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Magnetic Induction

by Tony Shing

Overview:

- Lorentz force
- Faraday's law, motional/transformer EMF & Lenz's law

In electromagnetism, theoretically every problem can be solved through a set of PDEs called the **Maxwell Equations**.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\longrightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

However, a *system of PDEs* is too complicated to be solved. So we need to learn different "tricks" to avoid them, which are enough for some simple scenarios.

Electrostatics only concerns the 3$^{\text{rd}}$ equation of the set - Faraday's law.

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0 A Brief History of Electromagnetism

Electromagnetic induction is likely the most confusing topic in beginner E&M. In my opinion, taking reference of the history is helpful to unify the concepts you have learnt.

Year	Advancement
Before 1500s	Different electrostatics phenomena were known. But they were not unified or explained at all.
1600	William Gilbert was the first person to use the word "electrical" to describe electrostatics phenomena. Also the first to propose that electrical effect is due to flows of particles.
1750	Benjamin Franklin developed a one "fluid" theory of electricity, and called this fluid "charge".
1784	Charles-Augustin de Coulomb experimentally showed that force between charged objects $\propto \frac{1}{r^2}$. (Coulomb's law $F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}$)
1800	Alessandro Volta Made the first battery from electro-chemistry. (First time to have steady current.)

1820 Hans Christian Ørsted discovered that current wire can deflect compress.
(First time to relate electric and magnetic phenomena.)

1820 André-Marie Ampère formulated and verified the force between current wires.
($F = I\vec{l}_1 \times \frac{\mu_0 I}{2\pi r}\vec{l}_2$)

1831 Michael Faraday discovered magnetic induction. Experiments include:
- Across iron core: Reading appears at the instant when switch is on/off.
- Moving frame: Reading appears when wireframe moves, changes shape or when magnetic field change.

insertFig

1834 Emil Lenz Explained direction of induced current by energy conservation.
(Lenz's Law)

1860 James Clerk Maxwell unified past discoveries into 20 equations, and used field description for the first time.
This was the first time **E** and **B** appeared in Physics.
Before Maxwell, everything was described in terms of force.

1893 Oliver Heaviside combined Maxwell's 20 equations into 4, by vector calculus.
(This is the version of Maxwell's equation we now know.)

1895 Hendrik Lorentz derive the correct force on charges under both $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$.
(Lorentz force formula)

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1 Motion under Lorentz Force

Nowadays we know that Lorentz force is the fundamental explanation to magnetic induction. But interestingly, it was formulated correctly only in the very late history of classical E&M.

$$\boxed{m\vec{\boldsymbol{a}} = \vec{\boldsymbol{F}}_E + \vec{\boldsymbol{F}}_B = q\vec{\boldsymbol{E}} + q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}}$$

In general, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ may be functions of time and position. For simplicity, we will only discuss the special case when the fields are constant. By symmetry, we can assume that $\vec{\boldsymbol{B}}$ is only in $\hat{\boldsymbol{z}}$ direction, i.e. $\vec{\boldsymbol{B}} = B_z\hat{\boldsymbol{z}}$. Then the Newton's 2^{nd} law then writes as

$$m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + q\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix}$$

which turns into a system of 3 1st order linear ODEs.

$$\begin{cases} m\dfrac{\mathrm{d}v_x}{\mathrm{d}t} = qE_x + qv_yB_z \\ m\dfrac{\mathrm{d}v_y}{\mathrm{d}t} = qE_y - qv_xB_z \\ m\dfrac{\mathrm{d}v_z}{\mathrm{d}t} = qE_z \end{cases}$$

This system is simple enough that we do not need to use any matrix methods.

- **z component**
  Motion in z direction is independent of x/y. Can be solved alone.

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = qE_z \qquad \Rightarrow \qquad v_z(t) = \frac{qE_z}{m}t + v_{z0}$$

$$z(t) = \frac{1}{2}\frac{qE_z}{m}t^2 + \underbrace{v_{z0}}_{\text{Initial z velocity}}t + \underbrace{z_0}_{\text{Initial z coordinate}}$$

  which is just a constant acceleration motion.

- **x/y components**
  They can be solved by differentiating one of them, then substitute into the other.

$$\begin{aligned} \frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} &= \frac{qB_z}{m}\frac{\mathrm{d}v_y}{\mathrm{d}t} \\ &= \frac{qB_z}{m}\left(\frac{qE_y}{m} - \frac{qB_z}{m}v_x\right) \\ &= -\frac{q^2B_z^2}{m}\left(v_x - \frac{E_y}{B_z}\right) \end{aligned}$$

  which is once again the familiar ODE of SHM.

$$v_x(t) = -C'\sin\left(\frac{qB_z}{m}t + \phi\right) + \frac{E_y}{B_z}$$

$$\Rightarrow \quad x(t) = \underbrace{C'\left(\frac{m}{qB_z}\right)}_{\text{constant}}\cos\left(\frac{qB_z}{m}t + \phi\right) + \frac{E_y}{B_z}t + x_0$$

$$= \underbrace{C}_{\text{Radius}}\cos\left(\frac{qB_z}{m}t + \underbrace{\phi}_{\text{Phase}}\right) + \frac{E_y}{B_z}t + \underbrace{x_0}_{\text{Initial x coordinate}}$$

And subsitute back to equation of v_y gives

$$\begin{aligned} v_y(t) &= \frac{m}{qB_z}\frac{\mathrm{d}v_x}{\mathrm{d}t} - \frac{E_x}{B_z} \\ &= -C'\cos\left(\frac{qB_z}{m}t+\phi\right) - \frac{E_x}{B_z} \\ \Rightarrow \qquad y(t) &= \underbrace{-C'\left(\frac{m}{qB_z}\right)}_{\text{constant}}\sin\left(\frac{qB_z}{m}t+\phi\right) - \frac{E_x}{B_z}t + y_0 \\ &= -\underset{\text{Radius}}{\underline{C}}\sin\left(\frac{qB_z}{m}t+\underset{\text{Phase}}{\underline{\phi}}\right) - \frac{E_y}{B_z}t + \underset{\substack{\text{Initial} \\ \text{y coordinate}}}{\underline{x_0}} \end{aligned}$$

The result x-y motion is a combination of circular motion + drifting.

(add figure here: lorentz drift)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{C\begin{pmatrix} \cos\left(\frac{qB_z}{m}t+\phi\right) \\ -\sin\left(\frac{qB_z}{m}t+\phi\right) \end{pmatrix}}_{\text{Circular motion}} + \underbrace{\frac{1}{B_z}\begin{pmatrix} E_y \\ -E_x \end{pmatrix}t}_{\text{Drifting}} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

1. Note that the drifting direction is not intuitive:
   - With E_x only, drifting direction $= -\hat{\boldsymbol{y}}$
   - With E_y only, drifting direction $= \hat{\boldsymbol{x}}$

   In general, the drifting direction is along direction of $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ with a speed $\dfrac{|\vec{\boldsymbol{E}}|}{|\vec{\boldsymbol{B}}|}$

2. The rotation radius and speed maintain a constant ratio:

$$\text{Radius} = R \qquad \Rightarrow \qquad \text{Speed} = \frac{qB_z}{m}R$$

   i.e. Angular velocity is fixed at $\omega = \dfrac{qB_z}{m}$, independent of the charge's initial velocity.
   (One can also derive the same result by $\dfrac{mv^2}{r} = qvB$)

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2 Magnetic Induction

2.1 EMF - Force or Voltage?

The term **electromotive force** (EMF) was invented by Alessandro Volta in 1801, for explaining observations in electrochemistry.

Metal electrodes in electrolyte

$\Rightarrow$ Current generates spontaneously.

$\Rightarrow$ There must be some kind of "force" pushing the current!

(add figure here: electrolyte)

In early 1800s, scientists tended to describe things like mechanical system, i.e. any motions of objects must be driven by some kind of "force". So when there is a steady current, there must be a "force" that keep pushing the charges forward. namely the "electromotive force".

- However, EMF took the unit of volt because voltage measurement was the only way to quantify the magnitude of EMF.
- Later when Michael Faraday discovered the phenomena of magnetic induction, he used the same word to refer to the source of the induced current.

Today we still keep the name as a "force", even though we already know much better how current is generated in different sources. Maybe because people are already used to refer the same term "EMF" as the general name for ANY kind of current source, rather than discriminating them by their origin of energy.

(add figure here: i dun care)

Below we only focus on the two kinds of EMF that relate to magnetic induction phenomena.

- **Motional EMF** : Source of energy = Motion of the charges' "container". Can be explained via Lorentz force.
- **Transformer EMF** : Source of energy = Change in magnetic field. Explanation requires relativity.

2.2 Motional EMF

Under Lorentz force, free charges move in circles in a constant B-field. But if the charges' movements are restricted, charge distribution becomes uneven by building up potential difference.

2.2.1 A Moving Battery

If there is only one charge in the middle of a rod, the charge is free to move in a circlar trajectory. (And the rod will be dragged along by the charge. Even if the rod is restricted to move horizonally, it simply undergoes SHM.)

(add figure here: pipe SHM)

On the other hand, for charges that are at the ends of the rod, they will be pushed against at the end walls and can no longer move.

(add figure here: pipe potential)

This results in a charge accumulation at the end of the rod, building up a potential difference, and storing energy that can be released once the rod stop moving.

The pipe can be used to drive current when it is kept moving - It is a moving battery!

We can calculate the the potential built up in the rod by looking into the energy conservation process.

1. Suppose the B-field is constant and is in $\hat{\boldsymbol{z}}$ direction, i.e. $\vec{\boldsymbol{B}} = B_z\hat{\boldsymbol{z}}$. While the work done on a charge by Lorentz force of B-field is 0 (because $\vec{\boldsymbol{F}}_B$ always perpendicular to $\vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} \sim \mathrm{d}\vec{\boldsymbol{r}}$), we can split it into the work done in two directions:

$$
\begin{aligned}
\text{W.D.} &= \vec{\boldsymbol{F}}_B \cdot \mathrm{d}\vec{\boldsymbol{r}} = 0 \\
&= q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{r}} \\
&= q\left[(\vec{\boldsymbol{v}}_x + \vec{\boldsymbol{v}}_y) \times \vec{\boldsymbol{B}}\right] \cdot (\mathrm{d}\vec{\boldsymbol{x}} + \mathrm{d}\vec{\boldsymbol{y}}) \\
&= \underbrace{\cancelto{0}{q(\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{x}}}}_{\substack{\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}} = v_x B_z(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}}) \\ = v_x B_z(-\hat{\boldsymbol{y}}) \\ \text{Dotting with } \hat{\boldsymbol{x}} \text{ gives } 0}} + q(\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{y}} + q(\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{x}} + \underbrace{\cancelto{0}{q(\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{y}}}}_{\substack{\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}} = v_y B_z(\hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}) \\ = v_y B_z(\hat{\boldsymbol{x}}) \\ \text{Dotting with } \hat{\boldsymbol{y}} \text{ gives } 0}} \\
&= \underbrace{q(\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{y}}}_{\substack{\text{W.D. along } \hat{\boldsymbol{y}} \text{ direction} \\ \sim F_y \mathrm{d}y}} + \underbrace{q(\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{x}}}_{\substack{\text{W.D. along } \hat{\boldsymbol{x}} \text{ direction} \\ \sim F_x \mathrm{d}x}}
\end{aligned}
$$

   Note that the W.D. along the two directions will always carry opposite sign. i.e. This is an energy conservation between the rod's KE in its travelling direction ($\hat{\boldsymbol{x}}$) and KE of charges along the rod ($\hat{\boldsymbol{y}}$).

   (add figure here: WD by B = 0)

2. On the other hand, as charge accumulates at the ends of the rod, the charges will reach the equilibrium state when Lorentz force of B-field is balanced by built-up potential from the charge accumulation. The net forces, and thus W.D. to travel against the potential is:

$$
\begin{aligned}
q\vec{\boldsymbol{E}} &= q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} \\
\Rightarrow \quad q\,\mathrm{d}\underset{\substack{\uparrow \\ \epsilon = \text{EMF}}}{\epsilon} &\equiv q(\vec{\boldsymbol{E}} \cdot \mathrm{d}\hat{\boldsymbol{y}}) = q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\hat{\boldsymbol{y}}
\end{aligned}
$$

   i.e. The KE of charges along the rod ($\hat{\boldsymbol{y}}$) turns into energy stored in the built-up potential.

(add figure here: free body diag)

That is, the phenomenon of motional EMF is no more than an energy conservation process from the KE of a rod of charges to the potential built-up by its charges.

2.2.2 Geometrical Relation with Flux

$$\begin{aligned} q\,\mathrm{d}\epsilon &= \vec{\boldsymbol{F}}_{\text{Lorentz}} \cdot \mathrm{d}\vec{\boldsymbol{y}} \\ &= q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{y}} \\ &= q(\mathrm{d}\vec{\boldsymbol{y}} \times \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{B}} \\ &= -q(\vec{\boldsymbol{v}} \cdot \mathrm{d}\vec{\boldsymbol{y}}) \cdot \vec{\boldsymbol{B}} \end{aligned}$$

2.3 Transformer EMF

2.4 Maxwell-Faraday Equation

2.5 Lenz's Law

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3 Basic Problems in Magnetic Induction

— The End —