Problem 1

In this section we'll explore different ways of written matrix products. Note that any matrix, $A \in \mathbb{R}^{m\times n}$ can be written in "multiple ways". **Remember that we always write vectors as column vectors, i.e.** $v \in \mathbb{R}^n$ **is a column vector of size** $n \times 1$**.**

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}, \tag{1}$$

this is how we write a matrix element by element (which in some cases is not very useful. Another way of writing the same matrix is:

$$A = \begin{bmatrix} a_1 & a_1 & \dots & a_n \end{bmatrix}, \tag{2}$$

where **each** $a_i \in \mathbb{R}^m$ and they are the **columns** of A, i.e. $a_i = \begin{bmatrix} a_{1,i} \\ a_{2,i} \\ \vdots \\ a_{m,i} \end{bmatrix}$ and there are **n of those m sized vectors**. But we can also write this matrix as:

$$A = \begin{bmatrix} a_1^{(1)\top} \\ a^{(2)\top} \\ \vdots \\ a^{(m)\top} \end{bmatrix}, \tag{3}$$

where **each** $a^j \in \mathbb{R}^n$ and $a^{(j)\top}$ are the **rows** of A and there are **m of those n sized vectors**. Meaning that $a^{(j)\top} = \begin{bmatrix} a_{j,1} & a_{j,2} & & a_{j,n} \end{bmatrix}$.

Work in teams of size 2 (or 3) and answer the following questions. Let $A, B \in \mathbb{R}^{n\times n}$,

- One of you expand the following product of matrices written as

$$AB = \begin{bmatrix} a_1 & a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1^{(1)\top} \\ b^{(2)\top} \\ \vdots \\ b^{(n)\top} \end{bmatrix},$$

  this can be seen as an "outer product summation".

- The other one of you expand the following product of matrices written as

$$AB = \begin{bmatrix} a_1^{(1)\top} \\ a^{(2)\top} \\ \vdots \\ a^{(n)\top} \end{bmatrix} \begin{bmatrix} b_1 & b_1 & \dots & b_n \end{bmatrix},$$

this can be seen as a matrix of inner products.

- Compare your results. Prove (together) that they are in fact, equal (using the definition of each entry of a matrix-matrix multiplication). When do you think each of these representations is useful?

Problem 2

In this section you will also work in teams of size 2 (or 3). Let

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Both of you are going to apply the power method (just calculate 3 iterations). Draw the resulting vector after each iteration.

- One of you will start with $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- The other one will start with $v_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Now calculate the exact Eigenvalues and Eigenvectors, compare both of your iterations with the exact result.

Problem 3

A matrix $Q \in \mathbb{R}^{n\times n}$ is said to be orthogonal if and only if $Q^\top Q = I$, the identity matrix of size $n \times n$.

- Prove that $det(Q) = \pm 1$.
- Let $u \in \mathbb{R}^n$, write the matrix $uu^\top$ in the "element by element" way (as in 1).

- Prove that $uu^\top$ is a range 1 matrix. Describe (geometrically) its kernel or null space. Describe (geometrically) its range.
- Prove that $Q = I - 2u^{u\top}$ with $\|u\|_2 = 1$ is an orthogonal matrix.

Problem 4

Let $u, v, w, z \in \mathbb{R}^n$. Define the following matrices:

$$A = uv^\top + wz^\top \quad \hat{A} = \begin{bmatrix} v^\top u & v^\top w \\ z^\top u & z^\top w \end{bmatrix}.$$

- What is the size of $\hat{A}$?
- What is the size of A? What are the possible values of its range (assuming all of u, v, w, z are nonzero)?
- If λ_1 and λ_2 are Eigenvalues of $\hat{A}$ prove that $0, \lambda_1$, and λ_2 are Eigenvalues of A.

Problem 5

Let A be an $n \times n$ matrix and such that the sum of each of its columns equals 1. Meaning that if a_i is the i-th column of A then $\sum_{j=1}^n a_{j,i} = 1$.

- Prove that A has at least one Eigenvalue which is equal to 1.
- Prove that if λ is an Eigenvalue of A then $|\lambda| \leq 1$.