Absurd Algebra: Extending Number System with Division by Zero

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Abstract

In this work, we tackle the problem of division by zero by introducing a new algebraic structure in which all numbers, including zero, have multiplicative inverses. We define this structure carefully to avoid contradictions that typically arise from such a modification, by strategically limiting or extending certain aspects of the traditional field framework. Our approach parallels the extension from real numbers to complex numbers—where imaginary units were introduced to resolve limitations—and we propose a further extension of the number system by introducing the multiplicative inverse of zero, which we term the absurd unit, acknowledging the perceived absurdity and undefined nature of this operation both in mathematics and societal perception.

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1 Introduction

The problem of division by zero has been matter of discussion mathematicians across different eras and civilizations, representing one of the most persistent, fundamental and infamous challenges in mathematical foundations. Ancient mathematical systems like those of the Babylonians and Egyptians avoided the issue entirely by not having a formal zero concept at all, though they developed complicated calculation methods for astronomy and engineering. The first serious confrontation with this mathematical dilemma started in 7th century India when Brahmagupta treated zero as a legitimate number in its own right within his revolutionary number system. In his seminal work *Brahmasphutasiddhanta*, he proposed three groundbreaking rules: first, that division by zero results in a fraction with zero denominator ($\frac{a}{0}$); second, that zero divided by any number equals zero ($\frac{0}{a} = 0$); and third, controversially, that zero divided by zero equals zero ($\frac{0}{0} = 0$) [1]. These rules represented an important first step toward formalizing arithmetic with zero, though they left the actual meaning of fractions with zero denominators unresolved and contained inherent contradictions that would trouble mathematicians for centuries.

Later Indian mathematicians tackled with these definitions in various ways that revealed the depth of the problem. In the 9th century, Mahāvīra made the interesting claim in his *Ganita Sara Samgraha* that division by zero leaves numbers unchanged ($\frac{a}{0} = a$), an approach that clearly contradicted basic arithmetic operations since it would imply $1 = 2$ through simple algebraic manipulation [2]. Bhāskara II in the 12th century took a more sophisticated path in his *Lilavati* by suggesting division by zero yields an infinite quantity ($\frac{a}{0} = \infty$), which worked for some cases but failed to properly address the problematic $\frac{0}{0}$ case [3]. These early attempts revealed the fundamental complexity of the problem without providing satisfactory solutions, showing how division by zero resists simple arithmetic treatment.

During the Islamic Golden Age, mathematicians like Al-Khwarizmi in his *The Compendious Book on Calculation by Completion and Balancing* carefully structured their algebraic works to avoid division by zero entirely, recognizing it as an operation that could lead to logical inconsistencies [4]. European mathematics followed a similar path until the Renaissance, typically treating division by zero as either undefined or meaningless in their algebraic systems. The development of calculus in the 17th century by Newton and Leibniz brought new need to the issue, as their work with infinitesimals and limits constantly approached division by zero without ever quite reaching it. Newton's method of fluxions involved ratios of vanishing quantities like $\frac{\dot{y}}{\dot{x}}$, while Leibniz's differential notation $\frac{dy}{dx}$ implicitly considered division by infinitesimals [5]. This indirect approach through limits became the standard way to handle such problems, though it left the direct operation of division by zero formally undefined and raised philosophical questions about the nature of infinitesimals.

Modern mathematics has developed several sophisticated but imperfect approaches to manage division by zero, each with significant trade-offs. Projective geometry extends the real number line $\mathbb{R}$ by adding a single point at infinity ($\mathbb{R} \cup \{\infty\}$), allowing definition of $\frac{a}{0} = \infty$ for $a \neq 0$ while leaving $\frac{0}{0}$ undefined [6]. Complex analysis employs the Riemann sphere $\mathbb{C} \cup \{\infty\}$ to similar effect, providing a geometric model where $\frac{z}{0} = \infty$ for all non-zero complex numbers z, though it cannot resolve the essential singularity at zero where $\frac{0}{0}$ remains indeterminate [7]. Algebraic structures like wheels introduce new elements such as $\perp = \frac{0}{0}$ to make division a total operation, but at the cost of abandoning familiar arithmetic properties since expressions like $x + \perp = \perp$ and $x \cdot \perp = \perp$ lead to an algebraic structure where many computations collapse to this undefined value [8]. In practical computing, the IEEE floating-point standard defines division by zero to return special values like $\pm\infty$ or NaN (Not a Number), which propagate through calculations in ways that often mask rather than resolve the underlying mathematical issues, sometimes leading to silent errors in numerical computations [9].

The fundamental problems with division by zero stem from deep mathematical principles that resist simple solutions. In any standard number system satisfying field axioms, the equation $0 \cdot x = 0$ for all x directly contradicts the requirement that division should undo multiplication, since defining $\frac{1}{0}$ would imply $1 = 0 \cdot \frac{1}{0} = 0$ [10]. The indeterminate form $\frac{0}{0}$ presents additional challenges, as the relation $0 \cdot x = 0$ suggests it could equal any number x while maintaining formal consistency. These issues manifest practically in computer systems where division by zero errors can crash programs or produce meaningless results that propagate through subsequent calculations. Theoretically, they appear in mathematical analysis where elementary functions like $f(x) = \frac{1}{x}$ show important discontinuities at zero that cannot be removed by simple algebraic redefinition, and in calculus where limits of the form $\lim_{x \to 0} \frac{f(x)}{g(x)}$ require tools like L'Hôpital's rule to evaluate properly when both numerator and denominator approach zero.

This paper presents a new approach to division by zero that introduces a new number system that preserves most standard arithmetic operations while giving consistent meaning to division by zero. Unlike previous attempts, our method allows operations very similar to standard arithmetic, while offering a mechanism for preserving and retrieving information when dividing or multiplying by zero. We demonstrate how this system provides consistent solutions to previously singular equations while remaining backward-compatible with standard arithmetic.

In this work, we approach the problem by introducing a novel number system featuring 0^{-1}, the multiplicative inverse of zero, which we term an *absurd unit* and denote as $z = 0^{-1}$ or equivalently $z = \frac{1}{0}$. This system, which we call *Absurd Algebra*, modifies traditional field axioms in careful ways to accommodate z while preserving most standard arithmetic operations. This approach successfully avoids the classical contradictions that arise when attempting to incorporate division by zero in conventional number systems.

The development of Absurd Algebra draws significant inspiration from the historical introduction of complex numbers. Just as $i = \sqrt{-1}$ expanded the real number system to solve previously unsolvable equations, our absurd unit $z = 0^{-1}$ extends conventional arithmetic to handle division by zero consistently. Several important parallels emerge between these systems: absurd units correspond to imaginary units, Absurd Algebra mirrors the complex field structure, and absurd numbers (numbers of the form $a + bz$ where $a, b \in \mathbb{C}$) resemble complex numbers in their algebraic behavior. These analogies provide valuable intuition for mathematicians familiar with complex analysis while suggesting potential applications in similar domains.

2 Algebraic Definition

2.1 Fields

A **field** is a set F equipped with two binary operations: addition $(+)$ and multiplication $(\cdot)$, satisfying a specific set of axioms. The structure $(F^+, +)$ forms an abelian (commutative) group under addition, referred to as the *additive group* of the field. Additionally, the set $F \setminus \{0\}$, under multiplication, forms an abelian group $(F^\times, \cdot)$, commonly called the *multiplicative group* of the field.

Fields generalize the familiar arithmetic of numbers. Examples of fields include the set of real numbers $(\mathbb{R})$, rational numbers $(\mathbb{Q})$, and complex numbers $(\mathbb{C})$. These structures are foundational in algebra and analysis, representing some of the most important algebraic systems in mathematics.

One critical aspect of fields is the existence of multiplicative inverses for all non-zero elements. This distinguishes fields from more general structures such as rings. In a field, every non-zero element $a \in F$ has an inverse $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.

The requirement that $1 \neq 0$ ensures that the multiplicative identity is distinct from the additive identity, which is essential to avoid trivial structures where the entire set collapses to a single element, since $1 \neq 0$ is part of the definition of a field. Division by zero is not defined in a field because it leads to contradictions, such as violating the uniqueness of multiplicative inverses and breaking the group structure of $(F \setminus \{0\}, \cdot)$. Specifically:

- If division by zero were allowed, the equation $a = a \cdot 1 = a \cdot (0^{-1} \cdot 0)$ would imply $a = 0$ for all $a \in F$, and that an inverse of zero exists.
- This would collapse the field to a singleton set, contradicting the definition of a field.
- It would destroy the injectivity of multiplication by non-zero elements, which is required in the multiplicative group.

Therefore, the restriction against division by zero is not arbitrary but a necessary condition to maintain consistency within the field axioms.

2.2 Absurd Algebra

In this work, we introduce a new algebraic structure termed *absurd algebra*, a field-like but unconventional system constructed to allow division by zero in an algebraically consistent manner. The primary goal is to preserve maximal field-like behavior while extending the applicability of algebraic frameworks beyond the limitations of classical field theory. Unlike traditional attempts to resolve division by zero by undefined or partial structures, absurd algebra embraces the operation and embeds it meaningfully within a redefined algebraic system.

The term *absurd* reflects the historical perception of division by zero as a nonsensical or paradoxical operation. However, in this context, we reinterpret its "absurdity" as an opportunity to innovate. By redefining core operations and selectively restricting certain axioms, absurd algebra introduces a new paradigm in which division by zero becomes algebraically sound and consistent.

Formally, an **absurd algebra** is a set $\mathbb{A}$ equipped with two binary operations: addition $(+)$ and multiplication $(\cdot)$, where both operations are defined over all of $\mathbb{A}$:

$$+,\ \cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}.$$

The addition operation creates $(\mathbb{A}, +)$ with the structure of a commutative (abelian) group. The multiplication operation follows a modified structure with a set of axioms designed to preserve group-like behavior, while extending it to include zero:

- **Closure**: $\forall a, b \in \mathbb{A}, \quad a \cdot b \in \mathbb{A}$.
- **Associativity**: $\forall a, b, c \in \mathbb{A} \setminus \{0\}, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Commutativity**: $\forall a, b \in \mathbb{A}, \quad a \cdot b = b \cdot a$.
- **Identity**: $\exists 1 \in \mathbb{A},\ \forall a \in \mathbb{A}, \quad a \cdot 1 = a$.
- **Inverses**: $\forall a \in \mathbb{A},\ \exists a^{-1} \in \mathbb{A}, \quad a \cdot a^{-1} = 1$.
- **Non-absorbing Zero**: $\forall a \in \mathbb{A} \setminus \{0, 1\}, \quad a \cdot 0 \neq 0$.

Additionally, the structure satisfies a modified distributive law linking addition and multiplication, restricted to non-zero elements:

- **Distributivity**: $\forall a, b, c \in \mathbb{A} \setminus \{0\}, \quad a \cdot (b + c) = a \cdot b + a \cdot c.$

To handle edge cases where zero is involved in cases excluded from traditional axiomatic framework — we introduce a set of additional rules known as **zero-handler rules**. These ensure algebraic consistency when zero participates in operations normally restricted to non-zero elements.

Zero-Handler Rules for Associativity:

$$\forall a, b \in \mathbb{A} \setminus \{0\}, \quad (a \cdot 0) \cdot b = a \cdot (0 \cdot b) \tag{1}$$
$$\forall a \in \mathbb{A} \setminus \{0\}, \quad (0 \cdot 0) \cdot a = 0 \cdot a \tag{2}$$

Zero-Handler Rules for Distributivity:

$$\forall a, b \in \mathbb{A} \setminus \{0\}, \quad (a + 0) \cdot b = a \cdot b + b \cdot 0 \tag{3}$$
$$\forall a \in \mathbb{A} \setminus \{0\}, \quad (0 + 0) \cdot a = 0 \cdot a \tag{4}$$

These rules serve as extensions to the conventional axioms, ensuring that the structure remains closed and well-defined even when zero is involved in operations that would otherwise lead to contradictions in classical algebra.

Key distinctions from fields:

- Associativity and distributivity of multiplication are restricted to non-zero elements.
- The zero element has a well-defined multiplicative inverse, which is not the case in any conventional field.
- Zero is not an absorbing element; $a \cdot 0 \neq 0$ for most $a \in \mathbb{A} \setminus \{0, 1\}$.

These foundational changes allow absurd algebra to meaningfully integrate division by zero, while still maintaining a rigorous internal logic. In the next subsection, we examine the implications of these rules, particularly the properties of zero's multiplicative inverse.

2.3 Absurd Numbers

We define a new number system, extending complex numbers called *absurd numbers*, denoted by the set:

$$\mathbb{A} = \{(a, b, c, d) \mid a, b, c, d \in \mathbb{R}\},$$

where:

- The pair (a, b) termed *complex part*, follows the field operations of complex numbers $\mathbb{C}$.
- The pair (c, d) termed *absurd part*, follows the rules of absurd algebra, as previously defined.
- The additive identity is $(0, 0, 0, 0)$.
- The multiplicative identity is defined as $(1, 0, 0, 0)$.

The extension of complex numbers happens by introducing new special element $z \in \mathbb{A}$ called the *absurd unit*, given by:

$$z := (0, 0, 1, 0) \tag{5}$$

This unit is constructed such that:

$$z \cdot 0 = (0, 0, 1, 0) \cdot (0, 0, 0, 0) = (1, 0, 0, 0)$$
$$\Rightarrow z \cdot 0 = 1$$

Therefore, z behaves as the multiplicative inverse of zero:

$$z := 0^{-1} \tag{6}$$

This is consistent with absurd algebra's **non-absorbing zero** axiom, where zero is no longer an absorbing element under multiplication. Non-absorbing zero axiom and generally absurd algebra only works for the absurd part of the absurd number and complex part follows standard field axioms.

Each absurd number can be represented in the form:

$$(a, b, c, d) = a + bi + (c + di)z \tag{7}$$

where $a, b, c, d \in \mathbb{R}$, i is the imaginary unit satisfying $i^2 = -1$, and $z = 0^{-1}$ is the absurd unit defined above.

Addition: Defined component-wise as:

$$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = \tag{8}$$
$$(a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2,\ d_1 + d_2) \tag{9}$$

or equivalently:

$$(n_1 + m_1 z) + (n_2 + m_2 z) = (n_1 + n_2) + (m_1 + m_2) z$$

where $a_k, b_k, c_k, d_k \in \mathbb{R}$ and $n, m \in \mathbb{C}$.

Multiplication: Defined by distributivity over:

$$(n_1 + m_1 z) \cdot (n_2 + m_2 z) = n_1 n_2 + (n_1 m_2 + n_2 m_1) z + m_1 m_2 z^2$$

Where, absurd parts follow the distributivity axiom and rules of absurd algebra, and everything else behaves field-wise. Imaginary unit and absurd unit are commutative $iz = zi$ to be mentioned and z^n where $n \in \mathbb{Z}^-$ is equal to zero, for example $z^{-1} = (0^{-1})^{-1} = 0$, and z^n with $n \in \mathbb{Z}^+$ creating polynomial ring-like structure over the absurd algebra, which we will discuss later in the paper.

Example Calculations:

- $1 + 0i + (0 + 0i)z = (1, 0, 0, 0) \Rightarrow$ Multiplicative identity
- $0 + 0i + (1 + 0i)z = (0, 0, 1, 0) = z$
- $z \cdot 0 = 1$
- $2 + 3i + (4 - i)z = (2, 3, 4, -1) \in \mathbb{A}$
- $\big(2 + 3i + (4 - i)z\big) \cdot 0 = 4 - i = (4, -1) \in \mathbb{C}$

Absurd numbers can also be represented in the Euler form. Let $w \in \mathbb{A}$, $w = a + bi + (c + di)z, \quad a, b, c, d \in \mathbb{R}$:

$$w = a + bi + (c + di)z = r_1 e^{i\alpha} + r_2 e^{i\beta} z \tag{10}$$

Where $r_1 = \sqrt{a^2 + b^2}, r_2 = \sqrt{c^2 + d^2}$, e is the Euler number and α, β are defined as:

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) \tag{11}$$
$$\beta = \tan^{-1}\left(\frac{d}{c}\right) \tag{12}$$

Complex numbers are isomorphically embedded to absurd numbers:

$$\phi : \mathbb{C} \to \mathbb{A}_0, \quad \phi(a + bi) = (a + bi) z^0 = (a, b) \in \mathbb{R}^2.$$

1. **Operation Preservation:**
$$\begin{aligned} \phi((a + bi) + (c + di)) &= \\ (a + c, b + d) &= \phi(a + bi) + \phi(c + di), \\ \phi((a + bi)(c + di)) &= (ac - bd, ad + bc) \\ &= \phi(a + bi) \cdot \phi(c + di). \end{aligned}$$
2. **Injectivity:** $\phi(a + bi) = \phi(c + di) \implies a = c, b = d$.
3. **Surjectivity:** Every $(a, b) \in \mathbb{A}_0$ corresponds to $a + bi \in \mathbb{C}$.

Since $\mathbb{C} = \{(a, b) | a, b \in \mathbb{R}\}$ is isomorphic to $\mathbb{R}^2$, and is $\mathbb{C}$ is isomorphically embedded in $\mathbb{A}$, we establish that:

$$\mathbb{A}_n = \{(a_1, b_1, a_2, b_2, \ldots, a_n, b_n) | a_k, b_k \in \mathbb{R}\} \tag{13}$$

is isomorphic to $\mathbb{R}^{2(n+1)}$.

2.4 Polynomial Ring

Now we construct the polynomial ring over the absurd algebra $\mathbb{A}$ to handle expressions like z^n and define quotient ring to enforce algebraic relations. Let $\mathbb{A}$ be the absurd algebra, then the polynomial ring over $\mathbb{A}$ is:

$$\mathbb{A}[z] = \left\{ \sum_{k=0}^{n} a_k z^k | a_k \in \mathbb{A}, z = 0^{-1} \in \mathbb{A}, n \in \mathbb{N} \right\} \tag{14}$$

Where addition and multiplication follow standard polynomial operations. **Addition**:

$$\left(\sum a_k z^k\right) + \left(\sum b_k z^k\right) = \sum (a_k + b_k) z^k \tag{15}$$

Multiplication:

$$\left(\sum a_k z^k\right) \cdot \left(\sum b_k z^k\right) = \sum_n \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n \tag{16}$$

Scalar Multiplication:

$$c \in \mathbb{C}, c \cdot (\sum a_k z^k) = \sum (c a_k) z^k \tag{17}$$

And now for the quotient ring, to enforce relations like $z \cdot 0 = 1$, we define an ideal I and form the quotient ring. Let I be the ideal generated by:

$$I = \langle z \cdot 0 - 1 \rangle \tag{18}$$

where z is the absurd unit or inverse of zero equivalently, and that's what first relationship $z \cdot 0 - 1$ encodes.

The quotient ring from this ideal is:

$$R = \mathbb{A}[z]/I \tag{19}$$

where elements are equivalence classes of polynomials modulo I, and the basis of R is $\{1, z, z^2, \dots\}$ subject to $z \cdot 0 = 1$, and arithmetic operations following the axioms of absurd algebra, defined above in the paper.

Now that polynomial ring over $\mathbb{A}$ is constructed, we can show basic rules of polynomial arithmetics in absurd algebra with absurd units:

$$\begin{aligned} z^0 &= 1 \\ z^n \cdot 0 &= z^{n-1} \\ z^n \cdot z &= z^{n+1} \\ z^n \cdot z^m &= z^{n+m} \\ (z^n)^m &= z^{nm} \end{aligned}$$

and

$$\begin{aligned} w^0 &= 1 \\ w^n \cdot w^m &= w^{n+m} \\ (w^n)^m &= w^{nm} \end{aligned}$$

Where w is an absurd number $w \in \mathbb{A}$ and $n, m \in \mathbb{R}$.

2.5 Linear Algebra

In this section, we develop linear algebra for the absurd number system.

An absurd vector is an ordered tuple:

$$v = (v_1, v_2, \dots, v_n), \quad v_k \in \mathbb{A} \tag{20}$$

For example: $v = (1+z, 4i-z) \in \mathbb{A}^2$. Absurd vectors add component-wise:

$$v + w = (v_1 + w_1, \dots, v_n + w_n), \quad v_k, w_k \in \mathbb{A} \tag{21}$$

And scalar multiplication is defined as followed:

$$\alpha v = (\alpha v_1, \dots, \alpha v_n), \quad \alpha \in \mathbb{A} \tag{22}$$

. The core difference from non-absurd vectors, is that if scalar is zero $\alpha = 0$, scalar multiplication doesn't yield to zero vector $0 = (0, \dots, 0)$. Additive inverse of vector v is $-v = (-v_1, \dots, -v_n)$ respectively. In standard linear algebra, $0 \cdot v = 0$, but in absurd algebra $0 \cdot v$ transforms vector into the complex based vector or reduces the power of the highest absurd unit by one. For example: $0 \cdot (a_0 + b_0, a_1 + b_1 z, \dots, a_n + b_n z^n) = (0, b_1, \dots, b_n z^{n-1})$. As for the dot product (inner product) of two absurd vectors $u, v \in \mathbb{A}$, the absurd inner product is defined as:

$$u \cdot v = \sum_{k=1}^{n} u_k \overline{v}_k \tag{23}$$

Where $\overline{v_k}$ is the complex conjugate of v_k, and **complex conjugate** of an absurd number $w = a + bz, \quad a, b \in \mathbb{C}, z = 0^{-1}, w \in \mathbb{A}$, denoted as $\overline{w}$ is defined as:

$$\overline{w} = \overline{a + bz} = \overline{a} + \overline{b}z \tag{24}$$

Where $\overline{a}$ and $\overline{b}$ are complex conjugates of a and b complex numbers respectively. It also should be mentioned that there is other kind of conjugate called **absurd conjugate**, which is defined as:

$$w^* = a - bz \tag{25}$$

Similar to complex conjugate, but instead of the complex coefficient, sign of the absurd coefficient is flipped.

As in standard spaces, the property of $v \cdot v = 0 \implies v = 0$, often referred to as non-degeneracy property, still holds true, and zero divisors exist as well, for example, vectors like $(1, z)$ and $(z, -1)$ are orthogonal:

$$(1, z) \cdot (z, -1) = z - z = 0 \tag{26}$$

As for the cross product, for $u, v \in \mathbb{A}^3$, the **cross product** is:

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{27}$$

where i, j, k are basis vectors.

An $m \times n$ absurd matrix has entries in $\mathbb{A}$:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m_1} & \dots & a_{mn} \end{pmatrix}, \quad a_{ij} \in \mathbb{A} \tag{28}$$

with following operations of addition, scalar multiplication and matrix multiplication:

$$(A+B)_{ij} = A_{ij} + B_{ij}$$
$$(\alpha A)_{ij} = \alpha A_{ij}$$
$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

The special property of absurd matrices is that all matrices have inverses, including singular matrices due to the allowance of division by zero by the number system. Special matrices include: zero matrix $0_{ij} = 0$; Identity matrix $I_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$; And singular matrices, with determinant equal to zero might not be considered special because they behave similar to other matrices when it comes to having an inverse, meaning that all matrices are invertible.

Determinants of absurd matrices are calculated in the same way as non-absurd matrices, for example with Leibniz formula, for $A \in M_n(\mathbb{A})$, we define $\det(A)$ as:

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

For example:

$$\det \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = 1 \cdot 1 - z \cdot 0 = 1$$

Just the existence of absurd matrices, allows complex valued singular matrices to have inverses just in the absurd form, for example:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}$$

Where $\det(A) = ad - bc = 0$, the inverse of this singular matrix is:

$$A^{-1} = \frac{1}{0} \begin{pmatrix} d & c \\ b & a \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} z = \begin{pmatrix} dz & cz \\ bz & az \end{pmatrix}$$

Giving singular matrix an inverse but only as an absurd matrix, handing ruling axioms to the absurd number system.

3 Other theories

3.1 Riemann Sphere

The Riemann sphere provides a formal, compact way to handle division by zero in complex analysis by extending the complex plane $\mathbb{C}$ with a single point at infinity, denoted ∞. It is the one-point compactification of $\mathbb{C}$, and the Riemann sphere $\hat{\mathbb{C}}$ is defined as:

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \tag{29}$$

Division by zero on the Riemann sphere, for any nonzero $u \in \mathbb{C}$, is defined as:

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0, \tag{30}$$

$$\frac{u}{0} = \infty, \quad \frac{u}{\infty} = 0 \tag{31}$$

However,

$$\frac{0}{0} \text{ and } \frac{\infty}{\infty} \tag{32}$$

are undefined—left to limits or contextual interpretation. The Riemann sphere works because, by treating ∞ as a single added point, all holomorphic functions (except at isolated poles) become continuous at their poles when extended to $\hat{\mathbb{C}}$, and division by zero yields ∞, corresponding to reaching these points.

Our approach in this work is different. Instead of adding a single point to the complex plane, we introduce a new plane—extending the complex plane—where division by zero is defined in an algebraically consistent way, and linking these planes via division by zero. Division by zero in our framework, for any nonzero $u \in \mathbb{C}$ and $z = 0^{-1} \in \mathbb{A}$, is:

$$\frac{1}{0} = z, \quad \frac{1}{z} = 0, \tag{33}$$

$$\frac{u}{0} = uz, \quad \frac{u}{z} = 0, \tag{34}$$

$$\frac{0}{0} = \frac{z}{z} = 1 \tag{35}$$

Unlike the Riemann sphere, division by zero preserves information in our framework. Instead of infinity, the result resides in a higher dimension, eliminating the need for singularities. Zero divided by zero is also defined, and numbers resulting from division by zero have more algebraic flexibility than infinity, making this work a strong replacement for the Riemann sphere.

3.2 Wheel Theory

A wheel is an algebraic structure that generalizes fields by allowing division by zero. It modifies the usual algebraic axioms to make division total and constructs a commutative ring with identity $(W, +, \cdot, -, /, 0, 1, \perp)$, where $/ : W \times W \to W$ is a total division operation and $\perp \in W$ is a special element called the error element or bottom. Wheel theory handles division by zero in the following manner for $a \in W$:

$$\frac{1}{0} = \perp, \quad \frac{1}{\perp} = \perp, \tag{36}$$

$$\frac{a}{0} = \perp, \quad \frac{a}{\perp} = \perp, \tag{37}$$

$$\frac{\perp}{\perp} = \frac{0}{0} = \perp \tag{38}$$

With additional rules for error propagation:

$$\perp + a = \perp, \quad \perp \cdot a = \perp, \quad -\perp = \perp \tag{39}$$

It is apparent that wheel theory does not offer the preservation of information either and lacks algebraic flexibility for interaction with real or complex numbers, unlike our framework. It should also be noted that while both theories attempt to handle division by zero by introducing a new algebraic structure, our structure differs in its axioms and the properties of its special element.

3.3 Other theories

Other significant theories include **Meadows**, where a commutative ring with total inverse is introduced, but instead of $0^{-1} = z$, we have $0^{-1} = 0$, which is another approach to handling division by zero algebraically. However, it does not offer preservation of information or much algebraic flexibility. There is also **Transreal Arithmetic**, where division by zero simply results in $\pm\infty$, and $\frac{0}{0}$ results in NaN (Not a Number). Lastly, the traditional partial field method treats division by zero as undefined.

References

[1] Kaplan, R. (1999). *The Nothing That Is: A Natural History of Zero.* Oxford University Press.

[2] Plofker, K. (2009). *Mathematics in India.* Princeton University Press.

[3] Boyer, C. B. (1991). *A History of Mathematics.* Wiley.

[4] Berggren, J. L. (1986). *Episodes in the Mathematics of Medieval Islam.* Springer.

[5] Baron, M. E. (1969). *The Origins of the Infinitesimal Calculus.* Pergamon Press.

[6] Coxeter, H. S. M. (2003). *Projective Geometry.* Springer.

[7] Ahlfors, L. V. (1979). *Complex Analysis.* McGraw-Hill.

[8] Carlsson, P. (2004). Wheels - On Division by Zero. *Mathematical Structures in Computer Science.*

[9] Overton, M. L. (2001). *Numerical Computing with IEEE Arithmetic.* SIAM.

[10] Barukčić, J. (2008). Contraproof of the Field Axioms. *Journal of Mathematics.*

[11] Lang, S. (2002). *Algebra.* Springer.