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From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators

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Abstract

The Kuramoto model describes a large population of coupled limit-cycle oscillators whose natural frequencies are drawn from some prescribed distribution. If the coupling strength exceeds a certain threshold, the system exhibits a phase transition: some of the oscillators spontaneously synchronize, while others remain incoherent. The mathematical analysis of this bifurcation has proved both problematic and fascinating. We review 25 years of research on the Kuramoto model, highlighting the false turns as well as the successes, but mainly following the trail leading from Kuramoto's work to Crawford's recent contributions. It is a lovely winding road, with excursions through mathematical biology, statistical physics, kinetic theory, bifurcation theory, and plasma physics. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the 1990s, Crawford wrote a series of papers about the Kuramoto model of coupled oscillators [1–3]. At first glance, the papers look technical, maybe even a bit intimidating.

For instance, take a look at "Amplitude expansions for instabilities in populations of globally coupled oscillators", his first paper on the subject [1]. Here, Crawford racks up 200 numbered equations as he calmly plows through a center manifold calculation for a nonlinear partial integro-differential equation.

Technical, yes, but a technical tour de force. Beneath the surface, there is a lot at stake. In his modest, methodical way, Crawford illuminated some problems that had appeared murky for about two decades.

My goal here is to set Crawford's work in context and to give a sense of what he accomplished. The larger setting is the story of the Kuramoto model [4–9]. It is an ongoing tale full of twists and turns, starting with Kuramoto's ingenious analysis in 1975 (which raised more questions than it answered) and culminating with Crawford's insights. Along the way, I will point out some problems that remain unsolved to this day, and tell a few stories about the various people who have worked on the Kuramoto model, including how Crawford himself got hooked on it.

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2. Background

The Kuramoto model was originally motivated by the phenomenon of collective synchronization, in which an enormous system of oscillators spontaneously locks to a common frequency, despite the inevitable differences in the natural frequencies of the individual oscillators [10–13]. Biological examples include networks of pacemaker cells in the heart [14,15]; circadian pacemaker cells in the suprachiasmatic nucleus of the brain (where the individual cellular frequencies have recently been measured for the first time [16]); metabolic synchrony in yeast cell suspensions [17,18]; congregations of synchronously flashing fireflies [19,20]; and crickets that chirp in unison [21]. There are also many examples in physics and engineering, from arrays of lasers [22,23] and microwave oscillators [24] to superconducting Josephson junctions [25,26].

Collective synchronization was first studied mathematically by Wiener [27,28], who recognized its ubiquity in the natural world, and who speculated that it was involved in the generation of alpha rhythms in the brain. Unfortunately Wiener's mathematical approach based on Fourier integrals [27] has turned out to be a dead end.

A more fruitful approach was pioneered by Winfree [10] in his first paper, just before he entered graduate school. He formulated the problem in terms of a huge population of interacting limit-cycle oscillators. As stated, the problem would be intractable, but Winfree intuitively recognized that simplifications would occur if the coupling were weak and the oscillators nearly identical. Then one can exploit a separation of timescales: on a fast timescale, the oscillators relax to their limit cycles, and so can be characterized solely by their phases; on a long timescale, these phases evolve because of the interplay of weak coupling and slight frequency differences among the oscillators. In a further simplification, Winfree supposed that each oscillator was coupled to the collective rhythm generated by the whole population, analogous to a mean-field approximation in physics. His model is

$$\dot{\theta}_i = \omega_i + \left(\sum_{j=1}^{N} X(\theta_j) \right) Z(\theta_i), \qquad i = 1, \ldots, N,$$

where θ_i denotes the phase of oscillator i and ω_i its natural frequency. Each oscillator j exerts a phase-dependent influence $X(\theta_j)$ on all the others; the corresponding response of oscillator i depends on its phase θ_i, through the sensitivity function $Z(\theta_i)$.

Using numerical simulations and analytical approximations, Winfree discovered that such oscillator populations could exhibit the temporal analog of a phase transition. When the spread of natural frequencies is large compared to the coupling, the system behaves incoherently, with each oscillator running at its natural frequency. As the spread is decreased, the incoherence persists until a certain threshold is crossed — then a small cluster of oscillators suddenly freezes into synchrony.

This cooperative phenomenon apparently made a deep impression on Kuramoto. As he wrote in a paper with his student Nishikawa ([8], p. 570):

> "... Prigogine's concept of time order [29], which refers to the spontaneous emergence of rhythms in nonequilibrium open systems, found its finest example in this transition phenomenon... It seems that much of fresh significance beyond physiological relevance could be derived from Winfree's important finding (in 1967) after our experience of the great advances in nonlinear dynamics over the last two decades."

Kuramoto himself began working on collective synchronization in 1975. His first paper on the topic [4] was a brief note announcing some exact results about what would come to be called the Kuramoto model. In later years, he would keep wrestling with that analysis, refining and clarifying the presentation each time, but also raising thorny new questions too [5–9].

3. Kuramoto model

3.1. Governing equations

Kuramoto [5] put Winfree's intuition about phase models on a firmer foundation. He used the perturbative method of averaging to show that for any system of weakly coupled, nearly identical limit-cycle oscillators, the long-term dynamics are given by phase equations of the following universal form:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), \quad i = 1, \ldots, N.$$

The interaction functions Γ_{ij} can be calculated as integrals involving certain terms from the original limit-cycle model (see Section 5.2 of [5] for details).

Even though the reduction to a phase model represents a tremendous simplification, these equations are still far too difficult to analyze in general, since the interaction functions could have arbitrarily many Fourier harmonics and the connection topology is unspecified — the oscillators could be connected in a chain, a ring, a cubic lattice, a random graph, or any other topology.

Like Winfree, Kuramoto recognized that the mean-field case should be the most tractable. The *Kuramoto model* corresponds to the simplest possible case of equally weighted, all-to-all, purely sinusoidal coupling:

$$\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N}\sin(\theta_j - \theta_i),$$

where $K \geq 0$ is the coupling strength and the factor $1/N$ ensures that the model is well behaved as $N \to \infty$.

The frequencies ω_i are distributed according to some probability density $g(\omega)$. For simplicity, Kuramoto assumed that $g(\omega)$ is unimodal and symmetric about its mean frequency Ω, i.e., $g(\Omega + \omega) = g(\Omega - \omega)$ for all ω, like a Gaussian distribution. Actually, thanks to the rotational symmetry in the model, we can set the mean frequency to $\Omega = 0$ by redefining $\theta_i \to \theta_i + \Omega t$ for all i, which corresponds to going into a rotating frame at frequency Ω. This leaves the governing equations

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N \tag{3.1}$$

invariant, but effectively subtracts Ω from all the ω_i and therefore shifts the mean of $g(\omega)$ to zero. So from now on,

$$g(\omega) = g(-\omega)$$

for all ω, and the ω_i denote deviations from the mean frequency Ω. We also suppose that $g(\omega)$ is nowhere increasing on $[0,\infty)$, in the sense that $g(\omega) \geq g(v)$ whenever $\omega \leq v$; this formalizes what we mean by "unimodal".

3.2. Order parameter

To visualize the dynamics of the phases, it is convenient to imagine a swarm of points running around the unit circle in the complex plane. The complex order parameter [5]

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j} \tag{3.2}$$

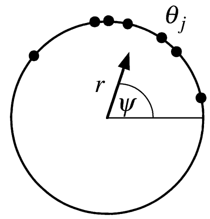

Fig. 1. Geometric interpretation of the order parameter (3.2). The phases θ_j are plotted on the unit circle. Their centroid is given by the complex number $r\,e^{i\psi}$, shown as an arrow.

is a macroscopic quantity that can be interpreted as the collective rhythm produced by the whole population. It corresponds to the centroid of the phases. The radius $r(t)$ measures the phase coherence, and $\psi(t)$ is the average phase (Fig. 1).

For instance, if all the oscillators move in a single tight clump, we have $r \approx 1$ and the population acts like a giant oscillator. On the other hand, if the oscillators are scattered around the circle, then $r \approx 0$; the individual oscillations add incoherently and no macroscopic rhythm is produced.

Kuramoto noticed that the governing equation

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

can be rewritten neatly in terms of the order parameter, as follows. Multiply both sides of the order parameter equation by $e^{-i\theta_i}$ to obtain

$$r\,e^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N}e^{i(\theta_j-\theta_i)}.$$

Equating imaginary parts yields

$$r\sin(\psi - \theta_i) = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i).$$

Thus (3.1) becomes

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), \quad i = 1, \dots, N. \tag{3.3}$$

In this form, the mean-field character of the model becomes obvious. Each oscillator appears to be uncoupled from all the others, although of course they are interacting, but only through the mean-field quantities r and ψ. Specifically, the phase θ_i is pulled toward the mean phase ψ, rather than toward the phase of any individual oscillator. Moreover, the effective strength of the coupling is proportional to the coherence r. This proportionality sets up a positive feedback loop between coupling and coherence: as the population becomes more coherent, r grows and so the effective coupling Kr increases, which tends to recruit even more oscillators into the synchronized pack. If the coherence is further increased by the new recruits, the process will continue; otherwise, it becomes self-limiting. Winfree [10] was the first to discover this mechanism underlying spontaneous synchronization, but it stands out especially clearly in the Kuramoto model.

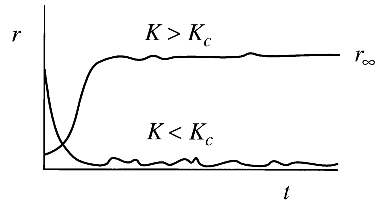

Fig. 2. Schematic illustration of the typical evolution of $r(t)$ seen in numerical simulations of the Kuramoto model (3.1).

3.3. Simulations

If we integrate the model numerically, how does $r(t)$ evolve? For concreteness, suppose we fix $g(\omega)$ to be a Gaussian or some other density with infinite tails, and vary the coupling K. Simulations show that for all K less than a certain threshold K_c, the oscillators act as if they were uncoupled: the phases become uniformly distributed around the circle, starting from any initial condition. Then $r(t)$ decays to a tiny jitter of size $O(N^{-1/2})$, as expected for any random scatter of N points on a circle (Fig. 2).

But when K exceeds K_c, this *incoherent state* becomes unstable and $r(t)$ grows exponentially, reflecting the nucleation of a small cluster of oscillators that are mutually synchronized, thereby generating a collective oscillation. Eventually $r(t)$ saturates at some level $r_\infty < 1$, though still with $O(N^{-1/2})$ fluctuations.

At the level of the individual oscillators, one finds that the population splits into two groups: the oscillators near the center of the frequency distribution lock together at the mean frequency Ω and co-rotate with the average phase $\psi(t)$, while those in the tails run near their natural frequencies and drift relative to the synchronized cluster. This mixed state is often called *partially synchronized*. With further increases in K, more and more oscillators are recruited into the synchronized cluster, and r_∞ grows as shown in Fig. 3.

The numerics further suggest that r_∞ depends only on K, and not on the initial condition. In other words, it seems there is a globally attracting state for each value of K.

3.4. Puzzles

These numerical results cry out for explanation. A good theory should provide formulas for the critical coupling K_c and for the coherence $r_\infty(K)$ on the bifurcating branch. The theory should also explain the apparent stability of the zero branch below threshold and the bifurcating branch above threshold. Ideally, one would like to formulate and prove *global* stability results, since the numerical simulations give no hint of any other attractors beyond those seen here. Even more ambitiously, can one formulate and prove some convergence results as $N \to \infty$?

As we will see below, the first few of these problems have been solved, while the rest remain open. Specifically, Kuramoto derived exact results for K_c and $r_\infty(K)$, Mirollo and I solved the linear stability problem for the zero

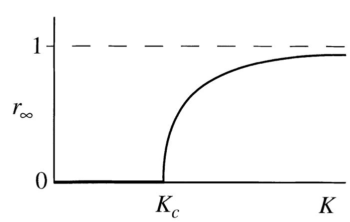

Fig. 3. Dependence of the steady-state coherence r_∞ on the coupling strength K.

branch, and Crawford extended those results to the weakly nonlinear case. But we still do not know how to show that the bifurcating branch is linearly stable along its entire length (if it truly is), and nobody has even touched the problems of global stability and convergence.

4. Kuramoto's analysis

In his earliest work, Kuramoto analyzed his model without the benefit of simulations — he guessed the correct long-term behavior of the solutions in the limit $N \to \infty$, using symmetry considerations and marvelous intuition. Specifically, he sought steady solutions, where $r(t)$ is constant and $\psi(t)$ rotates uniformly at frequency Ω. By going into the rotating frame with frequency Ω and choosing the origin of this frame correctly, one can set $\psi \equiv 0$ without loss of generality.

Then the governing equation (3.3) becomes

$$\dot{\theta}_i = \omega_i - Kr \sin\theta_i, \quad i = 1, \ldots, N. \tag{4.1}$$

Since r is assumed constant in (4.1), all the oscillators are effectively independent — that is the beauty of steady solutions. The strategy now is to solve for the resulting motions of all the oscillators (which will depend on r as a parameter). These motions in turn imply values for r and ψ which must be consistent with the values originally assumed. This *self-consistency* condition is the key to the analysis.

The solutions of (4.1) exhibit two types of long-term behavior, depending on the size of $|\omega_i|$ relative to Kr. The oscillators with $|\omega_i| \leq Kr$ approach a stable fixed point defined implicitly by

$$\omega_i = Kr \sin\theta_i, \tag{4.2}$$

where $|\theta_i| \leq \frac{1}{2}\pi$. These oscillators will be called "locked" because they are phase-locked at frequency Ω in the original frame. In contrast, the oscillators with $|\omega_i| > Kr$ are "drifting" — they run around the circle in a nonuniform manner, accelerating near some phases and hesitating at others, with the inherently fastest oscillators continually lapping the locked oscillators, and the slowest ones being lapped by them. The locked oscillators correspond to the center of $g(\omega)$ and the drifting oscillators correspond to the tails, as expected.

At this stage, Kuramoto has neatly explained why the population splits into two groups. But before we get too complacent, notice that the existence of the drifting oscillators would seem to contradict the original assumption that r and ψ are constant. How can the centroid of the population remain constant with all those drifting oscillators buzzing around the circle?

Kuramoto deftly avoided this problem by demanding that the drifting oscillators form a stationary distribution on the circle. Then the centroid stays fixed even though individual oscillators continue to move. Let $\rho(\theta, \omega)\, \mathrm{d}\theta$ denote the fraction of oscillators with natural frequency ω that lie between θ and $\theta + \mathrm{d}\theta$. Stationarity requires that $\rho(\theta, \omega)$ be inversely proportional to the speed at θ; oscillators pile up at slow places and thin out at fast places on the circle. Hence

$$\rho(\theta, \omega) = \frac{C}{|\omega - Kr \sin\theta|}. \tag{4.3}$$

The normalization constant C is determined by $\int_{-\pi}^{\pi} \rho(\theta, \omega)\, \mathrm{d}\theta = 1$ for each ω, which yields

$$C = \frac{1}{2\pi} \sqrt{\omega^2 - (Kr)^2}.$$

Next, we invoke the self-consistency condition: the constant value of the order parameter must be consistent with that implied by (3.2). Using angular brackets to denote population averages, we have

$$\langle \mathrm{e}^{\mathrm{i}\theta}\rangle = \langle \mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{lock}} + \langle \mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{drift}}.$$

Since $\psi = 0$ by assumption, $\langle \mathrm{e}^{\mathrm{i}\theta}\rangle = r\,\mathrm{e}^{\mathrm{i}\psi} = r$. Thus,

$$r = \langle \mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{lock}} + \langle \mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{drift}}.$$

We evaluate the locked contribution first. In the locked state, $\sin\theta^* = \omega/Kr$ for all $|\omega| \le Kr$. As $N \to \infty$, the distribution of locked phases is symmetric about $\theta = 0$ because $g(\omega) = g(-\omega)$; there are just as many oscillators at θ^* as at $-\theta^*$. Hence $\langle \sin\theta\rangle_{\mathrm{lock}} = 0$ and

$$\langle \mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{lock}} = \langle \cos\theta\rangle_{\mathrm{lock}} = \int_{-Kr}^{Kr} \cos\theta(\omega)\, g(\omega)\,\mathrm{d}\omega,$$

where $\theta(\omega)$ is defined implicitly by (4.2). Changing variables from ω to θ yields

$$\langle \mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{lock}} = \int_{-\pi/2}^{\pi/2} \cos\theta\, g(Kr\sin\theta) Kr\cos\theta\,\mathrm{d}\theta = Kr\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(Kr\sin\theta)\,\mathrm{d}\theta.$$

Now, consider the drifting oscillators. They contribute

$$\langle \mathrm{e}^{\mathrm{i}\theta}\rangle_{\mathrm{drift}} = \int_{-\pi}^{\pi}\int_{|\omega|>Kr} \mathrm{e}^{\mathrm{i}\theta}\rho(\theta,\omega)\, g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta.$$

It turns out that this integral vanishes. This follows from $g(\omega) = g(-\omega)$ and the symmetry $\rho(\theta+\pi, -\omega) = \rho(\theta,\omega)$ implied by (4.3).

Therefore, the self-consistency condition reduces to

$$r = Kr\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(Kr\sin\theta)\,\mathrm{d}\theta. \tag{4.4}$$

Eq. (4.4) always has a trivial *zero solution* $r = 0$, for any value of K. This corresponds to a completely incoherent state with $\rho(\theta,\omega) = 1/2\pi$ for all θ, ω. A second branch of solutions, corresponding to partially synchronized states, satisfies

$$1 = K\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(Kr\sin\theta)\,\mathrm{d}\theta. \tag{4.5}$$

This branch bifurcates continuously from $r = 0$ at a value $K = K_{\mathrm{c}}$ obtained by letting $r \to 0^+$ in (4.5). Thus,

$$K_{\mathrm{c}} = \frac{2}{\pi g(0)},$$

which is Kuramoto's exact formula for the critical coupling at the onset of collective synchronization. By expanding the integrand in (4.5) in powers of r, we find that the bifurcation is supercritical if $g''(0) < 0$ (the generic case for smooth, unimodal, even densities $g(\omega)$) and it is subcritical if $g''(0) > 0$. Near onset, the amplitude of the bifurcating branch obeys the square-root scaling law:

$$r \approx \sqrt{\frac{16}{\pi K_{\mathrm{c}}^3}}\sqrt{\frac{\mu}{-g''(0)}}, \tag{4.6}$$

where

$$\mu = \frac{K - K_{\mathrm{c}}}{K_{\mathrm{c}}}$$

is the normalized distance above threshold. For the special case of a Lorentzian or Cauchy density

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)}, \tag{4.7}$$

Kuramoto [4,5] integrated (4.5) exactly to obtain

$$r = \sqrt{1 - \frac{K_c}{K}}$$

for all $K \geq K_c$. This formula was later shown to match the results of numerical simulations [6,7].

5. Two unsolved problems

5.1. Finite-N fluctuations

In the last of her three Bowen lectures at Berkeley in 1986, Kopell pointed out that Kuramoto's argument contained a few intuitive leaps that were far from obvious — in fact, they began to seem paradoxical the more one thought about them — and she wondered whether one could prove some theorems that would put the analysis on firmer footing. In particular, she wanted to redo the analysis rigorously for large but finite N, and then prove a convergence result as $N \to \infty$.

But it would not be easy. Whereas Kuramoto's approach had relied on the assumption that r was strictly constant, Kopell emphasized that nothing like that could be strictly true for any finite N. Think about the simple case $K = 0$. Then $\dot{\theta}_i = \omega_i$ and every trajectory is dense on the N-torus, at least for the generic case where the frequencies are rationally independent. But then $r(t)$ eventually passes through every possible value between 0 and 1, completely unlike the constant value $r \equiv 0$ implied by Kuramoto's argument! Admittedly, $r(t)$ would spend nearly all its time very close to zero, at $r = \mathrm{O}(N^{-1/2}) \ll 1$, and only blip up extremely rarely — in that sense $r \equiv 0$ is practically correct. But how can this rough idea be made precise? When $K \neq 0$, the situation would become still more difficult, because now there would be *three* subpopulations of oscillators — locked and drifting ones as in Kuramoto's analysis, but also some fuzzy oscillators between them, determined by the ever-fluctuating boundary $\omega_i \approx Kr(t)$.

Kopell's suggestion was to try to prove something like this: For large N, for most initial conditions, and for most realizations of the ω_i, the coherence $r(t)$ approaches the Kuramoto value $r_\infty(K)$ and stays within $\mathrm{O}(N^{-1/2})$ of it for a large fraction of the time. Around the same time, Daido [30–33], and Kuramoto and Nishikawa [8,9] began exploring the finite-N fluctuations using computer simulations and physical arguments. It appears that the fluctuations are indeed $\mathrm{O}(N^{-1/2})$ except very close to K_c, where they may be amplified [30–33].

Still, the issue of fluctuations remains wide open mathematically. As of March 2000, there are no rigorous convergence results about the finite-N behavior of the Kuramoto model.

5.2. Stability

The other major issue left unresolved by Kuramoto's analysis concerns the stability of the steady solutions. It was in this arena that Crawford ultimately contributed so much, and so we will focus on it for the rest of this paper. Kuramoto was well aware of the stability problem; he writes [5] (p. 74):

> "One may expect that negative μ (i.e., weaker coupling) makes the zero solution stable, and positive μ (i.e., stronger coupling) unstable. Surprisingly enough, this seemingly obvious fact seems difficult to prove. The difficulty here comes from the fact that an infinitely large number of phase configurations $\{\theta_i, i = 1, \ldots, N\}$ belong to an identical "macroscopic" state specified by a given value of r."

He also remarks that it “appears to be difficult to prove” that the branch of partially synchronized states is stable when the bifurcation is supercritical, and unstable when it is subcritical.

6. Stability theories of Kuramoto and Nishikawa

Kuramoto and Nishikawa [8,9] were the first to tackle the stability problem. They proposed two different theories, both based on plausible physical reasoning, but neither of which ultimately turned out to be correct. Nevertheless, it is interesting to look back at their pioneering ideas, partly because they came tantalizingly close to the truth, and partly to remind us how subtle the stability problem appeared at the time.

6.1. First theory

In their first approach, Kuramoto and Nishikawa [8] tried to derive an evolution equation for $r(t)$ in closed form, a dynamical extension of the earlier self-consistency equation (4.5). The hope was that this might be possible close to the bifurcation, where $r(t)$ would be expected to evolve extremely slowly compared to the relaxation time of the individual oscillators. Then each oscillator would follow the order parameter almost adiabatically, allowing these rapid variables to be eliminated and causing a great reduction in the dynamics.

To push this strategy through, Kuramoto and Nishikawa [8] made several approximations whose validity was uncertain. As in the steady-state theory, they separated the population into locked and drifting groups; such a sharp division should be possible if $r(t)$ varies slowly enough. The characteristic timescale of the locked oscillators was argued to be of order $(Kr)^{-1}$, which is very slow since $r(t) \ll 1$ near the bifurcation. The theory also suggested that the drifting oscillators make a negligible contribution to the dynamics of $r(t)$.

In the end, they were led to the following unconventional equation (see Eq. (3.36) in [8]):

$$\dot{r} \approx \frac{K}{\xi_s}(\mu r^2 - \beta r^4), \tag{6.1}$$

where ξ_s is an O(1) constant that arises in their theory, $\mu = (K - K_c)/K_c$ as before, and $\beta = -\frac{1}{16}\pi K_c{}^3 g''(0)$. Note the peculiar extra factor of r on the right-hand side as compared to the usual normal form near a pitchfork bifurcation. Eq. (6.1) predicts that the zero solution is stable below threshold ($\mu < 0$), but with anomalously slow algebraic decay

$$r(t) = \mathrm{O}(t^{-1})$$

as $t \to \infty$. Above threshold, the zero solution is unstable, though weakly so: $r(t)$ initially grows only linearly in t, then eventually relaxes exponentially fast to $r_\infty = \sqrt{\mu/\beta}$.

6.2. Second theory

Kuramoto and Nishikawa soon realized that something was wrong. Two years later, they revisited the problem [9] and stated with admirable candor, “In the past, we seem to have held an erroneous view about the onset of collective oscillation. . . ”. They now believed that the drifting oscillators are *not* negligible throughout the whole evolution of $r(t)$ — rather, these oscillators play a decisive dynamical role in the earliest stages, thanks to their rapid response to fluctuations in $r(t)$, though in the long run they still do not affect the steady value of r.

Kuramoto and Nishikawa [9] also proposed a new strategy for deriving an evolution equation for $r(t)$. In the governing equation

$$\dot{\theta}_i = \omega_i - Kr(t)\sin\theta_i,$$

they pretend that $r(t)$ is an external force, say $h(t)$, and then derive the responses of the individual oscillators to $h(t)$, restricting attention to the linear regime where $h(t) \ll 1$. These individual responses (which depend on the whole history of $h(t)$) can then be combined to yield the response of $r(t)$. On general grounds, and without giving a derivation, Kuramoto and Nishikawa [9] guessed that $r(t)$ should be a linear functional of $h(t)$ of the form

$$r(t) = \int_0^\infty M(\tau) h(t-\tau)\,\mathrm{d}\tau,$$

where M is a memory function to be determined. But since h is really r in disguise, the equation must be

$$r(t) = \int_0^\infty M(\tau) r(t-\tau)\,\mathrm{d}\tau. \tag{6.2}$$

To calculate the kernel M, they consider the response to a step function

$$h(t) = \begin{cases} h_0, & t \le 0, \\ 0, & t > 0, \end{cases}$$

and find that, for example, $M(t) = \mathrm{e}^{-t}$ when the distribution is the Lorentzian $g(\omega) = [\pi(\omega^2+1)]^{-1}$. (The calculation of M is straightforward. The oscillators are initially distributed according to the stationary density $\rho(\theta,\omega)$ found in Section 4, where h_0 plays the role of r in the earlier formulas. The density ρ is smooth in θ for the drifting oscillators and a delta function in θ for the locked oscillators. Then, since $h(t) = 0$ for $t > 0$, all the oscillators and their corresponding densities rotate rigidly and independently at their natural frequencies. The corresponding evolution of $r(t)$ can be found by integrating $\mathrm{e}^{\mathrm{i}\theta}$ with respect to these rotating densities, weighted by $g(\omega)$, and then $M(t)$ can be extracted from the result.)

Within this revised framework, Kuramoto and Nishikawa [9] now found that $r(t)$ grows exponentially above threshold, and decays exponentially below threshold. In other words, the zero solution was now predicted to change stability in the most standard way — it goes from linearly stable to linearly unstable as K increases through K_c.

But, should one really believe this prediction? Remember, the integral equation (6.2) was not derived in any systematic way from the governing equation (3.1). On the other hand, the intuitive argument for (6.2) looked plausible, and maybe even convincing.

7. Continuum limit of the Kuramoto model

It was against this confusing backdrop that Mirollo and I began thinking about the stability problem. At the time, it was unclear how to formulate the problem mathematically. We did not even know how to write down an infinite-N version of the Kuramoto model, let alone analyze the stability of its steady solutions.

We eventually realized that the continuum limit should be phrased in terms of *densities*, just as in traffic flow, kinetic theory, or fluid mechanics [34]. For each natural frequency ω, imagine a continuum of oscillators distributed on the circle. Let $\rho(\theta,t,\omega)\,\mathrm{d}\theta$ denote the fraction of these oscillators that lie between θ and $\theta + \mathrm{d}\theta$ at time t. Then ρ is nonnegative, 2π-periodic in θ, and satisfies the normalization

$$\int_0^{2\pi} \rho(\theta,t,\omega)\,\mathrm{d}\theta = 1 \tag{7.1}$$

for all t and ω. The evolution of ρ is governed by the continuity equation

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial\theta}(\rho v) \tag{7.2}$$

which expresses conservation of oscillators of frequency ω. Here the velocity $v(\theta, t, \omega)$ is interpreted in an Eulerian sense as the instantaneous velocity of an oscillator at position θ, given that it has natural frequency ω. From (3.3), that velocity is

$$v(\theta, t, \omega) = \omega + Kr\sin(\psi - \theta), \tag{7.3}$$

where $r(t)$ and $\psi(t)$ are now given by

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \int_0^{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\theta}\rho(\theta, t, \omega)g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta, \tag{7.4}$$

which follows from the law of large numbers applied to (3.2). Equivalently, these equations can be combined to yield a single equation for ρ in closed form:

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial\theta}\left[\rho\left(\omega + K\int_0^{2\pi}\int_{-\infty}^{\infty}\sin(\theta' - \theta)\rho(\theta', t, \omega')g(\omega')\,\mathrm{d}\omega'\,\mathrm{d}\theta'\right)\right]. \tag{7.5}$$

The expression in parentheses is $v(\theta, t, \omega)$, written as the infinite-N version of (3.1).

Eq. (7.5) is the continuum limit of the Kuramoto model [34]. It is a nonlinear partial integro-differential equation for ρ. The virtue of (7.5) is that all questions about existence, stability, and bifurcation of various kinds of solutions can now be addressed systematically.

For instance, the stationary states of (7.5) are precisely the steady solutions that Kuramoto [4,5] wrote down intuitively. To see this, note that $\partial\rho/\partial t = 0$ implies $\rho v = C(\omega)$, where $C(\omega)$ is constant with respect to θ. If $C(\omega) \neq 0$, we recover the stationary density (4.3) for the drifting oscillators; if $C(\omega) = 0$, we find that ρ is a delta function in θ, based at the locked phase found earlier.

The simplest state is the uniform *incoherent state*

$$\rho_0(\theta, \omega) \equiv \frac{1}{2\pi},$$

or what we earlier called the zero solution. As we will see in Section 8, its linear stability properties turn out to be stranger than anyone had expected.

Eqs. (7.2)–(7.5) had been studied previously by Sakaguchi [35], who extended the Kuramoto model to allow rapid stochastic fluctuations in the natural frequencies. The governing equations are

$$\dot{\theta}_i = \omega_i + \xi_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \dots, N, \tag{7.6}$$

where the variables $\xi_i(t)$ are independent white noise processes that satisfy

$$\langle\xi_i(t)\rangle = 0, \qquad \langle\xi_i(s)\xi_j(t)\rangle = 2D\delta_{ij}\delta(s - t).$$

Here $D \geq 0$ is the noise strength and the angular brackets denote an average over realizations of the noise. Sakaguchi argued intuitively that since (7.6) is a system of Langevin equations with mean-field coupling, as $N \to \infty$ the density $\rho(\theta, t, \omega)$ should satisfy the Fokker–Planck equation

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2\rho}{\partial\theta^2} - \frac{\partial}{\partial\theta}(\rho v), \tag{7.7}$$

where $v(\theta, t, \omega)$, $r(t)$, and $\psi(t)$ are given by (7.3) and (7.4). Thus Sakaguchi's Fokker–Planck equation reduces to the continuum limit of the Kuramoto model when $D = 0$.

However, Sakaguchi [35] did not present a stability analysis of his model. Instead he solved for the stationary densities, and then extended Kuramoto's self-consistency argument to determine where a branch of partially synchronized states bifurcates from the incoherent state. In this way he showed that the critical coupling is

$$K_{\mathrm{c}} = 2\left[\int_{-\infty}^{\infty} \frac{D}{D^2+\omega^2} g(\omega)\,\mathrm{d}\omega\right]^{-1}, \tag{7.8}$$

which reduces to Kuramoto's formula $K_{\mathrm{c}} = 2/\pi g(0)$ as $D \to 0^+$.

8. Stability of the incoherent state

The linear stability problem for the incoherent state of Sakaguchi's model was solved in [34]. Here is an outline of the approach and the results (for consistency with the rest of this paper, we will restrict attention to the Kuramoto model, where $D=0$). Let

$$\rho(\theta, t, \omega) = \frac{1}{2\pi} + \varepsilon\eta(\theta, t, \omega), \tag{8.1}$$

where $\varepsilon \ll 1$ and we write the perturbation η as a Fourier series in θ:

$$\eta(\theta, t, \omega) = c(t, \omega)\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} + \eta^{\perp}(\theta, t, \omega). \tag{8.2}$$

Here c.c. denotes complex conjugate, and $\eta^{\perp}$ contains the second and higher harmonics of η. (Note that η automatically has zero mean, because of (7.1).) We write the perturbation in this way because it turns out that the linearized amplitude equation for the first harmonic, $c(t, \omega)$, is the only one with nontrivial dynamics; that's essentially because of the pure sinusoidal coupling in the Kuramoto model. Substituting for ρ into (7.5) yields

$$\frac{\partial c}{\partial t} = -\mathrm{i}\omega c + \frac{K}{2}\int_{-\infty}^{\infty} c(t, \omega')g(\omega')\,\mathrm{d}\omega'. \tag{8.3}$$

The right-hand side of (8.3) defines a linear operator A, given by

$$Ac \equiv -\mathrm{i}\omega c + \frac{K}{2}\int_{-\infty}^{\infty} c(t, \omega')g(\omega')\,\mathrm{d}\omega'. \tag{8.4}$$

The spectrum of A has both continuous and discrete parts, as shown in [34]. Its continuous spectrum is pure imaginary, $\{\mathrm{i}\omega\colon \omega \in \mathrm{support}(g)\}$, corresponding to a continuous family of *neutral* modes. These modes can be understood intuitively by imagining an initial perturbation $\eta(\theta, \omega, t=0)$ supported on a sliver of exactly one frequency, say $\omega=\omega_0$. In other words, we disturb the slice of the oscillator population with intrinsic frequency ω_0 and leave the rest alone in their perfectly incoherent state. The corresponding amplitude $c(0, \omega)$ would then take the form $c(0, \omega)=0$ for all $\omega \neq \omega_0$ (oscillators at those frequencies are not disturbed). As for $\omega=\omega_0$, we can choose $c(0, \omega_0)=1$ without loss of generality, since (8.4) is linear. The key point is that the integral in (8.4) vanishes for this strange sliver perturbation, and so (8.4) reduces to $Ac=\mathrm{i}\omega_0 c$. Hence, $c(0, \omega)$ is (morally speaking) an eigenfunction with pure imaginary eigenvalue $\mathrm{i}\omega_0$, and that explains the form of the continuous spectrum. Of course, this argument is not strictly correct, because this sliver perturbation is equivalent in L^2 to the zero perturbation, and so is not a valid eigenmode. But the intuition is right, and it agrees with the rigorous calculations given in [34].

To find the discrete spectrum of A, let

$$c(t, \omega) = b(\omega)\,\mathrm{e}^{\lambda t}.$$

Then

$$\lambda b = -\mathrm{i}\omega b + \frac{K}{2}\int_{-\infty}^{\infty} b(\omega')g(\omega')\,\mathrm{d}\omega'. \tag{8.5}$$

The integral is just a constant to be determined self-consistently. Thus, let

$$B = \frac{K}{2}\int_{-\infty}^{\infty} b(\omega')g(\omega')\,\mathrm{d}\omega'. \tag{8.6}$$

Solving (8.5) for b yields

$$b(\omega) = \frac{B}{\lambda + \mathrm{i}\omega}.$$

Substituting this b back into (8.5) gives the characteristic equation

$$1 = \frac{K}{2}\int_{-\infty}^{\infty} \frac{g(\omega)\,\mathrm{d}\omega}{\lambda + \mathrm{i}\omega}. \tag{8.7}$$

Now suppose that $g(\omega)$ is even and nowhere increasing on $[0,\infty)$, in the sense that $g(\omega) \geq g(v)$ whenever $\omega \leq v$; this is the case originally considered by Kuramoto. Then one can prove that (8.7) has at most one solution for λ, and if it exists, it is real [36]. Hence (8.7) becomes

$$1 = \frac{K}{2}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2 + \omega^2}\, g(\omega)\,\mathrm{d}\omega. \tag{8.8}$$

Eq. (8.8) shows that any eigenvalue must satisfy $\lambda \geq 0$, since otherwise the right-hand side of (8.8) is negative. Hence there can never be any negative eigenvalues!

So our analysis has yielded a surprise: the incoherent state of the Kuramoto model can never be linearly stable — it is either unstable or neutrally stable.

To find the borderline coupling K_c between these two cases, consider the limit $\lambda \to 0^+$ in (8.8). Then $\lambda/(\lambda^2 + \omega^2)$ becomes more and more sharply peaked about $\omega = 0$, yet its integral over $-\infty < \omega < \infty$ remains equal to π. Hence $\lambda/(\lambda^2 + \omega^2) \to \pi\delta(\omega)$, and so (8.8) tends to

$$1 = \tfrac{1}{2}K_c \pi g(0),$$

which gives a new derivation of the K_c found by Kuramoto [4,5].

Eq. (8.8) also provides explicit formulas for the growth rate λ, if $g(\omega)$ is a sufficiently simple density. For instance, the uniform density $g(\omega) = 1/2\gamma$ with $-\gamma \leq \omega \leq \gamma$ gives

$$\lambda = \gamma \cot\left(\frac{2\gamma}{K}\right) \tag{8.9}$$

and the Lorentzian density (4.7) gives

$$\lambda = \tfrac{1}{2}K - \gamma. \tag{8.10}$$

These eigenvalues match the growth rates seen in numerical simulations for $K > K_c$ [34].

In summary, the linearization about the incoherent state of the Kuramoto model has a purely imaginary continuous spectrum for $K < K_c$, and the discrete spectrum is empty. As K increases, a real eigenvalue λ emerges from the continuous spectrum and moves into the right half plane for $K > K_c$ (Fig. 4).

These results confirm Kuramoto's conjecture [5] that the incoherent state becomes unstable when $K > K_c$. But the shocker is that incoherence is linearly neutrally stable for all $K < K_c$.

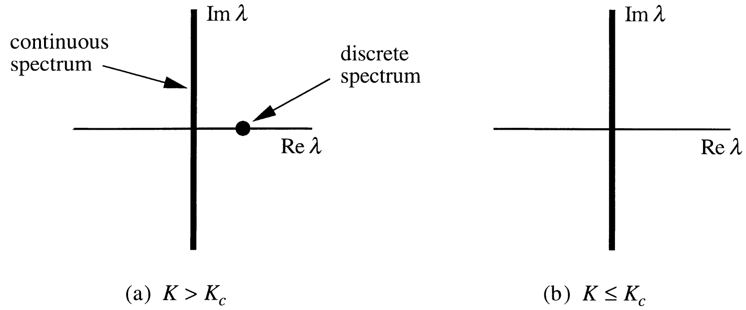

Fig. 4. Spectrum of the linear operator (8.4) that governs the linear stability of the incoherent state $\rho_0 \equiv 1/2\pi$. (a) For $K > K_c$, the incoherent state is unstable, thanks to the discrete eigenvalue $\lambda > 0$. This eigenvalue pops out of the continuous spectrum at $K = K_c$. (b) For $K \le K_c$, the discrete spectrum is empty and the incoherent state is neutrally stable.

9. Landau damping

Mirollo and I were novices at continuous spectra, and we were bewildered by its effects on the discrete spectrum. We expected that as K decreases through K_c, the eigenvalue λ should move toward the continuous spectrum, collide with it, then pop out the back. But it did not — it just disappeared. Where did it go? Another weird thing was that explicit formulas for λ like (8.9) and (8.10) look perfectly innocuous for $K < K_c$. They give no hint that λ is doomed; they simply predict, incorrectly, that λ goes negative.

Matthews, then an applied math instructor at MIT, became interested in this issue and we all began working on it together. The mystery deepened when Matthews ran some simulations for $K < K_c$ that seemed to show exponential decay of the coherence $r(t)$ — and the decay rate was exactly the negative λ predicted by the formulas, in the regime where they were not supposed to hold. Spooky!

9.1. The long-sought integral equation

But maybe $r(t)$ could decay exponentially even if $\eta(\theta, t, \omega)$ does not? We needed to find an equation governing the evolution of $r(t)$. Recall that this is what Kuramoto and Nishikawa [8,9] had been searching for too, as discussed in Section 6. Fortunately it was now possible to derive such an equation systematically, as follows [37]. Eqs. (8.1), (8.2) and (7.4) yield

$$r(t) = 2\pi\varepsilon \left| \int_{-\infty}^{\infty} c(t,\omega) g(\omega)\, \mathrm{d}\omega \right|. \tag{9.1}$$

Notice that the integral in (9.1) also appears in the linearized amplitude equation (8.4). Since (9.1) reveals an intimate relationship between that integral and $r(t)$, let us introduce the notation

$$R(t) = \int_{-\infty}^{\infty} c(t,\omega) g(\omega)\, \mathrm{d}\omega. \tag{9.2}$$

Eq. (8.3) is a first-order linear ordinary differential equation for $c(t,\omega)$, and hence is easily solved in terms of $R(t)$ and the initial condition $c_0(\omega) \equiv c(0,\omega)$. Inserting the result for $c(t,\omega)$ into (9.2) gives the linear integral equation

$$R(t) = (\widehat{c_0 g})(t) + \frac{K}{2} \int_0^t R(t-\tau)\hat{g}(\tau)\, \mathrm{d}\tau, \tag{9.3}$$

where the hat denotes Fourier transform:

$$\hat{g}(t) = \int_{-\infty}^{\infty} g(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega.$$

The structure of (9.3) is reminiscent of (6.2), the equation guessed by Kuramoto and Nishikawa [9], with $\hat{g}$ playing the role of their memory function M. In particular, $\hat{g}(t) = \mathrm{e}^{-t}$ when the density is the Lorentzian $g(\omega) = [\pi(\omega^2+1)]^{-1}$, in agreement with their finding that $M(t) = \mathrm{e}^{-t}$ in this case. The main differences are that (9.3) is an equation for R, not r, and (9.3) includes a variable upper limit of integration and the $c_0 g$ term.

To solve (9.3), use Laplace transforms and then apply the inversion formula to obtain the integral representation

$$R(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{(c_0 g)^*(s)}{1 - \frac{1}{2} K g^*(s)} \mathrm{e}^{st}\, \mathrm{d}s. \tag{9.4}$$

Here the contour Γ is a vertical line to the right of any singularities of the integrand, and the asterisk denotes an operation related to the Hilbert transform:

$$f^*(s) \equiv \int_{-\infty}^{\infty} \frac{f(\omega)\, \mathrm{d}\omega}{s + \mathrm{i}\omega}.$$

From (8.7), we see that the denominator in (9.4) vanishes precisely when s is in the discrete spectrum of A. Hence for $K < K_\mathrm{c}$, the denominator *never* vanishes.

Some explicit solutions of (9.4) are possible. For the extremely special initial condition $c_0(\omega) \equiv 1$, the exact solution is

$$R(t) = \exp\left[\left(\tfrac{1}{2}K - \gamma\right) t\right], \quad t \geq 0,$$

when $\hat{g}(t) = \mathrm{e}^{-|\gamma t|}$, corresponding to a Lorentzian $g(\omega)$. So exponential decay of $R(t)$, and hence $r(t)$, is possible for $K < K_\mathrm{c} = 2\gamma$, even though the incoherent state ρ_0 is neutrally stable! On the other hand, for the uniform density $g(\omega) = 1/2\gamma$ on $[-\gamma, \gamma]$, asymptotic analysis of the inversion integral (9.4) gives the much slower decay

$$R(t) \sim \left(\frac{-16\gamma}{K^2}\right) \frac{\sin \gamma t}{t \ln^2 t} \quad \text{as } t \to \infty$$

for $K < K_\mathrm{c}$.

More generally, Matthews, Mirollo, and I found that for $K < K_\mathrm{c}$, the asymptotic behavior of $R(t)$ depends crucially on whether $g(\omega)$ is supported on a finite interval $[-\gamma, \gamma]$ or the whole real line (these are the only possibilities, by our hypotheses that g is even and nowhere increasing for $\omega > 0$). For the case of compact support, we proved that $R(t) \to 0$ as $t \to \infty$, but the decay is always slower than exponential at long times, in agreement with numerics [37]. If $g(\omega)$ is supported on the whole line, the asymptotic behavior of $R(t)$ can be much wilder: any $R(t) \in L^2$ can be contrived by an appropriate choice of $c_0 \in L^2$. But in the best-behaved case where $g(\omega)$ and $c_0(\omega)$ are entire functions, $R(t)$ is merely a sum of decaying exponentials.

Finally, the integral representation (9.4) allowed us to understand the exponential decay that Matthews had seen at intermediate times in his simulations. The decay is caused by a pole in the left half plane — a pole not of the integrand but of its *analytic continuation* (as required for the validity of the usual contour manipulations). This pole coincides with the eigenvalue λ in the right half plane, but not in the left!

9.2. A lesson from Rowlands

In February 1991, Matthews gave a lecture at Warwick where he described the various bizarre features of our stability problem: the continuous spectrum on the imaginary axis; the disappearance of the unstable eigenvalue into the continuous spectrum at threshold; the need for tricky analytic continuation arguments; the fact that the macroscopic variable r can decay exponentially even though the density perturbation η does not.

Rowlands was in the audience, and he told Matthews that something just like this had been seen before in plasma physics, where it is called "Landau damping". For the next several months, we devoured whatever we could find on the subject, and soon realized that Landau damping was a fascinating, confusing story in its own right, starting with brilliant but not entirely rigorous work by Landau in 1946, followed by two decades worth of controversy [38–46].

Rowlands was right. There definitely was a link between Landau damping and the relaxation phenomena we were seeing. It was awe-inspiring: the same mathematics describes the violent world of plasmas and the silent, hypnotic pulsing of fireflies perched along a riverbank.

We spent a few months trying to get the mathematical story straight, and gradually we began writing a paper on what we had found. But before it was done, I took a few days off to attend Dynamics Days in Austin, in January 1992.

10. A lunch with Crawford

As usual at Dynamics Days, there was a big table in the hall where people had left piles of reprints. A paper caught my eye: "Amplitude equations on unstable manifolds: singular behavior from neutral modes", by Crawford [47].

Whoa — neutral modes! Heart beating fast, I skimmed the abstract and there it was: "The Vlasov equation for a collisionless plasma is the second model; in this case there are an infinite number of neutral modes corresponding to the van Kampen continuous spectrum". Yep, that confirms it. He's thinking about the same kind of things that we are. I had heard of Crawford and I knew that he was supposed to be a brilliant young guy and a great applied mathematician. Apparently he knows a lot about plasmas and continuous spectra — maybe he can clarify some things about Landau damping and tell me if our ideas about the Kuramoto model seem right.

So I asked around, and it seemed everybody but me knew who Crawford was. Mary Silber, Emily Stone, and Kurt Wiesenfeld all tried to describe him to me, but we could not find him anywhere.

Eventually our paths crossed. I was struck by his combination of seriousness and pleasantness. He seemed different from the rest of the gang, maybe more reserved, maybe just better manners? Anyway, I told him what I had hoped to discuss, and he seemed to like the idea, so we wandered off to have lunch together and ended up at a hamburger joint somewhere, a dark woody place, perfect for thinking about math.

I told him about the crazy behavior of the unstable eigenvalue and how it got absorbed by the continuous spectrum on the imaginary axis, but before I could get very far, he gave me a reassuring nod. He seemed to know the whole story without me telling him. Yes, all these things were familiar and standard in the context of collisionless plasmas [38–48]. Not only that, he explained, but similar phenomena occur in many other parts of science, in connection with instabilities of ideal shear flows [49–51], solitary waves [52,53], bubbly fluids [54], and resonance poles in atomic systems [55]. Wow — I was talking to the right guy.

He went on to explain some of his own work. He was trying to write amplitude equations for a weakly unstable mode in a Vlasov plasma, but the difficulty was that the coefficients in those equations become *singular* as the unstable eigenvalue approaches the neutral continuous spectrum, reflecting unusually strong nonlinear effects [47,48]. Whereas normally the saturated amplitude of the bifurcating mode grows like $\sqrt{\sigma}$ (where $\sigma = \mathrm{Re}\,\lambda$ is the linear growth rate), in these situations the nonlinear interactions lead to a much smaller amplitude ($\mathrm{O}(\sigma^2)$, in the Vlasov case).

Hold on, I said. In the Kuramoto model, we can find the amplitude of the bifurcating mode exactly, and we do see the usual square-root scaling; that follows from (4.6) and the fact that $\sigma \sim K - K_{\mathrm{c}}$ near threshold. That got

Crawford's attention. I showed him Kuramoto's classic analysis (Section 4) and yes, he agreed, something different seemed to be going on here. For some reason, the Kuramoto model was not showing signs of the singularities that afflicted the Vlasov problem. Crawford realized that this could be an instructive case. If he could derive the amplitude equations for the Kuramoto model, they should not turn out to be singular — and maybe that would shed some light on the plasma problem, as well as giving more general insight into the effects of the neutral continuum on the scaling of unstable modes.

That is how Crawford got started on the Kuramoto model.

11. Crawford's work on coupled oscillators

Crawford's first paper on coupled oscillators [1] contains the decisive step. He showed how to approach the local stability analysis of the Kuramoto model in a systematic way, using the tools of center manifold theory and equivariant bifurcation theory.

At the time, Crawford developed this approach almost in passing. What really grabbed his attention was a paper by Bonilla et al. [56] that had just appeared. Those authors were the first to attempt a nonlinear stability analysis of the Kuramoto model, and they noticed that Hopf bifurcations became possible if the frequency distribution $g(\omega)$ were allowed to be bimodal. But when Crawford saw their analysis, he instantly felt that something was amiss. It seemed to him that Bonilla et al. had unfortunately omitted half of the unstable eigenvectors that would generically be forced by the O(2) symmetry of the system. He wondered whether some nonlinear traveling and standing wave solutions had been overlooked. That turned out to be the case. So part of Crawford's paper [1] is devoted to a careful re-analysis of the dynamics for bimodal $g(\omega)$.

More significantly, Crawford presented the first derivation and analysis of the amplitude equations for both steady-state and Hopf bifurcations from the incoherent state $\rho_0(\theta,\omega) \equiv 1/2\pi$. He worked with Sakaguchi's generalization of the Kuramoto model:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}\left[\rho\left(\omega + K\int_0^{2\pi}\int_{-\infty}^{\infty}\sin(\theta' - \theta)\rho(\theta', t, \omega')g(\omega')\,\mathrm{d}\omega'\,\mathrm{d}\theta'\right)\right], \tag{11.1}$$

where the density $g(\omega)$ is assumed to be even, as before, but is no longer restricted to be unimodal.

With noise strength $D > 0$, the continuous spectrum lies safely in the left half plane, so center manifold reduction can be applied. Crawford exploits the system's O(2) symmetry to constrain the form of the center manifold and the vector field on it, yet the calculation is still daunting. Eventually he arrives at an equation ([1], Eq. (108)), that, in our notation, is equivalent to

$$\dot{r} = \lambda r + a r^3 + \mathrm{O}(r^5).$$

Recall from Section 6 that Kuramoto and Nishikawa [8] had been looking for an amplitude equation like this. Crawford finally found it. In Eq. (138) of [1], he works out the value of the coefficient a and confirms that as $D \to 0^+$, it agrees with the value found by Kuramoto's self-consistency approach. The amplitude equation also strongly suggests that the bifurcating branch is locally stable, at least at onset. Still, it is not a proof, as Crawford notes: "However, when $D = 0$, center manifold theory no longer justifies our reduction to two dimensions; the qualitative agreement at $D = 0$ between numerical simulations [6] and our amplitude equation may be fortuitous".

There is one other important result in that first paper. Just as Crawford had suspected at our lunch in Austin, the coefficients in the amplitude equation do indeed remain finite as $D \to 0^+$, in striking contrast to the singular behavior that occurs in the corresponding expansions for the Vlasov plasma problem. In both problems, the unstable modes correspond to an eigenvalue emerging from a neutral continuous spectrum at onset. So why are the amplitude equations singular in one case and not in the other?

An intriguing clue was provided by the work of Daido [57–60]. He investigated what happens when the sinusoidal coupling in the Kuramoto model is replaced by a general periodic function

$$f(\phi) = \sum_{n=-\infty}^{\infty} f_n\, \mathrm{e}^{\mathrm{i}n\phi}.$$

As before, the system exhibits incoherence for sufficiently small coupling, then bifurcates to a partially synchronized state as the coupling is increased past a critical value. So at first glance the generalized model seems to show nothing qualitatively new.

But upon closer inspection, it turns out that one aspect of the model — its scaling behavior near threshold — is altered in an essential way. Following Kuramoto's original calculation, Daido sought steady solutions and studied their bifurcations by imposing a self-consistency condition. He generalized Kuramoto's order parameter (which is tailored to sinusoidal coupling) by extending it to an "order function" H. Using a suitable norm of H to measure the amplitude of the bifurcating solution, Daido showed that

$$\|H\| \sim (K - K_{\mathrm{c}})^{\beta},$$

where the scaling exponent $\beta = 1$ generically. That was a big surprise — the obvious guess was that $\beta = \frac{1}{2}$, the square-root scaling familiar from pitchfork and Hopf bifurcations and most mean-field models, including the original Kuramoto model.

Crawford loved this result, because it meant that something singular must be happening in the amplitude equations. Time for another monstrous center manifold reduction! That is the topic of Crawford's next two papers [2,3]. Replacing the sine function in (11.1) with a general f yields

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}\left[\rho\left(\omega + K\int_0^{2\pi}\int_{-\infty}^{\infty} f(\theta' - \theta)\rho(\theta', t, \omega')g(\omega')\,\mathrm{d}\omega'\,\mathrm{d}\theta'\right)\right].$$

This evolution equation always has SO(2) symmetry. If f is odd and g is even, as in the original Kuramoto model, the symmetry is O(2).

In [2], Crawford computed the amplitude equations through third order and verified that they could become singular, depending on the harmonic content of f. His main result is that the saturated amplitude of an unstable mode $\mathrm{e}^{\mathrm{i}l\theta}$ with mode number l typically scales like

$$|\alpha_\infty| \sim \sqrt{\sigma(\sigma + l^2 D)}, \tag{11.2}$$

where σ is the linear growth rate and D the noise strength. The unusual factor $\sigma + l^2 D$ arises from a singularity in the amplitude equation; it is generic in the sense that it occurs for any coupling function $f(\phi)$ with

$$f_{2l} \neq 0.$$

To clarify this result, let us see why the original Kuramoto model gives no hint of the generic scaling (11.2). As discussed in Section 8, the $l = 1$ harmonic of the perturbation $\eta(\theta, t, \omega)$ is the only one that can go unstable; that is why it was sufficient to concentrate on the dynamics of its amplitude $c(t, \omega)$ and ignore the evolution of the higher harmonics in η. However, we see now that the square-root scaling (4.6) found in that case is nongeneric, because $f(\phi) = \sin\phi$ and hence $f_2 = 0$; the Kuramoto model has no second harmonic in the coupling.

For $D = 0$, (11.2) generically yields the scaling exponent $\beta = 1$ found earlier by Daido [59], but Crawford's analysis goes further by including stability information and the effects of noise. For instance, when $D > 0$, (11.2) shows that the scaling $|\alpha_\infty| \sim \sigma$ crosses over to the traditional result $|\alpha_\infty| \sim \sqrt{\sigma}$ sufficiently close to onset ($\sigma \to 0^+$), or when the noise becomes sufficiently strong.

The recent paper by Crawford and Davies [3] is an even deeper exploration of these issues. Now the singularity structure of the amplitude equations is calculated to *all orders*, but all the earlier conclusions still hold. This paper also contains a rigorous derivation of Sakaguchi's equation (7.7), starting from a Fokker–Planck equation for the coupled Langevin equation (7.6) on the N-torus and taking the limit $N \to \infty$.

In summary, Crawford made several important contributions to the analysis of the Kuramoto model, including:

1. The first systematic formulation of the weakly nonlinear stability problem for the incoherent state, using center manifold theory and equivariant bifurcation theory [1].
2. The first derivation of an evolution equation for $r(t)$, in the neighborhood of the incoherent state [1].
3. The first proof that the bifurcating branch of partially synchronized states is locally stable, near the synchronization threshold and in the presence of weak noise [1].
4. The first exploration of the effects of the neutral continuous spectrum on the scaling of unstable modes [1], using ideas that he had developed earlier in his work on the Vlasov model of collisionless plasmas [47,48], thereby forging a link between these two previously separate fields.
5. The discovery that the amplitude equations for the Kuramoto model are nonsingular, in contrast to those for the Vlasov model, and the explanation of this difference: the Kuramoto model has nongeneric singularity structure due to the lack of a second harmonic in the coupling function [2,3].
6. The first study of the singularity structure of the amplitude equations for a generalized Kuramoto model in which all harmonics are included [2,3].

Contributions 1–3 cracked some problems that had resisted solution for about two decades. Contributions 4–6 opened up a completely new line of inquiry, with implications not only for oscillator synchronization, but also for plasma physics, fluid mechanics, kinetic theory, and other fields where instabilities are created by unstable modes emerging from a continuous spectrum.

12. Epilog

The last time I saw Crawford was in spring 1998, at the Pattern Formation meeting at the Institute for Mathematics and its Applications. It was his first conference after many bouts of chemotherapy, and although he was a little weak, he was all smiles and his manner was as gracious as ever. We enjoyed some fun times together that week, especially during a dinner with Mirollo. Over pizza and a few beers, the three of us discussed the linear stability problem for the entire branch of partially synchronized states in the Kuramoto model. It is still unsolved, 25 years after Kuramoto first posed it, but we thought we had some ideas about how to proceed, and we hoped to collaborate on it after the conference. With Crawford on our team, I bet we could have done it.

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