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Magnetic Induction

by Tony Shing

Overview:

- Motion under Lorentz force
- Faraday's law: Motional & transformer EMF
- Lenz's law: Direction of induced EMF
- Appendix: A brief history of electromagnetism

In electromagnetism, theoretically every problem can be solved through a set of PDEs called the **Maxwell Equations**.

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \frac{\rho}{\epsilon_0} \qquad \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = 0$$

$$\longrightarrow \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \qquad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t}$$

However, a *system of PDEs* is too complicated to be solved. So we need to learn different "tricks" to avoid them, which are enough for some simple scenarios.

Magnetic induction concerns the 3^{rd} equation of the set - Faraday's law.

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1 Motion under Lorentz Force

Nowadays we know that Lorentz force is the fundamental explanation to magnetic induction. But interestingly, it was formulated correctly only in the very late history of classical E&M.

$$\begin{aligned} \vec{\boldsymbol{F}} &= \vec{\boldsymbol{F}}_E + \vec{\boldsymbol{F}}_B \\ &\boxed{= q\vec{\boldsymbol{E}} + q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}} \end{aligned}$$

In general, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ can be functions of time and position, but here we will only discuss the special case when the fields are constant.

By symmetry, we can assume that $\vec{\boldsymbol{B}}$ is only in $\hat{\boldsymbol{z}}$ direction, i.e. $\vec{\boldsymbol{B}} = B_z\hat{\boldsymbol{z}}$. Then the Newton's 2^{nd} law can be expanded as

$$m\vec{\boldsymbol{a}} = m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + q\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix}$$

which is a system of 3 first order linear ODEs.

$$\begin{cases} m\dfrac{\mathrm{d}v_x}{\mathrm{d}t} = qE_x + qv_yB_z \\ m\dfrac{\mathrm{d}v_y}{\mathrm{d}t} = qE_y - qv_xB_z \\ m\dfrac{\mathrm{d}v_z}{\mathrm{d}t} = qE_z \end{cases}$$

This system is simple enough that we do not need to use any matrix methods.

– **Equation for z component**

Motion in z direction is independent of x/y. It can be solved alone.

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = qE_z \qquad \Rightarrow \qquad v_z(t) = \frac{qE_z}{m}t + v_{z0}$$

$$\Rightarrow \qquad z(t) = \frac{1}{2}\frac{qE_z}{m}t^2 + \underbrace{v_{z0}}_{\text{Initial z velocity}}t + \underbrace{z_0}_{\text{Initial z coordinate}}$$

which is just a constant acceleration motion with an acceleration $\dfrac{qE_z}{m}$.

– **Equation for x/y components**

They can be solved by first differentiating one of them, then substitute into the other.

$$\begin{aligned} \frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} &= \frac{qB_z}{m}\frac{\mathrm{d}v_y}{\mathrm{d}t} \\ &= \frac{qB_z}{m}\left(\frac{qE_y}{m} - \frac{qB_z}{m}v_x\right) \\ &= -\frac{q^2B_z^2}{m}\left(v_x - \frac{E_y}{B_z}\right) \end{aligned}$$

which is the familiar ODE of SHM.

$$\begin{aligned} v_x(t) &= -C\sin\left(\frac{qB_z}{m}t + \phi\right) + \frac{E_y}{B_z} \\ \Rightarrow \quad x(t) &= \underbrace{C\left(\frac{m}{qB_z}\right)}_{\text{A constant}}\cos\left(\frac{qB_z}{m}t + \phi\right) + \frac{E_y}{B_z}t + x_0 \\ &= \underbrace{R}_{\text{Radius}}\cos\left(\frac{qB_z}{m}t + \underbrace{\phi}_{\text{Phase}}\right) + \frac{E_y}{B_z}t + \underbrace{x_0}_{\text{Initial x coordinate}} \end{aligned}$$

And subsitute back to equation of v_y gives

$$
\begin{aligned}
v_y(t) &= \frac{m}{qB_z}\frac{\mathrm{d}v_x}{\mathrm{d}t} - \frac{E_x}{B_z} = -C\cos\left(\frac{qB_z}{m}t+\phi\right) - \frac{E_x}{B_z} \\
\Rightarrow \quad y(t) &= \underbrace{-C\left(\frac{m}{qB_z}\right)}_{\text{A constant}}\sin\left(\frac{qB_z}{m}t+\phi\right) - \frac{E_x}{B_z}t + y_0 \\
&= -\underset{\text{Radius}}{\underline{R}}\sin\left(\frac{qB_z}{m}t+\underset{\text{Phase}}{\underline{\phi}}\right) - \frac{E_y}{B_z}t + \underset{\text{Initial y coordinate}}{\underline{x_0}}
\end{aligned}
$$

The result x/y motion is a combination of circular motion + drifting.

$$
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{R\begin{pmatrix} \cos\left(\frac{qB_z}{m}t+\phi\right) \\ -\sin\left(\frac{qB_z}{m}t+\phi\right) \end{pmatrix}}_{\text{Circular motion}} + \underbrace{\frac{1}{B_z}\begin{pmatrix} E_y \\ -E_x \end{pmatrix}}_{\text{Drifting}} t + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
$$

1. Be careful that the drifting direction is not intuitive:
   - With E_x only, drifting direction $= -\hat{\boldsymbol{y}}$
   - With E_y only, drifting direction $= \hat{\boldsymbol{x}}$

   In general, the drifting is along the direction of $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ with a speed $\dfrac{|\vec{\boldsymbol{E}}|}{|\vec{\boldsymbol{B}}|}$.

2. The rotation radius and speed maintain a constant ratio:

$$
\text{Radius} = R \qquad \Rightarrow \qquad \text{Speed} = \frac{qB_z}{m}R
$$

   i.e. Angular velocity is always $\omega = \dfrac{qB_z}{m}$, independent of the charge's initial velocity. You can also arrive at the same result from equation of circular motion:

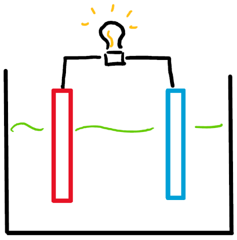
$$
\begin{aligned}
\frac{mv^2}{R} &= qvB \\
v &= \frac{qB}{m}R
\end{aligned}
$$

   (But how do you argue that it is a circular motion without solving ODEs?)

2 The Theory of Magnetic Induction

2.1 EMF - Force or Voltage?

The term **electromotive force** (EMF) was invented by Alessandro Volta in 1801, for explaining observations in electrochemistry.



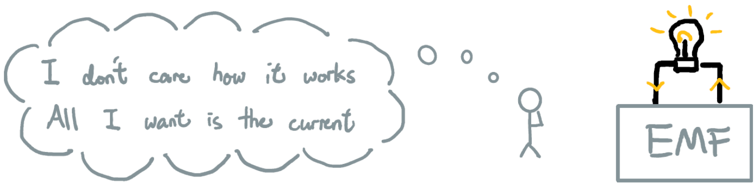
⇒ Current generates spontaneously.

⇒ There must be some "force" reponsible for it!

In early 1800s, scientists tended to describe things like a mechanical system, i.e. any motions of objects must be driven by some kind of "force" - When there is a flow current, there must be some "force" that keeps pushing the charges forward, namely the "electromotive force".

- However, EMF took the unit of volt because voltage measurement was the only way to quantify the magnitude of EMF.
- When Michael Faraday discovered the phenomena of magnetic induction, he also used the same wordings to refer to the source of the induced current.

Today we still keep its name as a "force", even though we already know much better how current is generated in different sources. Maybe because people are already used to refer the same term "EMF" as the general name for ANY kind of current source, rather than differentiating them by the origin of energy.



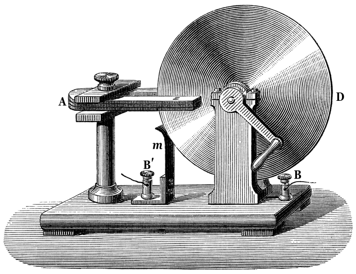
The discussions of magnetic induction involves two kinds of EMF:

- <u>**Motional EMF**</u>
  - Source of energy = Initial motion of the charges.
  - Can be explained by Lorentz magnetic force.
- <u>**Transformer EMF**</u>
  - Source of energy = Change in magnetic field.
  - Explanation requires relativity.

Historically, they were first discovered as two different phenomena. Then after a few decades, they were unified mathematically as a complete theory of magnetic induction.

2.2 Motional EMF

Motional EMF was discovered the first time by Faraday, with the device which is today known as Faraday disc or Faraday wheel.



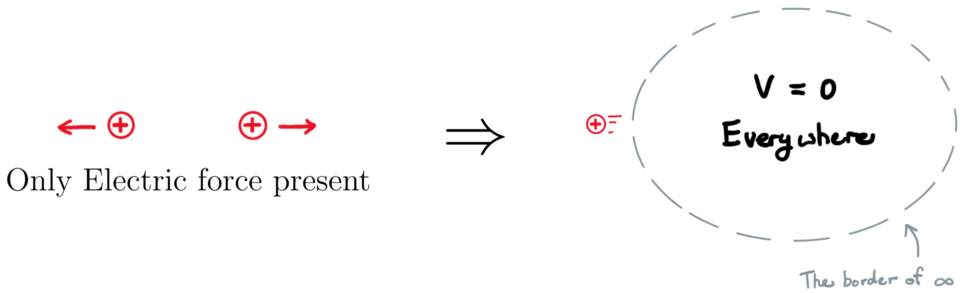
(Image from Wiki)

- D = A metal disc
- A = A Magnet creating B-field through the disc
- Current can conduct through B → Disc → m → B'

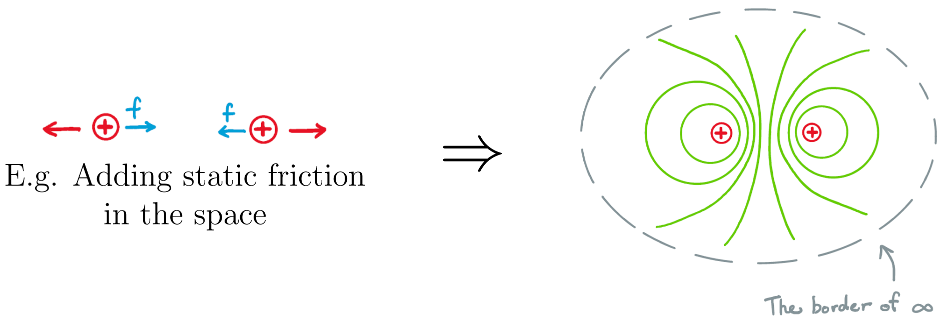
When the disc turns, EMF is observed between B and B'.

To understand how motional EMF is created, we can first think of what happens if charges are subject to a second kind of force in addition to the electric forces between each other.

- If there is only electric force, charges freely distribute themselves until net electric force is 0. i.e. Electric potential is the same everywhere.



- With a second kind of force in the space, charges distribute themselves to new equilibrium positions - making electric potential no longer a constant of position. i.e. Creating electric potential difference.



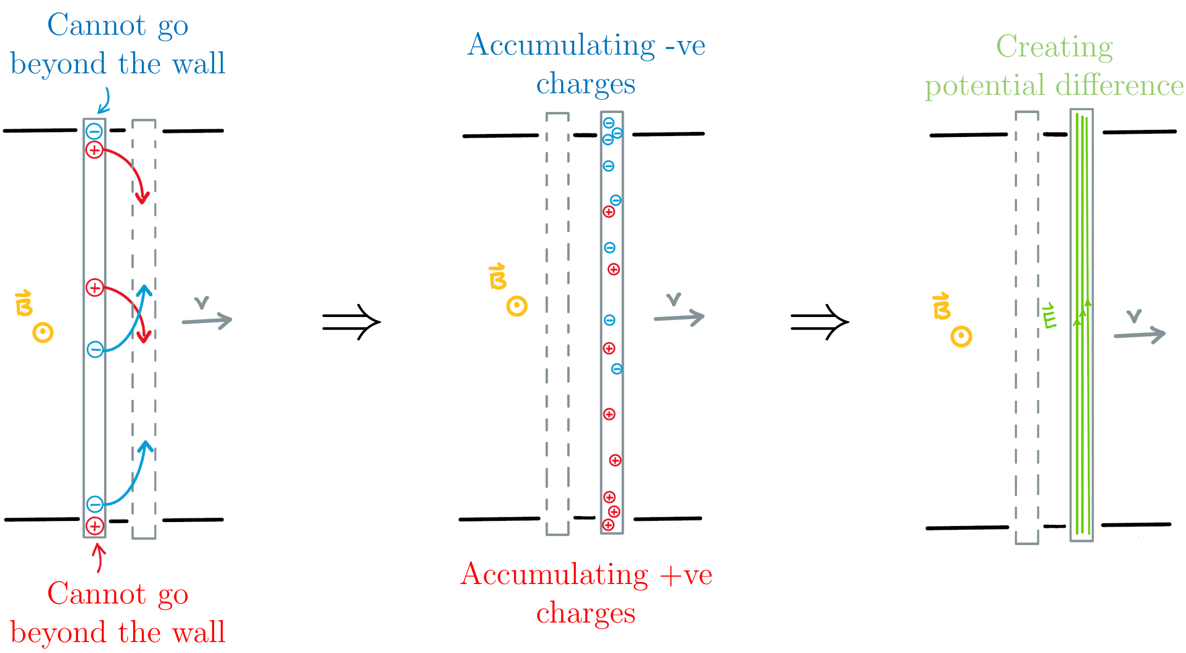
The situation becomes weird **if the second force is the Lorentz magnetic force**, whose magnitude is proportional to the charges' velocities. **The potential difference can be maintained only if the charges are moving**.

2.2.1 The Moving Rod Model

(This is probably the most common model for studying motional EMF for the first time.)

Consider a finite long conducting rod that contains some charges. We give the rod a small horizontal push initially.

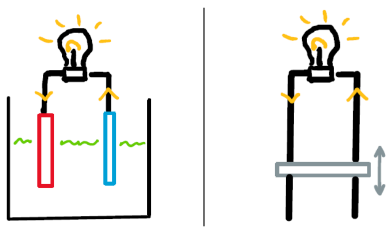
1. The charges cannot escape from the rod - they must move at the same horizontal velocity as the rod.
2. According to the direction of Lorentz force, the charges will receive accelerations in the direction along the rod.
3. However because the rod is closed off at its ends, charges near the ends will be pushed against the end walls and cannot move. Very soon, charge will accumulate.
4. The accumulated charges build up a E-field (i.e. potential difference) between the ends of the rod. All charges stop moving vertically when the force from the built-up E-field balance the magnetic force, i.e. when $q|E| = q|v||B|$.



Note that such "equilibrium" is only along the vertical direction. The rod and charges must maintain some horizontal motion to keep the magnetic force "alive". If the rod suddenly stop moving, the built-up potential difference, and thus the stored energy, will be released.

The rod can be used to drive current.
It is a EMF source - but only when it is moving!

If the EMF source comes from electrochemistry, it is called a battery.



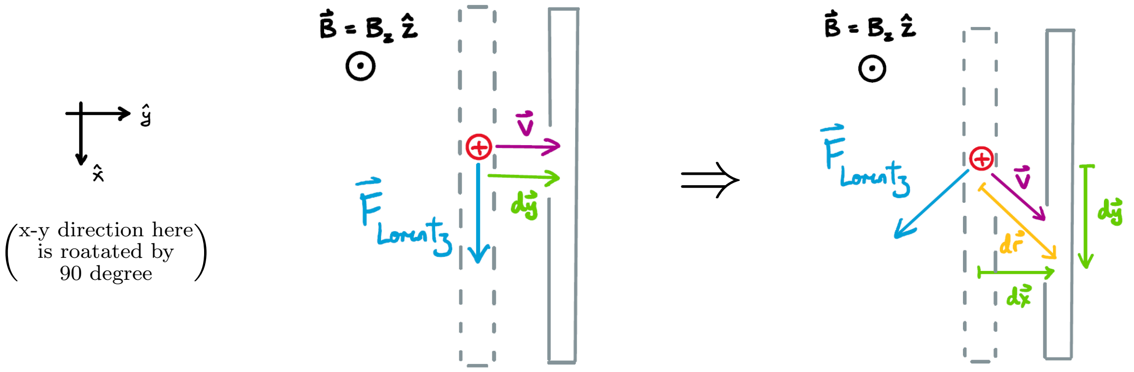
If the EMF source comes from moving conductor, it is called a generator.

2.2.2 Energy Conservation in Moving Wire

An alternative explanation of motional EMF is that, it is a result of energy conservation given that the only W.D. comes from Lorentz magnetic force.

Let the B-field be constant and in $\hat{\boldsymbol{z}}$ direction, i.e. $\vec{\boldsymbol{B}} = B_z\hat{\boldsymbol{z}}$. When the charge on its mid-way to its new equilibrium position,

- The charge's velocity is non-zero in both directions - $+\hat{\boldsymbol{y}}$ motion from the rod's initial motion, and $+\hat{\boldsymbol{x}}$ motion from acceleration by Lorentz force.
- Lorentz force is then in the direction of $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$, composed of a $+\hat{\boldsymbol{x}}$ and a $-\hat{\boldsymbol{y}}$ component.



Although Lorentz force is accelerating the charge's x motion, at the same time it is decelerating the charge's y motion. We can show that energy is conserved under this change in the charge's motion by looking at the W.D. under a small displacement $\mathrm{d}\vec{\boldsymbol{r}}$.

1. Always remember: Work done on a charge by magnetic force is always 0, because $\vec{\boldsymbol{F}}_B$ is perpendicular to the travelling direction $\mathrm{d}\vec{\boldsymbol{r}}$, (i.e. same direction as $\vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t}$).

$$\text{W.D.} = \vec{\boldsymbol{F}}_B \cdot \mathrm{d}\vec{\boldsymbol{r}} = q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{r}} = 0$$

2. Meanwhile we can separate this W.D. into contributions along x and y direction:

$$\begin{aligned}
\text{W.D.} &= q\Big[(\vec{\boldsymbol{v}}_x + \vec{\boldsymbol{v}}_y) \times \vec{\boldsymbol{B}}\Big] \cdot (\mathrm{d}\vec{\boldsymbol{x}} + \mathrm{d}\vec{\boldsymbol{y}}) \\
&= \underbrace{\cancelto{0}{q(\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{x}}}}_{\substack{\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}} = v_x B_z(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}}) \\ = v_x B_z(-\hat{\boldsymbol{y}}) \\ \text{Dotting with } \hat{\boldsymbol{x}} \text{ gives } 0}} + q(\vec{\boldsymbol{v}}_x \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{y}} + q(\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{x}} + \underbrace{\cancelto{0}{q(\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{y}}}}_{\substack{\vec{\boldsymbol{v}}_y \times \vec{\boldsymbol{B}} = v_y B_z(\hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}) \\ = v_y B_z(\hat{\boldsymbol{x}}) \\ \text{Dotting with } \hat{\boldsymbol{y}} \text{ gives } 0}} \\
&= (-q v_x B_z \hat{\boldsymbol{y}}) \cdot \mathrm{d}\vec{\boldsymbol{y}} + (q v_y B_z \hat{\boldsymbol{x}}) \cdot \mathrm{d}\vec{\boldsymbol{x}} \\
&= \underbrace{(-q v_x B_z)\,\mathrm{d}y}_{\substack{\text{W.D. on charge} \\ \text{in } \hat{\boldsymbol{y}} \text{ direction} \\ \sim\ F_y \mathrm{d}y}} + \underbrace{(q v_y B_z)\,\mathrm{d}x}_{\substack{\text{W.D. on charge} \\ \text{in } \hat{\boldsymbol{x}} \text{ direction} \\ \sim\ F_x \mathrm{d}x}} \\
&\equiv 0 \qquad \text{(Required by magnetic force)}
\end{aligned}$$

To always hold true, these two W.D. must have equal magnitude but opposite sign.

3. When charges accumulate at the ends of the rod, equilibrium happens when $q\vec{\boldsymbol{E}} = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$. So when a charge climbs up the E-field for a small distance $\mathrm{d}\vec{\boldsymbol{x}}$, it will gain a PE of:

$$\underset{\substack{\uparrow \\ \text{The potential gain} \\ \text{i.e. EMF}}}{q\,\mathrm{d}\epsilon} \equiv (q\vec{\boldsymbol{E}}) \cdot \mathrm{d}\hat{\boldsymbol{x}} = q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\hat{\boldsymbol{x}} = (qv_y B_z)\,\mathrm{d}x$$

We can see that these 3 terms are equal:

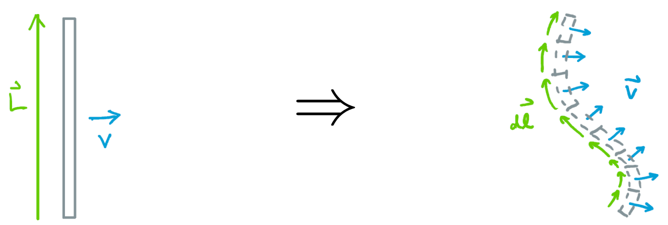
$$\underset{\substack{\text{Built-up} \\ \text{electric PE}}}{\underline{q\,\mathrm{d}\epsilon}} = \underset{\substack{\text{W.D. on charge} \\ \text{in } \hat{\boldsymbol{x}} \text{ direction}}}{\underline{(qv_y B_z)\,\mathrm{d}x}} = \underset{\substack{\text{W.D. on charge} \\ \text{in } \hat{\boldsymbol{y}} \text{ direction}}}{\underline{(qv_x B_z)\,\mathrm{d}y}}$$

which implies an energy conservation process:

- To build up the potential in the rod, We must do work to the charges along the rod (x direction) to drive the charge against the built-up potential.
- Lorentz magnetic force supplies this energy to do work to the charges along the rod, by doing work against the charge's motion in the y direction.

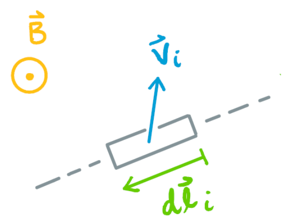
2.2.3 Magnetic Flux - A Geometrical Relation

The idea of magnetic flux comes from generalizing motional EMF by vector calculus. To begin with, transform the rod into a wire with arbitiary shape which moves in an arbitiary fashion.



The total EMF generated can be regarded as the sum of contribution by many infinitestimal segments - For the i^{th}segment,

- has a (length) + (orientation) labeled by $\mathrm{d}\vec{\boldsymbol{l}}_i$.
- is moving in velocity $\vec{\boldsymbol{v}}_i$.

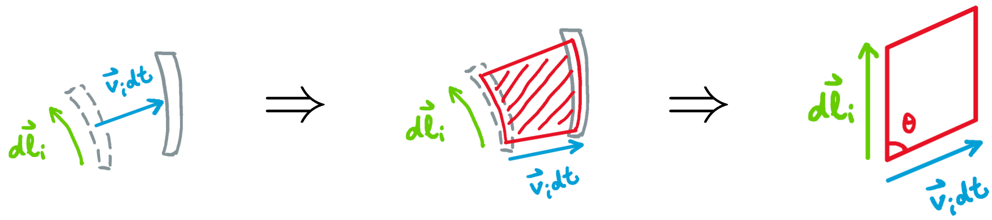


Then with some algebra, the EMF generated in each segment can be written as

$$\begin{aligned}
\mathrm{d}\epsilon_i &= (\vec{\boldsymbol{v}}_i \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{l}}_i \\
&= (\mathrm{d}\vec{\boldsymbol{l}}_i \times \vec{\boldsymbol{v}}_i) \cdot \vec{\boldsymbol{B}} \longleftarrow \begin{array}{c}\text{Vector identity:} \\ (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot \vec{\boldsymbol{c}} = (\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}) \cdot \vec{\boldsymbol{b}} = (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) \cdot \vec{\boldsymbol{a}} \\ (\text{i.e. } a \to b \to c \to a \text{ forming a cycle})\end{array} \\
&= -(\vec{\boldsymbol{v}}_i \times \mathrm{d}\vec{\boldsymbol{l}}_i) \cdot \vec{\boldsymbol{B}} \longleftarrow \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = -\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}}
\end{aligned}$$

In a short duration $\mathrm{d}t$, the displacement of the segment is $\vec{\boldsymbol{v}}_i\,\mathrm{d}t$. So we can approximate the swept area by the segment as a parallelogram:

$$
\begin{aligned}
\begin{pmatrix}\text{Swept}\\ \text{Area}\end{pmatrix} &= \begin{pmatrix}\text{Area of parallelogram}\\ \text{made by } \vec{\boldsymbol{v}}_i\,\mathrm{d}t \;\&\; \mathrm{d}\vec{\boldsymbol{l}}_i\end{pmatrix} \\
&\approx |\vec{\boldsymbol{v}}_i||\,\mathrm{d}\vec{\boldsymbol{l}}_i\,|\sin\begin{pmatrix}\text{Angle between}\\ \vec{\boldsymbol{v}}_i\,\mathrm{d}t \;\&\; \mathrm{d}\vec{\boldsymbol{l}}_i\end{pmatrix} \\
&= (\vec{\boldsymbol{v}}_i\,\mathrm{d}t) \times (\mathrm{d}\vec{\boldsymbol{l}}_i) \\
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\text{Swept}\\ \text{Area}\end{pmatrix} &= \vec{\boldsymbol{v}}_i \times \mathrm{d}\vec{\boldsymbol{l}}_i
\end{aligned}
$$



We can now relate EMF generated in the whole wire with swept area by the wire as:

$$
\begin{aligned}
\sum \epsilon = -\sum_{\substack{\text{All}\\ \text{segments}}} \left[(\vec{\boldsymbol{v}}_i \times \mathrm{d}\vec{\boldsymbol{l}}_i)\cdot\vec{\boldsymbol{B}}\right] &= -\sum_{\substack{\text{All}\\ \text{segments}}} \left[\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\text{Swept}\\ \text{area}\end{pmatrix}_i \cdot\vec{\boldsymbol{B}}\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\substack{\text{All}\\ \text{segments}}} \left[\begin{pmatrix}\text{Swept}\\ \text{area}\end{pmatrix} \cdot\vec{\boldsymbol{B}}\right]
\end{aligned}
$$

When the segments are infinitestimally short, the sum becomes integral.

$$
\begin{aligned}
\epsilon = -\int (\vec{\boldsymbol{v}} \times \mathrm{d}\vec{\boldsymbol{l}}_i)\cdot\vec{\boldsymbol{B}} &= -\frac{\mathrm{d}}{\mathrm{d}t}\left[\iint_{\substack{\text{Swept}\\ \text{area}}} \vec{\boldsymbol{B}}\cdot\mathrm{d}\vec{\boldsymbol{s}}\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\text{Magnetic flux through}\\ \text{the swept area}\end{pmatrix}
\end{aligned}
$$

Pay attention to the depedendence to t on the RHS:

- The B-field $\vec{\boldsymbol{B}}$ is static. It is not a function of t.
- $\mathrm{d}\vec{\boldsymbol{s}}$ is just a notation saying that this is a flux integral.
- The only thing that depends on t is the integration range.

So to be more accurate, the equation of motional EMF should be written as

$$
\boxed{\epsilon_{\text{(motional)}} = \int_{\substack{\text{along a wire}\\ \text{of shape of } l(t)}} (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}})\cdot\mathrm{d}\vec{\boldsymbol{l}} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\substack{\text{Area } S(t)\\ \text{swept by wire}}} \vec{\boldsymbol{B}}\cdot\mathrm{d}\vec{\boldsymbol{s}}}
$$

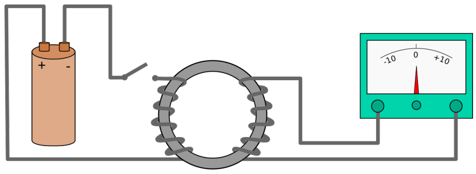
to emphasize that it is the wire's shape / swept area varying with time.

2.3 Transformer EMF

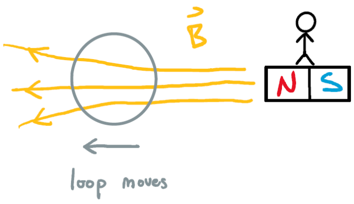
Transformer EMF was also discovered by Faraday, with a primitive toroidal transformer.

(Image from wiki)

When the current on one side is switch on / switch off, EMF is observed on the other side of the transformer.

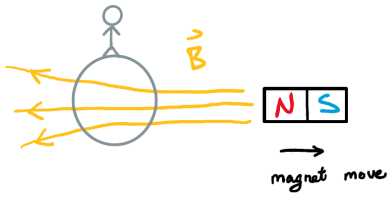


In modern explanation, transformer EMF can only be explained by relativity - motional EMF and transformer EMF are the same phenomenon being observed in different reference frame.



EMF generated due to motion of the ring

$\Leftrightarrow$



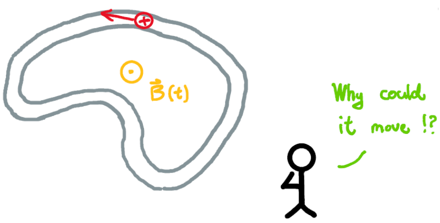
EMF generated due to changing magnetic field strength

2.3.1 Induced E-field

Imagine that we are in a reference frame which we only observe transformer EMF, i.e. the charge is stationary while B-field is the only thing changing with time.

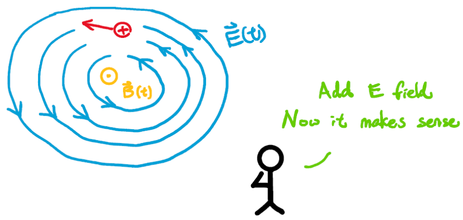
- The direct interaction between the charge and B-field - Lorentz magnetic force must be 0.
- But we somehow observe the charge moving (due to some potential difference).

This implies there is a second way how B-field can interact with charges. To complete the theory, an induced E-field is proposed.



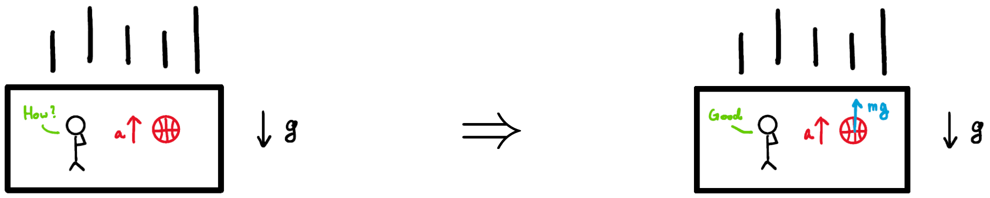
Observation:
Charge moves without force

$\Rightarrow$



Add induced E-field
to fix Newton's 2^{nd} law

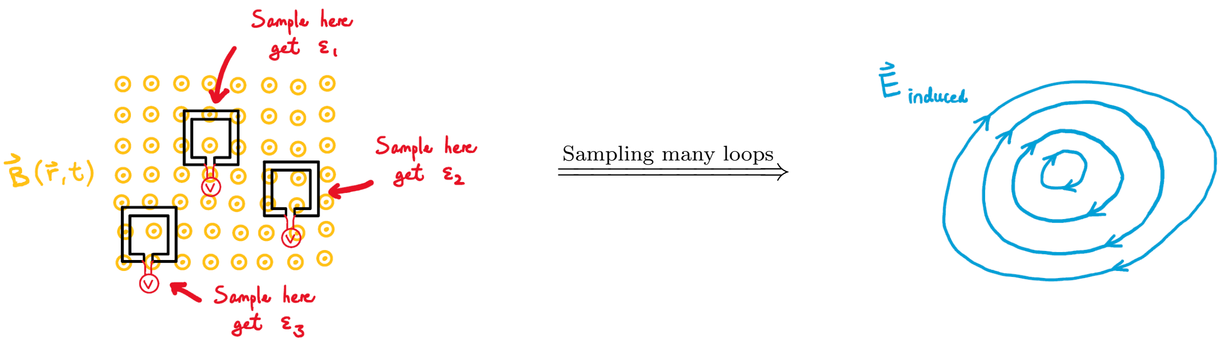
As an analogy, the addition of induced E-field is very similar to the reason of adding frictious force to accelerating frame.



Observation:
Ball rises up without force

Add ficticious force
to fix Newton's 2$^{\text{nd}}$ law

It is experimentally possible to determine the distribution of this E-field. For example, we can sample the induced EMF using different conductor loops:



With direct measurements, it was found that

- The induced E-field must be in form of loops, i.e. non-conservative. Otherwise charges will not run indefinitely in the conductor loops, and so cannot be used to drive current.
- The induced EMF is proportional to the rate of change of B-field and area of the loop. i.e.

$$\text{EMF} \;\propto\; \left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{B}}(t)\right)\cdot\left(\begin{matrix}\text{Loop's}\\ \text{area}\end{matrix}\right)$$

These observations suggest a mathematical relation in vector calculus:

$$\boxed{\epsilon_{(\text{transformer})} = \oint_{\substack{\text{The}\\ \text{loop}}} \vec{\boldsymbol{E}}_{\text{induced}}\cdot \mathrm{d}\vec{\boldsymbol{l}} = -\iint_{\substack{\text{The loop's}\\ \text{area}}} \left(\frac{\partial \vec{\boldsymbol{B}}}{\partial t}\right)\cdot \mathrm{d}\vec{\boldsymbol{s}}}$$

The minus sign in front of the surface integral is added to match the sign in motional EMF.

2.3.2 Maxwell-Faraday Equation

We have now learnt two ways of creating E-field: 1. emitted by charge or 2. induced by changing B-field. In theory, any E-field distribution must be a combination of these two sources.

$$\vec{\boldsymbol{E}}_{\text{total}} = \vec{\boldsymbol{E}}_{\text{charge}} + \vec{\boldsymbol{E}}_{\text{induced}}$$

Moreover, recall that E-field created by charges is always conservative. i.e.

$$\oint_{\substack{\text{Any}\\\text{loop}}} \vec{\boldsymbol{E}}_{\text{charge}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = 0$$

So we can add this term into the formula of transformer EMF and get the integral form of **Maxwell-Faraday's equation**, the 3rd equation from the set of Maxwell's equation:

$$\oint_{\substack{\text{The}\\\text{loop}}} \vec{\boldsymbol{E}}_{\text{induced}} + \vec{\boldsymbol{E}}_{\text{charge}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \oint_{\substack{\text{The}\\\text{loop}}} \vec{\boldsymbol{E}}_{\text{total}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = -\iint_{\substack{\text{The loop's}\\\text{area}}} \left(\frac{\partial \vec{\boldsymbol{B}}}{\partial t}\right) \cdot \mathrm{d}\vec{\boldsymbol{s}}$$

or simply

$$\boxed{\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = -\iint \left(\frac{\partial \vec{\boldsymbol{B}}}{\partial t}\right) \cdot \mathrm{d}\vec{\boldsymbol{s}}}$$

With Stoke's theorem, it can be converted into its differential form:

$$\oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} \overset{\text{Stokes' Theorem}}{=} \iint (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}) \cdot \mathrm{d}\vec{\boldsymbol{s}} = -\iint \left(\frac{\partial \vec{\boldsymbol{B}}}{\partial t}\right) \cdot \mathrm{d}\vec{\boldsymbol{s}}$$

$$\boxed{\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}}$$

2.4 Summary: Magnetic Flux & Faraday's Law

After such lengthy discussion, here again lists the formula of the two EMF:

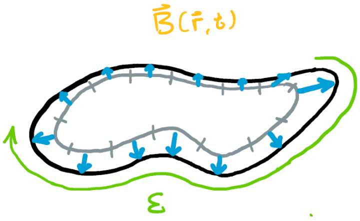
– **Motional EMF**: Wire moving, constant B-field

$$\epsilon_{(\text{motional})} = \int_{\substack{\text{along a wire}\\\text{of shape of } l(t)}} (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{l}} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\substack{\text{Area } S(t)\\\text{swept by wire}}} \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}}$$

– **Transformer EMF**: Stationary loop , changing B-field

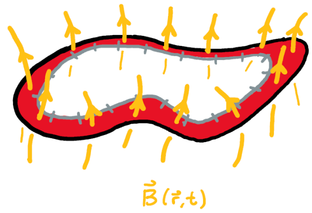
$$\epsilon_{(\text{transformer})} = \oint_{\substack{\text{The}\\\text{loop}}} \underset{\substack{\uparrow\\\text{No longer need to distinguish}\\\text{induced E or total E}}}{\vec{\boldsymbol{E}}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = -\iint_{\substack{\text{The loop's}\\\text{area}}} \left(\frac{\partial}{\partial t}\vec{\boldsymbol{B}}(t)\right) \cdot \mathrm{d}\vec{\boldsymbol{s}}$$

We can see a similiarity - both of the EMF can be expressed as a combination of time derivative and flux integral of B-field. If we close the wire in motional EMF to form a loop, let it sweep some area, and at the same time vary the B-field, we will observe the EMF contributed from both effects.



When the loop changes its shape and B-field changes happen together

$\Rightarrow$



The effective EMF is the rate of change of the magnetic flux in the red area

While mathematically, the two EMF can be merged like product rule of differentiation.

$$\epsilon_{\text{(motional)}} + \epsilon_{\text{(transformer)}} = -\left(\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\substack{\text{Area } S(t) \\ \text{swept by loop}}} \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}} \;+\; \iint_{\substack{\text{The loop's} \\ \text{area}}} \left(\frac{\partial}{\partial t} \vec{\boldsymbol{B}}(t) \right) \cdot \mathrm{d}\vec{\boldsymbol{s}} \right)$$

$\frac{\mathrm{d}}{\mathrm{d}t}$ only to integration range

B-field keep constant

integration range keep constant

$\frac{\mathrm{d}}{\mathrm{d}t}$ only to B-field

$$= -\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\left(\iint_{\substack{\text{Area } S(t) \\ \text{swept by loop}}} \vec{\boldsymbol{B}}(t) \cdot \mathrm{d}\vec{\boldsymbol{s}} \right)}_{\text{Total magnetic flux through the loop}}$$

(Just like $\mathrm{d}(uv) = u\mathrm{d}v + v\mathrm{d}u$
But no rigorous proof here)

This is exactly the description of **Faraday's law of induction**:

Any induced EMF appear	$\Leftrightarrow$	There are time-varying magnetic flux

In addition, the line integral part shows the origin of the EMF's energy:

$$\epsilon_{\text{(motional)}} + \epsilon_{\text{(transformer)}} = \oint_{\substack{\text{along a loop} \\ \text{of shape of } l(t)}} (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{l}} + \oint_{\substack{\text{The} \\ \text{loop}}} \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}}$$

$$\sim \oint (\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{l}}$$

$$q\epsilon_{\text{total}} = \oint (q\vec{\boldsymbol{E}} + q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \cdot \mathrm{d}\vec{\boldsymbol{l}}$$

which is the result of the work done by Lorentz force.

---

3 Lenz's Law

The formula of Faraday's law is only useful to tell the magnitude of the induced EMF, but not its direction.

$$\epsilon = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{\boldsymbol{B}}\cdot \mathrm{d}\vec{\boldsymbol{s}}$$

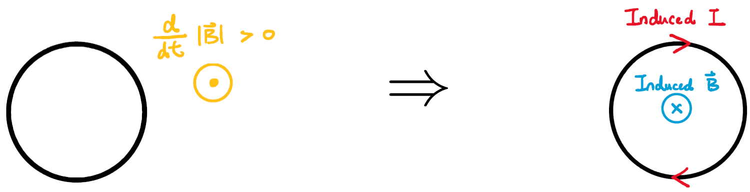
This minus sign is useless
for telling the direction of EMF

To determine the EMF's direction, we can apply **Lenz's law**, which is in principle,

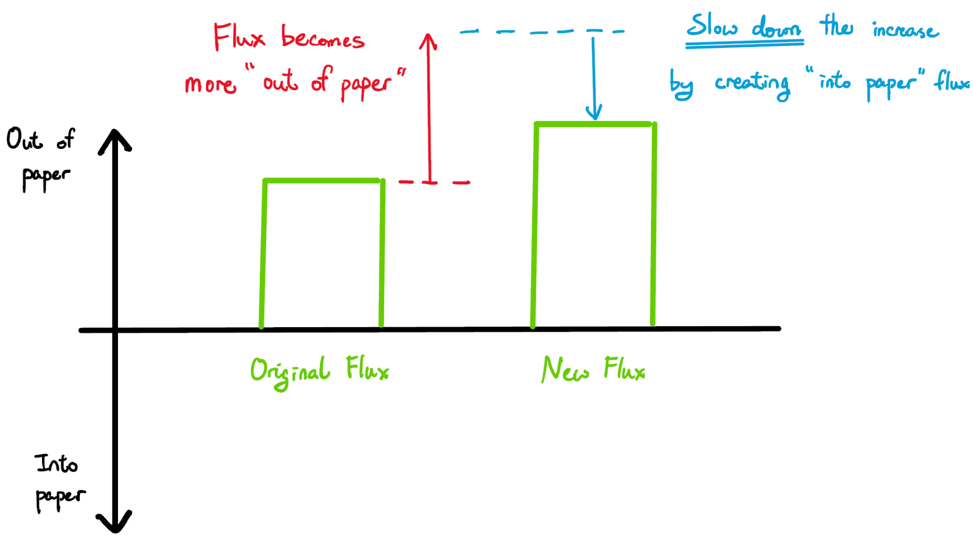
$$\boxed{\textbf{Induced EMF always want to "oppose" change in magnetic flux}}$$

Here are a few examples for you to get familiar with it. Right hand grip rule is all you need. (As a practice, you may also try to explain how energy is conserved in these examples.)

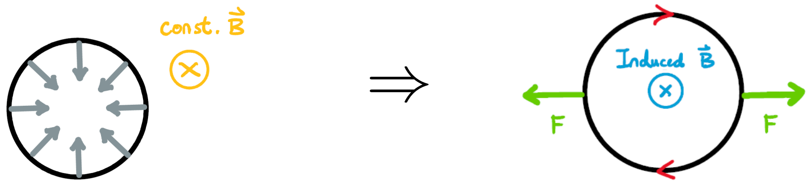
Example 3.1. Consider a ring inside a magnetic field whose magnitude is increasing in the out-of-paper direction.



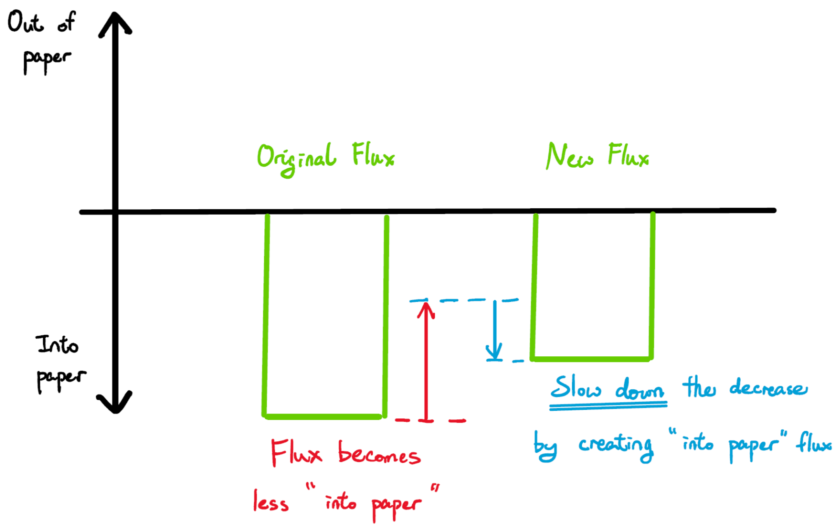
1. Magnetic flux is getting "more out-of-paper" due to increase in B-field's magnitude.
2. Induced EMF want to "oppose" this magnetic flux change.
3. To oppose an increasing out-of-paper flux, one needs to decrease it, i.e. create an into-paper flux to compensate the increase.
4. By right hand rule, into-paper flux can be created if a current flow clockwise in the ring. So the induced EMF must be clockwise.



Example 3.2. Consider a ring inside a constant into-paper B-field but is shrinking in radius.



1. Magnetic flux is getting "less into-paper" due to decrease in area.
2. Induced EMF want to "oppose" this magnetic flux change.
3. To oppose a decreasing into-paper flux, one needs to increase it, i.e. create an into-paper flux to compensate the decrease.
4. By right hand rule, into-paper flux can be created if a current flow clockwise in the ring. So the induced EMF must be clockwise.
5. You can also notice that the forces on induced current due to induced B field are pointing outward, i.e. trying to stop the loop from contracting.



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4 Solving Problems in Magnetic Induction

Generally speaking, there are only two levels of questions related to magnetic induction.

- Find EMF from the change in magnetic flux.
- Find induced E-field from the change in magnetic flux.

The difference is in the difficulties - EMF is just a single number, the same everywhere in the coil. But the E-field, when involving magnetic induction, is a vector function (distribution) that depends on position AND time.

4.1 Finding EMF

This is a straightforward calculation to the surface integral of Faraday's law.

- **For high school level** - B-field is usually given as a constant of position, so that the surface integral reduces to a multiplication.

$$\epsilon = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}} \sim -\frac{\mathrm{d}}{\mathrm{d}t}\Big(\vec{\boldsymbol{B}}(t) \cdot (\mathrm{Area})(t)\Big)$$

  And because you are not expected to have studied differentiation chain rule, you will never see a situation where both $\vec{\boldsymbol{B}}(t)$ and $\mathrm{Area}(t)$ are changing. It is always given the rate of change of one of them and the other is fixed.

$$\underbrace{\epsilon = -\vec{\boldsymbol{B}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Area}(t))}_{\text{When the question is about motional EMF}} \qquad \text{or} \qquad \underbrace{\epsilon = -\frac{\mathrm{d}\vec{\boldsymbol{B}}(t)}{\mathrm{d}t} \cdot (\mathrm{Area})}_{\text{When the question is about transformer EMF}}$$

- **For university level** - You are assumed to have already learnt Ampere's law, so more likely you are given the current to find the B-field, before finding EMF.

$$\text{First} \quad \underbrace{\oint \vec{\boldsymbol{B}}(t) \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I(t)}_{\text{Ampere's law for } \vec{\boldsymbol{B}}}, \quad \text{then} \quad \underbrace{\epsilon = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathrm{Area}(t)} \vec{\boldsymbol{B}}(t) \cdot \mathrm{d}\vec{\boldsymbol{s}}}_{\text{Faraday's law for } \epsilon}$$

And asking for the EMF's direction is pretty common, because applying Lenz's law does not involve any maths. All you need is your right hand.

4.2 Finding induced E-field

This is in fact the task of solving the PDE of Maxwell-Faraday equation.

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \qquad \Rightarrow \qquad \vec{\boldsymbol{E}}(t) = \text{Some function of } \vec{\boldsymbol{B}}(t)$$

Similar to how we deal with Gauss's law or Ampere's law, we can avoid solving PDE in some very symmetrical cases. If these conditions are satisfied:

1. E-field is of the same magnitude everywhere on the loop.
2. E-field make the same angle with each line segment of the loop.

Then we can reduce the integral form of Maxwell-Faraday equation in to multiplication.

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{s} &= \oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} \\
&= \oint \underline{|\vec{\boldsymbol{E}}|\,|\,\mathrm{d}\vec{\boldsymbol{l}}\,|\cos\theta} \longleftarrow \text{Just dot product } \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|\cos\theta \\
&= \underline{|\vec{\boldsymbol{E}}|}\ \underline{\cos\theta}\ \oint |\,\mathrm{d}\vec{\boldsymbol{l}}\,|
\end{aligned}
$$

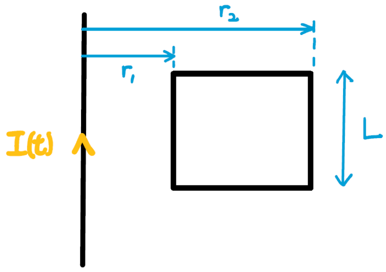
Same magnitude everywhere
Can move out of integral!

Form same angle everywhere
Can move out of integral!

$$= |\vec{\boldsymbol{E}}|\ \cos\theta\ (\text{Perimeter of loop})$$

$$\boxed{|\vec{\boldsymbol{E}}| = \frac{\left(\text{Flux of } \vec{\boldsymbol{B}}\right)}{(\text{Perimeter of loop})\cos\theta}}$$

Example 4.1. Consider a static rectangular loop next to an infinitely long wire, which carries a time-varying current $I(t)$, same magnitude everywhere along the wire.

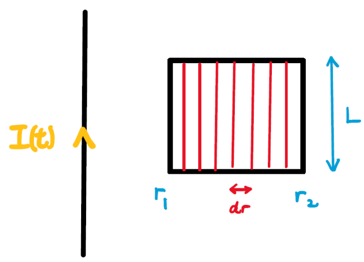


1. Always start with finding the magnetic flux through the loop. By Ampere's law, the B-field by an infinitely long wire is

$$\vec{\boldsymbol{B}}(r,t) = \frac{\mu_0}{2\pi}\frac{I(t)}{r}$$

The magnetic flux through the loop can be calculated by first dividing the loop's area into strips, then integrate the flux of all strips.

$$
\begin{aligned}
\Phi_B &= \iint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{s}} \\
&= \int_{r_1}^{r_2} \frac{\mu_0 I(t)}{2\pi r}(L\,\mathrm{d}r) \\
&= \frac{\mu_0 I(t) L}{2\pi}[\ln(r_2) - \ln(r_1)]
\end{aligned}
$$



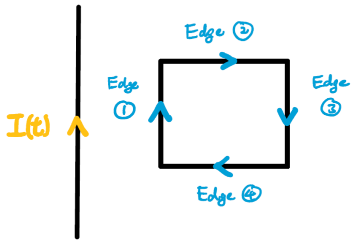
2. The total EMF generated in the loop is just the differentiation of the above. i.e.

$$\epsilon = \frac{\mu_0 L}{2\pi}\left(\frac{\mathrm{d}I(t)}{\mathrm{d}t}\right)[\ln(r_2) - \ln(r_1)]$$

Direction of the EMF depends on how the current varies. For example, if $\frac{\mathrm{d}I(t)}{\mathrm{d}t} > 0$, i.e. magnetic flux increasing in the into-paper direction. To oppose the change, EMF must be in anti-clockwise direction to produce an out-of-paper direction flux.

3. On the other hand, to compute the E-field distribution, all we only know is

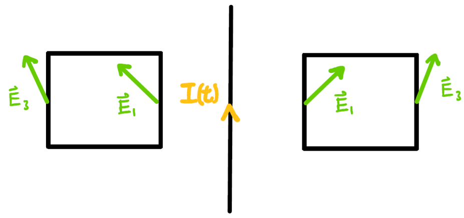
$$
\begin{aligned}
|\epsilon| &= \oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{l}} \\
&= E_1^{||}L + E_2^{||}(r_2 - r_1) + E_3^{||}L + E_4^{||}(r_2 - r_1)
\end{aligned}
$$



Note that after taking dot product, only the component parallel to the edge is left.

To find the E on each edge, we need symmetry arguments.

- By translational symmetry along the wire, $E_2^{||}$ and $E_4^{||}$ must be the same. Their contributions of dot product along the loop are cancelled.
- By rotation symmetry about the wire, the E-field is only a function of radial distance from the wire. We can claim that $E_1^{||}$ and $E_3^{||}$ must have the same function form $E^{||}(r)$,



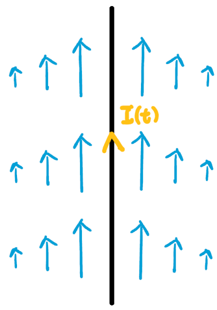
So the E-field relation to EMF is reduced to

$$
\begin{aligned}
|\epsilon| &= E_3^{||}L - E_1^{||}L \\
&= E^{||}(r_2)L - E^{||}(r_1)L \\
&\equiv \frac{\mu_0 L}{2\pi}\left(\frac{\mathrm{d}I(t)}{\mathrm{d}t}\right)[\ln(r_2) - \ln(r_1)]
\end{aligned}
$$

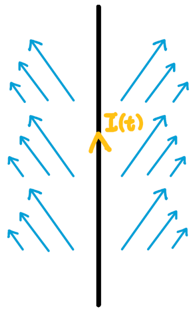
Therefore we can claim that $\vec{\boldsymbol{E}}$'s component parallel to the wire is

$$
E^{||}(r) = \frac{\mu_0 L}{2\pi}\left(\frac{\mathrm{d}I(t)}{\mathrm{d}t}\right)\ln(r)
$$

4. Note that in the above analysis, we cannot determine if the E-field has components perpendicular to the wire.

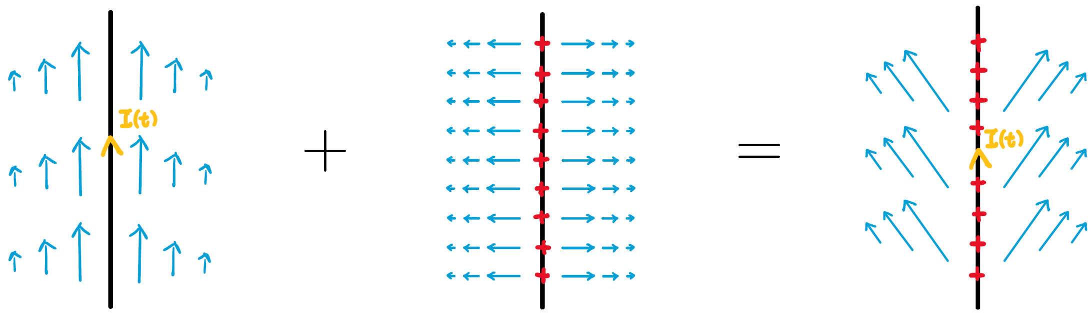


If the field only has component parallel to the wire



If the field has both components parallel and normal to the wire

Recall from Gauss's law, E-field can be perpendicular to the wire if the wire carries a static line charge density. So when the wire carries BOTH net charge and time-varying current, the total E-field will be in diagonal directions.



— The End —

Appendix: A Brief History of Electromagnetism

Electromagnetic induction is likely the most confusing topic in beginner E&M. In my opinion, taking reference of the history is helpful to unify the concepts you have learnt.

Year	Advancement
Before 1500s	Different electrostatics phenomena were known. But they were not unified or explained at all.
1600	William Gilbert was the first person to use the word "electrical" to describe electrostatics phenomena. Also the first to propose that electrical effect is due to flows of particles.
1750	Benjamin Franklin developed a one "fluid" theory of electricity, and called this fluid "charge".
1784	Charles-Augustin de Coulomb experimentally showed that force between charged objects $\propto \frac{1}{r^2}$. (Coulomb's law $F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}$)
1800	Alessandro Volta Made the first battery from electro-chemistry. (First time to have steady current.)
1820	Hans Christian Ørsted discovered that current wire can deflect compress. (First time to relate electric and magnetic phenomena.)
1820	André-Marie Ampère formulated and verified the force between current wires. ($F = I\vec{l}_1 \times \frac{\mu_0 I}{2\pi r}\vec{l}_2$)
1831	Michael Faraday discovered magnetic induction - Both motional and transformer EMF.
1834	Emil Lenz Explained direction of induced current by energy conservation. (Lenz's Law)
1860	James Clerk Maxwell unified past discoveries into 20 equations, and used field description for the first time. This was the first time **E** and **B** appeared in Physics. Before Maxwell, everything was described in terms of force.
1893	Oliver Heaviside combined Maxwell's 20 equations into 4, by vector calculus. (This is the version of Maxwell's equation we now know.)
1895	Hendrik Lorentz derive the correct force on charges under both $\vec{E}$ and $\vec{B}$. (Lorentz force formula)
After 1900s	Entering the quantum era. Not classical anymore.