Problem 1

Recall that, from given nodes $[x_0, ..., x_n]$ and function evaluations at those points, $[y_0, ..., y_n]$ we have the unique Lagrange polynomial:

$$P_n(x) = \sum_{i=o}^{n} y_i l_i(x),$$

where:

$$l_i = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}.$$

Also recall Rolle's theorem:

Theorem 1. *Let $f(x)$ be continuous on $[a, b]$, differentiable on (a, b), and suppose $f(a) = f(b) = 0$. Then there exists a point $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

- Find the Lagrange polynomial of degree 16 solving the interpolation problem $p(x_i) = 1$, for all $0 \leq i \leq 16$. (By hand)
- Prove that:

$$\sum_{i=0}^{n} l_i(x) = 1.$$

- Let $f \in \mathcal{C}^{n+1}([a, b])$ and let $p(x)$ be the Lagrange polynomial of degree n interpolating $f(x)$ at $a = x_0 < x_1 < ... < x_n = b$. Prove that for each $x \in [a, b]$ there exists a $\xi \in (a, b)$ such that:

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n).$$

- Let $h = \max_i(x_{i+1} - x_i)$, for any $x \in [x_0, x_n]$. Prove that:

$$|w(x)| = |(x - x_0)\cdots(x - x_n)| \leq \frac{n! h^{n+1}}{4}.$$

- Conclude that an error bound for the Lagrange polynomial interpolation is given by:

$$\|f - p\| \leq \frac{\|f^{n+1}\| h^{n+1}}{4(n+1)}.$$

Problem 2

In this section we're going to explore more the Runge's phenomenon (and code). Suppose we want to interpolate in $[-1,1]$ the following function:

$$f(x) = \frac{1}{1+25x^2}.$$

For every "task" try to estimate the error of the interpolation with the ∞ or Chebyschev norm:

$$\|g\|_\infty = \max_{t\in[-1,1]} g(t).$$

- First suppose we have equally spaced point in the interval $[-1,1]$:

  $$x_j = -1 + \frac{2j}{n}, \quad j = 0, 1, ..., n.$$

  For $n = 2, 5, 8, 14$ find the interpolating polynomial (using Python + your desired built in functions) and plot these interpolations.

- Now suppose we put more nodes (or points x_j at the ends of the interval). Generate a points in x such that $n/3$ of those points are in the section $[-1,-0.6]$, $n/3$ points are in the section $[-0.6, 0.6]$ and the rest $n/3$ points are in the section $[0.6, 1]$. For $n = 3, 6, 9, 15$ find the interpolating polynomial (using Python + your desired built in functions) and plot these interpolations.

- Now suppose we have nodes that are given by:

  $$x_j = \cos\left(\frac{2j-1}{2n}\pi\right), \quad j = 1, ..., n.$$

  For $n = 3, 6, 9, 15$ find the interpolating polynomial (using Python + your desired built in functions) and plot these interpolations. These nodes are the *very* famous Chebyshev nodes and they have *very* nice properties. Using problem set 1, how large must n be to guarantee $\|f - p\| \leq 0.01$?

Problem 3

We in groups of 3 and discuss the following:

Don't do everything in "automatic mode"

- Suppose we have the following 4 points: $(1, 8.8)$, $(-2.5, -8.7)$, $(4.2, 24.8)$, and $(168.3, 845.3)$. Don't do any calculations yet or try to sketch these points. Which is the "obvious" degree of the polynomial needed to interpolate these points?
- Try interpolating these points with a polynomial of degree of your choice. (Use a Vandermonde matrix)
- Sketch these points out. (If you can use Python/Matlab/Wolfram) What do you notice?
- Which is the "right" degree to use? How could you have noticed this without sketching these points out?

Vandermonde matrices

- Calculate the determinant for the following Vandermonde matrix:

$$V_1 = \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix}.$$

- Calculate the determinant for the following Vandermonde matrix:

$$V_2 = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}.$$

- Calculate the determinant for the following Vandermonde matrix:

$$V_3 = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}.$$

- Do you notice a pattern? What do you think might be the problem with this matrix as its size gets bigger?