EE263 Homework 3

Summer 2022

4.570. Orthogonal matrices.

a) Show that if U and V are orthogonal, then so is UV.

b) Show that if U is orthogonal, then so is U^{-1}.

c) Suppose that $U \in \mathbb{R}^{2\times 2}$ is orthogonal. Show that U is either a rotation or a reflection. Make clear how you decide whether a given orthogonal U is a rotation or reflection.

4.580. Projection matrices. A matrix $P \in \mathbb{R}^{n\times n}$ is called a *projection matrix* if $P = P^\mathsf{T}$ and $P^2 = P$.

a) Show that if P is a projection matrix then so is $I - P$.

b) Suppose that the columns of $U \in \mathbb{R}^{n\times k}$ are orthonormal. Show that UU^T is a projection matrix. (Later we will show that the converse is true: every projection matrix can be expressed as UU^T for some U with orthonormal columns.)

c) Suppose $A \in \mathbb{R}^{n\times k}$ is full rank, with $k \leq n$. Show that $A(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ is a projection matrix.

d) Show that $x - Px$ is orthogonal to range(P) for any x. (Aside: this means that that Px is the closest point in range(P) to x.)

4.610. Householder reflections. A *Householder matrix* is defined as

$$Q = I - 2uu^\mathsf{T},$$

where $u \in \mathbb{R}^n$ is normalized, that is, $u^\mathsf{T}u = 1$.

a) Show that Q is orthogonal.

b) Show that $Qu = -u$. Show that $Qv = v$, for any v such that $u^\mathsf{T}v = 0$. Thus, multiplication by Q gives reflection through the plane with normal vector u.

c) Given a vector $x \in \mathbb{R}^n$, find a unit-length vector u for which Qx lies on the line through e_1. *Hint:* Try a u of the form $u = v/\|v\|$, with $v = x + \alpha e_1$ (find the appropriate α and show that such a u works ...) Compute such a u for $x = (3, 2, 4, 1, 5)$. Apply the corresponding Householder reflection to x to find Qx.

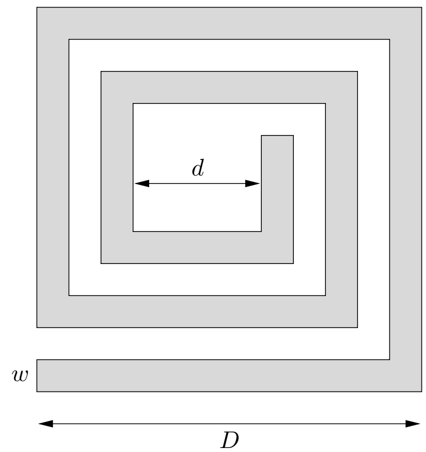
Note: Multiplication by an orthogonal matrix has very good numerical properties, in the sense that it does not accumulate much roundoff error. For this reason, Householder reflections are used as building blocks for fast, numerically sound algorithms.

5.680. Least-squares residuals. Suppose A is skinny and full-rank. Let x_{ls} be the least-squares approximate solution of $Ax = y$, and let $y_{\mathrm{ls}} = Ax_{\mathrm{ls}}$. Show that the residual vector $r = y - y_{\mathrm{ls}}$ satisfies

$$\|r\|^2 = \|y\|^2 - \|y_{\mathrm{ls}}\|^2.$$

Also, give a brief geometric interpretation of this equality (just a couple of sentences, and maybe a conceptual drawing).

6.720. Approximate inductance formula. The figure below shows a *planar spiral inductor*, implemented in CMOS, for use in RF circuits.



The inductor is characterized by four key parameters:

- n, the number of turns (which is a multiple of $1/4$, but that needn't concern us)
- w, the width of the wire
- d, the inner diameter
- D, the outer diameter

The inductance L of such an inductor is a complicated function of the parameters n, w, d, and D. The inductance L can be found by solving Maxwell's equations, which takes considerable computer time, or by fabricating the inductor and measuring the inductance. In this problem you will develop a simple approximate inductance model of the form

$$\hat{L} = \alpha n^{\beta_1} w^{\beta_2} d^{\beta_3} D^{\beta_4},$$

where $\alpha, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}$ are constants that characterize the approximate model. (since L is positive, we have $\alpha > 0$, but the constants $\beta_2, \ldots, \beta_4$ can be negative.) This simple approximate model, if accurate enough, can be used for design of planar spiral inductors.

The file `inductor_data.json` on the course web site contains data for 50 inductors. (The data is real, not that it would affect how you solve the problem ...) For inductor i, we give parameters n_i, w_i, d_i, and D_i (all in μm), and also, the inductance L_i (in nH) obtained from measurements. (The data are organized as vectors of length 50. Thus, for example, w_{13} gives the wire width of inductor 13.) Your task, *i.e.*, the problem, is to find α, $\beta_1, \ldots, \beta_4$ so that

$$\hat{L}_i = \alpha n_i^{\beta_1} w_i^{\beta_2} d_i^{\beta_3} D_i^{\beta_4} \approx L_i \qquad \text{for } i = 1, \ldots, 50.$$

Your solution must include a clear description of how you found your parameters, as well as their actual numerical values. Note that we have not specified the criterion that you use to judge the approximate model (*i.e.*, the fit between $\hat{L}_i$ and L_i); we leave that to your engineering judgment. But be sure to tell us what criterion you use. We define the *percentage error* between $\hat{L}_i$ and L_i as

$$e_i = 100|\hat{L}_i - L_i|/L_i.$$

Find the average percentage error for your model, *i.e.*, $(e_1 + \cdots + e_{50})/50$. (We are only asking you to find the average percentage error for your model; we do not require that your model minimize the average percentage error.) *Hint:* you might find it easier to work with $\log L$.