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Matrices and System of ODEs

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Overview:

- Matrix and determinant arithmetics.
- Theory of system of linear equations
- Eigenvalue & eigenvector
- Solving system of ODE - Coupled harmonic oscillators

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1 Matrices & Determinants

Matrix is a mathematical tool to express an array of values.

Usually use bold font capital letter to represent matrix $\longrightarrow$

$$\mathbf{A} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}}_{3} \Big\updownarrow 2$$

Dimension of this matrix $= 2 \times 3$

Each entry is called an "element"

We call the shape ((# of rows) $\times$ (# of columns)) of a matrix as its **dimension**. Depending on its shape, they may aquire special name:

<u>Column matrix</u>

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

<u>Row matrix</u>

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

<u>Square matrix</u>
(# of rows = # of columns)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

1.1 Matrix Arithmatics

Matrices calculations share similar operations as calculating number, but with different definitions.

1. **Addition / Subtraction**

   Can only be carried out between matrices of the same dimension. Each element is added / subtracted independently.

$$\mathbf{A} \pm \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \end{pmatrix}$$

2. **Multiplication with Constants**

   Multiplying a constant = Multiply every element with that constant

$$k\mathbf{A} = k \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{pmatrix}$$

3. **Transpose**

   Flip the matrix diagonally. Denote using the $\intercal$ symbol.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{A}^\intercal = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

4. **Matrices Multiplication**

   Given 2 matrices, one with dimension $\underline{m \times n}$ and the other $\underline{n \times p}$,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad , \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

   Their multiplication yield a product matrix of dimension $\underline{m \times p}$.

$$\mathbf{C} = \underline{\mathbf{AB}} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

   We don't write × sign
   in matrix multiplication

The computation of each element c is a sum-product of a and b:

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} \end{aligned}$$

$$\begin{pmatrix} & & \vdots & \big\downarrow \text{i}^{\text{th}}\text{row} & \\ & & \vdots & & \\ \cdots & \cdots & c_{ij} & \cdots & \cdots \\ \xrightarrow[\text{j}^{\text{th}}\text{column}]{} & & \vdots & & \\ & & \vdots & & \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \big\downarrow \text{i}^{\text{th}}\text{row} & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\ & & \xrightarrow[\text{Take every element}]{} & & \\ \vdots & \vdots & & & \vdots \end{pmatrix} \begin{pmatrix} \cdots & \cdots & b_{1j} & \big\downarrow \text{Take every element} & \cdots \\ \cdots & \cdots & b_{2j} & & \cdots \\ \xrightarrow[\text{j}^{\text{th}}\text{column}]{} & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & b_{nj} & & \cdots \end{pmatrix}$$

<u>Caution:</u> Matrices multiplication is not *commutative* in general. i.e. $\mathbf{AB} \neq \mathbf{BA}$.

Matrices can multiply only if (# of columns in 1st matrix) = (# of rows in 2nd matrix). For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad , \quad \mathbf{B} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \mathbf{AB} &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \times 7 + 2 \times 8 + 3 \times 9 \\ 4 \times 7 + 5 \times 8 + 6 \times 9 \end{pmatrix} \\ &= \begin{pmatrix} 50 \\ 122 \end{pmatrix} \end{aligned}$$

$$\text{But } \mathbf{BA} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ cannot be calculated.}$$

<u>Note:</u> Multiplication order can be "reversed" only after we take transpose, i.e. flipping the rows and columns.

$$(\mathbf{AB})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal$$

$$\left[\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}\begin{pmatrix}7\\8\\9\end{pmatrix}\right]^\intercal = \begin{pmatrix}7\\8\\9\end{pmatrix}^\intercal \begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}^\intercal$$

$$= \begin{pmatrix}7 & 8 & 9\end{pmatrix}\begin{pmatrix}1 & 4\\2 & 5\\3 & 6\end{pmatrix}$$

5. **Zero Matrix**

   They are matrices with all elements = 0.

$$\mathbf{0}_2 = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \quad , \quad \mathbf{0}_3 = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$$

   Note: Algebraically, "zero" is the element used to define *additive inverse* in an algebra, i.e. if $a + b = 0$, then b is the additive inverse of a. For real number, we have the number 0. For matrices, we have the zero matrices.

6. **Identity Matrix**

   They are **square matrices** whose diagonal elements = 1 and anything else = 0.

$$\mathbf{I}_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad , \quad \mathbf{I}_3 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

   A special property about identity matrices is that any square matrix multiply to it yields the original square matrix.

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

   For example,

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1\times1+2\times0 & 1\times0+2\times1\\3\times1+4\times0 & 3\times0+4\times1\end{pmatrix} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$$

   Note: Algebraically, "identity I" is the element used to define *multiplicative inverse* in an algebra, i.e. if $a \times b = I$, then b is the multiplicative inverse of a. For real number, we have the number 1. For matrices, we have the identity matrices.

7. <u>**Multiplicative Inverse**</u>

Matrix's multiplicative inverse is only well-defined for square matrices. We denote the multiplicative inverse of a matrix $\mathbf{A}$ as $\mathbf{A}^{-1}$. i.e. $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

There are two well-known way to compute a matrix's inverse. The first formula appears more frequently in textbooks, but is extremely complicate to compute.

$$\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})}$$

The other method is in fact the more efficient method - join the matrix with identity and carry **row operations**:

- Exchange row i and row j. ($R_i \leftrightarrow R_j$)
- Multiply the whole row i with a constant k. ($R_i \leftarrow kR_i$)
- Add a multiple of a row j to row i. ($R_i \leftarrow R_i + kR_j$)

Example 1.1. Here we demonstrate by finding the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

First join the matrix with identity of the same size on the right half.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

Then resolve with row operations to turn the left half into identity matrix.

<u>Turn bottom left "3" to 0</u> <u>(Row 2) $\leftarrow$ (Row 2) $-$ 3$\times$(Row 1)</u>

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3-3(1) & 4-3(2) & 0-3(1) & 1-3(0) \end{array}\right)$$
$$= \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

<u>Turn upper right "2" to 0</u> <u>(Row 1) $\leftarrow$ (Row 1) + (Row 2)</u>

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|cc} 1+0 & 2-2 & 1-3 & 0+1 \\ 0 & -2 & -3 & 1 \end{array}\right)$$
$$= \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

Turn bottom right "-2" to 1 $\quad$ (Row 2) $\leftarrow -\frac{1}{2}$(Row 2)

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow -\frac{1}{2}R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0(-\frac{1}{2}) & -2(-\frac{1}{2}) & -3(-\frac{1}{2}) & 1(-\frac{1}{2}) \end{array}\right)$$
$$= \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

At this point, the left half becomes identity. **Then the right half is exactly the inverse.**

$$\left(\begin{array}{c|c} \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}} & \boxed{\begin{array}{cc} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{array}} \end{array}\right)$$

Identity $\qquad$ The inverse

Double checking here:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1(-2) + 2(\frac{3}{2}) & 1(1) + 2(-\frac{1}{2}) \\ 3(-2) + 4(\frac{3}{2}) & 3(1) + 4(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Caution: Not all matrices have multiplicative inverse. Those that do not have multiplication inverse are called **singular matrices**. For example,

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

It is not possible to convert the first 2 elements of the bottom row to $(1, 0)$.

1.2 Determinant

Determinant is a unique operation that convert square matrices into a number.

The function is labelled "det(·)" $\longrightarrow$
$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$
$\longleftarrow$ Use vertical bars $|\cdot|$ instead of $(\cdot)$

Side Note:

(For your interest only) The technical definition of determinant is the alternating sum to all products of elements a_{ij}, generating by permuting indices.

$$\det(\mathbf{A}) = \sum_{\sigma_i \in S_n} \operatorname{sgn}(\sigma_i) a_{1,\sigma_i(1)} a_{2,\sigma_i(2)} \cdots a_{n,\sigma_i(n)}$$

Here explains the notations:

- σ_i represents a possible permutation. For example, if there are 3 items A, B, C, we can generate a total $3! = 6$ different permutations:

  $$(ABC), (ACB), (BAC), (BCA), (CAB), (CBA)$$

  So we can use 6 sigmas $\sigma_1, \sigma_2, ...\sigma_6$ to represent each permutation.

- S_n represents the set of all possible permutations for n objects. Note that in this expression, we are permuting the matrix's indices, spanning from number 1 to number n. Then the sum is over every possible permutation.

- $\sigma_i(n)$ returns the n^{th}object under the permutation σ_i. Like if the permutation is (ABCDE) $\xrightarrow{\sigma_i}$ (ACDBE), then $\sigma_i(4) =$ the 4^{th} item = B.

- $\operatorname{sgn}(\sigma_i)$ is the "sign" of the permutation σ_i. This "sign" is defined by how many swaps between items are required to construct such permutation

  - If (# of swap) = even number, it is an even permutation. $\operatorname{sgn}(\sigma_i) = +1$.
  - If (# of swap) = odd number, it is an old permutation. $\operatorname{sgn}(\sigma_i) = -1$.

  For example, (ABCDE) $\xrightarrow{\sigma_i}$ (ACDBE) can be constructed by

  $$(ABCDE) \xrightarrow{B \leftrightarrow C} (ACBDE) \xrightarrow{B \leftrightarrow D} (ACDBE)$$

  Total 2 swaps. So this is an even permutation. $\operatorname{sgn}(\sigma_i) = +1$.

1.2.1 Rule of Sarrus

For 2×2 and 3×3 matrices, there are shortcuts for remembering how to compute their determinants, called "**Rule of Sarrus**".

$$2\times 2 \quad : \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +(a_{11}\cdot a_{22}) - (a_{21}\cdot a_{12})$$

$$3\times 3 \quad : \quad \left|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \quad = \quad \begin{matrix} +(a_{11}\cdot a_{22}\cdot a_{33}) + (a_{12}\cdot a_{23}\cdot a_{31}) + (a_{13}\cdot a_{21}\cdot a_{32}) \\ -(a_{31}\cdot a_{22}\cdot a_{13}) - (a_{32}\cdot a_{23}\cdot a_{11}) - (a_{33}\cdot a_{21}\cdot a_{12}) \end{matrix}$$

This method does not work at all for 4×4 matrices or above. For example, for 4×4 you can only draw 8 arrows using this method, but there should be $4! = 24$ different terms (counted by permutation).

1.2.2 Standard Method for computing $n\times n$ Determinant

One must walk through the hard way for determinant with arbiturary size - minor & cofactor. Let $\mathbf{A}$ be an $n\times n$ matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

<u>**Minor of a_{ij}**</u> - Remove the i^{th}row and j^{th}column, then *take its determinant.*

$$M_{ij} = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2j} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & a_{i2} & \cdots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \cdots & a_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \cdots & a_{nn} \end{pmatrix} \quad \text{Remove } i^{\text{th}}\text{row}$$

Remove j^{th}column

Cofactor of a_{ij} - Multiply $(-1)^{(i+j)}$ to Minor of a_{ij}.

$$\operatorname{cof}(a_{ij}) = (-1)^{(i+j)} M_{ij}$$

Finally, the determinant's value can be computed by either

$$\sum_{\substack{\text{all } a_{ij} \\ \text{on the same} \\ \text{row}}} a_{ij} \operatorname{cof}(a_{ij}) \qquad \text{or} \qquad \sum_{\substack{\text{all } a_{ij} \\ \text{on the same} \\ \text{column}}} a_{ij} \operatorname{cof}(a_{ij})$$

This method works for 2×2 and 3×3 too, but no one would use it because rules of Sarrus exist.

Example 1.2. Compute the determinant of the following matrix using the standard method.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We can choose any row or column and compute the cofactor for each of the element. For example, here chooses the first row.

$$a_{11} = 1 \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \Rightarrow \quad \operatorname{cof}(a_{11}) = (-1)^{(1+1)} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3$$

$$a_{12} = 2 \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \Rightarrow \quad \operatorname{cof}(a_{12}) = (-1)^{(1+2)} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 6$$

$$a_{13} = 3 \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \Rightarrow \quad \operatorname{cof}(a_{13}) = (-1)^{(1+3)} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$$

So the value of determinant is

$$\begin{aligned} \det(\mathbf{A}) &= a_{11} \operatorname{cof}(a_{11}) + a_{12} \operatorname{cof}(a_{12}) + a_{13} \operatorname{cof}(a_{13}) \\ &= 1 \times (-3) + 2 \times (6) + 3 \times (-3) \\ &= 0 \end{aligned}$$

1.2.3 More Properties of Determinant

Here are some useful properties that allow faster calculation to a matrix's determinant.

1. (Antisymmetry) If any of the columns are swapped, the determinant gains a $-$ve sign.

$$\begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

   As a consequence, if any two columns are identical, swapping them makes the determinant gains a $-$ve sign but the matrix is unchanged. So the determinant must be 0.

2. (Multilinearity) Determinant is linear to each of the matrix's columns:
   (Note: In algebra, a function is linear if and only if $f(ka+b) = kf(a) + f(b)$)

   – Multiplying a constant k to each element of a column results in the determinant's value multiplying k.

$$\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

   – Sums of elements on the same column can be splited into individual matrices. The determinant's value is equal to the sum of determinant of each matrix.

$$\begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

   As a consequence, adding a multiple of another column does not change the value of the determinant.

$$\begin{vmatrix} a_{11}+a_{12} & a_{12} & a_{13} \\ a_{21}+a_{22} & a_{22} & a_{23} \\ a_{31}+a_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + 0$$

3. (Transpose) Determinant of the transpose of a matrix is the same as the original. This also implies that **every property that applies to column will also apply to rows**.

$$\det(\mathbf{A}) = \det(\mathbf{A}^\intercal)$$

4. (Muliplication) Determinant of a product between two matrics equal to the product of the determinat of each of the matrices.

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

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2 Theory of System of Linear Equation

We can always rewrite a system of linear equation into matrix form, which simplify the notations.

$$\left\{\begin{array}{rrrcr} & 2y & -z & = & 1 \\ x & -y & +z & = & 0 \\ 2x & +y & -z & = & -2 \end{array}\right. \quad \Leftrightarrow \quad \underbrace{\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}_{\mathbf{b}}$$

2.1 Solving Equations with Matrix Inverse

Notice that if $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$, then we can do

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{A}^{-1}(\mathbf{Ax}) &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \end{aligned}$$

So if we successfully find $\mathbf{A}$'s inverse, the solution to the system of equation is exactly $\mathbf{A}^{-1}\mathbf{b}$.

Example 2.1. Consider the system of equation:

$$\underbrace{\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}_{\mathbf{b}}$$

Computing inverse of $\mathbf{A}$:

$$\begin{aligned} \left(\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) &\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \\ &\xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow \frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & -3 & 0 & -2 & 1 \end{array}\right) \\ &\xrightarrow[R_3 \leftarrow R_3 - 3R_2]{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & -2 & 1 \end{array}\right) \end{aligned}$$

$$\xrightarrow{R_3 \leftarrow -\frac{2}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & \frac{4}{3} & -\frac{2}{3} \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow R_2 + \frac{1}{2}R_3]{R_1 \leftarrow R_1 - \frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 1 & \frac{4}{3} & -\frac{2}{3} \end{array}\right)$$

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{2}{3} & -\frac{1}{3} \\ 1 & \frac{4}{3} & -\frac{2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 4 & -2 \end{pmatrix}$$

Therefore, the solution to this system of equation is

$$\mathbf{A}^{-1}\mathbf{b} = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 \\ 5 \\ 7 \end{pmatrix} \quad \Rightarrow \quad x = -\frac{2}{3}\ ,\ y = \frac{5}{3}\ ,\ z = \frac{7}{3}$$

However, this method has a big problem: Not every matrix has a multiplicative inverse. In order to succefully carry out the above steps, we need to first check if the inverse exists.

To proceed, we first introduce these two theorems without rigorous proofs:

- Theorem without Rigorous Proof 1:

  Recall the inverse formula that nobody uses:

  $$\mathbf{A}^{-1} = \frac{\mathrm{adj}(\mathbf{A})}{\det(\mathbf{A})}$$

  where $\mathrm{adj}(\mathbf{A}) = [\mathrm{cof}(a_{ij})]^{\intercal}$ is a matrix composed by cofactor of *every elements* of $\mathbf{A}$, and then take its transpose. We can observe that if $\det(\mathbf{A}) = 0$, then $\mathbf{A}^{-1}$ is undefined.

  $$\boxed{\det(\mathbf{A}) \neq 0 \quad \Leftrightarrow \quad \mathbf{A}^{-1} \text{ exists}}$$

- Theorem without Rigorous Proof 2:

  If a matrix's inverse exists, the inverse is unique, i.e. It is impossible to have $\mathbf{AX} = \mathbf{AY} = \mathbf{I}$ but $\mathbf{X} \neq \mathbf{Y}$. So for a system of equation $\mathbf{Ax} = \mathbf{b}$,

  $$\boxed{\mathbf{A}^{-1} \text{ exists} \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \text{ is a unique solution}}$$

2.2 Homogeneity of Linear Equation

Recall in the note about ODE, we introduced the concept of homogeneity - If all terms in the equation contain the unknown, it is a homogeneous equation. Otherwise it is a non-homogeneous equation. The same applies to system of linear equations:

– Homogeneous

$$\mathbf{A}\underbrace{\mathbf{x}}_{\text{the unknown } \mathbf{x}} = \underline{\mathbf{0}} \longleftarrow \text{Every term contains } \mathbf{x} \text{ because there are no other terms}$$

– Non-homogeneous

$$\mathbf{A}\underbrace{\mathbf{x}}_{\text{the unknown } \mathbf{x}} = \underline{\mathbf{b}} \longleftarrow \text{There is a term that does not contain } \mathbf{x}$$

In ODE, the unknown $f(t)$ is a function, so the equation can contain terms of $f(t)$'s derivatives; In system of linear equations, the unknown $\mathbf{x}$ are only numbers, so the only possible term is $\mathbf{Ax}$.

2.2.1 Case 1: Homogeneous ($\mathbf{Ax} = \mathbf{0}$)

Obviously, $\mathbf{x} = \mathbf{0}$ is a solution (anything multiply $0 = 0$). This is called the **"trivial solution"**.

1. If $\det(\mathbf{A}) \neq 0$,

$$\mathbf{A}^{\text{-}1} \text{ exists } \Rightarrow \text{ Solution must be unique } \Rightarrow \boxed{\mathbf{x} = \mathbf{0} \text{ is the only solution}}$$

2. If $\det(\mathbf{A}) = 0$,

$$\mathbf{A}^{\text{-}1} \text{ not exists } \Rightarrow \text{ Solution is not unique } \Rightarrow \boxed{\text{Infinitely many solutions}}$$

For example,

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x + y = 0 \\ 0 + 0 = 0 \end{cases}$$

We can choose x to be anything (and then $y = -x$).

2.2.2 Case 2: Non-homogeneous

Just like in non-homogeneous ODE, the solution can be separated into 2 parts:

$$\mathbf{x} = \underbrace{\mathbf{x}_c}_{\text{Complementary solution}} + \underbrace{\mathbf{x}_p}_{\text{Particular solution}}$$

– Complementary solution $\mathbf{x}_c$: The solution to the homogeneous counterpart, i.e. $\mathbf{Ax} = \mathbf{0}$.

– Particular solution $\mathbf{x}_p$: Its presence is to cancel the non-homogeneous term.

1. If $\det(\mathbf{A}) \neq 0$,
   - The only solution to $\mathbf{A}\mathbf{x}_c = \mathbf{0}$ is $\mathbf{x}_c = \mathbf{0}$.
   - Since $\mathbf{A}\mathbf{x}_p = \mathbf{b}$, we can find $\mathbf{x}_p = \mathbf{A}^{-1}\mathbf{b}$.

$$\boxed{\text{The unique soluion is } \mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = \mathbf{A}^{-1}\mathbf{b}}$$

2. If $\det(\mathbf{A}) = 0$,
   - $\mathbf{x}_c$ has infinitely many solutions.
   - $\mathbf{x}_p$ may or may not exist. Need manual solving.

$$\boxed{\begin{array}{rl} \mathbf{x}_p \text{ exists} & \Rightarrow \ \mathbf{x} = \mathbf{x}_c\text{(Infinitely many)} + \mathbf{x}_p\text{(Exists)} = \text{(Infinitely many)} \\ \mathbf{x}_p \text{ not exists} & \Rightarrow \ \mathbf{x} = \mathbf{x}_c\text{(Infinitely many)} + \mathbf{x}_p\text{(Not Exists)} = \text{(Not Exists)} \end{array}}$$

Example 2.2.

$$\begin{array}{rrrcl} x & +2y & +3z & = & 0 \\ 2x & +3y & +8z & = & 0 \\ 3x & +2y & +17z & = & 0 \end{array} \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 3 & 2 & 17 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{b}}$$

This is a homogeneous equation. First we check if $\det(\mathbf{A}) = 0$:

$$\begin{array}{ccc|cc} & & & - & - \ - \\ 1 & 2 & 3 & 1 & 2 \\ 2 & 3 & 8 & 2 & 3 \\ 3 & 2 & 17 & 3 & 2 \\ & & & + & + \ + \end{array} = \begin{array}{c} (1)(3)(17) + (2)(8)(3) + (3)(2)(2) \\ -(3)(3)(3) - (2)(8)(1) - (17)(2)(2) \end{array} = 0$$

So there must be infinitely many solutions. We shall solve it exactly by row operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 8 & 0 \\ 3 & 2 & 17 & 0 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -4 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow -R_2]{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Transforming back to $x/y/z$, it reads

$$\begin{cases} x + 7z = 0 \\ y - 2z = 0 \\ \qquad 0 = 0 \end{cases}$$

where z can be any constant. So the solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7z \\ 2z \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} z$$

Example 2.3. Now changed the previous system to be non-homogeneous.

$$\begin{array}{cccccc} x & +2y & +3z & = & 3 \\ 2x & +3y & +8z & = & 4 \\ 3x & +2y & +17z & = & 1 \end{array} \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 3 & 2 & 17 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}}_{\mathbf{b}}$$

$\mathbf{A}$ is the same as the last example so we already know $\det(\mathbf{A}) = 0$. But $\mathbf{b}$ is non-zero, so the solution must be either infinitely many or not exist. Solve again with row operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 3 & 2 & 17 & 1 \end{array}\right) \xrightarrow[]{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -4 & 8 & -8 \end{array}\right)$$

$$\xrightarrow[]{\substack{R_3 \leftarrow R_3 - 4R_2 \\ R_2 \leftarrow -R_2}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[]{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Transforming back to $x/y/z$, it reads

$$\begin{cases} x + 7z = -1 \\ y - 2z = 2 \\ \qquad 0 = 0 \end{cases}$$

where z can be any constant. So the solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7z - 1 \\ 2z + 2 \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} z}_{\text{Complementary soln.}} + \underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{\text{Particular soln.}}$$

Example 2.4. Changed the non-homogeneous term in the last row.

$$\begin{array}{rrrcl} x & +2y & +3z & = & 3 \\ 2x & +3y & +8z & = & 4 \\ 3x & +2y & +17z & = & 2 \end{array} \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 3 & 2 & 17 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}}_{\mathbf{b}}$$

Solving with row operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 3 & 2 & 17 & 2 \end{array}\right) \xrightarrow[]{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -4 & 8 & -7 \end{array}\right)$$

$$\xrightarrow[]{\substack{R_3 \leftarrow R_3 - 4R_2 \\ R_2 \leftarrow -R_2}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Transforming back to $x/y/z$, the last row reads $0 + 0 + 0 = 1$. So there must be no solution.

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3 The Eigenvalue Problem

3.1 Definitions

Given a square matrix $\mathbf{A}$, we want to find any non-zero column matrix $\mathbf{x}$ and number λ satisfying $\mathbf{Ax} = \lambda\mathbf{x}$. Then

- $\mathbf{x}$ = Eigenvector of $\mathbf{A}$.
- λ = Eigenvalue of $\mathbf{A}$.

Theorem: λ is an eigenvalue of $\mathbf{A} \Leftrightarrow \det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Proof

$$\begin{aligned} \mathbf{Ax} &= \lambda\mathbf{x} = \lambda\mathbf{Ix} \\ \mathbf{Ax} - \lambda\mathbf{Ix} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}$$

□

Note that the above is a homogeneous system of linear equations.

- If $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$, The only solution is $\mathbf{x} = \mathbf{0}$ (reject)
- If $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, There are infinitely many solutions.

In fact, if $\mathbf{x}$ is an eigenvector that satisfies $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, for any k =constant,

$$\mathbf{A}(k\mathbf{x}) = k(\mathbf{A}\mathbf{x}) = k(\lambda\mathbf{x}) = \lambda(k\mathbf{x})$$

So $k\mathbf{x}$ is also an eigenvector to $\mathbf{A}$. This is why there are infinitely many solutions.

<u>Mathematically we treat $\mathbf{x}$ and $k\mathbf{x}$ as the same eigenvector.</u>

3.2 Standard Steps in Finding Eigenvectors

1. Solve the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to obtain some λ.
2. For <u>each</u> λ, substitute back into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ and solve its corresponding $\mathbf{x}$.

Example 3.1. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

1. Solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= \det\left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)(1-\lambda) - (4)(1) \\
&= \lambda^2 - \lambda - 3 = 0 \\
\lambda &= 3 \text{ or } -1
\end{aligned}$$

   Two different eigenvalues.

2. Substitute back <u>each</u> λ into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ and solve $\mathbf{x}$:

   – For $\lambda = 3$

$$\begin{aligned}
(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

   Read as $\begin{cases} -2x_1 + x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{cases} \Rightarrow\ x_2 = 2x_1$. So the eigenvector is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \underline{x_1}$$

x_1 can be any constant

– For $\lambda = -1$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - (-1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Read as $\begin{cases} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{cases} \Rightarrow x_2 = -2x_1$. So the eigenvector is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \underline{x_1}$$

x_1 can be any constant

Conclusion: $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ has 2 eigenvectors:

λ	3	-1
$\mathbf{x}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

<u>Note 1:</u> Eigenvalues and eigenvectors can be complex numbers. E.g.

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \Rightarrow$$

λ	-1	$1-2i$	$1+2i$
$\mathbf{x}$	$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1-i \\ -2i \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1+i \\ 2i \\ 1 \end{pmatrix}$

<u>Note 2:</u> An $n \times n$ matrix must have n eigenvalues, but some eigenvalues may have the same values. E.g.

$$\begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \Rightarrow \lambda = 3 \text{ only, and } \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \underline{s} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \underline{t}$$

s & t can be any constants

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4 System of Linear ODEs

We will only deal with the cases: Linear + Homogeneous + Constant Coefficients.

Otherwise they are too difficult to solve by hand!

4.1 1^{st} Order Linear Homogeneous with Constant Coefficient

For simplicity, let's deal with a system with 2 unknown functions only.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x_1(t) = ax_1(t) + bx_2(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}x_2(t) = cx_1(t) + dx_2(t) \end{cases}$$

The standard steps for solving such system:

1. Rewrite into matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A}\mathbf{x}$$

2. Use the $e^{\lambda t}$ trick, but this time we let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1e^{\lambda t} \\ k_2e^{\lambda t} \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$$

where k_1, k_2 are some constants. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \lambda \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$$

3. Substitute the above into the equation. Note that it has become an eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \lambda \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \cancel{e^{\lambda t}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \cancel{e^{\lambda t}} = \mathbf{A}\mathbf{x}$$

$$\lambda\mathbf{k} = \mathbf{A}\mathbf{k}$$

To be the solution, we require

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x} = \mathbf{k}e^{\lambda t} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t} \qquad \text{with} \qquad \begin{cases} \lambda \text{ to be } \mathbf{A}\text{ 's eigenvalue} \\ \mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \text{ to be } \mathbf{A}\text{ 's eigenvector} \end{cases}$$

4. After solving for λ and $\mathbf{k}$, the general solution is then the superposition of all its eigenvectors. In this system, a 2×2 matrix has at most 2 eigenvectors, so there are at most 2 terms in the general solution.

$$\mathbf{x} = \underset{\text{Constant 1}}{C_1} \underbrace{\begin{pmatrix} k_{11} \\ k_{12} \end{pmatrix}}_{\text{Eigenvector 1}} e^{\underset{\text{Eigenvalue 1}}{\lambda_1} t} + \underset{\text{Constant 2}}{C_2} \underbrace{\begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix}}_{\text{Eigenvector 2}} e^{\underset{\text{Eigenvalue 2}}{\lambda_2} t}$$

The C_1, C_2 shall be matched by initial conditions, e.g. knowing $\mathbf{x}(0)$ when $t = 0$.

Example 4.1.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x_1 = x_1 + x_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t} x_2 = 4x_1 + x_2 \end{cases} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We have already demonstrated with this matrix in the last section and found its eigenvalues and eigenvectors:

λ	3	-1
$\mathbf{x}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

So the general solution to the above system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

with C_1, C_2 being some constants to be matched with initial condition.

Exercise 4.1. Write down the general solution for this system. The eigenvalues and eigenvectors are already solved for you in the last section.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x_2 = x_1 + x_2 - 2x_3 \\ \dfrac{\mathrm{d}}{\mathrm{d}t} x_2 = x_2 - 4x_3 \\ \dfrac{\mathrm{d}}{\mathrm{d}t} x_3 = 2x_1 - x_3 \end{cases}$$

4.2 2^{nd} Order Linear Homogeneous with Constant Coefficient

For simplicity, let's deal with a system with 2 unknown functions only.

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} x_1(t) = a \dfrac{\mathrm{d}}{\mathrm{d}t} x_1(t) + b \dfrac{\mathrm{d}}{\mathrm{d}t} x_2(t) + c x_1(t) + d x_2(t) \\ \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} x_2(t) = p \dfrac{\mathrm{d}}{\mathrm{d}t} x_1(t) + q \dfrac{\mathrm{d}}{\mathrm{d}t} x_2(t) + r x_1(t) + s x_2(t) \end{cases}$$

If we naively write it in terms of some 2×2 matrices, like

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ p & q \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} c & d \\ r & s \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x} = \mathbf{A}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} + \mathbf{B}\mathbf{x}$$

Substitute $\mathbf{x} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$ does NOT gives you the form of eigenvalue problem.

$$\lambda^2 \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \cancel{e^{\lambda t}} = \lambda \begin{pmatrix} a & b \\ p & q \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \cancel{e^{\lambda t}} + \begin{pmatrix} c & d \\ r & s \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \cancel{e^{\lambda t}}$$

$$\lambda^2\mathbf{k} = \lambda\mathbf{A}\mathbf{k} + \mathbf{B}\mathbf{k} \longleftarrow \text{Don't know how to solve}$$

Instead, we can convert it to a 1^{st} order system by letting

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = u_1(t) \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}x_2 t = u_2(t)$$

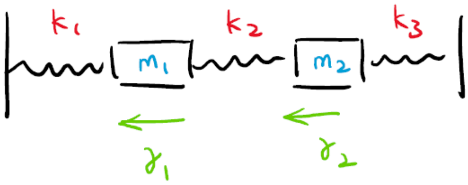
which then it reduces to a system of 1^{st} order ODEs with 4 unknowns.

$$\begin{array}{rl}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_1 = \frac{\mathrm{d}}{\mathrm{d}t}u_1 \longrightarrow & \left\{\begin{array}{l}
\frac{\mathrm{d}}{\mathrm{d}t}u_1(t) = au_1(t) + bu_2(t) + cx_1(t) + dx_2(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}u_2(t) = pu_1(t) + qu_2(t) + rx_1(t) + sx_2(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = u_1(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \qquad\qquad u_2(t)
\end{array}\right. \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_2 = \frac{\mathrm{d}}{\mathrm{d}t}u_2 \longrightarrow & \\
\text{Join with the definitions of } u_1, u_2 \longrightarrow &
\end{array}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ p & q & r & s \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix}$$

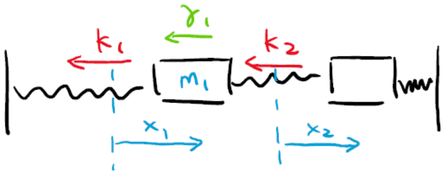
Now we can solve it as a eigenvalue problem. But solving by hand is a nightmare

4.3 Application: Coupled Spring-mass System



In this configuration, let the displacement of mass 1 $= x_1(t)$, displacement of mass 2 $= x_2(t)$.

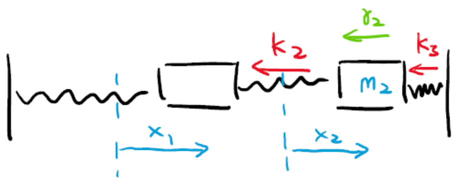
- <u>Newton's 2nd Law for m_1</u>:



  - Left spring (k_1) change in length $= |x_1|$
  - Right spring (k_2) change in length $= |x_2 - x_1|$

$$\therefore\ F = m_1 \frac{\mathrm{d}^2}{\mathrm{d}t^2} x_1 = -k_1 x_1 - k_2(x_1 - x_2) - \gamma_1 \frac{\mathrm{d}}{\mathrm{d}t} x_1$$

- <u>Newton's 2nd Law for m_2</u>:



  - Left spring (k_2) change in length $= |x_2 - x_1|$
  - Right spring (k_3) change in length $= |x_2|$

$$\therefore\ F = m_1 \frac{\mathrm{d}^2}{\mathrm{d}t^2} x_2 = -k_3 x_2 - k_2(x_2 - x_1) - \gamma_2 \frac{\mathrm{d}}{\mathrm{d}t} x_2$$

Group the 2 equations and transform it into a 1$^{\text{st}}$ order system:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u_1 = -\dfrac{\gamma_1}{m_1} u_1 - \dfrac{k_1 + k_2}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t} u_2 = -\dfrac{\gamma_2}{m_2} u_2 + \dfrac{k_2}{m_2} x_1 - \dfrac{k_2 + k_3}{m_2} x_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}t} x_1 = u_1 \\ \dfrac{\mathrm{d}}{\mathrm{d}t} x_2 = u_2 \end{cases}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{\gamma_1}{m_1} & 0 & -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ 0 & -\frac{\gamma_2}{m_2} & \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ x_1 \\ x_2 \end{pmatrix}$$

(Absolutely not recommend to waste time on solving it)

4.4 Special Case: Coupled Symmetric System without Damping

Here we only look into the results of a simplified case:

Let $k_1 = k_3,\ \gamma_1 = \gamma_2 = 0,\ m_1 = m_2$

The the Newton's 2$^{\text{nd}}$ Law reduce to

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}x_1 = -\dfrac{k_1+k_2}{m}x_1 + \dfrac{k_2}{m}x_2 \\ \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}x_2 = +\dfrac{k_2}{m}x_1 - \dfrac{k_1+k_2}{m}x_2 \end{cases}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{k_1+k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_1+k_2}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We do not have to break it into 4 equations. We can already subsitute $\begin{pmatrix} x1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\omega t}$:

$$\underset{\lambda}{\omega^2}\underbrace{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}}_{\mathbf{x}}\cancel{e^{\omega t}} = \underbrace{\begin{pmatrix} -\frac{k_1+k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_1+k_2}{m} \end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}}_{\mathbf{x}}\cancel{e^{\omega t}}$$

which is an eigenvalue problem to a 2×2 matrix, with eigenvalues $= \omega^2$. On solving we can find

ω^2	$\frac{k_1}{m}$	$-\frac{k_1+2k_2}{m}$
$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k_1}{m}}t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\sqrt{\frac{k_1}{m}}t} + C_3\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{k_1+2k_2}{m}}t} + C_4\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{\frac{k_1+2k_2}{m}}t}$$

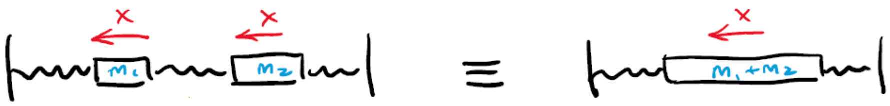
Physical Interpretation

The general solution is a superposition of 2 kinds of vibration of their own frequencies. (i.e. 2 modes of vibrations).

- Angular frequency 1: $\sqrt{\frac{k_1}{m}}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k_1}{m}}t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\sqrt{\frac{k_1}{m}}t} = \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos\left(\sqrt{\frac{k_1}{m}}t + \phi_1\right)}$$

In this pattern of vibration, $x_1 = x_2 = A\cos\left(\sqrt{\frac{k_1}{m}}t + \phi\right)$

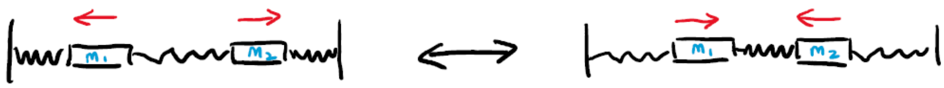


m_1, m_2 always move together $\Rightarrow$ Call this the "in phase" mode.

- Angular frequency 2: $\sqrt{\frac{k_1+2k_2}{m}}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{k_1+2k_2}{m}}t} + C_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{\frac{k_1+2k_2}{m}}t} = \boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 \cos\left(\sqrt{\frac{k_1+2k_2}{m}}t + \phi_2\right)}$$

In this pattern of vibration, $x_1 = -x_2 = A\cos\left(\sqrt{\frac{k_1+2k_2}{m}}t + \phi\right)$



m_1, m_2 always move in opposite direction $\Rightarrow$ Call this the "out of phase" mode.

— The End —