Geometry

1. A **convex polytope** is defined as an intersection of a finite number of half-spaces. Prove that the following set is a convex polytope:

$$D = \{x \in \mathbb{R}^n \,:\, Ax \leq b\},$$

for some $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

2. The following set is a convex polytope:

$$D = \left\{(x, y) \in \mathbb{R}^2 \,:\, x - y = 2, x, y \geq 0\right\}.$$

Write this polytope as intersection of half-spaces. Draw it. Is this set bounded?

3. Let C be a convex set in $\mathbb{R}^n$. Prove that $\hat{x}$ is an extreme point of C if and only if $C \setminus \{\hat{x}\}$ is convex as well.

4. Suppose we have the following LP:

$$\begin{aligned} \min_x \; & c^\top x \\ \text{s.t. } & Ax \leq 0 \\ & x \geq 0 \end{aligned}$$

Prove that the set of solutions is a convex set.

Standard form

Convert the following linear programs to standard form:

1.

$$\begin{aligned} \min\ & x + 2y + 3z \\ \text{s.t. }\ & 2 \leq x + y \leq 3 \\ & 4 \leq x + z \leq 5 \\ & x \geq 0, y \geq 0, z \leq 0 \end{aligned}$$

2.

$$\begin{aligned} \max\ & x + y + z \\ \text{s.t. } & x + 2y + 3x = 10 \\ & x \geq 1 \\ & y \geq 2 \\ & z \geq 1 \end{aligned}$$

3. Also solve geometrically the following problem

$$\begin{aligned} \max\ & x_1 + 4x_2 + x_3 \\ \text{s.t.}\ & 2x_1 - 2x_2 + x_3 = 4 \\ & x_1 - x_3 = 1 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \end{aligned}$$