# EE263 Homework 1

Summer 2022

**2.100. A mass subject to applied forces.** Consider a unit mass subject to a time-varying force $f(t)$ for $0 \le t \le n$. Let the initial position and velocity of the mass both be zero. Suppose that the force has the form $f(t) = x_j$ for $j-1 \le t < j$ and $j = 1, \ldots, n$. Let $y_1$ and $y_2$ denote, respectively, the position and velocity of the mass at time $t = n$.

a) Find the matrix $A \in \mathbb{R}^{2\times n}$ such that $y = Ax$.

b) For $n = 4$, find a sequence of input forces $x_1, \ldots, x_n$ that moves the mass to position 1 with velocity 0 at time $n$.

**2.150. Gradient of some common functions.** Recall that the gradient of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, at a point $x \in \mathbb{R}^n$, is defined as the vector

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix},$$

where the partial derivatives are evaluated at the point $x$. The first order Taylor approximation of $f$, near $x$, is given by

$$\hat{f}_{\text{tay}}(z) = f(x) + \nabla f(x)^\mathsf{T}(z - x).$$

This function is affine, *i.e.*, a linear function plus a constant. For $z$ near $x$, the Taylor approximation $\hat{f}_{\text{tay}}$ is very near $f$. Find the gradient of the following functions. Express the gradients using matrix notation.

a) $f(x) = a^\mathsf{T}x + b$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

b) $f(x) = x^\mathsf{T}Ax$, for $A \in \mathbb{R}^{n\times n}$.

c) $f(x) = x^\mathsf{T}Ax$, where $A = A^\mathsf{T} \in \mathbb{R}^{n\times n}$. (Yes, this is a special case of the previous one.)

**2.160. Some matrices from signal processing.** We consider $x \in \mathbb{R}^n$ as a signal, with $x_i$ the (scalar) value of the signal at (discrete) time period $i$, for $i = 1, \ldots, n$. Below we describe several transformations of the signal $x$, that produce a new signal $y$ (whose dimension varies). For each one, find a matrix $A$ for which $y = Ax$.

a) $2\times$ *up-conversion with linear interpolation.* We take $y \in \mathbb{R}^{2n-1}$. For $i$ odd, $y_i = x_{(i+1)/2}$. For $i$ even, $y_i = (x_{i/2} + x_{i/2+1})/2$. Roughly speaking, this operation doubles the sample rate, inserting new samples in between the original ones using linear interpolation.

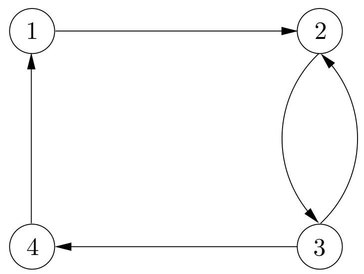
b) $2\times$ *down-sampling.* We assume here that $n$ is even, and take $y \in \mathbb{R}^{n/2}$, with $y_i = x_{2i}$.

c) $2\times$ *down-sampling with averaging.* We assume here that $n$ is even, and take $y \in \mathbb{R}^{n/2}$, with $y_i = (x_{2i-1} + x_{2i})/2$.

**2.180. Paths and cycles in a directed graph.** We consider a directed graph with $n$ nodes. The graph is specified by its *node adjacency matrix* $A \in \mathbb{R}^{n\times n}$, defined as

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from node } j \text{ to node } i \\ 0 & \text{otherwise.} \end{cases}$$

Note that the edges are *oriented*, *i.e.*, $A_{34} = 1$ means there is an edge from node 4 to node 3. For simplicity we do not allow self-loops, *i.e.*, $A_{ii} = 0$ for all $i$, $1 \le i \le n$. A simple example illustrating this notation is shown below.




The node adjacency matrix for this example is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In this example, nodes 2 and 3 are connected in both directions, *i.e.*, there is an edge from 2 to 3 and also an edge from 3 to 2. A *path* of length $l > 0$ from node $j$ to node $i$ is a sequence $s_0 = j, s_1, \ldots, s_l = i$ of nodes, with $A_{s_{k+1},s_k} = 1$ for $k = 0, 1, \ldots, l-1$. For example, in the graph shown above, $1, 2, 3, 2$ is a path of length 3. A *cycle* of length $l$ is a path of length $l$, with the same starting and ending node, with no repeated nodes other than the endpoints. In other words, a cycle is a sequence of nodes of the form $s_0, s_1, \ldots, s_{l-1}, s_0$, with

$$A_{s_1,s_0} = 1, \quad A_{s_2,s_1} = 1, \quad \ldots \quad A_{s_0,s_{l-1}} = 1,$$

and

$$s_i \neq s_j \text{ for } i \neq j, \quad i, j = 0, \ldots, l-1.$$

For example, in the graph shown above, $1, 2, 3, 4, 1$ is a cycle of length 4. The rest of this problem concerns a specific graph, given in the file `directed_graph.json` on the course web site. For each of the following questions, you must give the answer explicitly (for example, enclosed in a box). You must also explain clearly how you arrived at your answer.

a) What is the length of a shortest cycle? (Shortest means minimum length.)

b) What is the length of a shortest path from node 13 to node 17? (If there are no paths from node 13 to node 17, you can give the answer as 'infinity'.)

c) What is the length of a shortest path from node 13 to node 17, that *does not* pass through node 3?

d) What is the length of a shortest path from node 13 to node 17, that *does* pass through node 9?

e) Among all paths of length 10 that start at node 5, find the most common ending node.

f) Among all paths of length 10 that end at node 8, find the most common starting node.

g) Among all paths of length 10, find the most common pair of starting and ending nodes. In other words, find $i, j$ which maximize the number of paths of length 10 from $i$ to $j$.

**2.200. Quadratic extrapolation of a time series.** We are given a series $z$ up to time $t$. Using a quadratic model, we want to extrapolate, or predict, $z(t+1)$ based on the three previous elements of the series, $z(t)$, $z(t-1)$, and $z(t-2)$. We'll denote the predicted value of $z(t+1)$ by $\hat{z}(t+1)$. More precisely, you will find $\hat{z}(t+1)$ as follows.

a) Find the quadratic function $f(\tau) = a_2\tau^2 + a_1\tau + a_0$ which satisfies $f(t) = z(t)$, $f(t-1) = z(t-1)$, and $f(t-2) = z(t-2)$. Then the extrapolated value is given by $\hat{z}(t+1) = f(t+1)$. Show that

$$\hat{z}(t+1) = c \begin{bmatrix} z(t) \\ z(t-1) \\ z(t-2) \end{bmatrix},$$

where $c \in \mathbb{R}^{1\times 3}$, and does not depend on $t$. In other words, the quadratic extrapolator is a linear function. Find $c$ explicitly.

b) Use the following Julia code to generate a time series $z$:

```
t = collect(1:1000);
z = 5*sin.(t/10 .+ 2)  + 0.1 * sin.(t) + 0.1*sin.(2*t .- 5);
```

Use the quadratic extrapolation method from part (a) to find $\hat{z}(t)$ for $t = 4, \ldots, 1000$. Find the relative root-mean-square (RMS) error, which is given by

$$\left(\frac{(1/997)\sum_{j=4}^{1000}(\hat{z}(j) - z(j))^2}{(1/997)\sum_{j=4}^{1000} z(j)^2}\right)^{1/2}.$$