Recitation week 12
December 2, 2022

1. *Interior points method.* Recall the following for the interior point method

$$F(x, \lambda, s) = \begin{bmatrix} A^\top \lambda + s - c \\ Ax - b \\ XS\mathbb{1} - \mu\mathbb{1} \end{bmatrix}$$

Where λ, s are the Lagrange multipliers, $X = \operatorname{diag}(x_1, ..., x_n)$, $S = \operatorname{diag}(s_1, s_2, ..., s_n)$, $\mathbb{1} = [1, 1, ..., 1] \in \mathbb{R}^n$. Consider the following LP:

$$\begin{aligned} \min\ & x_1 \\ \text{such that } & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) Show that the primal-dual solution (x^*, λ^*, s^*) is given by:

$$x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \lambda^* = 0,\ s^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(b) Suppose that someone implemented the interior point method and got certain solution (x, λ, s). They claim that is the correct solution to the LP. Let that (x, λ, s) be:

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \lambda = 1,\ s = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Show that this solution satisfies $F(x, \lambda, s) = 0$.

(c) Show that that solution, (x, λ, s) is not dual feasible.

(d) What happened? Is this an optimal solution? What might have gone wrong during the implementation?

Writing down the Lagrange function:

$$L(x_1, x_2, \lambda, s_1, s_2) = x_1 + \lambda(x_1 + x_2 - 1) - s_1 x_1 - s_2 x_2.$$

We have that:

$$\begin{aligned} \frac{\delta L}{\delta x_1} &= 1 + \lambda - s_1 \\ \frac{\delta L}{\delta x_2} &= \lambda - s_2 \end{aligned}$$

Thus it must happen that:

$$
\begin{aligned}
1 + \lambda - s_1 &= 0 \\
\lambda - s_2 &= 0 \\
x_1 + x_2 &= 1 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
s_1 x_1 &= 0 \\
s_2 x_2 &= 0 \\
s_1 &\geq 0 \\
s_2 &\geq 0
\end{aligned}
$$

Notice that it must happen that $\lambda = s_2$, thus if s_2 is zero then $s_1 = 1$, implying that $x_1 = 0$. This means that $x_2 = 1$. Then the optimum of the function would be 0.

On the other hand, if $s_2 \neq 0$ then it must happen that $x_2 = 0$. Thus $x_1 = 1$. Then $s_1 = 0$. Then we have that $x_2 = 0$. In this case the function value would be 1.

Then the optimum is 0 with the following optimal variables:

$$
x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \lambda^* = 0,\ z^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
$$

Substituting the claimed solution to F we get:

$$
\begin{aligned}
A^\top \lambda + s - c &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \\
Ax - b &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 = 0 \\
XS\mathbb{1} - \mu\mathbb{1} &= 0 - 0 = 0
\end{aligned}
$$

Thus $F(x, \lambda, s) = 0$.

The dual problem is the following:

$$
\begin{aligned}
\min\ & -\lambda \\
\text{such that } & \lambda \leq 1 \\
& \lambda \leq 0
\end{aligned}
$$

Which can be written as:

$$\begin{aligned} \min\ & -\lambda \\ \text{such that } & \lambda \leq 0 \end{aligned}$$

Notice that in the proposed solution $\lambda = 1$ thus this solution is not dual feasible.

This can't be an optimal solution because it's not dual feasible. Also notice that $s_2 = 1 \leq 0$. In the interior points method it should always happen that $(x_k, s_k) \geq 0$. Then checking for this condition might be the problem in the implementation.

2. *Lagrangian and dual function.* Consider the optimization problem:

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

with variable $x \in \mathbb{R}$.

(a) Give the feasible set, the optimal value, and the optimal solution. The feasible set is:

$$X_f = \{x|2 \leq x \leq 4\}.$$

The optimal value is $x^{*2}+1=5$ at $x^*=2$.

(b) Plot the objective function x^2+1 versus x. On the same plot show the feasible set, optimal point and value, and plot the Lagrangian $\mathcal{L}(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \underset{x}{inf}\mathcal{L}(x,\lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

The Lagrange function is the following:

$$\mathcal{L}(x,\lambda) = x^2+1+\lambda((x-2)(x-4)).$$

The Lagrange dual function g is the following:

$$\begin{aligned} g(\lambda) &= \underset{x}{inf}\mathcal{L}(x,\lambda) \\ &= \underset{x}{inf}\ x^2+1+\lambda((x-2)(x-4)) \end{aligned}$$

Then we calculate that infimum when $2 \leq x \leq 4$:

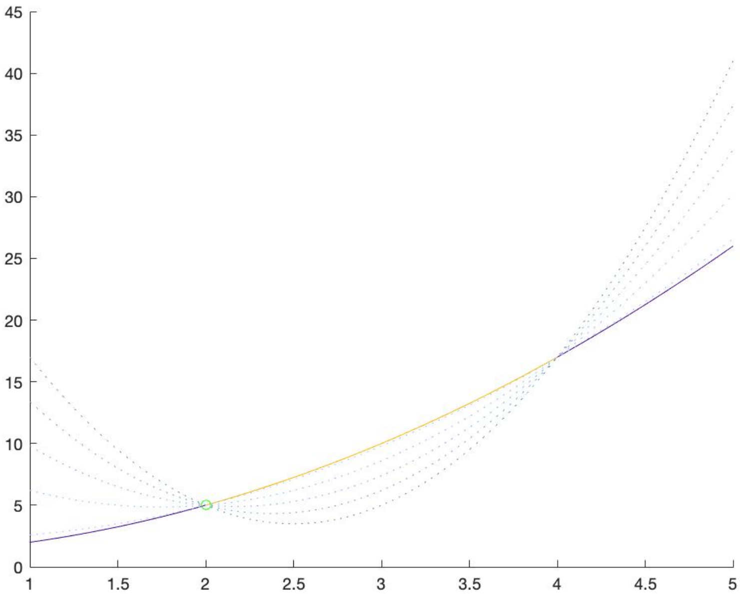
$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial x} = 0 \\ \iff\ & 0 = 2x+2\lambda x-6\lambda = x(2+2\lambda)-6\lambda \\ &\Rightarrow x = \frac{6\lambda}{2+2\lambda}. \end{aligned}$$

Thus we have that:

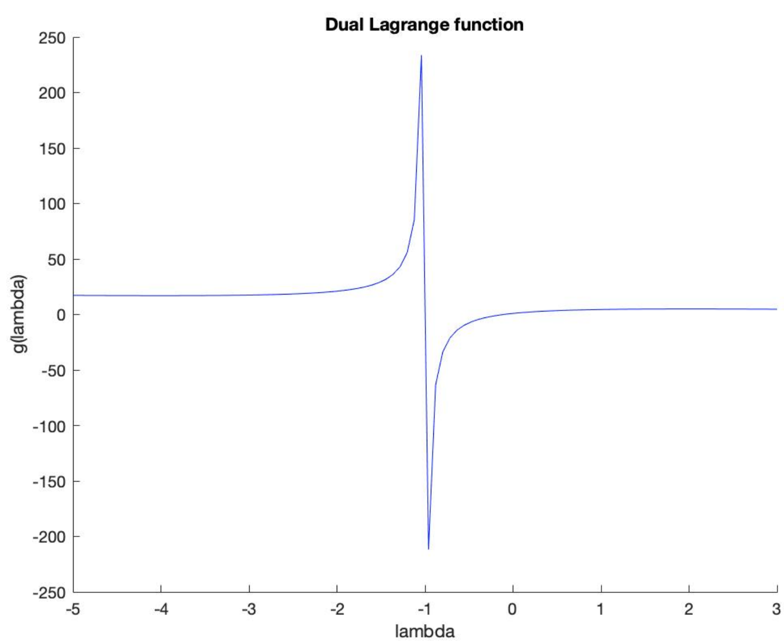
$$g(\lambda) = \left(\frac{6\lambda}{2+2\lambda}\right)^2 + 1 + \lambda\left(\frac{6\lambda}{2+2\lambda}-2\right)\left(\frac{6\lambda}{2+2\lambda}-4\right).$$

But the domain of this *lambda* is restricted thanks to the restriction that $2 \leq x \leq 4$. Since $\lambda \geq 0$, then our domain for the lambdas has to be $0 \leq \lambda \leq 2$.

In Figure (a) we can see the optimal point in green. In dark purple we have the objective function. In orange there is the feasible set. The dotted functions are the Lagrange function for different values of λ.



(a) Feasible set, optimal point, Lagrangian



(b) Lagrange dual function

(c) Does strong duality hold? YES!

3. *Strong duality.* Consider the one-variable optimization problem:

$$\begin{aligned} &\min e^x \\ &\text{s.t. } x \geq 0 \end{aligned}$$

Obviously, the primal solution is $x = 0$ with $p^* = 1$.

(a) Write down the Lagrangian and determine the Lagrange dual function $g(\lambda)$ for all $\lambda \in \mathbb{R}$. Either sketch it by hand or use Python to do so. (sketch it for points between 0 and 2)

(b) Solve the Lagrange dual problem by maximizing $g(\lambda)$ over $\lambda \geq 0$. Does strong duality hold?

The Lagrangian is:

$$\mathcal{L}(x, \lambda) = e^x - \lambda x.$$

Then the Lagrange dual function is:

$$\begin{aligned} g(\lambda) &= \inf_{x \geq 0} \mathcal{L}(x, \lambda) \\ &= \inf_{x \geq 0} e^x - \lambda x \\ &= \lambda - \lambda \log(\lambda). \end{aligned}$$

Plotting this:

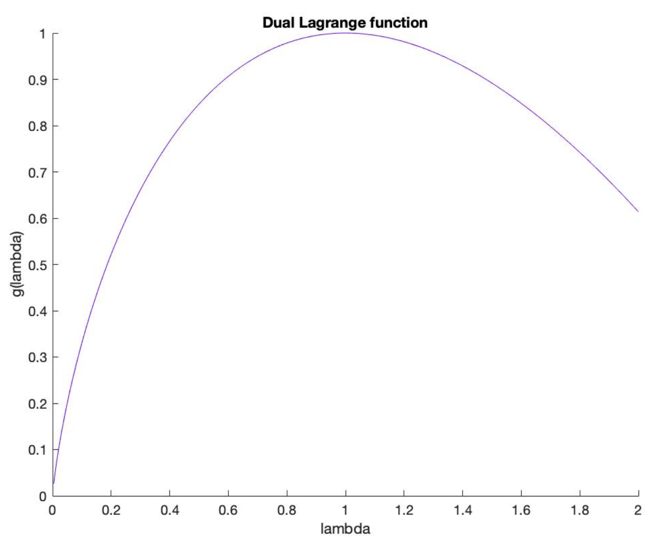


Figure 2: Lagrange dual function, $g(\lambda)$

To solve this we just have to calculate the derivative and make it equal to zero.

A plot

$$\begin{aligned}
\nabla g\left(\lambda\right) &= 0 \\
\iff 0 &= 1 - \log\left(\lambda\right) - \frac{\lambda}{\lambda} \\
&= \log(\lambda) \\
\iff 1 &= \lambda \\
\Rightarrow g\left(\lambda^*\right) &= 1 = p^*.
\end{aligned}$$

So strong duality does hold.

4. *Week duality.* Consider the following optimization problem:

$$\begin{aligned} &\min e^{-x} \\ &\text{s.t. } x^2/y \leq 0, \end{aligned}$$

with variables x and y in the domain $D = \{(x, y) : y \geq 0\}$.

(a) Verify that this is a convex optimization problem and find the optimal value.

(b) Give the Lagrange dual problem and find the optimal solution λ^* and optimal value d^* of the dual problem.

(c) Does strong duality hold? What is the optimal duality gap? Why do you think this happens?

Notice that $x^2 \geq 0$ for all x thus if $(x, y) \in \mathcal{D}$ then it must happen that $x = 0$. Thus the feasible set is $(0, y)$ for $y \geq 0$. This is a convex set. The function e^{-x} is a convex function. Thus this is a convex optimization problem.

Since the the feasible set is $(0, y)$ for $y \geq 0$ and the objective function only depends on x we have that the optimal value has to be $e^0 = 1$. This problem is the same as:

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x = 0 \end{array}$$

The Lagrange function is:

$$\mathcal{L}(x, \lambda) = e^{-x}$$

Thus:

$$g(\lambda) = 0.$$

Thus $d^* = 0$. Then the optimal duality gap is 1, strong duality doesn't hold.