Edited 2025

Angular Momentum and Rotational KE

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Overview:

- Review: Center of Mass
- Mathematical Origin of $\begin{cases} \text{Angular Momemtum} \\ \text{Moment of Inertia} \\ \text{Parallel Axis Theorem} \end{cases}$
- Rotational KE

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1 Review: Center of Mass

Given a set of point objects with individual masses $\{m_1, m_2, ..., m_n\}$ at positions $\{\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2, ..., \vec{\boldsymbol{r}}_n\}$, the coordinate of center of mass of these objects can be calculate with the formula:

$$\boxed{\vec{\boldsymbol{r}}_{CM} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + ... + m_n\vec{\boldsymbol{r}}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum_i^n m_i\vec{\boldsymbol{r}}_i}{\sum_i^n m_i}}$$

In case of a continuous distribution of masses, it becomes an integration:

$$\boxed{\vec{\boldsymbol{r}}_{CM} = \frac{\int \vec{\boldsymbol{r}}\,\mathrm{d}m}{\int \mathrm{d}m} \sim \frac{\int \vec{\boldsymbol{r}}\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{M_{\text{total}}}}$$

with ρ being the density as a function of position.

Proof

The formula can be proven by mathematical induction.

- Case $n = 2$:
  Starting with 2 masses m_1 and m_2 at positions $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$ respectively, the **center of mass** is defined as the position where the net torque under gravity $= 0$. Assume both masses are on the x axis to simplify our visualization:

  (add figure here: two mass on x axis)

  Let the x coordinates of the masses be x_1 and x_2, and the coordinate of the center of mass be x_{CM}. Consider the torques with pivot at x_{CM}:

  $$\begin{aligned} \circlearrowleft: \quad m_1 g(x_{CM} - x_1) &= m_2 g(x_2 - x_{CM}) \\ (m_1 + m_2)x_{CM} &= m_1 x_1 + m_2 x_2 \\ x_{CM} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \end{aligned}$$

- Case $n = k + 1$:
  Assume the formula works for k masses, i.e.

  $$x_{CM}^{(k)} = \frac{m_1 x_1 + m_2 x_2 + ... + m_k x_k}{m_1 + m_2 + ... + m_k}$$

  The center of mass is where the net torque under gravity $= 0$ by the two objects - [Group of the first k masses] and the $[(k+1)^{\text{th}}$ mass]

  $$\begin{aligned} \circlearrowleft: \quad (m_1 + m_2 + ... + m_k)g(x_{CM}^{(k+1)} - x_{CM}^{(k)}) &= m_{k+1} g(x_{k+1} - x_{CM}^{(k+1)}) \\ (m_1 + m_2 + ... + m_k + m_{k+1})x_{CM}^{(k+1)} &= (m_1 + m_2 + ... + m_k)x_{CM}^{(k)} + m_{k+1}x_{k+1} \\ &= (m_1 x_1 + m_2 x_2 + ... + m_k x_k) + m_{k+1}x_{k+1} \\ x_{CM}^{(k+1)} &= \frac{m_1 x_1 + m_2 x_2 + ... + m_{k+1} x_{k+1}}{m_1 + m_2 + ... + m_{k+1}} \end{aligned}$$

  So the formula also works for $k + 1$ masses. Because the same operation can be done on y and z coordinates, we can promote this formula to the vector expression

  $$\vec{\boldsymbol{r}}_{CM} = \frac{m_1 \vec{\boldsymbol{r}}_1 + m_2 \vec{\boldsymbol{r}}_2 + ... + m_n \vec{\boldsymbol{r}}_n}{m_1 + m_2 + ... + m_n}$$

  $\square$

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2 Torque, Angular Momentum & Moment of Inertia

2.1 Mathematical Origin

Recall that if $\mathrm{d}\vec{\boldsymbol{r}}\,/\vec{\boldsymbol{v}}/\vec{\boldsymbol{a}}$ are given, we can isolate their angular components $\mathrm{d}\theta\,/\omega/\alpha$ by taking $\vec{\boldsymbol{r}}\times$. We can do the same to force $\vec{\boldsymbol{F}}$ to arrive at the definition of **torque**.

$$\boxed{\vec{\boldsymbol{\tau}} \stackrel{\text{def}}{=} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}}$$

Continue deriving,

$$\begin{aligned}
\vec{\boldsymbol{\tau}} &\stackrel{\text{def}}{=} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} \\
&= r\hat{\boldsymbol{r}} \times m\vec{\boldsymbol{a}} \\
&= rma_\theta \hat{\boldsymbol{z}} \\
&= rm\left[2\frac{\mathrm{d}r}{\mathrm{d}t}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} + r\cdot\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right]\hat{\boldsymbol{z}} && \text{(Just the Corriolis and Euler terms)} \\
&= m\left[\frac{\mathrm{d}r^2}{\mathrm{d}t}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} + r^2\cdot\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right]\hat{\boldsymbol{z}} && \left(1^{\text{st}}\text{ term: } 2r\mathrm{d}r \to \mathrm{d}(r^2)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{\boldsymbol{z}} && \text{(product rule)}
\end{aligned}$$

Comparing with relations between force and momentum, we can define a similar quantity that represents the angular component of momentum, i.e. the **angular momentum** L,

$$\vec{\boldsymbol{F}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{\boldsymbol{v}}) = \frac{\mathrm{d}}{\mathrm{d}t}(\text{momentum}) \qquad \text{v.s.} \qquad \vec{\boldsymbol{\tau}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{z}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\text{Something}\\ \text{momentum-like}\end{pmatrix}$$

$$\boxed{\vec{\boldsymbol{L}} \stackrel{\text{def}}{=} \vec{\boldsymbol{r}} \times (m\vec{\boldsymbol{v}}) = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{z}}}$$

And since momentum is the product of an inertia quantity (mass) and velocity, we can propose that the angular momentum is also a product of some new inertial quantity and the velocity in rotation (angular velocity). So we arrive at the definition of **moment of inertia** I.

$$\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}} = (\text{inertia})\cdot(\text{velocity}) \qquad \text{v.s.} \qquad \vec{\boldsymbol{L}} = (mr^2)\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = \begin{pmatrix}\text{something}\\ \text{related to}\\ \text{inertia}\end{pmatrix}\cdot\begin{pmatrix}\text{Angular}\\ \text{velocity}\end{pmatrix}$$

$$\boxed{I \stackrel{\text{def}}{=} mr^2}$$

2.2 Rules of Using Rotational Quantities

Unlike their linear motion counterparts, the rotational motion quantities are not universal.

1. **Angular quantites dependent on the choice of coordinate system**

(add figure here: moment of inertia coordinate dependent)

As illustrated, observers from different coordinate system may describe the same object with different position vectors, so their observed angular velocity, torque, angular momentum, and moment of inertia are all different.

	Observed by A	Observed by B
$\vec{\omega}$	$\vec{r}_A \times \vec{v}$	$\vec{r}_B \times \vec{v}$
$\vec{\tau}$	$\vec{r}_A \times \vec{F}$	$\vec{r}_B \times \vec{F}$
$\vec{L}$	$m\vec{r}_A \times \vec{v}$	$m\vec{r}_B \times \vec{v}$
I	$m\lvert\vec{r}_A\rvert^2$	$m\lvert\vec{r}_B\rvert^2$

The only special case is when the observers share the same origin - then they will observe the same moment of inertia. Therefore you should use an origin at a fixed point.

2. **I is not always well-defined in multi-body system**

(add figure here: many body for I)

Suppose we have many objects moving with their individual velocities. As we know that the net torque can be found by summing each object's contribution, so does the angular momentum.

$$\vec{\tau}_{\text{total}} = \sum_i \vec{\tau}_i \quad \Rightarrow \quad \vec{L}_{\text{total}} = \sum_i (m_i \vec{r}_i \times \vec{v}_i) = \sum_i \left(m_i |\vec{r}_i|^2 \vec{\omega}_i\right)$$

But in general cases, there is no way to take out $m|\vec{r}|^2$ from the summation, and thus the moment of inertia is not well-defined. The only exception is when all objects move with the same ω about the same center, i.e. the case of rigid body, then we can define a total moment of inertia.

$$\vec{L}_{\text{total}} = \sum_i \left(m_i |\vec{r}_i|^2 \vec{\omega}\right) = \left(\sum_i m_i |\vec{r}_i|^2\right) \vec{\omega} = I_{\text{total}}\, \vec{\omega}$$

Example 2.1. Compute the angular momentum as a function of time for an object moving in a circular trajectory, which is centered at (x_0, y_0) with radius R. Let the object's mass be m and moving at constant velocity v. At time $t = 0$ the object is at the north of the circle.

(add figure here: circular track)

By parametrizing the trajectory, we have the position vector of the mass as

$$\vec{r}(t) = (x(t), y(t)) \quad \text{with} \quad \begin{cases} x(t) = x_0 + R\cos\left(\dfrac{v}{R}t + \dfrac{\pi}{2}\right) \\ y(t) = y_0 + R\sin\left(\dfrac{v}{R}t + \dfrac{\pi}{2}\right) \end{cases}$$

Differentiation gives its velocity vector:

$$\vec{\boldsymbol{v}}(t) = (v_x(t), v_y(t)) \quad \text{with} \quad \begin{cases} v_x(t) = -v\sin\left(\dfrac{v}{R}t + \dfrac{\pi}{2}\right) \\ v_y(t) = v\cos\left(\dfrac{v}{R}t + \dfrac{\pi}{2}\right) \end{cases}$$

Its angular momentum is then

$$\begin{aligned}
|\vec{\boldsymbol{L}}| &= |m\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}| \\
&= m(x(t)v_y(t) - y(t)v_x(t)) \\
&= m\left[\left(x_0 + R\cos\left(\frac{v}{R}t + \frac{\pi}{2}\right)\right)\left(v\cos\left(\frac{v}{R}t + \frac{\pi}{2}\right)\right)\right. \\
&\qquad \left. - \left(y_0 + R\sin\left(\frac{v}{R}t + \frac{\pi}{2}\right)\right)\left(-v\sin\left(\frac{v}{R}t + \frac{\pi}{2}\right)\right)\right] \\
&= mx_0v\cos\left(\frac{v}{R}t + \frac{\pi}{2}\right) + my_0v\sin\left(\frac{v}{R}t + \frac{\pi}{2}\right) + mRv
\end{aligned}$$

which contains addition terms other than the usual form $L = mRv$. Please beware of that $L = mRv$ is true only if the origin is chosen to be the center of the circular trajectory.

2.3 Motions Relative to Center of Mass

In many scenerios, rotation of objects are not about a fixed center. E.g.

- Planetary motion: The Moon is rotating around the Earth, which the Earth is moving around the sun.
- Rolling without slipping: The cylinder/sphere is rotating about its center, which its center is also moving along the floor.

Remember, the first rule of analyzing rotation motions is that you must use a fixed origin. So in the above examples, you should still choose the sun / the floor as the origin.

(add figure here: sun / floor as origin)

However this will cause a lot of complication because the position vector $\vec{\boldsymbol{r}}$ and velocity vector $\vec{\boldsymbol{v}}$ to each point are not perpendicular. Calculating the total torque or angular momentum requires integration.

$$\vec{\boldsymbol{\tau}} = \sum_i \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i \;\rightarrow\; \int \vec{\boldsymbol{r}} \times \mathrm{d}\vec{\boldsymbol{F}}$$

$$\vec{\boldsymbol{L}} = \sum_i m_i \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{v}}_i = \sum_i m_i |\vec{\boldsymbol{r}}_i|^2 \omega_i \;\rightarrow\; \int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\,\mathrm{d}m = \int |\vec{\boldsymbol{r}}|^2 \omega\,\mathrm{d}m$$

A proper treatment is to separate the position vectors (and velocity vectors) of the objects into two parts - the position vector of the center of mass $\vec{\boldsymbol{r}}_{CM}$, and the displacement vectors of each object relative to the center of mass $\vec{\boldsymbol{R}}_i$.

$$\begin{cases} \vec{\boldsymbol{r}}_i = \vec{\boldsymbol{r}}_{CM} + \vec{\boldsymbol{R}}_i \\ \vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{CM} + \vec{\boldsymbol{V}}_i \quad \left(= \dfrac{\mathrm{d}\vec{\boldsymbol{r}}_i}{\mathrm{d}t} = \dfrac{\mathrm{d}\vec{\boldsymbol{r}}_{CM}}{\mathrm{d}t} + \dfrac{\mathrm{d}\vec{\boldsymbol{R}}_i}{\mathrm{d}t}\right) \end{cases}$$

(add figure here: r cm and R i)

Substitute them into the definition of angular momentum, we get

$$
\begin{aligned}
\vec{\boldsymbol{L}} &= \sum_i (m_i \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{v}}_i) \\
&= \sum_i \Big[m_i (\vec{\boldsymbol{r}}_{\text{CM}} + \vec{\boldsymbol{R}}_i) \times (\vec{\boldsymbol{v}}_{\text{CM}} + \vec{\boldsymbol{V}}_i) \Big] \\
&= \sum_i m_i \Big[\vec{\boldsymbol{r}}_{\text{CM}} \times \vec{\boldsymbol{v}}_{\text{CM}} + \vec{\boldsymbol{r}}_{\text{CM}} \times \vec{\boldsymbol{V}}_i + \vec{\boldsymbol{R}}_i \times \vec{\boldsymbol{v}}_{\text{CM}} + \vec{\boldsymbol{R}}_i \times \vec{\boldsymbol{V}}_i \Big] \\
&= \left(\sum_i m_i \right) [\vec{\boldsymbol{r}}_{\text{CM}} \times \vec{\boldsymbol{v}}_{\text{CM}}] + \vec{\boldsymbol{r}}_{\text{CM}} \times \underbrace{\left[\cancelto{0}{\sum_i m_i \vec{\boldsymbol{V}}_i} \right]}_{\substack{\text{Total velocity} \\ \text{relative to CM} = 0}} + \underbrace{\left[\cancelto{0}{\sum_i m_i \vec{\boldsymbol{R}}_i} \right]}_{\substack{\text{Total displacement} \\ \text{from CM} = 0}} \times \vec{\boldsymbol{v}}_{\text{CM}} + \sum_i \Big[m_i \vec{\boldsymbol{R}}_i \times \vec{\boldsymbol{V}}_i \Big] \\
&= \underbrace{\left(\sum_i m_i \right) [\vec{\boldsymbol{r}}_{\text{CM}} \times \vec{\boldsymbol{v}}_{\text{CM}}]}_{\substack{\text{Angular momentum of} \\ \text{CM's motion about origin}}} + \underbrace{\sum_i \Big[m_i \vec{\boldsymbol{R}}_i \times \vec{\boldsymbol{V}}_i \Big]}_{\substack{\text{Angular momentum of} \\ \text{all object's motion about CM}}} \\
&= \underbrace{\Big(M_{\text{total}} |\vec{\boldsymbol{r}}_{\text{CM}}|^2 \Big)}_{\substack{\text{Moment of inertia of} \\ \text{a point mass } M_{\text{total}} \\ \text{relative to origin}}} \vec{\boldsymbol{\omega}}_{\text{CM}} + \sum_i \Big[\underbrace{m_i |\vec{\boldsymbol{R}}_i|^2}_{\substack{\text{Moment of inertia of} \\ \text{each object relative to CM}}} \vec{\boldsymbol{\Omega}}_i \Big] \\
&= M_{\text{total}} |\vec{\boldsymbol{r}}_{\text{CM}}|^2 \vec{\boldsymbol{\omega}}_{\text{CM}} + \sum_i \Big[I_{i,\text{CM}} \, \vec{\boldsymbol{\Omega}}_i \Big]
\end{aligned}
$$

So we can write a system's angular momentum by combining the center of mass's motion and objects's motion around the center of mass, which simplifies the calculation in many problems.

Example 2.2. (Satellite motion)

The Mars has two satellites Phobos and Deimos. The Mars is also revolving around the sun. Given the following data:

- Mass of Sun, Mars, Phobos and Deimos are m_S, m_M, m_P, m_D. Assume $m_S \gg m_M \gg m_P \approx m_D$
- Assume all orbits are circle. Distance between Mars & Sun $= r_{MS}$, between Mars & Phobos $= r_{PM}$, between Mars & Deimos $= r_{DM}$.
- Mars, Phobos and Deimos are self-rotating with angular velocity ω_M, ω_P, ω_D.
- Moment of inertia of Mars, Phobos and Deimos about their center are I_M, I_P, I_P.
- Assume the Sun does not move or rotate at all.

(add figure here: mars + satellite)

By gravity, the traveling speed of objects in circular motion is

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \qquad \Rightarrow \qquad v = \sqrt{\frac{GM}{r}}$$

So the revolving speed for Mars, Phobos and Deimos are

$$v_M = \sqrt{\frac{GM_S}{r_M}} \quad , \quad v_P = \sqrt{\frac{GM_M}{r_P}} \quad , \quad v_D = \sqrt{\frac{GM_M}{r_D}}$$

Taking the sun's center as the origin. Because $m_M \gg m_P \approx m_D$, the system's center of mass can be assumed at the Mars's center. We can write down the total angular momentum as

$$\begin{aligned}
L &= M_{\text{total}} r_{\text{CM}} v_{\text{CM}} + \sum_i L_{i,\text{CM}} \\
&= \underbrace{(m_M + m_P + m_D) r_{MS} v_M}_{\substack{\text{Angular momentum of a point mass} \\ \text{of mass } (m_M + m_P + m_D)}} + \underbrace{L_{(\text{Mars,CM})} + L_{(\text{Phobos,CM})} + L_{(\text{Deimos, CM})}}_{\substack{\text{Angular momentum of} \\ \text{anything rotating around Mars' center} \\ = \text{Mars's body + Phobos + Deimos}}} \\
&= (m_M + m_P + m_D) r_{MS} v_M + \underbrace{I_M \omega_M}_{\substack{\text{Mars' body} \\ \text{is in} \\ \text{rigid body} \\ \text{rotation}}} + \underbrace{m_P r_{PM} v_P + \sum_i L_{(i,\text{Phobos})}}_{\substack{\text{Angular momentum of Phobos'} \\ \text{center rotating around Mars} \\ \text{+ anything rotating about} \\ \text{Phobos' center}}} + \underbrace{m_D r_{DM} v_D + \sum_i L_{(i,\text{Deimos})}}_{\substack{\text{Angular momentum of Deimos'} \\ \text{center rotating around Mars} \\ \text{+ anything rotating about} \\ \text{Deimos' center}}} \\
&= (m_M + m_P + m_D) r_{MS} v_M + I_M \omega_M + m_P r_{PM} v_P + \underbrace{I_P \omega_P}_{\substack{\text{Things rotating around Phobos} \\ = \text{its own body mass} \\ \text{in rigid body rotation}}} + m_D r_{DM} v_D + \underbrace{I_D \omega_D}_{\substack{\text{Things rotating around Deimos} \\ = \text{its own body mass} \\ \text{in rigid body rotation}}}
\end{aligned}$$

Caution: We can only use the center of mass for such breakdown.

$\sum_i \left(m_i \vec{\boldsymbol{R}}_i\right)$ and $\sum_i \left(m_i \vec{\boldsymbol{V}}_i\right) = 0$ is true only if we take CM as the reference point. We can prove $\sum_i \left(m_i \vec{\boldsymbol{R}}_i\right) = 0$ using the CM formula. Differentiate to arrive $\sum_i \left(m_i \vec{\boldsymbol{V}}_i\right) = 0$.

$$\begin{aligned}
\sum_i \left(m_i \vec{\boldsymbol{R}}_i\right) &= \sum_i \left[m_i (\vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_{\text{CM}})\right] \\
&= \sum_i (m_i \vec{\boldsymbol{r}}_i) - \left(\sum_i m_i\right) \vec{\boldsymbol{r}}_{\text{CM}} \\
&= \sum_i (m_i \vec{\boldsymbol{r}}_i) - \cancel{\left(\sum_i m_i\right)} \left(\frac{\sum_i m_i \vec{\boldsymbol{r}}_i}{\cancel{\sum_i m_i}}\right) \\
&= 0
\end{aligned}$$

2.4 Parallel Axis Theorem

The second rule of using angular quantities is that moment of inertia is well-defined only when all masses rotate at the same angular velocity about the same center. This is In such case, we can always choose the rotation center as the origin.

(add figure here: rigid body motion)

The angular momentum writes as:

$$
\begin{aligned}
\vec{\boldsymbol{L}} &= \sum_i (m_i \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{v}}_i) \\
&= \sum_i (m_i |\vec{\boldsymbol{r}}_i|^2) \underline{\vec{\boldsymbol{\omega}}_O} \longleftarrow \begin{array}{c}\text{All objects have} \\ \text{the same angular velocity} \\ \text{relative to } O\end{array} \\
&= \sum_i (m_i |\vec{\boldsymbol{r}}_{\text{CM}} + \vec{\boldsymbol{R}}_i|^2) \vec{\boldsymbol{\omega}}_O \\
&= \left[\sum_i \left(m_i |\vec{\boldsymbol{r}}_{\text{CM}}|^2\right) + \sum_i \left(m_i |\vec{\boldsymbol{R}}_i|^2\right) + \underbrace{\cancelto{0}{2\sum_i \left(m_i \vec{\boldsymbol{r}}_{\text{CM}} \cdot \vec{\boldsymbol{R}}_i\right)}}_{= \vec{\boldsymbol{r}}_{CM} \cdot \left(\sum_i m_i \vec{\boldsymbol{R}}_i\right) = \vec{\boldsymbol{r}}_{CM} \cdot 0} \right] \vec{\boldsymbol{\omega}}_O \\
&= \left[\underbrace{M_{\text{total}} |\vec{\boldsymbol{r}}_{\text{CM}}|^2}_{\begin{array}{c}\text{Moment of inertia of} \\ \text{a point with mass } M_{\text{total}} \\ \text{relative to origin}\end{array}} + \underbrace{\sum_i \left(m_i |\vec{\boldsymbol{R}}_i|^2\right)}_{\begin{array}{c}\text{Moment of inertia} \\ \text{relative to CM}\end{array}} \right] \vec{\boldsymbol{\omega}}_O \\
&= \left[\underline{I_{\text{CM},O}} + \underline{I_{i,\text{CM}}} \right] \vec{\boldsymbol{\omega}}_O
\end{aligned}
$$

This tells us a special case where two moment of inertia relative to different points can be directly - One of them must be chosen as the rotation center and the other must be the center of mass.

$$\boxed{I_{\text{equiv}} = I_{CM,O} + I_{i,CM}}$$

Similar to the previous, we can only use the center of mass for such breakdown. $\vec{\boldsymbol{R}}_i$ must be a straight line from the CM.

(add figure here: must be a straight line connect to CM)

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3 Rotational KE

3.1 Mathematical Origin

There is always only 1 expression to kinetic energy - $\frac{1}{2}m|\vec{\boldsymbol{v}}|^2$. It is only the matter of which coordinate system we used to expand $\vec{\boldsymbol{v}}$. With $\{x, y\}$ coordinate, it is in the familar form:

$$\vec{\boldsymbol{v}} = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\boldsymbol{x}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\boldsymbol{y}} \quad \Rightarrow \quad \frac{1}{2}m|\vec{\boldsymbol{v}}|^2 = \frac{1}{2}m\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right] = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2$$

But with $\{r, \theta\}$ coordinate, it becomes:

$$\vec{\boldsymbol{v}} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\boldsymbol{r}} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{\theta}} \quad \Rightarrow \quad \frac{1}{2}m|\vec{\boldsymbol{v}}|^2 = \frac{1}{2}m\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right] = \frac{1}{2}mv_r^2 + \frac{1}{2}mr^2\omega^2$$

We can spot the familiar term $\frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$ in textbook. $\frac{1}{2}I\omega^2$ is in fact only about the angular components in the total KE. You should always write BOTH the radial and angular KE, unless you are sure that the whole object in pure rotation (which then radial KE = 0).

3.2 Breaking down by Center of Mass

Similar to angular momentum, we can separate KE for the center of mass's movement and individual objects' motions relative to center of mass.

(add figure here: same pic as in angular momentum?)

$$\begin{aligned}
\mathrm{KE} &= \sum_i \left(\frac{1}{2}m_i|\vec{\boldsymbol{v}}_i|^2\right) \\
&= \sum_i \left(\frac{1}{2}m_i|\vec{\boldsymbol{v}}_{\mathrm{CM}} + \vec{\boldsymbol{V}}_i|^2\right) \\
&= \sum_i \left(\frac{1}{2}m_i|\vec{\boldsymbol{v}}_{\mathrm{CM}}|^2\right) + \sum_i \left(\frac{1}{2}m_i|\vec{\boldsymbol{V}}_i|^2\right) + \underbrace{\cancelto{0}{\sum_i \left(m_i\vec{\boldsymbol{v}}_{\mathrm{CM}} \cdot \vec{\boldsymbol{V}}_i\right)}}_{= \vec{\boldsymbol{v}}_{\mathrm{CM}} \cdot \left(\sum_i m_i\vec{\boldsymbol{V}}_i\right) = \vec{\boldsymbol{v}}_{\mathrm{CM}} \cdot 0} \\
&= \underbrace{\frac{1}{2}M_{\mathrm{total}}|\vec{\boldsymbol{v}}_{\mathrm{CM}}|^2}_{\text{KE of a point mass } M_{\mathrm{total}}} + \underbrace{\sum_i \left(\frac{1}{2}m_i|\vec{\boldsymbol{V}}_i|^2\right)}_{\text{KE of individual object using velocity relative to CM}} \\
&= \frac{1}{2}M_{\mathrm{total}}|\vec{\boldsymbol{v}}_{\mathrm{CM}}|^2 + \sum_i \left[\underbrace{\frac{1}{2}m_i\left(\frac{\mathrm{d}R_i}{\mathrm{d}t}\right)^2}_{\text{Radial KE of individual object relative to CM}} + \underbrace{\frac{1}{2}m_iR_i^2\left(\frac{\mathrm{d}\Theta_i}{\mathrm{d}t}\right)^2}_{\text{Angular KE of individual object relative to CM}}\right]
\end{aligned}$$

Similar to angular momentum and moment of inertia, we can only use the center of mass for such breakdown.

Example 3.1. Repeat the example for Sun-Mars-Phobos-Deimos system, but write down its

KE this time.

$$\mathrm{KE} = \frac{1}{2} M_{\text{total}} v_{\text{CM}}^2 + \sum_i (\mathrm{KE}_{i,\text{CM}})$$

$$= \underbrace{\frac{1}{2}(m_M + m_P + m_D) v_M^2}_{\substack{\text{KE of a point mass} \\ \text{of mass } (m_M + m_P + m_D)}} + \underbrace{\mathrm{KE}_{(\text{Mars,CM})} + \mathrm{KE}_{(\text{Phobos,CM})} + \mathrm{KE}_{(\text{Deimos,CM})}}_{\substack{\text{KE of anything} \\ \text{rotating around Mars' center} \\ = \text{Mars's body} + \text{Phobos} + \text{Deimos}}}$$

$$= \frac{1}{2}(m_M + m_P + m_D) v_M^2 + \underbrace{\frac{1}{2} I_M \omega_M^2}_{\substack{\text{Mar's body} \\ \text{is in} \\ \text{rigid body} \\ \text{rotation}}} + \underbrace{\frac{1}{2} m_P v_P^2 + \sum_i \mathrm{KE}_{(i,\text{Phobos})}}_{\substack{\text{KE of Phobos' center} \\ \text{rotating around Mars} \\ + \text{ anything rotating about} \\ \text{Phobos' center}}} + \underbrace{\frac{1}{2} m_D v_D^2 + \sum_i \mathrm{KE}_{(i,\text{Deimos})}}_{\substack{\text{KE of Deimos' center} \\ \text{rotating around Mars} \\ + \text{ anything rotating about} \\ \text{Deimos' center}}}$$

$$= \frac{1}{2}(m_M + m_P + m_D) v_M^2 + \frac{1}{2} I_M \omega_M^2 + \frac{1}{2} m_P v_P^2 + \underbrace{\frac{1}{2} I_P \omega_P^2}_{\substack{\text{Things rotating around Phobos} \\ = \text{its own body mass} \\ \text{in rigid body rotation}}} + \frac{1}{2} m_D v_D^2 + \underbrace{\frac{1}{2} I_D \omega_D^2}_{\substack{\text{Things rotating around Deimos} \\ = \text{its own body mass} \\ \text{in rigid body rotation}}}$$

— The End —