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Magnetic Material & Magnetization

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Overview:

- Model of magnetic dipole
- How to describe dipole arrangement: Magnetization field $\vec{\boldsymbol{M}}$, bound currents $\vec{\boldsymbol{J}}_b$, $\vec{\boldsymbol{K}}_b$
- Material under external $\vec{\boldsymbol{B}}$ field:
  - A special case of material model: Linear magnetic material
  - Describe external field: Free currents $\vec{\boldsymbol{J}}_f$, $\vec{\boldsymbol{K}}_f$, auxiliary field $\vec{\boldsymbol{H}}$

(If linear magnetic material is all you need, feel free to skip all vector calculus)

1 Magnetic Dipole

1.1 Electron as Magnetic Source

In atoms, interactions with B-field mainly come from electrons. They behaves as magnetic dipoles in 2 ways:

- Orbital motion - Electrons orbit around the nucleus, which is equivalent to a current loop.

  (add figure here: orbital motion)

- Spin - Electrons carry a fundamental quantum property called **spin**, which determines how it interacts with B-field. It cannot be described fully by any classical picture.

  (add figure here: electron spin)

  Electron's spin is usually depicted as a rotating charged arrow, although electron is a *point* particle.

To study the magnetic properties of materials, we can view the atoms as many **magnetic dipoles**, arranged according to the material structure.

(add figure here: stacking dipoles)

Since magnetic dipoles emit B-field, theoretically all materials are sources of B-field. However the dipoles in real life are at atomic scale ($\sim 10^{-10}$ m) and are usually packed in random orientations, the net field can be treated as zero if we view the dipoles from a very large distance.

(add figure here: view at short distance vs long distance)

So magnetic properties of materials are normally unobservable, unless we apply an external B-field to align the dipoles in some direction, i.e. we **magnetize** the material.

(add figure here: many dipole from random direction flip to the same direction)

In the classical model, atoms as magnetic dipole can interact with external B-field in opposite ways, depend on which of the electron behavior dorminates:

	Diamagnetism	**Paramagnetism**
Dominated Behavior	Orbital Motion	Spin
Response	Dipoles align opposite to applied B-field	Dipoles align the same direction as applied B-field
Relative Magnitude	Weak	Much stronger
Found in	Most materials, where electrons with opposite spin all pair up such that spin effect is cancelled	Materials with unusual bondings which contain unpaired electrons.

The formal explanation requires quantum mechanics. But we may briefly explain why they lead to opposite behavior with classical pictures:

- Electron are points objects - they can flip their spins easily to align with B-field
- Orbital motion are real current loops - induced B-field is generated to oppose the external B-field, which is equivalent to creating dipoles opposite to the external B-field.

(add figure here: classical pic of dia/para)

Side Note:

Textbooks usually describe **ferromagnetism** as the 3rd kind of magnetic response. However its origin is very different from dia-/paramagnetism:

- Dia-/paramagnetism comes from individual atoms' response to B-field. They can exist even if there is only one atom.
- Ferromagnetism is a collective behavior of many neighbouring atoms. Its discussion also requires statistical mechanics.

On ferromagnetic materials, atoms tend to align to the same direction naturally, dividing the material into "domains" with the same dipole direction (typical size $\sim 10^{-6}$ m).

(add figure here: expanding domain)

When placed under an external B-field, the domains which aligns with the field expand to merge other domains, causing a strong and unify magnetization.

1.2 Fields of Magnetic Dipole

Since magnetic dipole is the basic unit of magnetic material, we can first analyze the field emitted by a single dipole, then sum the contributions from all dipoles according to the material structure. The classical model of magnetic dipole is as follows:

(add figure here: magnetic dipole model)

- A circular loop of radius R, carrying current I.
- We are only interested in the potential/field far away from the dipole, i.e. $|\vec{\boldsymbol{r}}| \gg R$. The position vector $\vec{\boldsymbol{r}}$ is referenced from the center of the dipole.

1.2.1 Potential by Electric Dipole

Deriving magnetic potential from a dipole is straightforward by Biot-Savart law, while applying Taylor approximation in the middle step.

1. The total potential contributions towards position $\vec{\boldsymbol{r}}$ by two current elements at opposite position on the loop is

$$\mathrm{d}\vec{\boldsymbol{A}}\left(\vec{\boldsymbol{r}}\right) = \frac{\mu_0}{4\pi}\left(\frac{I\,\mathrm{d}\vec{\boldsymbol{l}}_+}{|\vec{\boldsymbol{r}}_+|} + \frac{I\,\mathrm{d}\vec{\boldsymbol{l}}_-}{|\vec{\boldsymbol{r}}_-|}\right)$$

By geometry, $|\,\mathrm{d}\vec{\boldsymbol{l}}\,| = R\,\mathrm{d}\theta$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{R\,\mathrm{d}\theta}{|\vec{\boldsymbol{r}}_+|}(\hat{\boldsymbol{\theta}}) + \frac{R\,\mathrm{d}\theta}{|\vec{\boldsymbol{r}}_-|}(-\hat{\boldsymbol{\theta}})\right)$$

And they point in opposite direction

$$= \frac{\mu_0 I R\,\mathrm{d}\theta}{4\pi}\left(\frac{1}{\sqrt{|\vec{\boldsymbol{r}}|^2 + R^2 - 2|\vec{\boldsymbol{r}}|R\cos\phi}} - \frac{1}{\sqrt{|\vec{\boldsymbol{r}}|^2 + R^2 + 2|\vec{\boldsymbol{r}}|R\cos\phi}}\right)\hat{\boldsymbol{\theta}}$$

These are just cosine law

$$= \frac{\mu_0 I R\,\mathrm{d}\theta}{4\pi|\vec{\boldsymbol{r}}|}\left(\frac{1}{\sqrt{1 + \frac{R^2}{|\vec{\boldsymbol{r}}|^2} - \frac{2R}{|\vec{\boldsymbol{r}}|}\cos\phi}} - \frac{1}{\sqrt{1 + \frac{R^2}{|\vec{\boldsymbol{r}}|^2} + \frac{2R}{|\vec{\boldsymbol{r}}|}\cos\phi}}\right)\hat{\boldsymbol{\theta}}$$

(add figure here: show geometry, opposite current elements and position vectors)

2. Taylor expansion "$(1+x)^n \approx 1 + nx$ for $x \ll 1$" is always applied at this step. Because we are only interested in the potential far from the dipole, $\frac{R}{|\vec{\boldsymbol{r}}|} \ll 1$,

   - $\frac{R^2}{|\vec{\boldsymbol{r}}|^2} \approx 0$
   - $\frac{1}{\sqrt{1-\frac{R}{|\vec{\boldsymbol{r}}|}\cos\phi}} = \left(1 - 2\frac{R}{|\vec{\boldsymbol{r}}|}\cos\phi\right)^{-\frac{1}{2}} \approx 1 + \frac{R}{|\vec{\boldsymbol{r}}|}\cos\phi$

   So the potential becomes

$$\begin{aligned}\mathrm{d}\vec{\boldsymbol{A}}\left(\vec{\boldsymbol{r}}\right) &= \frac{\mu_0 I R\,\mathrm{d}\theta}{4\pi|\vec{\boldsymbol{r}}|}\left[\left(1 + \frac{R}{|\vec{\boldsymbol{r}}|}\cos\phi\right) - \left(1 - \frac{R}{|\vec{\boldsymbol{r}}|}\cos\phi\right)\right]\hat{\boldsymbol{\theta}} \\ &= \frac{\mu_0 I R\,\mathrm{d}\theta}{4\pi|\vec{\boldsymbol{r}}|}\cdot 2\frac{R}{|\vec{\boldsymbol{r}}|}\cos\phi\cdot\hat{\boldsymbol{\theta}}\end{aligned}$$

3. Before doing any integration, we first need to express every non-constant in terms of θ:

   – Recall what we learnt about polar coordinate, the angular unit vector $\hat{\boldsymbol{\theta}}_{\{r,\theta\}}$ is a function of angle θ.

$$\hat{\boldsymbol{\theta}} = -(\sin\theta)\hat{\boldsymbol{x}} + (\cos\theta)\hat{\boldsymbol{y}}$$

   – ϕ is the angle between the target's position $\vec{\boldsymbol{r}}$ and position vector of $I\,\mathrm{d}\vec{\boldsymbol{l}}_+$. Expressing them in x-y-z coordinate:

     – Target's position: $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$
     – Position vector of $I\,\mathrm{d}\vec{\boldsymbol{l}}_+$: $\vec{\boldsymbol{R}} = (R\cos\theta)\hat{\boldsymbol{x}} + (R\sin\theta)\hat{\boldsymbol{y}}$

   So that $\cos\phi$ can be calculated as their dot product:

$$\cos\phi = \frac{\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{R}}}{|\vec{\boldsymbol{r}}||\vec{\boldsymbol{R}}|} = \frac{x\cos\theta + y\sin\theta}{|\vec{\boldsymbol{r}}|}$$

4. Finally integrate over θ from 0 to π to find the total contribution of the loop.

$$\begin{aligned}
\int \mathrm{d}\vec{\boldsymbol{A}}\,(\vec{\boldsymbol{r}}) &= \int_0^{\pi} \frac{\mu_0 I R\,\mathrm{d}\theta}{4\pi|\vec{\boldsymbol{r}}|}\cdot 2\frac{R}{|\vec{\boldsymbol{r}}|}\cos\phi\cdot\hat{\boldsymbol{\theta}} \\
&= \frac{\mu_0 I R^2}{2\pi|\vec{\boldsymbol{r}}|^2}\int_0^{\pi}\left(\frac{x\cos\theta + y\sin\theta}{|\vec{\boldsymbol{r}}|}\right)(-(\sin\theta)\hat{\boldsymbol{x}} + (\cos\theta)\hat{\boldsymbol{y}})\,\mathrm{d}\theta \\
&= \frac{\mu_0 I R^2}{2\pi|\vec{\boldsymbol{r}}|^3}\int_0^{\pi}(-x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}})\underbrace{\sin\theta\cos\theta}_{\int_0^\pi \sin\theta\cos\theta\,\mathrm{d}\theta\,=\,0} + (x\hat{\boldsymbol{y}})\underbrace{\cos^2\theta}_{\int_0^\pi \cos^2\theta\,\mathrm{d}\theta\,=\,\frac{\pi}{2}} - (y\hat{\boldsymbol{x}})\underbrace{\sin^2\theta}_{\int_0^\pi \sin^2\theta\,\mathrm{d}\theta\,=\,\frac{\pi}{2}}\,\mathrm{d}\theta \\
&= \frac{\mu_0 I R^2}{2\pi|\vec{\boldsymbol{r}}|^3}\cdot\frac{\pi}{2}\cdot(x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}})
\end{aligned}$$

   It happens that the term in the bracket is the cross product of $\hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{r}}$, since

$$\hat{\boldsymbol{z}}\times\vec{\boldsymbol{r}} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = -y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}}$$

   So that the potential becomes

$$\begin{aligned}
\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) &= \frac{\mu_0 I R^2}{4|\vec{\boldsymbol{r}}|^2}\left(\frac{x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}}{|\vec{\boldsymbol{r}}|}\right) \\
&= \frac{\mu_0 I R^2}{4|\vec{\boldsymbol{r}}|^2}\frac{\hat{\boldsymbol{z}}\times\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|} \\
&= \frac{\mu_0}{4\pi}\frac{I\pi R^2}{|\vec{\boldsymbol{r}}|^2}(\hat{\boldsymbol{z}}\times\vec{\boldsymbol{r}})
\end{aligned}$$

5. To emphasize that we now treat a magnetic dipole as "one unit of source", We define the **magnetic dipole moment $\vec{m}$**,

$$\boxed{\vec{m} \stackrel{\text{def}}{=} I \cdot \binom{\text{Loop's}}{\text{Area}} \cdot \binom{\text{Unit vector}}{\text{normal to loop}}}$$

In our case of a loop in x-y plane, $\vec{m} = I(\pi R^2)\hat{z}$. Finally we reach the standard formula of magnetic potential from a magnetic dipole:

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{|\vec{r}|^2}}$$

As a comparison with Biot-Savart law formula from current element,

$$\mathrm{d}\vec{A}\,(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{l}}{|\vec{r}|}$$

(add figure here: V from point charge source)

- $\vec{A}$ is always along the same direction as the current element.
- Dependance on distance is $\frac{1}{r}$.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{|\vec{r}|^2}$$

(add figure here: V from dipole source)

- Magnitude of potential depends on the current's magnitude AND the angle (cross product) between $\vec{m}$ and $\hat{r}$.
- Dependance on distance is $\frac{1}{r^2}$.

1.2.2 B-field from Magnetic Dipole

The standard derivation of B-field from dipole is through the relation $\vec{B} = \vec{\nabla}\times\vec{A}$. It is nothing more than some boring vector calculus. Here I quote the final result:

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{|\vec{r}|^3}}$$

And this formula is rarely used because vector calculation is annoying.

The boring derivation:

1. Derivation is possible with high school calculus if we turn all vectors into x-y-z form: $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, $|\vec{\boldsymbol{r}}| = \sqrt{x^2+y^2+z^2}$ and $\vec{\boldsymbol{m}} = m_x\hat{\boldsymbol{x}} + m_y\hat{\boldsymbol{y}} + m_z\hat{\boldsymbol{z}}$:

$$\begin{aligned}
\frac{\vec{\boldsymbol{m}} \times \hat{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|^2} &= \frac{\vec{\boldsymbol{m}}}{|\vec{\boldsymbol{r}}|^2} \times \frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|} = \frac{1}{|\vec{\boldsymbol{r}}|^3}(\vec{\boldsymbol{m}} \times \vec{\boldsymbol{r}}) = \frac{1}{|\vec{\boldsymbol{r}}|^3}\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ m_x & m_y & m_z \\ x & y & z \end{vmatrix} \\
&= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\left[(m_y z - m_z y)\hat{\boldsymbol{x}} + (m_z x - m_x z)\hat{\boldsymbol{y}} + (m_x y - m_y x)\hat{\boldsymbol{z}}\right]
\end{aligned}$$

2. First the x term in curl, $\left(\dfrac{\partial \bullet_z}{\partial y} - \dfrac{\partial \bullet_y}{\partial z}\right)\hat{\boldsymbol{x}}$, can be calculated by:

$$\begin{aligned}
\frac{\partial \bullet_z}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{(m_x y - m_y x)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right] \\
&= \frac{m_x (x^2+y^2+z^2)^{\frac{3}{2}} - (m_x y - m_y x)\cdot 3y(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3} \\
&= \frac{m_x|\vec{\boldsymbol{r}}|^2 - 3(m_x y - m_y x)y}{|\vec{\boldsymbol{r}}|^5} \\
\frac{\partial \bullet_y}{\partial z} &= \frac{\partial}{\partial z}\left[\frac{(m_z x - m_x z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right] \\
&= \frac{-m_x (x^2+y^2+z^2)^{\frac{3}{2}} - (m_z x - m_x z)\cdot 3z(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3} \\
&= \frac{-m_x|\vec{\boldsymbol{r}}|^2 - 3(m_z x - m_x z)z}{|\vec{\boldsymbol{r}}|^5} \\
\Rightarrow \left(\frac{\partial \bullet_z}{\partial y} - \frac{\partial \bullet_y}{\partial z}\right)\hat{\boldsymbol{x}} &= \frac{2m_x|\vec{\boldsymbol{r}}|^2 - 3(m_x y - m_y x)y + 3(m_z x - m_x z)z}{|\vec{\boldsymbol{r}}|^5}\hat{\boldsymbol{x}} \\
&= \frac{2m_x|\vec{\boldsymbol{r}}|^2 - 3m_x y^2 + 3m_y xy + 3m_z xz - 3m_x z^2 + 3m_x x^2 - 3m_x x^2}{|\vec{\boldsymbol{r}}|^5}\hat{\boldsymbol{x}} \\
&= \frac{\underline{2m_x|\vec{\boldsymbol{r}}|^2} - \underline{3m_x(x^2+y^2+z^2)} + 3m_x x^2 + 3m_y xy + 3m_z xz}{|\vec{\boldsymbol{r}}|^5}\hat{\boldsymbol{x}} \\
&= \frac{-m_x|\vec{\boldsymbol{r}}|^2 + 3\underline{(m_x x + m_y y + m_z z)}x}{|\vec{\boldsymbol{r}}|^5}\hat{\boldsymbol{x}} \\
&= \frac{-m_x|\vec{\boldsymbol{r}}|^2 + 3\underline{(\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{r}})}x}{|\vec{\boldsymbol{r}}|^5}\hat{\boldsymbol{x}}
\end{aligned}$$

3. Similar for y and z terms:

$$\left(\frac{\partial \bullet_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} = \frac{-m_y|\vec{\boldsymbol{r}}|^2 + 3(\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{r}})y}{|\vec{\boldsymbol{r}}|^5}\hat{\boldsymbol{y}}$$

$$\left(\frac{\partial \bullet_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}} = \frac{-m_z|\vec{\boldsymbol{r}}|^2 + 3(\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{r}})z}{|\vec{\boldsymbol{r}}|^5}\hat{\boldsymbol{z}}$$

4. Summing all 3 components gives:

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

$$= \frac{\mu_0}{4\pi}\frac{3(\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{r}})\underline{(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}})} - \underline{(m_x\hat{\boldsymbol{x}} + m_y\hat{\boldsymbol{y}} + m_z\hat{\boldsymbol{z}})}|\vec{\boldsymbol{r}}|^2}{|\vec{\boldsymbol{r}}|^5}$$

$$= \frac{\mu_0}{4\pi}\frac{3(\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{r}})\underline{\vec{\boldsymbol{r}}} - \underline{\vec{\boldsymbol{m}}}|\vec{\boldsymbol{r}}|^2}{|\vec{\boldsymbol{r}}|^5}$$

Take out magnitude to become unit vector

$$= \frac{\mu_0}{4\pi}\frac{3(\vec{\boldsymbol{m}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,|\vec{\boldsymbol{r}}|^2 - \vec{\boldsymbol{m}}|\vec{\boldsymbol{r}}|^2}{|\vec{\boldsymbol{r}}|^5}$$

$$\boxed{\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{3(\vec{\boldsymbol{m}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \vec{\boldsymbol{m}}}{|\vec{\boldsymbol{r}}|^3}}$$

2 Describing Dipole Arrangement

In general, all materials are made of many tiny magnetic dipoles. To analyze the material's magnetic properties, we can begin with the property of a single dipole, then sum the contributions of all dipoles according to the dipole arrangement in the material.

$$\underbrace{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{\vec{\boldsymbol{m}}_i}{|\vec{\boldsymbol{r}}|^2}\times\left(\frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|}\right)}_{\text{By a single dipole from origin}} \quad \Rightarrow \quad \underbrace{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\sum_{\text{All dipoles } i}\frac{\vec{\boldsymbol{m}}_i}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_i|^2}\times\left(\frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_i}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_i|}\right)}_{\text{By many dipoles at different positions } \vec{\boldsymbol{r}}_i}$$

(add figure here: single dipole point to r vs many dipoles at different ri to r)

Here we introduce two quantities that describe magnetic dipole arrangements in materials.

- **Magnetization field** - $\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}})$
- **Bound current distributions** - $\vec{\boldsymbol{J}}_b(\vec{\boldsymbol{r}})$ and $\vec{\boldsymbol{K}}_b(\vec{\boldsymbol{r}})$

2.1 Magnetization Field

When the dipoles in the material are so dense such that we can treat them as a continuous distribution, we can replace

- Summmation of all dipoles $\xRightarrow{\text{become}}$ Volume integral over the whole object.
- Discrete dipoles source $\xRightarrow{\text{become}}$ A vector distribution called **magnetization field** $\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}})$.

$$\boxed{\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \iiint\limits_{\substack{\text{Whole}\\\text{material}}} \frac{\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^2} \times \left(\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) \mathrm{d}^3\vec{\boldsymbol{r}}'}$$

(add figure here: dispole arrangement is similar to some vector field in the object)

The magnetization field $\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}})$ can also be interpreted as **magnetic dipole density** because its usage is similar to charge density $\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}})$. Comparing with the Biot-Savart law for magnetic potential:

- When the source is made of current elements:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{\boldsymbol{J}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} \mathrm{d}^3\vec{\boldsymbol{r}}' \quad \sim \quad \frac{\mu_0}{4\pi} \sum \frac{(\text{current density})}{(\text{distance})}$$

- When the source is made of dipoles:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^2} \times \left(\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}\right) \mathrm{d}^3\vec{\boldsymbol{r}}' \quad \sim \quad \frac{\mu_0}{4\pi} \sum \frac{(\text{dipole density})}{(\text{distance})^2} \times \left(\substack{\text{unit}\\\text{vector}}\right)$$

(add figure here: charge density vs dipole density in the same material)

2.2 Bound Current Distribution

Ultimately, magnetic dipoles are just loops of currents. If the dipoles are aligned non-uniformly in a material, some regions may appear to have a higher flow of current in one direction than the other way.

(add figure here: dipole loops non uniform)

These regions with extra current flow are described as **bound current** distribution in the material - they are always "bounded" to regions where the current flow is not cancelled. So bound current distribution can be used to describe magnetic dipole arrangement in the material.

2.2.1 Mathematical Origin

With vector calculus, the potential formula can be rewritten into a "current densities form".

$$\begin{aligned}
\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) &= \frac{\mu_0}{4\pi} \underset{\substack{\text{Whole}\\ \text{material}}}{\iiint} \frac{\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|^2} \times \left(\frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right) \mathrm{d}^3\vec{\boldsymbol{r}}' \\
&= (\cdots \textit{ After more boring vector calculus } \cdots) \\
&= \frac{\mu_0}{4\pi} \underset{\substack{\text{Whole}\\ \text{material}}}{\iiint} \frac{\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|} \mathrm{d}^3\vec{\boldsymbol{r}}' + \frac{\mu_0}{4\pi} \underset{\substack{\text{Surface}\\ \text{of material}}}{\oiint} \frac{\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|} \times \mathrm{d}^2\vec{\boldsymbol{r}}'
\end{aligned}$$

By comparing with the Biot-Savart law for electric potential $\vec{\boldsymbol{A}} \sim \frac{\mu_0}{4\pi}\sum \frac{\text{(current density)}}{\text{(distance)}}$, we identify the 2 source terms as the **bound current densities**:

- **<u>The 1st term:</u>**

$$\frac{\mu_0}{4\pi} \underset{\substack{\text{Whole}\\ \text{material}}}{\iiint} \frac{\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|} \mathrm{d}^3\vec{\boldsymbol{r}}' \quad \sim \quad \frac{\mu_0}{4\pi} \sum_{\substack{\text{Inside}\\ \text{material}}} \frac{(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}})}{\text{(distance)}}$$

$\underline{\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}}}$ appears as some current density distributed <u>inside</u> the material. Therefore it is defined as the **volume bound current density** $\vec{\boldsymbol{J}}_b$.

$$\boxed{\vec{\boldsymbol{J}}_b(\vec{\boldsymbol{r}}) \overset{\text{def}}{=} \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}}(\vec{\boldsymbol{r}})}$$

- **<u>The 2nd term:</u>**

$$\frac{\mu_0}{4\pi} \underset{\substack{\text{Surface}\\ \text{of material}}}{\oiint} \frac{\vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|} \times \mathrm{d}^2\vec{\boldsymbol{r}}' \quad \sim \quad \frac{\mu_0}{4\pi} \sum_{\substack{\text{On material}\\ \text{surface}}} \frac{(\text{Surface cross product of } \vec{\boldsymbol{M}})}{\text{(distance)}}$$

The <u>surface cross product of $\vec{\boldsymbol{M}}$</u> means that we are taking cross product between $\vec{\boldsymbol{M}}$ and normal vector of the surface. It appears as some current density distributed <u>on the surface</u> of the material. So it is defined as the **surface bound current density** $\vec{\boldsymbol{K}}_b$.

$$\boxed{\vec{\boldsymbol{K}}_b(\vec{\boldsymbol{r}}) \overset{\text{def}}{=} \vec{\boldsymbol{M}}(\vec{\boldsymbol{r}}) \times \hat{\boldsymbol{n}}}$$

where $\hat{\boldsymbol{n}}$ is the (outward) unit normal vector on the material surface.

Finally, the expression of the "current densities" form is nothing more than saying that the magnetic potential from a material is the result of the two kinds of current distributions.

$$\begin{aligned}
\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) &= \frac{\mu_0}{4\pi} \underset{\substack{\text{Whole}\\ \text{material}}}{\iiint} \frac{\vec{\boldsymbol{J}}_b(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|} \mathrm{d}^3\vec{\boldsymbol{r}}' + \frac{\mu_0}{4\pi} \underset{\substack{\text{Surface}\\ \text{of material}}}{\oiint} \frac{\vec{\boldsymbol{K}}_b(\vec{\boldsymbol{r}}')}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|} \mathrm{d}^2\vec{\boldsymbol{r}}' \\
&= \begin{pmatrix}\text{Contribution}\\ \text{by currents inside}\\ \text{the material}\end{pmatrix} + \begin{pmatrix}\text{Contribution}\\ \text{by currents on}\\ \text{material's surface}\end{pmatrix}
\end{aligned}$$

The boring derivation:

1. Deriving the "current densities form" begins with a vector calculus identity:

$$\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \left(\hat{x}\frac{\partial}{\partial \underline{x'}} + \hat{y}\frac{\partial}{\partial \underline{y'}} + \hat{z}\frac{\partial}{\partial \underline{z'}}\right)\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)$$

Note: Differentiation is w.r.t. x', y', z'

$$= \frac{\hat{x}(x-x')+\hat{y}(y-y')+\hat{z}(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}$$

$$= \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

This part is just a unit vector

$$= \frac{1}{|\vec{r}-\vec{r}'|^2}\underline{\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\right)}$$

(This is basically $\frac{\mathrm{d}}{\mathrm{d}r'}\left(\frac{1}{r-r'}\right)=\frac{1}{(r-r')^2}$ but in vector version)

2. So the potential formula can be rewritten as:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|^2}\times\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\right)\mathrm{d}^3\vec{r}'$$

$$= \frac{\mu_0}{4\pi}\iiint \underbrace{\vec{M}(\vec{r}')\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)}_{\vec{G}\times\vec{\nabla}f\quad(\text{Note the order of } f \text{ and } \vec{G})}\mathrm{d}^3\vec{r}'$$

$$= \frac{\mu_0}{4\pi}\iiint \underbrace{-\vec{\nabla}\times\left(\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)}_{-\vec{\nabla}\times(f\vec{G})} + \underbrace{\frac{\vec{\nabla}\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}}_{f\vec{\nabla}\times\vec{G}}\mathrm{d}^3\vec{r}'$$

Here we used the product rule of curl $\boxed{\vec{\nabla}\times(f\vec{G}) = f\vec{\nabla}\times\vec{G}+(\vec{\nabla}f)\times\vec{G}}$, where f is a scalar function (like $\frac{1}{|\vec{r}-\vec{r}'|}$) and $\vec{G}$ is a vector function (like $\vec{M}$).

3. Finally use the "divergence theorem for curl" to convert the 1st term's volume integral into a surface integral over the volume's surface:

$$\vec{A}(\vec{r}) = \underline{\frac{\mu_0}{4\pi}\iiint\vec{\nabla}\times\left(\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\mathrm{d}^3\vec{r}'} + \frac{\mu_0}{4\pi}\iiint\frac{\vec{\nabla}\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\mathrm{d}^3\vec{r}'$$

Divergence theorem for curl

$$= \underline{\frac{\mu_0}{4\pi}\oiint\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\times\mathrm{d}^2\vec{r}'} + \frac{\mu_0}{4\pi}\iiint\frac{\vec{\nabla}\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\mathrm{d}^3\vec{r}'$$

$$= \begin{pmatrix}\text{Contribution}\\ \text{from material's}\\ \text{surface}\end{pmatrix} + \begin{pmatrix}\text{Contribution}\\ \text{from inside}\\ \text{the material}\end{pmatrix}$$

Remind that we are integrating regions where dipoles exist. So this volume integral corresponds to the whole material and surface integral corresponds to only the surface of the material.

2.2.2 Visualization

The bound current densities are related to $\vec{M}$ pretty much like normal current density are related to $\vec{B}$ in Ampere's law.

- **Volume bound current**: By drawing an Ampere's loop and check how many dipoles are pointing in the same direction as we travel along the loop,

More dipole align with the circle	$\Leftrightarrow$	Stronger current at the center

(add figure here: dipole 's current in toroid shape)

Recall that we can use sign of dot product line integral to identify positions with vector field in rotation pattern, and curl operator $\vec{\nabla}\times$ is equivalent to finding dot product line integral per area.

$$\begin{pmatrix}\text{Bound current}\\ \text{volume density}\end{pmatrix} \sim \vec{J}_b = \vec{\nabla} \times \vec{M} \sim \begin{pmatrix}\text{\# of dipoles per area}\\ \text{around a rotation center}\end{pmatrix}$$

- **Surface bound current**: On the material surface, we can see that

More dipole align in the same direction	$\Leftrightarrow$	More current flow on the material surface

(add figure here: surface out of paper dipole form current to left)

As we have defined the normal vector $\hat{n}$ of the material surface to be pointing outward, the direction of current is exacly along $\vec{M} \times \hat{n}$.

$$\begin{pmatrix}\text{Bound current}\\ \text{surface density}\end{pmatrix} \sim \vec{K}_b = \vec{M} \times \hat{n} \sim \begin{pmatrix}\text{Same direction dipole}\\ \text{per surface area}\end{pmatrix}$$

Side note:

For paramagnetic and diamagnetic materials, we almost never observe their $\vec{M}$ field or bound current because

- Magnitude of a dipole is too small. Typical magnetic dipole moment are of order 10^{-24} A·m^2.
- The dipoles are usually randomly arranged, or in patterns such that their effects cancel out. Especially when the material is large.

Their magnetization effect is neligible unless it is placed under an external B-field, which forces its dipoles to align and change their dipole moment's magnitude.

(add figure here: para, dia vs ferro)

But we see magnetization in ferromagnetic material because their dipoles are uniformly aligned at room temperature, making the material behaves like a giant dipole.

3 Material under External B-field

Recall that magnetic response of atoms is contributed by their electrons - by their orbital motion and spin. Depends on the contribution of each behavior and the material structure, dipole density in the material can vary by position. The final dipole alignment might not always align with the external B-field!

(add figure here: many dipoles under external B-field, not fully align)

In reality, fully model a material's response to B-field can be very complicated. The general theory assumes the magnetization field $\vec{\boldsymbol{M}}$ (i.e. new dipole alignment) as some function to the total B-field $\vec{B}_{\text{total}}$,

$$\vec{\boldsymbol{M}} \;=\; f(\vec{\boldsymbol{B}}_{\text{total}}) \;\sim\; \underbrace{\vec{\boldsymbol{a}}_i^{(1)} B_i + \vec{\boldsymbol{a}}_{ij}^{(2)} B_i B_j + \vec{\boldsymbol{a}}_{ijk}^{(3)} B_i B_j B_k + \cdots}_{\text{Like a Taylor expansion}}$$

This f function denotes a theoretical model that we have choosen to investigate the material. And the model parameters $\vec{\boldsymbol{a}}^{(1)}, \vec{\boldsymbol{a}}^{(2)}, \vec{\boldsymbol{a}}^{(3)}$... shall be determined from experiment.

(add figure here: random spherical dipole + regular external E-field = irregular align ellipse dipole)

3.1 Special Case: Linear Magnetic Material

Linear magnetic material is the simplest model of B-field response - where the dipoles are completely free to rotate, such that new dipole alignment is directly proportional to the external B-field's magnitude and direction. This assumption is applicable to most daily life materials.

(add figure here: dipoles uniform under external B-field)

The linear model only has 1 parameter to be determined from experiments - the proportionality constant between $\vec{\boldsymbol{M}}$ and $\vec{\boldsymbol{B}}_{\text{total}}$.

$$\vec{\boldsymbol{M}} = \left(\substack{\text{Some}\\\text{constant}}\right) \cdot \vec{\boldsymbol{B}}_{\text{total}}$$

Unlike using $\vec{\boldsymbol{P}}$ v.s. $\vec{\boldsymbol{E}}$ in linear dielectric model, linear magnetization model is almost always written by $\vec{\boldsymbol{M}}$ v.s. a new kind of field $\vec{\boldsymbol{H}}$. We shall explain what $\vec{\boldsymbol{H}}$ right after.

$$\boxed{\vec{\boldsymbol{M}} = \chi_m \vec{\boldsymbol{H}}}$$

Here the χ_m is called **magnetic susceptibility**, a pure number (no unit) whose value depends on the type of material.

- $\chi_m = 0$ for vacuum (by defintion), because there are no dipoles in a vacuum.
- $\chi_m < 0$ for diamagnetism, since their dipoles are opposite to the B-field's direction.
- $\chi_m > 0$ for paramagnetism, since their dipoles follow the B-field's direction.

3.2 Describing External Field

The total B-field is a result of the external B-field plus the field induced by dipole alignment. After we have chosen the model about dipole alignment, now we look at the external B-field.

3.2.1 Free Current Distribution

In order to create an external B-field around the material, we need a “setup” to build an external source of currents. For example, apply current through a long solenoid:

(add figure here: free charge vs bound charge in dielectric setup)

The currents that flow in the setup are given the name **free charges**, to distinguish from the bound currents (dipole alignment) in the material. The are ”free” because we can always control them by varying the setup, making them known quantities in calculation.

(add figure here: adjust solenoid current -¿ adjust free current amount)

$$\mu_0 n I_{\text{free}} = |\vec{\boldsymbol{B}}|$$

For calculation, free current densities are denoted like bound currents:

– **Volume free current density** $\vec{\boldsymbol{J}}_f(\vec{\boldsymbol{r}})$ - Current distribution inside the setup.

– **Surface free charge density** $\vec{\boldsymbol{K}}_f(\vec{\boldsymbol{r}})$ - Current on any surfaces of the setup.

In real practices, solenoid or a large U-shape electromagnet are the most common setup to create external B-field. The B-field can be directly control by controlling current. So line or surface free current are mostly all you need. No reason to make things complicated.

3.2.2 Auxiliary Field $\vec{\boldsymbol{H}}$

Previously, we have relate bound currents to a vector field quantity - the magnetization field $\vec{\boldsymbol{M}}$. Similarly, free current can be related with another vector field. From Ampere's law,

$$\begin{aligned}
\frac{1}{\mu_0}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}_{\text{total}} &= \vec{\boldsymbol{J}}_{\text{total}} = \begin{pmatrix}\text{All}\\ \text{currents}\end{pmatrix} \\
&= \vec{\boldsymbol{J}}_f + \vec{\boldsymbol{J}}_b = \begin{pmatrix}\text{Free currents}\\ \text{on setup}\end{pmatrix} + \begin{pmatrix}\text{Bound currents}\\ \text{on material}\end{pmatrix} \\
&= \vec{\boldsymbol{J}}_f + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{M}} \\
\vec{\boldsymbol{\nabla}} \times \left(\frac{1}{\mu_0}\vec{\boldsymbol{B}} - \vec{\boldsymbol{M}}\right) &= \vec{\boldsymbol{J}}_f
\end{aligned}$$

Here we define the **Auxiliary field** $\vec{\boldsymbol{H}}$:

$$\boxed{\vec{\boldsymbol{H}} \overset{\text{def}}{=} \frac{1}{\mu_0}\vec{\boldsymbol{B}} - \vec{\boldsymbol{M}}} \longleftarrow \text{This relation connects all 3 field quantities}$$

Such that it is related to the free charge density by:

$$\boxed{\begin{cases} \vec{\boldsymbol{J}}_f(\vec{\boldsymbol{r}}) \overset{\text{def}}{=} \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{H}}(\vec{\boldsymbol{r}}) \\ \vec{\boldsymbol{K}}_f(\vec{\boldsymbol{r}}) \overset{\text{def}}{=} \vec{\boldsymbol{H}}(\vec{\boldsymbol{r}}) \times \hat{\boldsymbol{n}} \end{cases}}$$

where $\hat{\boldsymbol{n}}$ is the (outward) unit normal vector on the equipment surface. Cross product with it shows the direction of the surface current relative to the $\vec{\boldsymbol{H}}$ field.

Notice the relations of $\vec{\boldsymbol{H}}$ with free currents are almost identical to $\vec{\boldsymbol{B}}$ with total currents in Ampere's law. In the special case of linear magnetic material, we can solve for $\vec{\boldsymbol{H}}$ from the given free currents like the Ampere's law integral form or with Biot-Savart law.

$$\begin{aligned} I_f &= \oint \vec{\boldsymbol{H}}(\vec{\boldsymbol{r}}) \cdot \mathrm{d}\vec{\boldsymbol{l}} \\ &= \oint \underline{|\vec{\boldsymbol{H}}||\,\mathrm{d}\vec{\boldsymbol{l}}\,|\cos\theta} \longleftarrow \begin{matrix}\text{Just dot product} \\ \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\cos\theta\end{matrix} \\ &= \underbrace{|\vec{\boldsymbol{H}}|}_{\substack{\text{Same magnitude everywhere} \\ \text{Can move out of integral}}} \; \underbrace{\cos\theta}_{\substack{\text{Form same angle everywhere} \\ \text{Can move out of integral}}} \oint |\,\mathrm{d}\vec{\boldsymbol{l}}\,| \\ &= |\vec{\boldsymbol{H}}| \; \cos\theta \; (\text{Perimeter of loop}) \end{aligned}$$

$$\boxed{|\vec{\boldsymbol{H}}| = \frac{I_f}{(\text{Perimeter of loop})\cos\theta}}$$

But note that this can be wrong for other models of magnetic material.

Side Note:

Although $\vec{\boldsymbol{H}}$ forms a PDE relationship to free current just like $\vec{\boldsymbol{B}}_{\text{total}}$ with total current, we cannot solve for $\vec{\boldsymbol{H}}$ exactly by free current if we don't know the material's model. This is because:

- $\vec{\boldsymbol{B}}_{\text{total}}$ is guarenteed to be "divergent-less", such that we can define a vector potential function $\vec{\boldsymbol{A}}$ and solve for $\vec{\boldsymbol{B}}_{\text{total}}$ uniquely.
- $\vec{\boldsymbol{M}}$ may not be "divergent-less" since the dipole arrangement can be arbituary, e.g. if they arrange into some vortex-like pattern. Then by $\vec{\boldsymbol{H}} = \frac{1}{\mu_0}\vec{\boldsymbol{B}}_{\text{total}} - \vec{\boldsymbol{M}}$, there is no guarenteed that $\vec{\boldsymbol{H}}$ is "divergent-less" either.

(add figure here: vortex dipole arrangement -¿ M field not curl 0)

But in the special case of linear magnetic material,

$$\boxed{\vec{\boldsymbol{M}} \text{ aligns with } \vec{\boldsymbol{B}}_{\text{total}}} \Rightarrow \boxed{\vec{\boldsymbol{M}} \text{ is "divergent-less"}} \Rightarrow \boxed{\begin{array}{c}\vec{\boldsymbol{H}} = \dfrac{1}{\mu_0}\vec{\boldsymbol{B}}_{\text{total}} - \vec{\boldsymbol{M}} \\ \text{is also "divergent-less"}\end{array}}$$

So it is fine to solve for $\vec{\boldsymbol{H}}$ like normal Ampere's law problems for linear magnetic material.

3.3 Measurement in Practice

In problems that involve magnetic material,

- The material's magnetization (dipole alignment) is the cause of all troubles. But usually we are only told which material model we need to apply. *(E.g. Given a linear magnetic material ...)*
- Actual calculation requires measurements from equipment setup. In magnetostatic experiments, usually the control/measure-able parameters are
  - Applied current - Directly connect with a power source. All you need is an ammeter.
  - Total B-field - Can be measured through magnetic force.

(add figure here: control set up with voltage or free charge procedure)

Then once we have chosen a model of the material, we can connect the two measurables by:

$$\vec{\boldsymbol{H}} = \frac{1}{\mu_0}\vec{\boldsymbol{B}}_{\text{total}} - f(\vec{\boldsymbol{B}}_{\text{total}})$$

Then the model parameters in $f(\cdots)$ can be determined by experiments. But in the special case of linear magnetic material, since there is only 1 constant parameter χ_m for each material,

$$\begin{aligned}
\vec{\boldsymbol{H}} &= \frac{1}{\mu_0}\vec{\boldsymbol{B}}_{\text{total}} - \chi_m \vec{\boldsymbol{H}} \\
(1+\chi_m)\vec{H} &= \frac{1}{\mu_0}\vec{\boldsymbol{B}}_{\text{total}} \\
\vec{\boldsymbol{H}} &= \frac{1}{\mu_0(1+\chi_m)}\vec{\boldsymbol{B}}_{\text{total}} \\
&= \frac{1}{(\text{A constant})} \cdot \vec{\boldsymbol{B}}_{\text{total}}
\end{aligned}$$

We usually use this new constant to represent the material's response to B-field, by defining

$$\boxed{\mu \overset{\text{def}}{=} \mu_0(1+\chi_m) \overset{\text{def}}{=} \mu_0\mu_r} \qquad \text{such that} \qquad \vec{\boldsymbol{H}} = \frac{1}{\mu}\vec{\boldsymbol{B}}_{\text{total}}$$

where

- μ = **Absolute permeability**, or **magnetic permeability**. Has the same unit as μ_0.
- μ_r = **Relative permeability**. A pure number. < 1 for diamagetism and > 1 for paramagnetism.

Note that $\vec{\boldsymbol{H}}$ only depends on applied current, i.e. can be easily controlled in experiments.

- For convenience, experimentalists always prefer using $\vec{\boldsymbol{H}}$ to describe material's magnetic response. This is why the linear model is written as $\vec{\boldsymbol{M}} \propto \vec{\boldsymbol{H}}$ instead of $\vec{\boldsymbol{M}} \propto \vec{\boldsymbol{B}}_{\text{total}}$.
- χ_m is more oftenly tabulated in material handbooks and used in theories. This is because most material has very small χ_m (e.g. 10^{-5} for aluminum). Tabulating $\mu_r = 1 + \chi_m$ only makes the values harder to read.

(add figure here: do you prefer reading table of chi m or mu r?)

3.4 Calculation Example

Here are some examples of questions you may see in the chapter of linear magnetic material. In principle, one of the information about $\vec{\boldsymbol{H}}$ (free current) or $\vec{\boldsymbol{B}}_{\text{total}}$ (force) will be given, then you are asked to find the other, and so as $\vec{\boldsymbol{M}}$ and bound currents.

— The End —