The bound for the number of iterations for the Bisection method assumes that the calculations are performed using infinite-digit arithmetic. When implementing the method on a computer, we need to consider the effects of round-off error. For example, the computation of the midpoint of the interval $[a_n, b_n]$ should be found from the equation

$$p_n = a_n + \frac{b_n - a_n}{2} \quad \text{instead of} \quad p_n = \frac{a_n + b_n}{2}.$$

The first equation adds a small correction, $(b_n - a_n)/2$, to the known value a_n. When $b_n - a_n$ is near the maximum precision of the machine, this correction might be in error, but the error would not significantly affect the computed value of p_n. However, when $b_n - a_n$ is near the maximum precision of the machine, it is possible for $(a_n + b_n)/2$ to return a midpoint that is not even in the interval $[a_n, b_n]$.

The Latin word *signum* means "token" or "sign". So the signum function quite naturally returns the sign of a number (unless the number is 0).

As a final remark, to determine which subinterval of $[a_n, b_n]$ contains a root of f, it is better to make use of the **signum** function, which is defined as

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

The test

$$\operatorname{sgn}(f(a_n)) \operatorname{sgn}(f(p_n)) < 0 \qquad \text{instead of} \qquad f(a_n)f(p_n) < 0$$

gives the same result but avoids the possibility of overflow or underflow in the multiplication of $f(a_n)$ and $f(p_n)$.

EXERCISE SET 2.1

1. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.
2. Let $f(x) = 3(x + 1)(x - \frac{1}{2})(x - 1)$. Use the Bisection method on the following intervals to find p_3.
   a. $[-2, 1.5]$ **b.** $[-1.25, 2.5]$
3. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval.
   a. $[0, 1]$ **b.** $[1, 3.2]$ **c.** $[3.2, 4]$
4. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on each interval.
   a. $[-2, -1]$ **b.** $[0, 2]$ **c.** $[2, 3]$ **d.** $[-1, 0]$
5. Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.
   a. $x - 2^{-x} = 0 \quad \text{for } 0 \le x \le 1$
   b. $e^x - x^2 + 3x - 2 = 0 \quad \text{for } 0 \le x \le 1$
   c. $2x\cos(2x) - (x + 1)^2 = 0 \quad \text{for } -3 \le x \le -2 \quad \text{and} \quad -1 \le x \le 0$
   d. $x\cos x - 2x^2 + 3x - 1 = 0 \quad \text{for } 0.2 \le x \le 0.3 \quad \text{and} \quad 1.2 \le x \le 1.3$
6. Use the Bisection method to find solutions, accurate to within 10^{-5} for the following problems.
   a. $3x - e^x = 0$ for $1 \le x \le 2$
   b. $2x + 3\cos x - e^x = 0 \quad \text{for } 0 \le x \le 1$
   c. $x^2 - 4x + 4 - \ln x = 0 \quad \text{for } 1 \le x \le 2 \quad \text{and} \quad 2 \le x \le 4$
   d. $x + 1 - 2\sin \pi x = 0 \quad \text{for } 0 \le x \le 0.5 \quad \text{and} \quad 0.5 \le x \le 1$

7. **a.** Sketch the graphs of $y = x$ and $y = 2\sin x$.

b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = 2\sin x$.

8. **a.** Sketch the graphs of $y = x$ and $y = \tan x$.

b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = \tan x$.

9. **a.** Sketch the graphs of $y = e^x - 2$ and $y = \cos(e^x - 2)$.

b. Use the Bisection method to find an approximation to within 10^{-5} to a value in $[0.5, 1.5]$ with $e^x - 2 = \cos(e^x - 2)$.

10. Let $f(x) = (x+2)(x+1)^2 x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

a. $[-1.5, 2.5]$ **b.** $[-0.5, 2.4]$ **c.** $[-0.5, 3]$ **d.** $[-3, -0.5]$

11. Let $f(x) = (x+2)(x+1)x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

a. $[-3, 2.5]$ **b.** $[-2.5, 3]$ **c.** $[-1.75, 1.5]$ **d.** $[-1.5, 1.75]$

12. Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the Bisection Algorithm. [*Hint:* Consider $f(x) = x^2 - 3$.]

13. Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-4} using the Bisection Algorithm.

14. Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$. Find an approximation to the root with this degree of accuracy.

15. Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

16. Let $f(x) = (x-1)^{10}$, $p = 1$, and $p_n = 1 + 1/n$. Show that $|f(p_n)| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

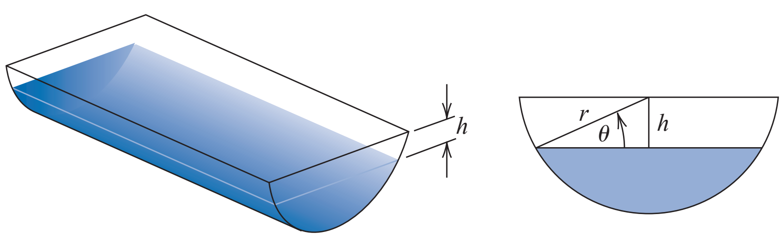
17. Let $\{p_n\}$ be the sequence defined by $p_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $\{p_n\}$ diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$.

18. The function defined by $f(x) = \sin \pi x$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the Bisection method converges to

a. 0, if $a + b < 2$ **b.** 2, if $a + b > 2$ **c.** 1, if $a + b = 2$

19. A trough of length L has a cross section in the shape of a semicircle with radius r. (See the accompanying figure.) When filled with water to within a distance h of the top, the volume V of water is

$$V = L\left[0.5\pi r^2 - r^2 \arcsin(h/r) - h(r^2 - h^2)^{1/2}\right].$$



Suppose $L = 10$ ft, $r = 1$ ft, and $V = 12.4$ ft^3. Find the depth of water in the trough to within 0.01 ft.

20. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$