Programme F3: Expressions & Equations

1. Expressions and equations
2. The evaluation process
3. Polynomial equations
4. Remainder theorem
5. Factorization of cubic polynomials
6. Factorization of fourth-order polynomials

1. Expressions and Equations

1. Evaluating expressions
2. Equations
3. Evaluating independent variables
4. Transposition of formulas

1. Evaluating expressions

When numerical values are assigned to the variables and constants in an algebraic expression, the expression itself assumes a numerical value that is obtained by following the usual precedence rules. This process is known as *evaluating* the expression.

For example; if $l = 2 \; and \; g = 9.81$ then the expression: $2\pi\sqrt{\frac{l}{g}} =$ is evaluated as:

$2\pi\sqrt{\frac{2}{9.81}} = 2.84 \; to \; 2dp$ Where $\pi = 3.14159$ ... so let's look at three examples

Example 1:

If $V = \frac{\pi h}{6}(3R^2 + h^2)$, determine the value of V where $h = 2.85, R = 6.24 \; and \; \pi = 3.142$?

$$V = \frac{3.142 * 2.85}{6}(3 * 6.24^2 + 2.85^2) = 186.46$$

Example 2:

If $R = \frac{R_1 R_2}{R_1 + R_2}$, evaluate R when $R_1 = 276 \; and \; R_2 = 145$.

$$R = \frac{276 * 145}{276 + 145} = 95.06$$

Example 3:

If $V = \frac{\pi b}{12}(D^2 + Dd + d^2)$, evaluate V to 3 Significant Figure when

$b = 1.46, D = 0.864, d = 0.517 \; and \; \pi = 3.142$

$$V = \frac{3.142 * 1.46}{12}(0.864^2 + 0.864 * 0.517 + 0.517^2) = 0.588 \; to \; 3 \; sig \; fig$$

2. Equations:

An equation is a statement of the equality of two expressions for example $r = 2s^3 + 3t$.

The variable r is called the dependent variable and the subject of the equation whose value depends on the values of the independent variables $s\ and\ t$.

There are different types of equation.

1. Conditional equation:

A *conditional equation*, usually just called an equation, is true only for certain values of the symbols involved. For example: the equation $x^2 = 4$ is an equation that is only true for each of the two values $x = +2\ and\ x = -2$

2. Identity:

An *identity* is a statement of the equality of two expressions that is true for all values of the symbols for which both expressions are defined. For example $2(5 - x) = 10 - 2x$

3. Defining Equation:

A *defining equation* is a statement of equality that defines an expression. For example:

$$a^2 \triangleq a \times a$$

4. Assigning Equation:

An *assigning equation* is a statement of equality that assigns a specific value to a variable.

For example; $P \coloneqq 4$

5. Formula:

A *formula* is a statement of equality that expresses a mathematical fact where all the variables, dependent and independent, are well-defined. For example; $A = \pi r^2$

3. Evaluating independent variables:-

Sometimes the numerical values assigned to the variables and constants in a formula include a value of the dependent variable and exclude a value of one of the independent variables. The exercise is then to find the corresponding value of that independent variable by transposing the equation.

Example 1: If $T = 2\pi\sqrt{\frac{l}{g}}$ where $\pi = 3.14\ and\ g = 9.81$

What is the length *l* that corresponds to $T = 1.03$? that is given:

$$1.03 = 2 * 3.14\sqrt{\frac{l}{9.81}} = \left(\frac{1.03}{6.28}\right)^2 = \left(\sqrt{\frac{l}{9.81}}\right)^2 = \left(\frac{1.03}{6.28}\right)^2 = \frac{l}{9.81} =$$

$$l = (9.81)\left(\frac{1.03}{6.28}\right)^2 = 0.264\ to\ 3\ Sig\ Fig.$$

Example 2: If $I = \frac{nE}{R+nr}$, when $n = 6, E = 2.01, R = 12\ and\ I = 0.98$ the corresponding value of $r =$?

$$0.98 = \frac{6 * 2.01}{12 + 6r} = (0.98)(12 + 6r) = 12.06$$

$$6r = \frac{12.06}{0.98} - 12 = r = \left(\frac{12.06}{0.98} - 12\right) * \frac{1}{6} \qquad r = 0.051$$

4. Transposition of Formulas:

A formula can be transposed even when values for the variables and constants have not been assigned.

Example 1: Transpose the formula $T = 2\pi\sqrt{\frac{l}{g}}$ to make l the subject?

$$\frac{T}{2\pi} = \sqrt{\frac{l}{g}} \rightarrow \left(\frac{T}{2\pi}\right)^2 = \left(\sqrt{\frac{l}{g}}\right)^2 \rightarrow \left(\frac{T}{2\pi}\right)^2 = \frac{l}{g} \rightarrow l = g * \left(\frac{T}{2\pi}\right)^2 = \frac{gT^2}{4\pi^2}$$

Example 2: Transpose the formula $a = \frac{2(ut-s)}{t^2}$ to make u the subject?

$$at^2 = 2ut - 2s$$

$$at^2 + 2s = 2ut$$

$$u = \frac{at^2 + 2s}{2t}$$

Example 3: Transpose the formula $d = 2\sqrt{h(2r-h)}$ to make r the subject?

$$\frac{d}{2} = \sqrt{h(2r-h)} = \left(\frac{d}{2}\right)^2 = \left(\sqrt{h(2r-h)}\right)^2$$

$$\frac{d^2}{4} = h(2r-h) \;\; Divide\ both\ sides\ by\ h$$

$$\frac{d^2}{4h} = 2r - h \quad Add\ h\ both\ sides$$

$$\frac{d^2}{4h} + h = 2r \;\; Multiply\ both\ sides\ by \frac{1}{2}$$

$$r = \left(\frac{1}{2}\right)\left(\frac{d^2}{4h} + h\right) \qquad\qquad r = \frac{d^2}{8h} + \frac{h}{2} = \frac{d^2 + 4h^2}{8h}$$

Example 4: Class Work

Transpose the formula $n = \frac{IR}{E-Ir}$ to make I the subject?

Example 5: Class Work

Transpose the formula $\frac{R}{r} = \sqrt{\frac{f+P}{f-P}}$ to make f the subject?

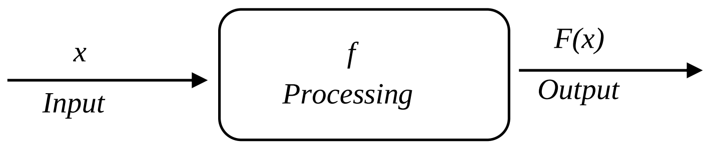
Example 6: Class Work

Transpose the formula $V = \frac{\pi h(3R^2+h^2)}{6}$ to make R the subject?

2. The Evaluation Process

System

A *system* is a process that is capable of accepting an *input*, *processing* the input and producing an *output*. We can use this idea of a system to describe the way we evaluate an algebraic expression.



We can describe the evaluation of 3*x* – 4 when *x* = 5 as *f processing* the *input* 5 to produce the *output* 11. The output *f(x)* is then the result of: ‘*f* acting on *x’*

Example 1: If $f(x) = 3x - 4$ then calculate $f(5) =?$ $f(5) = 3(5) - 5$ $f(5) = 11$

Example 2: If $f(x) = 4x^3 - \frac{6}{2x}$ then calculate $f(3), f(-4), f\left(\frac{2}{5}\right)$ and $f(-3.24) =?$

$$f(3) = 4(3)^3 - \frac{6}{2 * 3} = 108 - 1 = 107$$

$$f(-4) = 4(-4)^3 - \frac{6}{2(-4)} = -256 + 0.75 = -255.25$$

$$f\left(\frac{2}{5}\right) = f(0.4) = 4(0.4)^3 - \frac{6}{2 * 0.4} = 0.256 - 7.5 = -7.244$$

$$f(-3.24) = 4(-3.24)^3 - \frac{6}{2(-3.24)} = -136.05 + 0.93 = -135.12$$

3. Polynomial Equations:

1. Polynomial expressions
2. Evaluation of a polynomial by nesting

1. Polynomial Expressions:

A *polynomial* in *x* is an expression involving powers of *x*, normally arranged in descending (or sometimes ascending) powers. The degree of the *polynomial* is given by the highest power of *x* occurring in the expression. For example:

$5x^4 + 7x^3 + 3x -$ is a polynomial the 4th degree.

$2x^3 + 4x^2 - 2x + 7$ is a polynomial the 3th degree.

Polynomial of low degree often have alternative names:

1. $2x - 3$ Is a polynomial of the 1st degree or a linear expression?
2. $3x^2 + 4x + 2$ Is a polynomial of the 2nd degree or a quadratic expression?
3. A polynomial of the 3th degree is often referred to as a cubic expression.
4. A polynomial of the 4th degree is often referred to as a quartic expression.

2. Evaluation of a polynomial by nesting

To express a polynomial in nested form write down the coefficient and one factor *x* from the first term and add on the coefficient of the next term. Enclose in brackets, multiply by *x* and add on the next coefficient. Repeat the process. For example:

Example 1: Express the polynomial in nested form and evaluate the function for the given value? $f(x) = 5x^3 + 2x^2 - 3x + 6$ $\qquad f(4) = ?$

$$f(x) = \{[(5x + 2)x - 3]x + 6\}$$

$$f(4) = \{[(5 * 4 + 2)(4) - 3](4) + 6\} = 346$$

Example 2: Express the polynomial in nested form and evaluate the function for the given value? $f(x) = 4x^3 + 3x^2 + 2x - 4$ $\qquad f(2) = ?$

$$f(x) = \{[(4x + 3)x + 2]x - 4\}$$

$$f(2) = \{[(4 * 2 + 3)(2) + 2](2) - 4\} = 44$$

4. Remainder theorem

1. Remainder theorem:

The remainder theorem states that if a polynomial $f(x)$ is divided by $(x - a)$, the quotient will be a polynomial $g(x)$ of one degree less than the degree of $f(x)$, together with a remainder R still to be divided by $(x - a)$. That is:

$$\frac{f(x)}{(x-a)} = g(x) + \frac{R}{(x-a)}$$

Example 1: Apply the remainder theorem to determine the remainder in each of the following cases? $f(x) = (x^3 + 3x^2 - 13x - 10) \div (x - 3) = ?$

$$\begin{array}{r l}
 & x^2 + 6x + 5 \\
x - 3 \, \big) & \overline{x^3 + 3x^2 - 13x - 10} \\
 & \underline{\pm x^3 \mp 3x^2} \\
 & 6x^2 - 13x \\
 & \underline{\pm 6x^2 \mp 18x} \\
 & 5x - 10 \\
 & \underline{\pm 5x \mp 15} \\
 & 5 \rightarrow R
\end{array}$$

$$\frac{x^3 + 3x^2 - 13x - 10}{x - 3} = x^2 + 6x + 5 + \frac{5}{(x-3)}$$

Example 2: Apply the remainder theorem to determine the remainder in each of the following cases? $f(x) = (5x^3 - 4x^2 - 3x + 6) \div (x - 2) = ?$ $\quad Ans: 24$

$$\begin{array}{r}
5x^2 + 6x + 9 \\
x - 2\overline{\big)5x^3 - 4x^2 - 3x + 6} \\
\pm 5x^3 \mp 10x^2 \\
\hline
6x^2 - 3x \\
\pm 6x^2 \mp 12x \\
\hline
9x + 6 \\
\pm 9x \mp 18 \\
\hline
24 \to R
\end{array}$$

Example 3 C/W: Determine the remainder in each of the following cases?

1. $f(x) = (4x^3 - 3x^2 + 5x - 3) \div (x - 4)$ $\qquad Ans: 225$
2. $f(x) = (3x^3 - 11x^2 + 10x - 12) \div (x - 3)$ $\qquad Ans: 0$

2. Factor theorem:

If $f(x)$ is a polynomial and substituting $x = a$ gives a zero remainder, that is $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

Example 1: If $f(x) = x^3 + 2x^2 - 14x + 12 \ \ and \ \ x = 2$.

Calculate $f(2) = (2)^3 + 2(2)^2 - 14(2) + 12 = 0 \therefore x - 2 \ \to$ is a factor of $f(x)$.

The remaining factor can be found by using long division of $f(x)$ divided by $(x - 2)$

$$\begin{array}{r}
x^2 + 4x - 6 \\
x - 2\overline{\big)x^3 + 2x^2 - 14x + 12} \\
\pm x^3 \mp 2x^2 \\
\hline
4x^2 - 14x \\
\pm 4x^2 \mp 8x \\
\hline
-6x + 12 \\
\mp 6x \pm 12 \\
\hline
0 \to R
\end{array}$$

$\therefore f(x) \div (x - 2) = x^2 + 4x - 6$ $\quad$ So $f(x) = (x - 2)(x^2 + 4x - 6)$

The quadratic factor so obtained can sometimes be factorized further into two simple factors. So we apply the $b^2 - 4ac = k^2$

$\therefore x^2 + 4x - 6$ Where $a = 1, \ b = 4 \ \ and \ c = -6$

$b^2 - 4ac = k^2 \quad \to \quad 4^2 - 4(1)(-6) = 40 \quad \to$ is not a perfect square, so no simple factors exist, therefore, we cannot factorize further. $\quad \therefore f(x) = (x - 2)(x^2 + 4x - 6)$

Example 2: Test whether $(x-3)$ is a factor $f(x) = x^3 - 5x^2 - 2x + 24$ and if so, determine the remaining factors?

Calculate $f(3) = 3^3 - 5(3)^2 - 2(3) + 24$

$f(3) = 0,\ \ (x-3)\ is\ a\ factor\ of\ f(x)$ And long division now gives the remaining factors.

$$\begin{array}{r|l} & x^2 - 2x - 8 \\ x - 3 & \overline{x^3 - 5x^2 - 2x + 24} \\ & \pm x^3 \mp 3x^2 \\ \hline & \quad -2x^2 - 2x \\ & \quad \mp 2x^2 \pm 6x \\ \hline & \qquad -8x + 24 \\ & \qquad \mp 8x \pm 24 \\ \hline & \qquad\quad 0 \rightarrow R \end{array}$$

$\therefore\ f(x) \div (x-3) = x^2 - 2x - 8$. So $f(x) = (x-3)(x^2 - 2x - 8)$

Now calculate $x^2 - 2x - 8$ is a perfect square, where $a = 1, b = -2\ and\ c = -8$

$b^2 - 4ac =\ (-2)^2 - 4(1)(-8) = 36 = 6^2$ is a perfect square.

So calculate the quadratic factors $(x^2 - 2x - 8) = (x-4)(x+2)$

Collecting our results together:- $f(x) = x^3 - 5x^2 - 2x + 24 = (x-3)(x-4)(x+2)$

5. Factorization of quartic polynomials:-

The same method can be applied to polynomials of the fourth degree, provided that the given function has at least one simple factor.

Example 1: Factorize $f(x) = 2x^4 - x^3 - 8x^2 + x + 6$?

Calculate $f(1) = 0$; so $(x-1)$ is a factor. And determine the remaining factors of $f(x)$

$$\begin{array}{r|l} & 2x^3 + x^2 - 7x - 6 \\ x - 1 & \overline{2x^4 - x^3 - 8x^2 + x + 6} \\ & \pm 2x^4 \mp 2x^3 \\ \hline & \quad x^3 - 8x^2 \\ & \quad \pm x^3 \mp x^2 \\ \hline & \qquad -7x^2 + x \\ & \qquad \mp 7x^2 \pm 7x \\ \hline & \qquad\quad -6x + 6 \\ & \qquad\quad -6x + 6 \\ \hline & \qquad\qquad 0 \rightarrow R \end{array}$$

So $f(x) = 2x^4 - x^3 - 8x^2 + x + 6 = (x-1)(2x^3 + x^2 - 7x - 6) = (x-1)g(x)$

Now we can proceed to factorize $g(x)$ as we did with previous Cubics:

Now calculate $g(-1) = 0$. So $(x+1)$ is a factor of $g(x) = 2x^3 + x^2 - 7x - 6$

Long division shows that $g(x) = (x+1)(2x^2 - x - 6)$,

So $f(x) = (x-1)(x+1)(2x^2 - x - 6)$.

Calculate the quadratic factors of $f(x) = 2x^2 - x - 6$.

Test whether $f(x) = 2x^2 - x - 6$ is perfect square or not?

$b^2 - 4ac$ Where $a = 2, b = -1$ *and* $c = -6$

$(-1)^2 - 4(2)(-6) = 49$ Is a perfect square.

So calculate quadratic factors:

$$f(x) = 2x^2 - x - 6 = f(x) = 2x^2 - 4x + 3x - 6$$

$$= 2x(x-2) + 3(x-2)$$

$$= (2x+3)(x-2)$$

Finally $f(x) = (x-1)(x+1)(2x+3)(x-2)$

Example 2: Find the factors of $f(x) = x^4 + x^3 - 9x^2 + x + 10$?

Ans. $f(x) = (x+1)(x-2)(x^2 + 2x - 5)$.