Vectors in Polar Coordinates

by Tony Shing

Overview:

- The unit vectors in polar coordinate are not "constant" of position and time.

- Angular quantities $(\theta/\omega/\alpha) \sim$ Angular component of $(s/v/a)$.

- Relative angular velocity does not exists because it is only the angular component.

Notations:

2D coordinates are represented by (x, y) and (r, θ). Their conversion is according to the position vector's component:

$$\vec{r} = x\hat{x} + y\hat{y} = (r\cos\theta)\hat{x} + (r\sin\theta)\hat{y} = r\hat{r}$$

Note that $\vec{r}$ is the **position vector** and r is the **radial component**. But in polar coordinate, **length of position vector** $|\vec{r}| = r$ exactly.

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1 The Vector Expressions

1.1 Unit Vectors as function of coordinate

In x-y coordinate, every vector can be expressed in terms of the two unit vectors $\{\hat{x}, \hat{y}\}$ and their components.

$$\vec{s} = s_x\hat{x} + s_y\hat{y}$$

When switching into polar coordinate, we wish to do the same, but with two different unit vectors $\{\hat{r}, \hat{\theta}\}$.

$$\vec{s} = s_r\hat{r} + s_\theta\hat{\theta}$$

The problem about $\{\hat{r}, \hat{\theta}\}$ is that their directions depends on the coordinate (r, θ). We require:

- $\hat{r}$ should always be radially outward, i.e. extend from the origin to the current point.

- $\hat{\theta}$ should always be perpendicular to $\hat{r}$, like a anti-clockwise torque.

(add figure here: Compare two pair of unit vector in xy and polar respectively -¿ xy same, polar different)

From the figure, we can see that the pairs of unit vectors at different positions are pointing in different directions. Notation-wise we should write them as:

$$\hat{\boldsymbol{r}}_{\text{at } (r_1,\theta_1)} \neq \hat{\boldsymbol{r}}_{\text{at } (r_2,\theta_2)} \quad , \quad \hat{\boldsymbol{\theta}}_{\text{at } (r_1,\theta_1)} \neq \hat{\boldsymbol{\theta}}_{\text{at } (r_2,\theta_2)}$$

This makes a big difference in differentiation, because the unit vectors are essentially functions of the coordinates. Compare

- Vector in terms of $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}\}$
  $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ <u>always point in the same direction.</u> Never change by position $\Rightarrow$ Differentiation $= 0$.

$$\vec{\boldsymbol{s}} = s_x \hat{\boldsymbol{x}} + s_y \hat{\boldsymbol{y}}$$

$$\frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}t} = \frac{\mathrm{d}s_x}{\mathrm{d}t}\hat{\boldsymbol{x}} + s_x \underbrace{\frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}t}}_{=0} + \frac{\mathrm{d}s_y}{\mathrm{d}t}\hat{\boldsymbol{y}} + s_y \underbrace{\frac{\mathrm{d}\hat{\boldsymbol{y}}}{\mathrm{d}t}}_{=0}$$

- Vector in terms of $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\}$
  $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$ <u>are functions of r and θ, and (r,θ) are functions of t.</u> So differentiating against t is non-zero in general.

$$\vec{\boldsymbol{s}} = s_r \hat{\boldsymbol{r}} + s_\theta \hat{\boldsymbol{\theta}}$$

$$\frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}t} = \frac{\mathrm{d}s_r}{\mathrm{d}t}\hat{\boldsymbol{r}} + s_r \underbrace{\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t}}_{\substack{\hat{\boldsymbol{r}} \text{ contains} \\ r \ \& \ \theta}} + \frac{\mathrm{d}s_\theta}{\mathrm{d}t}\hat{\boldsymbol{\theta}} + s_\theta \underbrace{\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t}}_{\substack{\hat{\boldsymbol{\theta}} \text{ contains} \\ r \ \& \ \theta}}$$

<u>**Note 1**</u>: The above are just product rules, but applied on (component)$\times$(unit vector).

<u>**Note 2**</u>: Unit vectors are functions of coordinate for any non-rectangular coordinate.

(add figure here: curve space unit vectors)

1.2 Differentiation on Polar Unit Vectors

Because $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}\}$ never change by position, we usually call a rectangular coordinate as **"ambient coordinate"**, and use them as a reference to express other unit vectors. To tell how to differentiate the polar unit vectors, we can first express them in the $\bigcirc\hat{\boldsymbol{x}} + \square\hat{\boldsymbol{y}}$ form. Then differentiation is solely on the components.

(add figure here: polar coor -¿ xy coor)

The relations between $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\}$ and $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}\}$ are purely trigonometric:

$$\begin{cases} \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} = \cos\theta \cdot \hat{\boldsymbol{x}} + \sin\theta \cdot \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\theta}}_{\text{at } (r,\theta)} = -\sin\theta \cdot \hat{\boldsymbol{x}} + \cos\theta \cdot \hat{\boldsymbol{y}} \end{cases}$$

Obviously $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ are function to θ only. So

$$\boxed{\frac{\partial \hat{\boldsymbol{r}}}{\partial r} = \frac{\partial \hat{\boldsymbol{\theta}}}{\partial r} = 0}$$

P.2

and differentiating on θ gives

$$
\begin{cases}
\dfrac{\partial \hat{\boldsymbol{r}}}{\partial \theta} = -\sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{y}} = \hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)} \\[3mm]
\dfrac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\cos\theta\hat{\boldsymbol{x}} - \sin\theta\hat{\boldsymbol{y}} = -\hat{\boldsymbol{r}}_{\text{at }(r,\theta)}
\end{cases}
$$

The two unit vectors give each other upon differentiation by θ is purely a numerical coincident. (This is related to rotational symmetry.) Also worth mentioning, the differentiating against t is simply applying the (partial-D) chain rule:

$$
\begin{cases}
\dfrac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = \overset{0}{\cancel{\dfrac{\partial \hat{\boldsymbol{r}}}{\partial r}\dfrac{\mathrm{d}r}{\mathrm{d}t}}} + \dfrac{\partial \hat{\boldsymbol{r}}}{\partial \theta}\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)} \\[5mm]
\dfrac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = \overset{0}{\cancel{\dfrac{\partial \hat{\boldsymbol{\theta}}}{\partial r}\dfrac{\mathrm{d}r}{\mathrm{d}t}}} + \dfrac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta}\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\hat{\boldsymbol{r}}_{\text{at }(r,\theta)}
\end{cases}
$$

2 Kinematic Quantities in terms of (r,θ)

2.1 Position Vector

A position vector in the $\bigcirc\hat{\boldsymbol{x}} + \square\hat{\boldsymbol{y}}$ form has its components equal to the coordinate it is pointing to, i.e.

$$\text{Pointing at coordinate } (X,Y) \iff \text{ Expression } = X\hat{\boldsymbol{x}} + Y\hat{\boldsymbol{y}}$$

This is not true for vector expressed in other coordinates, e.g. You cannot write a vector pointing at polar coordinate (R,Θ) as $R\hat{\boldsymbol{r}} + \Theta\hat{\boldsymbol{\theta}}$. For proper conversion, we must first use the conversion between the unit vectors. Observed their relations can be written as a matrix:

$$
\begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{pmatrix}
$$

This square matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is known as the "**rotation matrix**", which when multiplied to a vector, will geometrically rotate the vector about the origin by an angle θ. Its inverse is trivial - by replacing θ to $-\theta$. (reverse of clockwise rotation = anticlockwise rotation) One can easily check that:

$$
\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Substitute this into a position vector tells us how to write a position vector properly by its polar coordinate (r, θ) and unit vectors $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\}$.

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{pmatrix}$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{pmatrix}$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}$$

$$= \begin{pmatrix} x\cos\theta + y\sin\theta & -x\sin\theta + y\cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}$$

$$= \begin{pmatrix} (r\cos\theta)\cos\theta + (r\sin\theta)\sin\theta & -(r\cos\theta)\sin\theta + (r\sin\theta)\cos\theta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}$$

$$= \begin{pmatrix} r & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix}$$

$$\boxed{\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}}}$$

So the proper way to write a vector in terms of its polar coordinate is ... just by its radial component. Isn't that weird?

Normally when we describe a point on a 2D plane, we need two information - its x and y components. Why is there only 1 information (r component) in the polar form? This is because the second piece of information is hidden in the unit vector $\hat{\boldsymbol{r}}$:

$$\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}} \qquad \xRightarrow{\text{writing more accurately}} \qquad \vec{\boldsymbol{r}} = r \cdot \hat{\boldsymbol{r}}_{\text{at }(r,\theta)}$$

which direction? in θ direction!

We cannot tell which direction $\hat{\boldsymbol{r}}$ is pointing to, if we do not know the "at (r, θ)" part. (Sadly, most text will omit writing this part.)

(add figure here: same expression $r\hat{\boldsymbol{r}}$, all in different direction, because θ is not labelled)

Unlike x-y coordinate, we do not care the two unit vectors much because we always know that $\hat{\boldsymbol{x}}$ is the one pointing horizontally and $\hat{\boldsymbol{y}}$ is the one pointing vertically.

$$\vec{\boldsymbol{r}} = x \cdot \hat{\boldsymbol{x}} + y \cdot \hat{\boldsymbol{y}}$$

always horiztonal   always vertical

(add figure here: x, y always horizontal/vertical, no matter anywhere)

2.2   Displacement Vector

Displacement vector is the subtraction between two position vector. In x-y coordinate, the subtraction is simply done within the components:

$$\vec{r}_2 - \vec{r}_1 = (x_2\hat{\boldsymbol{x}} + y_2\hat{\boldsymbol{y}}) - (x_1\hat{\boldsymbol{x}} + y_1\hat{\boldsymbol{y}}) = [x_2 - x_1]\hat{\boldsymbol{x}} + [y_2 - y_1]\hat{\boldsymbol{y}}$$

But in polar form, you cannot!

$$\vec{r}_2 - \vec{r}_1 = (r_2\hat{\boldsymbol{r}}) - (r_1\hat{\boldsymbol{r}}) \neq [r_2 - r_1]\hat{\boldsymbol{r}}$$

This is the fault of omitting the "at (r,θ)" part. Remember, $\hat{\boldsymbol{r}}$ at different position are differnt vectors. The true component subtraction will require a lot of trigonometry.

$$\vec{r}_2 - \vec{r}_1 = (r_2 \cdot \hat{\boldsymbol{r}}_{\text{at } (r_2,\theta_2)}) - (r_1 \cdot \hat{\boldsymbol{r}}_{\text{at } (r_1,\theta_1)}) = (\text{Awful!})$$

(add figure here: polar vector subtraction require nasty geometry)

Furthermore, if we really try to subtract by component, the result "vector" on the polar grid is not even a straightline. (This happens in every curved coordinate system.)

(add figure here: curved vector)

The only valid definition is the **infinitestimal displacement vector**, i.e. when we subtract two very close vector, their difference is approximately a straight line and we can take limit to its length to 0.

$$\mathrm{d}\vec{r} = \mathrm{d}\big(r\hat{\boldsymbol{r}}_{\text{at } (r,\theta)}\big)$$
$$= \mathrm{d}(r) \cdot \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} + r \cdot \mathrm{d}\big(\hat{\boldsymbol{r}}_{\text{at } (r,\theta)}\big)$$
$$\boxed{\mathrm{d}\vec{r} = \mathrm{d}(r) \cdot \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} + r \cdot \mathrm{d}(\theta) \cdot \hat{\boldsymbol{\theta}}_{\text{at } (r,\theta)}}$$

(add figure here: components in displacement vector)

The differential $\mathrm{d}(\hat{\boldsymbol{r}}_{\text{at } (r,\theta)})$ comes from $\dfrac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}\theta} = \hat{\boldsymbol{\theta}}$ which is derived previously. In textbooks you will usually find the form without "at (r,θ)", which is

$$\mathrm{d}\vec{r} = (\mathrm{d}r)\hat{\boldsymbol{r}} + (r\,\mathrm{d}\theta)\hat{\boldsymbol{\theta}}$$

2.3   Velocity Vector

Velocity is defined by displacement divided by period of time Δt, and then taking $\Delta t \to 0$. From the infinitestimal displacement, we immediately get

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\vec{r}(t)}{\Delta t}$$

$$= \frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t}$$

$$\boxed{\vec{v} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} + r \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \cdot \hat{\boldsymbol{\theta}}_{\text{at } (r,\theta)}}$$

We can identify the two components of a velocity vector. Notation-wise they may also be written as

$$\vec{v} = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)\hat{\boldsymbol{r}} + \left(r\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{\boldsymbol{\theta}} = v_r\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}} = v_r\hat{\boldsymbol{r}} + v_\theta\hat{\boldsymbol{\theta}}$$

– Radial component - $v_r = \dfrac{\mathrm{d}r}{\mathrm{d}t}$

– Angular component - $v_\theta = r\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = r\omega$

(add figure here: components in velocity vector)

2.4   Acceleration Vector

Acceleration is defined by differentiating the velocity once again. But with $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ being function of t as well, the full expansion gives a lengthy product rule.

$$\vec{\boldsymbol{a}}(t) = \frac{\mathrm{d}\vec{\boldsymbol{v}}(t)}{\mathrm{d}t}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\cdot\hat{\boldsymbol{r}}_{\text{at }(r,\theta)} + r\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)}\right)$$

$$= \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{r}}_{\text{at }(r,\theta)} + \frac{\mathrm{d}r}{\mathrm{d}t}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{r}}_{\text{at }(r,\theta)}\right) + \frac{\mathrm{d}r}{\mathrm{d}t}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)} + r\cdot\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)} + r\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)}\right)$$

$$= \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{r}}_{\text{at }(r,\theta)} + \frac{\mathrm{d}r}{\mathrm{d}t}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)} + \frac{\mathrm{d}r}{\mathrm{d}t}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)} + r\cdot\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\cdot\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)} + r\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\hat{\boldsymbol{r}}_{\text{at }(r,\theta)}$$

$$= \left[\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\cdot\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]\hat{\boldsymbol{r}}_{\text{at }(r,\theta)} + \left[2\frac{\mathrm{d}r}{\mathrm{d}t}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} + r\cdot\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right]\hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)}$$

There are 4 terms in total. Notation-wise we identify them as:

1. *(Along $\hat{\boldsymbol{r}}$)* <u>**Radial acceleration**</u> : $\dfrac{\mathrm{d}^2r}{\mathrm{d}t^2}$

2. *(Along $\hat{\boldsymbol{r}}$)* <u>**Centripetal acceleration**</u> : $-r\cdot\left(\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = -r\cdot\omega^2$
   Minus sign for pointing toward origin.

3. *(Along $\hat{\boldsymbol{\theta}}$)* <u>**Coriolis acceleration**</u> : $2\dfrac{\mathrm{d}r}{\mathrm{d}t}\cdot\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 2v_r\cdot\omega$
   This term appears only if radial distance is changing, i.e. $\frac{\mathrm{d}r}{\mathrm{d}t} \neq 0$.

4. *(Along $\hat{\boldsymbol{\theta}}$)* <u>**Euler acceleration**</u> : $r\cdot\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = r\alpha$
   This term appears only if the angular component is accelerating, i.e. $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \neq 0$.

They are also the 4 acceleration terms corresponding to the pseudo-forces in a rotating reference frame.

2.5   The Angular Quantities

Observe that if we perform cross product to a vector with its position vector, we essentially remove its radial component and only leave the angular compoenet. Because

$$\begin{cases} \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} = \underline{0} \;\longleftarrow\!\!-\!\!\text{Cross product with itself} = 0 \\ \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \hat{\boldsymbol{\theta}}_{\text{at } (r,\theta)} = \hat{\boldsymbol{z}} \;\longleftarrow\!\!\!-\!\!\!-\; \substack{\hat{\boldsymbol{z}} \text{ is independent} \\ \text{of position}} \end{cases}$$

So for a general vector $\vec{\boldsymbol{s}}$,

$$\begin{aligned} \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \vec{\boldsymbol{s}} &= \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \left(s_r \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} + s_\theta \hat{\boldsymbol{\theta}}_{\text{at } (r,\theta)} \right) \\ &= s_r (\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \hat{\boldsymbol{r}}_{\text{at } (r,\theta)}) + s_\theta (\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \hat{\boldsymbol{\theta}}_{\text{at } (r,\theta)}) \\ &= 0 + \underline{s_\theta} \hat{\boldsymbol{z}} \\ &\qquad\quad \uparrow \end{aligned}$$

<div align="center" style="color:red">Successfully filter out
the θ component</div>

Apply this on our displacement / velocity / acceleration vector will give the familiar definitions of angular displacement / angular velocity / angular acceleration.

1. **Angular position vector:**
   Such thing does not exist, because position vector does not have angular component.

   $$\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \vec{\boldsymbol{r}} = r (\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \hat{\boldsymbol{r}}_{\text{at } (r,\theta)}) = 0$$

2. **Angular displacement vector:**
   Only exists in the infinitestimal version.

   $$\begin{aligned} \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \mathrm{d}\vec{\boldsymbol{r}} &= (\mathrm{d}r)\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \overset{0}{\times \hat{\boldsymbol{r}}_{\text{at } (r,\theta)}} + (r\,\mathrm{d}\theta)\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \hat{\boldsymbol{\theta}}_{\text{at } (r,\theta)} \\ &= r\,\mathrm{d}\theta\,\hat{\boldsymbol{z}} \end{aligned}$$

   The infinitestimal angular displacement vector is formally defined in $\hat{\boldsymbol{z}}$ direction:

   $$\boxed{\; \mathrm{d}\vec{\boldsymbol{\theta}} = (\mathrm{d}\theta)\hat{\boldsymbol{z}} = \frac{\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \mathrm{d}\vec{\boldsymbol{r}}}{r} = \frac{(\text{Angular component of } \mathrm{d}\vec{\boldsymbol{r}})}{r}\hat{\boldsymbol{z}} \;}$$

   Its magnitude yields $\mathrm{d}\theta = \dfrac{|\mathrm{d}\vec{\boldsymbol{r}}|}{r} = \dfrac{\text{arc length}}{\text{radius}}$.

3. **Angular velocity vector:**

   $$\begin{aligned} \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \vec{\boldsymbol{v}} &= \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \overset{0}{\times \hat{\boldsymbol{r}}_{\text{at } (r,\theta)}} + r \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \hat{\boldsymbol{\theta}}_{\text{at } (r,\theta)} \\ &= r\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{z}} \end{aligned}$$

   The angular velocity vector is formally defined in $\hat{\boldsymbol{z}}$ direction:

   $$\boxed{\; \vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{z}} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{z}} = \frac{\hat{\boldsymbol{r}}_{\text{at } (r,\theta)} \times \vec{\boldsymbol{v}}}{r} = \frac{(\text{Angular component of } \vec{\boldsymbol{v}})}{r}\hat{\boldsymbol{z}} \;}$$

   Its magnitude yields our familiar definition $\omega = \dfrac{|\vec{\boldsymbol{v}}|}{r}$.

4. **Angular acceleration vector:**

$$\hat{\boldsymbol{r}}_{\text{at }(r,\theta)} \times \vec{\boldsymbol{a}} = \left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r \cdot \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2 \right] \hat{\boldsymbol{r}}_{\text{at }(r,\theta)} \times \hat{\boldsymbol{r}}_{\text{at }(r,\theta)} \overset{0}{\longrightarrow} + \left[2 \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} + r \cdot \frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} \right] \hat{\boldsymbol{r}}_{\text{at }(r,\theta)} \times \hat{\boldsymbol{\theta}}_{\text{at }(r,\theta)}$$

$$= \left[2 \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} + r \cdot \frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} \right] \hat{\boldsymbol{z}}$$

Different from the above, the angular acceleration is defined as the 2nd derivative to the angular displacement and ignore the Coriolis term. Therefore its definition is uglier. But most of its use cases are in pure rotation, which then the Coriolis term is 0.

$$\boxed{ \vec{\boldsymbol{\alpha}} = \alpha \hat{\boldsymbol{z}} = \frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} \hat{\boldsymbol{z}} = \frac{1}{r} \left(\hat{\boldsymbol{r}}_{\text{at }(r,\theta)} \times \vec{\boldsymbol{a}} - 2 \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \hat{\boldsymbol{z}} \right) }$$

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3   Relative Rotation?

A naïve example to talk about relative motion:

- B sees an object A in the same car, moving at velocity $\vec{\boldsymbol{V}}_{AB}$.

- C sees B in a car, moving at velocity $\vec{\boldsymbol{V}}_{BC}$.

- Then C sees object A moving at velocity $\vec{\boldsymbol{V}}_{AC} = \vec{\boldsymbol{V}}_{AB} + \vec{\boldsymbol{V}}_{BC}$.

(add figure here: moving car)

The above works in form of vector, i.e. it does not matter what trajectories are ABC actually moving in. We can break it down component-wise if we write the velocities in x/y coordinate.

$$\vec{\boldsymbol{V}}_{AC} = \vec{\boldsymbol{V}}_{AB} + \vec{\boldsymbol{V}}_{BC}$$
$$(V_{AC,x}\hat{\boldsymbol{x}} + V_{AC,y}\hat{\boldsymbol{y}}) = (V_{AB,x}\hat{\boldsymbol{x}} + V_{AB,y}\hat{\boldsymbol{y}}) + (V_{BC,x}\hat{\boldsymbol{x}} + V_{BC,y}\hat{\boldsymbol{y}})$$
$$= (V_{AB,x} + V_{BC,x})\hat{\boldsymbol{x}} + (V_{AB,y} + V_{BC,y})\hat{\boldsymbol{y}}$$

But in polar coordinate, note that <u>angular velocity is only the angular component in the velocity vector.</u> It does not make any sense if we add vectors just by one of their components.

(add figure here: traj different origin)

One must write

$$\vec{\boldsymbol{V}}_{AC} = \vec{\boldsymbol{V}}_{AB} + \vec{\boldsymbol{V}}_{BC}$$
$$\left(V_{AC,r}\hat{\boldsymbol{r}}_{AC} + V_{AC,\theta}\hat{\boldsymbol{\theta}}_{AC} \right) = \left(V_{AB,r}\hat{\boldsymbol{r}}_{AB} + V_{AB,\theta}\hat{\boldsymbol{\theta}}_{AB} \right) + \left(V_{BC,r}\hat{\boldsymbol{r}}_{BC} + V_{BC,\theta}\hat{\boldsymbol{\theta}}_{BC} \right)$$

You cannot directly add the components like "$V_{AC,\theta} = V_{AB,\theta} + V_{BC,\theta}$" because $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\}_{AC} \neq \{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\}_{AB} \neq \{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\}_{BC}$". Computing relative velocity in polar coordinate requires a mess of trigonometry.