Recitation week 10

November 18, 2022

1. For the following problems we are going to derive some of the "formulas" given in the table below:

	Primal		Dual
Criteria	Min	Criteria	Min
Variables	x_j	LHS of restriction	$A^\top y$
Objective function	$z = c^\top x$	RHS of restrictions	c
RHS of restrictions	b	Objective function	$w = -b^\top y$
Type of restriction	$A_l x \geq b$	Restriction on variables	$y_l \geq 0$
Type of restriction	$A_e x = b$	Restriction on variables	y_e free
Type of restriction	$A_g x \leq b$	Restriction on variables	$y_g \leq 0$
Restriction on variables	$x_l \leq 0$	Type of restriction	$A_l^\top y \geq c$
Restriction on variables	x_e free	Type of restrictions	$A_e^\top y = c$
Restriction on variables	$x_g \geq 0$	Type of restrictions	$A_g^\top y \leq c$

Let:

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

Consider the following LP:

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t.}\ & A_1 x \leq b_1 \\ & A_2 x = b_2 \\ & x \geq 0 \end{aligned}$$

According to the previous table, the dual of such problem should be:

$$\begin{aligned} \min\ & -b_1^\top \lambda_1 - b_2^\top \lambda_2 \\ \text{s.t.}\ & A^\top \lambda \leq c \\ & \lambda_1 \leq 0 \end{aligned}$$

Prove this using the KKT conditions as seen in lecture. The Lagrange function for the primal problem is given by:

$$\begin{aligned} L(x, \lambda_1, s_1, s_2) &= c^\top x + \lambda_1^\top (A_1 x - b_1) + s_1^\top (A_2 x - b_2) - s_2^\top x \\ &= (c^\top + \lambda_1^\top A_1 + s_1^\top A_2 - s_2^\top) x - \lambda_1^\top b_1 - s_1^\top b_2 \end{aligned}$$

Then the KKT conditions for this problem are the following:

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= c + A_1^\top \lambda_1 + A_2^\top s_1 - s_2 = 0 \\
A_1 x - b_1 &\leq 0 \\
A_2 x - b_2 &= 0 \\
-x &\leq 0 \\
\lambda_1 &\geq 0 \\
s_2 &\geq 0 \\
\lambda_1^\top (A_1 x - b_1) &= 0 \Rightarrow \lambda_1^\top A_1 x = \lambda_1^\top b_1
\end{aligned}
$$

Then under the KKT conditions:

$$
\begin{aligned}
c &= -A_1^\top \lambda_1 - A_2^\top s_1 + s_2 \\
L(\lambda_1, s_1) &= -\lambda_1^\top b_1 - s_1^\top b_2 \\
c^\top x &= (-A_1^\top \lambda_1 - A_2^\top s_1 + s_2)^\top x \\
&= -\lambda_1^\top b_1 - s_1^\top b_2
\end{aligned}
$$

Then the problem can be stated as follows:

$$
\begin{aligned}
\min\ & -\lambda_1^\top b_1 - s_1^\top b_2 \\
\text{s.t. } & c = -A_1^\top \lambda_1 - A_2^\top s_1 + s_2 \\
& \lambda_1 \geq 0 \\
& s_2 \geq 0
\end{aligned}
$$

Which is equivalent to

$$
\begin{aligned}
\min\ & -\lambda_1^\top b_1 - s_1^\top b_2 \\
\text{s.t. } & -A_1^\top \lambda_1 - A_2^\top s_1 \leq c \\
& \lambda_1 \geq 0
\end{aligned}
$$

Which is:

$$
\begin{aligned}
\min\ & -\lambda_1^\top b_1 - s_1^\top b_2 \\
\text{s.t. } & A_1^\top \lambda_1 + A_2^\top \lambda_2 \leq c \\
& \lambda_1 \leq 0
\end{aligned}
$$

2. Find the dual problem for the following LP

$$\begin{aligned} &\min\ c^\top x \\ &\text{s.t. } a \leq Ax \leq b \\ &\qquad l \leq x \leq u \quad i = 1, 2, ..., n \end{aligned}$$

Notice that we can write this problem as follows:

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t. } & Ax \leq b \\ & -Ax \leq -a \\ & x \leq u \\ & -x \leq -l \end{aligned}$$

Then we can define the following block matrix (which is of size $(2m + 2n) \times n$):

$$\hat{A} = \begin{bmatrix} A \\ -A \\ -I \\ I \end{bmatrix},$$

and the following block vector:

$$\hat{b} = \begin{bmatrix} b \\ -a \\ -l \\ u \end{bmatrix}.$$

Then our original problem can be written as:

$$\begin{aligned} &\min\ c^\top x` \\ &\text{s.t. } \hat{A}x \leq \hat{b}. \end{aligned}$$

Then using the table in the first page we get that the dual problem is:

$$\begin{aligned} &\min\ -b\lambda_1 + a\lambda_2 + l\lambda_3 - u\lambda_4 \\ &\text{s.t. } A^\top\lambda_1 - A^\top\lambda_2 - I\lambda_3 + I\lambda_4 = c \\ &\qquad \lambda_i \leq 0 \end{aligned}$$

3. Compute the dual of the following LP:

$$
\begin{aligned}
\min\ & 4x_1 + 2x_2 - x_3 \\
\text{s.t. } & x_1 + 2x_2 + 2x_3 \leq 6 \\
& x_1 - x_2 + 2x_3 = 8 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

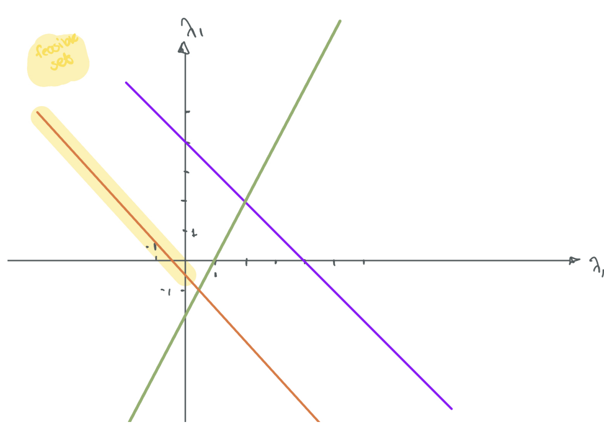
If you were asked to solve this graphically, which one would you solve? The primal or the dual? Why?
Define the following matrix:

$$
A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & 2. \end{bmatrix}
$$

Notice that in this case $b^\top = \begin{bmatrix} 6 & 8 \end{bmatrix}$ and $c^\top = \begin{bmatrix} 4 & 2 & -1 \end{bmatrix}$. We have two restrictions hence the dual problem is going to have two variables, call them λ_1 and λ_2. The first restriction is of the type $\leq$ thus we must impose that $\lambda_1 \leq 0$. The second restriction is of the type $=$ then λ_2 is a free variable. In the primal we have three variables thus in the dual we are going to have three restrictions. For the primal variables we have that $x_1 \geq 0$ and $x_2 \geq 0$ thus the first and second restrictions in the dual if of the type $\leq$. Then we have that x_3 is a free variable then the third restriction in the dual is of the type $=$. Putting this information together we get that the dual problem is:

$$
\begin{aligned}
\min\ & -6\lambda_1 - 8\lambda_2 \\
\text{s.t. } & \lambda_1 + \lambda_2 \leq 4 \\
& 2\lambda_1 - \lambda_2 \leq 2 \\
& 2\lambda_1 + 2\lambda_2 = -1 \\
& \lambda_1 \leq 0.
\end{aligned}
$$

Notice that the feasible set for the dual is part of the line highlighted in yellow below. The first restriction corresponds to the lower half plane below the purple line. The second restriction corresponds to the upper half plane from the green line, and the third restriction corresponds to the orange line.



We can see that this problem is unbounded, thus $\min_{\lambda_1,\lambda_2} 6\lambda_1 - 8\lambda_2 = -\infty$.
This means that the feasible set from the primal problem is the empty set. We can see this directly from the second restriction, as we try to solve for x_3 and then substituting back to the first restriction:

$$\begin{aligned} x_3 &= 4 - \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ &\Rightarrow x_1 + 2x_2 + 2\left(4 - \frac{1}{2}x_1 + \frac{1}{2}x_2\right) \leq 6 \\ &\Rightarrow 3x_2 \leq -2 \\ &\Rightarrow x_2 \leq -\frac{2}{3} < 0 \end{aligned}$$

Which can't happen because we have the restriction that x_2 should be positive.

4. Farkas' Lemma

Consider the separating hyperplane theorem:

Theorem 1 (Separating hyperplane theorem)**.** *Let A and B be two disjoint nonempty convex subsets of $\mathbb{R}^n$. Then there exists a nonzero vector b and a real number r such that*

$$y^\top b \geq r \text{ and } x^\top b \leq r$$

for all $y \in A$ and $x \in B$.

We are going to prove Farkas' Lemma:

Theorem 2 (Farkas' lemma)**.** *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following statements is true:*

- *There exists an $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.*
- *There exists a $y \in \mathbb{R}^m$ such that $A^\top y \geq 0$ and $b^\top y < 0$.*

Take the following steps

(a) Can it happen the following: there is an $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$ then there exists a $y \in \mathbb{R}^m$ such that $A^\top y \geq 0$ and $b^\top y < 0$? Why?

(b) Define the following set, called a conic hull:

$$C = \text{cone}\{a_1, ..., a_n\} = \left\{ \sum_{i=1}^{n} \alpha_i a_i \ : \ \alpha_i \geq 0 \right\},$$

is this set convex? Is it closed?

(c) If $b \notin C$, is b feasible?

(d) If $b \notin C$, can it be separated from C?

What is the geometric interpretation of Farkas' lemma?