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Electrostatics

by Tony Shing

Overview:

- Basic problems: Find $\vec{\boldsymbol{E}}$ and V by Coulomb's law with integration
- Gauss law, electric flux, divergence & divergence theorem
- Electrical potential & Poisson equation
- Image charge method

In electromagnetism, theoretically every problem can be solved through a set of PDEs called the **Maxwell Equations**.

$$\longrightarrow \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad\qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad\qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

However, a *system of PDEs* is just too complicate to be solved. So we need to learn different "tricks" to avoid them, which are enough for some simple scenarios.

Electrostatics only concerns the 1st equation of the set - Gauss's law.

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1 Basic Skill: Coulomb's Law with Integration

Promoting point charge to a distribution of charge is straightforward by

$$\begin{pmatrix}\text{Point}\\\text{charge}\end{pmatrix} = Q \quad\Longrightarrow\quad \int_{\substack{\text{whole}\\\text{space}}} \mathrm{d}Q \quad\sim \sum_{\text{everywhere}} \begin{pmatrix}\text{Infinitestimal}\\\text{charge units}\end{pmatrix}$$

Coulomb's law is then a sum of all E-field contribution from every infinitestimal charge:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\boldsymbol{r}} \quad\Longrightarrow\quad \int \mathrm{d}\vec{E} = \int_{\substack{\text{whole}\\\text{space}}} \frac{1}{4\pi\epsilon_0}\frac{\mathrm{d}Q}{r^2}\hat{\boldsymbol{r}}$$

And the formula for electrical potential follows:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad\Longrightarrow\quad \int \mathrm{d}V = \int_{\substack{\text{whole}\\\text{space}}} \frac{1}{4\pi\epsilon_0}\frac{\mathrm{d}Q}{r}$$

Notations:

We often use these symbols to represent charge density at different dimension:

- Charge per volume $= \rho \quad \Rightarrow \quad \mathrm{d}Q = \rho\, \mathrm{d}\tau = \binom{\text{Charge}}{\text{per volume}}\binom{\text{Unit}}{\text{volume}}$
- Charge per area $= \sigma \quad \Rightarrow \quad \mathrm{d}Q = \sigma\, \mathrm{d}s = \binom{\text{Charge}}{\text{per area}}\binom{\text{Unit}}{\text{area}}$
- Charge per length $= \lambda \quad \Rightarrow \quad \mathrm{d}Q = \lambda\, \mathrm{d}l = \binom{\text{Charge}}{\text{per length}}\binom{\text{Unit}}{\text{length}}$

Example 1.1. Suppose there is a rod lying on the x-axis, with its ends at $x = a$ and $x = b$. Let the total charge it carries be Q. What is the E-field / electric potential on an arbituary point on the z axis?

(add figure here: rod)

We can analyze by dividing the rod into infinitestimal pieces:

- Each segment has a length $\mathrm{d}x$
- Charge on each segment is thus $\lambda\, \mathrm{d}x = \frac{Q}{L}\, \mathrm{d}x$, where $L = b - a$.
- For the segment at position x, its distance from the targeted point is $\sqrt{z^2 + x^2}$.

(add figure here: infinte element)

Thus we can calculate V and $\vec{\boldsymbol{E}}$:

1. Electrical potential does not concern directions. So we can directly write

$$V = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{\frac{Q}{L}\, \mathrm{d}x}{\sqrt{z^2 + x^2}}$$

2. Electric field concerns directions. So we need to resolve the direction's component from the segment to the target point. By the triangle:

(add figure here: component)

So the E-field's vertical component (z) should be integrated by

$$E_z = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{\frac{Q}{L}\, \mathrm{d}x}{z^2 + x^2} \sin\theta = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{\frac{Q}{L}\, \mathrm{d}x}{z^2 + x^2} \frac{z}{\sqrt{z^2 + x^2}}$$

Similarly for the horizontal component (x):

$$E_x = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{\frac{Q}{L}\, \mathrm{d}x}{z^2 + x^2} \cos\theta = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{\frac{Q}{L}\, \mathrm{d}x}{z^2 + x^2} \frac{x}{\sqrt{z^2 + x^2}}$$

2 Gauss's Law

The Gauss's Law has two different expressions:

$$\oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \frac{Q}{\epsilon_0} \qquad \text{(Integral form)}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0} \qquad \text{(Differential form)}$$

It is easier to study the physical meaning and visualize by the integral form. After that we can generalize to the differential form by introducing an operator called **divergence**.

2.1 Flux

The literal description in Gauss's law integral form is

$$\begin{pmatrix}\text{Flux of electric field} \\ \text{on a closed surface}\end{pmatrix} = \oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \frac{Q}{\epsilon_0} = \frac{\text{(Charge enclosed)}}{\text{(Constant)}}$$

We first need to understand what **flux** is, and what that weird integral sign even means.

2.1.1 An Analogy: A Water Pipe

To begin with, we can make an analogy using a water pipe. Along a normal pipe, we expect that the amount of water flowing in should equal to the amount of water flowing out.

(add figure here: normal pipe)

- If we somehow find that the amount of water flowing in > amount flowing out, we know there is something absorbing water in the pipe! (A sink)

(add figure here: sink)

- If we somehow find that the amount of water flowing in < amount flowing out, we know there is something producing water in the pipe! (A source)

(add figure here: source)

How can we quantitatively tell if there is a source / sink in the pipe? We can measure by the volume flowing in / out within a short time interval Δt:

- Volume flowing in $= v_{\text{in}} \cdot \Delta t \cdot A_{\text{in}} = \begin{pmatrix}\text{Velocity at} \\ \text{entrance}\end{pmatrix} \cdot \Delta t \cdot \begin{pmatrix}\text{Area of} \\ \text{entrance opening}\end{pmatrix}$
- Volume flowing out $= v_{\text{out}} \cdot \Delta t \cdot A_{\text{out}} = \begin{pmatrix}\text{Velocity at} \\ \text{exit}\end{pmatrix} \cdot \Delta t \cdot \begin{pmatrix}\text{Area of} \\ \text{exit opening}\end{pmatrix}$

Then we can define a measure $\Phi = (v_{\text{out}} A_{\text{out}} - v_{\text{in}} A_{\text{in}})$ such that

$$\begin{cases} \text{if } \Phi > 0 & \Rightarrow \text{ There is a source} \\ \text{if } \Phi < 0 & \Rightarrow \text{ There is a sink} \end{cases}$$

2.1.2 Generalizing to Field

Now use your imagination to expand the

1. The flow of water should be continuous - We can describe the flow of water by a continuous vector field $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$.

   (add figure here: flow to field)

2. Our water pipe may be of any irregular shape - We can "twist" the pipe into an arbituary closed surface.

   (add figure here: peanut pipe)

   A "closed" surface needs to be well-distinguished between its "inner surface" and "outer surface".

Under these circumstances, in/out-flow are not restricted just flowing through the entrance / exit opening, but can appear on anywhere on the surface - each of the small grid on the surface can have a different field vector poking through it.

- Poking from inner to outer surface = Out-flow
- Poking from outer to inner surface = In-flow

(add figure here: messy flow)

How can we describe the flow direction mathematically? We can first define a normal vector $\vec{\boldsymbol{s}}$ for each grid. By convention, this $\vec{\boldsymbol{s}}$

- has a magnitude equal to the surface's area of the grid.
- points outward of the surface (from inner to outer).

Then we can take the dot product between the field vector $\vec{\boldsymbol{F}}$ and the surface normal vector $\vec{\boldsymbol{s}}$:

- If $\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{s}} > 0$, they are more or less in similar direction $\Rightarrow \vec{\boldsymbol{F}}$ is pointing outward!
- If $\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{s}} < 0$, they are more or less in opposite direction $\Rightarrow \vec{\boldsymbol{F}}$ is pointing inward!

Finally, we define "flux" Φ over the closed surface:

$$\boxed{\Phi \quad = \sum_{\substack{\text{All small grids } i \\ \text{on the closed surface}}} \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{s}}_i \quad \longrightarrow \quad \oiint \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{s}}}$$

A circle on double integral
= The surface integral is over a closed surface

(add figure here: sum of flux)

We can use the sign of Φ to tell if there are more in-flow / out-flow of field lines through the surface, thus telling if there are sources / sinks enclosed by the surface.

(add figure here: phi > 0 < 0 = 0)

2.1.3 Calculation Example

Recall that in line integral, we have to parametrize the line before we can actually do the integral. In calculation of flux, it is even more painful - it is an area integral and we have to parametrize a surface, which is generally an impossible task without very advanced calculus.

Because this is just a physics course, here only introduces some surfaces with simple parametrization. **And in most cases we do not to calculate them when using Gauss's law.**

Example 2.1. (Flux over a spherical surface)
An easy parametrization is making use of the spherical coordinate. Positions on the surface can be located by the 2 anglular variables (θ, ϕ):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

where (x_0, y_0, z_0) indicates the center of the sphere and r is the radius of the sphere.

(add figure here: sphere + surface unit)

The infinitestimal area is then $(r\,\mathrm{d}\theta) \times (r\sin\theta\,\mathrm{d}\phi)$ and normal in $\hat{\boldsymbol{r}}$ direction.

$$\mathrm{d}\vec{\boldsymbol{s}} = \hat{\boldsymbol{r}} r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

So the flux is simply a double integral over the whole sphere surface.

$$\Phi = \oiint \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \vec{\boldsymbol{F}}(x, y, z) \cdot \hat{\boldsymbol{r}} r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Example 2.2. (Flux over a plane)
We are free to choose any two perpendicular unit vectors $\hat{\boldsymbol{u}}$,$\hat{\boldsymbol{v}}$ to form a 2D coordinate system on the plane, expressing a position using 2 length quantities (u, v):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + u \underbrace{\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}}_{\hat{\boldsymbol{u}}} + v \underbrace{\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}}_{\hat{\boldsymbol{v}}}$$

(add figure here: uv coordinate)

The infinitestimal area is then $(\mathrm{d}u) \times (\mathrm{d}v)$ and normal must be in $(\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}})$ direction.

$$\mathrm{d}\vec{\boldsymbol{s}} = (\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}})\,\mathrm{d}u\,\mathrm{d}v$$

So the flux is simply a double integral over a region on the plane.

$$\Phi = \oiint \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \int_{v=c}^{v=d} \int_{u=a}^{u=b} \vec{\boldsymbol{F}}(x, y, z) \cdot (\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}})\,\mathrm{d}u\,\mathrm{d}v$$

2.2 Divergence

However there is a problem in using flux - if we choose the surface too arbituarily, the calculated flux is to ambiguous to tell where the sink/source are.

(add figure here: arbituary surface)

E.g. If we choose a very big surface and calculate the flux ≈ 0, where are the sinks and sources?

To tackle this problem, we need to introduce the **divergence** operator:

$$\underbrace{\vec{\boldsymbol{\nabla}}\cdot}_{\substack{\text{Like gradient operator}\\ \text{but with a dot}}} \bullet \overset{\text{def}}{=} \frac{\partial \bullet}{\partial x} + \frac{\partial \bullet}{\partial y} + \frac{\partial \bullet}{\partial z} \overset{\text{def}}{=} \underbrace{\text{div}}_{\substack{\text{Sometimes we}\\ \text{just write "div"}}} \bullet$$

The divergence operator can be applied on a vector function, and will return a scalar (number) function.

$$\begin{aligned}\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}} &= \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \\ &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= (\text{A number})\end{aligned}$$

The divergence of a vector field is related to its **total flux through an infinitestimally small but closed surface.**

2.2.1 Geometrical Interpretation

To visualize, we can draw an infinitestimal small cube around a point (x, y, z). Although a cube has 6 faces, because of symmetry in all 3 directions, it suffices to just analyze by the 2 faces that are parallel to the x-y plane.

(add figure here: infinitestmal cube + 2 face)

- Normal direction of these 2 faces $= \hat{\boldsymbol{z}} \Rightarrow$ The surface normal vector $= (\Delta x \Delta y)\hat{\boldsymbol{z}}$.
- Center point of top surface $= \left(x, y, z + \frac{\Delta z}{2}\right) \Rightarrow$ The field pokes through it $= \vec{\boldsymbol{F}}\left(x, y, z + \frac{\Delta z}{2}\right)$.
- Center point of bottom surface $= \left(x, y, z - \frac{\Delta z}{2}\right) \Rightarrow$ The field pokes through it $= \vec{\boldsymbol{F}}\left(x, y, z - \frac{\Delta z}{2}\right)$.

Therefore the total flux through the 2 planes is

$$\begin{pmatrix}\text{Total Out-flux through}\\ \text{surfaces}/\!/\text{x-y plane}\end{pmatrix} = \begin{pmatrix}\text{Out-flux through}\\ \text{top face}\end{pmatrix} - \begin{pmatrix}\text{In-flux through}\\ \text{bottom face}\end{pmatrix}$$

$$\begin{aligned}
\mathrm{d}\Phi_{xy} &= \vec{\boldsymbol{F}}\left(x, y, z + \tfrac{\Delta z}{2}\right) \cdot \hat{\boldsymbol{z}}(\Delta x \Delta y) - \vec{\boldsymbol{F}}\left(x, y, z - \tfrac{\Delta z}{2}\right) \cdot \hat{\boldsymbol{z}}(\Delta x \Delta y) \\
&= \left(\frac{\vec{\boldsymbol{F}}\left(x, y, z + \frac{\Delta z}{2}\right) - \vec{\boldsymbol{F}}\left(x, y, z - \frac{\Delta z}{2}\right)}{\Delta z}\right) \cdot \hat{\boldsymbol{z}}(\Delta x \Delta y \Delta z) \\
&= \underbrace{\boxed{\left(\frac{\vec{\boldsymbol{F}}\left(x, y, \boxed{z + \frac{\Delta z}{2}}\right) - \vec{\boldsymbol{F}}\left(x, y, \boxed{z - \frac{\Delta z}{2}}\right)}{\boxed{\Delta z}}\right)}}_{\text{This is exactly partial z}} \cdot \hat{\boldsymbol{z}}(\Delta x \Delta y \Delta z) \\
&= \frac{\partial}{\partial z} \underline{\vec{\boldsymbol{F}}(x, y, z) \cdot \hat{\boldsymbol{z}}}(\Delta x \Delta y \Delta z)
\end{aligned}$$

Dot product to $\hat{\boldsymbol{z}}$ = Only take z component

$$\begin{aligned}
&= \frac{\partial}{\partial z} \underline{F_z}(x, y, z)\ (\Delta x \Delta y \Delta z) \\
&= \begin{pmatrix}\text{Divergence's}\\ z \text{ term}\end{pmatrix} \begin{pmatrix}\text{Unit}\\ \text{volume}\end{pmatrix}
\end{aligned}$$

We can expect the similar results in the other 2 directions. Gather them together:

$$\begin{aligned}
\begin{pmatrix}\text{Total flux through}\\ \text{the volume}\end{pmatrix} &= \begin{pmatrix}\text{Total flux through}\\ \text{surfaces}/\!/\text{y-z plane}\end{pmatrix} + \begin{pmatrix}\text{Total flux through}\\ \text{surfaces}/\!/\text{x-z plane}\end{pmatrix} + \begin{pmatrix}\text{Total flux through}\\ \text{surfaces}/\!/\text{x-y plane}\end{pmatrix} \\
&= \left[\begin{pmatrix}\text{Divergence's}\\ x \text{ term}\end{pmatrix} + \begin{pmatrix}\text{Divergence's}\\ y \text{ term}\end{pmatrix} + \begin{pmatrix}\text{Divergence's}\\ z \text{ term}\end{pmatrix}\right] \begin{pmatrix}\text{Unit}\\ \text{volume}\end{pmatrix} \\
\mathrm{d}\Phi &= \left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right] (\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z) \\
&= \left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}\right) (\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z)
\end{aligned}$$

Therefore we can geometrically interpret divergence as

$$\boxed{(\textbf{Divergence}) = \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}} \sim \frac{\mathrm{d}\Phi}{\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z} = \frac{(\text{Flux through a closed surface})}{(\text{Volume enclosed by the surface})} \sim (\textbf{Flux } \underline{\textbf{density}})}$$

This density is by volume

2.2.2 Divergence Theorem

With the geometrical interpretation, we can directly state (without proof) a convenient formula related to divergence - the **divergence theorem**:

$$\boxed{\oiint \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \iiint (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}})\, \mathrm{d}\tau}$$

which is basically

$$\begin{pmatrix}\text{Total}\\ \text{Flux}\end{pmatrix} \sim \sum_{\text{All volumes}} \begin{pmatrix}\text{Flux}\\ \text{per volume}\end{pmatrix} \times (\text{Volume})$$

2.3 Gauss's Law - Explanation

The Gauss's law is purely an observation about the relation between E-field and charges:

Total flux of E-field on a closed surface $\neq 0$	$\Leftrightarrow$	There are charges inside the surface

The two forms of Gauss's law are describing this same observation:

- Integral form:

$$(\vec{\boldsymbol{E}}\text{'s flux}) \sim \oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \frac{Q}{\epsilon_0} \sim (\text{Charge})$$

- Differential form:

$$\begin{pmatrix}\vec{\boldsymbol{E}}\text{'s flux}\\ \text{density}\end{pmatrix} \sim \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0} \sim \begin{pmatrix}\text{Charge}\\ \text{density}\end{pmatrix}$$

And the two form can be inter-converted by divergence theorem.

$$\begin{array}{ccc} \oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} & = & \dfrac{Q}{\epsilon_0} \\ \text{Divergence Theorem} \downarrow & & \downarrow \text{Charge to charge density} \\ \iiint (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}})\, \mathrm{d}\tau & = & \dfrac{1}{\epsilon_0} \iiint \rho \, \mathrm{d}\tau \end{array}$$

2.4 Applying Gauss's Law Integral Form

In beginner electromagnetism, there is only one type of Gauss's law related problems:

Given the charge distribution, find the E-field everywhere by Gauss's law integral form in some very symmetrical scenarios.

which is basically asking you to *revert* the flux calculation:

$$\oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \frac{Q}{\epsilon_0} \quad \Rightarrow \quad \vec{\boldsymbol{E}} = \text{??? in terms of } Q$$

If Q has a very ugly distribution, there is nothing we can do except solving some partial differential equations. But if Q distributes very symmetrically, $\vec{\boldsymbol{E}}$ should also be symmetrical, such that the flux integral can be broken into multiplications.

In these cases, we can choose a "Gaussian" surface to to be integrated where

1. $\vec{\boldsymbol{E}}$ has constant magnitude everywhere on the surface.

2. $\vec{\boldsymbol{E}}$ forms the same angle with the surface normal vector everywhere on the surface

Only then, the flux integral can be broken down as

$$\oiint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{s}} = \oiint \underline{|\vec{\boldsymbol{E}}||\,\mathrm{d}\vec{\boldsymbol{s}}\,|\cos\theta} \longleftarrow \begin{array}{c}\text{Just dot product}\\ \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\cos\theta\end{array}$$

$$= \underline{|\vec{\boldsymbol{E}}|}\ \underline{\cos\theta} \oiint |\,\mathrm{d}\vec{\boldsymbol{s}}\,|$$

Same magnitude everywhere
Can move out of integral!

Form same angle everywhere
Can move out of integral!

$$= |\vec{\boldsymbol{E}}|\ \cos\theta\ (\text{Total surface area})$$

such that we can find the magnitude of $\vec{\boldsymbol{E}}$ with simple division

$$\boxed{|\vec{\boldsymbol{E}}| = \frac{(\text{Total flux})}{(\text{Total surface area})\cos\theta} = \frac{Q/\epsilon_0}{(\text{Total surface area})\cos\theta}}$$

In fact, there are not many of these "very symmetrical" cases. These examples below, with their respective Gaussian surface, are basically all the variations you can find in textbooks.

(add figure here: different gaussian surface)

Example 2.3. Given a solid sphere with uniform charge density ρ and radius R. By rotational symmetry, the E-field must satisfy:

- Only point in radial direction.
- Magnitude does not depend on angular directions θ, ϕ.

(add figure here: spherical symm)

Therefore we can choose the Gaussian surface to be a sphere of radius r to find the magnitude of E-field at distance r from the sphere center.

$$\begin{aligned} |\vec{\boldsymbol{E}}| &= \frac{Q/\epsilon_0}{(\text{Total surface area})\cos\theta} \\ &= \frac{Q}{\epsilon_0}\cdot\frac{1}{(4\pi r^2)}\cdot\frac{1}{\cos \underline{0^\circ}} \longleftarrow \begin{array}{l}\text{E-field = radial}\\ \therefore \text{Normal to surface}\end{array} \\ &= \frac{Q}{4\pi\epsilon_0 r^2} \\ \Rightarrow \quad \vec{\boldsymbol{E}} &= \frac{Q}{4\pi\epsilon_0 r^2}\underline{\hat{\boldsymbol{r}}} \longleftarrow \text{You have to manually add the unit vector} \end{aligned}$$

1. For radial distance $r < R$, the total charge enclosed in the gaussian surface is only the core of the sphere, up to radius r. So we should take $Q = \frac{4}{3}\pi r^3 \rho$.

$$\vec{\boldsymbol{E}} = \frac{\frac{4}{3}\pi r^3 \rho}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} = \frac{\rho r}{3\epsilon_0}\hat{\boldsymbol{r}}$$

(add figure here: inside)

2. For radial distance $r > R$ the total charge enclosed in the gaussian surface is the whole sphere, So we should take $Q = \frac{4}{3}\pi R^3 \rho$.

$$\vec{\boldsymbol{E}} = \frac{\frac{4}{3}\pi R^3 \rho}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{\boldsymbol{r}}$$

(add figure here: outside)

Example 2.4. Given an infinitely long hollow cylinder with inner radius $= a$ and outer radius $= b$, and its charge density is proportional to distance from center r, i.e. $\rho(\vec{\boldsymbol{r}}) = kr$

For cylinder, we can claim by rotational symmetry around the axis and translation symmetry along the axis, that the E-field must satisfy:

- Only point in r direction.
- Magnitude does not depend on θ or z.

(add figure here: cylinder symm)

Therefore we can choose the Gaussian surface to be a cylindrical sheet radius r and an arbitiary length L (which will be cancelled later) to find the magnitude of E-field at distance r from the rotation axis.

$$
\begin{aligned}
|\vec{\boldsymbol{E}}| &= \frac{Q/\epsilon_0}{(\text{Total surface area})\cos\theta} \\
&= \frac{Q}{\epsilon_0}\cdot\frac{1}{\underline{(2\pi r L)}}\cdot\frac{1}{\cos \underline{0^\circ}} \longleftarrow \begin{array}{c}\text{E-field = radial} \\ \therefore \text{Normal to curved surface}\end{array} \\
& \qquad \begin{array}{c}\text{E-field = radial} \\ \therefore \text{Only go through the curved surface} \\ \text{Top/bottom surface has no flux}\end{array} \\
&= \frac{Q}{2\pi\epsilon_0 r L} \\
\Rightarrow \quad \vec{\boldsymbol{E}} &= \frac{Q}{2\pi\epsilon_0 r L}\underline{\hat{\boldsymbol{r}}} \longleftarrow \text{You have to manually add the unit vector}
\end{aligned}
$$

(add figure here: not side surface cuz flux = 0)

This time the charge density depends on position, so the total charge enclosed by the surface needs to be computed by integration.

1. For radial distance $r < a$, there is no charge enclosed because the cylinder is hollow. So $Q = 0$ implying $\vec{\boldsymbol{E}} = 0$.

(add figure here: inner)

2. For radial distance $a < r < b$, total enclosed charge are distributed from radius $= a$ to radius $= r$, which calculates as

$$Q = \int_a^r \rho \cdot 2\pi r L \,\mathrm{d}r = 2\pi k L \int_a^r r^2 \,\mathrm{d}r = \frac{2\pi k L}{3}(r^3 - a^3)$$

So the E-field is

$$\vec{\boldsymbol{E}} = \frac{Q}{2\pi\epsilon_0 r L}\hat{\boldsymbol{r}} = \frac{k}{3\epsilon_0}\left(r^2 - \frac{a^3}{r}\right)\hat{\boldsymbol{r}}$$

(add figure here: middle)

3. For radial distance $r > b$, total enclosed charge are distributed from radius $= a$ to radius $= b$, which calculates as

$$Q = \int_a^b \rho \cdot 2\pi r L \,\mathrm{d}r = 2\pi k L \int_a^r r^2 \,\mathrm{d}r = \frac{2\pi k L}{3}(b^3 - a^3)$$

So the E-field is

$$\vec{\boldsymbol{E}} = \frac{Q}{2\pi\epsilon_0 r L}\hat{\boldsymbol{r}} = \frac{k}{3\epsilon_0 r}(b^3 - a^3)\hat{\boldsymbol{r}}$$

(add figure here: outer)

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3 Electric Potential

3.1 Mathematical Origin

The reason to create a electrical potential function $V(\vec{\boldsymbol{r}})$ is rather mathematical:

- Observation: E-field by static charge never forms loops. $\Rightarrow$ Static E-field is conservative.
- Mathematical fact: Any conservative field can be expressed as the gradient of some scalar function (i.e. potential).

Therefore we can define a scalar function $V(\vec{\boldsymbol{r}})$ such that

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}} V(\vec{\boldsymbol{r}})}$$

And the reverse can be calculated by

$$\boxed{V(\vec{\boldsymbol{r_0}}) = -\int_{\substack{\text{any path}\\ \text{from } \infty \text{ to } \vec{\boldsymbol{r_0}}}} \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{r}} = -\int_\infty^{\vec{r_0}} \vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})\,\mathrm{d}\vec{\boldsymbol{r}}}$$

3.2 Poisson Equation

If we substitute $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}} V$ into the Gauss's law, we arrive at a new equation:

$$\begin{aligned}
\frac{\rho}{\epsilon_0} &= \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} \\
&= \vec{\boldsymbol{\nabla}} \cdot \left(-\vec{\boldsymbol{\nabla}} V\right) \\
&= -\vec{\boldsymbol{\nabla}} \cdot \left(\frac{\partial V}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial V}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial V}{\partial z}\hat{\boldsymbol{z}}\right) \\
&= -\left(\underbrace{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}\right) \overset{\text{def}}{=} -\nabla^2 V
\end{aligned}$$

This is called **Laplacian Operator**

$$\boxed{\nabla^2 V(\vec{\boldsymbol{r}}) = -\frac{\rho(\vec{\boldsymbol{r}})}{\epsilon_0}}$$

This equation belongs to a class of PDE called **Poisson equation**, which is one of the earliest studied PDEs in history. It is the general method to relate potential between charge distribution. that works for any configurations of potential or charge distribution.

(add figure here: find one and other, add Q = lap V and V = ??? Q with arrow)

However, being general does not mean it is always easy to solve:

- $V(\vec{\boldsymbol{r}})$ to $\rho(\vec{\boldsymbol{r}})$: The Laplacian operator is just a sum of 2^{nd} order derivatives. Easy!
- $\rho(\vec{\boldsymbol{r}})$ to $V(\vec{\boldsymbol{r}})$: Need to solve the Laplacian equation, which is a 2^{nd} order non-homogeneous linear PDE. Awful!

Unfortunately in realistic problems, it is more frequent to ask for $V(\vec{\boldsymbol{r}})$ from $\rho(\vec{\boldsymbol{r}})$, because we can usually confine the charge distribution in a small region by using a very small test object; but for potential, it is always everywhere.

(add figure here: charge in small object -> can treat like point charge. but V spread everywhere)

We are not going to discuss its general solution here - it can take several book chapters to derive and analyze the solution forms at different boundary conditions.

In 1D wave equation, the boundary conditions are just about the 2 end points. Either fixed or free.

(add figure here: wave eq vs possion bc)

In Poisson equation, the boundary conditions are about the line/face boundary of the region of interest. Too many variations.

(add figure here: wave eq vs possion bc)

Although the general case is terribly complicated, we have already learnt the solution in one very special case - When the region of interest is infinitely large + potential vanishes at the boundary, i.e. $V(\vec{\boldsymbol{r}}=\infty)=0$, the solution is exactly the Coulomb's law.

$$\boxed{V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \iiint\limits_{\substack{\text{infinitely large}\\ \text{space}}} \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\,\mathrm{d}^3\vec{\boldsymbol{r}}'}$$

$$\sim \frac{1}{4\pi\epsilon_0} \sum_{\text{everywhere}} \frac{(\text{charge})}{(\text{distance})}$$

$$\equiv \underset{\text{(But written in a fancier vector form)}}{\text{Coulomb's law for electric potential}}$$

(add figure here: infinitely large region + potential element plot)

3.3 Finding $\vec{\boldsymbol{E}}$ from Q

On the other hand, Poisson equation provides an alternative to calculate E-field distribution from charge distribution. If we compare the Gauss's law and Poisson equation:

- Poisson equation $\left(\nabla^2 V = -\frac{\rho}{\epsilon_0}\right)$: $V(\vec{\boldsymbol{r}})$ is a scalar function. Only 1 function $V(\vec{\boldsymbol{r}})$ to be solved.
- Gauss's law $\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0}\right)$: $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ is a vector function with 3 components $E_x(\vec{\boldsymbol{r}})$, $E_y(\vec{\boldsymbol{r}})$, $E_z(\vec{\boldsymbol{r}})$, which are 3 inter-depending functions to be solved.

Obviously, there is no reason to try to solve the more difficult PDE of $\vec{\boldsymbol{E}}$ if alternatively we can solve the easier PDE of V, and then take gradient to get $\vec{\boldsymbol{E}}$ (i.e. via $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V$).

(add figure here: triangle)

Similar to potential, if the given boundary condition is $V(\vec{\boldsymbol{r}}=\infty)=0$, the solution for the Gauss's law as a PDE should also be the Coulomb's law.

$$\boxed{\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = -\vec{\boldsymbol{\nabla}}V(\vec{\boldsymbol{r}}) = \frac{1}{4\pi\epsilon_0} \iiint\limits_{\substack{\text{infinitely large}\\ \text{space}}} \frac{\rho(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|^2}\left[\frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right]\mathrm{d}^3\vec{\boldsymbol{r}}'}$$

$$\sim \frac{1}{4\pi\epsilon_0} \sum_{\text{everywhere}} \frac{(\text{charge})}{(\text{distance})^2}\left(\substack{\text{unit}\\ \text{vector}}\right)$$

$$\equiv \underset{\text{(But written in a fancier vector form)}}{\text{Coulomb's law for electric field}}$$

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4 Image Charge Method

4.1 Induced Charge & Equipotential on Conductors

Suppose we want to solve an electrostatics problem with conductors present. Because charges can freely flow in conductors, the presence of external charge sources can *induce* new charge in the conductors.

- The induced charge's distribution completely depends on the positions of external charge, and also the shape of the conductor.
- **Measuring the induced charge distribution is impossible**, because any charge probe will disturb it.

(add figure here: conductor charge run away)

In previous sections, we are solving for $\vec{\boldsymbol{E}}$ and V where the charge distribution ρ is known everywhere. But this time we do not know the exact distribution of induced charge! Luckily, if the induced charges are distributed on conductors, there is one property that the induced charges need to satisfy:

Charges on conductor must distribute s.t. the whole conductor is of equi-potential.

This is intuitive - because charges are allowed to move freely, they will spontaneously distribute themselves until net electric force on them is 0, i.e. the electric potential will be the same everywhere on the conductor.

(add figure here: conductor rod)

4.2 Image charges

In the viewpoint of solving PDE, requiring "conductor = equipotential" only means an additional boundary condition of V over the conductor's surface.

(add figure here: just BC)

From mathematical studies of Poisson equation, there is the **uniqueness theorem of Poisson equation** (proof on wiki) that if the boundary conditions is already given,

- Any solutions you can obtain will only be different by a constant.
- So the E-field, gradient of potential, must be unique.

$$\binom{\text{Find a}}{\text{Solution}} = V(\vec{\boldsymbol{r}}) + C \quad \Longrightarrow \quad \vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}V = -\vec{\boldsymbol{\nabla}}V(\vec{\boldsymbol{r}}) + 0 = \binom{\text{Same for}}{\text{any } C}$$

This theorem allows us to skip calculating where the induced charge are - we may find another charge configurations that creates the same potential boundary condition but easier to calculate. By the uniquessness theorem, the E-field by these two configurations must be the same.

This approach is called the **image charge method**, and the charges that we used to create the alternative configuration are called **image charge**. They are called so because the alternative configurations usually look like a "reflection" of the external charge sources through the conductor surface.

Example 4.1. (”Reflection” by plane)

Consider a point charge q at distance d from an infinitely large metal surface which is maintained at $V = 0$. We expect an induced charge distribution forming on the surface and contribute to the potential/E-field on top of the surface.

(add figure here: plate + induced charge)

But we already know another similar configuration that creates a flat equipotential surface of $V = 0$ and satisfy $V(\infty = 0)$ - when there is an additional charge of $-q$ at distance d under the metal surface.

(add figure here: plate + image charge)

The principle of image charge method tells us that the potential and E-field (on top of the metal surface) in both cases must be identical because they satisfy the same boundary condition of V. Therefore the potential is

$$\begin{aligned}
V_{\binom{\text{top}}{\text{half}}}(x,y,z) &= \begin{pmatrix}\text{Contribution by} \\ \text{original charge } q\end{pmatrix} + \begin{pmatrix}\text{Contribution by} \\ \text{image charge } -q\end{pmatrix} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+(z+d)^2}}
\end{aligned}$$

and the E-field is

$$\begin{aligned}
\vec{\boldsymbol{E}}_{\binom{\text{top}}{\text{half}}}(x,y,z) &= -\vec{\boldsymbol{\nabla}} V_{\binom{\text{top}}{\text{half}}}(x,y,z) \\
&= -\vec{\boldsymbol{\nabla}}\left(\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+(z-d)\hat{\boldsymbol{z}}}{(x^2+y^2+(z-d)^2)^{\frac{3}{2}}} - \frac{x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+(z+d)\hat{\boldsymbol{z}}}{(x^2+y^2+(z+d)^2)^{\frac{3}{2}}}\right)
\end{aligned}$$

While the V and $\vec{\boldsymbol{E}}$ on the bottom half must be 0 because it is in the metal.

After getting $\vec{\boldsymbol{E}}$, we can also find the induced charge distribution using Gauss's law. By drawing a Gaussian box with area A on the surface at position $(x, y, 0)$,

$$\begin{aligned}
\begin{pmatrix}\text{Total} \\ \text{flux}\end{pmatrix} &= \begin{pmatrix}\text{Flux through} \\ \text{top surface}\end{pmatrix} \\
&= \vec{\boldsymbol{E}}_{\binom{\text{top}}{\text{half}}}(x,y,0)\cdot A\hat{\boldsymbol{z}} \\
&= \frac{qA}{4\pi\epsilon_0}\frac{-2d}{(x^2+y^2+d^2)^{\frac{3}{2}}} \\
&\equiv \frac{Q_{\text{induced}}}{\epsilon}
\end{aligned}$$

(add figure here: induced charge box)

$$\Rightarrow \begin{pmatrix}\text{Induced charge}\\ \text{surface density}\end{pmatrix} = \sigma_{\text{induced}} \equiv \frac{Q_{\text{induced}}}{A} = -\frac{q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}}$$

Example 4.2. ("Reflection" by Sphere)

Consider a point charge q at distance a from the center of a metal sphere of radius R which is maintained at $V = 0$. We expect an induced charge distribution forming on the surface and contribute to the potential/E-field on top of the surface.

(add figure here: sphere + induced charge)

In the spherical case, we can substitute the induced charge by a single point charge $-q'$ at distance b from the sphere's center, which can be calculated by

$$\boxed{\begin{cases} b = \dfrac{R^2}{a} \\ q' = -\dfrac{R}{a}q \end{cases}}$$

(add figure here: show q', b, coordinate theta)

<u>*Proof*</u>

Choose the origin to be the center of the sphere and direction to q to be $\theta = 0$. Since it requires the potential on the sphere to be $V(r = R) = 0$, we can write the total V as

$$\begin{aligned}
V(R,\theta,\phi) = 0 &= \begin{pmatrix}\text{Contribution by}\\ \text{original charge } q\end{pmatrix} + \begin{pmatrix}\text{Contribution by}\\ \text{image charge } -q'\end{pmatrix} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}} - \frac{q'}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2 + b^2 - 2Rb\cos\theta}} \\
\Rightarrow \quad \left(\frac{-q'}{q}\right) &= \frac{R^2 + b^2 - 2Rb\cos\theta}{R^2 + a^2 - 2Ra\cos\theta} \\
0 &= \underbrace{\left[1 - \left(\frac{q'}{q}\right)^2\right]R^2 + \left[b^2 - \left(\frac{q'}{q}\right)^2 a^2\right]}_{\text{This part is independent of } \theta} - \underbrace{2R\left[b - \left(\frac{q'}{q}\right)^2 a\right]}_{\text{Coefficient of } \cos\theta}\cos\theta
\end{aligned}$$

This relation should hold for any θ.

1. Therefore the coefficient of $\cos\theta$ must be 0.

$$\begin{aligned}
0 &= 2R\left[b - \left(\frac{q'}{q}\right)^2 a\right] \\
\Rightarrow \quad \frac{b}{a} &= \left(\frac{q'}{q}\right)^2
\end{aligned}$$

2. Then the remaining term must also become 0.

$$\begin{aligned}
0 &= \left[1 - \left(\frac{q'}{q}\right)^2\right]R^2 + \left[b^2 - \left(\frac{q'}{q}\right)^2 a^2\right] \\
&= \left(1 - \frac{b}{a}\right)R^2 + \left(b^2 - a^2 \cdot \frac{b}{a}\right) \\
&= \left(1 - \frac{b}{a}\right)R^2 + ab\left(\frac{b}{a} - 1\right) \\
&= \left(1 - \frac{b}{a}\right)(R^2 - ab) \\
\Rightarrow \quad &\text{Either} \quad b = a \text{ (reject)} \quad \text{or} \quad \underline{b = \frac{R^2}{a}}
\end{aligned}$$

Finally substitute back into $\frac{b}{a} = \left(\frac{q'}{q}\right)^2$ to get

$$q' = \pm\sqrt{\frac{b}{a}}q = -\frac{R}{a}q$$

The image charge must be negative to be physical

$\square$

Exercise 4.1. How do you add image charges in these situations, such that the potential on the metal surface satisfy $V = 0$?

(add figure here: 4 different variations)

Here we shall summarize the methods of solving electrostatics problems:

1. Very symmetric configurations $\Rightarrow$ Gauss's law integral form. No calculus required.
2. Not so symmetric but satisfies $V(\vec{\boldsymbol{r}} = \infty) = 0 \Rightarrow$ Multiple integral with Coulomb's law.
3. $V(\vec{\boldsymbol{r}}) = 0$ on some nice surfaces with induced charge $\Rightarrow$ Try image charge method.
4. All the above do not apply $\Rightarrow$ Solve Poisson equation explicitly. PDE hell.

— The End —