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Wave Equation

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Overview:

The wave equation is a partial differential equation

$$\frac{\partial^2}{\partial x^2}y(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}y(x,t)$$

is an equation of $y(x,t)$ - the wave's magnitude as a function of position x and time t. We are going to derive and study its solutions.

- Derive wave equation - transverse and longitudinal wave
- Initial value problem *(Not so important)*
- Boundary value problem - What are "Modes" of standing wave *(Main focus)*

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1 Model of Transverse Wave

We usually use an elastic string to visualize transverse wave travel.

- When the string lies flat - Each string segment has a width Δx.

(add figure here: flat string)

- When the string shakes - The segment jumps up and down, horizontal length remains the same, but gain a vertial length.

(add figure here: curve string)

Transverse wave is height of string segments at different position / time, which is described by the function $y(x,t)$.

Deriving wave eqauation

1. Equations of forces by Newton's 2$^{\text{nd}}$ Law

(add figure here: tension)

Tension $\vec{F}$ must be a function of x because it must be different everywhere along the string.

Separate the tensions' horizontal and vertical components:

$$\begin{cases} \rightarrow: & F_{\rightarrow}(x+\Delta x,t) - F_{\rightarrow}(x,t) = \underline{0} \longleftarrow \\ \uparrow: & F_{\uparrow}(x+\Delta x,t) - F_{\uparrow}(x,t) = (\underline{\mu}\Delta x)a_{\uparrow} \end{cases}$$

Horizontal acceleration $= 0$ because the string segment only jump up & down

$\mu =$ Density per unit length
$\Rightarrow \mu\Delta x =$ Mass of the string segment

<u>Note</u>: There must be no gravity, or the 2nd law becomes

$$F_{\uparrow}(x+\Delta x,t) - F_{\uparrow}(x,t) - \underbrace{(\mu\Delta x)g}_{\text{Extra term}} = (\mu\Delta x)a_{\uparrow}$$

2. Analysis by the string's geometry

(add figure here: string geometry)

Tension $\vec{F}$ must be parallel to the slope at the 2 end points of the segment.

Beause the graph of string's height variation = the graph of $y(x,t)$ at some fix time t,

$$\text{Slope of the graph} = \frac{\partial}{\partial x}y(x,t)$$

And the inclination of tension $\vec{F}$ can be calculated as $\dfrac{F_{\uparrow}}{F_{\rightarrow}}$.

$$\Rightarrow \text{ Relation at end points : } \begin{cases} \dfrac{F_{\uparrow}(x+\Delta x,t)}{F_{\rightarrow}(x+\Delta x,t)} = \dfrac{\partial}{\partial x}y(x,t)\Big|_{\text{at } x+\Delta x} \\ \dfrac{F_{\uparrow}(x,t)}{F_{\rightarrow}(x,t)} = \dfrac{\partial}{\partial x}y(x,t)\Big|_{\text{at } x} \end{cases}$$

3. Substitute the above two results:

$$\begin{aligned} (\mu\Delta x)a_{\uparrow} &= F_{\uparrow}(x+\Delta x,t) - F_{\uparrow}(x,t) \\ &= \underline{F_{\rightarrow}(x+\Delta x,t)}\left[\frac{\partial}{\partial x}y(x,t)\Big|_{\text{at } x+\Delta x}\right] - \underline{F_{\rightarrow}(x,t)}\left[\frac{\partial}{\partial x}y(x,t)\Big|_{\text{at } x}\right] \\ &= \underline{F_{\leftrightarrow}}\left[\frac{\partial}{\partial x}y(x,t)\Big|_{\text{at } x+\Delta x} - \frac{\partial}{\partial x}y(x,t)\Big|_{\text{at } x}\right] \end{aligned}$$

Can be grouped together because they are equal.
This is from the Newton's 2nd Law for horizontal direction

$$\mu \underline{a_{\uparrow}} = F_{\leftrightarrow}\underbrace{\left[\frac{\frac{\partial}{\partial x}y(x,t)\Big|_{\boxed{\text{at } x+\Delta x}} - \frac{\partial}{\partial x}y(x,t)\Big|_{\boxed{\text{at } x}}}{\boxed{\Delta x}}\right]}$$

Vertical acceleration = 2nd derivative of segment's height over t

$$\underline{\mu\frac{\partial^2}{\partial t^2}y(x,t)} = F_{\leftrightarrow}\underline{\frac{\partial^2}{\partial x^2}y(x,t)} \longleftarrow$$

This is just the derivative $\frac{f(x+\Delta x)-f(x)}{\Delta x}$
$\Rightarrow$ become 2nd derivative over x

$$\boxed{\frac{\mu}{F_{\leftrightarrow}}\frac{\partial^2}{\partial t^2}y(x,t) = \frac{\partial^2}{\partial x^2}y(x,t)}$$

Compare with the general form of wave equation $\frac{1}{v^2}\frac{\partial^2}{\partial t^2}y(x,t) = \frac{\partial^2}{\partial x^2}y(x,t)$, we can identify the wave speed of transverse wave as

$$v = \sqrt{\frac{F_{\leftrightarrow}}{\mu}} = \sqrt{\frac{\text{Horizontal Tension}}{\text{Mass per Length}}}$$

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2 Model of Longitudinal Wave

We usually use a slinky to visualize longitudinal wave travel.

- When the slinky is static - Each peak are of equal spacing.

(add figure here: static slinky)

- When the slinky shakes - the peaks become unevenly distributed.

(add figure here: shake slinky)

Longitudinal wave is displacement of slinky segments at different position / time, which is described by the function $\Psi(x,t)$.

Deriving wave eqauation

1. Although the slinky is continuous, we can divide it into many very small segments and only look at the motions of the center of these segments - using their centers as nodes to represent the motion of the segment. Under this approximation, we assume

   - The segment is "rigid". The whole segment moves at the same velocity / acceleration as its center node.

     (add figure here: rigid segment)

   - Elastic forces are only between the center nodes. Think of it as a spring-mass system composed of many nodes.

     (add figure here: force is only between node)

2. Consider 3 neighbouring nodes. When a longitudinal wave is travelling through, their displacements are described by the function $\Psi(x,t)$. We can compute the change of separation between nodes.

Node's position	Left node	Center node	Right node
When static	$x - \Delta x$	x	$x + \Delta x$
When a wave is travelling through	$x - \Delta x + \Psi(x - \Delta x, t)$	$x + \Psi(x,t)$	$x + \Delta x + \Psi(x + \Delta x, t)$

Their distance in-between increases by $\Psi(x,t) - \Psi(x - \Delta x, t)$

Their distance in-between increases by $\Psi(x + \Delta x, t) - \Psi(x,t)$

3. The elastic force on the center node is proportional to the separation change with its neighbouring nodes, just like the elastic force in spring, $F = -k(\Delta L)$. But here we express the elastic force using **Young's modulus**:

$$(\text{Elastic force}) = F = -Y \cdot \left(\frac{\Delta L}{L}\right) = -Y \cdot \left(\frac{\text{Change in length}}{\text{Original length}}\right)$$

So the elastic forces on the two sides of the center node are:

$$\begin{cases} F_L = -Y \cdot \dfrac{\Psi(x,t) - \Psi(x-\Delta x, t)}{\Delta x} \\[2ex] F_R = -Y \cdot \dfrac{\Psi(x+\Delta x,t) - \Psi(x, t)}{\Delta x} \end{cases}$$

4. The Newton's 2nd Law on the center segment's center is therefore

(add figure here: free body center segment)

Horizontal acceleration
= 2nd derivative of
segment's displacement over t

$$m\underline{a_{\rightarrow}} = F_L - F_R$$

$$(\mu\Delta x)\underline{\frac{\partial^2}{\partial t^2}\Psi(x,t)} = Y\left[\underline{\frac{\Psi(x+\Delta x,t) - \Psi(x,t)}{\Delta x}} - \underline{\frac{\Psi(x,t) - \Psi(x-\Delta x,t)}{\Delta x}}\right]$$

μ = Density per unit length
$\Rightarrow \mu\Delta x$ = Mass of the segment

1st derivative of x

$$= Y\left[\frac{\partial}{\partial x}\Psi(x,t)\bigg|_{\text{at } x+\Delta x} - \frac{\partial}{\partial x}\Psi(x,t)\bigg|_{\text{at } x}\right]$$

$$\mu\frac{\partial^2}{\partial t^2}\Psi(x,t) = \underbrace{Y\left[\frac{\frac{\partial}{\partial x}\Psi(x,t)\Big|_{\boxed{\text{at } x+\Delta x}} - \frac{\partial}{\partial x}\Psi(x,t)\Big|_{\boxed{\text{at } x}}}{\boxed{\Delta x}}\right]}$$

$$= Y\frac{\partial^2}{\partial x^2}\underline{\Psi(x,t)}$$

This is just the derivative $\frac{f(x+\Delta x)-f(x)}{\Delta x}$
$\Rightarrow$ become 2nd derivative over x

$$\boxed{\frac{\mu}{Y}\frac{\partial^2}{\partial t^2}\Psi(x,t) = \frac{\partial^2}{\partial x^2}\Psi(x,t)}$$

Compare with the general form of wave equation $\dfrac{1}{v^2}\dfrac{\partial^2}{\partial t^2}y(x,t) = \dfrac{\partial^2}{\partial x^2}y(x,t)$, we can identify the wave speed of longitudinal wave as

$$v = \sqrt{\frac{\mu}{Y}} = \sqrt{\frac{\text{Mass per Length}}{\text{Young's modulus}}}$$

Side note:

Spring constant k depends on the length of the material. For example, we can compute the equivalent spring constants for two springs in series to be half of the original.

(add figure here: spring in series)

If we stack more springs in series, we can see that $k \propto \left(\frac{1}{\text{Length of material}}\right)$. To remove this dependency on length, we define the Young's modulus, which is a property that only depends on the material's type.

$$\boxed{F = -k(\Delta L) = -\frac{Y}{L}(\Delta L)}$$

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3 Wave Equation & Initial Value Problem

The initial value problem is asking the follow: If we are told the state of a system at the start, how will system evolve at later time?

For example in wave propagation, given that at $t = 0$, a string is hold to a shape described by the function $\Psi(x, 0)$. After releasing, how will the waveform evolves?

(add figure here: release string)

i.e. We would like to solve $\underbrace{\Psi(x,t)}_{\substack{\text{the waveform}\\\text{in the future}}}$ by the given $\underbrace{\Psi(x,0)}_{\substack{\text{the waveform}\\\text{at the start}}}$ and $\underbrace{\frac{\partial}{\partial t}\Psi(x,t)\bigg|_{t=0}}_{\substack{\text{velocity at each point}\\\text{at the start}}}$.

3.1 General Solution to Wave Equation

The wave equation is a **partial differential equation** (PDE):

$$\frac{\partial^2}{\partial x^2}\Psi(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\Psi(x,t)$$

We can show that the general solution is

$$\boxed{\Psi(x,t) = f(x+vt) + g(x-vt)}$$

where $f(\cdots)$ and $g(\cdots)$ are any single variable function, and then we substitute $x+vt$ or $x-vt$ as the inputs. For example,

$$f(u) = \sin u + u^2 \quad \Rightarrow \quad f(x+vt) = \sin(x+vt) + (x+vt)^2$$

Proof

By differentiation with chain rule.

L.H.S.

$$\frac{\partial}{\partial x}f(x+vt) = \frac{\partial f(u)}{\partial u}\bigg|_{u=x+vt}\frac{\partial(x+vt)}{\partial x}$$
$$= \frac{\partial f(u)}{\partial u}\bigg|_{u=x+vt}\cdot 1$$

$$\frac{\partial^2}{\partial x^2}f(x+vt) = \frac{\partial^2 f(u)}{\partial u^2}\bigg|_{u=x+vt}\frac{\partial(x+vt)}{\partial x}$$
$$= \frac{\partial^2 f(u)}{\partial u^2}\bigg|_{u=x+vt}\cdot 1$$

R.H.S.

$$\frac{1}{v^2}\frac{\partial}{\partial t}f(x+vt) = \frac{1}{v^2}\frac{\partial f(u)}{\partial u}\bigg|_{u=x+vt}\frac{\partial(x+vt)}{\partial t}$$
$$= \frac{1}{v^2}\frac{\partial f(u)}{\partial u}\bigg|_{u=x+vt}\cdot v$$

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}f(x+vt) = \frac{1}{v}\frac{\partial^2 f(u)}{\partial u^2}\bigg|_{u=x+vt}\frac{\partial(x+vt)}{\partial t}$$
$$= \frac{1}{v}\frac{\partial^2 f(u)}{\partial u^2}\bigg|_{u=x+vt}\cdot v$$

Obviously L.H.S = R.H.S.. You can also prove the same for $g(x-vt)$. $\square$

Physical Interpretation

- $f(x+vt)$ = A waveform travelling to the left ($-x$ direction).
- $g(x-vt)$ = A waveform travelling to the right ($+x$ direction).

When they are added together, they form superposition.

(add figure here: wave superposition)

3.2 Solution to Initial Value Problem

(The derivation is long but not important. Welcome to skip to the result.)

1. From the initial conditions, break them down by $\Psi = f + g$.

$$\Psi(x,0) = f(x+0) + g(x-0) \tag{1}$$

$$\frac{\partial}{\partial t}\Psi(x,t)\bigg|_{t=0} = v\frac{\mathrm{d}f(u)}{\mathrm{d}u}\bigg|_{u=x+0} - v\frac{\mathrm{d}g(u)}{\mathrm{d}u}\bigg|_{u=x-0} \tag{2}$$

2. Differentiate Eq.(1) with respect to x:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,0) = \frac{\mathrm{d}f(u)}{\mathrm{d}u}\bigg|_{u=x+0} + \frac{\mathrm{d}g(u)}{\mathrm{d}u}\bigg|_{u=x-0} \tag{3}$$

3. Isolate $\frac{\mathrm{d}f(u)}{\mathrm{d}u}\big|_{u=x+0}$ and $\frac{\mathrm{d}g(u)}{\mathrm{d}u}\big|_{u=x-0}$ from Eq.(3) and Eq.(2).

$$\text{Eq.(3)} + \frac{1}{v}(\text{Eq.(2)}) \quad \Rightarrow \quad \frac{\mathrm{d}f(u)}{\mathrm{d}u}\bigg|_{u=x+0} = \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,0) + \frac{1}{v}\frac{\partial}{\partial t}\Psi(x,t)\bigg|_{t=0}\right] \tag{4}$$

$$\text{Eq.(3)} - \frac{1}{v}(\text{Eq.(2)}) \quad \Rightarrow \quad \frac{\mathrm{d}g(u)}{\mathrm{d}u}\bigg|_{u=x-0} = \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,0) - \frac{1}{v}\frac{\partial}{\partial t}\Psi(x,t)\bigg|_{t=0}\right] \tag{5}$$

4. Integrate both Eq.(4) and Eq.(5).

Eq.(4) $\Rightarrow$

$$\int \left.\frac{\mathrm{d}f(u)}{\mathrm{d}u}\right|_{u=x+0} \mathrm{d}x = \frac{1}{2}\int \left[\underline{\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,0)} + \frac{1}{v}\frac{\partial}{\partial t}\Psi(x,t)\Big|_{t=0}\right]\mathrm{d}x$$

C_1 = some integration constant

$$\int \frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\mathrm{d}x = \frac{1}{2}\left[\underline{\Psi(x,0) + C_1} + \frac{1}{v}\int\left[\frac{\partial}{\partial t}\Psi(x,t)\Big|_{t=0}\right]\mathrm{d}x\right]$$

$$f(x) = \frac{1}{2}\left[\Psi(x,0) + C_1 + \frac{1}{v}\underbrace{\int_{s=0}^{s=x}\left[\frac{\partial}{\partial t}\Psi(s,t)\Big|_{t=0}\right]\mathrm{d}s}_{\substack{s\,=\,\text{A dummy variable to replace } x \\ \text{For convenience in later steps}}}\right]$$

Eq.(5) $\Rightarrow$

$$\int \left.\frac{\mathrm{d}g(u)}{\mathrm{d}u}\right|_{u=x-0} \mathrm{d}x = \frac{1}{2}\int \left[\underline{\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,0)} - \frac{1}{v}\frac{\partial}{\partial t}\Psi(x,t)\Big|_{t=0}\right]\mathrm{d}x$$

C_2 = some integration constant

$$\int \frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x = \frac{1}{2}\left[\underline{\Psi(x,0) + C_2} - \frac{1}{v}\int\left[\frac{\partial}{\partial t}\Psi(x,t)\Big|_{t=0}\right]\mathrm{d}x\right]$$

$$g(x) = \frac{1}{2}\left[\Psi(x,0) + C_2 - \frac{1}{v}\underbrace{\int_{s=0}^{s=x}\left[\frac{\partial}{\partial t}\Psi(s,t)\Big|_{t=0}\right]\mathrm{d}s}_{\substack{s\,=\,\text{A dummy variable to replace } x \\ \text{For convenience in later steps}}}\right]$$

5. Replace the "x" in $f(x)$ by "$x+vt$", and the "x" in $g(x)$ by "$x-vt$".

$$f(x+vt) = \frac{1}{2}\left[\Psi(x+vt,0) + C_1 + \frac{1}{v}\int_{s=0}^{s=x+vt}\left[\frac{\partial}{\partial t}\Psi(s,t)\Big|_{t=0}\right]\mathrm{d}s\right]$$

$$g(x-vt) = \frac{1}{2}\left[\Psi(x-vt,0) + C_1 - \frac{1}{v}\int_{s=0}^{s=x-vt}\left[\frac{\partial}{\partial t}\Psi(s,t)\Big|_{t=0}\right]\mathrm{d}s\right]$$

6. Add these two expression together to yield $\Psi(x,t)$

$$\begin{aligned}\Psi(x,t) &= f(x+vt)+g(x-vt)\\ &= \frac{1}{2}[\Psi(x+vt,0)+\Psi(x-vt,0)]+\frac{1}{2}(C_1+C_2)\\ &\quad+\frac{1}{2v}\int_{s=0}^{s=x+vt}\left[\frac{\partial}{\partial t}\Psi(s,t)\bigg|_{t=0}\right]\mathrm{d}s\underbrace{-\frac{1}{2v}\int_{s=0}^{s=x-vt}}_{\substack{\text{Can switch}\\ \text{upper/lower bound}\\ \text{and change sign}}}\left[\frac{\partial}{\partial t}\Psi(s,t)\bigg|_{t=0}\right]\mathrm{d}s\\ &= \frac{1}{2}[\Psi(x+vt,0)+\Psi(x-vt,0)]+\frac{1}{2}(C_1+C_2)\\ &\quad+\frac{1}{2v}\int_{s=0}^{s=x+vt}\left[\frac{\partial}{\partial t}\Psi(s,t)\bigg|_{t=0}\right]\mathrm{d}s+\frac{1}{2v}\int_{s=x-vt}^{s=0}\left[\frac{\partial}{\partial t}\Psi(s,t)\bigg|_{t=0}\right]\mathrm{d}s\\ &= \frac{1}{2}[\Psi(x+vt,0)+\Psi(x-vt,0)]+\frac{1}{2}(C_1+C_2)+\frac{1}{2v}\int_{s=x-vt}^{s=x+vt}\left[\frac{\partial}{\partial t}\Psi(s,t)\bigg|_{t=0}\right]\mathrm{d}s\end{aligned}$$

Combine integral by their bounds

7. Finally, substitute $t=0$ to find out what C_1+C_2 is.

$$\begin{aligned}\Psi(x,0) &= \frac{1}{2}[\Psi(x+0,0)+\Psi(x-0,0)]+\frac{1}{2}(C_1+C_2)+\frac{1}{2v}\int_{s=x-0}^{s=x+0}\left[\frac{\partial}{\partial t}\Psi(s,t)\bigg|_{t=0}\right]\mathrm{d}s\\ &= \frac{1}{2}[\Psi(x,0)+\Psi(x,0)]+\frac{1}{2}(C_1+C_2)+0\end{aligned}$$

$$C_1+C_2=0$$

Finally, we reach the solution to the initial value problem of wave equation.

$$\boxed{\underbrace{\Psi(x,t)}_{\substack{\text{the waveform}\\ \text{in the future}}} = \underbrace{\frac{1}{2}[\Psi(x+vt,0)+\Psi(x-vt,0)]}_{\substack{\text{Derived from the}\\ \text{initial waveform }\Psi(x,0)}}+\underbrace{\frac{1}{2v}\int_{s=x-vt}^{s=x+vt}\left[\frac{\partial}{\partial t}\Psi(s,t)\bigg|_{t=0}\right]\mathrm{d}s}_{\substack{\text{Derived from the}\\ \text{initial velocity }\frac{\partial}{\partial t}\Psi(s,t)\big|_{t=0}}}}$$

Example 3.1. Given the initial waveform of the string as

$$\Psi(x,0)=\begin{cases} b-\dfrac{b|x|}{a} & \text{for } |x|\le a\\ 0 & \text{for } |x|>a\end{cases}$$

and knowing that the string is static at the beginning (i.e. velocity = 0 everywhere).

(add figure here: ivp eg)

We can find how the wave will evolve by direct substituting these info into the general solution.

$$\Psi(x,t) = \frac{1}{2}[\Psi(x+vt)+\Psi(x-vt)] + \frac{1}{2v}\int_{x-vt}^{x+vt} \underbrace{\cancel{\left[\frac{\partial}{\partial t}\Psi(s,t)\Big|_{t=0}\right]}}_{\text{Because no initial velocity}}^{\;0} \mathrm{d}s$$

$$= \underbrace{\frac{1}{2}\left(b - \frac{b|x+vt|}{a}\right)}_{\substack{\text{This is a function of } x+vt \\ \text{i.e. the waveform travelling} \\ \text{in } -\text{ve direction}}} + \underbrace{\frac{1}{2}\left(b - \frac{b|x-vt|}{a}\right)}_{\substack{\text{This is a function of } x-vt \\ \text{i.e. the waveform travelling} \\ \text{in } +\text{ve direction}}} + 0$$

(add figure here: ivp eg result)

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4 Boundary Value Problem & Standing Wave

The boundary value problem is asking the follow: If we are only given the value of the function at the end points and the initial state, how will system evolve at later time?

(add figure here: standing wave)

In a standing wave configuraton, a string is usually tied at both ends (or free ends). It is very important to know: what kinds of "vibration pattern" are allowed?

(add figure here: question mark standing wave)

And if the standing wave starts in a shape described by the function $\Psi(x,0)$. After releasing, how will the waveform evolves?

4.1 The Method of Separation of Variables

Here introduces an alternative method to solve the wave equation - **method of separation of variables**. We first assume the solution to be able to be written as a product of 2 single variable function, one as a function position x and the other as a function of time t.

$$\Psi(x,t) = X(x)T(t)$$

This part only depends on x This part only depends on t

Substitute into the equation,

$$\frac{\partial^2}{\partial x^2}[X(x)T(t)] = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}[X(x)T(t)]$$

$$T(t)\frac{\partial^2}{\partial x^2}[X(x)] = \frac{1}{v^2}X(x)\frac{\partial^2}{\partial t^2}[T(t)]$$

$T(t)$ not depends on x. Can be taken out of $\frac{\partial^2}{\partial x^2}$ $X(x)$ not depends on t. Can be taken out of $\frac{\partial^2}{\partial t^2}$

Rearrange so that the L.H.S is a function of x only, and R.H.S. is a function of t only.

$$\underbrace{\frac{1}{X(x)}\frac{\partial^2}{\partial x^2}[X(x)]}_{\text{only contain } x} = \underbrace{\frac{1}{v^2}\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}[T(t)]}_{\text{only contain t}} = \left(\begin{matrix}\text{Some}\\ \text{Constant}\end{matrix}\right) = -k^2$$

The only possibility for two functions of different variables to equal is when both equal to a constant

The constant is written as $-k^2$ is just for convenience. We can now split the part with x and the part with t, forming two independent 2^{nd} order linear ODEs.

$$\begin{cases} \dfrac{1}{X(x)}\dfrac{\partial^2}{\partial x^2}[X(x)] = -k^2 & \Rightarrow \qquad \dfrac{\partial^2}{\partial x^2}X(x) + k^2 X(x) = 0 \\ \dfrac{1}{v^2}\dfrac{1}{T(t)}\dfrac{\partial^2}{\partial t^2}[T(t)] = -k^2 & \Rightarrow \qquad \dfrac{\partial^2}{\partial t^2}T(t) + v^2 k^2 T(t) = 0 \end{cases}$$

You should be very farmiliar with this kind of ODE - the equation of motion of SHM. Their solutions are

$$\begin{cases} X(x) = C\cos(kx) + D\sin(kx) \\ T(t) = A\cos(kvt) + B\sin(kvt) \end{cases}$$

where A, B, C, D are constants to be determined from the boundary conditions and initial conditions.

4.2 Boundary Conditions & Solutions

In standing wave, the boundary condition at the end of the string is either being fixed or free to move up / down. Here introduces two most common boundary conditions, and the corresponding solutions.

4.2.1 Dirichlet Condition

The **Dirichlet condition** in standing wave is essentially having **2 fixed ends**, i.e. requires the magnitude at both ends to be fixed at 0 (at any time t). If the string is of length L, then it writes:

$$\boxed{\begin{cases} \Psi(0,t) = 0 \\ \Psi(L,t) = 0 \end{cases}}$$

(add figure here: Dirichlet)

From the conditions, we must have $X(0) = X(L) = 0$.

$n = 0$ is technically an answer, but it gives $X(x) = D\sin 0 = 0$, meaning the string cannot shake at all.

- At $x = 0$, $X(0) = C\cos(0) + D\sin(0) = 0$. It holds only if $C = 0$.
- At $x = L$, $X(L) = D\sin(kL) = 0$. It holds only if $kL = n\pi$, with $n = 0, 1, 2, 3, \ldots$.

So we require $k = \dfrac{n\pi}{L}$. Substitute it into $X(x)$ and $T(t)$,

$$\begin{cases} X(x) = D \sin\left(\dfrac{n\pi}{L}x\right) \\ T(t) = A\cos\left(\dfrac{n\pi}{L}vt\right) + B\sin\left(\dfrac{n\pi}{L}vt\right) \end{cases}$$

For each integer $n = 1, 2, 3...$, we can construct one set of $\Psi(x,t)$:

$$\begin{aligned} \Psi_n(x,t) &= X_n(x)T_n(t) \\ &= \left[\sin\left(\frac{n\pi}{L}x\right)\right]\left[A_n\cos\left(\frac{n\pi}{L}vt\right) + B_n\sin\left(\frac{n\pi}{L}vt\right)\right] \end{aligned}$$

Then by the superposition property of linear equations, the general solution is the (linear) combination of all possible solutions.

$$\boxed{\Psi(x,t) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi}{L}x\right)\right]\left[A_n\cos\left(\frac{n\pi}{L}vt\right) + B_n\sin\left(\frac{n\pi}{L}vt\right)\right] \qquad \text{(Dirichlet Condition)}}$$

All the constants A_n, B_n shall be determined only after an initial condition is given.

4.2.2 Neumann Condition

The **Neumann condition** in standing wave is essentially having **2 free ends** - the end segments are free to move up and down following string body's motion.

To achieve so, the ends' holdings must not exert any vertical force at the end segment (otherwise the ends are not freely moving with the string body.) Having only horizontal force on the end segments means that the slope at both ends to be 0 (at any time t). If the string is of length L, then it writes:

$$\boxed{\begin{cases} \left.\dfrac{\partial}{\partial x}\Psi(x,t)\right|_{x=0} = 0 \\ \left.\dfrac{\partial}{\partial x}\Psi(x,t)\right|_{x=L} = 0 \end{cases}}$$

(add figure here: Neumann ends)

From the conditions, we must have $\left.\dfrac{\mathrm{d}X(x)}{\mathrm{d}x}\right|_{x=0} = \left.\dfrac{\mathrm{d}X(x)}{\mathrm{d}x}\right|_{x=L} = 0$.

- At $x = 0$, $\left.\dfrac{\mathrm{d}X(x)}{\mathrm{d}x}\right|_{x=0} = -C\sin(0) + D\cos(0) = 0$. It holds only if $D = 0$.
- At $x = L$, $\left.\dfrac{\mathrm{d}X(x)}{\mathrm{d}x}\right|_{x=L} = -C\sin(kL) = 0$. It holds only if $kL = n\pi$, with $n = 0, 1, 2, 3, \ldots$.

This time we can keep $n = 0$, because it gives $X(x) = C\cos 0 = C$. Motion is retained in $T(t)$.

So we require $k = \dfrac{n\pi}{L}$. Notice that

- when $k \neq 0$, we can substitute it into $X(x)$ and $T(t)$,

$$\begin{cases} X(x) = C\cos\left(\frac{n\pi}{L}x\right) \\ T(t) = A\cos\left(\frac{n\pi}{L}vt\right) + B\sin\left(\frac{n\pi}{L}vt\right) \end{cases}$$

- but when $k = 0$, the solution of $X(t)$ and $T(t)$ become

$$X(x) = C \qquad \text{and} \qquad \begin{aligned} &\frac{\partial^2}{\partial t^2}T(t) = 0 \\ \Rightarrow\quad &T(t) = At + Bs \end{aligned}$$

So for each integer $n = 0, 1, 2, 3...$, we can construct one set of $\Psi(x,t)$:

$$\begin{aligned} \Psi_n(x,t) &= X_n(x)T_n(t) \\ &= \begin{cases} \Big[1\Big]\Big[A_0 t + B_0\Big] & \text{for } n = 0 \\ \Big[\cos\left(\frac{n\pi}{L}x\right)\Big]\Big[A_n\cos\left(\frac{n\pi}{L}vt\right) + B_n\sin\left(\frac{n\pi}{L}vt\right)\Big] & \text{for } n > 0 \end{cases} \end{aligned}$$

Then by the superposition property of linear equations, the general solution is the (linear) combination of all possible solutions.

$$\boxed{\Psi(x,t) = \Big[A_0 t + B_0\Big] + \sum_{n=1}^{\infty}\Big[\cos\left(\frac{n\pi}{L}x\right)\Big]\Big[A_n\cos\left(\frac{n\pi}{L}vt\right) + B_n\sin\left(\frac{n\pi}{L}vt\right)\Big] \qquad \text{(Neumann Condition)}}$$

All the constants A_n, B_n shall be determined only after an initial condition is given.

Exercise 4.1. We can carry out similar steps to obtain the general solution in other combination of boundary conditions.

1. Use $x = 0$'s condition to elimiate one of C or D.
2. Use $x = L$'s condition to determine what values of k can be.
3. Substitute k's value into $\Psi(x,t) = X(x)T(t)$.
4. The general solution is the linear combination of $\Psi(x,t)$ of all possible k.

As a practice, you may try to derive for the case with one fixed end and one open end.

(add figure here: 1 fix 1 open)

You should get

$$\boxed{\Psi(x,t) = \sum_{n=1}^{\infty}\Big[\sin\left(\frac{2n-1}{2}\frac{\pi}{L}x\right)\Big]\Big[A_n\cos\left(\frac{2n-1}{2}\frac{\pi}{L}vt\right) + B_n\sin\left(\frac{2n-1}{2}\frac{\pi}{L}vt\right)\Big]}$$

4.3 Modes of Standing Wave

Observing that the general solution is a superposition of all simplier solutions of different n. Each n has its corresponding $X_n(x)$ and $T_n(t)$.

$$\Psi(x,t) = \sum_n \Psi_n(x,t) = \sum_n \left[X_n(x)T_n(t)\right]$$

- $X_n(x)$ is only about variation by position x - Carry info about the waveform.
- $T_n(t)$ is only about variation by time t - Carry info about the time evolution.

Each $\Psi_n(x,t)$ is **a unique set of vibration pattern** in standing wave, and they evolve independently from each other. Therefore we call them the **normal modes** of standing wave. i.e.

$$\begin{pmatrix}\text{The n}^{\text{th}}\text{mode}\\ \Psi_n(x,t)\end{pmatrix} = \begin{pmatrix}\text{Waveform of}\\ \text{the n}^{\text{th}}\text{mode}\\ X_n(x)\end{pmatrix} \times \begin{pmatrix}\text{Time evolution}\\ \text{of the n}^{\text{th}}\text{mode}\\ T_n(t)\end{pmatrix}$$

We can draw the graphs fro each $\Psi_n(x,t)$ as shown:

	Dirichlet (2 fixed ends)	Neumann (2 free ends)	1 fixed-1 free end
n=0			
n=1			
n=2			

$\vdots$

and so on.

Here I shall emphasize: **The combination of standing wave's pattern (waveform) and frequency (time evolution) is fixed** - It is impossible to let the string vibrate in one the above pattern but with a faster/slower frequency.

When we encounter a wave pattern that is not in one of the normal wave's waveform, we must first break it down into a sum of different normal modes. For example

(add figure here: sum of mode)

Because each mode has its own vibration frequency, the resulted wave will not maintain a regular shape like the initial waveform.

	$t = 0$	$t = \dfrac{L}{2v}$	$t = \dfrac{L}{v}$
1st mode Period $= \dfrac{2\pi}{\frac{\pi}{L}v} = \frac{2L}{v}$			
2nd mode Period $= \dfrac{2\pi}{\frac{2\pi}{L}v} = \frac{L}{v}$			
Sum			

4.4 Fourier Series

So if we are given some arbituary pattern as the initial wave form, how can we break it down into normal modes mathematically? The tool is called **Fourier Series**.

Denote a periodic function as $f(x) = f(x + L)$, which has a period L,

(add figure here: periodic func)

it can be expanded as a series of sin and cosine terms:

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{L} x\right) + b_n \sin\left(\frac{2\pi n}{L} x\right) \right]}$$

The **Fourier coefficients** a_n (including a_0) and b_n can be calculated by these integrals:

$$\boxed{\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cdot \cos\left(\frac{2\pi n}{L} x\right) \mathrm{d}x \\ b_n &= \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{2\pi n}{L} x\right) \mathrm{d}x \end{aligned}}$$

Proof

The above formulas work thanks to these integral properties. For any integer m, n,

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_0^{2\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \begin{cases} 0 & \text{if } m = n = 0 \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_0^{2\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0$$

i.e. the integral $= 0$ whenever $m \neq n$ or the sin / cos does not match. Let's have a demonstration using $\cos\left(\frac{2\pi n}{L}x\right)$ with $n \geq 1$.

$$\begin{aligned}
&\int_0^L f(x)\cos\left(\frac{2\pi n}{L}x\right)\mathrm{d}x \\
&= \int_0^L \left[\frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos\left(\frac{2\pi m}{L}x\right) + b_m \sin\left(\frac{2\pi m}{L}x\right)\right]\right]\cos\left(\frac{2\pi n}{L}x\right)\mathrm{d}x \\
&= \int_0^L \left[\underbrace{\frac{a_0}{2}\cos\left(\frac{2\pi n}{L}x\right)}_{\substack{\text{Integrate cos} \\ \text{for 1 period} \\ =0}} + \sum_{m=1}^{\infty} \underbrace{a_m \cos\left(\frac{2\pi m}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)}_{\substack{\text{cos and cos} \\ \text{Integral } \neq 0 \text{ only if } m=n}} + \sum_{m=1}^{\infty} \underbrace{b_m \sin\left(\frac{2\pi m}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)}_{\substack{\text{sin and cos} \\ \text{Integral always gives 0}}}\right]\mathrm{d}x \\
&= \int_0^L a_n \cos\left(\frac{2\pi n}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)\mathrm{d}x \\
&= a_n \cdot \frac{L}{2}
\end{aligned}$$

$$\Rightarrow \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n}{L}x\right)\mathrm{d}x$$

As an exercise, you can also prove the same formula for b_n.

□

Example 4.1. Evolution of square wave

Given the function of a periodic square wave as

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{L}{2} \\ -1 & \text{for } \frac{L}{2} < x < L \end{cases}$$

(add figure here: square wave)

The Fourier coefficients can be directly computed:

$$a_n = \frac{2}{L}\int_0^{\frac{L}{2}} 1 \cdot \cos\left(\frac{2\pi n}{L}x\right) \mathrm{d}x + \frac{2}{L}\int_{\frac{L}{2}}^{L} -1 \cdot \cos\left(\frac{2\pi n}{L}x\right) \mathrm{d}x$$
$$= 0$$

$$b_n = \frac{2}{L}\int_0^{\frac{L}{2}} 1 \cdot \sin\left(\frac{2\pi n}{L}x\right) \mathrm{d}x + \frac{2}{L}\int_{\frac{L}{2}}^{L} -1 \cdot \sin\left(\frac{2\pi n}{L}x\right) \mathrm{d}x$$
$$= \frac{2}{n\pi}[1-(-1)^n] = \begin{cases} 0 & \text{for } n = \text{even} \\ \frac{4}{n\pi} & \text{for } n = \text{odd} \end{cases}$$

So its Fourier series write as

$$f(x) = \frac{4}{\pi}\left[\underbrace{\underbrace{\underbrace{\sin\left(\frac{2\pi}{L}\cdot x\right)}_{\text{First mode}} + \frac{1}{3}\sin\left(\frac{2\pi}{L}\cdot 3x\right)}_{\text{First 2 modes}} + \frac{1}{5}\sin\left(\frac{2\pi}{L}\cdot 5x\right)}_{\text{First 3 modes}} + \dots\right]$$

(add figure here: square modes)

At this point, we have already obtained each $X_n(x)$ as

$$X_n(x) = \frac{4}{n\pi}\sin\left(\frac{2\pi n}{L}x\right) \qquad \text{(odd } n \text{ only)}$$

To complete the nthnormal mode, multiply $T_n(t)$ of the corresponding n.

$$\Psi_n(x,t) = X_n(x)T_n(t) = \left[\frac{4}{n\pi}\sin\left(\frac{2\pi n}{L}x\right)\right]\left[A_n\cos\left(\frac{2\pi n}{L}vt\right) + B_n\sin\left(\frac{2\pi n}{L}vt\right)\right]$$

Finally, the general evolution is the sum of all normal modes.

$$\boxed{\begin{aligned}\Psi(x,t) &= \sum_{n=1}^{\infty}\Psi_n(x,t) = \sum_{n=1}^{\infty}X_n(x)T_n(t) \\ &= \sum_{\text{odd } n \text{ only}}^{\infty}\left[\frac{4}{n\pi}\sin\left(\frac{2\pi n}{L}x\right)\right]\left[A_n\cos\left(\frac{2\pi n}{L}vt\right) + B_n\sin\left(\frac{2\pi n}{L}vt\right)\right]\end{aligned}}$$

To exactly determine the constants A_n, B_n, an initial condition is required. For example, <u>if we specify that the string is static before released</u>,

$$\Psi(x,0) = \sum_{\text{odd } n \text{ only}}^{\infty}\left[\frac{4}{n\pi}\sin\left(\frac{2\pi n}{L}x\right)\right]\left[A_n\cdot 1 + B_n\cdot 0\right] \equiv f(x) = \sum_{\text{odd } n \text{ only}}^{\infty}\left[\frac{4}{n\pi}\sin\left(\frac{2\pi n}{L}x\right)\right]$$

$$\Rightarrow \qquad \text{all } A_n = 1$$

$$\frac{\partial}{\partial t}\Psi(x,t)\bigg|_{t=0} = \sum_{\text{odd } n \text{ only}}^{\infty} \left[\frac{4}{n\pi}\sin\left(\frac{2\pi n}{L}x\right)\right]\left[-A_n \cdot 0 + B_n \cdot 1\right] \equiv 0$$

$$\Rightarrow \qquad \text{all } B_n = 0$$

— The End —