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Magnetic Induction

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Overview:

- Motion under Lorentz force
- Faraday's law: Motional & transformer EMF
- Lenz's law: Direction of induced EMF
- Appendix: A brief history of electromagnetism

In electromagnetism, theoretically every problem can be solved through a set of PDEs called the **Maxwell Equations**.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\longrightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

However, a *system of PDEs* is too complicated to be solved. So we need to learn different "tricks" to avoid them, which are enough for some simple scenarios.

Magnetic induction concerns the 3$^{\text{rd}}$ equation of the set - Faraday's law.

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1 Motion under Lorentz Force

Nowadays we know that Lorentz force is the fundamental explanation to magnetic induction. But interestingly, it was formulated correctly only in the very late history of classical E&M.

$$\vec{F} = \vec{F}_E + \vec{F}_B$$
$$\boxed{= q\vec{E} + q\vec{v} \times \vec{B}}$$

In general, $\vec{E}$ and $\vec{B}$ can be functions of time and position, but here we will only discuss the special case when the fields are constant.

By symmetry, we can assume that $\vec{B}$ is only in $\hat{z}$ direction, i.e. $\vec{B} = B_z\hat{z}$. Then the Newton's 2$^{\text{nd}}$ law can be expanded as

$$m\vec{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix}$$

which is a system of 3 first order linear ODEs.

$$\begin{cases} m\dfrac{\mathrm{d}v_x}{\mathrm{d}t} = qE_x + qv_yB_z \\ m\dfrac{\mathrm{d}v_y}{\mathrm{d}t} = qE_y - qv_xB_z \\ m\dfrac{\mathrm{d}v_z}{\mathrm{d}t} = qE_z \end{cases}$$

This system is simple enough that we do not need to use any matrix methods.

– **Equation for z component**

Motion in z direction is independent of x/y. It can be solved alone.

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = qE_z \qquad \Rightarrow \qquad v_z(t) = \frac{qE_z}{m}t + v_{z0}$$

$$\Rightarrow \qquad z(t) = \frac{1}{2}\frac{qE_z}{m}t^2 + \underbrace{v_{z0}}_{\text{Initial z velocity}}t + \underbrace{z_0}_{\text{Initial z coordinate}}$$

which is just a constant acceleration motion with an acceleration $\dfrac{qE_z}{m}$.

– **Equation for x/y components**

They can be solved by first differentiating one of them, then substitute into the other.

$$\begin{aligned} \frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} &= \frac{qB_z}{m}\frac{\mathrm{d}v_y}{\mathrm{d}t} \\ &= \frac{qB_z}{m}\left(\frac{qE_y}{m} - \frac{qB_z}{m}v_x\right) \\ &= -\frac{q^2B_z^2}{m}\left(v_x - \frac{E_y}{B_z}\right) \end{aligned}$$

which is the familiar ODE of SHM.

$$v_x(t) = -C\sin\left(\frac{qB_z}{m}t + \phi\right) + \frac{E_y}{B_z}$$

$$\Rightarrow \quad x(t) = \underbrace{C\left(\frac{m}{qB_z}\right)}_{\text{A constant}}\cos\left(\frac{qB_z}{m}t + \phi\right) + \frac{E_y}{B_z}t + x_0$$

$$= \underbrace{R}_{\text{Radius}}\cos\left(\frac{qB_z}{m}t + \underbrace{\phi}_{\text{Phase}}\right) + \frac{E_y}{B_z}t + \underbrace{x_0}_{\text{Initial x coordinate}}$$

And subsitute back to equation of v_y gives

$$v_y(t) = \frac{m}{qB_z}\frac{\mathrm{d}v_x}{\mathrm{d}t} - \frac{E_x}{B_z} = -C\cos\left(\frac{qB_z}{m}t + \phi\right) - \frac{E_x}{B_z}$$

$$\Rightarrow \quad y(t) = \underbrace{-C\left(\frac{m}{qB_z}\right)}_{\text{A constant}}\sin\left(\frac{qB_z}{m}t + \phi\right) - \frac{E_x}{B_z}t + y_0$$

$$= -\underbrace{R}_{\text{Radius}}\sin\left(\frac{qB_z}{m}t + \underbrace{\phi}_{\text{Phase}}\right) - \frac{E_y}{B_z}t + \underbrace{x_0}_{\text{Initial y coordinate}}$$

The result x/y motion is a combination of circular motion + drifting.

(add figure here: lorentz drift)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{R\begin{pmatrix} \cos\left(\frac{qB_z}{m}t + \phi\right) \\ -\sin\left(\frac{qB_z}{m}t + \phi\right) \end{pmatrix}}_{\text{Circular motion}} + \underbrace{\frac{1}{B_z}\begin{pmatrix} E_y \\ -E_x \end{pmatrix}}_{\text{Drifting}} t + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

1. Be careful that the drifting direction is not intuitive:
   - With E_x only, drifting direction $= -\hat{\boldsymbol{y}}$
   - With E_y only, drifting direction $= \hat{\boldsymbol{x}}$

   In general, the drifting is along the direction of $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ with a speed $\dfrac{|\vec{\boldsymbol{E}}|}{|\vec{\boldsymbol{B}}|}$.

2. The rotation radius and speed maintain a constant ratio:

   $$\text{Radius} = R \qquad \Rightarrow \qquad \text{Speed} = \frac{qB_z}{m}R$$

   i.e. Angular velocity is always $\omega = \dfrac{qB_z}{m}$, independent of the charge's initial velocity. You can also arrive at the same result from equation of circular motion:

   $$\frac{mv^2}{R} = qvB$$
   $$v = \frac{qB}{m}R$$

   (But how do you argue that it is a circular motion without solving ODEs?)

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2 Magnetic Induction

2.1 EMF - Force or Voltage?

The term **electromotive force** (EMF) was invented by Alessandro Volta in 1801, for explaining observations in electrochemistry.

Metal electrodes in electrolyte

⇒ Current generates spontaneously.

⇒ There must be some kind of "force" pushing the current!

(add figure here: electrolyte)

In early 1800s, scientists tended to describe things like a mechanical system, i.e. any motions of objects must be driven by some kind of "force" - When there is a flow current, there must be some "force" that keeps pushing the charges forward, namely the "electromotive force".

- However, EMF took the unit of volt because voltage measurement was the only way to quantify the magnitude of EMF.
- Later when Michael Faraday discovered the phenomena of magnetic induction, he used the same word to refer to the source of the induced current.

Today we still keep its name as a "force", even though we already know much better how current is generated in different sources. Maybe because people are already used to refer the same term "EMF" as the general name for ANY kind of current source, rather than differentiating them by the origin of energy.

(add figure here: i dun care)

In discussions of magnetic induction, there are two kinds of EMF relate to this phenomenon.

- **Motional EMF** : Source of energy = Motion of the charges' "container". Can be explained via Lorentz force.
- **Transformer EMF** : Source of energy = Change in magnetic field. Explanation requires relativity.

2.2 Motional EMF

Under Lorentz force, free charges move in circles in a constant B-field. But if the charges' movements are restricted, charge distribution becomes uneven by building up potential difference.

2.2.1 A Moving Battery

Suppose there is only ONE charge in the middle of a conducting rod. When the charge is given some initial velocity perpendicular to the rod (e.g. we give the rod a push), the charge will start to move in a circlar trajectory, and the rod will be dragged along by the charge.

(If the rod is restricted to move horizonally, it becomes SHM.)

(add figure here: pipe SHM)

But for charges that are at the ends of the rod, they will be pushed against the end walls and cannot move. Very soon, charge will accumulate at the end of the rod, build up a E-field (i.e. potential difference) between the ends of the rod. The central charges stop moving once the force from the built-up E-field balance the magnetic force, i.e. when $q\vec{\boldsymbol{E}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$.

(add figure here: pipe potential)

Note that if the rod suddenly stop moving, the built-up potential difference, and thus the stored energy will be released and lost.

The rod can be used to drive current.
It is a battery - but only when it keeps moving!

We can analyze how the EMF arise from its energy conservation process. Let the B-field be constant and in $\hat{\boldsymbol{z}}$ direction, i.e. $\vec{\boldsymbol{B}} = B_z\hat{\boldsymbol{z}}$.

1. Note that the work done on a charge by magnetic force is always 0, because $\vec{\boldsymbol{F}}_B$ always perpendicular to the travelling direction $\mathrm{d}\vec{\boldsymbol{r}}$, (i.e. the same direction as $\vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t}$).

$$\text{W.D.} = \vec{\boldsymbol{F}}_B \cdot \mathrm{d}\vec{\boldsymbol{r}} = q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{r}} = 0$$

2. Meanwhile we can separate the W.D. into contributions along x and y direction:

$$\begin{aligned}
\text{W.D.} &= q\Big[(\vec{\boldsymbol{v}}_x + \vec{\boldsymbol{v}}_y) \times \vec{\boldsymbol{B}}\Big] \cdot (\mathrm{d}\vec{\boldsymbol{x}} + \mathrm{d}\vec{\boldsymbol{y}}) \\
&= \underbrace{\cancelto{0}{q(\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{x}}}}_{\substack{\vec{\boldsymbol{v}}_x\times\vec{\boldsymbol{B}}=v_xB_z(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{z}}) \\ =v_xB_z(-\hat{\boldsymbol{y}}) \\ \text{Dotting with } \hat{\boldsymbol{x}} \text{ gives } 0}} + q(\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{y}} + q(\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{x}} + \underbrace{\cancelto{0}{q(\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{y}}}}_{\substack{\vec{\boldsymbol{v}}_y\times\vec{\boldsymbol{B}}=v_yB_z(\hat{\boldsymbol{y}}\times\hat{\boldsymbol{z}}) \\ =v_yB_z(\hat{\boldsymbol{x}}) \\ \text{Dotting with } \hat{\boldsymbol{y}} \text{ gives } 0}} \\
&= (-qv_xB_z\hat{\boldsymbol{y}}) \cdot \mathrm{d}\vec{\boldsymbol{y}} + (qv_yB_z\hat{\boldsymbol{x}}) \cdot \mathrm{d}\vec{\boldsymbol{x}} \\
&= \underbrace{(-qv_xB_z)\,\mathrm{d}y}_{\substack{\text{W.D. in } \hat{\boldsymbol{y}} \text{ direction} \\ \sim F_y\mathrm{d}y}} + \underbrace{(qv_yB_z)\,\mathrm{d}x}_{\substack{\text{W.D. in } \hat{\boldsymbol{x}} \text{ direction} \\ \sim F_x\mathrm{d}x}} \\
&\equiv 0 \quad \text{(Required by magnetic force)}
\end{aligned}$$

To always hold true, these two terms must always have equal magnitude but opposite sign. This implies an energy conservation between:

- W.D. in x direction = Any energy that cause the charge moving horizontally. Must be indirectly applied on the charge through the movement of the rod.

- W.D. in y direction = Any energy that cause the charge moving vertically. Can be independent of the rod's movement because charge can flow freely along the rod.

And this conversion is possible all because of the "weird" direction of magnetic force (always perpendicular to v).

(add figure here: WD by B = 0)

F_{Lorentz} is in the same direction as $\mathrm{d}y$, opposite

3. When charges accumulate at the ends of the rod, equilibrium happens when $q\vec{\boldsymbol{E}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$. So when a charge climbs up the E-field for a small distance $\mathrm{d}\vec{\boldsymbol{y}}$, it will gain a PE of:

$$q\,\underset{\substack{\uparrow \\ \text{The potential gain} \\ \text{i.e. EMF}}}{\mathrm{d}\underline{\epsilon}} \;\equiv\; (q\vec{\boldsymbol{E}}) \cdot \mathrm{d}\hat{\boldsymbol{y}} = q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\hat{\boldsymbol{y}} = -(qv_x B_z)\,\mathrm{d}y$$

To summarize, motional EMF is the result of energy conservation between

$$\underset{\substack{\text{W.D. along } \hat{\boldsymbol{x}} \text{ direction} \\ \sim F_x \mathrm{d}x}}{\underline{(qv_y B_z)\,\mathrm{d}x}}$$

2.2.2 Generalization to Arbituary Wire

2.2.3 Magnetic Flux - A Geometrical Relation

Consider a wire with a changing shape in under a static B-field. Each segment i of the wire has a length + orientation $\mathrm{d}l_i$ and is moving in velocity v_i. With some algebra, the EMF generated in each segment can be re-written as

$$\begin{aligned} \mathrm{d}\epsilon &= (\vec{\boldsymbol{v}}_i \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{l}}_i \\ &= (\mathrm{d}\vec{\boldsymbol{l}}_i \times \vec{\boldsymbol{v}}_i) \cdot \vec{\boldsymbol{B}} \\ &= -(\vec{\boldsymbol{v}}_i \cdot \mathrm{d}\vec{\boldsymbol{l}}_i) \cdot \vec{\boldsymbol{B}} \end{aligned}$$

Meanwhile, within a short duration $\mathrm{d}t$, the displacement of the segment is $\vec{\boldsymbol{v}}_i\,\mathrm{d}t$. Thus we can approximate the swept area by the segment as a parallelogram:

$$\begin{aligned} \begin{pmatrix} \text{Swept} \\ \text{Area} \end{pmatrix} &= \begin{pmatrix} \text{Area of parallelogram} \\ \text{made by } v_i\,\mathrm{d}t \;\;\&\;\; \mathrm{d}l_i \end{pmatrix} \\ &\approx |\vec{\boldsymbol{v}}_i||\,\mathrm{d}\vec{\boldsymbol{l}}_i\,| \sin \begin{pmatrix} \text{Angle between} \\ \vec{\boldsymbol{v}}_i\,\mathrm{d}t \;\;\&\;\; \mathrm{d}l_i \end{pmatrix} \\ &= (\vec{\boldsymbol{v}}_i\,\mathrm{d}t) \times (\mathrm{d}\vec{\boldsymbol{l}}_i) \frac{\mathrm{d}}{\mathrm{d}t} v_i\,\mathrm{d}t \;\;\&\;\; \mathrm{d}l_i \qquad\qquad = \vec{\boldsymbol{v}}_i \times \mathrm{d}l_i \end{aligned}$$

(add figure here: swept area)

We can now relate EMF generated in the whole wire with swept area by the wire as:

$$\begin{aligned} \sum \epsilon = -\sum_{\substack{\text{All} \\ \text{segments}}} \left[(\vec{\boldsymbol{v}}_i \times \mathrm{d}\vec{\boldsymbol{l}}_i) \cdot \vec{\boldsymbol{B}} \right] &= -\sum_{\substack{\text{All} \\ \text{segments}}} \left[\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \text{Swept} \\ \text{area} \end{pmatrix}_i \cdot \vec{\boldsymbol{B}} \right] \\ &= -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{\substack{\text{All} \\ \text{segments}}} \left[\begin{pmatrix} \text{Swept} \\ \text{area} \end{pmatrix} \cdot \vec{\boldsymbol{B}} \right] \end{aligned}$$

When the segments are infinitestimally short, the sum becomes integral.

$$\epsilon = -\int (\vec{\boldsymbol{v}} \times \mathrm{d}\vec{\boldsymbol{l}}_i) \cdot \vec{\boldsymbol{B}} = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\iint_{\substack{\text{Swept}\\\text{area}}} \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}}\right]$$
$$= -\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\text{Magnetic flux through}\\\text{the swept area}\end{pmatrix}$$

Pay attention to the depedendence to t on the RHS:

- The B-field $\vec{\boldsymbol{B}}$ is static. It is not a function of t.
- $\mathrm{d}\vec{\boldsymbol{s}}$ is just a notation saying that this is a flux integral.
- The only thing that depends on t is the integration range.

So to be more accurate, the equation of motional EMF may be written as

$$\epsilon = -\int_{\substack{\text{along a wire}\\\text{of shape of } l(t)}} (\vec{\boldsymbol{v}} \times \mathrm{d}\vec{\boldsymbol{l}}) \cdot \vec{\boldsymbol{B}} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\substack{\text{Area } S(t)\\\text{swept by wire}}} \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}}$$

to emphasize that it is the wire's shape / swept area varying with time.

2.3 Transformer EMF

The modern explanation to transformer EMF is by relativity - motional EMF and transformer EMF are the same phenomenon being observed in different reference frame.

(add figure here: different ref frame)

EMF generated due to motion of the ring EMF generated due to changing magnetic field strength Consider a reference frame where the charge is static to the observer. Because the charge is not moving, Lorentz force by magnetic field $q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$ must be 0. But if a time-varying B-field is applied, experiments show that a transformer EMF will create, i.e. some mysterious forces appear on the charges, forcing them to form a potential difference.
How to explain this mysterious force? Add an induced E-field, and claim it to be generated by time-varying B-field.

(add figure here: fix newton 2nd law)

2.4 Maxwell-Faraday Equation

2.5 Lenz's Law

The formula of Faraday's law, in particular the integral form, can only be used to tell the magnitude of the induced EMF, but not its direction.

$$\epsilon = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}}$$

This minus sign is not useful for determining the direction of EMF at all.
To determine the EMF's direction, we can apply **Lenz's law**, which is in principle,

Induced EMF always want to "oppose" change in magnetic flux

And right hand grip rule is the only thing you need. We can look at a few example to get familiar with it.

Example 2.1. Consider a ring inside a magnetic field whose magnitude is increasing in the out-of-paper direction.

1. Magnetic flux is getting "more out-of-paper" due to increase in B-field's magnitude.
2. Induced EMF want to "oppose" this magnetic flux change.
3. To oppose an increasing out-of-paper flux, one needs to decrease it, i.e. create an into-paper flux to compensate the increase.
4. By right hand rule, into-paper flux can be created if a current flow clockwise in the ring. So the induced EMF must be clockwise.

(add figure here: B field direction)

(add figure here: induce flux direction)

(add figure here: bar chart)

Example 2.2. Consider a ring inside a constant into-paper B-field. The ring is shrinking in radius.

1. Magnetic flux is getting "less into-paper" due to decrease in area.
2. Induced EMF want to "oppose" this magnetic flux change.
3. To oppose a decreasing into-paper flux, one needs to increase it, i.e. create an into-paper flux to compensate the decrease.
4. By right hand rule, into-paper flux can be created if a current flow clockwise in the ring. So the induced EMF must be clockwise.

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3 Solving Problems in Magnetic Induction

Generally speaking, there are only two levels of questions related to magnetic induction.

- Find EMF from the change in magnetic flux.
- Find induced E-field from the change in magnetic flux.

The difference is in the difficulties - EMF is just a single number, the same everywhere in the coil. But the E-field, when involving magnetic induction, is a vector function which depends on position AND time.

3.1 Finding EMF

This is a straightforward calculation to the surface integral of Faraday's law.

- **For high school level** - B-field is usually given as a constant of position, so that the surface integral reduces to a multiplication.

$$\epsilon = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}} \;\sim\; -\frac{\mathrm{d}}{\mathrm{d}t}\Big(\vec{\boldsymbol{B}}(t) \cdot (\text{Area})(t)\Big)$$

And because you are not expected to have studied differentiation chain rule, you will never see a situation where both $\vec{\boldsymbol{B}}(t)$ and Area(t) are changing. It is always given the rate of change of one of them and the other is fixed.

$$\underbrace{\epsilon = -\vec{\boldsymbol{B}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\text{Area}(t))}_{\text{When the question is about motional EMF}} \qquad \text{or} \qquad \underbrace{\epsilon = -\frac{\mathrm{d}\vec{\boldsymbol{B}}(t)}{\mathrm{d}t} \cdot (\text{Area})}_{\text{When the question is about transformer EMF}}$$

- **For university level** - You are assumed to have already learnt Ampere's law, so more likely you are given the current and need to find B-field before EMF.

$$\text{First} \quad \underbrace{\oint \vec{\boldsymbol{B}}(t) \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I(t)}_{\text{Ampere's law for } \vec{\boldsymbol{B}}}\;, \quad \text{then} \quad \underbrace{\epsilon = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\text{Area}(t)} \vec{\boldsymbol{B}}(t) \cdot \mathrm{d}\vec{\boldsymbol{s}}}_{\text{Faraday's law for } \epsilon}$$

Asking for the EMF's direction is pretty common, because applying Lenz's law does not involve any maths. All you need is your right hand.

3.2 Finding induced E-field

This is in fact the task of solving the PDE of Maxwell-Faraday equation.

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \qquad \Rightarrow \qquad \vec{\boldsymbol{E}}(t) = \text{Some function of } \vec{\boldsymbol{B}}(t)$$

Similar to how we deal with Gauss's law or Ampere's law, we can avoid solving PDE in some very symmetrical cases. If these conditions are satisfied:

1. E-field is of the same magnitude everywhere on the loop.
2. E-field make the same angle with each line segment of the loop.

Then we can reduce the integral form of Maxwell-Faraday equation in to multiplication.

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{\boldsymbol{B}}\cdot \mathrm{d}\vec{s} &= \oint \vec{\boldsymbol{E}}\cdot \mathrm{d}\vec{\boldsymbol{l}} \\
&= \oint \underline{|\vec{\boldsymbol{E}}||\,\mathrm{d}\vec{\boldsymbol{l}}\,|\cos\theta} \longleftarrow \begin{matrix}\text{Just dot product}\\ \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\cos\theta\end{matrix} \\
&= \underbrace{|\vec{\boldsymbol{E}}|}_{\substack{\text{Same magnitude everywhere}\\ \text{Can move out of integral!}}}\ \underbrace{\cos\theta}_{\substack{\text{Form same angle everywhere}\\ \text{Can move out of integral!}}}\ \oint |\,\mathrm{d}\vec{\boldsymbol{l}}\,| \\
&= |\vec{\boldsymbol{E}}|\ \cos\theta\ (\text{Perimeter of loop})
\end{aligned}
$$

$$
\boxed{|\vec{\boldsymbol{E}}| = \frac{\left(\text{Flux of }\vec{\boldsymbol{B}}\right)}{(\text{Perimeter of loop})\cos\theta}}
$$

Example 3.1. Consider a static rectangular loop next to an infinitely long wire, which carries a time-varying current $I(t)$, same magnitude everywhere along the wire.

(add figure here: loop next to wire)

1. Always start with finding the magnetic flux through the loop. By Ampere's law, the B-field by an infinitely long wire is

$$
\vec{\boldsymbol{B}}(r,t) = \frac{\mu_0}{2\pi}\frac{I(t)}{r}
$$

   The magnetic flux through the loop can be calculated by first dividing the loop's area into strips, then integrate the flux of all strips.

$$
\begin{aligned}
\Phi_B &= \iint \vec{\boldsymbol{B}}\cdot \mathrm{d}\vec{\boldsymbol{s}} \\
&= \int_{r_1}^{r_2} \frac{\mu_0 I(t)}{2\pi r}(L\,\mathrm{d}r) \\
&= \frac{\mu_0 I(t) L}{2\pi}[\ln(r_2) - \ln(r_1)]
\end{aligned}
$$

(add figure here: split into strips)

2. The total EMF generated in the loop is just the differentiation of the above. i.e.

$$
\epsilon = \frac{\mu_0 L}{2\pi}\left(\frac{\mathrm{d}I(t)}{\mathrm{d}t}\right)[\ln(r_2) - \ln(r_1)]
$$

   Direction of the EMF depends on how the current varies. For example, if $\frac{\mathrm{d}I(t)}{\mathrm{d}t} > 0$, i.e. magnetic flux increasing in the into-paper direction. To oppose the change, EMF must be in anti-clockwise direction to produce an out-of-paper direction flux.

3. On the other hand, to compute the E-field distribution, all we know is

$$|\epsilon| = \oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = E_1^{\|} L + E_2^{\|}(r_2 - r_1) + E_3^{\|} L + E_4^{\|}(r_2 - r_1)$$

(add figure here: show label of edge only parallel)

After taking dot product, only the component parallel to the edge is left.

To find the E on each edge, we need symmetry arguments.

- By rotation symmetry, the E-field is only a function of radial distance from the wire. We can claim that $E_1^{\|}$ and $E_3^{\|}$ must have the same function form $\vec{\boldsymbol{E}}(r)$,

  (add figure here: rotation sym)

- By translational symmetry, $E_2^{\|}$ and $E_4^{\|}$ must be the same. Their contributions of dot product along the loop are cancelled.

So the E-field relation to EMF is reduced to

$$\begin{aligned} |\epsilon| &= E_3^{\|} L - E_1^{\|} L \\ &= E^{\|}(r_2) L - E^{\|}(r_1) L \\ &\equiv \frac{\mu_0 L}{2\pi} \left(\frac{\mathrm{d}I(t)}{\mathrm{d}t} \right) [\ln(r_2) - \ln(r_1)] \end{aligned}$$

We can claim that the $\vec{\boldsymbol{E}}$'s component parallel to the wire is

$$E^{\|}(r) = \frac{\mu_0 L}{2\pi} \left(\frac{\mathrm{d}I(t)}{\mathrm{d}t} \right) \ln(r)$$

4. Note that in the above analysis, we cannot determine if the E-field has components perpendicular to the wire.

(add figure here: parallel + perpendicular)

If there is only parallel component
If there is also perpendicular component. Is it physical?

Recall from Gauss's law, E-field can be perpendicular to the wire if the wire carries a static line charge density. So when the wire carries BOTH net charge and time-varying current, the total E-field will be in diagonal directions.

(add figure here: sum of horizontal + vertical contribution = diagonal)

Appendix: A Brief History of Electromagnetism

Electromagnetic induction is likely the most confusing topic in beginner E&M. In my opinion, taking reference of the history is helpful to unify the concepts you have learnt.

Year	Advancement
Before 1500s	Different electrostatics phenomena were known. But they were not unified or explained at all.
1600	William Gilbert was the first person to use the word "electrical" to describe electrostatics phenomena. Also the first to propose that electrical effect is due to flows of particles.
1750	Benjamin Franklin developed a one "fluid" theory of electricity, and called this fluid "charge".
1784	Charles-Augustin de Coulomb experimentally showed that force between charged objects $\propto \frac{1}{r^2}$. (Coulomb's law $F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}$)
1800	Alessandro Volta Made the first battery from electro-chemistry. (First time to have steady current.)
1820	Hans Christian Ørsted discovered that current wire can deflect compress. (First time to relate electric and magnetic phenomena.)
1820	André-Marie Ampère formulated and verified the force between current wires. ($F = I\vec{l}_1 \times \frac{\mu_0 I}{2\pi r}\vec{l}_2$)
1831	Michael Faraday discovered magnetic induction. Experiments include: - Across iron core: Reading appears at the instant when switch is on/off. - Moving frame: Reading appears when wireframe moves, changes shape or when magnetic field change. insertFig
1834	Emil Lenz Explained direction of induced current by energy conservation. (Lenz's Law)
1860	James Clerk Maxwell unified past discoveries into 20 equations, and used field description for the first time. This was the first time **E** and **B** appeared in Physics. Before Maxwell, everything was described in terms of force.
1893	Oliver Heaviside combined Maxwell's 20 equations into 4, by vector calculus. (This is the version of Maxwell's equation we now know.)
1895	Hendrik Lorentz derive the correct force on charges under both $\vec{E}$ and $\vec{B}$. (Lorentz force formula)

— The End —