Newton -Raphson Method

This method is one of the most powerful method and well known methods, used for finding a root of f(x)=0 the formula many be derived in many ways the simplest way to derive this formula is by using the first two terms in Taylor's series expansion of the form,

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

$$setting\ f(x_{n+1}) = 0\ gives,$$

$$f(x_n) + (x_{n+1} - x_n) f'(x_n) = 0$$

$$thus\ on\ simplification, we\ get,$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\ for\ n = 0,1,2...$$

Geometrical interpretation

Let the curve f(x)=0 meet the x-axis at x=α meet the x axis at x=α .it means that α is the original root of the f(x)=0.Let x_0 be the point near the root α of the equation f(x)=0 then the equation of the tangent $P_0[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

This cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

This is the first approximation to the root α .if $P_1[x_1, f(x_1)]$ is the point corresponding to x_1 on the curve then the tangent at P_1 is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

This cuts the x-axis at $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

This is the second approximation to the root α .Repeating this process we will get the root α with better approximations quite rapidly.

Note:

1. When $f'(x)$ very large .i.e. is when the slope is large, then h will be small (as assumed) and hence, the root can be calculated in even less time.
2. If we choose the initial approximation x_0 close to the root then we get the root of the equation very quickly.
3. The process will evidently fail if $f'(x) = 0$ is in the neighborhood of the root. In such cases the regula falsi method should be used.

4. If the initial approximation to the root is not given, choose two say, a and b, such that f(a) and f(b) are of opposite signs. If |f(a)|<|f(b)| then take a as the initial approximation.
5. Newton's raphson method is also referred to as the method of tangents.

Example

Find a real root of the equation $x3 - x - 1 = 0$ using Newton - Raphson method, correct to four decimal places.

Solution

$f(x)=x^3 - x - 1$

$f(1)=1^3 - 1 - 1 =-1<0$

$f(2)=2^3 - 2 - 1 = 8-2-1=5>0$

so the root lies between 1 *and* 2

$here\ f'(x)=3x^2-1 \quad and \quad f''(x)=6x$

$f'(1)=3*1^2-1=2$

$f'(x)=3*2^2-1=11$

here

$f''(1)=6$

$f''(2)=6(2)=12$

here $f(2)$ *and* $f''(2)$ *have the same signs so* $x_0=2$

The second approximation is computed using Newton-Raphson method as

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{5}{11} = 1.54545$$

f(1.54545)=1.54545^3 - 1.54541 - 1 =3.691177-1.54541-1=1.14576

$$f'(x) = 3x^2 - 1 = 3(1.54541)^2 - 1 = 3(2.38829) - 1 = 7.16487 - 1 = 6.16487$$

$$x_2 = 1.54545 - \frac{1.14576}{6.16525} = 1.35961$$

f(1.35961)=1.35961^3 - 1.35961 - 1 =3.691177-1.54541-1=1.14576

$$f'(x) = 3x^2 - 1 = 3(1.54541)^2 - 1 = 3(2.38829) - 1 = 7.16487 - 1 = 6.16487$$

$$x_2 = 1.54545 - \frac{1.14576}{6.16525} = 1.35961$$

$$x_3 = 1.35961 - \frac{0.15369}{4.54562} = 1.32579, \qquad f(x_3) = 4.60959 \times 10^{-3}$$

$$x_4 = 1.32579 - \frac{4.60959 \times 10^{-3}}{4.27316} = 1.32471, \qquad f(x_4) = -3.39345 \times 10^{-5}$$

$$x_5 = 1.32471 + \frac{3.39345 \times 10^{-5}}{4.26457} = 1.324718, \qquad f(x_5) = 1.823 \times 10^{-7}$$

Hence, the required root is 1.3247

Note!

Methods such as bisection method and the false position method of finding roots of a nonlinear equation *f(x) = 0* require bracketing of the root by two guesses. Such methods are called bracketing methods. These methods are always convergent since they are based on reducing the interval between the two guesses to *zero in* on the root.
In the Newton-Raphson method, the root is not bracketed. Only one initial guess of the root is needed to get the iterative process started to find the root of an equation. Hence, the method falls in the category of open methods.
Newton - Raphson method is based on the principle that if the initial guess of the root of *f(x) = 0* is at xi, then if one draws the tangent to the curve at *f(xi)*, the point xi+1 where the tangent crosses the x-axis is an improved estimate of the root

Draw backs of N-R Method

Divergence at inflection points:

If the selection of a guess or an iterated value turns out to be close to the inflection point of $f(x)$,
[where $f''(x) = 0$],
the roots start to diverge away from the root.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Division of zero or near zero:

If an iteration, we come across the division by zero or a near-zero number, then we get a large magnitude for the next value, x_{i+1}.

Root jumping:

In some case where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

Oscillations near local maximum and minimum:

Results obtained from N-R method may oscillate about the local max or min without converging on a root but converging on the local max or min. Eventually, it may lead to division to a number close to zero and may diverge.

Convergence of N-R Method

Let us compare the N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with the general iteration formula

$$x_{n+1} = \phi(x_n),$$

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

The iteration method converges if

$$|\phi'(x)| < 1.$$

Therefore, N-R formula converges, provided

$$|f(x)f''(x)| < |f'(x)|^2$$

in the interval considered.

Newton-Raphson formula therefore converges, provided the initial approximation *x*0 is chosen sufficiently close to the root and are continuous and bounded in any small interval containing the root.

Definition

Let $x_n = \alpha + \varepsilon_n, x_{n+1} = \alpha + \varepsilon_{n+1}$

where α is a root of f (x) = 0.

If we can prove that $\varepsilon_{n+1} = K\varepsilon_n^p$,

where K is a constant and ε_n is the error involved at the n - th step, while finding the root by an iterative method, then the rate of convergence of the method is *p*.

The N-R method converges quadratically

$$x_n = \alpha + \varepsilon_n,$$

$$x_{n+1} = \alpha + \varepsilon_{n+1}$$

where α is a root of $f(x) = 0$ and ε_n is the error involved at the *n*-th step, while finding the root by N-R formula

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon n)} = \frac{\varepsilon_n f'(\alpha + \varepsilon_n) - f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

Using Taylor's expansion, we get

$$\varepsilon_{n+1} = \frac{1}{f'(\alpha)+\varepsilon_n f''(\alpha)+\cdots}\left\{\varepsilon_n[f'(\alpha)+\varepsilon_n f''(\alpha)+\cdots]-\left[f(\alpha)+\varepsilon_n f'(\alpha)+\frac{\varepsilon_n^2}{2}f''(\alpha)+\cdots\right]\right\}$$

Since α is a root, $f(\alpha)=0$. Therefore, the above expression simplifies to

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2}f''(\alpha)\frac{1}{f'(\alpha)+\varepsilon_n f''(\alpha)}$$

$$= \frac{\varepsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\left[1+\varepsilon_n\frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$= \frac{\varepsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\left[1-\varepsilon_n\frac{f''(\alpha)}{f'(\alpha)}\right]$$

$$= \frac{\varepsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\left[1-\varepsilon_n\frac{f''(\alpha)}{f'(\alpha)}\right]$$

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}+O(\varepsilon_n^3)$$

On neglecting terms of order ε_n^3 and higher powers, we obtain

$$\varepsilon_{n+1} = K\varepsilon_n^2$$

Where

$$K = \frac{f''(\alpha)}{2f'(\alpha)}$$

It shows that N-R method has second order convergence or converges quadratically.

Example

Set up Newton's scheme of iteration for finding the square root of a positive number *N*.

Solution

The square root of *N* can be carried out as a root of the equation

$x^2 - N = 0.$

Let

$f(x) = x^2 - N.$

By Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In this Problem

$f(x) = x^2 - N, f'(x) = 2x.$

Therefore

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n} \right)$$

Example

Evaluate $\sqrt{12}$, by Newton's formula.

Solution

Since

$\sqrt{9} = 3, \sqrt{16} = 4,$

We take

$x_0 = (3+4)/2 = 3.5.$

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0} \right) = \frac{1}{2}\left(3.5 + \frac{12}{3.5} \right) = 3.4643$$

$$x_2 = \frac{1}{2}\left(3.4643 + \frac{12}{3.4643} \right) = 3.4641$$

$$x_3 = \frac{1}{2}\left(3.4641 + \frac{12}{3.4641} \right) = 3.4641$$

Hence

$\sqrt{12} = 3.4641.$

Here in this solution we use the iterative scheme developed in the above example and simply replace the previous value in the next iteration and finally come to the result.

Example

Find the first three iteration of the equation $f(x) = x - 0.8 - 0.2\sin x$ in the interval $[0, \pi/2]$.

Solution

$$f(0)=0-0.8-0.2\sin(0)=0-0.8-0.2(0)=-0.8$$

$$f(1.57)=1.57-0.8-0.2\sin(1.75)$$
$$=1.57-0.8-0.2(0.99999)$$
$$=1.57-0.8-0.199998=0.570002$$

$$f'(x)=1-0.2\cos x$$

$$f'(0)=1-0.2\cos(0)=1-0.2=0.8$$

$$here\,|f(0)|\,is\ greater\ then\ x_0=0$$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=0-\frac{-0.8}{0.8}=1$$

now

$$f(1)=1-0.8-0.2\sin(1)$$
$$=1-0.8-0.1683$$
$$=0.0317$$

$$f'(x)=1-0.2\cos x$$

$$f'(1)=1-0.2\cos(1)=1-0.1081=0.8919$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=1-\frac{0.0317}{0.8919}=1-0.0355=0.9645$$

$$f(0.9645)=0.9645-0.8-0.2\sin(0.9645)$$
$$=0.9645-0.8-0.1645$$
$$=0.0002$$

$$f'(x)=1-0.2\cos x$$

$$f'(0.9645)=1-0.2\cos(0.9645)=1-0.11396=0.88604$$

$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}=0.9645-\frac{0.0002}{0.88604}=0.9645-0.00022=0.9643$$

NOTE: In the above question the function is a transcendental function and we have to perform all the calculation in the radians mode and the value of pi should be taken as 3.14

Example

Consider the equation $f(x)=4x\cos x-(x-2)^2$ find the root of the equation in the range $0\le x\le 8$

Solution

$$f(x)=4x\cos x-(x-2)^2$$

here

$$f(0)=4(0)\cos 2(0)-(0-2)^2=-4$$

$$f(8)=4(8)\cos 2(8)-(8-2)^2$$

$$=32\cos 16-(6)^2=-30.6451-36=-66.6451$$

$$f'(x)=4\cos 2x-8x\sin 2x-2(x-2)$$

$$=4\cos 2x-8x\sin 2x-2(x-2)$$

$$f'(8)=4\cos 16-64\sin 16-2(8-2)$$

$$=-3.8306+18.4250-12=2.5952$$

$\sin ce\,|f(8)|\,is\ greater\ so\ x_0=8$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=8-\frac{(-66.6451)}{2.5952}=33.6801$$

$$f(33.6801)=4(33.6801)\cos 2(33.6801)-(33.6801-2)^2$$

$$=-24.6545-1003.6=-1028.25$$

$$f'(33.6801)=4\cos 2(33.6801)-8(33.6801)\sin 2(33.6801)-2(33.6801-2)$$

$$=-0.7320+264.89=-63.3602$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=33.6801+\frac{1028.25}{200.79}=38.8011$$

$$f(38.8011)=4(38.8011)\cos 2(38.8011)-(38.8011-2)^2$$

$$=-91.8361-1354.3=1446.14$$

$$f'(38.8011)=4\cos 2(38.8011)-8(38.8011)\sin 2(38.8011)-2(38.8011-2)$$

$$=-2.3668-250.236-73.6022=-326.205$$

$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}=38.8011+\frac{1446.14}{-326.205}=38.8011+4.4332=43.2343$$

Example

Perform three iteration of the equation $\ln(x-1)+\cos(x-1)=0\ when\ 1.2\le x\le 2$ .Use Newton Raphson method to calculate the root.

Solution

Here

$$\ln(x-1)+\cos(x-1)=0\ when\ 1.2\le x\le 2$$

$$f(x)=\ln(x-1)+\cos(x-1)$$

$$f(x) = \ln(x-1) + \cos(x-1)$$
$$f(1.2) = \ln(1.2-1) + \cos(1.2-1)$$
$$= -1.6094 + 0.9801 = -0.6293$$
$$f(2) = \ln(2-1) + \cos(2-1)$$
$$= 0 + 0.5403 = 0.5403$$
$$now\ f(x) = \ln(x-1) + \cos(x-1)$$
$$f'(x) = \frac{1}{x-1} - \sin(x-1)$$
$$f'(1.2) = \frac{1}{1.2-1} - \sin(1.2-1)$$
$$= 5 - 0.1986 = 4.8014$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \frac{-0.6293}{4.8014} = 1.2 + 0.1311 = 1.3311$$
$$f(1.311) = \ln(1.3311-1) + \cos(1.3311-1)$$
$$= -1.1053 + 0.9457 = -0.1596$$
$$f'(1.3311) = \frac{1}{1.3311-1} - \sin(1.3311-1)$$
$$= 3.0202 - 0.3251 = 2.6951$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3311 - \frac{-0.1596}{2.6951} = 1.3311 + 0.0592 = 1.3903$$
$$f(1.3903) = \ln(1.3903-1) + \cos(1.3903-1)$$
$$= -0.9408 + 0.9248 = -0.016$$
$$f'(1.3903) = \frac{1}{1.3903-1} - \sin(1.3903-1)$$
$$= 2.5621 - 0.3805 = 2.1816$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.3903 - \frac{-0.016}{2.1816} = 1.3903 + 0.0073 = 1.3976$$