EE263 Homework 7

Summer 2022

13.1950. Minimum energy control. Consider the discrete-time linear dynamical system

$$x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, 2, \ldots$$

where $x(t) \in \mathbb{R}^n$, and the input $u(t)$ is a scalar (hence, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$). The initial state is $x(0) = 0$.

a) Find the matrix $\mathcal{C}_T$ such that

$$x(T) = \mathcal{C}_T \begin{bmatrix} u(T-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}.$$

b) For the remainder of this problem, we consider a specific system with $n = 4$. The dynamics and input matrices are

$$A = \begin{bmatrix} 0.5 & 0.7 & -0.9 & -0.5 \\ 0.4 & -0.7 & 0.1 & 0.3 \\ 0.7 & 0.0 & -0.6 & 0.1 \\ 0.4 & -0.1 & 0.8 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Suppose we want the state to be x_{des} at time T. Consider the desired state

$$x_{\text{des}} = \begin{bmatrix} 0.8 \\ 2.3 \\ -0.7 \\ -0.3 \end{bmatrix}.$$

What is the smallest T for which we can find inputs $u(0), \ldots, u(T-1)$, such that $x(T) = x_{\text{des}}$? What are the corresponding inputs that achieve x_{des} at this minimum time? What is the smallest T for which we can find inputs $u(0), \ldots, u(T-1)$, such that $x(T) = x_{\text{des}}$ for any $x_{\text{des}} \in \mathbb{R}^4$? We'll denote this T by T_{min}.

c) Suppose the energy expended in applying inputs $u(0), \ldots, u(T-1)$ is

$$E(T) = \sum_{t=0}^{T-1} (u(t))^2.$$

For a given T (greater than T_{min}) and x_{des}, how can you compute the inputs which achieve $x(T) = x_{\text{des}}$ with the minimum expense of energy? Consider now the desired state

$$x_{\text{des}} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

For each T ranging from $T_{\min}$ to 30, find the minimum energy inputs that achieve $x(T) = x_{des}$. For each T, evaluate the corresponding input energy, which we denote by $E_{\min}(T)$. Plot $E_{\min}(T)$ as a function of T. (You should include in your solution a description of how you computed the minimum-energy inputs, and the plot of the minimum energy as a function of T. But you don't need to list the actual inputs you computed!)

d) You should observe that $E_{\min}(T)$ is non-increasing in T. Show that this is the case in general (*i.e.*, for any A, B, and x_{des}).

13.2130. Chasing a sea monster. A sea monster is loose in the Pacific Ocean! Your monster-chasing colleague has been measuring the sea monster's movements and has predicted it will surface at m positions $p_i \in \mathbb{R}^2$ at times s_i. Here p_i is the ith column of the matrix P given by

$$P = \begin{bmatrix} 1 & 1.75 & 2.4 & 2 & 0.5 & 0 \\ 0.75 & 0.6 & 1.2 & 2.3 & 0.75 & 0 \end{bmatrix}$$

and the times $s = (2, 5, 8, 11, 17, 20)$. You plan to observe the monster with a drone. Unfortunately the sea monster ate the last two drones you sent and you are almost out of research funding so your drone's sensors are not very good, and the drone must be exactly in the right position to observe the monster.

a) The dynamics of the drone are

$$\ddot{q} = u$$

where $q \in \mathbb{R}^2$ is the position of the drone, and $u \in \mathbb{R}^2$ is an input force. Write this as a linear dynamical system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where $y \in \mathbb{R}^2$ is the position of the drone.

b) We will use sample period h. Assume that the force input is piecewise constant on sample intervals, and construct the exact discretization

$$\begin{aligned} x_d(k+1) &= A_d x_d(k) + B_d u_d(k) \\ y_d(k) &= C_d x_d(k) \end{aligned}$$

where $x_d(k) = x(kh)$, and similarly for y_d and u_d.

c) The drone starts at the origin with zero velocity, and we would like to move the drone so that $y(s_i) = p_i$ for $i = 1, \ldots, m$. We will operate the drone on the time interval $[0, T]$ where $T = s_m$. For convenience, let $N = T/h$. Since drone batteries are limited, we would like to minimize

$$J = \sum_{k=0}^{N-1} \|u_d(k)\|^2$$

Explain in detail how you would solve this problem.

d) Use your method to compute the optimal input u, and plot u versus time. Use $h = 0.1$.

e) Report the optimal value of J that you obtained.

f) Plot the trajectory of the drone. Use axes q_1 and q_2, so that the plot shows the path followed by the drone. Mark on your plot the points p_i where the monster surfaces.

18.2890. Linear dynamical systems for portfolio management. We consider a portfolio of n financial assets (like stocks) and cash, which we manage over T time steps of unit length (e.g. one month). We call $x_t \in \mathbb{R}^{n+1}$ for $t = 1, \ldots, T$ our state vector. The first n elements are our positions in each of the assets, in dollars, and the last element is the dollar amount of cash we hold. Every element of x can be either positive (for long positions) and negative (for short or borrowing). For $t = 1, \ldots, T-1$, the transition from x_t to x_{t+1} is composed of two steps.

- First, the portfolio positions change value because of market returns. Let $\mu \in \mathbb{R}^n_{++}$ be the vector of returns, where $\mathbb{R}^n_{++}$ is the set of all vectors of length n with strictly positive entries. We define the post-return portfolio $\tilde{x}_t$ to be

$$(\tilde{x}_t)_i = \begin{cases} \mu_i (x_t)_i & i = 1, \ldots, n, \\ (x_t)_i & i = n+1. \end{cases}$$

  (Intuitively, cash is unchanged, and the asset positions are multiplied by the corresponding element of the vector of returns.) For simplicity we assume that the vector of returns does not change in time.

- Then we trade. We can exchange any amount of cash for the corresponding amount of any of the assets. Note that the *only* valid trades are cash for asset. If you wish to trade some amount of an asset with the same amount of another asset, you have to perform *two trades*: trade the first asset with cash, and then trade cash with the second asset. (Think carefully about this definition of trade when you formulate the transaction costs.) For example, if we buy $c > 0$ dollars of the first asset and sell $d > 0$ dollars of the second asset the state evolves as

$$x_{t+1} = \tilde{x}_t + \begin{bmatrix} c \\ -d \\ 0 \\ \vdots \\ 0 \\ -(c-d) \end{bmatrix}.$$

Finally, we define the portfolio value $v_t \in \mathbb{R}$ for $t = 1, \ldots, T$ to be

$$v_t = \mathbf{1}^T x_t.$$

a) Formulate the problem as a linear dynamical system of the form

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1.$$

The control vector u_t should have dimension n.

b) Assume that our trades incur quadratic transaction costs with parameter $\rho > 0$. For example, if at time t we buy $c > 0$ dollars of the first asset, and we sell $d > 0$ dollars of the second asset (the example above), then the transaction costs for the transition x_t to x_{t+1} are

$$\rho(c^2 + d^2).$$

(Be careful, they are **not** $\rho(c + d)^2$.) Explain how to solve the problem of maximizing the final value of the portfolio v_T minus the total transaction costs. (The sequence of controls $u_1, \dots, u_{T-1}$ that achieves the maximum should be a function of A, B, and ρ). Use methods from EE263.

c) Apply your method to the following data.

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T = 12;
x_1 = [1000, 1000, 0, 1000, 0, 0];
mu = [1.001, 1.003, 1.004, 1.006, 1.007];
rho = 0.0001;
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What is the final value v_T? What are the total transaction costs? Plot the trajectories of the portfolio positions x_t. (On the same plot you should draw $n+1$ lines, one for each of the assets and cash, with time on the x-axis.)

d) Now assume that we aim to *liquidate* an initial portfolio, which means that at time T we want to have zero positions in any of the n assets and only hold cash. We thus impose the constraint $(x_T)_i = 0$, for $i = 1, \dots, n$. Explain how to solve the problem of maximizing the final portfolio value (in this case, all cash) minus the transaction costs with this additional constraint. Use methods from EE263.

e) Apply your method to the data given above. What is the final value v_T? What are the total transaction costs? Plot the trajectories of the portfolio positions x_t. (On the same plot you should draw $n + 1$ lines, one for each of the assets and cash, with time on the x-axis.)