LU Decomposition Method

(Factorization Method)
This factorization method is also called Choleskey's method.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = LU$$

$LUX = B$ Let $UX = Y$

$LY = B$

In computing element of L & U

1. First row of U i.e u_{11}, u_{12}, u_{13}
2. First column of L i.e l_{21}, l_{31}
3. second row of U u_{21}, u_{23}
4. second column of L l_{32}
5. Thrid row i.e u_{33}

Q:- Solve the following system of equation by LU Decomposition method

$$x + 5y + z = 14$$
$$2x + y + 3z = 13$$
$$3x + y + 4z = 17$$

Sol:-

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$$A \qquad X = B$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = L \qquad U$$

$A = \begin{cases} u_{11} \\ l_{21} \cdot u_{11} + u_{12} \end{cases}$

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$$

$A = LU$

$$A = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}\cdot u_{11} & l_{21}\cdot u_{12} + u_{22} & l_{21}\cdot u_{13} + u_{23} \\ l_{31}\cdot u_{11} & l_{21}\cdot u_{12} + l_{32}\cdot u_{22} & l_{31}\cdot u_{13} + l_{32}\cdot u_{23} + u_{33} \end{bmatrix}$$

$\boxed{u_{11} = 1}$

$\boxed{u_{12} = 5}$

$\boxed{u_{13} = 1}$

$l_{21}\cdot u_{11} = 2 \Rightarrow l_{21}(1) = 2 \Rightarrow \boxed{l_{21} = 2}$

$l_{31}\cdot u_{11} = 3 \Rightarrow l_{31}(1) = 3 \Rightarrow \boxed{l_{31} = 3}$

$l_{21}\cdot u_{12} + u_{22} = 1$

$(2)(5) + u_{22} = 1$

$10 + u_{22} = 1$

$\boxed{u_{22} = -9}$

$l_{21}\cdot u_{13} + u_{23} = 3$

$(2)(1) + u_{23} = 3$

$u_{23} = 3 - 2 = 1$

$\boxed{u_{23} = 1}$

$$l_{31} \cdot u_{12} + l_{32} \cdot u_{22} = 1$$

$$(3)(5) + l_{32}(-9) = 1$$

$$l_{32} = \frac{1-15}{-9} = \frac{14}{9}$$

$$\boxed{l_{32} = \frac{14}{9}}$$

$$l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + u_{33} =$$

$$(3)(1) + \left(\frac{14}{9}\right)(1) + u_{33} = 4$$

$$u_{33} = 4 - 3 - \frac{14}{9}$$

$$\boxed{u_{33} = \frac{-5}{9}}$$

$$A = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & \frac{-5}{9} \end{bmatrix}$$

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$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 14/9 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix}$$

$AX = B$

$LUX = B$

Let $UX = Y$

$LY = B$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 14/9 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ 2y_1 + y_2 \\ 3y_1 + \frac{14}{9} y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$\boxed{y_1 = 14}$

$2y_1 + y_2 = 13$

$y_2 = 13 - 2y_1$

$y_2 = 13 - 2(14) =$

$\boxed{y_2 = -15}$

$$3y_1 + \frac{14}{9}y_2 + y_3 = 17$$

$$y_3 = 17 - 3y_1 - \frac{14}{9}y_2$$

$$y_3 = 17 - 3(14) - \frac{14}{9}(-15)$$

$$\boxed{y_3 = -\frac{5}{3}}$$

$$UX = Y$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 5x_2 + x_3 \\ -9x_2 + x_3 \\ -\frac{5}{9}x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{bmatrix}$$

$$-\frac{5}{9}x_3 = -\frac{5}{3}$$

$$\boxed{x_3 = -3}$$

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$$-9(x_2) + x_3 = -15$$

$$-9x_2 = -15 - x_3$$

$$x_2 = \frac{-15 - x_3}{-9}$$

$$x_2 = \frac{15 - 3}{9} = \frac{4}{3}$$

$$\boxed{x_2 = \frac{4}{3}}$$

$$x_1 + 5x_2 + x_3 = 14$$

$$x_1 = 14 - 5x_2 - x_3$$

$$x_1 = 14 - 5\left(\frac{4}{3}\right) - (-3)$$

$$x_1 =$$

$$UX = Y$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix}$$

$$x + 5y + z = 14$$

$$-9y + z = -15$$

$$\frac{-5}{9}z = \frac{-5}{3}$$

$$\frac{-8}{\cancel{3}}z = \frac{-8}{\cancel{3}}$$

$$\boxed{z = 3}$$

$$-9y + z = -15$$

$$y = \frac{-15 - z}{-9}$$

$$y = \frac{-15-3}{-9} = 2$$

$$\boxed{y = 2}$$

$$x + 5y + z = 14$$

$$x = 14 - 5y - z$$

$$x = 14 - 5(2) - (3)$$

$$\boxed{x = 1}$$

Check

$$3x + y + 4z = 17$$

$$3(1) + 2 + 4(3) = 17$$

$$3 + 2 + 12 = 17$$

$$17 = 17$$

— x — x —