theta \\ dx = r \sec^2 \theta d\theta \end{cases} \\ \int \sqrt{x^2 - r^2} dx &\Rightarrow \text{Subst. } x = r \sec \theta \Rightarrow \begin{cases} \sqrt{x^2 - r^2} = r \tan \theta \\ dx = r \sec \theta \tan \theta d\theta \end{cases} \end{aligned}$$

1.3.5 Fundamental Theorem of Calculus

Up to this point, we are in fact looking at two different use cases of the symbol \int :

1. \int_a^b represents definite integral, which represents the area under curve between $[a, b]$.

$$\int_a^b \sim \lim_{\Delta x \rightarrow 0} \sum_{i=0}^{n-1}$$

Calculating this limit of sum looks quite tedious.

2. \int represents indefinite integral, which is the opposite operation of differentiation.

$$\int f(x) dx = F(x) \quad \Leftrightarrow \quad \frac{d}{dx} F(x) = f(x)$$

Simplification formulas are straightly derived from reversing differentiation.

Of course, nowadays we know that the two's calculations are basically the same. (And that's why mathematicians assigned them the same symbol.) However, for rigourous mathematics needs, we need an extra step to show that they are intrinsically the same, which is why we have the **fundamental theorem of calculus**.

- 1st fundamental theorem of calculus (definite \rightarrow indefinite) :

An antiderivative $F(x)$ can be constructed from a definite integral on f over an interval with a varying upper bound, i.e.,

$$F(x) = \int_b^x f(t) dt$$

- 2nd fundamental theorem of calculus (indefinite \rightarrow definite) :

The definite integral on $f(x)$ over an interval can be computed by the subtraction between the antiderivatives F at the bounds, i.e.,

$$\int_a^b f(x) dx = F(b) - F(a)$$

The proofs can be found with clear explanation on wiki, so I shall omit them in this note.

https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus#Formal_statements

2 Calculus Relations in Mechanics

2.1 Relation between s, v, a

In secondary school textbook, we have the definitions:

- Displacement s
- Average velocity $v \sim \frac{\Delta s}{\Delta t} = \frac{\text{Change in displacement}}{\text{time}}$
- Average acceleration $a \sim \frac{\Delta v}{\Delta t} = \frac{\text{Change in velocity}}{\text{time}}$

But the term "average" is never accurate to describe things.

(add figure here: high varying distance)

For accurate description, we need to shorten the time separation between two measurements as much as possible, i.e. limit to 0.

\Rightarrow Instantaneous quantities = quantity measured at a "single" time point.

- Instantaneous velocity = $\lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{d}{dt}s(t)$
- Instantaneous acceleration = $\lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{d}{dt}v(t)$

$$s(t) \xrightleftharpoons[\int dt]{\frac{d}{dt}} v(t) \xrightleftharpoons[\int dt]{\frac{d}{dt}} a(t)$$

Example 2.1. Constant Acceleration Motion

By definition, the acceleration $a(t) = a$ is a constant. Finding velocity and displacement are then straightforward integration.

– Velocity : $v(t) = \int a \, dt = at + C_1$

At $t = 0$, $v(0) = C_1 =$ initial velocity, which is usually denoted as u in textbook.

$$\Rightarrow v(t) = v(0) + at$$

– Displacement : $s(t) = \int v(t) \, dt = v(0)t + \frac{1}{2}at^2 + C_2$

At $t = 0$, $s(0) = C_2 =$ initial displacement.

$$\Rightarrow s(t) = s(0) + v(0)t + \frac{1}{2}at^2$$

In fact, out of the 4 formulas we usually found in textbook, only 2 of them are independent.

$$\left\{ \begin{array}{l} v(t) = v(0) + at \\ s(t) = s(0) + v(0)t + \frac{1}{2}at^2 \\ s(t) - s(0) = \frac{v(t) - v(0)}{2}t \\ v(t)^2 - v(0)^2 = 2a[s(t) - s(0)] \end{array} \right.$$

Definition
nothing to explain

Can be derived by
substitution of the above 2

Having only 2 independent equations implies that we can at most solve 2 unknowns out of the 5 alphabets "SUVAT". So in such standard question, you should always able to find the info of at least 3 of the alphabets. Only then you can solve the remaining 2.

2.2 Force and Energy

Work done is the fundamental relation between force and energy:

$$\text{W.D.} = F \cdot \Delta s = \left(\frac{\text{Average}}{\text{Force}} \right) \cdot (\text{Displacement})$$

For a time varying force $F(t)$, consider the W.D. in a very short time interval $[t_i, t_{i+1}]$

$$\begin{aligned} \Delta \text{W.D.}_i &\approx \frac{F(t_i) + F(t_{i+1})}{2} \cdot [s(t_{i+1}) - s(t_i)] \\ &= \frac{F(t_i) + F(t_{i+1})}{2} \cdot \left[\frac{s(t_{i+1}) - s(t_i)}{t_{i+1} - t_i} \right] (t_{i+1} - t_i) \end{aligned}$$

The the total W.D. in a long period time

$$\begin{aligned} &= \text{Sum of all W.D. of many small time interval} \\ &= \sum_i \left[\frac{F(t_i) + F(t_{i+1})}{2} \right] \cdot \left[\frac{s(t_{i+1}) - s(t_i)}{t_{i+1} - t_i} \right] (t_{i+1} - t_i) \end{aligned}$$

Take limit to the time interval $\Delta t = t_{i+1} - t_i \rightarrow 0$, then $t_{i+1} \approx t_i$ and $F(t_{i+1}) \approx F(t_i)$.

$$\begin{aligned}\text{W.D} &\rightarrow \int \frac{F(t) + F(t)}{2} \cdot \left[\frac{ds(t)}{dt} \right] dt \\ &= \int F(t) \cdot v(t) dt \\ &= \int F(t) \cdot d[s(t)] \\ &= \int \underline{F^*[s(t)]} d[s(t)]\end{aligned}$$

More commonly, \uparrow force is provided
as a function of displacement

Example 2.2. Gravitational force & potential energy (PE)

$$F(\textcolor{red}{r}) = -\frac{GMm}{\textcolor{red}{r}^2}$$

\uparrow
Force is a function of distance from the masses

Then the W.D. by gravitational force from position r_1 to r_2 can be computed by integrating the force

$$\begin{aligned}\text{W.D.} &= \int_{r_1}^{r_2} -\frac{GMm}{r^2} dr \\ &= \left(\frac{GMm}{r_2} \right) - \left(\frac{GMm}{r_1} \right) \\ &= \textcolor{red}{-}(\text{GPE at } r_2 - \text{GPE at } r_1)\end{aligned}$$

\uparrow
W.D. by gravitation \Rightarrow loss in PE
W.D. against gravitation \Rightarrow gain in PE

Reversely, we take the **minus** differentiation to the PE formula to find the force due to gravitation.

$$\begin{aligned}\text{PE} &= -\frac{GMm}{r} \\ \text{Gravitational force} &= \textcolor{red}{-} \frac{d}{dr}(\text{PE}) = -\frac{GMm}{r^2}\end{aligned}$$

\uparrow
Add negative sign for attractive force

2.3 Momentum, Impulse and Newton 2nd Law

Momentum and impulse is in fact an equivalent description to Newton's 2nd Law.

| <u>Force & Acceleration</u> | <u>Impulse & Momentum</u> |
|---------------------------------|-------------------------------|
| $F = ma$ | $I = \Delta(mv)$ |
| Apply a force | Apply an impulse |
| Get acceleration | Change in momentum |

Recall that the definition of impulse as

$$\begin{aligned} I &\sim F \cdot \Delta t \\ &= (\text{Applied Force}) \cdot (\text{Duration}) \\ &= \text{Area under curve in F-t graph} \\ &= \text{Change in momentum} \end{aligned}$$

(add figure here: impulse, bar)

But in reality, when objects collide, the applied force is not constant during the whole time.

(add figure here: impulse, distribution)

So in general, force should be taken as a function of time.

Similar to how we computed W.D., we can first consider the change in momentum in a very short time interval $[t_i, t_{i+1}]$

$$\Delta(mv)_i = I_i \approx \frac{F(t_i) + F(t_{i+1})}{2} (t_{i+1} - t_i)$$

The the total change in momentum in a long period time

$$\begin{aligned} \sum_i \Delta(mv)_i &= \sum_i I_i = \text{Sum of all impulse of many small time interval} \\ &= \sum_i \left[\frac{F(t_i) + F(t_{i+1})}{2} \right] (t_{i+1} - t_i) \end{aligned}$$

Take limit to the time interval $\Delta t = t_{i+1} - t_i \rightarrow 0$, then $t_{i+1} \approx t_i$ and $F(t_{i+1}) \approx F(t_i)$.

$$\begin{aligned} \int d(mv) = I &\rightarrow \int \frac{F(t) + F(t)}{2} dt \\ &= \int F(t) dt \end{aligned}$$

i.e. The "real" impulse-momentum relation should be represented with integral. We can also differentiate both side by t to recover momentum's relation with force .

$$\frac{d}{dt}(mv) = F(t)$$

(Rate of change of momentum) = (Force)

In an advanced force problem (e.g. rocket problem), we may NOT assume mass to be time constant. The more general way to write Newton's 2nd Law should be

$$\begin{aligned} F(t) &= \frac{d}{dt}(m(t)v(t)) \\ &= m(t) \frac{dv(t)}{dt} + \frac{dm(t)}{dt} v(t) \\ &= \boxed{ma + \frac{dm}{dt} \cdot v} \end{aligned}$$

Only if we guarantee that the mass is a constant, it can be reduced to the common form $F = ma$.

— The End —