The bound on the magnitude of $g_4'(x)$ is much smaller than the bound (found in (**c**)) on the magnitude of $g_3'(x)$, which explains the more rapid convergence using g_4.

(e) The sequence defined by

$$g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

converges much more rapidly than our other choices. In the next sections we will see where this choice came from and why it is so effective. □

From what we have seen,

- Question: How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

might have

- Answer: Manipulate the root-finding problem into a fixed point problem that satisfies the conditions of Fixed-Point Theorem 2.4 and has a derivative that is as small as possible near the fixed point.

In the next sections we will examine this in more detail.

Maple has the fixed-point algorithm implemented in its *NumericalAnalysis* package. The options for the Bisection method are also available for fixed-point iteration. We will show only one option. After accessing the package using *with*(*Student*[*NumericalAnalysis*]): we enter the function

$$g := x - \frac{(x^3 + 4x^2 - 10)}{3x^2 + 8x}$$

and Maple returns

$$x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

Enter the command

FixedPointIteration(fixedpointiterator = *g*, *x* = 1.5, tolerance = 10^{-8}, *output* = *sequence*, *maxiterations* = 20)

and Maple returns

1.5, 1.373333333, 1.365262015, 1.365230014, 1.365230013

EXERCISE SET 2.2

1. Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

a. $g_1(x) = \left(3 + x - 2x^2\right)^{1/4}$ **b.** $g_2(x) = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2}$

c. $g_3(x) = \left(\dfrac{x+3}{x^2+2}\right)^{1/2}$ **d.** $g_4(x) = \dfrac{3x^4+2x^2+3}{4x^3+4x-1}$

2. **a.** Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.

b. Which function do you think gives the best approximation to the solution?

3. The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a. $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$ **b.** $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c. $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$ **d.** $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

4. The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a. $p_n = p_{n-1}\left(1 + \dfrac{7 - p_{n-1}^5}{p_{n-1}^2}\right)^3$ **b.** $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$

c. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$ **d.** $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$

5. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on [1, 2]. Use $p_0 = 1$.

6. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on [1, 2]. Use $p_0 = 1$.

7. Use Theorem 2.3 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

8. Use Theorem 2.3 to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-4}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-4} accuracy, and compare this theoretical estimate to the number actually needed.

9. Use a fixed-point iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 12 of Section 2.1.

10. Use a fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 13 of Section 2.1.

11. For each of the following equations, determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a. $x = \dfrac{2 - e^x + x^2}{3}$ **b.** $x = \dfrac{5}{x^2} + 2$

c. $x = (e^x/3)^{1/2}$ **d.** $x = 5^{-x}$

e. $x = 6^{-x}$ **f.** $x = 0.5(\sin x + \cos x)$

12. For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a. $2 + \sin x - x = 0$ use $[2, 3]$ **b.** $x^3 - 2x - 5 = 0$ use $[2, 3]$

c. $3x^2 - e^x = 0$ **d.** $x - \cos x = 0$

13. Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-4}.