Searching for Optimal Symplectic Maps

Faculty Mentor: Ely Kerman
Project Leader: Aline Leite, Yefei Zhang
IML Scholars: Eli Berry, Tianyang Ma, Alex Ware, Eleven Yan

University of Illinois at Urbana-Champaign

Illinois Mathematics Lab

ILLINOIS

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Basic Definitions

Definition

A smooth map $F : (M, \omega_M) \to (N, \omega_N)$ of symplectic manifolds is called **symplectic** if $F^* \omega_N = \omega_M$.

- For this project, we are working in $\mathbb{R}^2$, so we have the following equivalent definition.

Definition

A smooth invertible map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is **symplectic** if it preserves area.

Parameterizing Symplectic Maps in $\mathbb{R}^2$

For any smooth maps $f, g : \mathbb{R} \to \mathbb{R}$, we have symplectic maps

$$A : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x, y + \frac{\partial f}{\partial x}(x))$$

$$B : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x + \frac{\partial g}{\partial y}(y), y)$$

Then for $f_1, ..., f_k, g_1, ..., g_k \in C^\infty(\mathbb{R})$, the composition of maps

$$F_{AB} = A_1 \circ B_1 \circ \cdots \circ A_k \circ B_k$$

is also symplectic.

Symplectic Maps using Polynomials

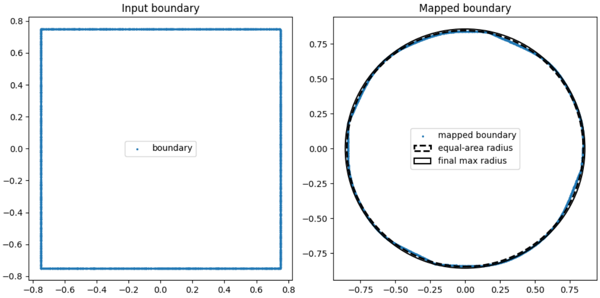
- A good choice of maps in $C^{\infty}(\mathbb{R})$ is polynomials of fixed degree, d.
- When d and/or k is large, this gives us a large family of maps to parameterize over.
- In fact, any symplectic map can be approximated using this method.

Project Goal

- Goal in $\mathbb{R}^2$: Given a region, search for symplectic maps that move the region into the smallest possible circle around origin.
- Method: Choose points $p_1, \ldots, p_L$ on the boundary of the region and use a gradient descent algorithm to minimize the function
$$\max_{\ell} \|F_{AB}(p_\ell)\|^2.$$

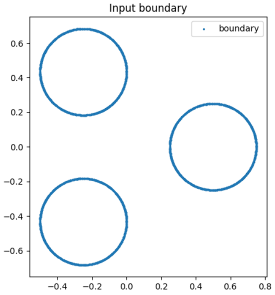
Square Example

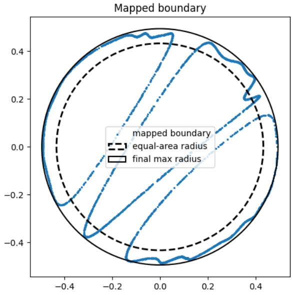
Square Animation



3 Circle Example

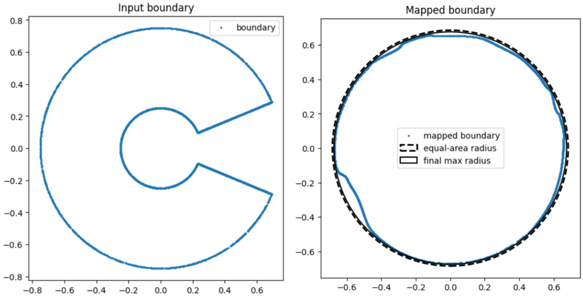
3 Circle Animation





Keyhole Example

Keyhole Animation



Higher Dimensions

- Symplectic maps have additional structure in higher dimensions.
- They still are volume preserving.
- We will look for symplectic maps in $\mathbb{R}^4$ that optimize for various constraints.

Symplectic Maps in $\mathbb{R}^4$

Concretely, a smooth map $F : \mathbb{R}^4 \to \mathbb{R}^4$ is symplectic if

$$(dF)^T \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} dF = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Obstructions to Embedding

Theorem

(Gromov) If $r > 1$, then there is no symplectic embedding of $B^4(r)$ into $Z(1) = \{x_1 + y_1 \leq 1\} \subset \mathbb{R}^4$.

working note/figs/Nonsqueezing.png

$\mathbb{R}^4$ Case

$\mathbb{R}^4$ Animation