Lecture 29

Warshall’s Algorithm, and Floyd’s Algorithm for the All-Pairs Shortest-Paths Problem.

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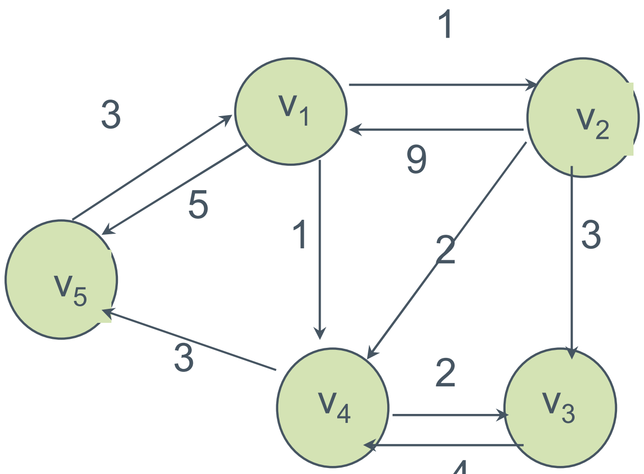
All pairs shortest path

- *The problem: find the shortest path between every pair of vertices of a graph*
- *The graph: may contain negative edges but no negative cycles*
- *A representation: a weight matrix where*
  W(i,j)=0 if i=j.
  W(i,j)=¥ if there is no edge between i and j.
  W(i,j)="weight of edge"
- *Note: we have shown principle of optimality applies to shortest path problems*

The principle of optimality applies if the optimal solution to a problem can be obtained by combining the optimal solutions to all subproblems.

The weight matrix and the graph

i / j	1	2	3	4	5
1	0	1	∞	1	5
2	9	0	3	2	∞
3	∞	∞	0	4	∞
4	∞	∞	2	0	3
5	3	∞	∞	∞	0



The sub problems

- *How can we define the shortest distance $d_{i,j}$ in terms of "smaller" problems?*
- *One way is to restrict the paths to only include vertices from a restricted subset.*
- *Initially, the subset is empty.*
- *Then, it is incrementally increased until it includes all the vertices.*

The sub problems

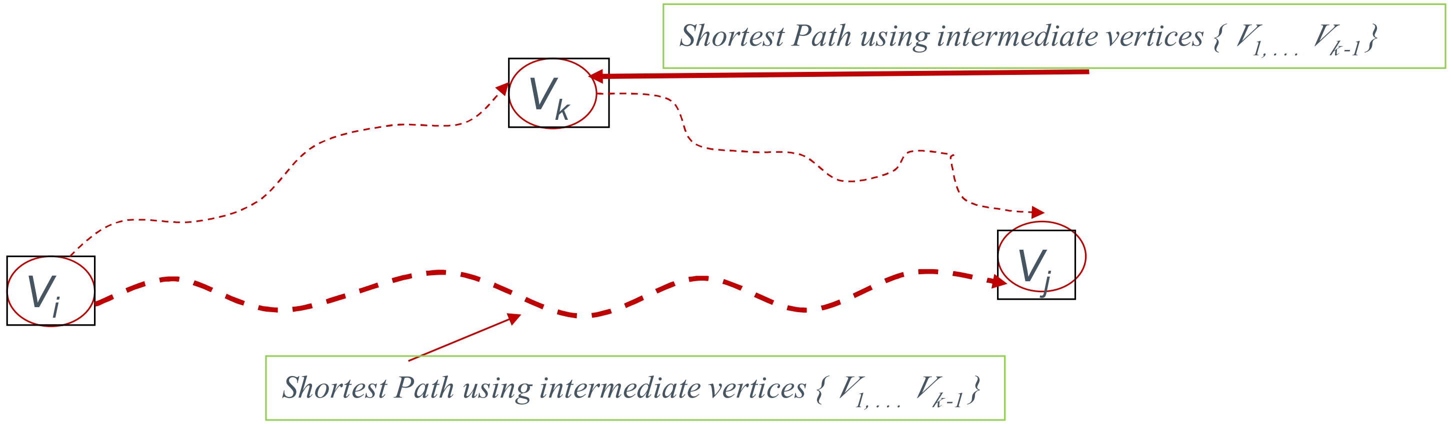
- *Let $D^{(k)}[i, j]$=weight of a shortest path from v_i to v_j using only vertices from $\{v_1, v_2, \ldots, v_k\}$ as intermediate vertices in the path*
  - $D^{(0)} = W$
  - $D^{(n)} = D$ *which is the goal matrix*

- *How do we compute $D^{(k)}$ from $D^{(k-1)}$?*

The Recursive Definition:

Case 1: *A shortest path from v_i to v_j restricted to using only vertices from $\{v_1, v_2, \ldots, v_k\}$ as intermediate vertices does not use v_k. Then $D^{(k)}[i,j] = D^{(k-1)}[i,j]$.*

Case 2: *A shortest path from v_i to v_j restricted to using only vertices from $\{v_1, v_2, \ldots, v_k\}$ as intermediate vertices does use v_k. Then $D^{(k)}[i,j] = D^{(k-1)}[i,k] + D^{(k-1)}[k,j]$.*



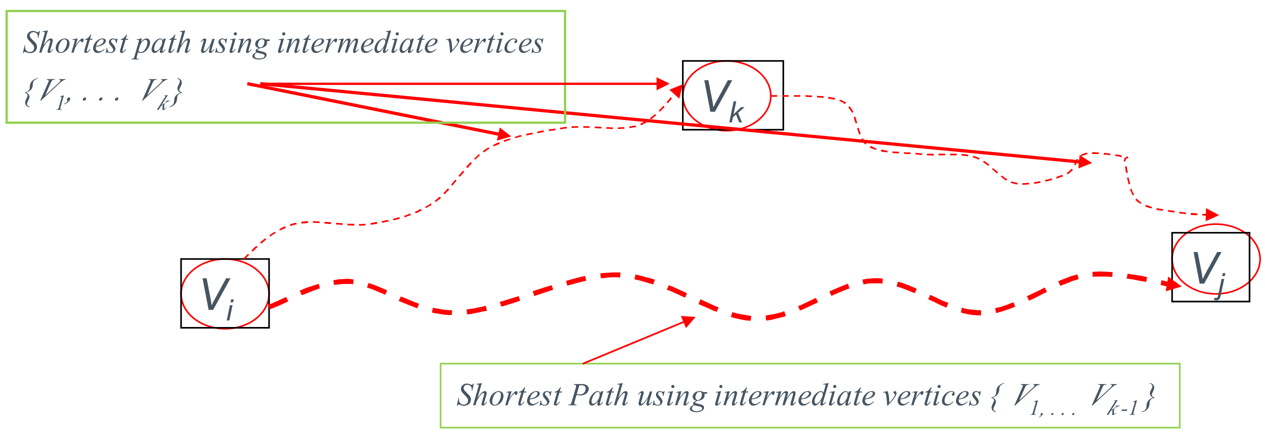
The recursive definition

Since $D^{(k)}[i,j] = D^{(k-1)}[i,j]$ or

$D^{(k)}[i,j] = D^{(k-1)}[i,k] + D^{(k-1)}[k,j]$.

We conclude:

$$D^{(k)}[i,j] = \min\{ D^{(k-1)}[i,j], D^{(k-1)}[i,k] + D^{(k-1)}[k,j] \}.$$



The pointer array P

- *Used to enable finding a shortest path*
- *Initially the array contains 0*
- *Each time that a shorter path from i to j is found the k that provided the minimum is saved (highest index node on the path from i to j)*
- *To print the intermediate nodes on the shortest path a recursive procedure that print the shortest paths from i and k, and from k to j can be used*

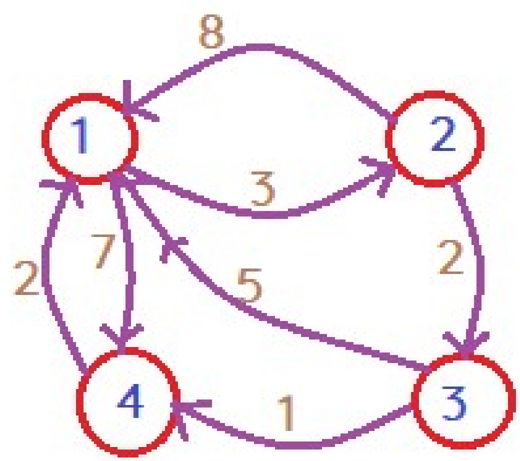
Floyd's Algorithm Using n+1 D matrices

Floyd //Computes shortest distance between all pairs of nodes, and

// saves P to enable finding shortest paths

1. $D^0 \leftarrow W$ *// initialize D array to W[ ]*
2. $P \leftarrow 0$ *// initialize P array to [0]*
3. FOR $k \leftarrow 1$ to n
4. DO FOR $i \leftarrow 1$ to n
5. DO FOR $j \leftarrow 1$ to n
6. IF $(D^{k-1}[\,i, j\,] > D^{k-1}[\,i, k\,] + D^{k-1}[\,k, j\,])$
7. THEN $D^k[\,i, j\,] \leftarrow D^{k-1}[\,i, k\,] + D^{k-1}[\,k, j\,]$
8. $P[\,i, j\,] \leftarrow k$;
9. ELSE $D^k[\,i, j\,] \leftarrow D^{k-1}[\,i, j\,]$

Example: Floyd's Algorithm – All pairs Shortest Path



Consider VERTEX 1 and the ROW 1 and Column 1 Will remain unchanged.

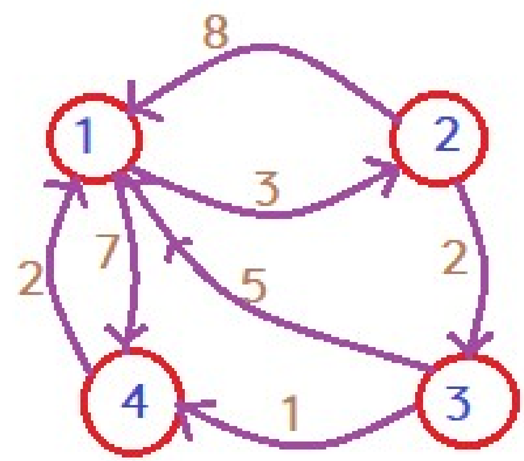
K=1

$A^1(i,j)=\text{Min }\{A^0(2,3),\ A^0(2,1)+A^0(1,3)\}=\min(2,\ 8+\infty)=2$
$A^1(i,j)=\text{Min }\{A^0(2,4),\ A^0(2,1)+A^0(1,4)\}=15$
$A^1(i,j)=\text{Min }\{A^0(3,2),\ A^0(3,1)+A^0(1,2)\}=8$

$$A^0 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & \infty \\ 5 & \infty & 0 & 1 \\ 2 & \infty & \infty & 0 \end{pmatrix} \end{array}$$

$$A^1 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & \infty & 0 \end{pmatrix} \end{array}$$

Example: Floyd-Warshal Algorithm – All pairs Shortest Path



Consider VERTEX 1 and the ROW 2 and Column 2
Will remain unchanged.

K=2

$A^2(i,j)$=Min $\{A^1(1,3), A^1(1,2)+A^1(2,3)\}$= min($\infty$, 3+2)=5
$A^2(i,j)$=Min $\{A^1(1,4), A^1(1,2)+A^1(2,4)\}$= 7

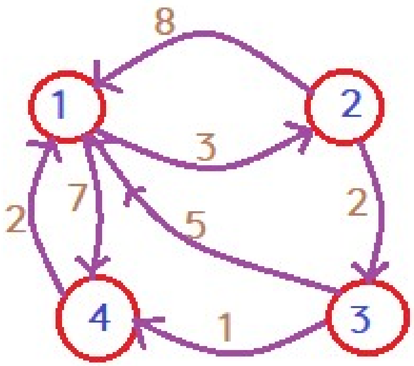
Similarly fill the remaining values

$$A^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & \infty \\ 5 & \infty & 0 & 1 \\ 2 & \infty & \infty & 0 \end{pmatrix} \end{array}$$

$$A^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & \infty & 0 \end{pmatrix} \end{array}$$

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 3 & 5 & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{pmatrix} \end{array}$$

Example: Floyd-Warshal Algorithm – All pairs Shortest Path



Consider VERTEX 1 and the ROW 3 and Column 3
Will remain unchanged.

K=2

$A^3(i,j)=\text{Min}\{A^2(1,2), A^2(1,3)+A^2(3,2)\}= \min(3, 5+8)=3$

Similarly fill the remaining values using the following formula having time complexity of $O(n^3)$

$$\boxed{A^k(i, j)=\min\{ A^{k-1}(i,j), A^{k-1}(i,k) + A^{k-1}(k, j) \}}$$

$$A^0 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & \infty \\ 5 & \infty & 0 & 1 \\ 2 & \infty & \infty & 0 \end{pmatrix} \end{array}$$

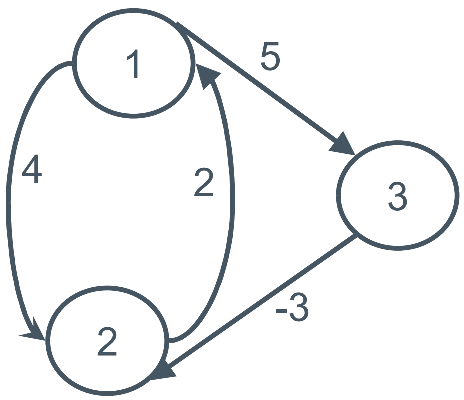
$$A^1 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & \infty & 0 \end{pmatrix} \end{array}$$

$$A^2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 3 & 5 & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{pmatrix} \end{array}$$

$$A^3 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 3 & 5 & 6 \\ 7 & 0 & 2 & 3 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{pmatrix} \end{array}$$

$$A^4 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 3 & 5 & 6 \\ 5 & 0 & 2 & 3 \\ 3 & 6 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{pmatrix} \end{array}$$

Example



$$W = D^0 = \begin{array}{c|ccc} & \multicolumn{3}{c}{j} \\ & 1 & 2 & 3 \\ \hline i\ 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \infty \\ 3 & \infty & -3 & 0 \end{array}$$

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}$$

1. $D^0 \leftarrow W$ // initialize D array to $W[\,]$

2. $P \leftarrow 0$ // initialize P array to [0]

Example

$$D^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \infty \\ 3 & \infty & -3 & 0 \end{array}$$

$$D^1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & 7 \\ 3 & \infty & -3 & 0 \end{array}$$

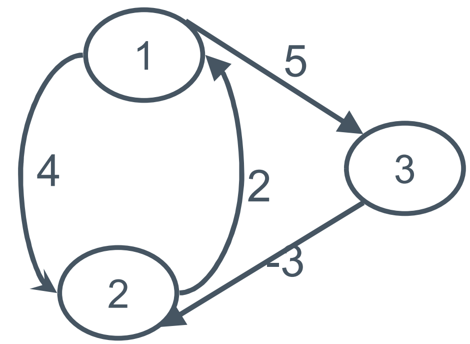
$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{array}$$

if $(D^{k-1}[i, j] > D^{k-1}[i, k] + D^{k-1}[k, j])$

then $D^k[i, j] \leftarrow D^{k-1}[i, k] + D^{k-1}[k, j]$

$P[i, j] \leftarrow k;$

else $D^k[i, j] \leftarrow D^{k-1}[i, j]$



$k = 1$

Vertex 1 can be intermediate node

$D^1[2,3] = \min(D^0[2,3], D^0[2,1]+D^0[1,3])$

$= \min(\infty, 7)$

$= 7$

$D^1[3,2] = \min(D^0[3,2], D^0[3,1]+D^0[1,2])$

$= \min(-3, \infty)$

$= -3$

Example

```
if (D^(k-1)[ i, j] > D^(k-1) [ i, k] + D^(k-1) [ k, j] )
        then D^k[ i, j] ← D^(k-1) [ i, k] + D^(k-1) [ k, j]
                P[ i, j] ← k;
        else D^k[ i, j] ← D^(k-1) [ i, j]
```

$D^1 =$

	1	2	3
1	0	4	5
2	2	0	7
3	∞	-3	0

$D^2 =$

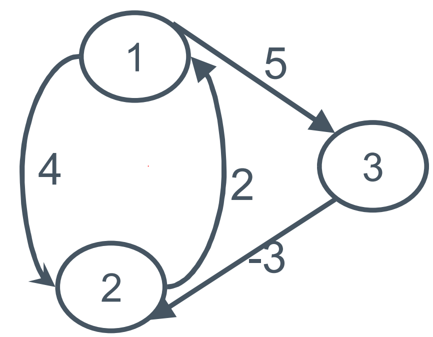
	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

P =

	1	2	3
1	0	0	0
2	0	0	1
3	2	0	0

k = 2

Vertices 1, 2 can be intermediate



$D^2[1,3] = \min(D^1[1,3], D^1[1,2]+D^1[2,3])$

$= \min (5, 4+7)$

$= 5$

$D^2[3,1] = \min(D^1[3,1], D^1[3,2]+D^1[2,1])$

$= \min (\infty, -3+2)$

$= -1$

Example

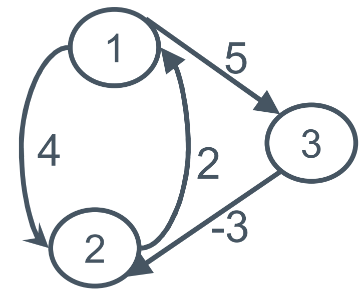
6. if ($D^{k-1}[i, j] > D^{k-1}[i, k] + D^{k-1}[k, j]$)
7. then $D^k[i, j] \leftarrow D^{k-1}[i, k] + D^{k-1}[k, j]$
8. $P[i, j] \leftarrow k$;
9. else $D^k[i, j] \leftarrow D^{k-1}[i, j]$

$D^2 =$

	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

k = 3

Vertices 1, 2, 3 can be intermediate



$D^3 =$

	1	2	3
1	0	2	5
2	2	0	7
3	-1	-3	0

$P =$

	1	2	3
1	0	3	0
2	0	0	1
3	2	0	0

$D^3[1,2] = min(D^2[1,2], D^2[1,3]+D^2[3,2])$

$= min(4, 5+(-3))$

$= 2$

$D^3[2,1] = min(D^2[2,1], D^2[2,3]+D^2[3,1])$

$= min(2, 7+(-1))$

$= 2$

Floyd's Algorithm: Using 2 D matrices

Floyd

1. $D \leftarrow W$ // initialize D array to W[ ]
2. $P \leftarrow 0$ // initialize P array to [0]
3. **for** $k \leftarrow 1$ **to** n

   // Computing D' from D
4. **do for** $i \leftarrow 1$ **to** n
5. **do for** $j \leftarrow 1$ **to** n
6. **if** $(D[i, j] > D[i, k] + D[k, j])$
7. **then** $D'[i, j] \leftarrow D[i, k] + D[k, j]$
8. $P[i, j] \leftarrow k$;
9. **else** $D'[i, j] \leftarrow D[i, j]$
10. **Move D' to D.**

Can we use only one D matrix?

- *$D[i, j]$ depends only on elements in the kth column and row of the distance matrix.*
- *We will show that the kth row and the kth column of the distance matrix are unchanged when D^k is computed*
- *This means D can be calculated in-place*

The main diagonal values

- *Before we show that kth row and column of D remain unchanged, we show that the main diagonal remains 0*

- $$D^{(k)}[j,j] = \min\{ D^{(k-1)}[j,j],\ D^{(k-1)}[j,k] + D^{(k-1)}[k,j] \}$$
$$= \min\{ 0,\ D^{(k-1)}[j,k] + D^{(k-1)}[k,j] \}$$
$$= 0$$

- *Based on which assumption?*

The kth column

- k*th column of* D^k *is equal to the* k*th column of* D^{k-1}

- *Intuitively true - a path from i to k will not become shorter by adding k to the allowed subset of intermediate vertices*

- *For all i,* $D^{(k)}[i,k] =$

$$
\begin{aligned}
&= \min\{ D^{(k-1)}[i,k],\ D^{(k-1)}[i,k] + D^{(k-1)}[k,k] \} \\
&= \min \{ D^{(k-1)}[i,k], D^{(k-1)}[i,k]+0 \} \\
&= D^{(k-1)}[i,k]
\end{aligned}
$$

The kth row

- kth row of D^k is equal to the kth row of D^{k-1}

$$
\begin{aligned}
\text{For all } j,\ D^{(k)}[k,j] &= \\
&= \min\{ D^{(k-1)}[k,j], D^{(k-1)}[k,k] + D^{(k-1)}[k,j] \} \\
&= \min\{ D^{(k-1)}[k,j], 0 + D^{(k-1)}[k,j] \} \\
&= D^{(k-1)}[k,j]
\end{aligned}
$$

Floyd's Algorithm using a single D

Floyd

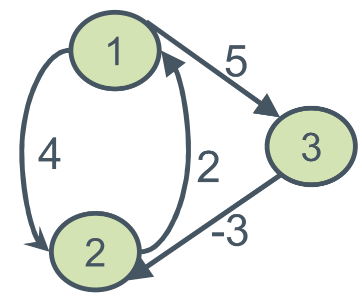
1. $D \leftarrow W$ // initialize D array to $W[\,]$
2. $P \leftarrow 0$ // initialize P array to [0]
3. **for** $k \leftarrow 1$ **to** n
4. **do for** $i \leftarrow 1$ **to** n
5. **do for** $j \leftarrow 1$ **to** n
6. **if** $(D[i, j] > D[i, k] + D[k, j])$
7. **then** $D[i, j] \leftarrow D[i, k] + D[k, j]$
8. $P[i, j] \leftarrow k;$

Printing intermediate nodes on shortest path from q to r

```
path(index q, r)
    if (P[ q, r ]!=0)
            path(q, P[q, r])
            println( "v"+ P[q, r])
            path(P[q, r], r)
            return;
    //no intermediate nodes
    else return
```

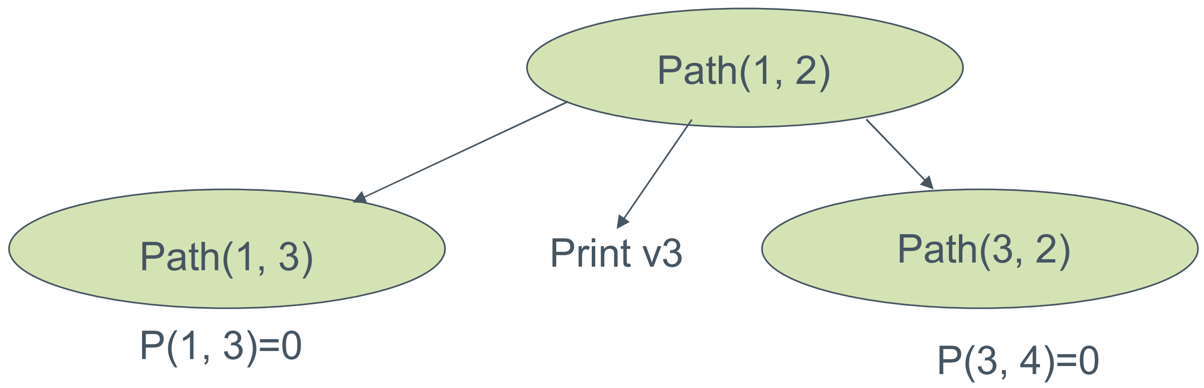
P =

	1	2	3
1	0	3	0
2	0	0	1
3	2	0	0



Before calling path check $D[q, r] < \infty$, and print node q, after the call to path print node r

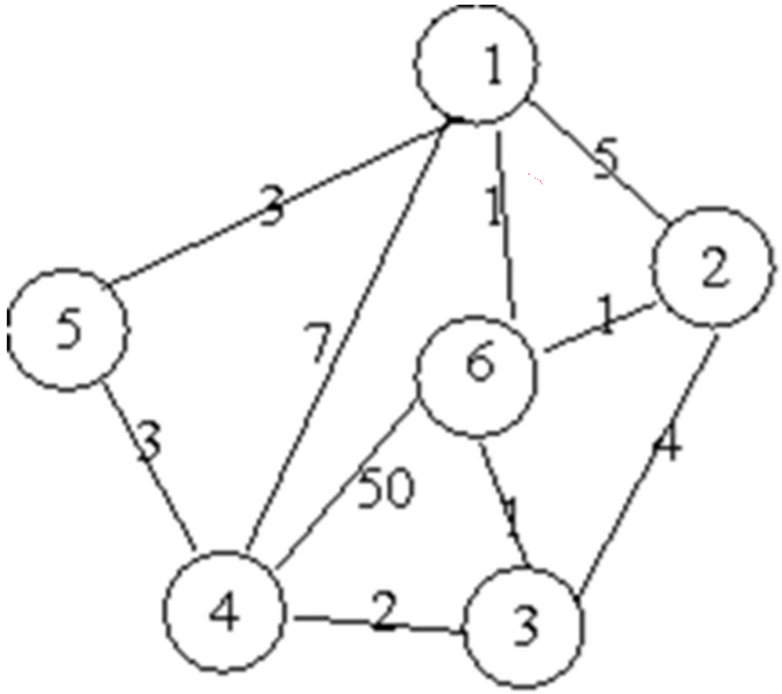
The call tree for Path(1, 3)



The intermediate nodes on the shortest path from 1 to 2 is v3.

The shortest path is v1, v3, v4.

Example

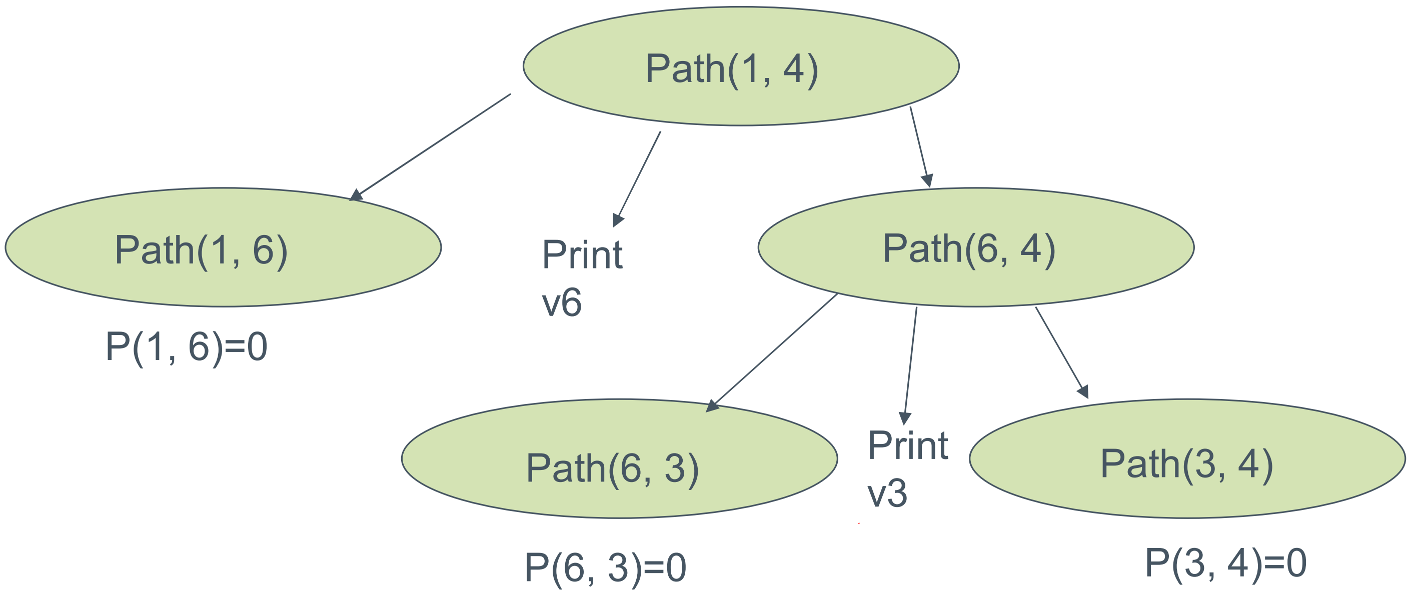


The final distance matrix and P

$$D^6 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 2(6) & 2(6) & 4(6) & 3 & 1 \\ 2 & 2(6) & 0 & 2(6) & 4(6) & 5(6) & 1 \\ 3 & 2(6) & 2(6) & 0 & 2 & 5(4) & 1 \\ 4 & 4(6) & 4(6) & 2 & 0 & 3 & 3(3) \\ 5 & 3 & 5(6) & 5(4) & 3 & 0 & 4(1) \\ 6 & 1 & 1 & 1 & 3(3) & 4(1) & 0 \end{array}$$

The values in parenthesis are the nonzero P values.

The call tree for Path(1, 4)



The intermediate nodes on the shortest path from 1 to 4 are v6, v3.

The shortest path is v1, v6, v3, v4.

Summary

- *Floyd algorithm help to find the shortest path between every pair of vertices of a graph.*

- *Floyd graph may contain negative edges but no negative cycles*

- *A representation of weight matrix where*
  W(i,j)=0 if i=j.
  W(i,j)=¥ if there is no edge between i and j.
  W(i,j)="weight of edge"

Thank You!!!

Have a good day