

Non-Cartesian Coordinate

by Tony Shing

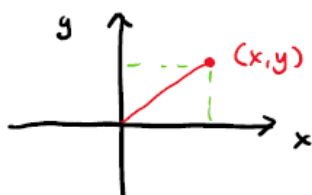
Overview:

- 2D: Polar coordinate
- 3D: Cylindrical coordinate & Spherical coordinate

1 Polar Coordinate

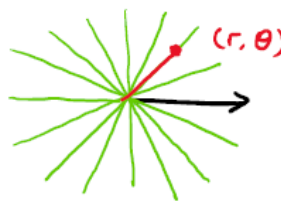
You should have already learnt what polar coordinate is from high school, and its conversion with rectangular coordinate.

From (x, y) to (r, θ)



$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left(\frac{y}{x} \right) \end{cases}$$

From (r, θ) to (x, y)



$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

What about double integral over an area that is represented by polar coordinate?

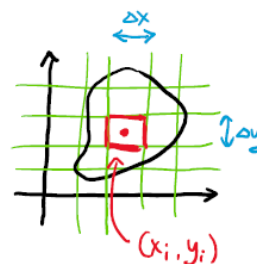
$$\boxed{\iint f(x, y) \, dx \, dy \quad \Leftrightarrow \quad \iint f(r, \theta) \, \underline{r} \, dr \, d\theta}$$

Caution: An extra r

The reason for this extra r comes from the dimension of the area element.

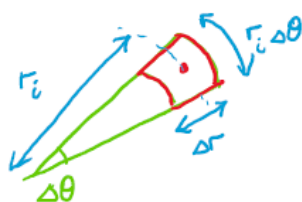
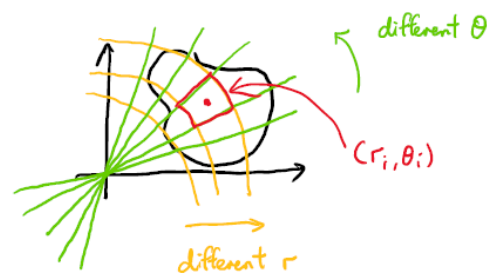
- In x/y coordinate, area of each grid is fixed.

$$I = \sum_i \underbrace{f(x_i, y_i)}_{\text{"Weight" assigned to the point } (x_i, y_i)} \cdot \underbrace{\Delta x \Delta y}_{\text{Each grid are of area } \Delta x \Delta y} \sim \iint f(x, y) \, dx \, dy$$



- In polar coordinate, area of the grids depends on its r coordinate. When $\Delta\theta$ is very small, the grid's area $\approx (\text{height}) \times (\text{width}) \approx \Delta r \times r_i \Delta\theta$. So

$$I = \sum_i \underbrace{f(r_i, \theta_i)}_{\text{"Weight" assigned to the point } (r_i, \theta_i)} \cdot \underbrace{r_i \cdot \Delta r \Delta\theta}_{\substack{\text{The grid's area} \\ \text{depends on } r \text{ coordinate} \\ = \Delta r \times r_i \Delta\theta}} \sim \iint f(r, \theta) r \, dr \, d\theta$$



Each grid's area depends on its r coordinate.

When $\Delta\theta$ is very small,
grid's area $\approx (\text{height}) \times (\text{width})$
 $\approx (\Delta r) \times (r \Delta\theta)$

2 Cylindrical Coordinate

Cylindrical coordinate is essentially polar coordinate + z-axis.

From (x, y, z) to (r, θ, z)

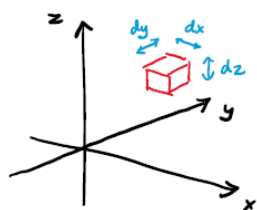
$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1} \left(\frac{y}{x} \right) \\ z = z \end{cases}$$

From (r, θ, z) to (x, y, z)

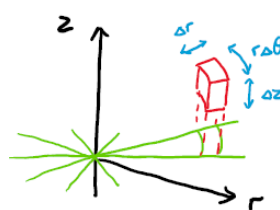
$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

Therefore the triple integral expression is very similar to the double integral in polar coordinate, with an extra r present in the volume element.

$$\iiint f(x, y, z) \, dx \, dy \, dz \quad \Leftrightarrow \quad \iiint f(r, \theta, z) \, \underline{r} \, dr \, d\theta \, dz$$

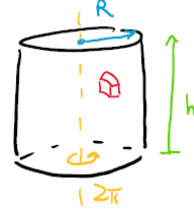


\Leftrightarrow



Example 2.1. Given a solid cylinder with mass density distribution $\rho(r, \theta, z) = r^2$, and dimension: radius = R , height = H . Making use of cylindrical coordinate,

- Mass of each volume element = (density) \times (volume) = $\rho(r, \theta, z) r \, dr \, d\theta \, dz$
- Total mass = $\iiint \rho(r, \theta, z) r \, dr \, d\theta \, dz$
- Upper/Lower bound for each dimension are:
 - Range of r : From $r = 0$ to $r = R$
 - Range of θ : From $\theta = 0$ to $\theta = 2\pi$ (whole circle)
 - Range of z : From $z = 0$ to $z = H$



The calculation of the total mass is then

$$\begin{aligned}
 & \int_{z=0}^{z=H} \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \overset{\text{density function } \rho(r, \theta, z) = r^2}{\underline{r^2}} \cdot r \, d\theta \, dr \, dz \\
 &= \int_{z=0}^{z=H} \int_{r=0}^{r=R} \underbrace{\left(\int_{\theta=0}^{\theta=2\pi} r^3 \, d\theta \right)}_{\text{First integrate } \theta} \, dr \, dz \\
 &= \int_{z=0}^{z=H} \int_{r=0}^{r=R} \left[r^3 \theta \right]_{\theta=0}^{\theta=2\pi} \, dr \, dz \\
 &= \int_{z=0}^{z=H} \underbrace{\left(\int_{r=0}^{r=R} 2\pi r^3 \, dr \right)}_{\text{Then integrate } r} \, dz \\
 &= \int_{z=0}^{z=H} \left[\frac{2\pi r^4}{4} \right]_{r=0}^{r=R} \, dz \\
 &= \underbrace{\int_{z=0}^{z=H} \frac{\pi R^4}{2} \, dz}_{\text{Finally integrate } z} \\
 &= \frac{\pi R^4 H}{2}
 \end{aligned}$$

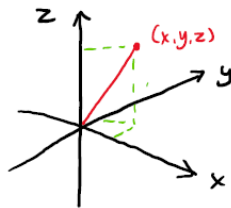
3 Spherical Coordinate

Spherical coordinate is a description of position on a sphere by 3 parameters:

- r = Radius, \sim Altitude
- θ = Polar angle, \sim Latitude
- ϕ = Azimuthal angle, \sim Longitude

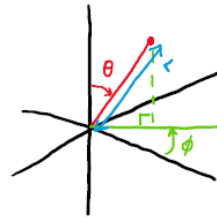
The conversion to rectangular coordinate is as follow:

From (x, y, z) to (r, θ, ϕ)



$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right) \\ \phi = \tan^{-1} \left(\frac{y}{x} \right) \end{cases}$$

From (r, θ, ϕ) to (x, y, z)

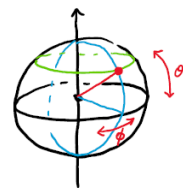


$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

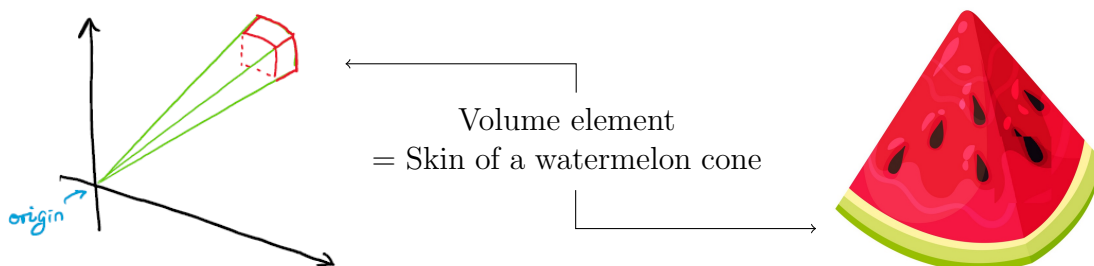
Caution 1: The notations here are adopting the **physics convention**, where ϕ is the angle on the x - y plane and θ is the inclination to the z -axis. This is the convention found in modern Physics textbooks. However in many mathematics textbook and older physics books, you may find the **mathematics convention** where the meaning of ϕ and θ are swapped.

Caution 2: The choice of the angles are not the same as we use geography. Note that

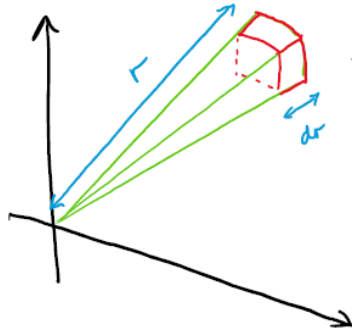
- Range of $\theta = [0, \pi]$. But in geography, latitude angle is $90^\circ - \theta$, which ranged between $[-90^\circ, 90^\circ]$.
- Range of $\phi = [0, 2\pi)$. But in geography, longitude is ranged between $(-180^\circ, 180^\circ]$.



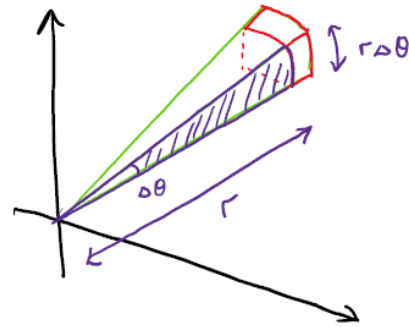
When dealing with triple integral, the unit volume in spherical coordinate is $\sim (\Delta r) \times (r \Delta \theta) \times (r \sin \theta \Delta \phi)$. We may visualize as follows:



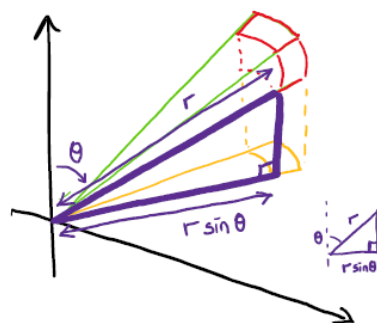
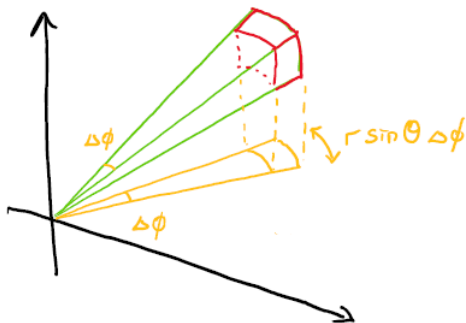
The side along r direction: dr



The side along θ direction: $r d\theta$



The side along ϕ direction: $r \sin \theta d\phi$



The radius is $r \sin \theta$, not r .
Because it is the radius's
projection

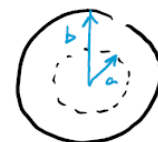
As a result, the triple integral has to be written as

$$\iiint f(x, y, z) dx dy dz \Leftrightarrow \iiint f(r, \theta, \phi) \underline{r^2 \sin \theta} dr d\theta d\phi$$

Caution: An extra $r^2 \sin \theta$

Example 3.1. Given a hollow but thick sphere with radius range from $r = a$ to $r = b$, and with mass density distribution $\rho(r, \theta, \phi) = r^4$. Making use of spherical coordinate,

- Mass of each volume element = (density) \times (volume) = $\rho(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$
- Total mass = $\iiint \rho(r, \theta, \phi) r^2 \sin \theta dr d\theta d\phi$
- Upper/Lower bound for each dimension are:
 - Range of r : From $r = a$ to $r = b$
 - Range of θ : From $\theta = 0$ to $\theta = \pi$
 - Range of ϕ : From $\phi = 0$ to $\phi = 2\pi$



The calculaton of the total mass is then

$$\begin{aligned}
 & \int_{r=a}^{r=b} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \text{density function } \rho(r, \theta, \phi) = r^4 \cdot r^2 \sin \theta \, d\theta \, d\phi \, dr \\
 &= \int_{r=a}^{r=b} \int_{\phi=0}^{\phi=2\pi} \underbrace{\left(\int_{\theta=0}^{\theta=\pi} r^6 \sin \theta \, d\theta \right)}_{\text{First integrate } \theta} d\phi \, dr \\
 &= \int_{r=a}^{r=b} \int_{\phi=0}^{\phi=2\pi} \left[-r^6 \cos \theta \right] \bigg|_{\theta=0}^{\theta=\pi} d\phi \, dr \\
 &= \int_{r=a}^{r=b} \underbrace{\left(\int_{\phi=0}^{\phi=2\pi} 2r^6 \, d\phi \right)}_{\text{Then integrate } \phi} dr \\
 &= \int_{r=a}^{r=b} \left[2r^6 \phi \right] \bigg|_{\phi=0}^{\phi=2\pi} dr \\
 &= \underbrace{\int_{r=a}^{r=b} 4\pi r^6 \, dr}_{\text{Finally integrate } r} \\
 &= \frac{4\pi}{7} (b^7 - a^7)
 \end{aligned}$$

— The End —