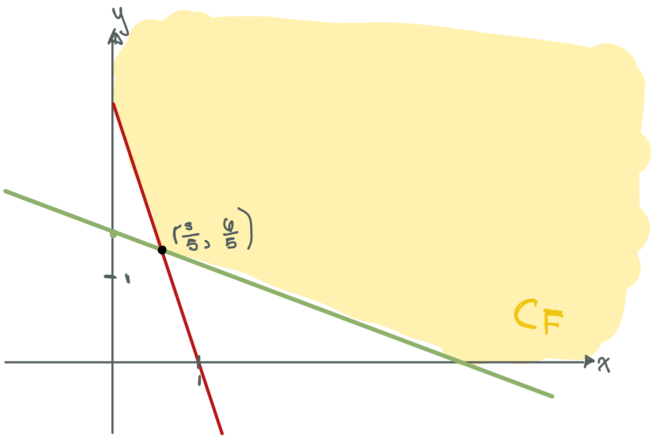
1a

$c^T x$

$3x + y \geq 3$

$x + 2y \geq 3$

$x, y \geq 0$

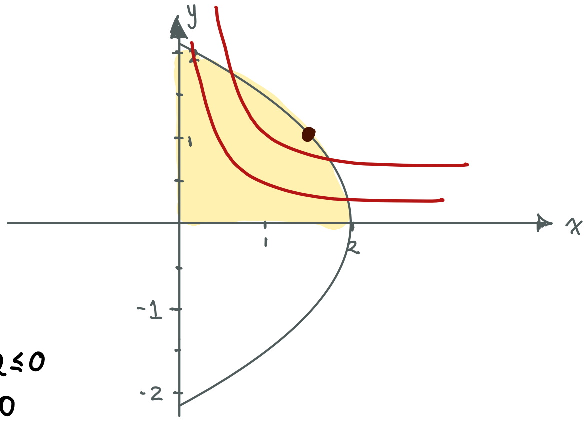


Some possible values for c:

- One optimal solution: $c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad x^* = \left(\frac{3}{5}, \frac{6}{5}\right)$
- A lot of optimal solutions $c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- Unbounded $c = \begin{bmatrix} -10 \\ -1 \end{bmatrix}$

2.

$$\min \; -xy$$
$$\text{s.t.} \;\; x+y^2 \leq 2$$
$$x, y \geq 0$$



We can write these conditions as:

$$\min \; f(x,y) = -xy$$
$$\text{s.t.} \;\; g_1(x,y) = x+y^2-2 \leq 0$$
$$g_2(x,y) = -x \leq 0$$
$$g_3(x,y) = -y \leq 0$$

The Lagrangian is

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3) = -xy + \lambda_1 (x+y^2-2) + \lambda_2(-x) + \lambda_3(-y)$$

Thus the KKT conditions can be written as:

1. $\frac{\delta \mathcal{L}}{\delta x} = 0 \iff -y+\lambda_1 - \lambda_2 = 0$ (stationarity)
2. $\frac{\delta \mathcal{L}}{\delta y} = 0 \iff -x + 2\lambda_1 y - \lambda_3 = 0$
3. $g_1(x,y) \leq 0 \iff x+y^2-2 \leq 0$ (primal feasibility)
4. $g_2(x,y) \leq 0 \iff -x \leq 0$
5. $g_3(x,y) \leq 0 \iff -y \leq 0$
6. $\lambda_1 \geq 0$ (dual feasibility)
7. $\lambda_2 \geq 0$
8. $\lambda_3 \geq 0$
9. $\lambda_1 g_1(x,y) = 0 \iff \lambda_1(x+y^2-2) = 0$ (complementary slackness)
10. $\lambda_2 g_2(x,y) = 0 \iff \lambda_2(-x) = 0$
11. $\lambda_3 g_3(x,y) = 0 \iff \lambda_3(-y) = 0$

CASE 1

Suppose $\lambda_1 = 0$. Then 1 tells us $y+\lambda_2 = 0$ and 2 tells us $x+\lambda_3 = 0$. Since each term is nonnegative, the only way that this can happen is if $x=y=\lambda_2=\lambda_3=0$. The KKT conditions are satisfied when $x=y=\lambda_1=\lambda_2=\lambda_3=0$. In this case $f(0,0)=0$

CASE 2

Suppose on the other hand that we're on the boundary of the parabola, $\lambda_1 \neq 0$ but it has to happen that $x+y^2=2$.

CASE 2A Suppose that $x>0$, then by 10 it has to happen that λ_2. Then by 1 we have that $\lambda_1 = y$. By 2 we have that $x-2\lambda_1 y + \lambda_3 = 2-3y^2+\lambda_3 = 0$. Then $3y^2 = 2+\lambda_3 > 0$ and $\lambda_3 = 0$ (by 11). Thus $y = \sqrt{2/3}$ and hence $x = 2-2/3 = 4/3$. In thus case $f(4/3, \sqrt{2/3}) = -\frac{4}{3}\sqrt{\frac{2}{3}}$.

CASE 2B On the other hand suppose that $x=0$ thus $y=\sqrt{2}$. Since $y>0$ it must happen that $\lambda_3=0$ by 11. Then by 2 it must happen that $\lambda_1 = 0$. But this is similar to case 1, it can't happen because of 1.

Then the possible optima are

$(0,0) \Rightarrow f(0,0)=0$

$(4/3, \sqrt{2/3}) \Rightarrow f(4/3, \sqrt{2/3}) = -\frac{4}{3}\sqrt{\frac{2}{3}} < 0$

Thus $(4/3, \sqrt{2/3})$ is the optimum.

3o

For:

$$\min \quad x^2+y^2-4x-4y$$
$$x^2 \le y$$
$$x+y \le 2$$

In this case

$$\min \ f(x,y) = x^2+y^2-4x-4y$$
$$\text{s.t.} \quad g_1(x,y) = x^2-y \le 0$$
$$g_2(x,y) = x+y-2 \le 0$$

The Lagrange function is

$$\mathcal{L}(x,y,\lambda_1,\lambda_2) = x^2+y^2-4x-4y+\lambda_1(x^2-y)+\lambda_2(x+y-2)$$

The KKT conditions are:

(1) $\frac{\delta \mathcal{L}}{\delta x} = 0 \iff 2x-4+2\lambda_1+\lambda_2=0$

(2) $\frac{\delta \mathcal{L}}{\delta y} = 0 \iff 2y-4-\lambda_1+\lambda_2=0$

(stationarity)

(3) $g_1(x,y) \le 0 \iff x^2-y \le 0$

(4) $g_2(x,y) \le 0 \iff x+y-2 \le 0$

(primal feasibility)

(5) $\lambda_1 \ge 0$

(6) $\lambda_2 \ge 0$

(dual feasibility)

(7) $\lambda_1 g_1(x,y) = 0 \iff \lambda_1(x^2-y)=0$

(8) $\lambda_2 g_2(x,y) = 0 \iff \lambda_2(x+y-2)=0$

(complementary slackness)

The global minimizer for the objective function is $(x,y)=(2,2)$. This point is not feasible. Thus both constraints are active. Then we can discard the case when $\lambda_1=\lambda_2=0$. Thus by (1), (2), (7), (8) we get the following system:

$$2x-4+2\lambda_1+\lambda_2=0$$
$$2y-4-\lambda_1+\lambda_2=0$$
$$x^2-y=0$$
$$x+y-2=0$$

We can solve this system. By (7) and (8) we get

$$0 = x^2+x-2 = (x+2)(x-1)$$

Then either $y=4$ or $y=1$.

CASE 1 We consider $(1,1)$. By (1) and (2) we get

$$2-4+2\lambda_1+\lambda_2=0 \quad \text{and} \quad 2-4-\lambda_1+\lambda_2=0$$
$$2\lambda_1+\lambda_2=2 \qquad\qquad -\lambda_1+\lambda_2=2$$

$$\begin{array}{r} 2\lambda_1+\lambda_2=2 \\ \lambda_1-\lambda_2=-2 \\ \hline 3\lambda_1=0 \end{array}$$

Thus $\lambda_1=0$ and $\lambda_2=2$. This $x, y, \lambda_1, \lambda_2$ fullfill (1)-(8). In this case $f(x,y)=f(1,1)=-6$.

CASE 2 We consider $(-2,4)$. By (1) and (2) we get

$$-4-4+2\lambda_1+\lambda_2=0 \quad \text{and} \quad 8-4-\lambda_1+\lambda_2=0$$
$$2\lambda_1+\lambda_2=8 \qquad\qquad -\lambda_1+\lambda_2=-4$$

$$\begin{array}{r} 2\lambda_1+\lambda_2=8 \\ \lambda_1-\lambda_2=4 \\ \hline 3\lambda_1=12 \end{array} \iff \lambda_1=4 \Rightarrow \lambda_2=0$$

In this case $f(-2,4) = 12$

Thus the optimal is $(1,1)$ with $f^*(x,y) = -6$.

4.

For

$$\begin{aligned} \min \quad & 2e^{x-1} + (y-x)^2 + z^2 \\ \text{s.t.} \quad & xyz \le 1 \\ & x + z \ge c \end{aligned}$$

$$\begin{aligned} \min \quad & f(x,y,z) = 2e^{x-1} + (y-x)^2 + z^2 \\ \text{s.t.} \quad & g_1(x,y,z) = xyz - 1 \le 0 \\ & g_2(x,y,z) = c - x - z \le 0 \end{aligned}$$

For which values of c does $x = \begin{bmatrix}1\\1\\1\end{bmatrix}$ with multipliers fulfill the KKT conditions?

We write the Lagrange function:

$$\mathcal{L}(x,y,z,\lambda_1,\lambda_2) = 2e^{x-1} + (y-x)^2 + z^2 + \lambda_1(xyz-1) + \lambda_2(c-x-z)$$

$$2e^{x-1} + y^2 - 2xy + x^2 + z^2 + \lambda_1(xyz-1) + \lambda_2(c-x-z)$$

The KKT conditions are:

(stationarity)

(1) $\frac{\delta \mathcal{L}}{\delta x} = 0 \iff 2e^{x-1} - 2y + 2x + \lambda_1 yz - \lambda_2 = 0$

(2) $\frac{\delta \mathcal{L}}{\delta y} = 0 \iff 2y - 2x + \lambda_1 xz = 0$

(3) $\frac{\delta \mathcal{L}}{\delta z} = 0 \iff 2z + \lambda_1 xy - \lambda_2 = 0$

(primal feasibility)

(4) $g_1(x,y,z) \le 0 \iff xyz - 1 \le 0$

(5) $g_2(x,y,z) \le 0 \iff c - x - z \le 0$

(dual feasibility)

(6) $\lambda_1 \ge 0$

(7) $\lambda_2 \ge 0$

(complementary slackness)

(8) $\lambda_1 g_1(x,y,z) = 0 \iff \lambda_1(xyz-1) = 0$

(9) $\lambda_2 g_2(x,y,z) = 0 \iff \lambda_2(c-x-z) = 0$

For $x = \begin{bmatrix}1\\1\\1\end{bmatrix}$ (1), (2), (3), (4), (5), (8), (9) are:

(1) $2 - 2 + 2 + \lambda_1 - \lambda_2 = 0 \iff \lambda_1 - \lambda_2 = -2$

(2) $2 - 2 + \lambda_1 = 0 \iff \lambda_1 = 0$

(3) $2 + \lambda_1 - \lambda_2 = 0 \iff$ SAME AS (1)

(4) $0 = xyz - 1 \le 0$

(5) $c - 2 \le 0$

(8) $\lambda_1(0) = 0$

(9) $\lambda_2(c-2) = 0 \iff \lambda_2(c-2) = 0$

Then we get that $\lambda_1 = 0$, $\lambda_2 = 2$. For primal feasibility we need $c \le 2$. But notice that if $c < 2$ then (9) is not fulfilled. Thus it must happen that $c = 2$.

In this case $f(1,1,1) = 2 + 1 = 3$.

For this value of c only the second constraint is active.