Mustafa MAJJI

Portfolio Optimization : Hierarchical Risk Parity

1 The Hierarchical Risk Parity

The Hierarchical Risk Parity (HRP) model is a portfolio optimization framework designed to balance risk contributions across a portfolio. It achieves this by constructing a tree-like structure, or dendrogram, that groups assets based on their similarity, typically measured using metrics such as correlations or other dependence measures.

Step1 : Hierarchical Tree Clustering

In this step, assets are grouped into hierarchical clusters using distance as the criterion to group assets within the same cluster. In fact, Marcos López de Prado, in his paper "Building Diversified Portfolios that Outperform Out-of-Sample", proposed the following metrics for grouping assets :

First, the correlation matrix is transformed into a correlation-distance D, where the distance between two assets is defined as :

$$D(i,j) = \sqrt{0.5 \cdot (1 - \rho(i,j))}$$

Second, we calculate another distance matrix $\overline{D}$ where :

$$\overline{D}(i,j) = \sqrt{\sum_{k=1}^{N} (D(k,i) - D(k,j))^2}$$

Third, we begin grouping assets based on the minimum distance. To combine different clusters, various types of linkage methods can be used, each of which can influence the outcome of the clustering process :

— Single linkage : the distance between two clusters is defined as the minimal distance of any two elements in the clusters

$$d_{C_i,C_j} = \min_{x \in C_i, y \in C_j} d(x,y)$$

— Complete linkage : the maximum distance between any two elements of the clusters

$$d_{C_i,C_j} = \max_{x \in C_i, y \in C_j} d(x,y)$$

— Average linkage : the average of the distance of any two elements in the examined clusters

$$d_{C_i,C_j} = \frac{1}{|C_i|}\frac{1}{|C_j|}\sum_{x\in C_i}\sum_{y\in C_j} d(x,y)$$

— Ward's linkage :

$$d_{C_i,C_j} = \frac{|C_i|\cdot|C_j|}{|C_i|+|C_j|}\|\mu_i-\mu_j\|^2$$

where μ_i and μ_j are the centers of clusters C_i and C_j.

Step 2 : Quasi-Diagonalization

After selecting the linkage method, the data should be reordered according to the structure of the dendrogram. As a result, similar investments are placed together, and dissimilar investments are placed far apart.

Step 3 : Recursive Bisection

Since inverse-variance allocations are the most optimal for a diagonal covariance matrix, and we are dealing with a quasi-diagonalized matrix, it makes sense to use the inverse-allocation weights to calculate the variance for this subcluster.

The Top-Down allocation algorithm is as follows :

— Initialize the weighting for each security to 1, $w_i = 1$, where $i \in [|1,n|]$.
— Bisect the portfolio into two sets, s_1 and s_2.
— Let V_i be the covariance matrix for set s_i.
— Let $W_i = \mathrm{diag}(V_i)^{-1} \times \frac{1}{\mathrm{tr}(\mathrm{diag}(V_i)^{-1})}$.
— Let $V_{\mathrm{adj},i} = W_i^T \times V_i \times W_i$.
— Let $a_1 = 1 - \frac{V_{\mathrm{adj},1}}{V_{\mathrm{adj},1}+V_{\mathrm{adj},2}}$, and $a_2 = 1 - a_1$.
— Adjust weightings for each set as $w_{s_i} = w_{s_i} \times a_i$.