

Multivariable Calculus

by Tony Shing

Overview:

- Comparison between single variable functions & multivariable functions
- Partial differentiation (on scalar function)
- Multiple integral (on scalar function)

1 Functions with Multiple Variables

To well-define a function $f(x)$ in advanced mathematics, we actually need to specify the function's **domain** and **image**.

- Domain = The set of values that be substitute into x .
- Image = The set of all possible output of $f(x)$.

E.g. Formal notation in math text to define $f(x) = \frac{1}{|x|}$:

$$f: \mathbb{R} \rightarrow \mathbb{R}^+ \\ x \mapsto \frac{1}{|x|}$$

Domain Image

We can classify functions by whether their domain/image are made of single number / tuple of numbers.

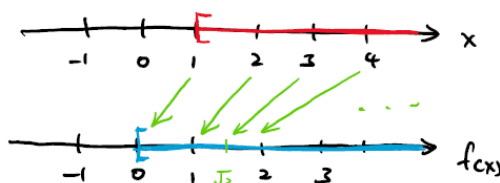
1.1 Single Variable Scalar Function

They are the functions that you have already learnt.

- Domain = A set of single number
- Image = A set of single number

For example,

$$f(x) = \sqrt{x-1} \quad \Rightarrow \quad \begin{cases} \text{Domain} = \text{Any real number } \geq 1 \\ \text{Image} = \text{Any real number } \geq 0 \end{cases}$$

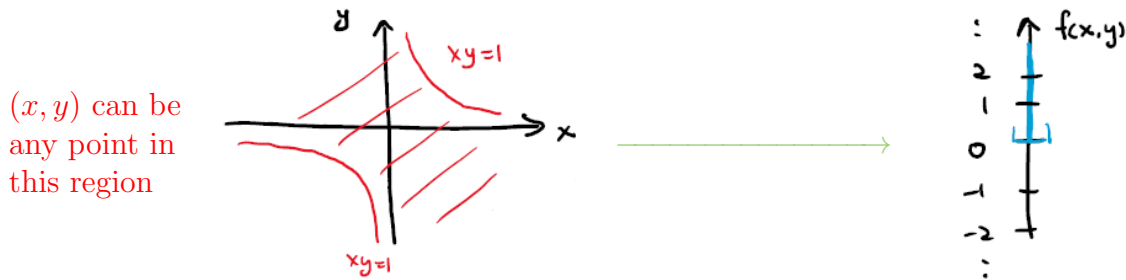


1.2 Multivariable Scalar Function

- Domain = A set of tuples of number, like $x = (1, 2, 3)$
- Image = A set of single number, like $f(x) = 5$

For example,

$$f(x, y) = \sqrt{1 - xy} \quad \Rightarrow \quad \begin{cases} \text{Domain} = \text{Any pair of values } x, y \text{ where } xy \leq 1 \\ \text{Image} = \text{Any real number } \geq 0 \end{cases}$$

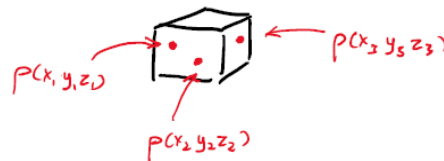


Example in Physics:

- Gravitational potential energy

$$U(x, y, z) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}}$$

- Density distribution in object $\rho(x, y, z)$

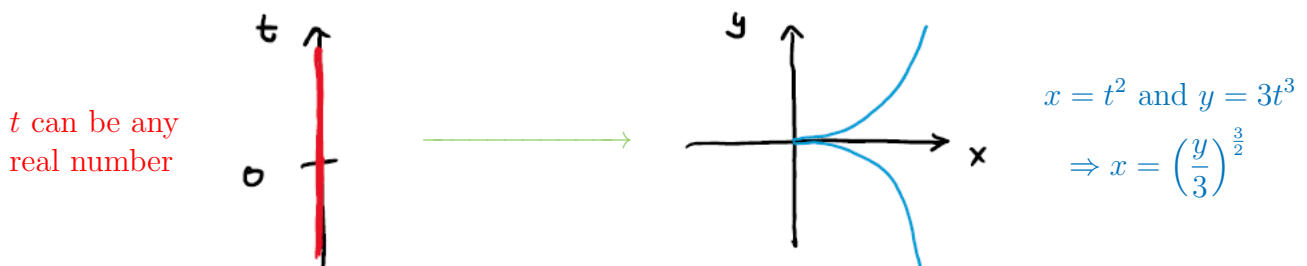


1.3 Single Variable Vector Function

- Domain = A set of single number
- Image = A set of tuple of numbers

For example,

$$\vec{r}(t) = (x(t), y(t)) = (t^2, 3t^3) \quad \Rightarrow \quad \begin{cases} \text{Domain} = \text{Any real number } t \\ \text{Image} = (x, y) \text{ pairs restricting on } x = \left(\frac{y}{3}\right)^{\frac{2}{3}} \end{cases}$$



Example in Physics:

- Displacement, velocity, acceleration

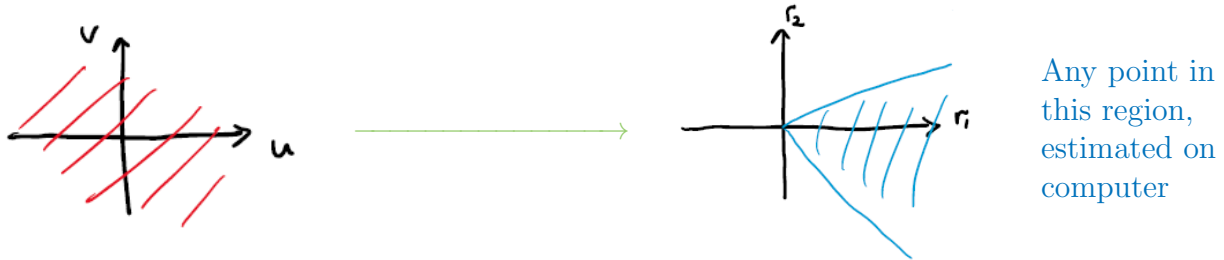
$$\begin{cases} \vec{s}(t) = (x(t), y(t), z(t)) \\ \vec{v}(t) = (v_x(t), v_y(t), v_z(t)) \\ \vec{a}(t) = (a_x(t), a_y(t), a_z(t)) \end{cases}$$

1.4 Multivariable Vector Function

- Domain = A set of tuple of numbers
- Image = A set of tuple of numbers

For example,

$$\vec{r}(u, v) = (r_1(u, v), r_2(u, v)) = (u^2 + v^2, u - 1 - v^2) \quad \Rightarrow \quad \begin{cases} \text{Domain} = \text{The whole } u\text{-}v \text{ plane} \\ \text{Image} = \text{Region depicted below} \end{cases}$$



Example in Physics:

- Gravitational force

$$\begin{aligned} \vec{F}(\vec{r}) &= \vec{F}(x, y, z) \\ &= -\frac{GMm}{|\vec{r}|^2} \cdot \left(\frac{\vec{r}}{|\vec{r}|} \right) \quad \leftarrow \text{Unit vector of } \vec{r} \\ &= -\frac{GMm}{x^2 + y^2 + z^2} \cdot \left(\frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} \right) \quad \leftarrow \text{Separate into components of } \hat{x}/\hat{y}/\hat{z} \\ &= \left[\frac{-GMmx}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] \hat{x} + \left[\frac{-GMmy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] \hat{y} + \left[\frac{-GMmz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right] \hat{z} \end{aligned}$$

1.5 Function Composition for multivariable functions

For single variable scalar function, you should be familiar with the what function composition is. For example, if $f(x) = \sin x$, $g(x) = e^x$, we can have these compositions:

$$f(f(x)) = \sin(\sin x) \quad , \quad f(g(x)) = \sin(e^x) \quad , \quad g(f(x)) = e^{\sin x} \quad , \quad g(g(x)) = e^{e^x}$$

Note that function composition is the key in chain rule.

$$\frac{d}{dx}f(g(x)) = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$$

However for multivariable functions, we can construct function composition only if the number of output matches the next function's number of input. For example, let's have

$$\left\{ \begin{array}{ll} f(p, q) = \sqrt{p+q} & \text{2 inputs, 1 output. Denote as } (2 \xrightarrow{f} 1) \\ \vec{g}(t) = (t-1, t^2) & \text{1 input, 2 outputs. Denote as } (1 \xrightarrow{g} 2) \\ \vec{h}(u, v) = (u^2+v, u-v) & \text{2 inputs, 2 outputs. Denote as } (2 \xrightarrow{h} 2) \end{array} \right.$$

We can have the following composition:

$$\begin{aligned} f(\vec{g}(t)) &= \sqrt{(t-1) + (t^2)} & (1 \xrightarrow{g} 2 \xrightarrow{f} 1) \\ f(\vec{h}(u, v)) &= \sqrt{(u^2+v) + (u-v)} & (2 \xrightarrow{h} 2 \xrightarrow{f} 1) \\ \vec{g}(f(p, q)) &= (\sqrt{p+q} - 1, p+q) & (2 \xrightarrow{f} 1 \xrightarrow{g} 2) \\ \vec{h}(\vec{g}(t)) &= ((t-1)^2 + (t^2), (t-1) - (t^2)) & (1 \xrightarrow{g} 2 \xrightarrow{h} 2) \\ \vec{h}(\vec{h}(t)) &= ((u^2+v)^2 + (u-v), (u^2+v) - (u-v)) & (2 \xrightarrow{h} 2 \xrightarrow{h} 2) \end{aligned}$$

But these are NOT allowed:

$$\begin{aligned} g(h(u, v)) &: (2 \xrightarrow{h} 2 \Rightarrow 1 \xrightarrow{g} 2) \\ h(f(p, q)) &: (2 \xrightarrow{f} 1 \Rightarrow 2 \xrightarrow{h} 2) \end{aligned}$$

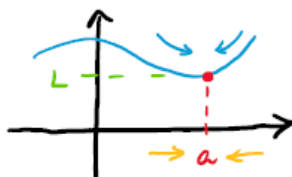
2 Limits on Multivariable Scalar Function

In single variable functions, $\lim_{x \rightarrow a} f(x) = L$ means when the input x is "close enough" to a value a , output of $f(x)$ must be "close" to some L . This idea can be extended to multivariable function, i.e.

$$\lim_{(x_1, x_2, \dots, x_n) \rightarrow (a_1, a_2, \dots, a_n)} f(x_1, x_2, \dots, x_n) = L$$

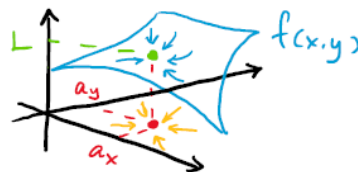
The idea requires all the inputs x_1, x_2, \dots to be "close enough" to some corresponding values a_1, a_2, \dots , only after then the output of $f(\dots)$ will be "close" enough to some L . We can visually compare it with single variable function as follow:

Single Variable Function



x can approach a
from either left or right

Multivariable Function



x can approach a
from every direction

Therefore, the "existence" of limit in multivariable functions has a much stricter requirement.

- Single variable function:
 - Input x must approach the point a from either left (x^-) or right (x^+).
 - "Existence" of limit only require showing both left/right limits approach to the same output L .
- Multivariable function, (e.g. functions with 2 inputs):
 - Inputs (x_1, x_2) can approach the point (a_1, a_2) along any trajectories on the plane.
 - "Existence" of limit require showing that along ALL trajectories.

Proving a limit exist rigorously is a lot harder in multivariable function. But in physics, we almost never need to deal with any strange functions that has limit only along certain trajectories. **We may assume that every function we encounter is well-behaved, and then calculation can be done like in single variable functions.** E.g.

$$\lim_{(x,y) \rightarrow (\frac{\pi}{2}, \frac{\pi}{2})} \sin x \cos y = \sin\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right)$$

3 Partial Differentiation

- Notation: $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots$
- Usually pronounced as "partial x", "partial y", etc.

Comparing with ordinary differentiation to single variable function, the notation difference is to emphasize that the **differentiation is only about 1 of the inputs**.

3.1 Definition & Geometrical Interpretation

The limit definition of partial differentiation of $f(x_1, x_2, \dots, x_n)$ at (a_1, a_2, \dots, a_n) in the i^{th} input (x_i)'s direction is defined as:

$$\begin{aligned} & \frac{\partial}{\partial x_i} f(x_1, x_2, \dots, x_i, \dots, x_n) \\ &= \lim_{\Delta x_i \rightarrow 0} \left[\frac{f(x_1, x_2, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, x_2, \dots, x_i, \dots, x_n)}{\Delta x_i} \right] \end{aligned}$$

Note that **the limit only acts on the i^{th} input**. Other inputs remains untouched.

Therefore in calculation, when doing partial differentiation over x_i , only x_i is differentiated (the same way we do in single variable differentiation), while the other x are treated as constants.

E.g. $f(x, y, z) = x^2 y \sin z$

$$\frac{\partial f}{\partial x} = 2x \cdot y \sin z$$

$$\left(\frac{d}{dx} x^2 = 2x, \text{ don't touch } y, z \right)$$

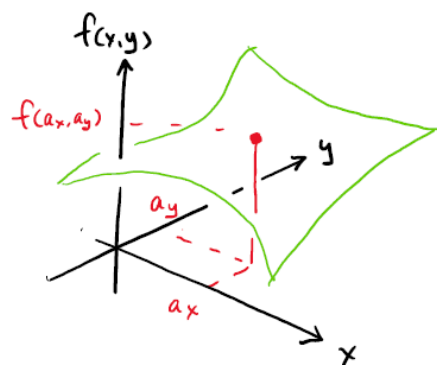
$$\frac{\partial f}{\partial y} = x^2 \cdot 1 \cdot \sin z$$

$$\left(\frac{d}{dy} y = 1, \text{ don't touch } x, z \right)$$

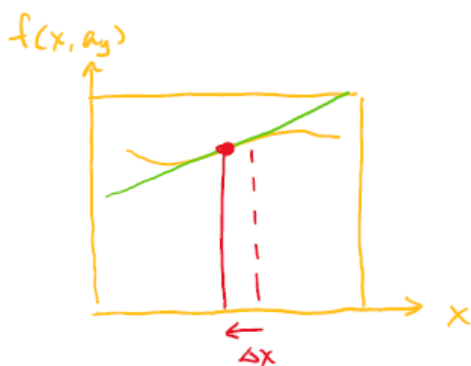
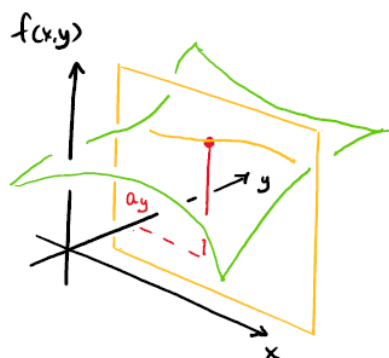
$$\frac{\partial f}{\partial z} = x^2 y \cdot \cos z$$

$$\left(\frac{d}{dz} \sin z \cos z, \text{ don't touch } x, y \right)$$

The visualization to partial differentiation is straightforward. Take a 2-inputs function $f(x, y)$ as example, we can demonstrate by the following illustrations:

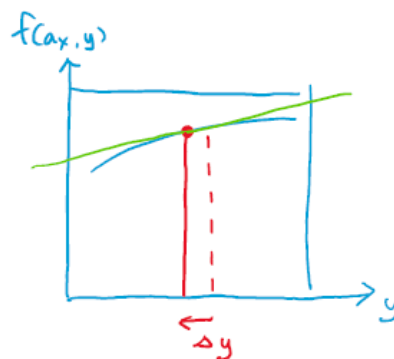
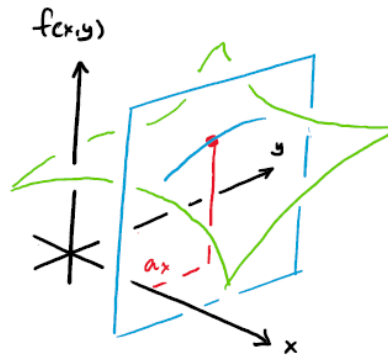


Make a slice at $y = a_y$



$\frac{\partial}{\partial x}$ = On the plane of constant y ,
find slope along x direction.

Make a slice at $x = a_x$



$\frac{\partial}{\partial y}$ = On the plane of constant x ,
find slope along y direction.

We can conclude:

$$\frac{\partial}{\partial x_i} = \text{Find slope / rate of change of function with respect to } x_i$$

3.2 Evaluating Partial Differentiation

Calculation rules for partial differentiation are all the same as in single variable functions except **chain rule**. Due to how function composition works in multivariable calculus, the multivariable version is the **sum of chain rule with respect to each of the input**.

$$\begin{aligned} \frac{\partial}{\partial x_i} f(\vec{g}(x_1, x_2, \dots, x_n)) &= \sum_j \frac{\partial}{\partial g_j} f(\vec{g}) \frac{\partial g_j}{\partial x_i} \\ &= \frac{\partial}{\partial g_1} f(\vec{g}) \frac{\partial}{\partial x_i} g_1(x_1, x_2, \dots, x_n) + \frac{\partial}{\partial g_2} f(\vec{g}) \frac{\partial}{\partial x_i} g_2(x_1, x_2, \dots, x_n) + \dots \end{aligned}$$

As for now you do not need to remember this formula. We will be able to write it in a more compact (and easier to remember) form after learning matrix.

As an example of calculation, suppose we start with two functions without knowing their exact expression:

$$f(p, q) \quad (2 \xrightarrow{f} 1)$$

$$\vec{h}(u, v) = (h_1(u, v), h_2(u, v)) = (h_1, h_2) \quad (2 \xrightarrow{h} 2)$$

And construct the following composition:

$$f(\vec{h}(u, v)) = f((h_1, h_2)) = f((h_1(u, v), h_2(u, v))) \quad (2 \xrightarrow{h} 2 \xrightarrow{f} 1)$$

Because $f(\vec{h}(u, v))$ takes 2 inputs u, v , there must be 2 partial differentiations (one for u and one for v). With chain rule, the partial differentiations write as

With respect to u :

$$\begin{aligned} \frac{\partial}{\partial u} f(\vec{h}(u, v)) &= \frac{\partial}{\partial u} f(\overbrace{h_1}^{u \text{ on } h_2}, \overbrace{h_2}^{u \text{ on } h_2}) \\ &= \underbrace{\left(\frac{\partial}{\partial h_1} f(h_1, h_2) \cdot \frac{\partial h_1}{\partial u} \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{\partial}{\partial h_2} f(h_1, h_2) \cdot \frac{\partial h_2}{\partial u} \right)}_{\text{Chain rule over } h_2 \text{ only}} \end{aligned}$$

With respect to v :

$$\begin{aligned}
 \frac{\partial}{\partial v} f(\vec{h}(u, v)) &= \frac{\partial}{\partial v} f(h_1, h_2) \\
 &\quad \begin{array}{c} \text{v on } h_2 \\ \text{v on } h_1 \end{array} \\
 &= \underbrace{\left(\frac{\partial}{\partial h_1} f(h_1, h_2) \cdot \frac{\partial h_1}{\partial v} \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{\partial}{\partial h_2} f(h_1, h_2) \cdot \frac{\partial h_2}{\partial v} \right)}_{\text{Chain rule over } h_2 \text{ only}}
 \end{aligned}$$

We may do straightforward substitution, if the functions' expressions are given. Let's say,

$$f(p, q) = \sqrt{p+q} \quad \text{and} \quad \vec{h}(u, v) = (u^2 + v, u - v) = (h_1, h_2)$$

Then

$$\begin{aligned}
 \frac{\partial}{\partial u} f(\vec{h}(u, v)) &= \underbrace{\left(\frac{\partial}{\partial h_1} f(h_1, h_2) \frac{\partial h_1}{\partial u} \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{\partial}{\partial h_2} f(h_1, h_2) \frac{\partial h_2}{\partial u} \right)}_{\text{Chain rule over } h_2 \text{ only}} \\
 &= \underbrace{\left(\frac{\partial}{\partial h_1} \sqrt{h_1 + h_2} \Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot \frac{\partial}{\partial u} (u^2 + v) \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{\partial}{\partial h_2} \sqrt{h_1 + h_2} \Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot \frac{\partial}{\partial u} (u - v) \right)}_{\text{Chain rule over } h_2 \text{ only}} \\
 &= \underbrace{\left(\frac{1}{2\sqrt{h_1 + h_2}} \Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot 2u \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{1}{2\sqrt{h_1 + h_2}} \Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot (1) \right)}_{\text{Chain rule over } h_2 \text{ only}} \\
 &= \frac{2u}{2\sqrt{u^2 + u}} + \frac{1}{2\sqrt{u^2 + u}} \\
 &= \frac{2u + 1}{2\sqrt{u^2 + u}}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial}{\partial v} f(\vec{h}(u, v)) &= \underbrace{\left(\frac{\partial}{\partial h_1} f(h_1, h_2) \frac{\partial h_1}{\partial v} \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{\partial}{\partial h_2} f(h_1, h_2) \frac{\partial h_2}{\partial v} \right)}_{\text{Chain rule over } h_2 \text{ only}} \\
 &= \underbrace{\left(\frac{\partial}{\partial h_1} \sqrt{h_1 + h_2} \Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot \frac{\partial}{\partial v} (u^2 + v) \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{\partial}{\partial h_2} \sqrt{h_1 + h_2} \Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot \frac{\partial}{\partial v} (u - v) \right)}_{\text{Chain rule over } h_2 \text{ only}} \\
 &= \underbrace{\left(\frac{1}{2\sqrt{h_1 + h_2}} \Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot (1) \right)}_{\text{Chain rule over } h_1 \text{ only}} + \underbrace{\left(\frac{1}{2\sqrt{h_1 + h_2}} \Big|_{\substack{h_1=u^2+v \\ h_2=u-v}} \cdot (-1) \right)}_{\text{Chain rule over } h_2 \text{ only}} \\
 &= \frac{1}{2\sqrt{u^2 + u}} + \frac{-1}{2\sqrt{u^2 + u}} \\
 &= 0
 \end{aligned}$$

We can also compute the composition directly for result checking:

$$f(\vec{h}(u, v)) = \sqrt{u^2 + v + u - v} = \sqrt{u^2 + u}$$

$$\Rightarrow \quad \frac{\partial f}{\partial u} = \frac{2u + 1}{2\sqrt{u^2 + u}} \quad \text{and} \quad \frac{\partial f}{\partial v} = 0$$

Exercise 3.1. Given the functions and their composition:

$$\begin{cases} f(p, q) = \sqrt{p + q} \\ \vec{g}(t) = (t - 1, t^2) \end{cases} \quad \Rightarrow \quad f(\vec{g}(t)) = \sqrt{t^2 + t - 1}$$

Compute the derivative $\frac{d}{dt}f(\vec{g}(t))$, by

1. directly differentiating against t
2. first differentiate via chain rule over (p, q)

(You should get equal results.)

4 Multiple Integral

The limit definition of multiple integral can be written as

$$\begin{aligned} & \int \cdots \int_{\text{(Some region)}} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ &= \lim_{\Delta x_1, \Delta x_2, \dots, \Delta x_n \rightarrow 0} \sum_{\substack{\text{all divisions} \\ \text{in the region}}} f(\xi_1, \xi_2, \dots, \xi_n) \Delta x_1 \Delta x_2 \dots \Delta x_n \end{aligned}$$

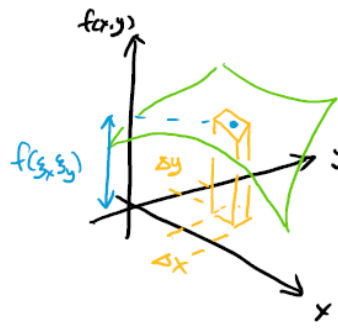
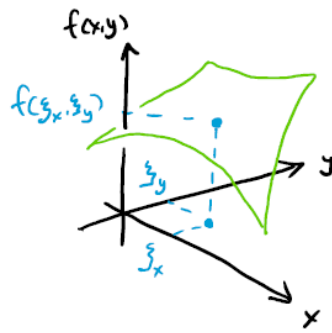
Recall that we have introduced 2 geometrical interpretations of integration. Here we can demonstrate them on the two most frequently used multiple integral.

4.1 Double Integral

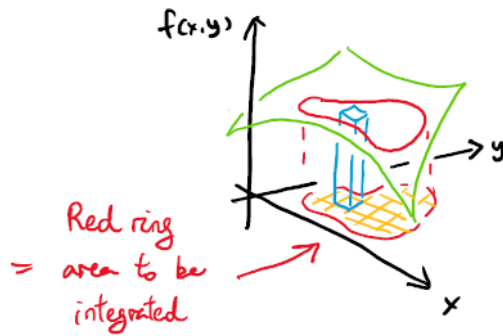
For functions with 2 inputs.

Interpretation 1: Volume under surface, bounded by base area

$$\iint_{\text{Some Area}} f(x, y) dx dy = \lim_{\Delta x, \Delta y \rightarrow 0} \sum_{\substack{\text{All divisions} \\ \text{in the area}}} \underbrace{f(\xi_x, \xi_y)}_{\substack{\text{Pillar's height} \\ \uparrow}} \underbrace{\Delta x \Delta y}_{\substack{\text{Pillar's base area} \\ \nwarrow}}$$



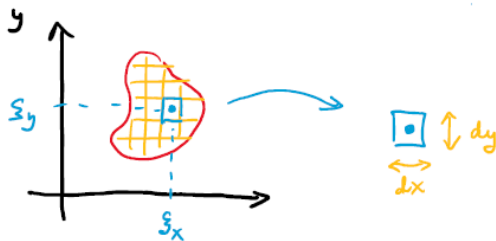
One pillar for every point



Double integral
= Sum the volumes of all pillars
within the red ring (region)

Interpretation 2: Weighted sum over an area

$$\iint_{\text{Some Area}} f(x, y) \, dx \, dy = \lim_{\Delta x, \Delta y \rightarrow 0} \sum_{\text{All divisions in the area}} \underbrace{f(\xi_x, \xi_y)}_{\text{Weight for the grid at } (\xi_x, \xi_y)} \underbrace{\Delta x \Delta y}_{\text{Area of each grid}}$$



Double integral
= For each grid of weight $f(\xi_x, \xi_y)$,
Sum all grids within the red ring (region)

For a physics example, the *area mass density distribution* $\sigma(x, y)$ may depends on position coordinate (x, y) .

- Each small grid has an area ($dx \, dy$)
- At position (ξ_x, ξ_y) , the grid has a density $\sigma(x, y)$

Thus,

Total mass = Sum of mass of all small grids

$$= \sum_{\substack{\text{all small grids} \\ \text{in the area}}} \left(\begin{array}{c} \text{density} \\ \text{of each grid} \end{array} \right) \left(\begin{array}{c} \text{area} \\ \text{of each grid} \end{array} \right)$$

$$= \sum_{\substack{\text{all small grids} \\ \text{in the area}}} \sigma(x, y) \cdot (\Delta x \Delta y)$$

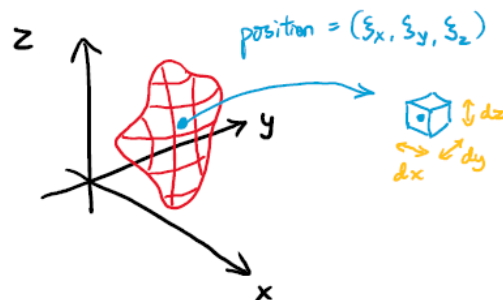
$$\approx \iint_{\text{the area}} \sigma(x, y) \, dx \, dy$$

4.2 Triple Integral

Interpretation 1: ??? under volume

(Sorry, we live in a 3D space. No idea how to draw 4D objects.)

Interpretation 2: Weighted sum over a volume

$$\iiint_{\text{Some Volume}} f(x, y, z) \, dx \, dy \, dz = \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \sum_{\substack{\text{All divisions} \\ \text{in the volume}}} \frac{f(\xi_x, \xi_y, \xi_z) \Delta x \Delta y \Delta z}{\text{Weight for the cube at } (\xi_x, \xi_y, \xi_z)} \quad \text{Volume of each cube}$$


Triple integral
= For each cube of weight $f(\xi_x, \xi_y, \xi_z)$,
Sum all cubes within the red region

Similar to double integral, if $\rho(x, y, z)$ is the *volume mass density distribution*, then,

Total mass = Sum of mass of all small cubes

$$= \sum_{\substack{\text{all small cubes} \\ \text{in the volume}}} \left(\begin{array}{c} \text{density} \\ \text{of each cube} \end{array} \right) \left(\begin{array}{c} \text{volume} \\ \text{of each cube} \end{array} \right)$$

$$= \sum_{\substack{\text{all small cubes} \\ \text{in the volume}}} \rho(x, y, z) \cdot (\Delta x \Delta y \Delta z)$$

$$\approx \iiint_{\text{the volume}} \rho(x, y, z) \, dx \, dy \, dz$$

4.3 Evaluating Multiple Integral

The difficulty in calculation mostly comes from determining the region to be integrated. Here are the main steps in your calculation:

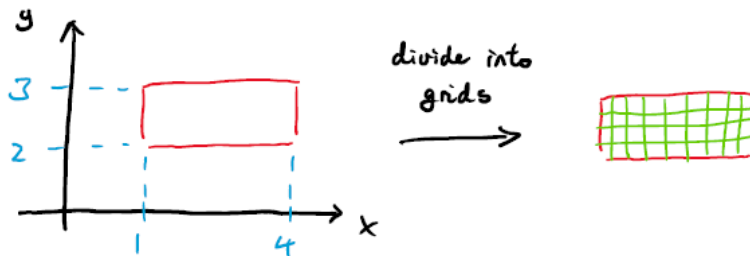
1. Decide the integration order, i.e. how to divide a region.
 - The integration order decide the expression. Follow the expression to integrate "from inside to outside".

$$\iiint f(x, y, z) dx dy dz = \int \left(\int \left(\int f(x, y, z) dx \right) dy \right) dz$$

outer = 3rd middle = 2nd inner = 1st

- Calculation is exactly how you do single variable integration, but do it multiple times.
 - While integrating one variable, treat the others as constants.
2. Derive the corresponding upper/lower bounds
 - It would be easier if you can draw out the region.
 - Note that if you switch the integration order, the bounds must change.

Example 4.1. Integrate $f(x, y) = x^2y - xy^3$ over the region bounded by $\begin{cases} x = 1 \\ x = 4 \end{cases}$ and $\begin{cases} y = 2 \\ y = 3 \end{cases}$



Integration order 1: First x , then y .

1. Integrate x = Sum all grid with the same y coordinate to form horizontal strips.

$$\Rightarrow \int_{x=1}^{x=4} f(x, y) dx$$

2. Integrate y = Sum all horizontal strips to form the integration region.

$$\Rightarrow \int_{y=2}^{y=3} \left[\int_{x=1}^{x=4} f(x, y) dx \right] dy$$

3. In the calculation, follow the expression's order: Integrate x first, then y . Note that before integrating y , you need to clear all x by substituting the given upper/lower bounds.

$$\begin{aligned}
 & \int_{y=2}^{y=3} \left[\int_{x=1}^{x=4} x^2 y - xy^3 dx \right] dy \\
 &= \int_{y=2}^{y=3} \left[\frac{x^3}{3} y - \frac{x^2}{2} y^3 \right] \Big|_{x=1}^{x=4} dy \\
 &= \int_{y=2}^{y=3} \underbrace{\left(\frac{64}{3} y - \frac{16}{2} y^3 \right)}_{\text{Subst. } x=4} - \underbrace{\left(\frac{1}{3} y - \frac{1}{2} y^3 \right)}_{\text{Subst. } x=1} dy \\
 &= \int_{y=2}^{y=3} 21y - \frac{15}{2} y^3 dy \\
 &= \left[\frac{21}{2} y^2 - \frac{15}{8} y^4 \right] \Big|_{y=2}^{y=3} \\
 &= \frac{-555}{8}
 \end{aligned}$$

Integration order 2: First y , then x .

1. Integrate y = Sum all grid with the same x coordinate to form vertical strips.

$$\begin{aligned}
 & \text{Diagram: A vertical strip of small squares with width } dx \text{ and height } dy. \\
 & \text{Diagram: A single vertical strip of height } y \text{ from } y=2 \text{ to } y=3. \\
 & \Rightarrow \int_{y=2}^{y=3} f(x, y) dy
 \end{aligned}$$

2. Integrate x = Sum all horizontal strips to form the integration region.

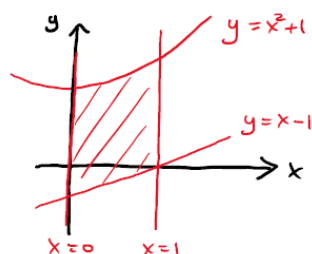
$$\begin{aligned}
 & \text{Diagram: Three vertical strips of width } dx. \\
 & \text{Diagram: A grid of width } x \text{ from } x=1 \text{ to } x=4 \text{ and height } y \text{ from } y=2 \text{ to } y=3. \\
 & \Rightarrow \int_{x=1}^{x=4} \left[\int_{y=2}^{y=3} f(x, y) dy \right] dx
 \end{aligned}$$

3. In the calculation, follow the expression's order: Integrate y first, then x . Note that before integrating x , you need to clear all y by substituting the given upper/lower bounds.

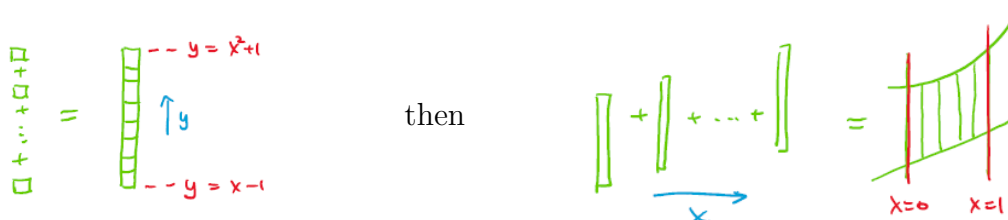
$$\begin{aligned}
 & \int_{x=1}^{x=4} \left[\int_{y=2}^{y=3} x^2 y - xy^3 \, dy \right] dx \\
 &= \int_{x=1}^{x=4} \left[\frac{1}{2} x^2 y^2 - \frac{1}{4} xy^4 \right] \bigg|_{y=2}^{y=3} dx \\
 &= \int_{x=1}^{x=4} \underbrace{\left(\frac{9}{2} x^2 - \frac{81}{4} x \right)}_{\text{Subst. } y=3} - \underbrace{(2x^2 - 4x)}_{\text{Subst. } y=2} dx \\
 &= \int_{x=1}^{x=4} \frac{5}{2} x^2 - \frac{65}{4} x \, dx \\
 &= \left[\frac{5}{6} x^3 - \frac{65}{8} x^2 \right] \bigg|_{x=1}^{x=4} \\
 &= \frac{-555}{8}
 \end{aligned}$$

However, if the boundaries of the region is ugly, some integration order make your life easier than the others.

Example 4.2. Consider integration over the below region (with an arbitrary $f(x, y)$):



Integration order 1: First y , then x .



This approach is easy because all vertical strips have the same bounds:

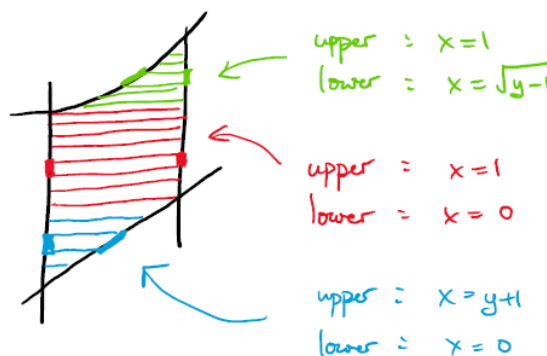
- Upper bound: The curve $y = x^2 + 1$
- Lower bound: The curve $y = x - 1$

We can write the integral expression as a single term.

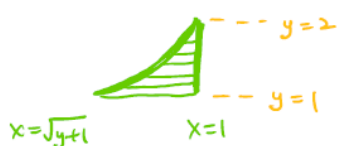
$$\int_{x=0}^{x=1} \int_{y=x-1}^{y=x^2+1} f(x, y) \, dy \, dx$$

Integration order 2: First x , then y .

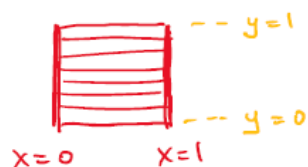
Note that the bounds of horizontal strips are different for different y :



So we need to integrate each region individually.



$$I_1 = \int_{y=1}^{y=2} \int_{x=\sqrt{y-1}}^{x=1} f(x, y) \, dx \, dy$$



$$I_2 = \int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x, y) \, dx \, dy$$



$$I_3 = \int_{y=0}^{y=-1} \int_{x=0}^{x=y+1} f(x, y) \, dx \, dy$$

And the final answer would be the sum to all 3 regions $I_1 + I_2 + I_3$. Although we should get the same value as we integrate y first then x , integrating x first then y takes a lot more effort.

— The End —