Displacement Current

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Overview:

- Definition of current, conservation of charge

- Fixing Ampere's law - displacement current

In electromagnetism, theoretically every problem can be solved through a set of PDEs called the **Maxwell Equations**.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad\qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \longrightarrow \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

However, a *system of PDEs* is too complicated to be solved. So we need to learn different "tricks" to avoid them, which are enough for some simple scenarios.

In this note, we will discuss why it is necessary to add the second term in Ampere's law.

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1 Charge Conservation

In magnetostatics, we describe everything by using current as the source. But in the dynamics part of E&M, charge is the only real source, while current has to be treated as moving charge.

(add figure here: pool of charge + current ¡-¿ pool of static charge + moving charge)

1.1 Definition of Current

To inter-convert the two pespectives, we need to apply the definition of current. Its literal definition is simply:

> Current = Rate of charge's flow into/out of a volume

Across textbooks, its definition can appear as many different formula.

1. **"Wire" description** - Consider the volume to flow through to be a line. Current can be defined by counting the total amount of charge Q through the wire.

$$\boxed{I = \frac{\mathrm{d}Q}{\mathrm{d}t} \qquad \text{or} \qquad I = \frac{Q}{t} \quad \left(\begin{smallmatrix}\text{for people} \\ \text{who can't do} \\ \text{calculus}\end{smallmatrix}\right)}$$

2. **"Cylinder" description** - Magnify the wire to become a straight cylinder. Current can be defined by counting the total amount of charge flowing into/out of the cylinder.

$$I = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\begin{smallmatrix}\text{Charge}\\ \text{per particle}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{\# of particle}\\ \text{per volume}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{Volume of}\\ \text{cylinder}\end{smallmatrix}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}t}(qnAL)$$

By taking velocity of charge $v \sim \dfrac{\mathrm{d}L}{\mathrm{d}t}$, it becomes the more common form in textbooks:

$$I = nAqv$$

3. **Microscopic definition** - The whole wire can be abstractly treated as a surface. Just like the water pipe we have learnt in Gauss's law.

This turns the flow in / out of condition becomes a flux integral.

$$I = \iint nq\vec{v} \cdot \mathrm{d}\vec{s}$$

$$nq = \left(\begin{smallmatrix}\text{\# of particle}\\ \text{per volume}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{charge per}\\ \text{particle}\end{smallmatrix}\right) \sim \text{Charge density}$$

$$= \iint \rho\vec{v} \cdot \mathrm{d}\vec{s}$$

ρ and $\vec{v}$ can be different at different position $\therefore$ should be taken as functions of position

$$= \iint \rho(\vec{r})\,\vec{v}(\vec{r}) \cdot \mathrm{d}\vec{s}$$

$$= \iint \vec{J} \cdot \mathrm{d}\vec{s}$$

At the end, we return to the microscopic definition of volume current density $\vec{J}(\vec{r}) = \rho(\vec{r})\vec{v}(\vec{r})$.

1.2   Continuity Equation

The simplest form of charge conservation is simply

$$\left(\begin{smallmatrix}\text{Current flow}\\ \text{into the region}\end{smallmatrix}\right) = I_{\text{in}} = \frac{\mathrm{d}}{\mathrm{d}t}Q_{\text{enc}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{smallmatrix}\text{Charge enclosed}\\ \text{in the region}\end{smallmatrix}\right)$$

We can also express this relation by the fancier density distributions:

- Charge → Volume charge density

$$Q(t) = \iiint_{\text{region}} \rho(\vec{r}, t) \, \mathrm{d}\tau$$

- Current → Volume current density

$$I(t) = \iint_{\text{surface}} \vec{\boldsymbol{J}}(\vec{r}, t) \cdot \mathrm{d}\vec{\boldsymbol{s}}$$

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2 Displacement Current

2.1 Derivation

2.2 Complication of Electromagnetic Induction

— The End —