Problem 7 Consider the following approximation to the integral of f on an interval $[a,b]$:

$$\int_a^b f(x)dx \approx f\left(\frac{a+b}{2}\right)(b-a).$$

Suppose that $b-a=\mathcal{O}(h)$. Derive an error estimate for this approximation using a Taylor expansion. What assumption(s) do you need to take into consideration?

If we define $\bar{x} := \frac{a+b}{2}$, then notice that the error can be written as:

$$E = \int_a^b f(x)dx - f(\bar{x})(b-a) = \int_a^a f(x) - f(\bar{x})dx.$$

Then notice that $f(x) = f((x-\bar{x})+\bar{x})$. Then we can expand $f(x)$ with Taylor's series such as:

$$f(x) = f((x-\bar{x})+\bar{x}) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + f''(\xi)\frac{(x-\bar{x})^2}{2}.$$

Then:

$$\begin{aligned}
f(x) &= \int_a^b f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + f''(\xi)\frac{(x-\bar{x})^2}{2} - f(\bar{x})dx \\
&= \int_a^b f'(\bar{x})(x-\bar{x}) + f''(\xi)\frac{(x-\bar{x})^2}{2}dx \\
&= f'(\bar{x})\int_a^b x-\bar{x}dx + \frac{f''(\xi)}{2}\int_a^b (x-\bar{x})^2dx \\
&= f'(\bar{x})\left(x^2/2 - \bar{x}x\right)\Big|_a^b + \frac{f''(\xi)(b-a)^3}{24} \\
&= \frac{f''(\xi)(b-a)^3}{24}.
\end{aligned}$$

Then we need to have the assumption that f'' is bounded in the interval $[a,b]$. If this happens, then we have that $|f''(\xi)| \leq M$. Then our error is:

$$\frac{M(b-a)^3}{24}.$$