Solution of Linear System of Equations and Matrix Inversion

Gauss–Jordon Elimination Method

This method is a variation of Gaussian elimination method. In this method, the elements above and below the diagonal are simultaneously made zero. That is a given system is reduced to an equivalent diagonal form using elementary transformations. Then the solution of the resulting diagonal system is obtained. Sometimes, we normalize the pivot row with respect to the pivot element, before elimination. Partial pivoting is also used whenever the pivot element becomes zero.

Example

Solve the system of equations using Gauss-Jordan elimination method:

$$\left.\begin{aligned} x+2y+z&=8 \\ 2x+3y+4z&=20 \\ 4x+3y+2z&=16 \end{aligned}\right\}$$

Solution

In matrix notation, the given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 20 \\ 16 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -5 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 4 \\ -16 \end{pmatrix} \qquad \text{(-2) R1 +R2 and (-4) R1+R3}$$

Now, we eliminate y from the first and third rows using the second row. Thus, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & -12 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 16 \\ 4 \\ -36 \end{pmatrix}$$

Before, eliminating z from the first and second row, normalize the third row with respect to the pivot element, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 16 \\ 4 \\ 3 \end{pmatrix}$$

Using the third row of above Equation, and eliminating z from the first and second rows, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

The solution is

x =1, y =2, z =3.

Example

Solve the system of equation by using the Gauss Jordan elimination method

$$10x+y+z=12$$

$$2x+10y+z=13$$

$$x+y+5z=7$$

Solution

$10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

the given system

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

the arg*umented matrix may be written as*

$$[A/B] = \begin{bmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 \,|& 13 \\ 1 & 1 & 5 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -8 & -44 & & -51 \\ 2 & 10 & 1 & | & 13 \\ 1 & 1 & 5 & & 7 \end{bmatrix} R_1 - 9R_3$$

$$\sim \begin{bmatrix} 1 & -8 & -44 & & -51 \\ 0 & 26 & 89 & | & 115 \\ 0 & 9 & 49 & & 58 \end{bmatrix} R_2 - 2R_1, R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & -8 & -44 & & -51 \\ 0 & 1 & 89 & | & 59 \\ 0 & 9 & 49 & & 58 \end{bmatrix} R_2 - 3R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 420 & & 421 \\ 0 & 1 & 58 & | & 59 \\ 0 & 0 & -473 & & -473 \end{bmatrix} R_1 + 8R_2, R_3 - 9R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 \,|& 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \frac{1}{473} R_3\,, R_1 - 420R_3, R_2 - 58R_2$$

thus the system reduces to reduced echlon form

so $x = y = z = 1$

Example

Solve the system of equations by Gauss Jordan method

$10x_1 + x_2 + x_3 = 12$

$x_1 + 10x_2 - x_3 = 10$

$x_1 - 2x_2 + 10x_3 = 9$

Solution

$$10x_1 + x_2 + x_3 = 12$$
$$x_1 + 10x_2 - x_3 = 10$$
$$x_1 - 2x_2 + 10x_3 = 9$$

the matrix form of the given system may be written as

$$\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}$$

the arg*umented mairix may be written as*

$$[A/B] = \begin{bmatrix} 10 & 1 & 1 & 12 \\ 1 & 10 & -1 \mid & 10 \\ 1 & -2 & 10 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 1 & 10 & -1 \mid & 88 \\ 1 & -2 & 10 & 87 \end{bmatrix} R_1 - 9R_2$$

$$\sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 \mid & 8 \\ 0 & 1 & 0 & 1 \end{bmatrix} R_2 - R_1, R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 \mid & 8 \\ 0 & 1 & 0 & 1 \end{bmatrix} \frac{R_2}{99}, \frac{R_3}{87}$$

$$\sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 0 & 1 & -1 \mid & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} R_2 - 8R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -79 & -78 \\ 0 & 1 & -1 \mid & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_1 - 89R_2, R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 \mid & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_1 + 79R_3, R_2 + R_3$$

so the system gives the values of all the three var*iables*

$$x_1 = x_2 = x_3 = 1$$

Example

Solve the system of equations by using Gauss Jordan method.

$$x + 2y + z - w = -2$$
$$2x + 3y - z + 2w = 7$$
$$x + y + 3z - 2w = -6$$
$$x + y + z + w = 2$$

Solution

$$x + 2y + z - w = -2$$
$$2x + 3y - z + 2w = 7$$
$$x + y + 3z - 2w = -6$$
$$x + y + z + w = 2$$

the system may be written as

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -6 \\ 2 \end{bmatrix}$$

the arg*umented matrix may be written as*

$$[A/B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 1 & 3 & -2 & -6 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 0 & 1 & 3 & -4 & 11 \\ 0 & -1 & 2 & -1 & -4 \\ 0 & -1 & 0 & 2 & 4 \end{array}\right] \quad R_2 - 2R_1, R_3 - R_1, R_4 - R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 7 & 20 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 3 & -2 & -7 \end{array}\right] \quad R_1 - 2R_2, \frac{1}{5}(R_3 + R_2), R_4 + R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad R_1 + 5R_3, R_2 - 3R_3, R_4 - 3R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad R_1 - 2R_4, R_2 + R_4, R_3 + R_4$$

the system maybe written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

the values of all the ariables are

$x = 1, y = 0, z = -1, w = 2$

Crout's ReductionMethod

Here the coefficient matrix [A] of the system of equations is decomposed into the product of two matrices [L] and [U], where [L] is a lower-triangular matrix and [U] is an upper-triangular matrix with 1's on its main diagonal.

For the purpose of illustration, consider a general matrix in the form

$[L][U] = [A]$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The sequence of steps for getting [L] and [U] are given below:
Step I: Multiplying all the rows of [L] by the first column of [U], we get

$$l_{11} = a_{11}, \qquad l_{21} = a_{21}, \qquad l_{31} = a_{31}$$

Thus, we observe that the first column of [L] is same as the first column of [A].

Step II: Multiplying the first row of [L] by the second and third columns of [U], we obtain

$$l_{11}u_{12} = a_{12}, \qquad l_{11}u_{13} = a_{13}$$

Or

$$u_{12} = \frac{a_{12}}{l_{11}}, \qquad u_{13} = \frac{a_{13}}{l_{11}}$$

Thus, the first row of [U] is obtained. We continue this process, getting alternately the column of [L] and a row of [U].

Step III: Multiply the second and third rows of [L] by the second column of [U] to get

$$l_{21}u_{12} + l_{22} = a_{22}, \qquad l_{31}u_{12} + l_{32} = a_{32}$$

This gives

$$l_{22} = a_{22} - l_{21}u_{12}, \qquad l_{32} = a_{32} - l_{31}u_{12}$$

Step IV: Now, multiply the second row of [L] by the third column of [U] which yields

$$l_{21}u_{13} + l_{22}u_{23} = a_{23}$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

Step V: Lastly, we multiply the third row of [L] by the third column of [U] and get

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}$$

This gives

$$l_{33} = a_{33} - l_{33}u_{13} - l_{32}u_{23}$$

Thus, the above five steps determine [L] and [U].
This algorithm can be generalized to any linear system of order n.
Consider a system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\}$$

In matrix notation as [A](X) = (B). Let [A] = [L] [U], then we get,

$$[L][U](X) = (B)$$

Substituting [U] (X) = (Z) in Eq. we obtain [L] (Z) = (B)

$$\left.\begin{aligned} l_{11}z_1 &= b_1 \\ l_{21}z_1 + l_{22}z_2 &= b_2 \\ l_{31}z_1 + l_{32}z_2 + l_{33}z_3 &= b_3 \end{aligned}\right\}$$

Having computed z1, z2 and z3, we can compute x1, x2, and x3 from equation [U] (X) = (Z) or from

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

This method is also known as Cholesky reduction method. This technique is widely used in the numerical solutions of partial differential equations.

This method is very popular from computer programming point of view, since the storages space reserved for matrix [A] can be used to store the elements of [L] and [U] at the end of computation.
This method fails if any

$a_{ii} = 0$

Example

Solve the following system of equations by Crout's reduction method

$$5x_1 - 2x_2 + x_3 = 4$$
$$7x_1 + x_2 - 5x_3 = 8$$
$$3x_1 + 7x_2 + 4x_3 = 10$$

Solution

Let the coefficient matrix [A] be written as [L] [U]. Thus,

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Step I: Multiply all the rows of [L] by the first column of [U], we get

$$l_{11} = 5, \qquad l_{21} = 7, \qquad l_{31} = 3$$

Step II: Multiply the first row of [*L*] by the second and third columns of [*U*], we have

$$l_{11}u_{12} = -2, \qquad l_{11}u_{13} = 1$$

$$u_{12} = -\frac{2}{5}, u_{13} = \frac{1}{5}$$

STEP III: Multiply the 2nd and 3rd rows of [*L*] by the 2nd column of [*U*], we get

$$\left.\begin{array}{lll} l_{21}u_{12} + l_{22} = 1 & or & l_{22} = 1 + \frac{14}{5} = \frac{19}{5} \\ l_{31}u_{12} + l_{32} = 7 & or & l_{32} = 7 + \frac{6}{5} = \frac{41}{5} \end{array}\right\}$$

STEP IV: Multiply the 2nd row of [L] by the 3rd column of [U]

$$l_{21}u_{13} + l_{22}u_{23} = -5$$

$$\frac{19}{5}u_{23} = -5 - \frac{7}{5}$$

$$u_{23} = -\frac{32}{19}$$

STEP V:Finally, multiply the 3rd row of [L] with the 3rd column of [U], we obtain

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4$$

$$l_{33} = \frac{327}{19}$$

Thus, the given system of equations takes the form [*L*][*U*][*X*] = (*B*).

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{32}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

That is,

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

Let [U](X) = (Z), then

$[L](Z) = (4 \quad 8 \quad 10)^T$

Or

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

Which gives utilizing these values of z, the Eq becomes

By back substitution method, we obtain

$$x_3 = \frac{46}{327}, \qquad x_2 = \frac{284}{327}, \qquad x_1 = \frac{366}{327}$$

This is the required solution.

Example

Solve the following system
$$\begin{aligned} 2x - 3y + 10z &= 3 \\ -x + 4y + 2x &= 20 \\ 5x + 2y + z &= -12 \end{aligned}$$

Solution

$$2x - 3y + 10z = 3$$
$$-x + 4y + 2x = 20$$
$$5x + 2y + z = -12$$

the given system is $AX = B$

$$A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

let $LU = A$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

here

$$u_{11}=2,\ u_{12}=-3\quad u_{13}=10$$

$$l_{21}u_{11}=-1\ \Rightarrow l_{21}=\frac{-1}{2}$$

$$l_{31}u_{11}=5\quad \Rightarrow l_{31}=\frac{5}{2}$$

$$l_{21}u_{12}+u_{22}=4\quad \Rightarrow u_{22}=4-l_{21}u_{12}=4-(\frac{-1}{2})(-3)=\frac{5}{2}$$

$$l_{21}u_{13}+u_{23}=2\quad \Rightarrow u_{23}=2-l_{21}u_{13}=2-(\frac{-1}{2})10=7$$

$$l_{31}u_{12}+l_{32}u_{22}=5\Rightarrow l_{32}=\frac{1}{u_{22}}[5-l_{31}u_{12}]=\frac{19}{5}$$

$$l_{31}u_{13}+l_{32}u_{23}+u_{33}\Rightarrow u_{33}=1-l_{31}u_{13}-l_{32}u_{23}=\frac{253}{5}$$

$$L=\begin{bmatrix}1 & 0 & 0\\ \frac{-1}{2} & 1 & 0\\ \frac{5}{2} & \frac{19}{5} & 1\end{bmatrix}\qquad U=\begin{bmatrix}2 & -3 & 10\\ 0 & \frac{5}{2} & 7\\ 0 & 0 & \frac{-253}{5}\end{bmatrix}$$

$$let\ UX=Y\quad where\ y=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix},then\ LY=B,$$

$$i.e\begin{bmatrix}1 & 0 & 0\\ \frac{-1}{2} & 1 & 0\\ \frac{5}{2} & \frac{19}{5} & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}3\\ 20\\ -12\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 0 & 0\\ \frac{-1}{2} & 1 & 0\\ \frac{5}{2} & \frac{19}{5} & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

now eqn (1) *implies*

$$y_1=3$$

$$\frac{-1}{2}y_1+y_2=20\Rightarrow y_2=\frac{43}{2}$$

$$\frac{5}{2}y_1+\frac{19}{5}y_2+y_3=-12\ \Rightarrow y_3=\frac{-506}{5}$$

$2x-3y+10z=3\ ,\frac{5}{2}y+7z=\frac{43}{2},-\frac{253}{5}z=\frac{-506}{3}$

by back substitution $x=-4, y=3 and\ z=2$

Example

Solve the following system

$x+3y+4z=4$

$x+4y+3z=-2$

$x+3y+4z=1$

Solution

$x+3y+4z=4$

$x+4y+3z=-2$

$x+3y+4z=1$

the given system is $AX=B$

$$A=\begin{bmatrix}1 & 3 & 8\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\quad \mathrm{B}=\begin{bmatrix}4\\ -2\\ 1\end{bmatrix}$$

let $LU=A$

$$L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\quad U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

$$\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}1 & 3 & 8\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$$

here

$u_{11}=1,\ u_{12}=3\quad u_{13}=8$

$l_{21}u_{11}=-1\ \Rightarrow l_{21}=1$

$l_{31}u_{11}=1\quad \Rightarrow l_{31}=1$

$$l_{21}u_{12}+u_{22}=4 \quad \Rightarrow u_{22}=4-l_{21}u_{12}=4-(\frac{-1}{2})(-3)=\frac{5}{2}$$

$$l_{21}u_{13}+u_{23}=2 \quad \Rightarrow u_{23}=4-l_{21}u_{13}=4-(1)3=1$$

$$l_{31}u_{12}+l_{32}u_{22}=3 \Rightarrow l_{32}=\frac{1}{u_{22}}[3-l_{31}u_{12}]=0$$

$$l_{31}u_{13}+l_{32}u_{23}+u_{33} \Rightarrow u_{33}=4-l_{31}u_{13}-l_{32}u_{23}=-4$$

$$L=\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix} \qquad U=\begin{bmatrix}1&3&8\\0&1&-5\\0&0&-4\end{bmatrix}$$

$$let\ UX=Y \quad where\ y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}, then\ LY=B,$$

$$i.e\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}4\\-2\\1\end{bmatrix}$$

and

$$\begin{bmatrix}1&0&0\\\frac{-1}{2}&1&0\\\frac{5}{2}&\frac{19}{5}&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

now eqn (1) *implies*

$y_1=4$

$y_2=-2$

$y_1+y_3=1 \Rightarrow y_3=1-y_1=-3$

we also have

$x+3y+8z=4$

$y-5z=-2$

$-4z=-3$

by back substitutiton

$$z=\frac{3}{4}$$

$$y=-2+5z=-2+5(\frac{3}{4})=\frac{7}{4}$$

$$x=4-3y-8z=4-3(\frac{7}{4})-8(\frac{3}{4})=\frac{29}{4}$$