EE263 Homework 6

Summer 2022

As of our July 28th lecture, we have not seen all of the linear dynamical systems part of this homework. We will discuss this material on Tuesday, August 2.

9.1360. A simple population model. We consider a certain population of fish (say) each (yearly) season. $x(t) \in \mathbb{R}^3$ will describe the population of fish at year $t \in \mathbb{Z}$, as follows:

- $x_1(t)$ denotes the number of fish less than one year old
- $x_2(t)$ denotes the number of fish between one and two years old
- $x_3(t)$ denotes the number of fish between two and three years

(We will ignore the fact that these numbers are integers.) The population evolves from year t to year $t+1$ as follows.

- The number of fish less than one year old in the next year $(t+1)$ is equal to the total number of offspring born during the current year. Fish that are less than one year old in the current year (t) bear no offspring. Fish that are between one and two years old in the current year (t) bear an average of 2 offspring each. Fish that are between two and three years old in the current year (t) bear an average of 1 offspring each.
- 40% of the fish less than one year old in the current year (t) die; the remaining 60% live on to be between one and two years old in the next year $(t+1)$.
- 30% of the one-to-two year old fish in the current year die, and 70% live on to be two-to-three year old fish in the next year.
- All of the two-to-three year old fish in the current year die.

Express the population dynamics as an autonomous linear system with state $x(t)$, *i.e.*, in the form $x(t+1) = Ax(t)$. **Remark:** this example is silly, but more sophisticated population dynamics models are very useful and widely used.

9.1460. Linear dynamical system with constant input. We consider the system $\dot{x} = Ax + b$, with $x(t) \in \mathbb{R}^n$. A vector x_e is an equilibrium point if $0 = Ax_e + b$. (This means that the constant trajectory $x(t) = x_e$ is a solution of $\dot{x} = Ax + b$.)

a) When is there an equilibrium point?

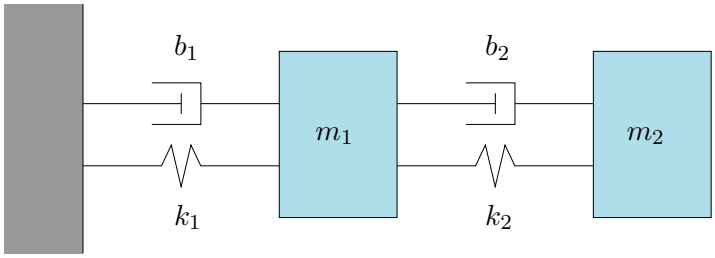
b) When are there multiple equilibrium points?

c) When is there a unique equilibrium point?

d) Now suppose that x_e is an equilibrium point. Define $z(t) = x(t) - x_e$. Show that $\dot{z} = Az$. From this, give a general formula for $x(t)$ (involving x_e, $\exp(tA)$, $x(0)$).

e) Show that if all eigenvalues of A have negative real part, then there is exactly one equilibrium point x_e, and for any trajectory $x(t)$, we have $x(t) \to x_e$ as $t \to \infty$.

9.1490. System in a box. You are given a mysterious box containing two unit masses connected via springs and dampers as follows.



The equations of motion for this system are

$$\ddot{q}_1 = -k_1 q_1 + k_2(q_2 - q_1) - b_1\dot{q}_1 + b_2(\dot{q}_2 - \dot{q}_1)$$
$$\ddot{q}_2 = -k_2(q_2 - q_1) - b_2(\dot{q}_2 - \dot{q}_1)$$

where $k_i > 0$ are spring constants, $b_i > 0$ are damping constants, and q_i is the displacement of mass i. Both masses are $m_i = 1$.

a) This mysterious box can be modeled as a continuous-time linear dynamical system

$$\dot{x} = Ax$$

Find A in terms of k_1, k_2, b_1, b_2. Use state $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)$.

b) Use the forward Euler discretization

$$x(t + h) = (I + hA)x(t)$$

to simulate this system, with parameters $k_1 = 1$, $k_2 = 2$, $b_1 = 1$, $b_2 = 3$. Set the initial conditions so that $q_1(0) = 1$, $q_2(0) = 2$, and $\dot{q}_1(0) = \dot{q}_2(0) = 0$. Use time step $h = 0.1$, and simulate on time interval $[0, T]$ with $T = 15$. Plot q_1, q_2, $\dot{q}_1$, and $\dot{q}_2$ (on one plot) as functions of time.

c) Unfortunately your dog ate the documentation for this system in a box so you do not know the parameters k_1, k_2, b_1, b_2. However, we have experimental data, consisting of measurements of $x(t)$ with sample period $h = 0.1$ on time interval $[0, T]$ where $T = 15$. These may be found in the file `box.json`. We will use this data to estimate k_1, k_2, b_1, b_2.

To do this, we will find the matrix A that minimizes

$$\sum_{k=0}^{N-1} \|(I + hA)x(kh) - x(kh + h)\|^2$$

where $N = T/h$, and x is the given data. Explain how you would do this.

d) Apply your method from part (c) to estimate A, and report the estimate you find.

e) Using initial conditions of $x(0)$ in the dataset, and again with $h = 0.1$ and $T = 15$, simulate the system using your estimate of A and plot q_1, q_2, $\dot{q}_1$, and $\dot{q}_2$ (on one plot) as functions of time.

15.2301. Matrix norms and singular values.

a) *Eigenvalues and singular values of a symmetric matrix.* Suppose $A \in \mathbb{R}^{n\times n}$ with $A = A^\top$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, and assume that the eigenvalues are ordered such that $|\lambda_1| \geq \cdots \geq \cdots |\lambda_n|$. Let $\sigma_1, \ldots, \sigma_n$ be the singular values of A; by definition the singular values are ordered such that $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$. How are the eigenvalues and singular values of A related?

b) Suppose $X \in \mathbb{R}^{n\times n}$. Is $\sigma_{\max}(X) \geq \max_{1\leq i\leq n} \sqrt{\sum_{1\leq j\leq n} |X_{ij}|^2}$? Prove or give a counterexample.

c) Suppose $X \in \mathbb{R}^{n\times n}$. Is $\sigma_{\min}(X) \geq \min_{1\leq i\leq n} \sqrt{\sum_{1\leq j\leq n} \mid X_{ij} \mid^2}$? Prove or give a counterexample.

d) Suppose $X \in \mathbb{R}^{n\times n}$. Is $\sigma_{\max}(XY) \leq \sigma_{\max}(X)\sigma_{\max}(Y)$? Prove or give a counterexample.

e) Suppose $X, Y \in \mathbb{R}^{n\times n}$. Is $\sigma_{\min}(XY) \geq \sigma_{\min}(X)\sigma_{\min}(Y)$? Prove or give a counterexample.

f) Suppose $X, Y \in \mathbb{R}^{n\times n}$. Is $\sigma_{\min}(X + Y) \geq \sigma_{\min}(X) - \sigma_{\max}(Y)$? Prove or give a counterexample.

g) Recall that the Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$ is defined to be

$$\|A\|_F = \sqrt{\operatorname{trace}(A^\mathsf{T} A)}.$$

Show that

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Thus, the Frobenius norm is simply the Euclidean norm of a matrix, when we think of the matrix as an element of $\mathbb{R}^{mn}$. Additionally, note that the Frobenius norm is much easier to compute than the spectral norm.

h) Show that if $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal, then

$$\|UA\|_F = \|AV\|_F = \|A\|_F.$$

Thus, multiplication by orthogonal matrices on the left or right does not change the Frobenius norm.

i) Show that

$$\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2},$$

where $\sigma_1, \ldots, \sigma_r$ are the nonzero singular values of A. Use this result to deduce that

$$\sigma_{\max}(A) \leq \|A\|_F \leq \sqrt{r}\sigma_{\max}(A).$$

In particular, we have that $\|Ax\| \leq \|A\|_F \|x\|$ for all $x \in \mathbb{R}^n$.

16.2710. The EE263 search engine. In this problem we examine how linear algebra and low-rank approximations can be used to find matches to a search query in a set of documents. Let's assume we have four documents: **A**, **B**, **C**, and **D**. We want to search these documents for three terms: *piano*, *violin*, and *drum*. We know that:
in **A**, the word *piano* appears 4 times, *violin* 3 times, and *drum* 1 time;
in **B**, the word *piano* appears 6 times, *violin* 1 time, and *drum* 0 times;
in **C**, the word *piano* appears 7 time, *violin* 4 times, and *drum* 39 times; and
in **D**, the word *piano* appears 0 times, *violin* 0 times, and *drum* 5 times.
We can tabulate this as follows:

	A	**B**	**C**	**D**
piano	4	6	7	0
violin	3	1	4	0
drum	1	0	39	5

This information is used to form a *term-by-document* matrix A, where A_{ij} specifies the frequency of the ith term in the jth document, *i.e.*,

$$A = \begin{bmatrix} 4 & 6 & 7 & 0 \\ 3 & 1 & 4 & 0 \\ 1 & 0 & 39 & 5 \end{bmatrix}.$$

Now let q be a *query vector*, with a non-zero entry for each term. The query vector expresses a criterion by which to select a document. Typically, q will have 1 in the entries corresponding to the words we want to search for, and 0 in all other entries (but other weighting schemes are possible.) A simple measure of how relevant document j is to the query is given by the inner product of the jth column of A with q:

$$a_j^T q.$$

However, this criterion is biased towards large documents. For instance, a query for *piano* ($q = [1\ 0\ 0]^T$) by this criterion would return document **C** as most relevant, even though document **B** (and even **A**) is probably much more relevant. For this reason, we use the inner product normalized by the norm of the vectors,

$$\frac{a_j^T q}{\|a_j\|\,\|q\|}.$$

Note that our criterion for measuring how well a document matches the query is now the cosine of the angle between the document and query vectors. Since all entries are non-negative, the cosine is in $[0, 1]$ (and the angle is in $[-\pi/2, \pi/2]$.) Define $\tilde{A}$ and $\tilde{q}$ as normalized versions of A and q (A is normalized column-wise, *i.e.*, each column is divided by its norm.) Then,

$$c = \tilde{A}^T \tilde{q}$$

is a column vector that gives a measure of the relevance of each document to the query. And now, the question. In the file `term_by_doc.json` you are given m search terms, n documents, and the corresponding term-by-document matrix $A \in \mathbb{R}^{m \times n}$. (They were obtained randomly from Stanford's *Portfolio* collection of internal documents from the 1990s.) The variables `term`

and `document` are lists of strings. The string `term[i]` contains the ith word. Each document is specified by its former URL, *i.e.*, the jth document used to be at the URL `document[j]`; the documents are no longer available online, but you might be able to find some of them on some internet archive (like the Wayback Machine) if you're curious. (You don't need to in order to solve the problem.) The matrix entry `A[i,j]` specifies how many times term i appears in document j.

When you specify documents in your results, please just specify the indices, that is, give us `j` rather than `document[j]`.

a) Compute $\tilde{A}$, the normalized term-by-document matrix. Compute and plot the singular values of $\tilde{A}$.

b) Perform a query for the word *students* ($i = 53$) on $\tilde{A}$. What are the 5 top results?

c) We will now consider low-rank approximations of $\tilde{A}$, that is

$$\hat{A}_r = \min_{\hat{A},\, \mathbf{rank}(\hat{A}) \leq r} \|\tilde{A} - \hat{A}\|.$$

Compute $\hat{A}_{32}$, $\hat{A}_{16}$, $\hat{A}_8$, and $\hat{A}_4$. Perform a query for the word *students* on these matrices. Comment on the results.

d) Are there advantages of using low-rank approximations over using the full-rank matrix? (You can assume that a very large number of searches will be performed before the term-by-document matrix is updated.)

Note: Variations and extensions of this idea are actually used in commercial search engines (although the details are closely guarded secrets . . .) Issues in real search engines include the fact that m and n are enormous and change with time. These methods are very interesting because they can recover documents that don't include the term searched for. For example, a search for *automobile* could retrieve a document with no mention of *automobile*, but many references to cars (can you give a justification for this?) For this reason, this approach is sometimes called *latent semantic indexing*.

Julia hints: You may find the command `sortperm` useful. It returns the index permutation that would sort its argument into an ascending order, or if the `rev=true` argument is supplied, into a descending order. Here's some sample code that prints the indices of the two largest elements of a vector `c`:

```
p = sortperm(c, rev=true)
@show p[1:2]                # indices of two largest elements
c_sorted = c[p]             # equivalent of sort(c, rev=true)
@show c[p[1:2]]             # two largest elements of c
```

16.2980. Smoothing. We have a discrete-time signal given by $x \in \mathbb{R}^n$. We get to measure $y \in \mathbb{R}^n$, given by

$$y_i = \sum_{k=-h}^{h} c_k x_{i+k} + w_i \qquad \text{for } i = 1, \ldots, n$$

where w_i is noise. Here we use the convention that $x_i = 0$ for $i < 1$ or $i > n$. That is, y is c convolved with x plus noise. In applications, very often the effect of convolution with c is to smooth or blur x, and we would like to undo this.

The file `regl_data.json` contains c, w and x.

a) In Julia, construct the $n \times n$ matrix such that $y = Ax + w$. Plot the singular values σ_k against k.

b) Plot the first 6 right singular vectors of A (i.e. plot V_{ij} against i for $j = 1, \ldots, 6$.) Explain what you see.

c) Find and plot the least-squares estimate of x given y_{meas}, computng y_{meas} using c, x and w given in `regl_data.json`. Explain what happens.

d) Many of the singular values of A are very small; this means that the measurement in the directions of the corresponding right singular vectors is being swamped by the noise.

If we believe these components are small, we can remove them from our estimate of x altogether by *truncating* the SVD of A and using the truncated SVD to compute the estimate. This is called the *truncated SVD regularization* of least-squares.

Suppose we decided only to keep the first r components. Then truncate by letting $\tilde{V}$ and $\tilde{U}$ be the first r columns of V and U, and letting $\tilde{\Sigma}$ be the top-left $r \times r$ submatrix of Σ. Then we can construct an estimator that ignores the noise components by

$$A_{\text{est}} = \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^T$$

and set

$$x_{\text{est}} = A_{\text{est}} y_{\text{meas}}$$

For values of r in $5, 10, 15, 30, 50$, compute and plot the corresponding estimates of x. Explain what you see.

e) For each r between 1 and 35, compute the norm of the error

$$\|x - x_{\text{est}}\|$$

Plot this against r. Explain what you see.

f) Pick the 'best' r and plot the corresponding estimate.

g) Another approach is to use Tychonov regularization. Find and plot the vector $x_{\text{reg}} \in \mathbb{R}^n$ that minimizes the function

$$\|Ax - y\|^2 + \mu\|x\|^2,$$

where $\mu > 0$ is the regularization parameter. Pick a value of μ that gives a good estimate, in your opinion.

h) The regularized solution is a linear function of y, so it can be expressed as $x_{\text{reg}} = By$ where $B \in \mathbb{R}^{n \times n}$. Express the SVD of B in terms of the SVD of A. To be more specific, let

$$B = \sum_{i=1}^{n} \tilde{\sigma}_i \tilde{u}_i \tilde{v}_i^T$$

denote the SVD of B. Express $\tilde{\sigma}_i$, $\tilde{u}_i$, $\tilde{v}_i$, for $i = 1, \ldots, n$, in terms of σ_i, u_i, v_i, $i = 1, \ldots, n$ (and, possibly, μ). Recall the convention that $\tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_n$.

i) Find the norm of B. Give your answer in terms of the SVD of A (and μ).

j) Find the worst-case relative inversion error, defined as

$$\max_{y \neq 0} \frac{\|ABy - y\|}{\|y\|}.$$

Give your answer in terms of the SVD of A (and μ).