Searching for Optimal Symplectic Maps

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Introduction

Goal: Find symplectic maps that embed regions into the smallest possible ball.

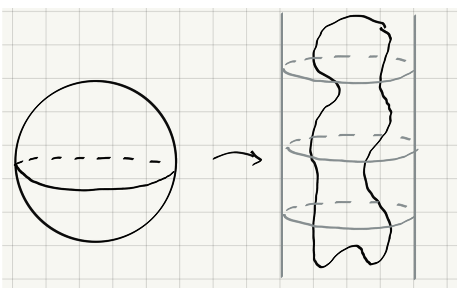
Definition. *A smooth invertible map $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is* ***symplectic*** *if*

$$F^*\left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

- Symplectic maps preserve volume. For $n > 1$, the set of symplectic maps is strictly smaller than the set of volume preserving ones.

Theorem (Gromov). *If $r > 1$, then there is no symplectic map of $\mathbb{R}^{2n}$ that takes $B^{2n}(r)$, the ball of radius r, into $Z(1) = \{x_1^2 + y_1^2 \leq 1\} \subset \mathbb{R}^{2n}$.*

On the other hand, it is easy to find volume preserving map that map $B^{2n}(r)$ into $Z(1)$ for any value of r.



- The following approximation result is crucial to our project.

Theorem (Turaev). *Every symplectic map in $\mathbb{R}^{2n}$ can be approximated by a composition of Hénon maps of the form*

$$(x, y) \mapsto (y + \alpha, -x + \nabla V(y)),$$

where $\alpha \in \mathbb{R}^n$ is a constant vector and $V : \mathbb{R}^n \to \mathbb{R}$ is a **polynomial**.

Optimization Framework. For a fixed k and fixed degree d we consider the landscape of symplectic maps that are compositions of k Hénon maps whose polynomial functions have degree d. The constants of the maps and coefficients of their polynomials are the weights we train, Given a region in $\mathbb{R}^{2n}$ we take N points on the boundary, apply a map in our landscape and evaluate a loss function which measures how close the points are to the origin. We then train the weights using the Adams optimization algorithm.

2D Embeddings

We first tested our optimization methods in dimension two. Here every domain should get mapped to the round disk with the same area.

Sample results

Example 1 (Keyhole). *Our 2D algorithm mapped the non-convex keyhole domain into an almost circular shape. The dashed circle is the theoretical equal-area radius; our mapped boundary aligns closely with it.*

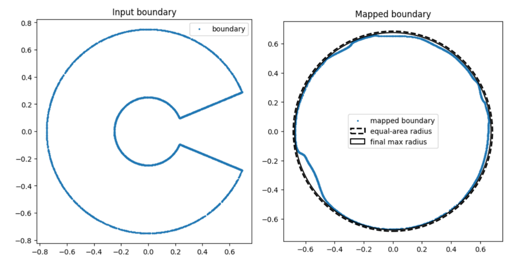


Figure 1: *Keyhole*

Example 2 (Three Disjoint Circles). *Our algorithm maps three disjoint circles into a nearly circular configuration while preserving area. The dashed circle shows the theoretical equal-area radius, and the mapped boundary closely matches it.*

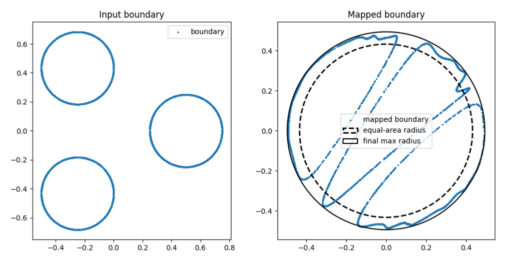


Figure 2: *Three Disjoint Circles*

4D Embeddings

Standard domains

In $\mathbb{R}^4$ we focused on the following types of domains:

Ellipsoids $\quad E(1, a) = \left\{\pi(x_1^2 + y_1^2) + \frac{\pi(x_2^2+y_2^2)}{a}) \leq 1\right\}$

Polydisks $\quad P(1, a) = \left\{\pi(x_1^2 + y_1^2) \leq 1, \pi(x_2^2 + y_2^2) \leq a\right\}$

Lagrangian tori $\quad L(1, a) = \left\{\pi(x_1^2 + y_1^2) = 1, \pi(x_2^2 + y_2^2) = a\right\}$

Goal

For each domain D our goal is to find a symplectic map which takes D into the smallest ball of the form $B(\lambda) = \{\pi(x_1^2 + y_1^2 + x_2^2 + y_2^2) \leq \lambda\}$. In other words, setting

$$c(D) = \inf\{\lambda \mid D \text{ symplectically embedds into } B(\lambda)\}$$

our goal is to find a symplectic map that takes D into $B(c(D))$. For many domains D, $c(D)$ is known and we have a target to shoot for.

4D Algorithm:

In 4D we made several refinements to the search algorithm. We still used the Adam Optimizer, but we rewrote it in the machine learning library JAX. We also redefined the loss function. Given a map Φ in our landscape and a sample of N points p_i on the boundary of D we replaced the loss function

$$L(\Phi) = \max_i \|\Phi(p_i)\|$$

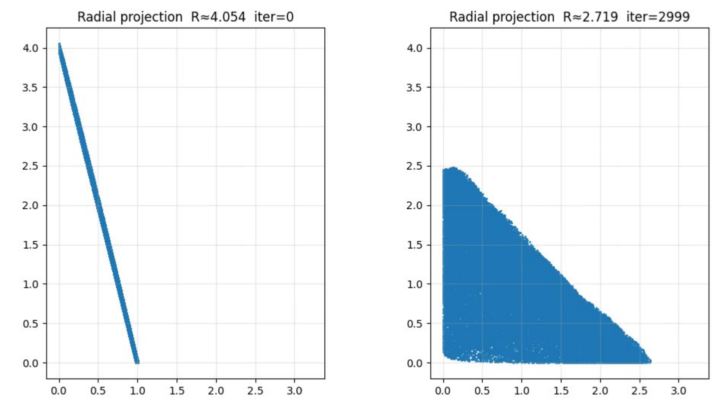
with the function

$$LSE(\Phi) = \frac{1}{\tau}\log\left(\sum_{i=1}^{n} \exp(\tau||\Phi(p_i)||)\right)$$

for some fixed $\tau > 0$. This is a smoothing of the max function which provides stable, usable gradients for optimization.

4D Results:

The numbers $c(E(1, a))$ were computed in a famous paper of McDuff and Schlenk. In particular, they showed that $c(E(1, 4)) = 2$ and $c(E(1, 6.25)) = 2.5$. Our algorithm found concrete symplectic maps which achieved values for these cases of 2.19 and 3.05, respectively. (The maps of McDuff and Schlenk are far from explicit.)



Future Directions

Our algorithm needs to be refined in order to approximate optimal symplectic embeddings of Lagrangian tori and polydisks.

Polydisks

For polydisks we have the following theorem as guidance.

Theorem (Schlenk). $c(P(1, a)) \leq \frac{a}{2} + 2$ *for* $2 \leq a \leq 6$.

Our current algorithm struggles to get close to these bounds.

$P(1, 6) \hookrightarrow B(6.18).$

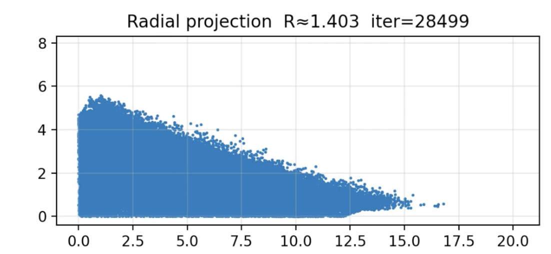


Figure: Radial projection for $P(1, 6)$

Lagrangian Tori

For Lagrangian we have the following complete theorem to shoot for.

Theorem (Hind-Opshtein). $c(L(1, a)) = 3$ *for all* $a \geq 2$.

Again, our current algorithm struggles to get close to these bounds.

$L(1, 6) \hookrightarrow B(7.02).$

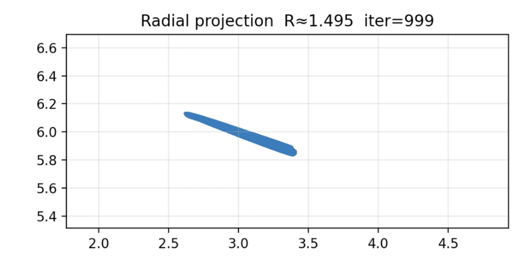


Figure: Radial projection for $L(1, 6)$

Future goals: Search for maps which take $P(1, 6)$ into $B(\lambda)$ for λ below the known upper bound of 5. Improve the algorithm so that it comes close to matching the performance of the Hind-Opshtein map.

References

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