Problem 1: Linear Algebra

Problems with triangular matrices

A unit lower triangular matrix, A of size $n \times n$ is a matrix that has the following form:

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_{21} & 1 & 0 \dots & 0 & \\ a_{31} & a_{32} & 1 \dots & 0 & \\ \vdots & \vdots & \vdots \ddots & \vdots & \\ a_{n1} & a_{n2} & a_{n3} \dots & 1 & \end{bmatrix}.$$

By definition, the Frobeinus norm of a matrix A is the following:

$$\|A\|_F^2 = \langle A, A \rangle = trace\left(A\right) = \sum_i^n \sum_j^n |a_{ij}|^n = \sum_i^n \lambda_i\left(A\right).$$

Recall that the product of two matrices, $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ is:

$$\left(AB\right)_{ij} = \sum_k^n a_{ik} b_{kj}.$$

1. Let B be a lower triangular matrix with zeros in its diagonal entries. Prove that N^n is the zero matrix.
2. Let $L = I - N$ be a unit lower triangular matrix of size $n \times n$. Show that its inverse is given by:

$$L^{-1} = I + N + N^2 + ... + N^{n-1}.$$

3. Calculate $\|L^{-1}\|_F$ if $N_{ij} = 1$ for all $i > j$.
4. Prove that the product of two lower triangular matrices is also lower triangular.

Problem 2: Numerical Linear Algebra

Suppose that we have two matrices $S, T \in \mathbb{R}^{n \times n}$ such that both of them are upper triangular. Suppose that $\left(ST - \lambda I\right) x = b$ is a nonsingular system. Give an $\mathcal{O}\left(n^2\right)$ algorithm for computing x.