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Vector Calculus

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Overview:

- Limit of vector functions
- Differentiation $\begin{cases} \text{To vectors} \\ \text{By vectors} \end{cases}$
- Curve parametrization & Line integral

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1 Review: Vector Geometry

Geometrically, vectors can be thought as objects that have both **magnitude** and **direction**.

- We usually visualized a vector as an arrow, pointing from origin to some coordinate

(add figure here: vector=coor)

- Vectors can be expressed by column matrices. Using row matrices is OK but mathematicians do interpret them differently.

$$\text{prefered} \rightarrow \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \qquad \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \leftarrow \text{Not prefered}$$

Here are some notations and operations that you (should have) learnt about geometrical vectors.

1. **Norm** (the magnitude)
   - Notation: $\|\vec{\boldsymbol{a}}\|$. Can also use $|\vec{\boldsymbol{a}}|$ informally.
     Some mathematicians are strict about that norm should use double vertical bars $\|\cdot\|$, while single vertical bar $|\cdot|$ is reserved for absolute value. But many books would just use $|\cdot|$ in both cases.
   - Geometrically the length of the vector. Calculation follows Pythagoras theorem.

   (add figure here: vector norm)

2. **Unit vector** (the direction)
   - Notation: $\hat{\boldsymbol{a}}$, replacing the vector arrow $\vec{}$ with a hat $\hat{}$.

- Formed by a vector divided by its norm (length) : $\hat{\boldsymbol{a}} \equiv \frac{\vec{\boldsymbol{a}}}{|\vec{\boldsymbol{a}}|}$, so that it is a vector that points in a given direction but always has norm = 1.

In x-y-z coordinate, the 3 basis vectors are conventionally represented as

$$\hat{\boldsymbol{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \hat{\boldsymbol{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

And so any vectors in the 3D space can be expressed as a linear combination of these 3 unit vectors:

$$\begin{aligned} \vec{\boldsymbol{a}} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} &= a_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}} + a_z\hat{\boldsymbol{z}} \end{aligned}$$

3. **Addition/Subtraction**
   - Operations are component-wise:

$$\begin{aligned} \vec{\boldsymbol{a}} \pm \vec{\boldsymbol{b}} &= (a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) \pm (b_x\hat{\boldsymbol{x}} + b_y\hat{\boldsymbol{y}} + b_z\hat{\boldsymbol{z}}) \\ &= (a_x \pm b_x)\hat{\boldsymbol{x}} + (a_y \pm b_y)\hat{\boldsymbol{y}} + (a_z \pm b_z)\hat{\boldsymbol{z}} \end{aligned}$$

   - Geometrically, it can be depicted as parallelogram rule.

(add figure here: parallelogram rule)

4. **Multiplication with constants**
   - Operations are component-wise:

$$\begin{aligned} k\vec{\boldsymbol{a}} &= k(a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) \\ &= (ka_x)\hat{\boldsymbol{x}} + (ka_y)\hat{\boldsymbol{y}} + (ka_z)\hat{\boldsymbol{z}} \end{aligned}$$

   - Geometrically, it can be depicted as extending/contracting the vector in the same/opposite direction.

(add figure here: scaling)

5. **Dot product** (Scalar product)

   A multiplication operation between 2 vectors, that yields a number.
   - With the components of both vectors, it can be calculated as

$$\begin{aligned} \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} &= a_xb_x + a_yb_y + a_zb_z \\ &= \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \end{aligned}$$

   Note that dot product is symmetric, i.e. $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = \vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{a}}$.

- Geometrically, dot product can be viewed as projection of one vector onto the other

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}| \cos\theta$$

(add figure here: dot prod)

where θ is the angle between the vectors. Specifically if one of them is a unit vector, computing dot product yields the component of another vector in this unit vector's direction.

$$\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} = |\vec{\boldsymbol{a}}|\underbrace{|\hat{\boldsymbol{b}}|}_{=1} \cos\theta = |\vec{\boldsymbol{a}}| \cos\theta$$

(add figure here: projection)

We can use this property to separate components of a vector. E.g.

$$\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{x}} = \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \cdot \hat{\boldsymbol{x}} = \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a_x = \text{ x component of } \vec{\boldsymbol{a}}$$

and write something like

$$\vec{\boldsymbol{a}} = (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{x}})\hat{\boldsymbol{x}} + (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{y}})\hat{\boldsymbol{y}} + (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{z}})\hat{\boldsymbol{z}}$$

6. **Cross product** (Vector product)

A multiplication operation between 2 vectors, that yields a new vector.

- Cross product is only defined in 3D space.

(There are definitions extended for vectors in higher dimension but they are totally out of our scope)

- With the components of both vectors, it can be calculated as

$$\vec{\boldsymbol{c}} = \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Note that cross product is antisymmetric, i.e. $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = -\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}}$.

- Geometrically, cross product outputs a vector that is perpendicular to the plane spanned by the original 2 vectors. This vector has a magnitude $|\vec{\boldsymbol{c}}| = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}| \sin\theta$.

(add figure here: cross prod)

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2 Limits on Vector Functions

On Single Variable Vector Function

The definition of limit implies that when the input t is getting closer to t_0, then the output of the function $\vec{\boldsymbol{F}}(t)$, as a vector, is getting closer to match on a "limiting" vector $\vec{\boldsymbol{L}}$.

$$\lim_{t \to t_0} \vec{\boldsymbol{F}}(t) = \vec{\boldsymbol{L}} \Leftrightarrow \text{ When } t \text{ approaches } t_0$$

(add figure here: vector limit)

Equivalently, it implies that the distance between $\vec{\boldsymbol{F}}(t)$ and $\vec{\boldsymbol{L}}$ need to be as small as possible

$$\begin{aligned}
&\lim_{t \to t_0} |\vec{\boldsymbol{F}}(t) - \vec{\boldsymbol{L}}| = 0 \\
\Leftrightarrow\ &\lim_{t \to t_0} \sqrt{(F_x(t) - L_x)^2 + (F_y(t) - L_y)^2 + (F_z(t) - L_z)^2} \\
\Leftrightarrow\ &\begin{cases} \lim_{t \to t_0} |F_x(t) - L_x| = 0 \\ \lim_{t \to t_0} |F_y(t) - L_y| = 0 \\ \lim_{t \to t_0} |F_z(t) - L_z| = 0 \end{cases}
\end{aligned}$$

It is the same requiring all 3 components to independently approach their own limits.

On Multivariable Vector Function

The definition and notation is similar (but picture is hard to visualize).

$$\lim_{\underline{(x_1, x_2, ..., x_n)} \to \underline{(a_1, a_2, ..., a_n)}} \underline{\vec{\boldsymbol{F}}(x_1, x_2, ..., x_n)} = \underline{\vec{\boldsymbol{L}}}$$

When every input x_i approach their target value a_i

Each components F_i will approach their limit value L_i

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3 Vector Differentiation

Notations:

In the following section, I will stick to the colour scheme that

- red = indices for inputs, e.g. $\vec{\boldsymbol{x}} = (x_1, x_2, ..., x_n)$
- blue = indices for outputs, e.g. $\vec{\boldsymbol{F}} = (F_1, F_2, ..., F_n)$

3.1 Differentiation on Vector Functions

3.1.1 On Single Variable Vector Function

Here we let the functions to have n components $F_1(t)$ to $F_n(t)$, using unit vectors $\hat{\boldsymbol{u}}_1$ to $\hat{\boldsymbol{u}}_n$.

$$\vec{\boldsymbol{F}}(t) = F_1(t)\hat{\boldsymbol{u}}_1 + F_2(t)\hat{\boldsymbol{u}}_2 + ... + F_n(t)\hat{\boldsymbol{u}}_n$$

The differention, by limit definition, is

$$\begin{aligned}
\frac{\mathrm{d}\vec{\boldsymbol{F}}}{\mathrm{d}t} &= \lim_{\Delta t \to 0}\left[\frac{\vec{\boldsymbol{F}}(t+\Delta t) - \vec{\boldsymbol{F}}(t)}{\Delta t}\right] \\
&= \lim_{\Delta t \to 0}\left[\frac{F_1(t+\Delta t) - F_1(t)}{\Delta t}\right]\hat{\boldsymbol{u}}_1 + \lim_{\Delta t \to 0}\left[\frac{F_2(t+\Delta t) - F_2(t)}{\Delta t}\right]\hat{\boldsymbol{u}}_2 + \dots \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}t}F_1(t)\right]\hat{\boldsymbol{u}}_1 + \left[\frac{\mathrm{d}}{\mathrm{d}t}F_2(t)\right]\hat{\boldsymbol{u}}_2 + \cdots + \left[\frac{\mathrm{d}}{\mathrm{d}t}F_n(t)\right]\hat{\boldsymbol{u}}_n \\
&= \begin{pmatrix} \frac{\mathrm{d}F_1}{\mathrm{d}t} \\ \frac{\mathrm{d}F_2}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}F_n}{\mathrm{d}t} \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix} = \text{Differentiate each component individually}
\end{aligned}$$

The rules for calculation are basically the same as in single variable functions.

- Addition/Subtraction : $\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{\boldsymbol{F}}(t) \pm \vec{\boldsymbol{G}}(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{F}}(t) \pm \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{G}}(t)$
- Scaling : $\frac{\mathrm{d}}{\mathrm{d}t}\left[k\vec{\boldsymbol{F}}(t)\right] = k\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{F}}(t)$
- Product rule : One product rule for each kind of muliplications between vectors.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{\boldsymbol{F}}(t) \cdot \vec{\boldsymbol{G}}(t)\right] = \left[\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{F}}(t)\right] \cdot \vec{\boldsymbol{G}}(t) + \vec{\boldsymbol{F}}(t) \cdot \left[\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{G}}(t)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{\boldsymbol{F}}(t) \times \vec{\boldsymbol{G}}(t)\right] = \underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{F}}(t)\right] \times \vec{\boldsymbol{G}}(t) + \vec{\boldsymbol{F}}(t) \times \left[\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{G}}(t)\right]}_{\text{"}F \times G\text{" must stay the same order because order matters in cross Product}}$$

(And there is no quotient rule, obviously)

3.1.2 On Multivariable Vector Function

The major difference is that there is one partial differentiation per input.

$$\begin{cases}
\boxed{\frac{\partial}{\partial x_1}\vec{\boldsymbol{F}}(x_1, ..., x_m)} = \frac{\partial}{\partial x_1}F_1(x_1, ..., x_m)\hat{\boldsymbol{u}}_1 + \frac{\partial}{\partial x_1}F_2(x_1, ..., x_m)\hat{\boldsymbol{u}}_2 + \cdots + \frac{\partial}{\partial x_1}F_n(x_1, ..., x_m)\hat{\boldsymbol{u}}_n \\
\qquad\vdots \qquad\qquad = \qquad \vdots \\
\boxed{\frac{\partial}{\partial x_m}\vec{\boldsymbol{F}}(x_1, ..., x_m)} = \frac{\partial}{\partial x_m}F_1(x_1, ..., x_m)\hat{\boldsymbol{u}}_1 + \frac{\partial}{\partial x_m}F_2(x_1, ..., x_m)\hat{\boldsymbol{u}}_2 + \cdots + \frac{\partial}{\partial x_m}F_n(x_1, ..., x_m)\hat{\boldsymbol{u}}_n
\end{cases}$$

To make their expression easier to read, we can write them as column matrices.

$$\frac{\partial \vec{F}}{\partial x_1} = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} \\ \dfrac{\partial F_2}{\partial x_1} \\ \vdots \\ \dfrac{\partial F_n}{\partial x_1} \end{pmatrix} \quad , \quad \frac{\partial \vec{F}}{\partial x_2} = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_2}{\partial x_2} \\ \vdots \\ \dfrac{\partial F_n}{\partial x_2} \end{pmatrix} \quad , \quad \cdots \quad , \quad \frac{\partial \vec{F}}{\partial x_m} = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_m} \\ \dfrac{\partial F_2}{\partial x_m} \\ \vdots \\ \dfrac{\partial F_n}{\partial x_m} \end{pmatrix}$$

And join each column into one big matrix:

$$\frac{\partial \vec{F}}{\partial \vec{x}} = \begin{array}{cc} \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_m} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} & \cdots & \dfrac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_n}{\partial x_1} & \dfrac{\partial F_n}{\partial x_2} & \cdots & \dfrac{\partial F_n}{\partial x_m} \end{pmatrix} & \updownarrow \begin{array}{c} n \text{ rows for} \\ \text{function with} \\ n \text{ components} \end{array} \\ \longleftrightarrow & \\ \begin{array}{c} m \text{ columns for} \\ \text{function with } m \text{ inputs} \end{array} & \end{array}$$

(This matrix is called **Jacobian matrix**)

3.2 Differentiaton by Vector - Gradient

3.2.1 On Multivariable Scalar Function

Recall the definition of partial differentiation

$$\begin{aligned} \frac{\partial}{\partial x} f(x,y) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \text{Slope in x direction} \\ \frac{\partial}{\partial y} f(x,y) &= \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \text{Slope in y direction} \end{aligned}$$

What if we want the slope in an arbituary direction?

(add figure here: arbi slope)

Suppose we want to find the slope along the direction of some vector $\vec{\boldsymbol{v}}$. We know $\vec{\boldsymbol{v}}$ can be decomposed into component form of $\hat{\boldsymbol{x}}/\hat{\boldsymbol{y}}$.

$$\begin{aligned} \vec{\boldsymbol{v}} &= u_x \hat{\boldsymbol{x}} + u_y \hat{\boldsymbol{y}} \\ &= (|\vec{\boldsymbol{v}}| \cos\theta)\hat{\boldsymbol{x}} + (|\vec{\boldsymbol{v}}| \sin\theta)\hat{\boldsymbol{y}} \end{aligned}$$

(add figure here: dir of v)

And the unit vector of $\hat{\boldsymbol{v}}$ can be expressed as

$$\hat{\boldsymbol{v}} = \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|} = (\cos\theta)\hat{\boldsymbol{x}} + (\sin\theta)\hat{\boldsymbol{y}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

To find the slope along $\vec{\boldsymbol{v}}$, first vary $f(x,y)$ to $f(x+v_x, y+v_y)$, and then divide by $|\vec{\boldsymbol{v}}|$. Finally limit $|\vec{\boldsymbol{v}}| \to 0$ to turn it into differentiation.

$$\underbrace{D_{\hat{\boldsymbol{v}}}}_{\text{Just a notation of differentiating in } \vec{\boldsymbol{v}}\text{'s direction}} f(x,y) = \lim_{|\vec{\boldsymbol{v}}|\to 0} \frac{f(x+v_x, y+v_y) - f(x,y)}{|\vec{\boldsymbol{v}}|}$$

$$= \lim_{|\vec{\boldsymbol{v}}|\to 0} \frac{f(x+v_x, y+v_y) - f(x, y+v_y)}{|\vec{\boldsymbol{v}}|} + \lim_{|\vec{\boldsymbol{v}}|\to 0} \frac{f(x, y+v_y) - f(x,y)}{|\vec{\boldsymbol{v}}|}$$

$$= \lim_{|\vec{\boldsymbol{v}}|\to 0} \frac{f(x+v_x, y+v_y) - f(x, y+v_y)}{v_x}\cos\theta + \lim_{|\vec{\boldsymbol{v}}|\to 0} \frac{f(x, y+v_y) - f(x,y)}{v_y}\sin\theta$$

$$= \underbrace{\boxed{\lim_{v_x\to 0} \frac{f(\boxed{x+v_x}, y+v_y) - f(\boxed{x}, y+v_y)}{\boxed{v_x}}}}_{\text{This is exactly partial x}} \cos\theta + \underbrace{\boxed{\lim_{v_y\to 0} \frac{f(x, \boxed{y+v_y}) - f(x, \boxed{y})}{\boxed{v_y}}}}_{\text{This is exactly partial y}} \sin\theta$$

$$= \left(\frac{\partial}{\partial x} f(x,y)\right)\cos\theta + \left(\frac{\partial}{\partial y} f(x,y)\right)\sin\theta$$

$$= \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} f(x,y) + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} f(x,y)\right) \cdot ((\cos\theta)\hat{\boldsymbol{x}} + (\sin\theta)\hat{\boldsymbol{y}})$$

$$= \underbrace{\begin{pmatrix} \frac{\partial}{\partial x} f(x,y) & \frac{\partial}{\partial y} f(x,y) \end{pmatrix}}_{\text{Some row vector independent of } \vec{\boldsymbol{u}}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \longleftarrow \begin{array}{c}\text{This is just } \hat{\boldsymbol{v}} \\ = (\cos\theta)\hat{\boldsymbol{x}} + (\sin\theta)\hat{\boldsymbol{y}}\end{array}$$

So the slope in $\vec{\boldsymbol{v}}$ direction can be computed by a row vector of partial D of the function, doing dot product with $\hat{\boldsymbol{v}}$. We give a special name to this row vector: **gradient vector**.

$$\vec{\boldsymbol{\nabla}} f \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_m} \end{pmatrix} = \frac{\partial f}{\partial \vec{\boldsymbol{x}}} \stackrel{\text{def}}{=} \text{grad } f$$

This symbol ∇ is usually pronounced "Del", although it is formally called "Nabla"

Sometimes we just write "grad"

The operation to compute the gradient vector of a function is called "**gradient**"

$$\vec{\boldsymbol{\nabla}}(\cdot) = \hat{\boldsymbol{x}}_1 \frac{\partial}{\partial x_1}(\cdot) + \hat{\boldsymbol{x}}_2 \frac{\partial}{\partial x_2}(\cdot) + \cdots + \hat{\boldsymbol{x}}_n \frac{\partial}{\partial x_n}(\cdot)$$

Here are two important facts about gradient vector:

1. **Gradient vector itself is NOT the slope.**

   It is just a "property"/"charateristic" of a function, which we can use to obtain the function's slope at any position in any direction. Remind that slope is a number, not a vector.

2. **Direction of gradient vector is the same as the maximum slope's direction.**

   Because $\vec{\boldsymbol{\nabla}} f \cdot \hat{\boldsymbol{v}} = |\vec{\boldsymbol{\nabla}} f||\hat{\boldsymbol{v}}|\cos\theta \leq |\vec{\boldsymbol{\nabla}} f|$, the maximum slope at any position is $|\vec{\boldsymbol{\nabla}} f|$ and this happens only if $\cos\theta = 1$, i.e. $\hat{\boldsymbol{v}}$ is parallel to $\vec{\boldsymbol{\nabla}} f$.

Gradient	Gradient Vector	Slope
$\vec{\boldsymbol{\nabla}}(\cdot)$	$\vec{\boldsymbol{\nabla}} f$	$\vec{\boldsymbol{\nabla}} f \cdot \hat{\boldsymbol{v}}$

3.2.2 On Multivariable Scalar Function

We can take gradient to each of the function's component:

$$\begin{cases} \boxed{\dfrac{\partial}{\partial \vec{\boldsymbol{x}}} F_1(x_1, ..., x_m)} = \hat{\boldsymbol{x}}_1 \dfrac{\partial}{\partial x_1} F_1(x_1, ..., x_m) + \hat{\boldsymbol{x}}_2 \dfrac{\partial}{\partial x_2} F_1(x_1, ..., x_m) + \cdots + \hat{\boldsymbol{x}}_m \dfrac{\partial}{\partial x_m} F_1(x_1, ..., x_m) \\ \qquad \vdots \qquad\qquad = \qquad\qquad \vdots \\ \boxed{\dfrac{\partial}{\partial \vec{\boldsymbol{x}}} F_n(x_1, ..., x_m)} = \hat{\boldsymbol{x}}_1 \dfrac{\partial}{\partial x_1} F_n(x_1, ..., x_m) + \hat{\boldsymbol{x}}_2 \dfrac{\partial}{\partial x_2} F_n(x_1, ..., x_m) + \cdots + \hat{\boldsymbol{x}}_m \dfrac{\partial}{\partial x_m} F_n(x_1, ..., x_m) \end{cases}$$

To make their expression easier to read, we can write them as row matrices.

$$\frac{\partial F_1}{\partial \vec{\boldsymbol{x}}} = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_m} \end{pmatrix}$$

$$\frac{\partial F_2}{\partial \vec{\boldsymbol{x}}} = \begin{pmatrix} \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} & \cdots & \dfrac{\partial F_2}{\partial x_m} \end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial F_n}{\partial \vec{\boldsymbol{x}}} = \begin{pmatrix} \dfrac{\partial F_n}{\partial x_1} & \dfrac{\partial F_n}{\partial x_2} & \cdots & \dfrac{\partial F_n}{\partial x_m} \end{pmatrix}$$

And join each row into one big matrix:

$$\frac{\partial \vec{\boldsymbol{F}}}{\partial \vec{\boldsymbol{x}}} = \underbrace{\begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_m} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} & \cdots & \dfrac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_n}{\partial x_1} & \dfrac{\partial F_n}{\partial x_2} & \cdots & \dfrac{\partial F_n}{\partial x_m} \end{pmatrix}}_{\substack{m \text{ columns for} \\ \text{function with } m \text{ inputs}}} \updownarrow \begin{matrix} n \text{ rows for} \\ \text{function with} \\ n \text{ components} \end{matrix}$$

(This is again the **Jacobian matrix**)

A Short Summary

	Single Variable	**Multivariable**
Scalar Function	$\frac{\mathrm{d}f}{\mathrm{d}t}$	$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_m} \end{pmatrix}$
Vector Function	$\begin{pmatrix} \frac{\mathrm{d}F_1}{\mathrm{d}t} \\ \frac{\mathrm{d}F_2}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}F_n}{\mathrm{d}t} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}$

3.3 Chain Rule in Matrix Form

Wtih matrix, writing chain rule for multivariable function is much cleaner. For example, let $\vec{\boldsymbol{g}}(\vec{\boldsymbol{f}}(\vec{\boldsymbol{x}}))$ be a function compositon of number of variables ($n \xrightarrow{f} m \xrightarrow{g} p$).

$$\begin{aligned} \vec{\boldsymbol{x}} &= \underbrace{(x_1, x_2, ..., x_n)} \\ \vec{\boldsymbol{f}}(\cdots) &= \underbrace{(f_1(\cdots), f_2(\cdots), ..., f_m(\cdots))} \\ \vec{\boldsymbol{g}}(\cdots) &= (g_1(\cdots), g_2(\cdots), ..., g_p(\cdots)) \end{aligned}$$

The chain rule can be written as a matrix multiplication.

$$\begin{aligned} \frac{\partial \vec{\boldsymbol{g}}}{\partial \vec{\boldsymbol{x}}} &= \frac{\partial \vec{\boldsymbol{g}}}{\partial \vec{\boldsymbol{f}}} \cdot \frac{\partial \vec{\boldsymbol{f}}}{\partial \vec{\boldsymbol{x}}} \\ \binom{p \times n}{\text{matrix}} &= \binom{p \times m}{\text{matrix}} \binom{m \times n}{\text{matrix}} \end{aligned}$$

$$\begin{pmatrix} & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & \frac{\partial g_i}{\partial x_j} & \cdots & \cdots \\ & & \vdots & & \\ & & \vdots & & \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \frac{\partial g_i}{\partial f_1} & \frac{\partial g_i}{\partial f_2} & \cdots & \cdots & \frac{\partial g_i}{\partial f_m} \\ \vdots & \vdots & & & \vdots \end{pmatrix} \begin{pmatrix} \cdots & \cdots & \frac{\partial f_1}{\partial x_j} & \cdots \\ \cdots & \cdots & \frac{\partial f_2}{\partial x_j} & \cdots \\ & & \vdots & \\ & & \vdots & \\ \cdots & \cdots & \frac{\partial f_m}{\partial x_j} & \cdots \end{pmatrix}$$

i^{th}row j^{th}column i^{th}row Take every element j^{th}column Take every element

As individual terms which is

$$\begin{aligned}\frac{\partial g_i}{\partial x_j} &= \sum_{k=1}^{m} \frac{\partial g_i}{\partial f_k} \cdot \frac{\partial f_k}{\partial x_j} \\ &= \frac{\partial g_i}{\partial f_1} \cdot \frac{\partial f_1}{\partial x_j} + \frac{\partial g_i}{\partial f_2} \cdot \frac{\partial f_2}{\partial x_j} + \cdots + \frac{\partial g_i}{\partial f_m} \cdot \frac{\partial f_m}{\partial x_j}\end{aligned}$$

Example 3.1. Recall this example we have seen in note of multivariable calculus,

$$\begin{cases} f(p,q) = \sqrt{p+q} \\ \vec{h}(u,v) = (u^2+v, u-v) \end{cases} \qquad \Rightarrow \qquad f(\boldsymbol{h}(\vec{\boldsymbol{u}}, \boldsymbol{v})) = \sqrt{u^2+u}$$

We can first express their derivatives in matrix form.

$$\begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{p+q}} & \frac{1}{2\sqrt{p+q}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & 1 \\ 1 & -1 \end{pmatrix}$$

The chain rule is therefore expressed as

$$\begin{aligned}\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} &= \begin{pmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \end{pmatrix} \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2\sqrt{p+q}} & \frac{1}{2\sqrt{p+q}} \end{pmatrix} \Bigg|_{\substack{p=u^2+v \\ q=u-v}} \begin{pmatrix} 2u & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2u+1}{2\sqrt{u^2+u}} & 0 \end{pmatrix}\end{aligned}$$

Exercise 3.1. Given the functions and their composition:

$$\begin{cases} f(p,q) = \sqrt{p+q} \\ \vec{g}(t) = (t-1, t^2) \end{cases} \qquad \Rightarrow \qquad f(\boldsymbol{g}(\vec{\boldsymbol{t}})) = \sqrt{t^2+t-1}$$

Compute the derivative $\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{\boldsymbol{g}}(t))$, this time by chain rule in the matrix expression.

---

4 Line Integral

4.1 Parametrizing Curves in Space

Recall that a single variable vector function is essentially describing a curve in a space.

$$\vec{\boldsymbol{r}} = (x(t), y(t))$$

(add figure here: r(t))

Any point on the curve only needs 1 input (t) to fully locate it. (Intuitively, Curve = 1D object = only 1 free variable.)

Here we introduce the idea of **parametrization**: Choose a lower-dimension coordinate system on the object to describe every point on it, rather than using the environmental x/y/z coordinate.

(add figure here: parametrization)

We can also do it to higher dimension objects, but the maths are way more complicated.

(add figure here: surface parametrization)

<u>Note</u>: Parametrization to an object is never unique, because there can be infinitely many ways to choose a coordinate system.

Example 4.1. Parametrizing the curve $y = 3x^{\frac{3}{2}}$.

- Choice 1: Let $x = t^2$, then $y = 3(t^2)^{\frac{3}{2}} = 3t^2 \quad \Rightarrow \quad$ Parametrize as $\vec{\boldsymbol{r}}(t) = (t^2, 3t^3)$.
- Choice 2: Let $x = t$, then $y = 3t^{\frac{3}{2}} \quad \Rightarrow \quad$ Parametrize as $\vec{\boldsymbol{r}}(t) = (t, 3t^{\frac{3}{2}})$

Here are some example parametrization to common objects.

- <u>Straight line</u>

$$\vec{\boldsymbol{r}}(t) = (x(t), y(t)) = (x_0 + td_x, y_0 + td_y)$$

(add figure here: straight line para)

- <u>Ellipse / Circle</u>

$$\vec{\boldsymbol{r}}(t) = (x(t), y(t)) = (x_0 + a\cos{(\omega t + \phi)}, y_0 + b\sin{(\omega t + \phi)})$$

(If $a = b$, it becomes a circle.)

(add figure here: circle para)

4.2 Line Integral on Scalar Functions

Recall that

- $\int f(x, y)\,\mathrm{d}x$ = Integrate along x-axis, at constant y.
- $\int f(x, y)\,\mathrm{d}y$ = Integrate along y-axis, at constant x.

(add figure here: intx inty)

What about integrating along an arbituary curve?

(add figure here: int arb curve)

Recall that in integration on single variable function, we first interpret it as a sum of area under curve. We can write something similar for integration along an arbituary line.

$$\int f(x)\,\mathrm{d}x = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} \underbrace{f(\xi_i)}_{\text{Height of strip}} \overbrace{\Delta x_i}^{\text{Width of strip}} \quad \Rightarrow \quad \int f(x,y)\,\mathrm{d}\vec{\boldsymbol{r}} = \lim_{|\Delta \vec{\boldsymbol{r}}_i| \to 0} \sum_{i=1}^{n} \underbrace{f(\xi_{i,x}, \xi_{i,y})}_{\text{Height of strip}} \overbrace{|\Delta \vec{\boldsymbol{r}}_i|}^{\text{Width of strip}}$$

While the heights of the strips are simply the function's values, the widths need to be estimated by Pythegoras theorem.

(add figure here: int para curve)

$$\begin{aligned}
\text{Width of interval } |\Delta \vec{\boldsymbol{r}}_i| &= \sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2} \\
&= |t_{i+1} - t_i| \sqrt{\left[\frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}\right]^2 + \left[\frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}\right]^2} \\
&= |\Delta t_i| \sqrt{\left[\frac{\Delta x_i}{\Delta t_i}\right]^2 + \left[\frac{\Delta y_i}{\Delta t_i}\right]^2} \\
\lim_{\Delta t_i \to 0} |\Delta \vec{\boldsymbol{r}}_i| &= \mathrm{d}t \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}
\end{aligned}$$

So for the line integral calculation:

$$\begin{aligned}
\boxed{\int f(x,y)\,\mathrm{d}\vec{\boldsymbol{r}}} &= \lim_{|\Delta \vec{\boldsymbol{r}}_i| \to 0} \sum_{i=1}^{n} f(\xi_{i,x}, \xi_{i,y}) |\Delta \vec{\boldsymbol{r}}_i| \\
&= \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} f(\xi_{i,x}, \xi_{i,y}) |\Delta t_i| \sqrt{\left[\frac{\Delta x_i}{\Delta t_i}\right]^2 + \left[\frac{\Delta y_i}{\Delta t_i}\right]^2} \\
&= \boxed{\int f(x(t), y(t)) \sqrt{\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right)^2}\,\mathrm{d}t}
\end{aligned}$$

The notation you can find in textbook

This is how you really calculate line integral e.g. along some line on x-y plane. (You have to decide how to parametrize the curve!)

In addition, We can also interpret line integral as a weighted sum, like in single variable integral.

$$\int f(x,y)\mathrm{d}\vec{\boldsymbol{r}} \stackrel{\text{def}}{=} \lim_{\Delta x_i \to 0} \underbrace{\sum_{i=0}^{n-1}}_{\text{Sum them all}} \underbrace{f(\xi_{i,x}, \xi_{i,y})}_{\substack{\text{"weight"} \\ \text{assigned to the interval} \\ [(x_i,y_i),(x_{i+1},y_{i+1})]}} \underbrace{|\Delta \vec{\boldsymbol{r}}_i|}_{\substack{\text{length of} \\ \text{interval} \\ [(x_i,y_i),(x_{i+1},y_{i+1})]}}$$

(add figure here: weight sum interpret)

More on notations

1. Extend to N-variables Functions
   The notation for line integral for general multivariable function is by writing the variables in f as a vector $\vec{\boldsymbol{r}}$

$$\boxed{\int f(\vec{\boldsymbol{r}})\,\mathrm{d}\vec{\boldsymbol{r}}} \xRightarrow{\text{When calculate}} \boxed{\int f(x_1,...,x_n)\sqrt{\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right)^2+...+\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2}\,\mathrm{d}t}$$

2. Integration Range
   Single variable integration only requires knowing the upper and lower bounds because the integration range is a straight line. However in line integral, we must describe the whole curve, e.g. provide the function of the curve, the starting and ending points, etc.

$$\int_a^b f(x)\,\mathrm{d}x \qquad \Rightarrow \qquad \int_{\substack{\ldots\ldots\ldots \\ \text{TL,DR}}} f(\vec{\boldsymbol{r}})\,\mathrm{d}\vec{\boldsymbol{r}}$$

   This is too many words to write under the integral sign. So conventionally, we just write a symbol C under the integral sign to indicate that this integral is along some curve, and then describe the curve in additional texts.

$$\int_C f(\vec{\boldsymbol{r}})\,\mathrm{d}\vec{\boldsymbol{r}} \qquad \text{with} \quad C = \text{Curve of XXX...}$$

3. Loop Integral
   It is possible that the curve to be integrated along forms a closed loop (e.g. a circle). A special symbol is assigned specifically for this use case:

$$\oint_C f(\vec{\boldsymbol{r}})\,\mathrm{d}\vec{\boldsymbol{r}}$$

(add figure here: loop int)

   This is because loop integral has some interesting properties and appears in many theorems. We shall encounter them later, especially in electrodynamics.

4.3 Line Integral on Vector Functions

Observe that $f(x,y)\mathrm{d}r$ is a muliplication between a number and a vector. If $f(x,y)$ is to be replaced by a vector function, we have 2 possibilities:

- Dot Product Line Integral

$$\int \vec{\boldsymbol{F}}(x,y) \cdot \mathrm{d}\vec{\boldsymbol{r}} = \lim_{\Delta\vec{\boldsymbol{r}}_i \to 0} \sum_{i=1}^{n} \vec{\boldsymbol{F}}(\xi_{i,x},\xi_{i,y}) \cdot |\Delta\vec{\boldsymbol{r}}_i|$$

- Cross Product Line Integral

$$\int \vec{\boldsymbol{F}}(x,y) \times \mathrm{d}\vec{\boldsymbol{r}} = \lim_{\Delta\vec{\boldsymbol{r}}_i \to 0} \sum_{i=1}^{n} \vec{\boldsymbol{F}}(\xi_{i,x}, \xi_{i,y}) \times |\Delta\vec{\boldsymbol{r}}_i|$$

(Because order matters in cross product, it is quite common to see $\int_C \mathrm{d}\vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$)

Calculation

Here demonstrates with dot product line integral. Cross product is exactly the same.

$$\int \vec{\boldsymbol{F}}(x,y) \cdot \mathrm{d}\vec{\boldsymbol{r}} = \lim_{\Delta\vec{\boldsymbol{r}}_i \to 0} \sum_{i=1}^{n} \boxed{\vec{\boldsymbol{F}}(\xi_{i,x}, \xi_{i,y}) \cdot |\Delta\vec{\boldsymbol{r}}_i|}$$

Notice that this is a summation of all dot product along a curve. If we express each dot product by the curve parameter t:

$$\begin{aligned}
\vec{\boldsymbol{F}}(\boldsymbol{x}(\boldsymbol{t_i}), \boldsymbol{y}(\boldsymbol{t_i})) \cdot \Delta\vec{\boldsymbol{r}}_i &= \vec{\boldsymbol{F}}(x(t_i), y(t_i)) \cdot [\vec{\boldsymbol{r}}(t_{i+1}) - \vec{\boldsymbol{r}}(t_i)] \\
&= \vec{\boldsymbol{F}}(x(t_i), y(t_i)) \cdot \left[\frac{\vec{\boldsymbol{r}}(t_{i+1}) - \vec{\boldsymbol{r}}(t_i)}{t_{i+1} - t_i}\right](t_{i+1} - t_i) \\
&= \vec{\boldsymbol{F}}(x(t_i), y(t_i)) \cdot \left(\frac{\Delta\vec{\boldsymbol{r}}_i}{\Delta t_i}\right)\Delta t_i \\
\lim_{\Delta t_i \to 0} \left(\vec{\boldsymbol{F}}(x(t_i), y(t_i)) \cdot \Delta\vec{\boldsymbol{r}}_i\right) &= \left(\vec{\boldsymbol{F}}(x,y) \cdot \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t}\right)\mathrm{d}t
\end{aligned}$$

So for the line integral calculation:

$$\begin{aligned}
\boxed{\int \vec{\boldsymbol{F}}(x,y) \cdot \mathrm{d}\vec{\boldsymbol{r}}} &= \int \vec{\boldsymbol{F}}(x(t), y(t)) \cdot \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\hat{\boldsymbol{x}} + \frac{\mathrm{d}y(t)}{\mathrm{d}t}\hat{\boldsymbol{y}}\right)\mathrm{d}t \\
&= \boxed{\int \left[F_x(x(t), y(t))\frac{\mathrm{d}x}{\mathrm{d}t} + F_y(x(t), y(t))\frac{\mathrm{d}y}{\mathrm{d}t}\right]\mathrm{d}t}
\end{aligned}$$

The notation you can find in textbook

This is how you really calculate line integral e.g. along some line on x-y plane. (You have to decide how to parametrize the curve!)

Geometrical intepretation

The function $vvecF(x,y)$ can be plotted as a "field of vectors", i.e. at each point (x,y), there is a vector (F_x, F_y)

(add figure here: vector field)

The quantity $\vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}}$ is like,

- Find the vector on each segment
- Find the interval of each insegment as a displacement vector

Then compute the such dot product for each segment and sum them all.

(add figure here: sum dot prod)

Example 4.2. An object is moving on a surface with a positional dependent friction force

$$\vec{\boldsymbol{F}}(x,y) = (x^2y, xy+1)$$

Suppose the object is moving in a circular trajectory that is

- Center at $(1,2)$, radius $= 3$
- Starts at $(1,5)$, ends at $(4,2)$
- Travelling in anticlockwise direction

Find the W.D. due to friction on this trajectory.

1. Begin with parametrizing the trajectory

   (add figure here: trajectory)

   Recall that parametrization for circle is usaully in the from

   $$\begin{cases} x(\theta) = x_0 + R\cos(\theta+\phi) \\ y(\theta) = y_0 + R\sin(\theta+\phi) \end{cases}$$

   From the given information, we can identify

   $$(x_0, y_0) = (1,2) \quad , \quad R = 3 \quad , \quad \phi = 0$$
   $$\Rightarrow \begin{cases} x(\theta) = 1 + 3\cos\theta \\ y(\theta) = 2 + 3\cos\theta \end{cases}$$

   The start/end correspond to

   $$\begin{cases} \theta = 0 & \Leftrightarrow & (x,y) = (1,5) \\ \theta = \dfrac{\pi}{2} & \Leftrightarrow & (x,y) = (2,4) \end{cases}$$

2. Substitute into the dot product line integral

   $$\begin{aligned} \text{W.D.} &= \int_C \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} \\ &= \int_C \vec{\boldsymbol{F}}(x,y) \cdot \frac{\mathrm{d}\vec{\boldsymbol{r}}(\theta)}{\mathrm{d}\theta}\,\mathrm{d}\theta \\ &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \begin{pmatrix} x^2y & xy+1 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\theta}\begin{pmatrix} x \\ y \end{pmatrix} \mathrm{d}\theta \\ &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[x^2y\frac{\mathrm{d}x}{\mathrm{d}\theta} + (xy+1)\frac{\mathrm{d}y}{\mathrm{d}\theta} \right] \mathrm{d}\theta \\ &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left[\underbrace{(1+3\cos\theta)^2}_{x^2}\underbrace{(2+3\sin\theta)}_{y}\underbrace{(-3\sin\theta)}_{\frac{\mathrm{d}x}{\mathrm{d}\theta}} + \left[\underbrace{(1+3\cos\theta)}_{x}\underbrace{(2+3\sin\theta)}_{y}\underbrace{+1}_{+1}\right]\underbrace{3\cos\theta}_{\frac{\mathrm{d}y}{\mathrm{d}\theta}} \right] \mathrm{d}\theta \\ &= \cdots \end{aligned}$$

   The remaining steps are just solving an annoying single variable integration, which you all should known how to do so.

4.4 Gradient Theorem

If $f(\vec{\boldsymbol{r}})$ is a <u>continuous scalar function</u>, and $\vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{r}})$ is its gradient vector (field), then the line integral

$$\int_{\substack{\text{start}\\\text{from } \vec{\boldsymbol{r}}_1}}^{\substack{\text{end}\\\text{at } \vec{\boldsymbol{r}}_2}} \vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}} = f(\vec{\boldsymbol{r}}_2) - f(\vec{\boldsymbol{r}}_1)$$

is <u>independent of what curve is integrated along</u>. (which means we can skip all the steps of curve parametrization.)

(add figure here: int all equal)

To illustrate, recall that $\vec{\boldsymbol{\nabla}} f \cdot \hat{\boldsymbol{u}} =$ slope in $\hat{\boldsymbol{u}}$'s direction. So,

$$\begin{aligned}
\vec{\boldsymbol{\nabla}} f \cdot \mathrm{d}\vec{\boldsymbol{r}} &= \vec{\boldsymbol{\nabla}} f \cdot \underbrace{\boxed{\frac{\mathrm{d}\vec{\boldsymbol{r}}}{|\,\mathrm{d}\vec{\boldsymbol{r}}\,|}}}_{\text{unit vector}} |\,\mathrm{d}\vec{\boldsymbol{r}}\,| \\
&= \begin{pmatrix} \text{slope in} \\ \mathrm{d}\vec{\boldsymbol{r}} \text{ direction} \end{pmatrix} \times \begin{pmatrix} \text{Base} \\ \text{length} \end{pmatrix} \\
&= \begin{pmatrix} \text{Change in height} \\ \text{along } \mathrm{d}\vec{\boldsymbol{r}} \end{pmatrix}
\end{aligned}$$

(add figure here: change in height)

Therefore,

$$\begin{aligned}
\int_{\substack{\text{start}\\\text{from } \vec{\boldsymbol{r}}_1}}^{\substack{\text{end}\\\text{at } \vec{\boldsymbol{r}}_2}} \vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}} &= \text{Sum of all } \vec{\boldsymbol{\nabla}} f \cdot \mathrm{d}\vec{\boldsymbol{r}} \text{ along a curve from } \vec{\boldsymbol{r}}_1 \text{ to } \vec{\boldsymbol{r}}_2 \\
&= \text{Net height change by travelling from } \vec{\boldsymbol{r}}_1 \text{ to } \vec{\boldsymbol{r}}_2
\end{aligned}$$

And this should be intuitive - when the landscape is continuous, the net height change should be always independent of which path is taken.

(add figure here: path taken)

4.5 Application: Conservative Force & Potential

Any vector function $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$ is **conservative** if it equals to the gradient vector of some scalar function $U(\vec{\boldsymbol{r}})$.

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}} U(\vec{\boldsymbol{r}})$$

Vector field
= the force

Have a minus sign
by definition

Scalar function
= the potential energy

If the force's vector field is conservative, it has a nice property:

$$\begin{aligned}\begin{matrix}\text{Total W.D. along}\\ \text{any path between } \vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2\end{matrix} &= \int_{\vec{r}_1 \to \vec{r}_2} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} \\ &= \int_{\vec{r}_1 \to \vec{r}_2} -\vec{\boldsymbol{\nabla}} U \cdot \mathrm{d}\vec{\boldsymbol{r}} \\ &= -(U(\vec{\boldsymbol{r}}_2) - U(\vec{\boldsymbol{r}}_1))\end{aligned}$$

Because of gradient theorem, all we need to know are just the start and end points. Taking any path will cost the same work done.

Example 4.3. We all know that gravitional force is a conservative vector function.

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = \frac{GMm}{|\vec{\boldsymbol{r}}|^2}$$

which is why we can compute the gravitational potential energy change by

$$\Delta U(\vec{\boldsymbol{r}}) = -\int_{\vec{r}_1 \to \vec{r}_2} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}} = -\left(\frac{GMm}{|\vec{\boldsymbol{r}}_2|^2} - \frac{GMm}{|\vec{\boldsymbol{r}}_2|^2}\right)$$

without EVER doing annoying line integral. (And so it can appear in your high school syllabus)

(add figure here: UFO)

<u>*Side note:*</u>

To prove that a force field is conservative, we have to show that its *curl = 0*, i.e.

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}} = 0$$

However we will not touch this devil until E&M.

— The End —