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Ordinary Differential Equation

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Overview:

Newton's 2$^{\text{nd}}$ Law is a differential equation

$$F(t) = ma(t) = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)$$

which is an equation of $x(t)$, but involves the 2$^{\text{nd}}$ derivative of $x(t)$.

In this note, I will introduce some basic techniques to solve differential equations that you will encounter in mechanics.

- Classification of differential equations
- 1$^{\text{st}}$ order linear ODE
- 2$^{\text{nd}}$ order linear ODE
- Linear ODE with non-homogenoeous terms

The final target is to familiar with solving equation for general harmonic oscillator, i.e.

$$-kx(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}t}x(t) + F(t) = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)$$

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1 Classification of Differential Equations

We can classify differential equations with these characteristics:

1. Number of variables in the wanted function
2. Order
3. Linearity
4. Type of Coefficients
5. Homogenity

1.1 Number of variables in the wanted function

- If the function to be solved is a single variable function, the equation is called an **Ordinary Differential Equation (ODE)**.
- If the function to be solved is a multivariable function, the equation is called a **Partial Differential Equation (PDE)**.

It is easy to identify PDE from ODE because they must involve partial derivatives. Solving PDE can be a lot more complicated than ODE. We will not deal with PDE in this note at all.

1.2 Order

Order = Finding the highest derivative of the wanted function in the equation. E.g.

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}f(t)}_{1^{\text{st}}} + \underbrace{f(t)}_{0^{\text{th}}} = \ln t \qquad \left(\begin{array}{c}\text{highest} = 1^{\text{st}}\text{ derivative} \\ \Rightarrow 1^{\text{st}}\text{ order}\end{array}\right)$$

$$\underbrace{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)}_{2^{\text{nd}}} + \underbrace{f(t)\frac{\mathrm{d}}{\mathrm{d}t}f(t)}_{0^{\text{th}} \times 1^{\text{st}}} = \sin t \qquad \left(\begin{array}{c}\text{highest} = 2^{\text{nd}}\text{ derivative} \\ \Rightarrow 2^{\text{nd}}\text{ order}\end{array}\right)$$

$$\underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right)^2}_{1^{\text{st}} \times 1^{\text{st}}} + \underbrace{(f(t))^2}_{0^{\text{th}} \times 0^{\text{th}}} = 1 \qquad \left(\begin{array}{c}\text{highest} = 1^{\text{st}}\text{ derivative} \\ \Rightarrow 1^{\text{st}}\text{ order}\end{array}\right)$$

1.3 Linearity

Linear = Whether any terms contain multiplication between derivaties. E.g.

$$\underbrace{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)}_{\text{power 1}} - e^t \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}f(t)}_{\text{power 1}} + \underbrace{f(t)}_{\text{power 1}} = 0 \qquad (\text{Power } \leq 1 \Rightarrow \text{ Linear})$$

$$\underbrace{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)}_{\text{power 1}} - \underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right)^2}_{\text{power 2}} = \sin t \qquad (\text{Power } > 1 \Rightarrow \text{Non-Linear})$$

$$\underbrace{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)}_{\text{power 1}} + \underbrace{f(t)\frac{\mathrm{d}}{\mathrm{d}t}f(t)}_{\text{power 1 + power 1}} = 5 \qquad (\text{Power } > 1 \Rightarrow \text{Non-Linear})$$

1.4 Type of Coefficients

If all the coefficients of the derivatives are constant (i.e. not function of t), The equation can be solved with much easier methods. E.g.

$$\underbrace{2}_{+2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t) \underbrace{-}_{-1}\frac{\mathrm{d}}{\mathrm{d}t}f(t) + \underbrace{2}_{+2}f(t) = \cos t \qquad (\text{All are constants})$$

$$\underbrace{(t^2-1)}_{(t^2-1)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t) \underbrace{-t}_{-t}\frac{\mathrm{d}}{\mathrm{d}t}f(t) + \underbrace{2}_{+2}f(t) = 0 \qquad (\text{Some are functions of } t)$$

1.5 Homogenity

Homogenity = Whether all terms contain the wanted functions or its derivatives. E.g.

$$\underbrace{\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)}_{\text{Yes}} + 2\cos(t)\underbrace{f(t)}_{\text{Yes}} = 0 \qquad \text{(All yes } \Rightarrow \text{Homogeneous)}$$

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}f(t)}_{\text{Yes}} + \underbrace{4f(t)}_{\text{Yes}} - \underbrace{\ln(t)}_{\text{No}} = 0 \qquad \text{(Some terms not having } f(t) \Rightarrow \text{Non-homogeneous)}$$

What ODEs can we solve analytically?

- Linear ODEs with any order
  - Constant coefficients
    - Homogeneous - The $e^{\lambda t}$ trick
    - Non-homogeneous - Method of undetermined coefficient
  - Non-constant coefficients - More complicated methods, E.g. $\begin{cases}\text{Integrating factor} \\ \text{Series expansion} \\ \text{Laplace/Fourier transform}\end{cases}$
- Non-linear ODEs - No general methods. Only case by case.

For a general harmonic oscillator problem, the Newton's 2$^{\text{nd}}$ Law writes

$$-kx(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}t}x(t) + F(t) = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)$$

where k, γ and m are usually given as constants, and $F(t)$ could be arbituary function of t. So this is a 2$^{\text{nd}}$ order linear constant coefficient non-homogeneous ODE.

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2 1$^{\text{st}}$ Order Linear Constant Coefficient Homogeneous ODE

This is the simplet kind of ODE

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}f(t) + \lambda f(t) = 0}$$

Here λ is a constant number. The solution is trivial by making use of the fact

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{at} = a \cdot e^{at}$$

which is exactly saying $f(t) = e^{at}$ is a solution to the equation $\frac{\mathrm{d}}{\mathrm{d}t}f(t) - af(t) = 0$. We can also observe that the relation still holds after multiplying any constant to $f(t)$. So we have

$$\boxed{\text{General Solution}: \ f(t) = \underset{\substack{\uparrow \\ C=\text{any constant number}}}{\underline{C}}e^{-\lambda t}}$$

In fact, we can show that this is the only solution to this ODE by solving it with integration:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}f(t) + \lambda f(t) &= 0 \\
\frac{1}{f(t)}\frac{\mathrm{d}f(t)}{\mathrm{d}t} + \lambda &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}[\ln f(t)] &= -\lambda \\
\ln f(t) &= \int -\lambda \, \mathrm{d}t = -\lambda t + C \\
f(t) &= e^{-\lambda t + C} \\
&= \underline{C'}e^{-\lambda t}
\end{aligned}$$

Take $C' = e^C$, which is still a constant

Example 2.1. The **decay equation** is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -kN(t)$$

where

- $N(t)$ = Number of particles
- $\frac{\mathrm{d}}{\mathrm{d}t}N(t)$ = Decay rate in number of particles

The equation theorizes phenenomena where rate of decay is proportional to the number of particles present, i.e. $\frac{\mathrm{d}N}{\mathrm{d}t} \propto N$. From above, we can tell the general solution to be

$$N(t) = Ce^{-kt}$$

where C can be any constant. How do we tell what number we should substitute into C in a scenario? **By matching an initial condition.**

For example, given that at $t = 0$, we are told that there are N_0 particles. Then by substitution,

$$N(0) = Ce^{-k\cdot 0} = C = N_0$$

which leads to a specific solution $N(t) = N_0e^{-kt}$.

Side note:

The expression for *half life* comes from this solution. By definition, At half life $t = \tau_{\frac{1}{2}}$, number of particles remain $= \frac{N_0}{2} = \frac{1}{2}$ the number at start $(t = 0)$. Thus

$$
\begin{aligned}
N\left(\tau_{\frac{1}{2}}\right) = N_0 e^{-k\tau_{\frac{1}{2}}} &= \frac{N_0}{2} \\
\tau_{\frac{1}{2}} &= \frac{\ln 2}{k}
\end{aligned}
$$

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3 2nd Order Linear Constant Coefficient Homogeneous ODE

The equation comes in the form

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f(t) + cf(t) = 0$$

where a, b, c are all constants. To solve it, we can apply the same trick as in 1st ODE - substitute $f(t) = e^{\lambda t}$:

$$
\begin{aligned}
a\frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{\lambda t} + b\frac{\mathrm{d}}{\mathrm{d}t}e^{\lambda t} + ce^{\lambda t} &= 0 \\
a\lambda^2\cancel{e^{\lambda t}} + b\lambda\cancel{e^{\lambda t}} + c\cancel{e^{\lambda t}} &= 0 \\
a\lambda^2 + b\lambda + c &= 0 \qquad \text{(a quadratic equation of } \lambda) \\
\Rightarrow \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

We can either take $f_1(t) = Ce^{\frac{-b+\sqrt{b^2-4ac}}{2a}t}$ or $f_2(t) = Ce^{\frac{-b-\sqrt{b^2-4ac}}{2a}t}$ as a solution.

3.1 Superposition of Solutions

However, **solely taking $f_1(t)$ or $f_2(t)$ is NOT the general solution** because linear ODE allows superpositions (linear combination), i.e.

If $f_1(t)$ and $f_2(t)$ are solutions to a linear homogeneous ODE, any superposition $C_1 f_1(t) + C_2 f_2(t)$ is also a solution to the ODE, for arbituary constants C_1, C_2.

$$\boxed{\text{General Solution : } f(t) = C_1 e^{\frac{-b+\sqrt{b^2-4ac}}{2a}t} + C_2 e^{\frac{-b-\sqrt{b^2-4ac}}{2a}t}}$$

Proof

Given that $f_1(t)$ and $f_2(t)$ are solutions:

$$\begin{cases} a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_1(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f_1(t) + cf_1(t) = 0 \\ a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_2(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f_2(t) + cf_2(t) = 0 \end{cases}$$

To test whether $C_1 f_1(t) + C_2 f_2(t)$ is a solution, we can do substitution:

$$\begin{aligned} \text{L.H.S.} &= a\frac{\mathrm{d}^2}{\mathrm{d}t^2}[C_1 f_1(t) + C_2 f_2(t)] + b\frac{\mathrm{d}}{\mathrm{d}t}[C_1 f_1(t) + C_2 f_2(t)] + c[C_1 f_1(t) + C_2 f_2(t)] \\ &= C_1 \cdot \left[a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_1(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f_1(t) + cf_1(t)\right] + C_2 \cdot \left[a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_2(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f_2(t) + cf_2(t)\right] \\ &= C_1 \cdot 0 + C_2 \cdot 0 \\ &= 0 \\ &= \text{R.H.S.} \end{aligned}$$

So $C_1 f_1(t) + C_2 f_2(t)$ is also a solution. □

Side note:

This superposition property can be easily extended to any Nthorder linear ODE.

1. If a linear ODE is of Nthorder, there must be N (linear) independent solution (Require rigorous proof from linear algebra):

$$f_1(t), f_2(t), ..., f_N(t)$$

2. The general solution is then any superposition (linear combination) of these N solutions:

$$f(t) = C_1 f_1(t) + C_2 f_2(t) + ... + C_N f_N(t)$$

with $C_1, C_2, ..., C_N$ being some constants.

3.2 3 Sub-cases of the Solution

We may further derive the general solution according to the value of $b^2 - 4ac$.

3.2.1 Case 1: $b^2 - 4ac > 0$

Both $\lambda_\pm = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are real number. Nothing can be further simplified. We would just keep the form

$$\boxed{f(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t} \qquad \text{(Both } \lambda \text{ real)}}$$

3.2.2 Case 2: $b^2 - 4ac < 0$

Both $\lambda_{\pm} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are complex number. We can separate their real and imaginary parts. Denote as

$$\mathrm{Re}[\lambda_{\pm}] = -\frac{b}{2a} \stackrel{\text{def}}{=} p \quad , \quad \mathrm{Im}[\lambda_{\pm}] = \pm\frac{\sqrt{4ac - b^2}}{2a} \stackrel{\text{def}}{=} \pm q$$

which is just a re-labelling to $\lambda_{\pm} \stackrel{\text{def}}{=} p \pm iq$.

Then we can apply the **Euler formula**

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Rewriting $f(t)$ as

$$\begin{aligned}
f(t) &= C_1 e^{(p+iq)t} + C_2 e^{(p-iq)t} \\
&= e^{pt}\left[C_1 e^{iqt} + C_2 e^{-iqt}\right] \\
&= e^{pt}[C_1(\cos qt + i\sin qt) + C_2(\cos qt - \sin qt)] \\
&= e^{pt}[\ \underbrace{(C_1 + C_2)}_{\substack{\text{Both are constants.} \\ \text{Can combine.}}}\ \cos qt + i\ \underbrace{(C_1 - C_2)}_{\substack{\text{Both are constants.} \\ \text{Can combine.}}}\ \sin qt]
\end{aligned}$$

$$\boxed{f(t) = e^{pt}[C_1' \cos qt + C_2' \sin qt] \qquad (\text{Both } \lambda \text{ complex})}$$

which is an expression without the imaginary i, so that we can use it to describe physics.

We can also construct another convenient form for physics by trigonometry. Combine the sin / cos into 1 sinusoidal function by change of variables:

$$\begin{cases} C_1' = A\cos\phi \\ C_2' = -A\sin\phi \end{cases} \Leftrightarrow \begin{cases} A = \sqrt{C_1'^2 + C_2'^2} \\ \phi = \tan^{-1}\left(\dfrac{-C_2'}{C_1'}\right) \end{cases}$$

Such that

$$\begin{aligned}
f(t) &= e^{pt}[C_1' \cos qt + C_2' \sin qt] \\
&= e^{pt}[(A\cos\phi)\cos qt + (-A\sin\phi)\sin qt]
\end{aligned}$$

$$\boxed{f(t) = e^{pt} \cdot A\cos(qt + \phi) \qquad (\text{Both } \lambda \text{ complex})}$$

Here we have used the cosine addition rule $\cos(a + b) = \cos a \cos b - \sin a \sin b$.

As a conclusion, we have reached 3 different forms of solution for the case $b^2 - 4ac < 0$, which all are convenient to use in some scenarios.

$$\boxed{f(t) = \begin{cases} C_1 e^{(p+iq)t} + C_2 e^{(p-iq)t} & (\text{Complex form}) \\ e^{pt}[C_1' \cos qt + C_2' \sin qt] & (\text{CS form}) \\ e^{pt} \cdot A\cos(qt + \phi) & (\text{Amplitude form}) \end{cases} \qquad (\text{Both } \lambda \text{ complex})}$$

Side note 1:

The **Euler formula** is an extension to the definition of sin / cos function to complex number inputs. It can be proven by Taylor series expansion:

$$\begin{array}{rcllllll} e^{ix} & = & 1 & +ix & +\frac{(ix)^2}{2!} & +\frac{(ix)^3}{3!} & +\frac{(ix)^4}{4!} & +\frac{(ix)^5}{5!} \\ \cos x & = & 1 & & -\frac{x^2}{2} & & +\frac{x^4}{4} & \\ \sin x & = & & +x & & -\frac{x^3}{3} & & +\frac{x^5}{5} \end{array}$$

It allows any complex number $p + iq$ to be expressed in polar form:

$$\begin{aligned} z &= p + iq \\ &= \sqrt{p^2 + q^2}\cos\theta + i\sqrt{p^2 + q^2}\sin\theta \\ &= \sqrt{p^2 + q^2}(cos\theta + i\sin\theta) \\ &= \sqrt{p^2 + q^2}e^{i\theta} \end{aligned}$$

(add figure here: complex to polar)

Side note 2:

The **Taylor series expansion** is a polynomial approximation to the any continuous functions $f(x)$, for finding the value of $f(k+x)$ given the value of $f(k)$.

If the following is a "good" approximation (applicable when x is small enough)

$$f(k+x) \approx a_0 + a_1(x) + a_2(x)^2 + a_3(x)^3 + ...a_n(x)^n$$

with k being a known value, then we can determine $a_0, a_1, ..., a_n$:

$$a_0 = f(k) \ , \ a_1 = \left.\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right|_{t=k} \ , \ a_2 = \left.\frac{1}{2!}\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right|_{t=k} \ , \ a_3 = \left.\frac{1}{3!}\frac{\mathrm{d}^3 f(t)}{\mathrm{d}t^3}\right|_{t=k} \ , \cdots$$

$$\boxed{a_n = \left.\frac{1}{n!}\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right|_{t=k}}$$

Proof

Let $t = k + x$. Differentiate against t and substitute $x = 0$ (so t becomes $k + 0 = k$),

$$\begin{aligned}
f(k+0) &= a_0 + a_1\cancel{(0)} + a_2\cancel{(0)}^2 + a_3\cancel{(0)}^3 + a_4\cancel{(0)}^4 + ...\\
\left.\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right|_{t=k} &= +a_1 + 2a_2\cancel{(0)} + 3a_3\cancel{(0)}^2 + 4a_4\cancel{(0)}^3 + ...\\
\left.\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right|_{t=k} &= +2a_2 + (3\cdot 2)a_3\cancel{(0)} + (4\cdot 3)a_4\cancel{(0)}^2 + ...\\
\left.\frac{\mathrm{d}^3 f(t)}{\mathrm{d}t^3}\right|_{t=k} &= +(3\cdot 2)a_3 + (4\cdot 3\cdot 2)a_4\cancel{(0)} + ...
\end{aligned}$$

And so on. □

3.2.3 Case 3: $b^2 - 4ac = 0$

This case is problematic in that $\lambda_+ = \lambda_- = \frac{b^2}{2a} \stackrel{\text{def}}{=} p$. We can only get 1 independent solution Ce^{pt} by using the $e^{\lambda t}$ trick.

But mathematicians say if the ODE is of Nthorder, the general solution must be made of N independent functions.

How to find the remaining independent function in our 2$^{\text{nd}}$ order ODE? The method is called **reduction of order**.

1. Let the other independent function be $v(t)e^{pt}$. The goal is to find a suitable $v(t)$ that help it form another solution. First substitute it into the original ODE.

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}[v(t)e^{pt}] + b\frac{\mathrm{d}}{\mathrm{d}t}[v(t)e^{pt}] + c[v(t)e^{pt}] = 0$$

2. Do product rule for each term:

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}[v(t)e^{pt}] &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t)e^{pt} + 2\frac{\mathrm{d}}{\mathrm{d}t}v(t)\frac{\mathrm{d}}{\mathrm{d}t}e^{pt} + v(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{pt} \\
&= e^{pt}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t) + 2p\frac{\mathrm{d}}{\mathrm{d}t}v(t) + p^2v(t)\right] \\
\frac{\mathrm{d}}{\mathrm{d}t}[v(t)e^{pt}] &= \frac{\mathrm{d}}{\mathrm{d}t}e^{pt} + v(t)\frac{\mathrm{d}}{\mathrm{d}t}e^{pt} \\
&= e^{pt}\left[\frac{\mathrm{d}}{\mathrm{d}t}v(t) + pv(t)\right]
\end{aligned}$$

3. Group terms by derivatives of $v(t)$ and solve it

$$\begin{aligned}
0 &= a\cancel{e^{pt}}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t) + 2p\frac{\mathrm{d}}{\mathrm{d}t}v(t) + p^2v(t)\right] + b\cancel{e^{pt}}\left[\frac{\mathrm{d}}{\mathrm{d}t}v(t) + pv(t)\right] + c\big[v(t)\cancel{e^{pt}}\big] \\
&= a\frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t) - 2\underbrace{(ap+b)}_{\substack{=0 \\ \text{because } p=\frac{-b}{2a}}}\frac{\mathrm{d}}{\mathrm{d}t}v(t) + \underbrace{(ap^2+bp+c)}_{\substack{=0 \\ \text{because } p=\frac{-b}{2a} \text{ is the soln.} \\ \text{to } ax^2+bx+c=0}} \\
&= a\frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t) \\
v(t) &= C_1t + C_2
\end{aligned}$$

where C_1, C_2 are some constants. Therefore we find the other independent function to be $(C_1t+C_2)e^{pt}$. Note that it already contains the first independent function Ce^{pt}. We may write

$$\boxed{f(t) = C_1e^{pt} + C_2te^{pt} \qquad (\lambda \text{ equal})}$$

Side note 3:

We can use the **Leibniz formula** to compute higher derivatives faster.

$$\begin{aligned}
\mathrm{d}(uv) &= \underbrace{\mathrm{d}u \cdot v}_{1} + \underbrace{u\,\mathrm{d}v}_{1} \\
\mathrm{d}^2(uv) &= \underbrace{\mathrm{d}^2u \cdot v}_{1} + \underbrace{2\,\mathrm{d}u \cdot \mathrm{d}v}_{2} + \underbrace{u \cdot \mathrm{d}^2v}_{1} \\
\mathrm{d}^3(uv) &= \underbrace{\mathrm{d}^3u \cdot v}_{1} + \underbrace{3\,\mathrm{d}^2u \cdot \mathrm{d}v}_{3} + \underbrace{3\,\mathrm{d}^2u \cdot \mathrm{d}^2v}_{3} + \underbrace{u \cdot \mathrm{d}^3v}_{1} \\
&\vdots \\
\mathrm{d}^n(uv) &= \sum_{r=0}^{n} C_r^n (\mathrm{d}^r u)(\mathrm{d}^{n-r} v)
\end{aligned}$$

The coefficients for each term are binomial coefficients $C_r^n = \dfrac{n!}{r!(n-r)!}$, which can be computed beforehand.

3.3 Application: Damped Harmonic Oscillator

Consider a spring-mass system with damping factor on the spring. When the spring is compressed by a displacement x, the forces on it are

- Spring's elastic force: $-kx$ (Require $k > 0$)
- Damping force: $-\gamma v$ (Require $\gamma > 0$)

(add figure here: damp shm)

In a naive model, the damping force is usually assumed proportional and opposite to the mass's velocity. Otherwise it will be a lot more difficult to calculate.

The Newton's 2nd Law writes:

$$\text{(total force)} = -kx - \gamma v = ma$$
$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}t}x(t) + kx(t) = 0$$

Substitute $x(t) = Ce^{\lambda t}$, we can find $\lambda_{\pm} = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$. Then further analyze by the 3 cases $\gamma^2 - 4mk \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases}$, which correspond to different physical behaviors.

3.3.1 Case 1: $\gamma^2 - 4mk > 0$ - Over-damped

Check the sign of $\lambda_\pm$. Since

$$\begin{aligned} \gamma^2 > \gamma^2 - 4mk > 0 \\ \gamma > \sqrt{\gamma^2 - 4mk} \\ 0 > \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} = \lambda_+ \end{aligned}$$

Also $\lambda_- = \dfrac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} < \lambda_+$. So both γ are negative. We may write

$$\begin{aligned} x(t) &= C_1 e^{-|\lambda_+|t} + C_2 e^{-|\lambda_-|t} \\ &= \text{A sum of 2 exponentially decaying functions} \end{aligned}$$

(add figure here: over damp)

It is called "over-damped" as the damping force is too large that the mass can never return to its original position.

3.3.2 Case 2: $\gamma^2 - 4mk < 0$ - Under-damped

Write the solution in amplitude form:

$$\begin{aligned} x(t) &= e^{pt} \cdot A \cos(qt + \phi) \\ &= e^{\frac{\gamma}{2m}t} \cdot A \cos\left(\frac{\sqrt{4mk - \gamma^2}}{2m} t + \phi \right) \\ &= (\text{Exponentially Decay}) \times (\text{Sinusoidal}) \end{aligned}$$

(add figure here: under damp)

It is called "under-damped" as the damping force is not strong enough to stop the mass. The mass will oscillate forever although the amplitude will decrease with time.

3.3.3 Case 3: $\gamma^2 - 4mk = 0$ - Critical-damped

The general solution is

$$x(t) = C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t}$$

(add figure here: crit damp)

It is called "critical-damped" because it is in between the other 2 cases. It looks like over-damped case but the mass can return to the original position.

(add figure here: summary)

4 Non-homogeneous Constant Coefficient Linear ODE

Now we consider non-homogeneous linear ODE. For example,

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f(t) + cf(t) = \underbrace{g(t)}_{\substack{\text{non-homogeneous term}\\ \text{not containing } f(t)}} \neq 0$$

The general solution to a non-homogeneous linear ODE is made of 2 parts:

$$f(t) = f_c(t) + f_p(t)$$

where

- $f_c(t)$ = **Complementary solution**. It is the general solution to the homogeneous counterpart of the ODE, i.e. the solution to

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_c(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f_c(t) + cf_c(t) = 0$$

- $f_p(t)$ = **Particular solution**. Its presence is to cancel the non-homogeneous term.

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_p(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f_p(t) + cf_p(t) = g(t)$$

Show by substitution to be clearer:

$$\begin{aligned} g(t) &= a\frac{\mathrm{d}^2}{\mathrm{d}t^2}[f_c(t) + f_p(t)] + b\frac{\mathrm{d}}{\mathrm{d}t}[f_c(t) + f_p(t)] + c[f_c(t) + f_p(t)] \\ &= \underbrace{\left[a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_c(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f_c(t) + cf_c(t)\right]}_{\substack{\text{Require the parts}\\ \text{constructed by } f_c(t)\\ \text{to become } 0}} + \underbrace{\left[a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f_p(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f_p(t) + cf_p(t)\right]}_{\substack{\text{Require the parts}\\ \text{constructed by } f_p(t)\\ \text{to become } g(t)}} \\ &= 0 + g(t) \end{aligned}$$

4.1 An Example of Particular Solution

Consider a spring-mass system that is subject to gravity. The Newton's 2$^{\text{nd}}$ Law writes:

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = -kx(t) - mg$$

(add figure here: shm gravity)

The simplest way to solve it is by grouping mg into $x(t)$.

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[x(t) + \underline{\frac{mg}{k}}\right] = -k\left[x(t) + \frac{mg}{k}\right]$$

Then substitute $y(t) = x(t) + \dfrac{mg}{k}$. We can see that this ODE of $y(t)$ is the same as equation of motion of a spring-mass system without gravity.

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) = -ky(t)$$

which the solution is already known: $y(t) = A\cos\left(\sqrt{\frac{k}{m}}t + \phi\right)$. So we can solve $x(t)$ and identify the particular solution.

$$\begin{aligned} x(t) &= y(t) - \frac{mg}{k} \\ &= \underline{A\cos\left(\sqrt{\frac{k}{m}}t + \phi\right)} - \underline{\frac{mg}{k}} \end{aligned}$$

The complementary soln. $f_c(t)$
i.e. the soln. of the homogeneous ODE
$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) = -ky(t)$$

The particular soln. $f_p(t)$
i.e. for canceling the
non-homogeneous term mg

4.2 Method of Undetermined Coefficients

Finding a suitable $f_p(t)$ for an arbituary non-homogeneous term $g(t)$ is hard. But in most applications, $g(t)$ appears as a combination of common functions. In these cases, we can make smart guess of what functions is $f_p(t)$ made of.

4.2.1 Families of common functions & their derivatives

Here we consider the function and the constituents of its derivatives as a family.

- Polynomial/Log: Derivaties of a polynomial must be made of polynomials of lower degree.
  - $t^n \to t^{n-1} \to ... \to t^2 \to t \to 1$ (+ve integral power)
  - $\ln t \to t^{-1} \to t^{-2} \to ...$ (-ve integral power)
  - $t^{\frac{1}{2}} \to t^{-\frac{1}{2}} \to t^{-\frac{3}{2}} \to ...$ (fractional power)
- Trigonometric: Derivatives of $\sin(kt)/\cos(kt)$ cycle between themselves.

$$\sin(kt) \to \cos(kt) \to \sin(kt) \to ...$$

- Exponential: Derivatives of e^{kt} always yield multiples of itself.

$$e^{kt} \to e^{kt} \to e^{kt} \to ...$$

$\frac{mg}{k}$ is a constant. So $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mg}{k}\right) = 0$

The product between different family yield a set of its own derivatives. For example,

$$t^2 \sin t \to \begin{cases} \mathrm{d}(t^2 \sin t) = 2t \sin t + t^2 \cos t \\ \mathrm{d}^2(t^2 \sin t) = 2 \sin t + 4t \cos t - t^2 \sin t \\ \mathrm{d}^3(t^3 \sin t) = 6 \cos t - 6t \sin t - t^2 \cos t \\ \vdots \end{cases}$$

One can observe that all its derivatives are made up of 6 different functions, which form a famaily of function:

$$t^2 \sin t \ , \ t^2 \cos t \ , \ t \sin t \ , \ t \cos t \ , \ \sin t \ , \ \cos t$$

4.2.2 Method of Undetermined Coefficient

According to the ODE,

$$\begin{aligned} g(t) &= a\frac{\mathrm{d}^2}{\mathrm{d}t^2} f_p(t) + b\frac{\mathrm{d}}{\mathrm{d}t} f_p(t) + c f_p(t) \\ &= \sum (\text{constant}) \cdot (\text{Derivatives of } f_p(t)) \end{aligned}$$

Conversely, we can make a good guess by assuming $f_p(t) = \dfrac{g(t)}{c} + (\text{Some Derivatives of } g(t))$. Then

$$\begin{array}{rrcl} & c \times [f_p(t)] & = & c \times \left[\dfrac{g(t)}{c} + (\text{Some Derivatives of } g(t))\right] \\ & b \times \left[\dfrac{\mathrm{d}}{\mathrm{d}t} f_p(t)\right] & = & b \times \left[\dfrac{1}{c}\dfrac{\mathrm{d}}{\mathrm{d}t} g(t) + \left(\text{Some Derivatives of } \dfrac{\mathrm{d}}{\mathrm{d}t} g(t)\right)\right] \\ +) & a \times \left[\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} f_p(t)\right] & = & a \times \left[\dfrac{1}{c}\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} g(t) + \left(\text{Some Derivatives of } \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} g(t)\right)\right] \\ \hline & a\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} f_p(t) + b\dfrac{\mathrm{d}}{\mathrm{d}t} f_p(t) + c f_p(t) & = & g(t) + 0 \end{array}$$

If we can find a combination of $g(t)$'s derivatives that make up the "(Some Derivatives of $g(t)$)" terms, such that all terms on R.H.S except $g(t)$ cancel one another, then we recover $f_p(t)$.

Example 4.1.

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) + b\frac{\mathrm{d}}{\mathrm{d}t} f(t) + c f(t) = t^2 + 2t$$

- Sine the ODE is of 2^{nd} order, the highest derivative that can be found in "(Some Derivatives of $g(t)$)" is at most $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} g(t)$. Otherwise it cannot be canceled.
- Derivatives of t^2 and t both belong to the "integral power polynomial family".

So we guess

$$\begin{aligned} f_p(t) &= (\text{Some combination of } t^2, t, 1) \\ &= At^2 + Bt + C \end{aligned}$$

for some constants A, B, C to be solved.

$$\begin{array}{rrcl} & c \times [At^2 + Bt + C] & = & c \times [At^2 + Bt + C] \\ & b \times \left[\dfrac{\mathrm{d}}{\mathrm{d}t}(At^2 + Bt + C)\right] & = & b \times [2At + B] \\ +) & a \times \left[\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(At^2 + Bt + C)\right] & = & a \times [2A] \\ \hline & a\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}f_p(t) + b\dfrac{\mathrm{d}}{\mathrm{d}t}f_p(t) + cf_p(t) & = & (c \cdot A)t^2 + (c \cdot B + b \cdot 2A)t + (c \cdot C + b \cdot B + a \cdot A) \\ & g(t) & = & 1 \times t^2 \; + \; 2t + 0 \end{array}$$

By matching coefficients of t^2, t and 1 respectively, we require

$$\begin{cases} c \cdot A = 1 \\ c \cdot B + b \cdot 2A = 2 \\ c \cdot C + b \cdot B + a \cdot A = 0 \end{cases}$$

which is a system of 3 equations with 3 unknowns A, B, C. (Leave the solving to you.)

Example 4.2.

$$a\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t) + b\frac{\mathrm{d}}{\mathrm{d}t}f(t) + cf(t) = \sin(2t)$$

- Sine the ODE is of 2$^{\text{nd}}$ order, the highest derivative that can be found in "(Some Derivatives of $g(t)$)" is at most $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}g(t)$. Otherwise it cannot be canceled.
- Derivatives of $\sin 2t$ will cycle between $\sin 2t$ and $\cos 2t$.

So we guess

$$\begin{aligned} f_p(t) &= (\text{Some combination of } \sin 2t, \cos 2t) \\ &= A \sin 2t + B \cos 2t \end{aligned}$$

for some constants A, B to be solved.

$$
\begin{aligned}
c \times \left[A \sin 2t + B \cos 2t\right] &= c \times \left[A \sin 2t + B \cos 2t\right] \\
b \times \left[\frac{\mathrm{d}}{\mathrm{d}t}(A \sin 2t + B \cos 2t)\right] &= b \times \left[2A \cos 2t - 2B \sin 2t\right] \\
+\Big) \quad a \times \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}(A \sin 2t + B \cos 2t)\right] &= a \times \left[-4A \sin 2t - 4B \cos 2t\right] \\
\hline
a\frac{\mathrm{d}^2}{\mathrm{d}t^2} f_p(t) + b\frac{\mathrm{d}}{\mathrm{d}t} f_p(t) + c f_p(t) &= (c \cdot A - b \cdot 2B - a \cdot 4A) \sin 2t + (c \cdot B + b \cdot 2A - a \cdot 4B) \cos 2t \\
g(t) &= 1 \times \sin 2t \; + \; 0 \times \cos 2t
\end{aligned}
$$

By matching coefficients of $\sin 2t$ and $\cos 2t$ respectively, we require

$$
\begin{cases}
c \cdot A - b \cdot 2B - a \cdot 4A = 1 \\
c \cdot B + b \cdot 2A - a \cdot 4B = 0
\end{cases}
$$

which is a system of 2 equations with 2 unknowns A, B. (Leave the solving to you.)

Example 4.3. What if particular solution has the same form as the complementary solution?

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) - k^2 f(t) = e^{kt}
$$

Its homogeneous counterpart is $\frac{\mathrm{d}^2}{\mathrm{d}t^2} f_c(t) - k^2 f_c(t) = 0$. We can immediately write its general solution as

$$
f_c(t) = C_1 e^{kt} + C_2 e^{-kt}
$$

But the inhomogeneous term $g(t) = e^{kt}$ is also in the same exponential family! How do we find a correct $f_p(t)$? By **Reduction of order** again.

1. Let $f_p(t) = v(t)e^{kt}$. The goal is to find a suitable $v(t)$ that helps it form a particular solution that does not belong to the same family as e^{kt}. First substitute it into the original ODE and break down by product rule.

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}[v(t)e^{kt}] - k^2 \frac{\mathrm{d}}{\mathrm{d}t}[v(t)e^{kt}] &= e^{kt} \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2} v(t) \cdot \cancel{e^{kt}} + 2\frac{\mathrm{d}}{\mathrm{d}t} v(t) \cdot k\cancel{e^{kt}} + \cancel{v(t) \cdot k^2 e^{kt}} - \cancel{k^2 v(t) e^{kt}} &= \cancel{e^{kt}} \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2} v(t) + 2k\frac{\mathrm{d}}{\mathrm{d}t} v(t) &= 1
\end{aligned}
$$

2. Because this equation has no 0^{th} order of $v(t)$, we can integrate it once to reduce the total order. We arrive at a 1^{st} order inhomogeneous ODE of $v(t)$.

$$
\frac{\mathrm{d}}{\mathrm{d}t} v(t) + 2kv(t) = t + C_3
$$

where C_3 is an arbituary integration constant. Now we have to carry out the standard procedure for solving non-homogeneous equation again:

$$
v(t) = v_c(t) + v_p(t)
$$

3. The complementary solution $v_c(t)$ is trivial:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_c(t) + 2kv_c(t) = 0$$
$$v_c(t) = C_4 e^{-2kt}$$

with C_4 being some constant.

4. Use method of undetermined coefficients for $v_p(t)$. The non-homogeneous term $t + C_3$ is a polynomial of degree 1. So its derivative can only be made of t and 1. Let $v_p(t) = At + B$.

$$\begin{array}{rrcl} & 2k \times [At + B] & = & 2k \times [At + B] \\ +) & 1 \times \left[\frac{\mathrm{d}}{\mathrm{d}t}(At + B)\right] & = & 1 \times [A] \\ \hline & \frac{\mathrm{d}}{\mathrm{d}t}v_c(t) + 2kv_c(t) & = & (2k \cdot A)t + (2k \cdot B + 1 \cdot A) \\ & g(t) & = & 1 \times t \ + \ C_3 \end{array}$$

By matching coefficients of t and 1 respectively, we require

$$\begin{cases} 2k \cdot A = 1 \\ 2k \cdot B + 1 \cdot A = C_3 \end{cases}$$

which gives $A = \frac{1}{2k}$ and $B = \frac{C_3}{2k} - \frac{1}{4k^2}$. So,

$$v_p(t) = \frac{t}{2k} + \left(\frac{C_3}{2k} - \frac{1}{4k^2}\right)$$

Finally we arrive at

$$v(t) = v_c(t) + v_p(t) = C_4 e^{-2kt} + \frac{t}{2k} + \left(\frac{C_3}{2k} - \frac{1}{4k^2}\right)$$

$$\Rightarrow f_p(t) = v(t)e^{kt} = C_4 e^{-kt} + \left(\frac{t}{2k}\right)e^{kt} + \underbrace{\left(\frac{C_3}{2k} - \frac{1}{4k^2}\right)}_{\substack{\text{All are constants.} \\ \text{Can combine.}}} e^{kt}$$

$$= \underbrace{C_4 e^{-kt} + C_3' e^{kt}}_{\text{Already in } f_c(t)} + \underbrace{\left(\frac{t}{2k}\right)e^{kt}}_{\text{The true } f_p(t)}$$

4.3 Application: Forced Harmonic Oscillator

Consider a spring-mass system driven by an external force. When the spring is compressed by a displacement x, the forces on it are

– Spring's elastic force: $-kx$ (Require $k > 0$)

- Damping force: $-\gamma v$ (Require $\gamma > 0$)
- External force: $F(t)$ (No restriction)

(add figure here: forced shm)

The Newton's 2nd Law writes:

$$\text{(total force)} = -kx - \gamma v + F(t) = ma$$
$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}t}x(t) + kx(t) = F(t)$$

which is an inhomogeneous 2nd order ODE. We have already learnt the complementary solution:

$$x_c(t) = \begin{cases} C_1 e^{\frac{-\gamma+\sqrt{\gamma^2-4mk}}{2m}t} + C_2 e^{\frac{-\gamma-\sqrt{\gamma^2-4mk}}{2m}t} & (\gamma^2 - 4mk > 0) \\ e^{\frac{\gamma}{2m}t} \cdot A\cos\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t + \phi\right) & (\gamma^2 - 4mk < 0) \\ C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t} & (\gamma^2 - 4mk = 0) \end{cases}$$

We cannot determine $x_p(t)$ unless we are given the expression of $F(t)$. One common form of external force is vibration, which can be assume as sinusoidal:

$$F(t) = F_0 \cos\omega t$$

By method of undetermined coefficient, we can guess $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$.

$$\begin{aligned}
k \times [A\cos(\omega t) + B\sin(\omega t)] &= k \times [A\cos(\omega t) + B\sin(\omega t)] \\
\gamma \times \left[\frac{\mathrm{d}}{\mathrm{d}t}(A\cos(\omega t) + B\sin(\omega t))\right] &= \gamma \times [-\omega A\sin(\omega t) + B\omega\cos(\omega t)] \\
+)\quad m \times \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}(-\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t))\right] &= k \times [A\cos(\omega t) + B\sin(\omega t)] \\
\hline
m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_p(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}t}x_p(t) + kx_p(t) &= (-\omega^2 Am + \omega B\gamma + Ak)\cos\omega t \\
&\quad +(-\omega^2 Bm - \omega A\gamma + Bk)\sin\omega t \\
F(t) &= F_0\cos 2t + 0
\end{aligned}$$

By matching coefficients of $\cos\omega t$ and $\sin\omega t$ respectively, we require

$$\begin{cases} -\omega^2 Am + \omega B\gamma + Ak = F_0 \\ -\omega^2 Bm - \omega A\gamma + Bk = 0 \end{cases}$$

Solving, yield

$$A = \frac{F_0(m\omega^2 - k)}{(\gamma\omega)^2 + (m\omega^2 - k)^2} \quad , \quad B = \frac{F_0\gamma\omega}{(\gamma\omega)^2 + (m\omega^2 - k)^2}$$

By combining $\sin / \cos$ into 1 sinusoidal, $x_p(t)$ becomes

$$\begin{aligned}
x_p(t) &= A\cos\omega t + B\sin\omega t \\
&= \sqrt{A^2+B^2}\cos\left[\omega t + \tan^{-1}\left(\frac{B}{A}\right)\right] \\
&= \frac{F_0}{\sqrt{(\gamma\omega)^2+(m\omega^2-k)^2}}\cos\left[\omega t + \tan^{-1}\left(\frac{\gamma\omega}{m\omega^2-k}\right)\right]
\end{aligned}$$

Special Case: Resonance

When the frequency of the external force equals to the natural frequency of harmonic oscillator $\sqrt{\frac{k}{m}}$, or $m\omega^2 - k = 0$. The displacement $x_p(t)$ becomes

$$\begin{aligned}
x_p(t) &= \frac{F_0}{\gamma\omega}\cos\left[\omega t - \frac{\pi}{2}\right] \\
&= \frac{F_0}{\gamma}\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)
\end{aligned}$$

In particular, if the damping force also diminish (i.e. $\gamma \to 0$), The amplitude of $x_p(t)$ will explode to infinity.

— The End —