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Magnetostatics

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Overview:

- Basic problems: Find $\vec{\boldsymbol{B}}$ by Biot-Savat law with integration
- Divergent-less of B-field
- Ampere's law, dot product line integral, curl & Stokes' theorem
- *(Optional reading)* Magnetic vector potential & Poisson equation

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In electromagnetism, theoretically every problem can be solved through a set of PDEs called the **Maxwell Equations**.

$$\begin{aligned}
\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} &= \frac{\rho}{\epsilon_0} & &\longrightarrow & \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{B}} &= 0 \\
\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{E}} &= -\frac{\partial\vec{\boldsymbol{B}}}{\partial t} & &\longrightarrow & \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} &= \mu_0\vec{\boldsymbol{J}} + \mu_0\epsilon_0\frac{\partial\vec{\boldsymbol{E}}}{\partial t}
\end{aligned}$$

However, a *system of PDEs* is too complicated to be solved. So we need to learn different "tricks" to avoid them, which are enough for some simple scenarios.

Magnetostatics only concerns the 2^{nd} and 4^{th} equation of the set - Gauss's law on B-field and Ampere's law.

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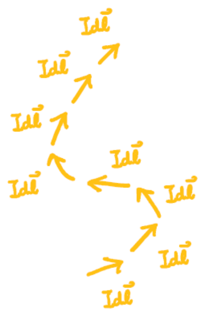
1 Basic Skill: Biot-Savat Law with Integration

Physically, currents are just moving charges. There is no such thing called "point current". However we can imagine a line of current being divided into many infinitestimally small segments such that each current segment "looks like" a vector point source.

$$\begin{pmatrix}\text{Current}\\\text{line}\end{pmatrix} = I\vec{\boldsymbol{L}} \quad\Longrightarrow\quad \int I\,\mathrm{d}\vec{\boldsymbol{l}} \sim \sum(\text{Current})\begin{pmatrix}\text{Unit}\\\text{length}\end{pmatrix}$$

$\Rightarrow$



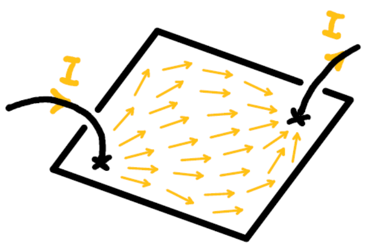
In reality, current sources must form a continuous line and cannot suddenly appear / disappear at nowhere. This is why Biot-Savat law must be written as an integral and so never be found in your high school textbooks.

$$\underbrace{\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\frac{I\vec{\boldsymbol{L}}}{r^2}\times\hat{\boldsymbol{r}}}_{\text{Always remember that You should not write this}} \qquad \Longrightarrow \qquad \underbrace{\int \mathrm{d}\vec{\boldsymbol{B}} = \int_{\substack{\text{whole}\\\text{line}}} \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{\boldsymbol{l}}}{r^2}\times\hat{\boldsymbol{r}}}_{\text{Biot-Savat law can only be written as an integral}}$$

Furthermore, because we are living in a 3D world, current does not always travel along a line segment, but may flow on a surface or through an object such that the current is position dependent. In these cases, we should describe current as a distribution of flow (i.e. vector field).

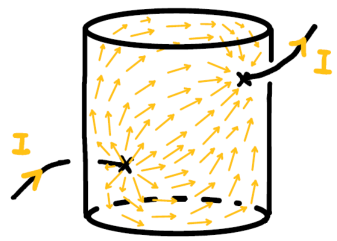
When current is travelling on a surface, the current flow is described as a 2D vector field called **surface current density $\vec{\boldsymbol{K}}$**.

$$I\,\mathrm{d}\vec{\boldsymbol{l}} \quad \Rightarrow \quad \vec{\boldsymbol{K}}\,\mathrm{d}s$$



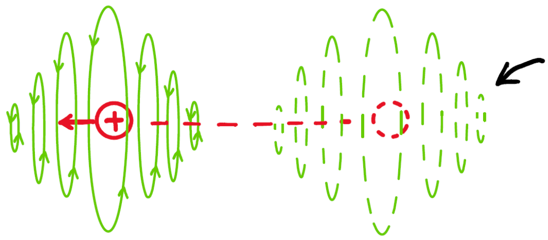
When current is travelling in a volume, the current flow is described as a 3D vector field called **volume current density $\vec{\boldsymbol{J}}$**.

$$I\,\mathrm{d}\vec{\boldsymbol{l}} \quad \Longrightarrow \quad \vec{\boldsymbol{J}}\,\mathrm{d}\tau$$



Caution:

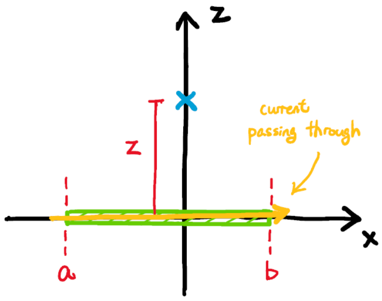
In magnetostatics problem, we require the current flow to be independent of time. Although you may have learnt that a moving point charge q traveling at velocity v acts like a point current source qv, we do not consider this as a static case because this current is time-varying.

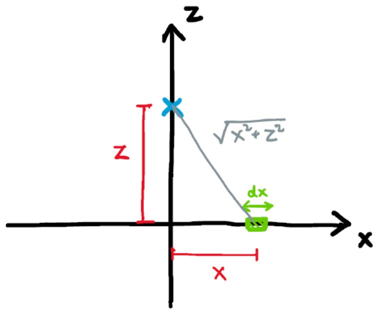


The B-field here is created **only for a short duration**. This B-field is NOT static. And neither is the current created by point charge.

Moreover, the true formula of B-field by point charge needs to consider the travelling time of B-field (light speed). But this is completely out of our scope.

Example 1.1. Suppose there is a wire lying on the x-axis, with its ends at $x = a$ and $x = b$. Let there be uniform current I flowing along it. What is the B-field on an arbituary point on the z axis?





We can analyze by dividing the rod into infinitestimal pieces:

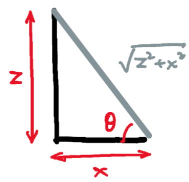
- Each segment has a length $\mathrm{d}x$
- Each unit of current segment is thus $I\,\mathrm{d}\vec{\boldsymbol{x}} = I\,\mathrm{d}x\,\hat{\boldsymbol{x}}$.
- For the segment at position x, its distance from the targeted point is $\sqrt{z^2+x^2}$.

Thus we can calculate $\vec{\boldsymbol{B}}$ by Biot-Savat law.

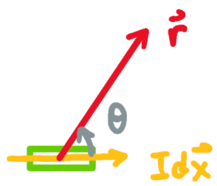
$$\vec{\boldsymbol{B}} = \frac{\mu_0}{4\pi}\int_a^b \frac{I\,\mathrm{d}x\,\hat{\boldsymbol{x}}}{z^2+x^2} \times \left(\substack{\text{directon to}\\ \text{target point}}\right)$$

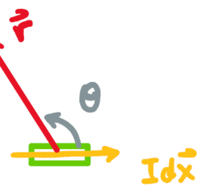
To do the cross product, we need to resolve the direction's component from the segment to the target point.

$$\hat{\boldsymbol{r}} = \frac{x}{\sqrt{z^2+x^2}}\hat{\boldsymbol{x}} + \frac{z}{\sqrt{z^2+x^2}}\hat{\boldsymbol{z}}$$
$$\Rightarrow \quad \hat{\boldsymbol{x}} \times \hat{\boldsymbol{r}} = \frac{z}{\sqrt{z^2+x^2}}(-\hat{\boldsymbol{y}})$$



In this situation, $-\hat{\boldsymbol{y}}$ is the out of paper direction.





You can also deduce by cross product $\vec{\boldsymbol{B}} \sim I\,\mathrm{d}\vec{\boldsymbol{x}} \times \hat{\boldsymbol{r}}$ - line elements on both sides of the rod create a B-field in out-of-paper $(-\hat{\boldsymbol{y}})$direction.

So the result B-field should be integrated as

$$B_y = -\frac{\mu_0}{4\pi}\int_a^b \frac{I\,\mathrm{d}x}{z^2+x^2}\frac{z}{\sqrt{z^2+x^2}}$$

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2 Divergent-less of B-field

Similar to E-field, there is the **Gauss's law for B-field**, which has two different expressions:

$$\oiint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = 0 \qquad \text{(Integral form)}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = 0 \qquad \text{(Differential form)}$$

This law is purely an observation to B-field, claiming that

$$\boxed{\begin{array}{c} \text{Magnetic point source} \\ \text{does not exist} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \text{Total flux of B-field} \\ \text{on a closed surface always} = 0 \end{array}}$$

Because so far no one has found any magnetic monopoles, we determine that B-field lines must exist as closed loops, and can never form diverging/converging patterns like E-field does.

This law is not as important as the other 3 in the Maxwell equation because it does not involve any source terms. It is only sometimes useful when we need to make symmetry claims or simplify derivations.

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3 Ampere's Law

The Ampere's Law (in magnetostatics) has two different expressions:

$$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I \qquad \text{(Integral form)}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}} \qquad \text{(Differential form)}$$

It is easier to study the physical meaning and visualize by the integral form. After that we can generalize to the differential form by introducing an operator called **curl**.

3.1 Revisit: Dot Product Line Integral

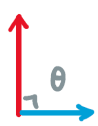
The literal description in Ampere's law integral form is

$$\begin{pmatrix} \text{Dot product line integral} \\ \text{of magnetic field along a loop} \end{pmatrix} = \oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I = \text{(Constant)(Current enclosed)}$$

Recall that we can use the sign of a dot product between 2 vectors to determine if the vectors are in similar / opposite directions.

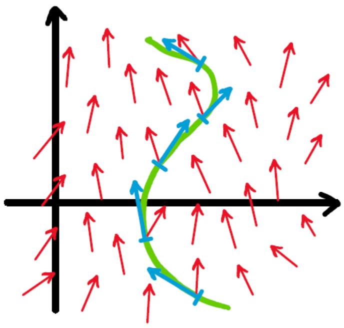
If $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} > 0$,
$\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are more or less in similar directions.

If $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} < 0$,
$\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are more or less in opposite directions.

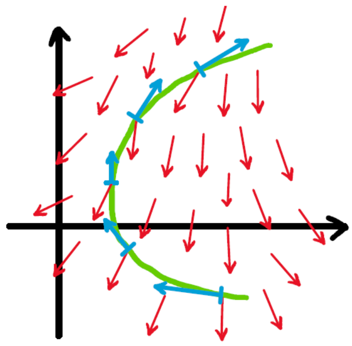


If $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = 0$,
$\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are perpendicular to each other.

Now consider that we are travelling in a vector field along some path. At each step, we can take note of field vector there and our travelling direction, then compute their dot product.



If the total of all the dot products > 0, we are travelling around the same direction as the field's flow.



If the total of all the dot products < 0, we are travelling around the opposite direction as the field's flow.

If we divide our path into infinitestimal small segments, then the sum become line integral.

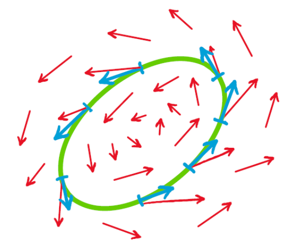
$$\int_{\text{path}} \vec{\boldsymbol{F}} \cdot \mathrm{d}l \quad \begin{cases} > 0 & \Rightarrow \quad \sim \text{Our path is along the flow} \\ < 0 & \Rightarrow \quad \sim \text{Our path is opposite to the flow} \end{cases}$$

3.2 Detection of Rotating Flow

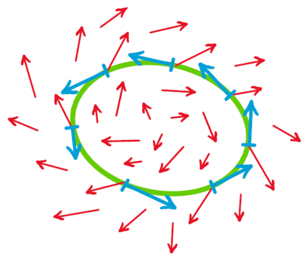
The situation becomes interesting if we choose our path to be a closed loop - this loop integral becomes an indicator whether there are rotation trends around the loop.

By convention, we always use an anti-clockwise loop (Just like right hand rule).

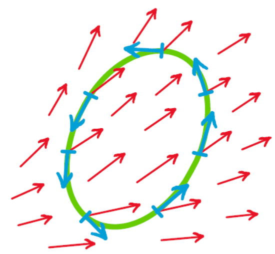
If $\oint_{\substack{\text{our}\\\text{loop}}} \vec{\boldsymbol{F}} \cdot \mathrm{d}l > 0$, the loop may have enclosed some rotation centers of anti-clockwise flow.



If $\oint_{\substack{\text{our}\\\text{loop}}} \vec{\boldsymbol{F}} \cdot \mathrm{d}l < 0$, the loop may have enclosed some rotation centers of clockwise flow.

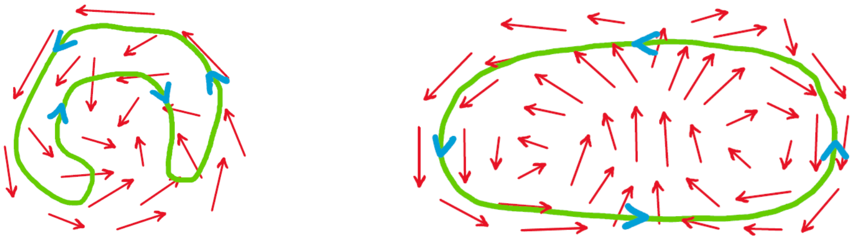


If $\oint_{\substack{\text{our}\\\text{loop}}} \vec{\boldsymbol{F}} \cdot \mathrm{d}l \approx 0$, the loop probably does not enclose any rotation centers.



3.3 Curl

However there is a problem in using loop integral of dot product - if we choose the loop too arbituarily, the calculated dot product is ambiguous to tell where the rotation centers are.



E.g. If we choose an irregular loop, it may not catch the positions of rotation centers.

To tackle this problem, we need to introduce the **curl** operator:

$$\underline{\vec{\nabla}\times}\bullet \overset{\text{def}}{=} \left(\frac{\partial \bullet_z}{\partial y}-\frac{\partial \bullet_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial \bullet_x}{\partial z}-\frac{\partial \bullet_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial \bullet_y}{\partial x}-\frac{\partial \bullet_x}{\partial y}\right)\hat{\boldsymbol{z}} \overset{\text{def}}{=} \underline{\text{curl}}\bullet$$

Like gradient operator but with a cross

Sometimes we just write "curl"

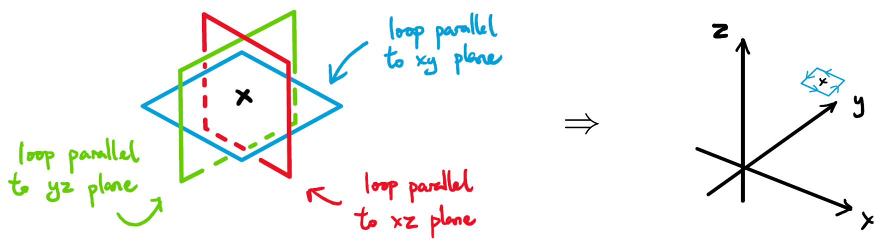
The curl operator can be applied on a vector function, and it returns another vector function.

$$\begin{aligned}
\vec{\nabla}\times F &= \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\times\begin{pmatrix}F_x & F_y & F_z\end{pmatrix}\\
&= \begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}\\
&= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}}\\
&= (\text{A vector})
\end{aligned}$$

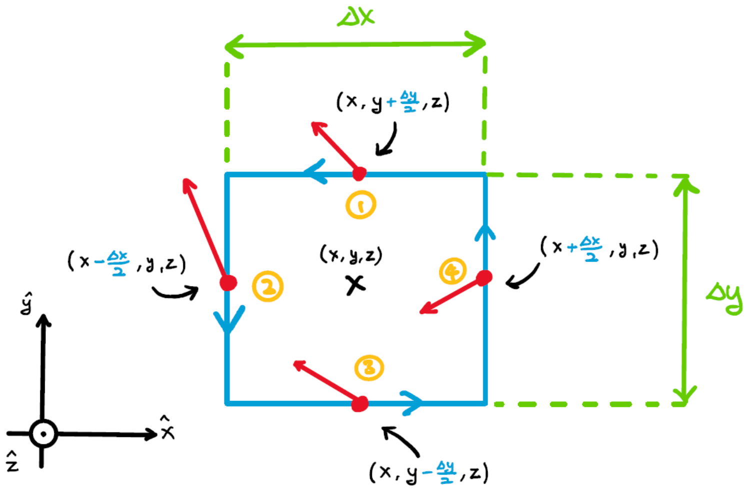
Each component of the curl of a vector field is related to its **loop integral along an infinitestimal small loop in each direction.**

3.3.1 Geometrical Interpretation

To visualize, we can draw 3 infinitestimal small loop around a point (x, y, z). Because of symmetry in all 3 directions, it suffices to just analyze the loop that is parallel to the x-y plane.

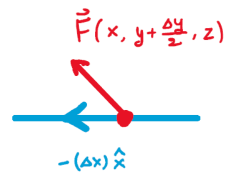


The dot product on each side of the loop are calculated as follow:



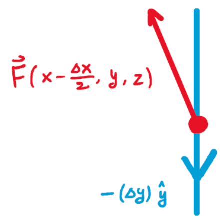
- Edge 1 :
  - Vector field on the edge center $= \vec{\boldsymbol{F}}\left(x, y + \frac{\Delta y}{2}, z\right)$
  - Edge length $= \Delta x$, in $-x$ direction

$\Rightarrow$ (Dot product) $= \vec{\boldsymbol{F}}\left(x, y + \frac{\Delta y}{2}, z\right) \cdot (\Delta x)(-\hat{\boldsymbol{x}}) = -F_{\underline{x}}\left(x, y + \frac{\Delta y}{2}, z\right)\Delta x$



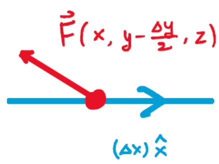
- Edge 2 :
  - Vector field on the edge center $= \vec{\boldsymbol{F}}\left(x - \frac{\Delta x}{2}, y, z\right)$
  - Edge length $= \Delta y$, in $-y$ direction

$\Rightarrow$ (Dot product) $= \vec{\boldsymbol{F}}\left(x - \frac{\Delta x}{2}, y, z\right) \cdot (\Delta y)(-\hat{\boldsymbol{y}}) = -F_{\underline{y}}\left(x - \frac{\Delta x}{2}, y, z\right)\Delta y$



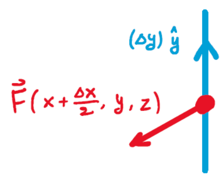
- Edge 3 :
  - Vector field on the edge center $= \vec{\boldsymbol{F}}\left(x, y - \frac{\Delta y}{2}, z\right)$
  - Edge length $= \Delta x$, in $+x$ direction

$\Rightarrow$ (Dot product) $= \vec{\boldsymbol{F}}\left(x, y - \frac{\Delta}{2}, z\right) \cdot (\Delta x)(+\hat{\boldsymbol{x}}) = F_{\underline{x}}\left(x, y - \frac{\Delta y}{2}, z\right)\Delta x$



- Edge 4 :
  - Vector field on the edge center $= \vec{\boldsymbol{F}}\left(x + \frac{\Delta x}{2}, y, z\right)$
  - Edge length $= \Delta y$, in $+y$ direction

$\Rightarrow$ (Dot product) $= \vec{\boldsymbol{F}}\left(x + \frac{\Delta x}{2}, y, z\right) \cdot (\Delta y)(+\hat{\boldsymbol{y}}) = F_{\underline{y}}\left(x + \frac{\Delta x}{2}, y, z\right)\Delta y$



Therefore the total dot product along the loop is

$$\underbrace{F_y\left(x+\tfrac{\Delta x}{2},y,z\right)\Delta y}_{\text{Edge 4}}-\underbrace{F_y\left(x-\tfrac{\Delta x}{2},y,z\right)\Delta y}_{\text{Edge 2}}-\underbrace{F_x\left(x,y+\tfrac{\Delta y}{2},z\right)\Delta x}_{\text{Edge 1}}+\underbrace{F_x\left(x,y-\tfrac{\Delta y}{2},z\right)\Delta x}_{\text{Edge 3}}$$

$$=\left(\frac{F_x\left(x+\frac{\Delta x}{2},y,z\right)-F_x\left(x-\frac{\Delta x}{2},y,z\right)}{\Delta x}-\frac{F_x\left(x,y+\frac{\Delta y}{2},z\right)-F_x\left(x,y-\frac{\Delta y}{2},z\right)}{\Delta y}\right)(\Delta x\Delta y)$$

$$=\left(\underbrace{\boxed{\frac{F_x\left(\boxed{x+\frac{\Delta x}{2}},y,z\right)-F_x\left(\boxed{x-\frac{\Delta x}{2}},y,z\right)}{\boxed{\Delta x}}}}_{\text{This is exactly partial x}}-\underbrace{\boxed{\frac{F_x\left(x,\boxed{y+\frac{\Delta y}{2}},z\right)-F_x\left(x,\boxed{y-\frac{\Delta y}{2}},z\right)}{\boxed{\Delta y}}}}_{\text{This is exactly partial y}}\right)(\Delta x\Delta y)$$

$$=\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)(\Delta x\Delta y)$$

$$=\begin{pmatrix}\text{Curl's}\\ z\text{ component}\end{pmatrix}\begin{pmatrix}\text{Unit area}\\ \text{parallel to xy plane}\end{pmatrix}$$

$$=\begin{pmatrix}\text{Curl's}\\ z\text{ component}\end{pmatrix}\begin{pmatrix}\text{Unit area}\\ \text{normal to z direction}\end{pmatrix}$$

We can expect the similar results in the other 2 directions. Gather them together:

$$\begin{cases}\begin{pmatrix}\text{Loop integral}\\ \text{normal to}\\ x\text{ direction}\end{pmatrix}=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)(\mathrm{d}y\,\mathrm{d}z)=(\vec{\nabla}\times\vec{F})_x(\mathrm{d}y\,\mathrm{d}z)=\begin{pmatrix}\text{Curl's}\\ x\text{ component}\end{pmatrix}\begin{pmatrix}\text{Unit area}\\ \text{normal to}\\ x\text{ direction}\end{pmatrix}\\ \begin{pmatrix}\text{Loop integral}\\ \text{normal to}\\ y\text{ direction}\end{pmatrix}=\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)(\mathrm{d}z\,\mathrm{d}x)=(\vec{\nabla}\times\vec{F})_y(\mathrm{d}z\,\mathrm{d}x)=\begin{pmatrix}\text{Curl's}\\ y\text{ component}\end{pmatrix}\begin{pmatrix}\text{Unit area}\\ \text{normal to}\\ y\text{ direction}\end{pmatrix}\\ \begin{pmatrix}\text{Loop integral}\\ \text{normal to}\\ z\text{ direction}\end{pmatrix}=\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)(\mathrm{d}x\,\mathrm{d}y)=(\vec{\nabla}\times\vec{F})_z(\mathrm{d}x\,\mathrm{d}y)=\begin{pmatrix}\text{Curl's}\\ z\text{ component}\end{pmatrix}\begin{pmatrix}\text{Unit area}\\ \text{normal to}\\ z\text{ direction}\end{pmatrix}\end{cases}$$

Therefore we can geometrically interpret curl as

$$\boxed{\begin{pmatrix}\textbf{Curl's}\\ \mathbf{i^{th}}\textbf{ component}\end{pmatrix}=(\vec{\nabla}\times\vec{F})_i=\frac{(\text{Loop integral normal to i}^{\text{th}}\text{ direction})}{(\text{Area enclosed by the loop})}\sim\begin{pmatrix}\textbf{loop integral}\\ \underbrace{\textbf{density}}_{\substack{\text{This density}\\ \text{is by area}}}\end{pmatrix}_{\substack{\mathbf{i}\\ \text{In i}^{\text{th}}\\ \text{direction}}}}$$

3.3.2 Stokes' Theorem

With the geometrical interpretation, we can directly state (without proof) a convenient formula related to curl - the **Stokes' theorem**:

$$\boxed{\oint\vec{F}\cdot\mathrm{d}\vec{l}=\iint(\vec{\nabla}\times\vec{F})\cdot\mathrm{d}\vec{s}}$$

which is basically

$$\begin{pmatrix}\text{Total}\\ \text{Loop integral}\end{pmatrix} \sim \sum_{\text{All area}} \begin{pmatrix}\text{Loop integral}\\ \text{per area}\end{pmatrix} \times (\text{Area})$$

3.4 Ampere's Law - Explanation

The Ampere's law is purely an observation about the relation between B-field and currents:

$$\boxed{\begin{matrix}\text{Total line integral of B-field}\\ \text{along a closed loop} \neq 0\end{matrix} \quad \Leftrightarrow \quad \begin{matrix}\text{There are currents}\\ \text{circled by the loop}\end{matrix}}$$

The two forms of Ampere's law are describing this same observation:

- Integral form:

$$(\vec{\boldsymbol{B}}\text{'s loop integral}) \sim \oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I \sim (\text{Current})$$

- Differential form:

$$\begin{pmatrix}\vec{\boldsymbol{B}}\text{'s loop integral}\\ \text{density}\end{pmatrix} \sim \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}} \sim \begin{pmatrix}\text{Current}\\ \text{density}\end{pmatrix}$$

And the two form can be inter-converted by Stokes' theorem.

$$\begin{array}{ccc}\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} & = & \mu_0 I \\ \Big\downarrow \text{Stokes' Theorem} & & \Big\downarrow \text{Current to Current density} \\ \iiint (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{s}} & = & \mu_0 \iint \vec{\boldsymbol{J}} \cdot \mathrm{d}\vec{\boldsymbol{s}}\end{array}$$

3.5 Applying Ampere's Law Integral Form

In beginner electromagnetism, there is only one type of problems related to Ampere's law:

Given the current distribution, find the B-field everywhere by Ampere's law integral form in some very symmetrical scenarios.

which is basically asking you to *revert* the loop integral calculation:

$$\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I \qquad \Rightarrow \qquad \vec{\boldsymbol{B}} = \text{Some function of } I$$

If I has a very ugly distribution, there is nothing we can do other than solving some PDEs. But if I distributes very symmetrically, $\vec{\boldsymbol{B}}$ should also be symmetrical, such that the loop integral can be broken into multiplications.

In these cases, we can choose an "Amperian" loop to to be integrated where

1. $\vec{\boldsymbol{B}}$ has constant magnitude everywhere along the loop.
2. $\vec{\boldsymbol{B}}$ forms the same angle with the tangent vector everywhere along the loop.

Only then, the loop integral can be broken down as

$$\oint \vec{\boldsymbol{B}}\cdot \mathrm{d}\vec{\boldsymbol{l}} = \oint \underline{|\vec{\boldsymbol{B}}||\,\mathrm{d}\vec{\boldsymbol{l}}\,|\cos\theta} \longleftarrow \text{Just dot product } \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\cos\theta$$

$$= \underline{|\vec{\boldsymbol{B}}|}\ \underline{\cos\theta}\ \oint |\,\mathrm{d}\vec{\boldsymbol{l}}\,|$$

Same magnitude everywhere
Can move out of integral!

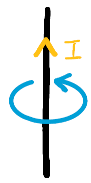
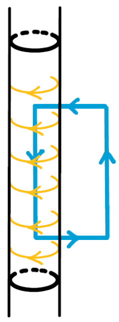
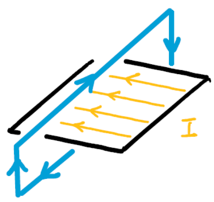
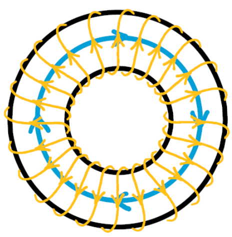
Form same angle everywhere
Can move out of integral!

$$= |\vec{\boldsymbol{B}}|\ \cos\theta\ (\text{Perimeter of loop})$$

such that we can find the magnitude of $\vec{\boldsymbol{B}}$ with simple division

$$\boxed{|\vec{\boldsymbol{B}}| = \frac{(\text{Total loop integral})}{(\text{Perimeter of loop})\cos\theta} = \frac{\mu_0 I}{(\text{Perimeter of loop})\cos\theta}}$$

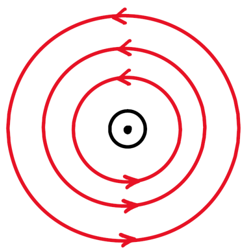
In fact, there are not many of these "very symmetrical" cases. These examples below, with their respective Amperian loops, are basically all the variations you can find in textbooks.

Current configuration (Assuming uniform density)		Good Amperian loop
Infinitely long wire		Circle
Infinitely long solenoid		Rectangle at cross-section
Infinitely large plane		Infinitely large rectangle wrapping it around
Toroidal solenoid		Circle through center of the core

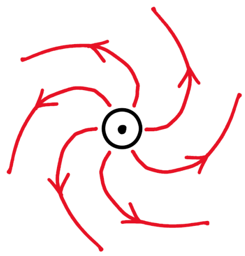
Example 3.1. Given an infinitely long wire with current I flowing along. By cylindrical symmetry, the B-field must satisfy:

- Unchange when translate along the wire (z-axis).
- Unchange when rotate about the wire (z-axis).

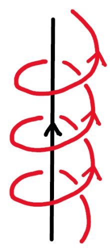
Note that cylindrical symmetry does not restrict the B-field to be circle loops. For example, all of these configurations satisfy cylindrical symmetry:



If $\vec{\boldsymbol{B}}$ only has $\hat{\boldsymbol{\theta}}$ component
$\Rightarrow$ Circular loops



If $\vec{\boldsymbol{B}}$ has $\hat{\boldsymbol{r}}$ component
$\Rightarrow$ Can spiral out

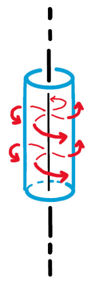


If $\vec{\boldsymbol{B}}$ has $\hat{\boldsymbol{z}}$ component
$\Rightarrow$ Can spiral up

We need to rule out the 2 latter cases before we can use Ampere's law to find $\vec{\boldsymbol{B}}$:

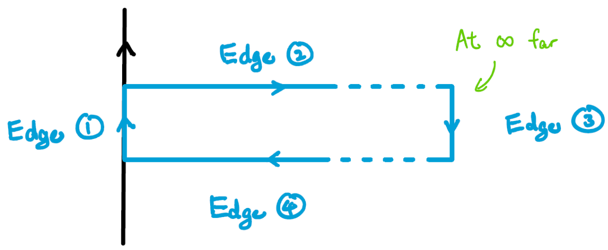
1. $\vec{\boldsymbol{B}}$ is divergent-less :

   We can draw a cylindrical Gaussian box around the wire.
   If the B-field has $\hat{\boldsymbol{r}}$ component, we would get the flux of $\vec{\boldsymbol{B}} \neq 0$.
   Then it contradicts to $\vec{\boldsymbol{B}}$ being divergent-less.



2. Another Amperian loop :

   We can draw an Amperian loop that is parallel to the wire. By Ampere's law, there is no current enclosed by this loop, so the loop integral should be 0.

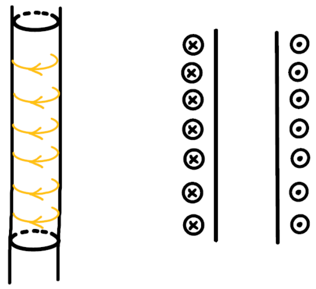


- By translational symmetry, the dot product at the top (edge 2) and bottom (edge 4) should cancel each other.
- First choose the loop which stretches to infinitly far. Because there should be no B-field at infinity, dot product on edge 3 should be 0.
- Then if we choose a loop which does not stretch to infinitly far, we can conclude that the dot product on edge 1 is 0. So this $\vec{\boldsymbol{B}}$ must not have $\hat{\boldsymbol{z}}$ component.

After all these symmetry arguments, we can finally conclude that B-field around a wire must be circular loops. We can choose a circular Amperian loop of radius r to find the magnitude of B-field at distance r from the wire.

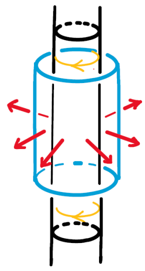
$$
\begin{aligned}
|\vec{\boldsymbol{B}}| &= \frac{\mu_0 I}{(\text{Perimeter of loop})\cos\theta} \\
&= \mu_0 I \cdot \frac{1}{(2\pi r)} \cdot \frac{1}{\cos \underline{0^\circ}} \longleftarrow \begin{array}{l}\text{B-field = angular} \\ \therefore \text{ Tangent to loop}\end{array} \\
&= \frac{\mu_0 I}{2\pi r} \\
\Rightarrow \quad \vec{\boldsymbol{B}} &= \frac{\mu_0 I}{2\pi r}\underline{\hat{\boldsymbol{\theta}}} \longleftarrow \text{You have to manually add the unit vector}
\end{aligned}
$$

Example 3.2. Given an infinitely long solenoid made of a uniform density of coils $\frac{N}{L}$, with a current I passiing through. This is another cylindrical symmetry case.



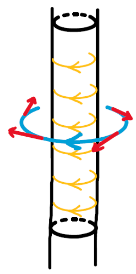
- Unchange when translate along the wire (z-axis).
- Unchange when rotate about the wire (z-axis).

Again, cylindrical symmetry does not restrict the B-field to be purely along z direction. We need to first argue that the B-field does not have radial or angular components before we can apply Ampere's law.



1. $\vec{\boldsymbol{B}}$ is divergent-less :

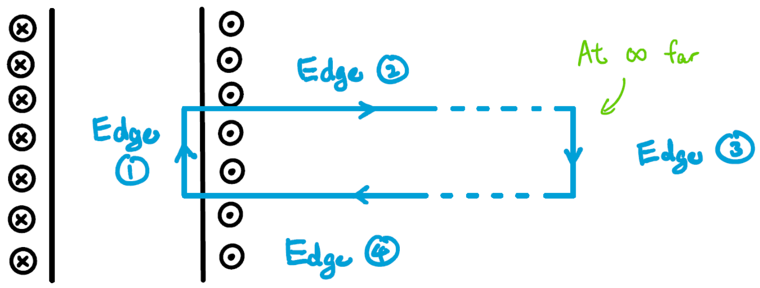
   We can draw a cylindrical Gaussian box around the solenoid.
   If the B-field has $\hat{\boldsymbol{r}}$ component, we would get the flux of $\vec{\boldsymbol{B}} \neq 0$.
   Then it contradicts to $\vec{\boldsymbol{B}}$ being divergent-less.



2. Another Amperian loop :

   We can draw a loop that circulates around the solenoid.
   Because there is no current enclosed by this loop,
   dot product integral around this loop must be 0.
   By Ampere's law, this B-field must not have $\hat{\boldsymbol{\theta}}$ component.

Now we can conclude that B-field created by a solenoid must be in z direction only. We can choose a circular Amperian loop over the cross-section of the solenoid, with one edge in the center of solenoid and another edge at infinitely far.



– We have already argued that this B-field has no radial component. Dot product on edge 2 and 4 must be 0.

– At infinitely far, there should be no B-field at infinity, dot product on edge 3 should be 0.

So only edge 1 can have a non-zero dot product. We can now apply Ampere's law:

$$\underbrace{|\vec{\boldsymbol{B}}|L\cos 0^\circ}_{\text{Edge 1}} + \underbrace{0+0+0}_{\text{Edge 2 \& 3 \& 4}} = \mu_0(NI)$$

$$|\vec{\boldsymbol{B}}| = \frac{\mu_0 NI}{L}$$

$$\vec{\boldsymbol{B}} = \frac{\mu_0 NI}{L}\underline{\hat{\boldsymbol{z}}} \longleftarrow \text{You have to manually add the unit vector}$$

---

4 Magnetic Vector Potential

4.1 Mathematical Origin

Magnetic field also has a corresponding potential function $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$. But unlike electric potential, the magnetic potential is a vector function. The reason to create it is purely mathematical:

– Observation: B-field always form loops because we have never observed any magnetic point source. $\Rightarrow$ B-field is divergent-less.

– Mathematical fact: Any divergent-less field can be expressed as the curl of some vector function (i.e. vector potential).

Therefore we can define a vector function $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$, called **magnetic vector potential**, such that

$$\boxed{\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})}$$

Unlike electric potential, there is no reverse formula for reversely computing $\vec{\boldsymbol{A}}$ from $\vec{\boldsymbol{B}}$.

4.2 Poisson Equation

If we substitute $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$ into the Ampere's law differential form, we arrive at a new equation:

$$
\begin{aligned}
\mu_0 \vec{\boldsymbol{J}} &= \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} \\
&= \vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}\right) \\
&= \vec{\boldsymbol{\nabla}} \underbrace{(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}})} - \nabla^2 \vec{\boldsymbol{A}} \\
&= -\nabla^2 \vec{\boldsymbol{A}}
\end{aligned}
$$

$\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}\right) \equiv \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}\right) - \nabla^2 \vec{\boldsymbol{F}}$
is a vector calculus identity
commonly used in derivations.

We can CHOOSE $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = 0$.
This freedom is similar to electric potential
where we can CHOOSE to add any constant to V.
The formal reason is called "gauge freedom".

$$
\boxed{\nabla^2 \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = -\mu_0 \vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})}
$$

This is again the **Poisson equation**, the same PDE that we have encountered in electric potential ($\nabla^2 V(\vec{\boldsymbol{r}}) = -\frac{\rho(\vec{\boldsymbol{r}})}{\epsilon_0}$). But for magnetic potential, because it is a vector function, this is actually 3 equations, one for each direction:

$$
\nabla^2 \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = -\mu_0 \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}
$$

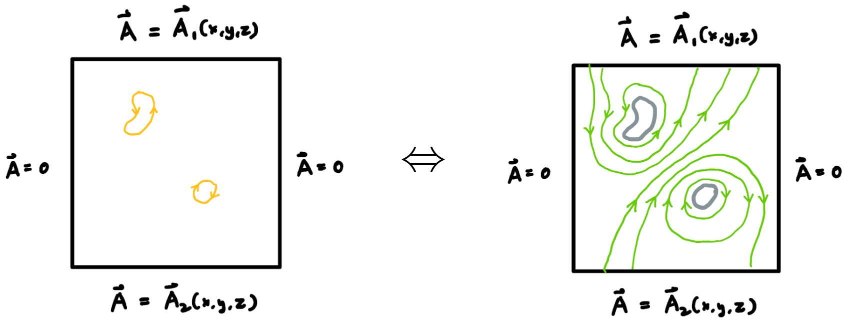
It is the most fundamental relation between magnetic vector potential and current. Given any configurations of current or vector potential, idealy we can find the other using this PDE.

- $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$ to $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})$: The Laplacian operator is just a sum of 2nd order derivatives. (And we just need to do it 3 times). Relatively easy.

- $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})$ to $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$: Need to solve the Poisson equation (for 3 times), which are 2nd order non-homogeneous linear PDEs. Awful!

Unfortunately in realistic problems, it is more frequent to ask for $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$ from $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})$, because we can usually confine the current distribution in a small region by using very small test objects; but for vector potential, it is always everywhere.

It is easy to
make $\vec{\boldsymbol{J}} = 0$
in most places



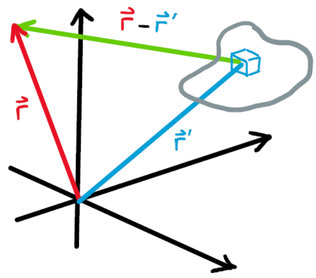
It is difficult
to know $\vec{\boldsymbol{A}}$
everywhere

Again we are not going to discuss how to solve it. Here I only provide you the solution in one very special case - When the region of interest is infinitely large + $\vec{\boldsymbol{A}}$ is chosen to be divergent-less, i.e. $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{A}} = 0$, the solution is the Biot-Savat law for magnetic vector potential.

$$\boxed{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \underset{\substack{\text{infinitely large} \\ \text{space}}}{\iiint} \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \mathrm{d}^3\vec{\boldsymbol{r}}'}$$

$$\sim \frac{\mu_0}{4\pi} \sum_{\text{everywhere}} \frac{(\text{current})}{(\text{distance})}$$

$$\equiv \underset{\text{(written in a fancy vector form)}}{\text{Biot-Savat law for magnetic vector potential}}$$



4.3 Finding $\vec{B}$ from $\vec{J}$

On the other hand, Poisson equation provides an alternative to calculate B-field distribution from current distribution. If we compare the Ampere's law and Poisson equation of $\vec{\boldsymbol{A}}$:

- Poisson equation :

  Although $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$ is a vector function, the Poisson equations for each compoenent $A_x(\vec{\boldsymbol{r}})$, $A_y(\vec{\boldsymbol{r}})$, $A_z(\vec{\boldsymbol{r}})$ are independent.

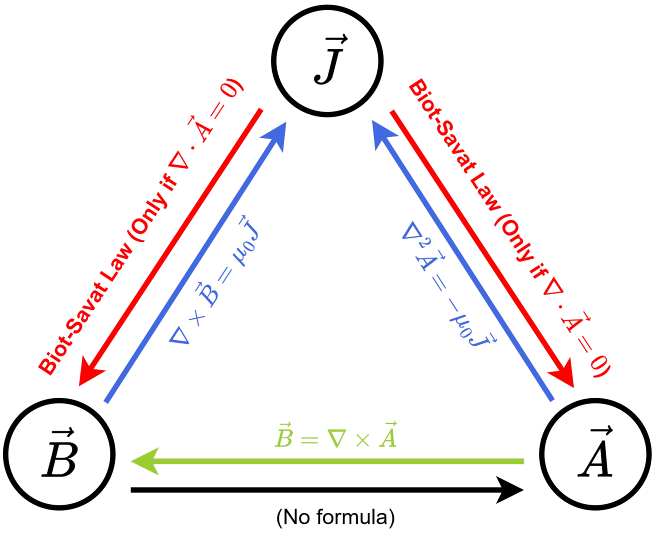
$$\nabla^2 \vec{\boldsymbol{A}} = -\mu_0 \vec{\boldsymbol{J}} \quad \Leftrightarrow \quad \begin{cases} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A_{\underline{x}} = -\mu_0 J_{\underline{x}} & \longleftarrow \text{Only involve x components} \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A_{\underline{y}} = -\mu_0 J_{\underline{y}} & \longleftarrow \text{Only involve y components} \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A_{\underline{z}} = -\mu_0 J_{\underline{z}} & \longleftarrow \text{Only involve z components} \end{cases}$$

- Ampere's law :

  $\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}})$ is a vector function with 3 components $B_x(\vec{\boldsymbol{r}})$, $B_y(\vec{\boldsymbol{r}})$, $B_z(\vec{\boldsymbol{r}})$, Because of the curl operation, the PDE for each component of $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{J}}$ all mix together.

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}} \quad \Leftrightarrow \quad \begin{cases} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 J_y \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_z}{\partial y} = \mu_0 J_z \end{cases}$$

In practice, there is no reason to try to directly solve the more difficult PDEs of $\vec{\boldsymbol{B}}$, if we can alternatively solve the easier PDEs of $\vec{\boldsymbol{A}}$, and then take curl to get $\vec{\boldsymbol{B}}$ (i.e. via $\vec{\boldsymbol{B}} = -\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$).



In this way, we can tell the solution of Ampere's law as a PDE, which is as expected, the Biot-Savat law for $\vec{\boldsymbol{B}}$.

$$\boxed{\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \underset{\substack{\text{infinitely large}\\ \text{space}}}{\iiint} \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^2} \times \left[\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right] \mathrm{d}^3\vec{\boldsymbol{r}}'}$$

$$\sim \frac{\mu_0}{4\pi} \sum_{\text{everywhere}} \frac{(\text{current})}{(\text{distance})^2} \times \left(\substack{\text{unit}\\ \text{vector}}\right)$$

$$\equiv \underset{\text{(But written in a fancier vector form)}}{\text{Biot-Savat law for magnetic field}}$$

Side note:

We have learnt 3 differential operator related to $\vec{\boldsymbol{\nabla}}$.

- Gradient $\vec{\boldsymbol{\nabla}} f$ - Scalar function $\overset{\vec{\boldsymbol{\nabla}}}{\Longrightarrow}$ Vector function
- Divergence $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}$ - Vector function $\overset{\vec{\boldsymbol{\nabla}}\cdot}{\Longrightarrow}$ Scalar function
- Curl $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}$ - Vector function $\overset{\vec{\boldsymbol{\nabla}}\times}{\Longrightarrow}$ Vector function

We can derive several 2nd derivatives relations of $\vec{\boldsymbol{\nabla}}$ (by brute force):

- Wrapped by gradient:

$$\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\hat{\boldsymbol{y}}$$
$$+ \frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\hat{\boldsymbol{z}}$$

Cannot simplify further than this form.

- Wrapped by divergence:

$$\vec{\nabla} \cdot \left(\vec{\nabla} f\right) \stackrel{\text{def}}{=} \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(Laplacian operator)}$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{F}\right) \equiv 0 \qquad \text{(Divergence of curl is always 0)}$$

- Wrapped by curl:

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0 \qquad \text{(Curl of gradient is always 0)}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) - \nabla^2 \vec{F} \qquad \text{(A useful identity)}$$

The two "$\equiv 0$" relations are the reasons why we can have potential functions.

1. In electrostatics, $\vec{E}$ is conservative $\Rightarrow$ Loop integral $= 0$. Then by Stoke's law,

$$\oint_{\substack{\text{ANY}\\ \text{loop}}} \vec{E} \cdot d\vec{l} = \iint_{\substack{\text{ANY}\\ \text{surface}}} (\vec{\nabla} \times \vec{E}) \cdot d\vec{s} = 0 \quad \Rightarrow \quad \vec{\nabla} \times \vec{E} = 0$$

   Because curl of gradient is always 0, we can always replace $\vec{E}$ with the gradient of some scalar function.

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{\nabla} \times \left(\vec{\nabla} V\right) = 0$$

2. Divergence of $\vec{B}$ is always 0, meanwhile divergence of curl is always 0. So we can always replace $\vec{B}$ with the curl of some vector function.

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$$

Here we shall summarize the methods of solving magnetostatics problems:

1. Very symmetric configurations $\Rightarrow$ Ampere's law integral form. No calculus required.
2. Not so symmetric but satisfies $\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow$ Multiple integral with Biot-Savat law.
3. All the above do not apply $\Rightarrow$ Solve Poisson equation explicitly. PDE hell.

— The End —