0.1. FUNCTIONS

DEFINITION OF A FUNCTION:

If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that **y is a function of x**. Functions is like computer machine, when we give input to computer it give us output. Similarly function is also work as computer.

$$x \xrightarrow{\text{Input}} \boxed{f} \xrightarrow{\text{Outp}} y \text{ or } f(x)$$

Four common methods for representing functions are:

1. Numerically by tables.
2. Geometrically by graphs.
3. Algebraically by formulas.
4. Verbally.

INDEPENDENT AND DEPENDENT VARIABLES:

For a given input ***x***, the output of a function ***f*** is called the value of ***f*** at ***x*** or the image of ***x*** under ***f***. Sometimes we will want to denote the output by a single letter, say ***y***, and write $y = f(x)$.This equation expresses ***y*** as a function of ***x***; the variable ***x*** is called the ***independent variable*** and the variable ***y*** is called the ***dependent variable***.

Example 1: the equation $y = 3x^2 - 4x + 2$ has the form of $y = f(x)$ in which the function f is given by the formula $f(x) = 3x^2 - 4x + 2$

Solution: Let $[x = 0, -1.7 \; and \; \sqrt{2}]$, then:

$$y = f(x) = 3(0)^2 - 4(0) + 2 = 2$$

$$y = f(x) = 3(-1.7)^2 - 4(-1.7) + 2 = 17.47$$

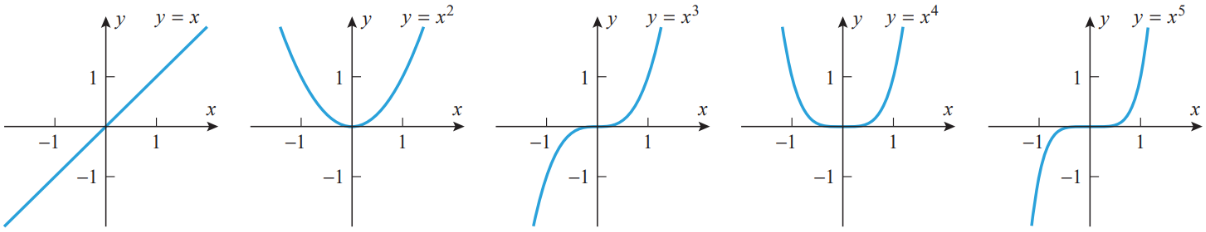
$$y = f(x) = 3\left(\sqrt{2}\right)^2 - 4\left(\sqrt{2}\right) + 2 = 8 - 4\sqrt{2}$$

GRAPHS OF FUNCTIONS

If ***f*** is a real-valued function of a real variable, then the graph of ***f*** in the *x*y-plane is defined to be the graph of the equation $y = f(x)$.

Example 2: The graph of the function $f(x) = x$ is the graph of the equation $y = f(x)$

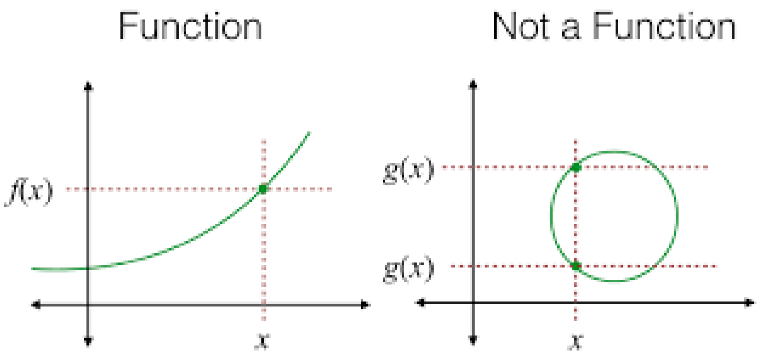
Solution: Let $y = x$, $y = x^2$ *and* $y = x^3$ respectively.



THE VERTICAL LINE TEST

Lines which are parallel to $y - axis$ is called **Vertical Line** and to check whether the graph in $xy - plane$ is graph of function or not that is called **Vertical Line Test**.

Definition: If a vertical line touch the curve in the $xy - plane$ two or more than two points, then we say the curve is not graph of function.



> *Rule:*
>
> *If any input appears with two or more different outputs → not a function.*
>
> *If every input appears with exactly one output → function.*

PIECEWISE-DEFINED FUNCTION

*A **piecewise-defined function** is a function that is described by **different formulas or expressions depending on the input value (the "piece" of the domain)**. In other words, instead of having one single rule for all inputs, the function is split into parts, and each part has its own rule that applies to a specific interval of the input.*

The absolute value function $f(x) = |x|$ is an example of a function that is defined piecewise in the sense that the formula for ***f*** changes, depending on the value of x.

Example 4: Evaluate Piecewise Function If $f(x) = \begin{cases} 2x - 3 & if \quad x \le 4 \\ 5 - x^2 & if \quad 4 < x \le 6 \\ 7 & if \quad x > 6 \end{cases}$

Find the value of $f(-3), f(6), and\ f(25)$

Solution:

$$f(-3) = 2x - 3 = 2(-3) - 3 = -6 - 3 = -9$$

$$f(6) = 5 - x^2 = 5 - 6^2 = 5 - 36 = -31$$

$$f(25) = 7$$

Example 5: If $f(x) = \begin{cases} x + 1, & x \le -2 \\ 3, & -2 < x \le 1 \\ -x^2 & 1 < x \end{cases}$ Evaluate$f(-4)$, $f(0)$, and $f(4)$?

Solution:

1. $f(-4) = x + 1 = -4 + 1 = -3$
2. $f(0) = 3$
3. $f(4) = -x^2 = -(4)^2 = -16$

DOMAIN AND RANGE

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are related by the equation $y = f(x)$, then the set of all allowable inputs ($\boldsymbol{x}$-values) is called the domain of $\boldsymbol{f}$, and the set of outputs ($\boldsymbol{y}$-values) that result when $\boldsymbol{x}$ varies over the domain is called the range of $\boldsymbol{f}$.

Definition: If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the ***natural domain*** of the function.

Example 6: Find the natural domain of:-

A. $f(x) = x^3$
B. $f(x) = \frac{1}{[(x-1)(x-3)]}$
C. $f(x) = \sqrt{x^2 - 5x + 6}$
D. $f(x) = \tan x$

Solution (A): The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

Solution (B): The function f has real values for all real x, except $x = 1$ and$x = 3$, where divisions by zero occur. Thus, the natural domain is $D: (-\infty, 1) \cup (1,3) \cup (3, +\infty)$

Solution (C): The function $\boldsymbol{f}$ has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers $\boldsymbol{x}$ such that

$$f(x) = \sqrt{x^2 - 5x + 6} = \sqrt{(x-3)(x-2)} \quad Domain: (-\infty, 2] \cup [3, +\infty)$$

Solution (D): Since $f(x) = \tan x = \frac{\sin x}{\cos x}$, the function f has real values except where cos x = 0, and this occurs when x is an odd integer multiple of π/2. Thus, the natural domain consists of all real numbers except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

3.1. NEW FUNCTION FROM OLD

ARITHMETIC OPERATIONS ON FUNCTIONS

Two functions, f and g, can be added, subtracted, multiplied, and divided in a natural way to form new functions.

A. $(f+g)(x)=f(x)+g(x)$
B. $(f-g)(x)=f(x)-g(x)$
C. $(fg)(x)=f(x)g(x)$
D. $(f/g)(x)=f(x)/g(x)$

Example-1: Let $f(x)=1+\sqrt{x-2}$ and $g(x)=x-3$. Find $\left[f+g, f-g, fg, \frac{f}{g}, and 7f\right]$?

4. $(f+g)(x)=f(x)+g(x)=\left(1+\sqrt{x-2}\right)+(x-3)=x-2+\sqrt{x-2}$
5. $(f-g)(x)=f(x)-g(x)=\left(1+\sqrt{x-2}\right)-(x-3)=4-x+\sqrt{x-2}$
6. $(fg)(x)=f(x)g(x)=\left(1+\sqrt{x-2}\right)(x-3)$
7. $(f/g)(x)=\frac{f(x)}{g(x)}=\frac{(1+\sqrt{x-2})}{(x-3)}$
8. $(7)f(x)=(7)\left(1+\sqrt{x-2}\right)=7+7\sqrt{x-2}$

COMPOSITION OF FUNCTIONS

We now consider an operation on functions, called composition, the operation of composition is performed by substituting some function for the independent variable of another function. For example, suppose that $f(x)=x^2$ and $g(x)=x+1$, if we substitute $g(x)$ for $\boldsymbol{x}$ in the formula for $\boldsymbol{f}$, we obtain a new function $f\left(g(x)\right)=\left(g(x)\right)^2=(x+1)^2$.

Given functions f and g, the composition of f with g, denoted by fog, is the function defined by $(fog)(x)=f\left(g(x)\right)$.

Example-2: Let $f(x)=x^2+3$ and $g(x)=\sqrt{x}$ Find

1. $(fog)(x)=?$
2. $(gof)(x)=?$

$(fog)(x)=f\left(g(x)\right)=[g(x)]^2+3=\left(\sqrt{x}\right)^2+3=x+3$

$(gof)(x)=g\left(f(x)\right)=\sqrt{f(x)}=\sqrt{x^2+3}$

Compositions can also be defined for three or more functions; for example, $(fogoh)(x)$ is computed as $(fogoh)(x)=f\left(g\left(h(x)\right)\right)$

Example-3: Let $f(x)=\sqrt{x},\ g(x)=\frac{1}{x},\ h(x)=x^3$ Find $(fogoh)(x)$ =?

$$(fogoh)(x)=f\left(g\big(h(x)\big)\right)=f\big(g(x^3)\big)=f(1/x^3)=\sqrt{\frac{1}{x^3}}=\frac{1}{x^{\frac{3}{2}}}$$

EXPRESSING A FUNCTION AS A COMPOSITION

Many problems in mathematics are solved by "decomposing" functions into compositions of simpler functions. For example, consider the function ***h*** given by $h(x)=(x+1)^2$.

To evaluate $h(x)$ for a given value of ***x***, we would first compute $x+1$ and then square the result. These two operations are performed by the functions $g(x)=x+1$ *and* $f(x)=x^2$. We can express ***h*** in terms of ***f*** and ***g*** by writing:-

$$h(x)=(x+1)^2=[g(x)]^2=f\big(g(x)\big)$$

So we have succeeded in expressing h as the composition $h=fog$.

Example-4: Express $\sin x^3$ as a composition of two functions.

To evaluate $\sin x^3$, we would first compute x^3 and then take the sine, so $g(x)=x^3$ is the inside function and $f(x)=\sin x$ the outside function. Therefore, $\sin x^3=f\big(g(x)\big)$

Table 0.2.1 gives some more examples of decomposing functions into compositions.

FUNCTION	g(x) INSIDE	f(x) OUTSIDE	COMPOSITION
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10}=f\big(g(x)\big)$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x=f\big(g(x)\big)$
$\sqrt{4-3x}$	$4-3x$	$\sqrt{x}$	$\sqrt{4-3x}=f\big(g(x)\big)$
$8+\sqrt{x}$	$\sqrt{x}$	$8+x$	$8+\sqrt{x}=f\big(g(x)\big)$
$\frac{1}{x+1}$	$x+1$	$\frac{1}{x}$	$\frac{1}{x+1}=f\big(g(x)\big)$

EXERCISE 0.2

1. If $f(x) = 2\sqrt{x-1}\ \ and\ g(x) = \sqrt{x-1}$ Calculate $(f+g), (f-g), (fg), and\ (f/g)$
2. If $f(x) = x^2$ and $g(x) = \sqrt{1-x}$ Calculate (fog) and (gof)?
3. If $f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{x^2+3}$ Calculate (fog) and (gof)?
4. Let $f(x) = x^2+1,\ \ g(x) = \frac{1}{x},\ \ and\ \ h(x) = x^3$. Find the composition of $(fogoh)$?
5. Express ***f*** as a composition of two functions; that is, find ***g*** and ***h*** such that $f = goh$.

   A. $f(x) = \sqrt{x+2}$ B. $f(x) = \sin^2 x$ C. $f(x) = \frac{1}{x-1}$

0.3. FAMILIES OF FUNCTIONS

Functions are often grouped into families according to the form of their defining formulas or other common characteristics.

FAMILY OF CURVES

The graph of a constant function $f(x) = c$ is the graph of the equation $y = c$, which is the horizontal line shown in Figure. If we vary $\boldsymbol{c}$, then we obtain a set or family of horizontal lines such as those in Figure. Constants that are varied to produce families of curves are called **parameters**.

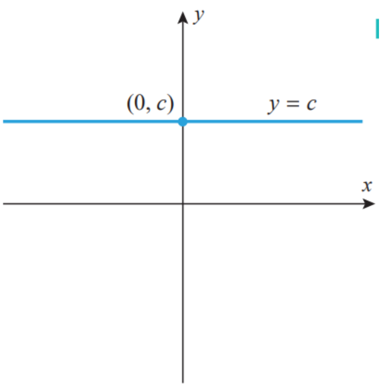


Figure-a

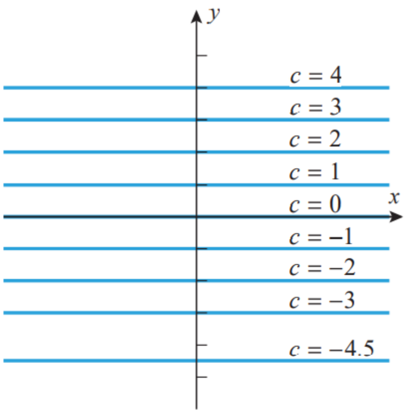
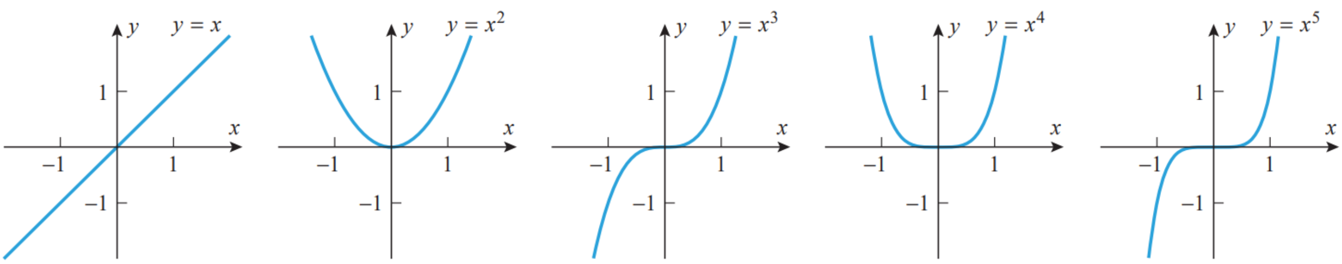


Figure-b

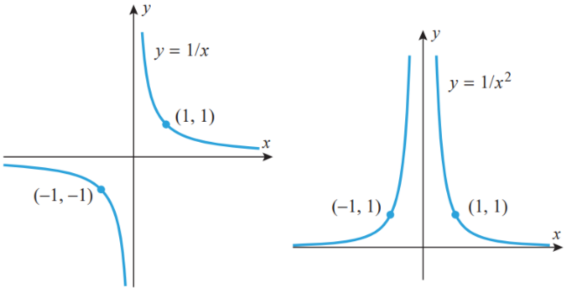
POWER FUNCTIONS: - THE FAMILY $y = x^n$

A function of the form $f(x) = x^p$, where $\boldsymbol{P}$ is constant, is called a power function. For the moment, let us consider the case where $\boldsymbol{P}$ is a positive integer, say $P = n$. The graphs of the curves $y = x^n$ for $n = 1,2,3,4$ *and* 5 are shown in Figure.



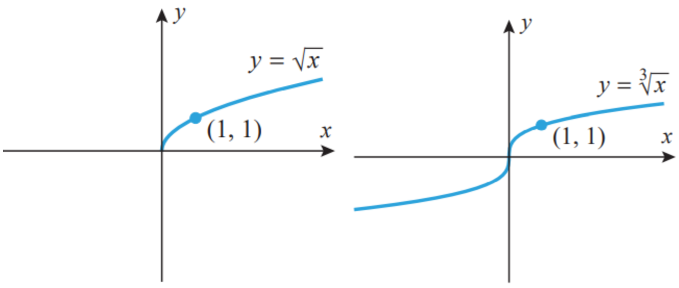
THE FAMILY $y = x^{-n}$

If P is a negative integer, say $P = -n$, then the power functions $f(x) = x^p$ have the form $f(x) = x^{-n} = \frac{1}{x^n}$. Figure below shows the graphs of $y = \frac{1}{x}$ and$y = \frac{1}{x^2}$. The graph of $y = \frac{1}{x}$ is called **an equilateral hyperbola**. The shape of the curve $y = \frac{1}{x^n}$ depends on whether n is even or odd:



POWER FUNCTIONS WITH NONINTEGER EXPONENTS

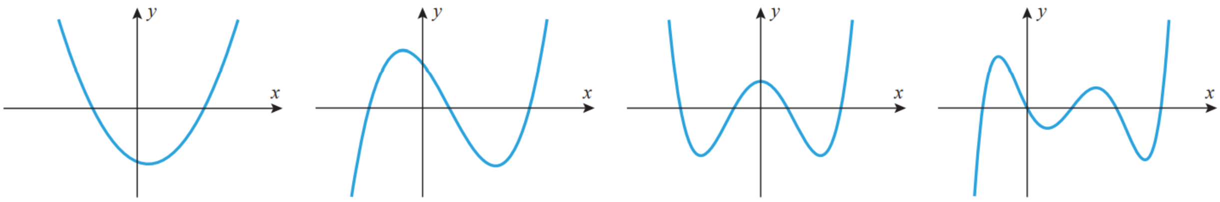
If $P = \frac{1}{n}$, where $\boldsymbol{n}$ is a positive integer, then the power functions $f(x) = x^P$ have the form $f(x) = x^{1/n} = \sqrt[n]{x}$. In particular, if $n = 2$, then $f(x) = \sqrt{x}$, and if $n = 3$, then $f(x) = \sqrt[3]{x}$. The graphs of these functions are shown in Figure (a and b)



(a) (b)

POLYNOMIALS

A polynomial in x is a function that is expressible as a sum of finitely many terms of the form Cx^n, where $\boldsymbol{C}$ is a constant and $\boldsymbol{n}$ is a nonnegative integer. Some examples of polynomials are:- $2x + 1$, $3x^2 + 5x - \sqrt{2}$, x^3, $4(= 4x^0)$, $5x^7 - x^4 + 3$

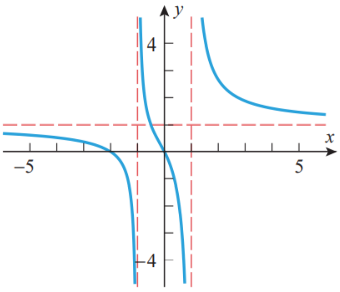


Degree 2 Degree 3 Degree 4 Degree 5

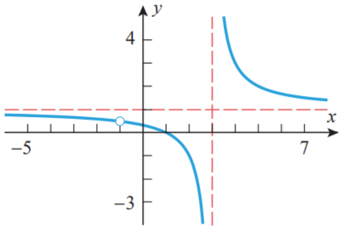
RATIONAL FUNCTIONS

A function that can be expressed as a ratio of two polynomials is called a rational function. If $P(x)$ and $Q(x)$ are polynomials, then $f(x) = \frac{P(x)}{Q(x)}$.

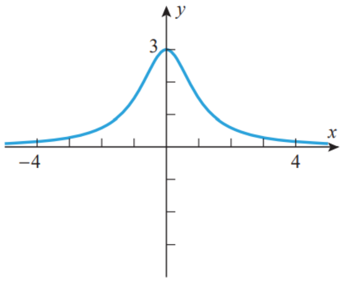
For example $f(x) = \frac{x^2+2x}{x^2-1}$, Where $Q(x) \neq 0$



$y = \dfrac{x^2 + 2x}{x^2 - 1}$



$y = \dfrac{x^2 - 1}{x^2 - 2x - 3}$



$y = \dfrac{3}{x^2 + 1}$

0.4. INVERSE FUNCTION; INVERSE TRIGONOMETRIC FUNCTION

INVERSE FUNCTIONS;

The idea of solving an equation $y = f(x)$ for $\boldsymbol{x}$ as a function of $\boldsymbol{y}$, say $x = g(x)$, is one of the most important ideas in mathematics. Sometimes, solving an equation is a simple process; for example, using basic algebra the equation

$$\boldsymbol{y = x^3 + 1} \qquad \boldsymbol{y = f(x)}$$

Can be solved for x as a function of $\boldsymbol{y}$:

$$\boldsymbol{x = \sqrt[3]{y-1}} \qquad \boldsymbol{x = g(x)}$$

If the functions f and g satisfy the two conditions

1. $g\big(f(x)\big) = x \ \rightarrow f^{-1}\big(f(x)\big) = x$
2. $f\big(g(x)\big) = y \ \rightarrow f^{-1}\big(f(y)\big) = y$

Then we say that f is an inverse of g and g is an inverse of f or that f and g are inverse functions.

If $\boldsymbol{f}$ is a function, then the -1 in the symbol f^{-1} always denotes an inverse and never an exponent. That is, $f^{-1}(x)$ never means $\frac{1}{f(x)}$.

Example 1: Find the inverse function if $f(x) = 2x - 7$?

$$\begin{aligned} f(x) &= 2x - 7 \\ y &= 2x - 7 \\ y + 7 &= 2x \\ x &= \frac{y+7}{2} \\ f^{-1}(y) &= \frac{y+7}{2} \end{aligned}$$

$$\begin{aligned} f(x) &= 2x - 7 \\ y &= 2x - 7 \\ x &= 2y - 7 \\ x + 7 &= 2y \\ y &= \frac{x+7}{2} \\ f^{-1}(x) &= \frac{x+7}{2} \end{aligned}$$

Example2: Find the inverse function if $f(x) = \sqrt{3x-2}$?

$$\begin{aligned} f(x) &= \sqrt{3x-2} \\ y &= \sqrt{3x-2} \\ y^2 &= 3x - 2 \\ y^2 + 2 &= 3x \\ x &= \frac{y^2+2}{3} \\ f^{-1}(y) &= \frac{y^2+2}{3} \end{aligned}$$

$$\begin{aligned} f(x) &= \sqrt{3x-2} \\ y &= \sqrt{3x-2} \\ x &= \sqrt{3y-2} \\ x^2 + 2 &= 3y \\ y &= \frac{x^2+2}{3} \\ f^{-1}(x) &= \frac{x^2+2}{3} \end{aligned}$$

INVERSE TRIGONOMETRIC FUNCTIONS

You have studied the trigonometric functions $\sin x$, $\cos x$, $\tan x$ can be used to find an unknown side length of a right-angle triangle, if one side length and an angle measure are known.

The inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ are used to find the unknown measure of an angle of a right-angle triangle when two side length are known.

Example (1):

The base of a ladder is placed $1.5m$ away from a $4m$ high wall, so that the top of the ladder meets the top of the wall. What is the measure of the angle formed by the ladder and the ground?

Solution:

$\tan^{-1}\left(\frac{4}{1.5}\right) \approx 69.44^{\circ}$

0.5. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this section we will review some properties of exponents and logarithms and then use our work on inverse functions to develop results about exponential and logarithmic functions.

IRRATIONAL EXPONENTS

If b is a nonzero real number, then nonzero integer powers of b are defined by

$$\boldsymbol{b^n = b * b * \ldots\ldots * b \text{ and } b^{-n} = \frac{1}{b^n}}$$

And if $n = 0$, then $b^0 = 1$. Also, if P/q is a positive rational number expressed in lowest terms, then

$$\boldsymbol{b^{P/q} = \sqrt[q]{b^P} = \left(\sqrt[q]{b}\right)^P \; And\, b^{-P/q} = \frac{1}{b^{P/q}}}$$

If b is negative, then some fractional powers of b will have imaginary values, the quantity $(-2)^{1/2} = \sqrt{-2}$, for example. To avoid this complication, we will assume throughout this section that $b > 0$, even if it is not stated explicitly.

There are various methods for defining irrational powers such as $\boldsymbol{\left(2^{\pi}, 3^{\sqrt{2}}, \pi^{-\sqrt{7}}\right)}$

THE FAMILY OF EXPONENTIAL FUNCTIONS

A function of the form $f(x) = b^x$, where $b > 0$, is called an exponential function with base b. Some examples are $f(x) = 2^x$, $\quad f(x) = \left(\frac{1}{2}\right)^x$, $\quad f(x) = \pi^x$

THE NATURAL EXPONENTIAL FUNCTION

Among all possible bases for exponential functions there is one particular base that plays a special role in calculus. That base, denoted by the letter e, is a certain irrational number whose value to six decimal places is $e \approx 2.718282$

The function $f(x) = e^x$ is called the **Natural Exponential Function**

LOGARITHMIC FUNCTIONS

A logarithm is an exponent, if $b > 0$ and $b \neq 1$, then for a positive value of x the expression $\log_b x$. (Read "the logarithm to the base b of x") denotes that exponent to which b must be raised to produce x. The function $f(x) = \log_b x$ the logarithmic function with base b.

Theorem (5.1): If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.

SOLVING EQUATIONS INVOLVING EXPONENTIALS AND LOGARITHMS

You should be familiar with the following properties of logarithms from your earlier studies.

Theorem (5.2): [Algebraic Properties of Logarithms]

If $b > 0$ and $b \neq 1, a > 0, c > 0, and\ r$ is any real number, then:

A. $\log_b(ac) = \log_b(a) + \log_b c$ → Product Property.
B. $\log_b(a/c) = \log_b(a) - \log_b c$ → Quotient Property.
C. $\log_b(a^r) = r\log_b a$ → Power Property.
D. $\log_b(1/c) = -\log_b c$ → Reciprocal Property.

These properties are often used to expand a single logarithm into sums, differences, and multiples of other logarithms.

Examples:

1. $\log\frac{xy^5}{\sqrt{z}} = \log xy^5 - \log\sqrt{z} = \log x + \log y^5 - \log z^{\frac{1}{2}} = \log x + \log y^5 - \frac{1}{2}\log z$
2. $5\log 2 + \log 3 - \log 8 = \log 2^5 + \log 3 - \log 8 = \log 32 + \log 3 - \log 8 = \log\frac{32*3}{8}$
3. $\frac{1}{3}\ln x - \ln(x^2-1) + 2\ln(x+3) = \ln x^{1/3} - \ln(x^2-1) + \ln(x+3)^2 = \ln\frac{\sqrt[3]{x}*(x+3)^2}{x^2-1}$