EE263 Homework 2

Summer 2022

3.240. Price elasticity of demand. The demand for n different goods is a function of their prices:

$$q = f(p),$$

where p is the price vector, q is the demand vector, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the demand function. The current price and demand are denoted p^* and q^*, respectively. Now suppose there is a small price change δp, so $p = p^* + \delta p$. This induces a change in demand, to $q \approx q^* + \delta q$, where

$$\delta q \approx Df(p^*)\delta p,$$

where Df is the derivative or Jacobian of f, with entries

$$Df(p^*)_{ij} = \frac{\partial f_i}{\partial p_j}(p^*).$$

This is usually rewritten in term of the *elasticity matrix* E, with entries

$$E_{ij} = \frac{\partial f_i}{\partial p_j}(p^*)\frac{1/q_i^*}{1/p_j^*},$$

so E_{ij} gives the relative change in demand for good i per relative change in price j. Defining the vector y of relative demand changes, and the vector x of relative price changes,

$$y_i = \frac{\delta q_i}{q_i^*}, \qquad x_j = \frac{\delta p_j}{p_j^*},$$

we have the linear model $y = Ex$.

Here are the questions:

a) What is a reasonable assumption about the diagonal elements E_{ii} of the elasticity matrix?

b) Goods i and j are called *substitutes* if they provide a similar service or other satisfaction (*e.g.*, train tickets and bus tickets, cake and pie, etc.). They are called *complements* if they tend to be used together (*e.g.*, automobiles and gasoline, left and right shoes, etc.). For each of these two generic situations, what can you say about E_{ij} and E_{ji}?

c) Suppose the price elasticity of demand matrix for two goods is

$$E = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Describe the nullspace of E, and give an interpretation (in one or two sentences). What kind of goods could have such an elasticity matrix?

3.250. Color perception. Human color perception is based on the responses of three different types of color light receptors, called *cones*. The three types of cones have different spectral-response characteristics, and are called L, M, and, S because they respond mainly to long, medium, and short wavelengths, respectively. In this problem we will divide the visible spectrum into 20 bands, and model the cones' responses as follows:

$$L_{\text{cone}} = \sum_{i=1}^{20} l_i p_i, \qquad M_{\text{cone}} = \sum_{i=1}^{20} m_i p_i, \qquad S_{\text{cone}} = \sum_{i=1}^{20} s_i p_i,$$

where p_i is the incident power in the ith wavelength band, and l_i, m_i and s_i are nonnegative constants that describe the spectral responses of the different cones. The perceived color is a complex function of the three cone responses, *i.e.*, the vector $(L_{\text{cone}}, M_{\text{cone}}, S_{\text{cone}})$, with different cone response vectors perceived as different colors. (Actual color perception is a bit more complicated than this, but the basic idea is right.)

a) *Metamers.* When are two light spectra, p and $\tilde{p}$, visually indistinguishable? (Visually identical lights with different spectral power compositions are called *metamers*.)

b) *Visual color matching.* In a color matching problem, an observer is shown a test light, and is asked to change the intensities of three primary lights until the sum of the primary lights looks like the test light. In other words, the observer is asked the find a spectrum of the form

$$p_{\text{match}} = a_1 u + a_2 v + a_3 w,$$

where u, v, w are the spectra of the primary lights, and a_i are the intensities to be found, that is visually indistinguishable from a given test light spectrum p_{test}. Can this always be done? Discuss briefly.

c) *Visual matching with phosphors.* A computer monitor has three phosphors, R, G, and B. It is desired to adjust the phosphor intensities to create a color that looks like a reference test light. Find weights that achieve the match or explain why no such weights exist. The data for this problem is in `color_perception_data.json`, which contains the vectors `wavelength`, `B_phosphor`, `G_phosphor`, `R_phosphor`, `L_coefficients`, `M_coefficients`, `S_coefficients`, and `test_light`.

d) *Effects of illumination.* An object's surface can be characterized by its reflectance (*i.e.*, the fraction of light it reflects) for each band of wavelengths. If the object is illuminated with a light spectrum characterized by I_i, and the reflectance of the object is r_i (which is between 0 and 1), then the reflected light spectrum is given by $I_i r_i$, where $i = 1, \ldots, 20$ denotes the wavelength band. Now consider two objects illuminated (at different times) by two different light sources, say an incandescent bulb and sunlight. Sally argues that if the two objects look identical when illuminated by a tungsten bulb, then they will look identical when illuminated by sunlight. Beth disagrees: she says that two objects can appear identical when illuminated by a tungsten bulb, but look different when lit by sunlight. Who is right? If Sally is right, explain why. If Beth is right give an example of two objects that appear identical under one light source and different under another. You can use the vectors `sunlight` and `tungsten` defined in the data file as the light sources.

Remark. Spectra, intensities, and reflectances are all nonnegative quantities, which the material of EE263 doesn't address. So just ignore this while doing this problem. These issues can be handled using the material of EE364a, however.

3.260. Halfspace. Suppose $a, b \in \mathbb{R}^n$ are two given points. Show that the set of points in $\mathbb{R}^n$ that are closer to a than b is a halfspace, *i.e.*:

$$\{x \mid \|x-a\| \leq \|x-b\| \ \} = \{ \ x \mid c^{\mathsf{T}}x \leq d\}$$

for appropriate $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Give c and d explicitly, and draw a picture showing a, b, c, and the halfspace.

3.340. Vector spaces over the Boolean field. In this course the *scalar field*, *i.e.*, the components of vectors, will usually be the real numbers, and sometimes the complex numbers. It is also possible to consider vector spaces over other fields, for example $\mathbb{Z}_2$, which consists of the two numbers 0 and 1, with Boolean addition and multiplication (*i.e.*, $1+1=0$). Unlike $\mathbb{R}$ or $\mathbb{C}$, the field $\mathbb{Z}_2$ is finite, indeed, has only two elements. A vector in $\mathbb{Z}_2^n$ is called a *Boolean vector*. Much of the linear algebra for $\mathbb{R}^n$ and $\mathbb{C}^n$ carries over to $\mathbb{Z}_2^n$. For example, we define a function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^m$ to be linear (over $\mathbb{Z}_2$) if $f(x+y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$ for every x, $y \in \mathbb{Z}_2^n$ and $\alpha \in \mathbb{Z}_2$. It is easy to show that every linear function can be expressed as matrix multiplication, *i.e.*, $f(x) = Ax$, where $A \in \mathbb{Z}_2^{m\times n}$ is a Boolean matrix, and all the operations in the matrix multiplication are Boolean, *i.e.*, in $\mathbb{Z}_2$. Concepts like nullspace, range, independence and rank are all defined in the obvious way for vector spaces over $\mathbb{Z}_2$. Although we won't consider them in this course, there are many important applications of vector spaces and linear dynamical systems over $\mathbb{Z}_2$. In this problem you will explore one simple example: block codes. *Linear block codes.* Suppose $x \in \mathbb{Z}_2^n$ is a Boolean vector we wish to transmit over an unreliable channel. In a linear block code, the vector $y = Gx$ is formed, where $G \in \mathbb{Z}_2^{m\times n}$ is the *coding matrix*, and $m > n$. Note that the vector y is 'redundant'; roughly speaking we have *coded* an n-bit vector as a (larger) m-bit vector. This is called an (n, m) code. The coded vector y is transmitted over the channel; the received signal $\hat{y}$ is given by

$$\hat{y} = y + v,$$

where v is a noise vector (which usually is zero). This means that when $v_i = 0$, the ith bit is transmitted correctly; when $v_i = 1$, the ith bit is changed during transmission. In a *linear decoder*, the received signal is multiplied by another matrix: $\hat{x} = H\hat{y}$, where $H \in \mathbb{Z}_2^{n\times m}$. One reasonable requirement is that if the transmission is perfect, *i.e.*, $v = 0$, then the decoding is perfect, *i.e.*, $\hat{x} = x$. This holds if and only if H is a left inverse of G, *i.e.*, $HG = I_n$, which we assume to be the case.

a) What is the practical significance of range(G)?

b) What is the practical significance of null(H)?

c) A one-bit *error correcting code* has the property that for any noise v with one component equal to one, we still have $\hat{x} = x$. Consider $n = 3$. Either design a one-bit error correcting linear block code with the smallest possible m, or explain why it cannot be done. (By design we mean, give G and H explicitly and verify that they have the required properties.)

Remark: linear decoders are never used in practice; there are far better nonlinear ones.

3.350. Right inverses. This problem concerns the specific matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

This matrix is full rank (*i.e.*, its rank is 3), so there exists at least one right inverse. In fact, there are many right inverses of A, which opens the possibility that we can seek right inverses that in addition have other properties. For each of the cases below, either find a specific matrix $B \in \mathbb{R}^{5\times 3}$ that satisfies $AB = I$ and the given property, or explain why there is no such B. In cases where there is a right inverse B with the required property, you must briefly explain how you found your B. You must also attach a printout of your Julia script or Jupyter notebook showing the verification that $AB = I$. (We'll be very angry if we have to type in your 5×3 matrix into Julia to check it.) When there is no right inverse with the given property, briefly explain why there is no such B.

a) The second row of B is zero.

b) The nullspace of B has dimension one.

c) The third column of B is zero.

d) The second and third rows of B are the same.

e) B is upper triangular, *i.e.*, $B_{ij} = 0$ for $i > j$.

f) B is lower triangular, *i.e.*, $B_{ij} = 0$ for $i < j$.