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Multivariable Calculus

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Overview:

- Comparison between single variable functions & multivariable functions
- Partial differentiation (on scalar function)
- Multiple integral (on scalar function)

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1 Functions with Multiple Variables

To well-define a function $f(x)$ in advanced mathematics, we actually need to specify the function's **domain** and **image**.

- Domain = The set of values that be substitute into x.
- Image = The set of all possible output of $f(x)$.

E.g. Formal notation in math text to define $f(x) = \frac{1}{|x|}$:

$$\begin{aligned} f: \underbrace{\mathbb{R}}_{\text{Domain}} &\longrightarrow \underbrace{\mathbb{R}^+}_{\text{Image}} \\ x &\longmapsto \frac{1}{|x|} \end{aligned}$$

We can classify functions by whether their domain/image are made of single number / tuple of numbers.

1.1 Single Variable Scalar Function

They are the functions that you have already learnt.

- Domain = A set of single number
- Image = A set of single number

For example,

$$f(x) = \sqrt{x-1} \qquad \Rightarrow \qquad \begin{cases} \text{Domain} &= \text{Any real number } \geq 1 \\ \text{Image} &= \text{Any real number } \geq 0 \end{cases}$$

(add figure here: number line map)

1.2 Multivariable Scalar Function

- Domain = A set of tuples of number, like $x = (1, 2, 3)$
- Image = A set of single number, like $f(x) = 5$

For example,

$$f(x,y) = \sqrt{1 - xy} \qquad \Rightarrow \qquad \begin{cases} \text{Domain} & = \text{Any pair of values } x, y \text{ where } xy \leq 1 \\ \text{Image} & = \text{Any real number } \geq 0 \end{cases}$$

(add figure here: 1-sqrt(xy))

<u>Example in Physics:</u>

- Gravitational potential energy

$$U(x,y,z) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}}$$

- Density distribution in object $\rho(x, y, z)$

(add figure here: density distribution)

1.3 Single Variable Vector Function

- Domain = A set of single number
- Image = A set of tuple of numbers

For example,

$$\vec{\boldsymbol{r}}(t) = (x(t), y(t)) = (t^2, 3t^3) \qquad \Rightarrow \qquad \begin{cases} \text{Domain} & = \text{Any real number } t \\ \text{Image} & = (x,y) \text{ pairs restricting on } x = \left(\frac{y}{3}\right)^{\frac{2}{3}} \end{cases}$$

(add figure here: t2, 3t3)

<u>Example in Physics:</u>

- Displacement, velocity, acceleration

$$\begin{cases} \vec{\boldsymbol{s}}(t) = (x(t), y(t), z(t)) \\ \vec{\boldsymbol{v}}(t) = (v_x(t), v_y(t), v_z(t)) \\ \vec{\boldsymbol{a}}(t) = (a_x(t), a_y(t), a_z(t)) \end{cases}$$

1.4 Multivariable Vector Function

- Domain = A set of tuple of numbers
- Image = A set of tuple of numbers

For example,

$$\vec{\boldsymbol{r}}(u,v) = (r_1(u,v), r_2(u,v)) = (u^2+v^2, u-1-v^2) \qquad \Rightarrow \qquad \begin{cases} \text{Domain} &= \text{The whole u-v plane} \\ \text{Image} &= \text{Region depicted below} \end{cases}$$

(add figure here: u2v2)

<u>Example in Physics:</u>

- Gravitational force

$$\begin{aligned}
\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) &= \vec{\boldsymbol{F}}(x,y,z) \\
&= -\frac{GMm}{|\vec{\boldsymbol{r}}|^2} \cdot \left(\frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|}\right) \quad \longleftarrow \text{ Unit vector of } \vec{\boldsymbol{r}} \\
&= -\frac{GMm}{x^2+y^2+z^2} \cdot \left(\frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\sqrt{x^2+y^2+z^2}}\right) \quad \longleftarrow \text{ Separate into components of } \hat{\boldsymbol{x}}/\hat{\boldsymbol{y}}/\hat{\boldsymbol{z}} \\
&= \left[\frac{-GMmx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{\boldsymbol{x}} + \left[\frac{-GMmy}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{\boldsymbol{y}} + \left[\frac{-GMmz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{\boldsymbol{z}}
\end{aligned}$$

1.5 Function Composition for multivariable functions

For single variable scalar function, you should be familiar with the what function composition is. For example, if $f(x) = \sin x, g(x) = e^x$, we can have these compositions:

$$f(f(x)) = \sin(\sin x) \quad , \quad f(g(x)) = \sin(e^x) \quad , \quad g(f(x)) = e^{sinx} \quad , \quad g(g(x)) = e^{e^x}$$

Note that function composition is the key in chain rule.

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = \frac{\mathrm{d}f(g(x))}{\mathrm{d}g(x)} \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

However for multivariable functions, we can construct function composition only if the number of output matches the next function's number of input. For example, let's have

$$\begin{cases} f(p,q) = \sqrt{p+q} & \text{2 inputs, 1 output. Denote as } (2 \xrightarrow{f} 1) \\ \vec{\boldsymbol{g}}(t) = (t-1, t^2) & \text{1 input, 2 outputs. Denote as } (1 \xrightarrow{g} 2) \\ \vec{\boldsymbol{h}}(u,v) = (u^2+v, u-v) & \text{2 inputs, 2 outputs. Denote as } (2 \xrightarrow{h} 2) \end{cases}$$

We can have the following composition:

$$\begin{aligned}
f(\vec{\boldsymbol{g}}(t)) &= \sqrt{(t-1)+(t^2)} \qquad (1 \xrightarrow{g} 2 \xrightarrow{f} 1)\\
f(\vec{\boldsymbol{h}}(u,v)) &= \sqrt{(u^2+v)+(u-v)} \qquad (2 \xrightarrow{h} 2 \xrightarrow{f} 1)\\
\vec{\boldsymbol{g}}(f(p,q)) &= (\sqrt{p+q}-1, p+q) \qquad (2 \xrightarrow{f} 1 \xrightarrow{g} 2)\\
\vec{\boldsymbol{h}}(\vec{\boldsymbol{g}}(t)) &= ((t-1)^2+(t^2),(t-1)-(t^2)) \qquad (1 \xrightarrow{g} 2 \xrightarrow{h} 2)\\
\vec{\boldsymbol{h}}(\vec{\boldsymbol{h}}(t)) &= ((u^2+v)^2+(u-v),(u^2+v)-(u-v)) \qquad (2 \xrightarrow{h} 2 \xrightarrow{h} 2)
\end{aligned}$$

But these are NOT allowed:

$$\begin{aligned}
g(h(u,v)) &: \quad (2 \xrightarrow{h} 2 \Rightarrow 1 \xrightarrow{g} 2)\\
h(f(p,q)) &: \quad (2 \xrightarrow{f} 1 \Rightarrow 2 \xrightarrow{h} 2)
\end{aligned}$$

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2 Limits on Multivariable Scalar Function

In single variable functions, $\lim_{x\to a} f(x) = L$ means when the input x is "close enough" to a value a, output of $f(x)$ must be "close" to some L. This idea can be extended to multivariable function, i.e.

$$\lim_{(x_1,x_2,\ldots,x_n)\to(a_1,a_2,\ldots,a_n)} f(x_1,x_2,\ldots,x_n) = L$$

The idea requires all the inputs $x_1, x_2, \ldots$ to be "close enough" to some corresponding values $a_1, a_2, \ldots$, only after then the output of $f(\ldots)$ will be "close" enough to some L. We can visually compare it with single variable function as follow:

(add figure here: limit sing var vs mul var)

However, the "existance" of limit in multivariable functions has a much stricter requrement.

- Single variable function:
  - Input x must approach a from either left (x^-) or right (x^+).
  - "Existance" of limit only require showing both left/right limit approach to the same output L.
- Multivariable function, (e.g. functions with 2 inputs):
  - Inputs (x_1, x_2) can approach the point (a_1, a_2) along any trajectories on the plane.
  - "Existance" of limit require showing that along ALL trajectories

(add figure here: 2 dir vs inf dir approach)

Proving a limit exist rigorously is a lot harder in multivariable function. But in physics, we almost never need to deal with any strange functions that has limit only along certain trajectories. We may assume that every function we encounter is well-behaved, and then calculation can be done like in single variable functions. E.g.

$$\lim_{(x,y)\to\left(\frac{\pi}{2},\frac{\pi}{2}\right)} \sin x \cos y = \sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)$$

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3 Partial Differentiation

- Notation: $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots$
- Usually pronouced as "partial x", "partial y", etc.

Comparing with ordinary differentiation to single variable function, the notation difference is to emphasize that the **differentiation is only about 1 of the inputs**.

3.1 Definition & Geometrical Interpretation

The limit definition of partial differentiation of $f(x_1, x_2, ..., x_n)$ at $(a_1, a_2, ..., a_n)$ in the ith input (x_i) 's direction is defined as:

$$\begin{aligned}
&\frac{\partial}{\partial x_i} f(x_1, x_2, ..., x_i, ..., x_n) \\
&= \lim_{\Delta x_i \to 0}\left[\frac{f(x_1, x_2, ..., x_i + \Delta x_i, ..., x_n) - f(x_1, x_2, ..., x_i, ..., x_n)}{\Delta x_i}\right]
\end{aligned}$$

Note that the limit only acts on the ith input. Other inputs remains untouched.

Therefore in calculation, when doing partial differentiation over x_i, only x_i is differentiated (the same way we do in single variable differentiation), while the other x are treated as constants.

E.g. $f(x, y, z) = x^2 y \sin z$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x \cdot y \sin z && \left(\frac{\mathrm{d}}{\mathrm{d}x} x^2 = 2x, \text{don't touch } y, z\right) \\
\frac{\partial f}{\partial y} &= x^2 \cdot 1 \cdot \sin z && \left(\frac{\mathrm{d}}{\mathrm{d}y} y = 1, \text{don't touch } x, z\right) \\
\frac{\partial f}{\partial z} &= x^2 y \cdot \cos z && \left(\frac{\mathrm{d}}{\mathrm{d}z} \sin z \cos z, \text{don't touch } x, y\right)
\end{aligned}$$

The visualization to partial differentiation is straightforward. Take a 2-inputs function $f(x, y)$ as example, we can draw the followings:

(add figure here: partial d geo interp)

$$\frac{\partial}{\partial x} = \text{On the plane of constant } y \text{, find slope along } x \text{ direction.}$$
$$\frac{\partial}{\partial y} = \text{On the plane of constant } x \text{, find slope along } y \text{ direction.}$$

We can conclude:

$$\frac{\partial}{\partial x_i} = \text{Find slope / rate of change of function with respect to } x_i$$

3.2 Evaluation

Calculation rules for partial differentation are the same as you have learnt in single variable differentiation. The only exception is chain rule, which is the**sum of chain rule with respect to each of the input.**

$$\begin{aligned}
\frac{\partial}{\partial x_i} f(\vec{\boldsymbol{g}}(x_1, x_2, ..., x_n)) &= \sum_j \frac{\partial}{\partial g_j} f(\vec{\boldsymbol{g}}) \frac{\partial g_j}{\partial x_i} \\
&= \frac{\partial}{\partial g_1} f(\vec{\boldsymbol{g}}) \frac{\partial}{\partial x_i} g_1(x_1, x_2, ..., x_n) + \frac{\partial}{\partial g_2} f(\vec{\boldsymbol{g}}) \frac{\partial}{\partial x_i} g_2(x_1, x_2, ..., x_n) + ...
\end{aligned}$$

As for now you do not need to remember this formula. We will be able to write it in a more compact (and easier to remember) form after learning matrix.

As an example of calculation, suppose we start with two functions without knowing their exact expression:

$$f(p, q) \qquad (2 \xrightarrow{f} 1)$$
$$\vec{\boldsymbol{h}}(u, v) = (h_1(u, v), h_2(u, v)) = (h_1, h_2) \qquad (2 \xrightarrow{h} 2)$$

And construct the following composition:

$$f(\vec{h}(u, v)) = f((h_1, h_2)) = f((h_1(u, v), h_2(u, v))) \qquad (2 \xrightarrow{h} 2 \xrightarrow{h} 1)$$

Because $f(\vec{h}(u, v))$ takes 2 inputs u, v, there must be 2 partial differentiations (one for u and one for v). With chain rule, the partial differentiations write as

With respect to u:

$$\frac{\partial}{\partial u} f(\vec{\boldsymbol{h}}(u, v)) = \frac{\partial}{\partial \underline{u}} f(\underline{h_1}, \underline{h_2})$$

u on h_2

u on h_1

$$= \underbrace{\left(\frac{\partial}{\partial \boxed{h_1}} f(\boxed{h_1}, h_2) \cdot \boxed{\frac{\partial}{\partial u} h_1} \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{\partial}{\partial \boxed{h_2}} f(h_1, \boxed{h_2}) \boxed{\frac{\partial}{\partial u} h_2} \right)}_{\text{Chain rule over } h_2 \text{ only}}$$

With respect to v:

$$\frac{\partial}{\partial v} f(\vec{\boldsymbol{h}}(u,v)) = \frac{\partial}{\partial v} f(h_1, h_2)$$

v on h_2

v on h_1

$$= \underbrace{\left(\frac{\partial}{\partial \boxed{h_1}} f(\boxed{h_1}, h_2) \cdot \boxed{\frac{\partial}{\partial v} h_1}\right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{\partial}{\partial \boxed{h_2}} f(h_1, \boxed{h_2}) \boxed{\frac{\partial}{\partial v} h_2}\right)}_{\text{Chain rule over } h_2 \text{ only}}$$

We may do straightforward substitution, if the functions' expressions are given. Let's say,

$$f(p,q) = \sqrt{p+q} \qquad \text{and} \qquad \vec{\boldsymbol{h}}(u,v) = (u^2+v, u-v) = (h_1, h_2)$$

Then

$$\begin{aligned}
\frac{\partial}{\partial u} f(\vec{\boldsymbol{h}}(u,v)) &= \left(\frac{\partial}{\partial h_1} f(h_1,h_2) \frac{\partial}{\partial u} h_1\right) + \left(\frac{\partial}{\partial h_2} f(h_1,h_2) \frac{\partial}{\partial u} h_2\right) \\
&= \left(\frac{\partial}{\partial h_1} \sqrt{h_1+h_2}\Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot \frac{\partial}{\partial u}(u^2+v)\right) + \left(\frac{\partial}{\partial h_2} \sqrt{h_1+h_2}\Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot \frac{\partial}{\partial u}(u-v)\right) \\
&= \left(\frac{1}{2\sqrt{h_1+h_2}}\Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot 2u\right) + \left(\frac{1}{2\sqrt{h_1+h_2}}\Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot (-1)\right) \\
&= \frac{2u}{2\sqrt{u^2+u}} + \frac{-1}{2\sqrt{u^2+u}} \\
&= \frac{2u-1}{2\sqrt{u^2+u}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial v} f(\vec{\boldsymbol{h}}(u,v)) &= \left(\frac{\partial}{\partial h_1} f(h_1,h_2) \frac{\partial}{\partial v} h_1\right) + \left(\frac{\partial}{\partial h_2} f(h_1,h_2) \frac{\partial}{\partial v} h_2\right) \\
&= \left(\frac{\partial}{\partial h_1} \sqrt{h_1+h_2}\Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot \frac{\partial}{\partial v}(u^2+v)\right) + \left(\frac{\partial}{\partial h_2} \sqrt{h_1+h_2}\Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot \frac{\partial}{\partial v}(u-v)\right) \\
&= \left(\frac{1}{2\sqrt{h_1+h_2}}\Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot (1)\right) + \left(\frac{1}{2\sqrt{h_1+h_2}}\Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot (-1)\right) \\
&= \frac{1}{2\sqrt{u^2+u}} + \frac{-1}{2\sqrt{u^2+u}} \\
&= 0
\end{aligned}$$

We can also compute the composition directly for result checking:

$$f(\vec{\boldsymbol{h}}(u,v)) = \sqrt{u^2+v+u-v} = \sqrt{u^2+u}$$

$$\Rightarrow \quad \frac{\partial f}{\partial u} = \frac{2u+1}{2\sqrt{u^2+u}} \qquad \text{and} \qquad \frac{\partial}{\partial f v} = 0$$

Exercise 3.1. Given the functions and their composition:

$$\begin{cases} f(p,q) = \sqrt{p,q} \\ \vec{\boldsymbol{g}}(t) = (t-1, t^2) \end{cases} \qquad \Rightarrow \qquad f(\vec{\boldsymbol{g}}(t)) = \sqrt{t^2+t-1}$$

Compute the derivative $\dfrac{\mathrm{d}f(\vec{\boldsymbol{g}}(t))}{\mathrm{d}t}$, by

1. directly differentiating against t
2. first differentiate via chain rule over (p,q)

(You should get equal results.)

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4 Multiple Integral

The limit definition of multiple integral can be written as

$$\int \cdots \int_{\binom{\text{Some}}{\text{region}}} f(x_1, x_2, ..., x_n)\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n$$
$$= \lim_{\Delta x_1, \Delta x_2, ..., \Delta x_n to 0} \sum_{\substack{\text{all divisions} \\ \text{in the region}}} f(\xi_1, \xi_2, ..., \xi_n)\Delta x_1 \Delta x_2 ... \Delta x_n$$

Recall that we have introduced 2 geometrical interpretations of integration. Here we can demonstrate them on the two most frequently used multiple integral.

4.1 Double Integral

For functions with 2 inputs.

Interpretation 1: Volume under surface, bounded by base area

$$\iint_{\substack{\text{Some}\\ \text{Area}}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\Delta x, \Delta y \to 0} \sum_{\substack{\text{All divisions}\\ \text{in the area}}} \underbrace{f(\xi_x, \xi_y)}_{\text{Pillar's height}} \underbrace{\Delta x \Delta y}_{\text{Pillar's base area}}$$

(add figure here: vol under surface)

Interpretation 2: Weighted sum over an area

$$\iint_{\substack{\text{Some}\\\text{Area}}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\Delta x,\Delta y\to 0} \sum_{\substack{\text{All divisions}\\\text{in the area}}} \underbrace{f(\xi_x,\xi_y)}_{\substack{\text{Weight for the}\\\text{grid at }(\xi_x,\xi_y)}}\underbrace{\Delta x\Delta y}_{\text{Area of each grid}}$$

(add figure here: area grid divi)

For a physics example, the *area mass density distribution* $\sigma(x,y)$ may depends on position coordinate (x,y).

- Each small grid has an area $\mathrm{d}x\,\mathrm{d}y$
- At position (ξ_x,ξ_y), the grid has a density $\sigma(x,y)$

Thus,

$$\begin{aligned}
\text{Total mass} &= \text{Sum of mass of all small grids}\\
&= \sum_{\substack{\text{all small grids}\\\text{in the area}}} \left({}^{\text{density}}_{\text{of each grid}}\right)\left({}^{\text{area}}_{\text{of each grid}}\right)\\
&= \sum_{\substack{\text{all small grids}\\\text{in the area}}} \sigma(x,y)\cdot(\Delta x\Delta y)\\
&\approx \iint_{\text{the area}} \sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

4.2 Triple Integral

Interpretation 1: ??? under volume

Sorry, we live in a 3D space. No idea how to draw 4D objects.

Interpretation 2: Weighted sum over a volume

$$\iiint_{\substack{\text{Some}\\\text{Volume}}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \lim_{\Delta x,\Delta y,\Delta z\to 0} \sum_{\substack{\text{All divisions}\\\text{in the volume}}} \underbrace{f(\xi_x,\xi_y,\xi_z)}_{\substack{\text{Weight for the}\\\text{cube at }(\xi_x,\xi_y,\xi_z)}}\underbrace{\Delta x\Delta y\Delta z}_{\text{Volume of each cube}}$$

Similar to double integral, if $\rho(x, y, z)$ is the *volume mass density distribution*, Thus,

$$\begin{aligned}
\text{Total mass} &= \text{Sum of mass of all small cubes} \\
&= \sum_{\substack{\text{all small cubes} \\ \text{in the volume}}} \left(\substack{\text{density} \\ \text{of each cube}}\right)\left(\substack{\text{volume} \\ \text{of each cube}}\right) \\
&= \sum_{\substack{\text{all small cubes} \\ \text{in the volume}}} \sigma(x, y) \cdot (\Delta x \Delta y) \\
&\approx \iiint_{\text{the volume}} \sigma(x, y)\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}$$

4.3 Evaluating Multiple Integral

The difficulty in calculation mostly comes from determining the region to be integrated. Here are the main steps in your calculation:

1. Decide the integration order, i.e. how to divide a region.
   - The integration order decide the expression. Follow the expression to integrate "from inside to outside".

$$\iiint f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int \left(\int \left(\int f(x, y, z)\, \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z$$

$\text{outer} = 3^{\text{rd}}$ $\quad$ $\text{middle} = 2^{\text{nd}}$ $\quad$ $\text{inner} = 1^{\text{st}}$

   - Calculation is just like how you do to single variable functions, but do it multiple times.
   - While integrating one variable, treat the others as constants.
2. Derive the corresponding upper/lower bounds
   - It would be easier if you can draw out the region.
   - Note that if you switch the integration order, the bounds must change.

Example 4.1. Integrate $f(x, y) = x^2y - xy^3$ over the region bounded by $\begin{cases} x &= 1 \\ x &= 4 \end{cases}$ and $\begin{cases} y &= 2 \\ y &= 3 \end{cases}$

(add figure here: rect int)

<u>Integration order 1:</u> First x, then y.

1. Integrate x = Sum all grid with the same y coordinate to form horizontal strips.

(add figure here: hoz strip)

$$\int_{x=1}^{x=4} f(x, y)\, \mathrm{d}x$$

2. Integrate y = Sum all horizontal strips to from the integration region.

(add figure here: hoz stip to region)

$$\int_{y=2}^{y=3}\left[\int_{x=1}^{y=4} f(x,y)\,\mathrm{d}x\right]\mathrm{d}y$$

3. In the calculation, follow the expression's order: Integrate x first, then y. Note that before integrating y, you need to clear all x by substituting the given upper/lower bounds.

$$\int_{y=2}^{y=3}\left[\int_{x=1}^{y=4} x^2y - xy^3\,\mathrm{d}x\right]\mathrm{d}y$$

$$=\int_{y=2}^{y=3}\left[\frac{x^3}{3}y - \frac{x^2}{2}y^3\right]\Bigg|_{x=1}^{x=4}\mathrm{d}y$$

$$=\int_{y=2}^{y=3}\underbrace{\left(\frac{64}{3}y - \frac{16}{2}y^3\right)}_{\text{Subst. } x=4} - \underbrace{\left(\frac{1}{3}y - \frac{1}{2}y^3\right)}_{\text{Subst. } x=1}\mathrm{d}y$$

$$=\int_{y=2}^{y=3} 21y - \frac{15}{2}y^3\,\mathrm{d}y$$

$$=\left[\frac{21}{2}y^2 - \frac{15}{8}y^4\right]\Bigg|_{y=2}^{y=3}$$

$$=\frac{-555}{8}$$

Integration order 2: First y, then x.

1. Integrate y = Sum all grid with the same x coordinate to form vertical strips.

(add figure here: vert strip)

$$\int_{y=3}^{y=2} f(x,y)\,\mathrm{d}y$$

2. Integrate y = Sum all horizontal strips to from the integration region.

(add figure here: hoz stip to region)

$$\int_{x=1}^{x=4}\left[\int_{y=2}^{y=3} f(x,y)\,\mathrm{d}y\right]\mathrm{d}x$$

3. In the calculation, follow the expression's order: Integrate y first, then x. Note that before integrating x, you need to clear all y by substituting the given upper/lower bounds.

$$\int_{x=1}^{x=4} \left[\int_{y=2}^{y=3} x^2 y - xy^3 \, \mathrm{d}y \right] \mathrm{d}x$$

$$= \int_{x=1}^{x=4} \left[\frac{1}{2} x^2 y^2 - \frac{1}{4} x y^4 \right] \Bigg|_{y=2}^{y=3} \mathrm{d}x$$

$$= \int_{x=1}^{x=4} \underbrace{\left(\frac{9}{2} x^2 - \frac{81}{4} x \right)}_{\text{Subst. } y=3} - \underbrace{\left(2x^2 - 4x \right)}_{\text{Subst. } y=2} \mathrm{d}x$$

$$= \int_{x=1}^{x=4} \frac{5}{2} x^2 - \frac{65}{4} x \, \mathrm{d}x$$

$$= \left[\frac{5}{6} x^3 - \frac{65}{8} x^2 \right] \Bigg|_{x=1}^{x=4}$$

$$= \frac{-555}{8}$$

However, if the boundaries of the region is ugly, some integration order make your life easier than the others. As a demonstration, consider integrating over the below region (with an arbituary $f(x, y)$):

(add figure here: ugly region)

Integration order 1: First y, then x.

(add figure here: vert bar than add)

This approach is easy because all vertical strips have the same bounds:

- Upper bound: The curve $y = x^2 + 1$
- Lower bound: The curve $y = x - 1$

We can write the integral expression as a single term.

$$\int_{x=0}^{x=1} \int_{y=x-1}^{y=x^2+1} f(x, y) \, \mathrm{d}y \, \mathrm{d}x$$

Integration order 2: First x, then y.

Note that the bounds of horizontal strips are different for different y:

(add figure here: different bound)

So we need to integrate each region individually.

$$I_1 = \int_{y=1}^{y=2} \int_{x=\sqrt{y+1}}^{x=1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$I_2 = \int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$I_3 = \int_{y=0}^{y=-1} \int_{x=0}^{x=y+1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

And the final answer would be the sum to all 3 regions $I_1 + I_2 + I_3$. Although we should get the same value as we integrate y first then x, integrating x first then y takes a lot more effort.

— The End —