Problem set 1

Solve the following sets of linear algebra problems (by hand), explain what you did in order to be able to solve them (don't calculate inverses of matrices). Write a set of "instructions" to solve those problems.

1. Solve the following system of equations:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ 12 & 9 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. Solve both of the systems:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

3. Solve the following system of equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 2 \\ 2 & 18 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 10 \end{bmatrix}.$$

4. Solve both of the systems:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 10 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -16 \end{bmatrix}.$$

5. Solve the following system of equations:

$$\begin{bmatrix} 1 & 2 & 8 \\ 2 & 0 & -1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \\ 0 \end{bmatrix}.$$

6. Solve both of the systems:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 14 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 14 \end{bmatrix}.$$

7. Perform the following matrix-matrix multiplication:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$$

Problem set 2

A permutation can be seen as a matrix that is "like" the identity matrix but with some of its rows (or columns) moved from their "original" place. There is another way to write this idea of "switching" things from their original place. One of those ideas is Cauchy's two-line notation. Here one lists the elements of I (the indices) in the first row, and for each one its image below it in the second row. For us right now those indices are the e_i vectors, the canonical vectors. These vectors have 0's on every one of their entries except on the $i-$th entry, which is a one. A particular permutation of the set $I = \{1, 2, 3\}$ can be written as:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

This permutation can also be written in matrix form:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

1. Write the permutation (in matrix form and in Cauchy's two-line form) for the following to be true:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 7 \end{bmatrix} = P \begin{bmatrix} 1 & 2 & 8 \\ 2 & 0 & -1 \\ 2 & 4 & 2 \end{bmatrix}$$

2. Write a permutation such that it interchanges the least amount of rows and such that when performing Gaussian elimination on A, non of the elements in the diagonal are zero. What do these permutations have in common? (And do Gaussian elimination)

   (a) $A = \begin{bmatrix} 0 & 1 & 1 \\ 4 & 2 & 4 \\ 1 & 2 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

Problem set 3

In this section we want to practice modelling with matrices. In stochastic processes (you don't need to know what they are) a continuous time Markov Chain can be characterized with its generator. We are going to explore such matrices.

Suppose that you're working in a coffee shop. Clients arrive at a rate of 1.5 clients per minute (whatever that means). If there are less than 5 clients, you take your time to prepare the coffee for them and even do some latte art. When this happens, you deliver their orders at a rate of 1 client per minute. Assume that as soon as the clients have their coffee they leave, they're not that friendly. But if there are 5 or more clients you start to get a bit anxious and don't have time to do all that latte art and be nice so you prepare the coffee faster and get the orders done at a rate of 2 clients per minute. This is a continuous time Markov Chain (don't worry with this fancy name). We can represent how clients arrive and leave with a matrix. In this case its an infinite matrix because we assume that there can be infinite amount of clients inside the coffee shop (because it's a really really big shop and there is no covid).

The entries of the generator G on the off diagonal represent the different rates of changing from state i to j (the rate of the change in the amount of clients). The diagonal entries, $-q_{ii}$ are the overall rates of leaving each state i.

1. What's the row sum of such matrix?
2. Which entries are positive?
3. Which entries are negative?
4. Write down part of such matrix. (We can't write the whole matrix because it's infinite)
5. Write down a code that calculates the entries of such matrix

6. Write down a code that outputs a “chopped” version of such matrix

7. Using the built in LU factorization from Python, calculate the matrices L, U, and P for a “chopped” 5×5 matrix. What do you expect from the P matrix?

8. Using the built in LU factorization from Python, calculate the matrices L, U, and P for a “chopped” 6×6 matrix. What do you expect from the P matrix?

A “mean first passage” can be seen as the average first time to reach certain number of clients if you start the day with certain other amount of clients. Suppose you start the day with 10 clients (of course, coffee is great, they were waiting for you). You want to know, in average how long it takes you to reach 0 clients. In order to solve this we need to solve the following system of linear equations:

$$Q'\tau = -1,$$

where -1 is the according size vector with all its entries equal to -1. Q' is the matrix Q but with the 0-th row and 0-th column deleted. The 10-th entry in τ lets you know this mean first passage time.

1. “Chop” the matrix so that we get a 25×25 size matrix.

2. With this approximation of the matrix, the time when you can take a break.