The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 3 (Week 4)

MATH2069/2969: Discrete Mathematics and Graph Theory — Semester 1, 2010

1. Prove by induction that, for all $n \geq 0$,

(a) $n^3 + 5n$ is a multiple of 3 (i.e. $n^3 + 5n = 3\ell$ for some integer ℓ).

Solution: The $n = 0$ case holds because $0^3 + 0 = 0$ is a multiple of 3 (it is 3×0). Suppose that $n \geq 1$ and that the result is known for $n - 1$, i.e.

$$(n-1)^3 + 5(n-1) \; = \; 3\ell, \text{ for some integer } \ell.$$

Then

$$3\ell \; = \; n^3 - 3n^2 + 3n - 1 + 5n - 5 \; = \; n^3 + 5n - 3(n^2 - n + 2) \,,$$

so $n^3 + 5n = 3(\ell + n^2 - n + 2)$ is a multiple of 3, establishing the inductive step and completing the proof.

(b) $5^n - 4n - 1$ is a multiple of 16.

Solution: The $n = 0$ case holds because $5^0 - 4 \times 0 - 1 = 0$ is a multiple of 16. Suppose that $n \geq 1$ and that the result is known for $n - 1$, i.e.

$$5^{n-1} - 4(n-1) - 1 \; = \; 16\ell, \text{ for some integer } \ell.$$

This equation can be rewritten as

$$5^{n-1} \; = \; 4n - 3 + 16\ell.$$

So

$$5^n - 4n - 1 \; = \; 5(4n - 3 + 16\ell) - 4n - 1 \; = \; 16(n - 1 + 5\ell) \,,$$

which is a multiple of 16, establishing the inductive step and completing the proof.

2. Use the characteristic polynomial to solve the following recurrence relations:

(a) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, where $a_0 = 2$, $a_1 = 5$.

Solution: The characteristic polynomial is $x^2 - 5x + 6 = (x-2)(x-3)$ with roots 2 and 3, so the general solution is $a_n = C_1 2^n + C_2 3^n$ for some constants C_1, C_2. In our case we have

$$2 = a_0 = C_1 + C_2 \qquad \text{and} \qquad 5 = a_1 = 2C_1 + 3C_2.$$

Solving yields $C_1 = C_2 = 1$, so the solution is

$$a_n = 2^n + 3^n.$$

(b) $a_n = 4a_{n-1} - 3a_{n-2}$ for $n \geq 2$, where $a_0 = -1$, $a_1 = 2$.

Solution: The characteristic polynomial is $x^2 - 4x + 3 = (x-1)(x-3)$ with roots 1 and 3, so the general solution is $a_n = C_1 1^n + C_2 3^n$ for some constants C_1, C_2. In our case we have

$$-1 = a_0 = C_1 + C_2 \qquad \text{and} \qquad 2 = a_1 = C_1 + 3C_2.$$

Solving yields $C_1 = -5/2$ and $C_2 = 3/2$, so the solution is

$$a_n = \frac{3^{n+1} - 5}{2}.$$

(c) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, where $a_0 = 3$, $a_1 = 8$.

Solution: The characteristic polynomial is $x^2 - 4x + 4 = (x-2)^2$ with repeated root 2, so the general solution is $a_n = C_1 2^n + C_2 n 2^n$ for some constants C_1, C_2. In our case we have

$$3 = a_0 = C_1 \qquad \text{and} \qquad 8 = a_1 = 2C_1 + 2C_2,$$

yielding $C_1 = 3$ and $C_2 = 1$, so the final solution is

$$a_n = 3 \times 2^n + n2^n = (n+3)2^n.$$

(d) $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, where $a_0 = 2$, $a_1 = -3$.

Solution: The characteristic polynomial is $x^2 - 6x + 9 = (x-3)^2$ with repeated root 3, so the general solution is $a_n = C_1 3^n + C_2 n 3^n$ for some constants C_1, C_2. In our case we have

$$2 = a_0 = C_1 \qquad \text{and} \qquad -3 = a_1 = 3C_1 + 3C_2,$$

yielding $C_1 = 2$ and $C_2 = -3$, so that the final solution is

$$a_n = 3^n(2 - 3n).$$

*(e) $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ for $n \geq 3$, where $a_0 = 3$, $a_1 = 5$, $a_2 = 11$.

Solution: The characteristic polynomial is $x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$ with roots $1, 2, 3$, so the general solution is

$$a_n = C_1 + C_2 2^n + C_3 3^n$$

for some constants C_1, C_2, C_3. In our case we have

$$3 = a_0 = C_1 + C_2 + C_3, \quad 5 = a_1 = C_1 + 2C_2 + 3C_3, \quad 11 = C_1 + 4C_2 + 9C_3,$$

yielding $C_1 = 2$, $C_2 = 0$ and $C_3 = 1$, so the final solution is

$$a_n = 3^n + 2.$$

*(f) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ for $n \geq 3$, where $a_0 = 2$, $a_1 = 4$, $a_2 = 16$.

Solution: The characteristic polynomial is $x^3 - 6x^2 + 12x - 8 = (x-2)^3$ with repeated root 2, so the general solution is

$$a_n = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n$$

for some constants C_1, C_2, C_3. In our case we have

$$2 = a_0 = C_1, \quad 4 = a_1 = 2C_1 + 2C_2 + 2C_3, \quad 16 = a_2 = 4C_1 + 8C_2 + 16C_3,$$

yielding $C_1 = 2$, $C_2 = -1$ and $C_3 = 1$, so the final solution is

$$a_n = 2^n(2 - n + n^2).$$

3. Companies A and B control the market for a certain product. From one year to the next, A retains 70% of its custom and loses to B the remaining 30%, while B retains 60% of its custom and loses to A the remaining 40%. Let a_n denote the market share of company A after n years (thus, that of company B is $1-a_n$).

(a) Write down a recurrence relation expressing a_n in terms of a_{n-1}, for $n \geq 1$.

Solution: For $n \geq 1$, we have

$$a_n \;=\; \frac{7}{10}a_{n-1} + \frac{4}{10}(1 - a_{n-1}) \;=\; \frac{3}{10}a_{n-1} + \frac{4}{10}.$$

(b) Solve the recurrence relation, in the sense of giving a closed formula for a_n, in terms of a_0.

Solution: Unravelling the recurrence relation, we get:

$$\begin{aligned}
a_n &= \frac{3}{10}a_{n-1} + \frac{4}{10} = \frac{3}{10}\left(\frac{3}{10}a_{n-2} + \frac{4}{10}\right) + \frac{4}{10} \\
&= \left(\frac{3}{10}\right)^2 a_{n-2} + \frac{3}{10}\frac{4}{10} + \frac{4}{10} = \left(\frac{3}{10}\right)^2 \left(\frac{3}{10}a_{n-3} + \frac{4}{10}\right) + \frac{3}{10}\frac{4}{10} + \frac{4}{10} \\
&= \left(\frac{3}{10}\right)^3 a_{n-3} + \left(\frac{3}{10}\right)^2 \frac{4}{10} + \frac{3}{10}\frac{4}{10} + \frac{4}{10} \\
&\;\;\vdots \\
&= \left(\frac{3}{10}\right)^n a_0 + \frac{4}{10}\left[\left(\frac{3}{10}\right)^{n-1} + \cdots + \frac{3}{10} + 1\right] \\
&= \left(\frac{3}{10}\right)^n a_0 + \frac{4}{10}\frac{\left(\frac{3}{10}\right)^n - 1}{\frac{3}{10} - 1} \\
&= \left(a_0 - \frac{4}{7}\right)\left(\frac{3}{10}\right)^n + \frac{4}{7}.
\end{aligned}$$

Here the second-last equality uses the formula for the sum of a geometric progression.

(c) Hence prove that the market share of company A in the long run (i.e. the limit of a_n as $n \to \infty$) is independent of its initial market share a_0.

Solution: As $n \to \infty$, the power $(\frac{3}{10})^n$ tends to 0, so $a_n \to \frac{4}{7}$. So whatever the initial situation, the market tends to a stable situation where company A has a $\frac{4}{7}$ market share and company B has a $\frac{3}{7}$ market share.

4. Let b_n be the number of ways of forming a line of n people distinguished only by whether they are male (M) or female (F), such that no two males are next to each other. For example, the possibilities with 3 people are FFF, FFM, FMF, MFF, and MFM, so $b_3 = 5$. Write down a recurrence relation for b_n. Do you recognize the sequence?

Solution: We have $b_0 = 1$, $b_1 = 2$, and $b_n = b_{n-1} + b_{n-2}$ if $n \geq 2$. To see this notice that in a line of n people with $n \geq 2$, if the last person is female then there are b_{n-1} possibilities for the line of the first $n-1$ people, whilst if the last person is male then the second last person must be female, so that there are b_{n-2} possibilities for the line of the first $n-2$ people. We get the Fibonacci sequence with the first two terms deleted, so $b_n = F_{n+2}$.

5. Define a sequence recursively by $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1}a_{n-2}$ for $n \geq 2$.

(a) Find a_2, a_3, a_4, a_5 and a_6.

Solution: $a_2 = 2$, $a_3 = 4 = 2^2$, $a_4 = 8 = 2^3$, $a_5 = 32 = 2^5$, $a_6 = 2^8$.

(b) Prove that $a_n = 2^{F_n}$, where $F_0, F_1, F_2, \cdots$ is the Fibonacci sequence.

Solution: As seen in lectures, we only need to show that 2^{F_n} satisfies the same initial conditions and recurrence relation as a_n. The initial conditions hold because $2^{F_0} = 2^0 = 1$ and $2^{F_1} = 2^1 = 2$. The recurrence relation holds because for $n \geq 2$,

$$2^{F_n} = 2^{F_{n-1}+F_{n-2}} = 2^{F_{n-1}}2^{F_{n-2}},$$

by the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$.

6. Imagine a $2^n \times 2^n$ array of equal-sized squares, where n is some positive integer. We want to cover this array with non-overlapping L-shaped tiles, each of which exactly covers three squares (one square and two of the adjacent squares, not opposite to each other). Since the number of squares is not a multiple of 3, we need to remove one square before we start. Prove by induction that no matter which square we remove, the remaining squares can be covered by these L-shaped tiles.

Solution: The base case is clear, because removing a square from a 2×2 array leaves 3 squares which can be covered by a single tile. We now prove the claim for $n \geq 2$, assuming its truth for $n-1$. Let G be the $2^n \times 2^n$ array with exactly one square missing. Denote the quarters of G by UL for upper left, UR for upper right, LL for lower left and LR for lower right. Each quarter is a $2^{n-1} \times 2^{n-1}$ array, except that one of the quarters has one square missing. By rotating G if necessary, we may suppose that the missing square is in LL. Let T be the L-shape formed by the lower-rightmost subsquare of UL, the lower-leftmost subsquare of UR and the upper-leftmost subsquare of LR. Then removing T from G produces a union of four $2^{n-1} \times 2^{n-1}$ arrays each with one square missing, and each of these can be tiled by L-shapes, by the induction hypothesis. Hence G is tiled by all these L-shapes together with T, establishing the inductive step and completing the proof.

*7. The following argument 'proves' that whenever a group of people is in the same room, they all have the same height. There must be an invalid step; find it.

> We argue by induction on the number n of people in the room. The $n = 1$ case is obviously true. Suppose that $n \geq 2$ and that the claim holds for rooms with $n - 1$ people. Let $P_1, P_2, \dots, P_n$ be the n people in this room. If P_n were to leave the room we would have a room with $n-1$ people, so by the inductive hypothesis, $P_1, P_2, \dots, P_{n-1}$ all have the same height. We can apply the same reasoning with P_1 leaving the room, so $P_2, \dots, P_{n-1}, P_n$ all have the same height. But P_2 is in both these collections, so all of $P_1, P_2, \dots, P_n$ have the same height. This establishes the inductive step, and so the claim holds for all n by induction.

Solution: Since the claim is false even when $n = 2$, the proof must fail already in this case; when you run through the argument with $n = 2$, the error emerges. The invalid step is the assertion that "P_2 is in both these collections", because this ignores the convention governing the way these collections were written out. When you start with n people $P_1, P_2, \dots, P_n$ and remove P_n, it is reasonable to list the remaining people as "$P_1, P_2, \dots, P_{n-1}$", but you have to bear in mind that if $n = 2$, this list will just consist of P_1 and will not in fact include P_2.

*8. For which n is the Fibonacci number F_n even, and for which n is F_n odd? Prove your answer by induction.

Solution: Examining the first few terms, one is led to guess that F_n is even when n is a multiple of 3, and odd when n is not a multiple of 3. To prove this by induction, we first observe that the $n = 0$ and $n = 1$ cases are true (because $F_0 = 0$ is even and $F_1 = 1$ is odd). Then in proving the result for $n \geq 2$, we can assume it for $n - 1$ and for $n - 2$. Recall that we have

$$F_n = F_{n-1} + F_{n-2}.$$

There are now three cases, depending on the remainder of n after division by 3.

If $n \equiv 0 \pmod 3$ (i.e. n is a multiple of 3), then $n - 1$ and $n - 2$ are not multiples of 3, so F_{n-1} and F_{n-2} are odd, so F_n is even as required.

If $n \equiv 1 \pmod 3$, then $n - 1$ is a multiple of 3 but $n - 2$ is not, so F_{n-1} is even and F_{n-2} is odd, so F_n is odd as required.

If $n \equiv 2 \pmod 3$ then $n - 1$ is not a multiple of 3 but $n - 2$ is, so F_{n-1} is odd and F_{n-2} is even, so F_n is odd as required.

This completes the inductive step, and the claim follows by induction.

**9. Suppose we want to solve a recurrence relation which is almost a kth-order homogeneous linear recurrence relation, but with an extra constant term C:

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} + C, \text{ for all } n \geq k.$$

Let $p(x) = x^k - r_1 x^{k-1} - \cdots - r_k$ be the characteristic polynomial of the homogeneous recurrence relation obtained by omitting C.

(a) Show that any solution a_n also satisfies the $(k+1)$th-order linear homogeneous recurrence relation with characteristic polynomial $(x-1)p(x)$.

Solution: Suppose that the sequence a_n is a solution of our recurrence relation. Then for any $n \geq k+1$, we have

$$\begin{aligned} a_n &= r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} + C \text{ and} \\ a_{n-1} &= r_1 a_{n-2} + r_2 a_{n-3} + \cdots + r_k a_{n-k-1} + C. \end{aligned}$$

Subtracting the second equation from the first gives

$$a_n - a_{n-1} = r_1(a_{n-1} - a_{n-2}) + r_2(a_{n-2} - a_{n-3}) + \cdots + r_k(a_{n-k} - a_{n-k-1})$$

for all $n \geq k+1$, which can be rearranged as

$$a_n = (r_1 - 1)a_{n-1} + (r_2 - r_1)a_{n-2} + \cdots + (r_k - r_{k-1})a_{n-k} + (-r_k)a_{n-k-1}.$$

This is the homogeneous recurrence relation with characteristic polynomial

$$\begin{aligned} &x^{k+1} - (r_1 - 1)x^k - (r_2 - r_1)x^{k-1} - \cdots - (r_k - r_{k-1})x + r_k \\ &\qquad = (x-1)(x^k - r_1 x^{k-1} - \cdots - r_{k-1}x - r_k) = (x-1)p(x), \end{aligned}$$

as claimed.

(b) Hence describe the general solution a_n in terms of the roots of $p(x)$. (The answer will depend on whether 1 is a root of $p(x)$ or not.)

Solution: Let the different roots of $p(x)$ be $\lambda_1, \cdots, \lambda_s$ with multiplicities $m_1, \cdots, m_s$ (where the multiplicity of a non-repeated root is 1).
First suppose that none of the λ_i's equals 1; then $(x-1)p(x)$ has roots $\lambda_1, \cdots, \lambda_s, 1$ with multiplicities $m_1, \cdots, m_s, 1$. By the general solution of homogeneous recurrence relations given in lectures, (a) implies that

$$\begin{aligned} a_n = (C_{11} + C_{12}n + \cdots + C_{1,m_1}n^{m_1-1})\lambda_1^n + \cdots \\ + (C_{s1} + C_{s2}n + \cdots + C_{s,m_s}n^{m_s-1})\lambda_s^n + D, \end{aligned}$$

for some constants C_{ij}, D. Conversely, any sequence a_n of this form is a solution of the homogeneous recurrence relation with characteristic polynomial $(x-1)p(x)$. This is not quite enough to imply that it satisfies our recurrence relation: but the additional requirement is just the $n = k$ case, namely

$$a_k = r_1 a_{k-1} + \cdots + r_{k-1}a_1 + r_k a_0 + C,$$

because, as we saw in the previous part, the difference between this equation and the $n = k+1$ case is a case of the homogenous recurrence relation, as is the difference between the $n = k+1$ case and the $n = k+2$ case, and so on.

It is easy to see that the $n = k$ case reduces to a constraint on the constant D, namely

$$D = r_1 D + \cdots + r_k D + C,$$

so the general solution is given by the above formula but with D specified to equal $\dfrac{C}{1 - r_1 - \cdots - r_k}$. (The denominator is $p(1)$, which we assumed to be nonzero.)

Now suppose that 1 is a root of $p(x)$; without loss of generality, $\lambda_s = 1$. Then $(x-1)p(x)$ has roots $\lambda_1, \cdots, \lambda_{s-1}, 1$ with multiplicities $m_1, \cdots, m_{s-1}, m_s+1$. Solving the homogeneous recurrence, we obtain

$$\begin{aligned} a_n = (C_{11} + C_{12}n + \cdots + C_{1,m_1}n^{m_1-1})\lambda_1^n + \cdots \\ + (C_{s-1,1} + C_{s-1,2}n + \cdots + C_{s-1,m_{s-1}-1}n^{m_{s-1}-1})\lambda_{s-1}^n \\ + (C_{s1} + C_{s2}n + \cdots + C_{s,m_s-1}n^{m_s-1} + Dn^{m_s}), \end{aligned}$$

for some constants C_{ij}, D. As in the previous case, we have one extra constraint on the constant D in order that the $n = k$ case of the desired recurrence relation should hold, namely

$$Dk^{m_s} = r_1 D(k-1)^{m_s} + \cdots + r_{k-1}D + C,$$

so the general solution is given by the above formula but with D specified to equal $\dfrac{C}{k^{m_s} - r_1(k-1)^{m_s} - \cdots - r_{k-1}}$. (The denominator is nonzero, because it is what you get when you substitute $x = 1$ in the polynomial obtained from $p(x)$ by applying m_s times the operator $x\frac{d}{dx}$; each application reduces the multiplicity of the root 1 by 1, so it is no longer a root at the end.)