Solution of Linear System of Equationsand Matrix Inversion

Jacobi's Method

This is an iterative method, where initial approximate solution to a given system of equations is assumed and is improved towards the exact solution in an iterative way.

In general, when the coefficient matrix of the system of equations is a sparse matrix (many elements are zero), iterative methods have definite advantage over direct methods in respect of economy of computer memory

Such sparse matrices arise in computing the numerical solution of partial differential equations

Let us consider

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \;\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_1 \end{array}\right\}$$

In this method, we assume that the coefficient matrix [A] is strictly diagonally dominant, that is, in each row of [A] the modulus of the diagonal element exceeds the sum of the off-diagonal elements.

We also assume that the diagonal element do not vanish. If any diagonal element vanishes, the equations can always be rearranged to satisfy this condition.

Now the above system of equations can be written as

$$\left.\begin{array}{l} x_1 = \dfrac{b_1}{a_{11}} - \dfrac{a_{12}}{a_{11}}x_2 - \cdots - \dfrac{a_{1n}}{a_{11}}x_n \\ x_2 = \dfrac{b_2}{a_{22}} - \dfrac{a_{21}}{a_{22}}x_1 - \cdots - \dfrac{a_{2n}}{a_{22}}x_n \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ x_n = \dfrac{b_n}{a_{nn}} - \dfrac{a_{n1}}{a_{nn}}x_1 - \cdots - \dfrac{a_{n(n-1)}}{a_{nn}}x_{n-1} \end{array}\right\}$$

We shall take this solution vector $(x_1, x_2, ..., x_n)^T$ as a first approximation to the exact solution of system. For convenience, let us denote the first approximation vector by $(x_1^{(1)}, x_2^{(1)}, ..., x_n^{(1)})$ got after taking as an initial starting vector.

Substituting this first approximation in the right-hand side of system, we obtain the second approximation to the given system in the form

$$\left.\begin{aligned}
x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\
x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\
\vdots \quad & \quad \vdots \qquad \vdots \qquad\qquad \vdots \\
x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \cdots - \frac{a_{n(n-1)}}{a_{nn}} x_{n-1}^{(1)}
\end{aligned}\right\}$$

This second approximation is substituted into the right-hand side of Equations and obtain the third approximation and so on.

This process is repeated and *(r+1)th* approximation is calculated

$$\left.\begin{aligned}
x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(r)} \\
x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(r)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(r)} \\
\vdots \quad & \quad \vdots \qquad \vdots \qquad\qquad \vdots \\
x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(r)} - \cdots - \frac{a_{n(n-1)}}{a_{nn}} x_{n-1}^{(r)}
\end{aligned}\right\}$$

Briefly, we can rewrite these Equations as

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(r)},$$

$$r = 1, 2, \ldots, \qquad i = 1, 2, \ldots, n$$

It is also known as method of simultaneous displacements,

since no element of $x_i^{(r+1)}$ is used in this iteration until every element is computed.

A sufficient condition for convergence of the iterative solution to the exact solution is

$\left|a_{ii}\right| > \sum_{\substack{j=1 \\ j\neq 1}}^{n} \left|a_{ij}\right|, \qquad i = 1, 2, \ldots, n$ When this condition (diagonal dominance) is true, Jacobi's method converges

Example

Find the solution to the following system of equations using Jacobi's iterative method for the first five iterations:

$$83x + 11y - 4z = 95$$

$$7x + 52y + 13z = 104$$

$$3x + 8y + 29z = 71$$

Solution

$$\left.\begin{aligned} x &= \frac{95}{83} - \frac{11}{83}y + \frac{4}{83}z \\ y &= \frac{104}{52} - \frac{7}{52}x - \frac{13}{52}z \\ z &= \frac{71}{29} - \frac{3}{29}x - \frac{8}{29}y \end{aligned}\right\}$$

Taking the initial starting of solution vector as $(0,0,0)^T$, from Eq. ,we have the first approximation as

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{pmatrix} = \begin{pmatrix} 1.1446 \\ 2.0000 \\ 2.4483 \end{pmatrix}$$

Now, using Eq. ,the second approximation is computed from the equations

$$\left.\begin{aligned} x^{(2)} &= 1.1446 - 0.1325y^{(1)} + 0.0482z^{(1)} \\ y^{(2)} &= 2.0 - 0.1346x^{(1)} - 0.25z^{(1)} \\ z^{(2)} &= 2.4483 - 0.1035x^{(1)} - 0.2759y^{(1)} \end{aligned}\right\}$$

Making use of the last two equations we get the second approximation as

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \end{pmatrix} = \begin{pmatrix} 0.9976 \\ 1.2339 \\ 1.7424 \end{pmatrix}$$

Similar procedure yields the third, fourth and fifth approximations to the required solution and they are tabulated as below;

Variables			
Iteration number r	x	y	z
1	1.1446	2.0000	2.4483
2	0.9976	1.2339	1.7424
3	1.0651	1.4301	2.0046

4	1.0517	1.3555	1.9435
5	1.0587	1.3726	1.9655

Example

Solve the system by jacobi's iterative method

$$8x-3y+2z=20$$

$$4x+11y-z=33$$

$$6x+3y+12z=35$$

(Perform only four iterations)

Solution

Consider the given system as

$$8x-3y+2z=20$$

$$4x+11y-z=33$$

$$6x+3y+12z=35$$

the system is diagonally dominant

$$x=\frac{1}{8}[20+3y-2z]$$

$$y=\frac{1}{11}[33-4x+z]$$

$$z=\frac{1}{12}[35-6x-3y]$$

we start with an initial aproximation $x_0=y_0=z_0=0$

substituting these

first iteration

$$x_1=\frac{1}{8}[20+3(0)-2(0)]=2.5$$

$$y_1=\frac{1}{11}[33-4(0)+0]=3$$

$$z_1=\frac{1}{12}[35-6(0)-3(0)]=2.916667$$

Second iteration

$$x_2=\frac{1}{8}[20+3(3)-2(2.9166667)]=2.895833$$

$$y_2=\frac{1}{11}[33-4(2.5)+2.9166667]=2.3560606$$

$$z_2=\frac{1}{12}[35-6(2.5)-3(3)]=0.9166666$$

third iteration

$$x_3 = \frac{1}{8}\left[20 + 3(2.3560606) - 2(0.9166666)\right] = 3.1543561$$

$$y_3 = \frac{1}{11}\left[33 - 4(2.8958333) + 0.9166666\right] = 2.030303$$

$$z_3 = \frac{1}{12}\left[35 - 6(2.8958333) - 3(2.3560606)\right] = 0.8797348$$

fourth iteration

$$x_4 = \frac{1}{8}\left[20 + 3(2.030303) - 2(0.8797348)\right] = 3.0419299$$

$$y_4 = \frac{1}{11}\left[33 - 4(3.1543561) + 0.8797348\right] = 1.9329373$$

$$z_4 = \frac{1}{12}\left[35 - 6(3.1543561) - 3(2.030303)\right] = 0.8319128$$

Example

Solve the system by jacobi's iterative method

$$3x + 4y + 15z = 54.8$$

$$x + 12y + 3z = 39.66$$

$$10x + y - 2z = 7.74$$

(Perform only four iterations)

Solution

Consider the given system as

$$3x + 4y + 15z = 54.8$$

$$x + 12y + 3z = 39.66$$

$$10x + y - 2z = 7.74$$

*the system is not diagonally do*min*ant we rearrange the system*

$$10x + y - 2z = 7.74$$

$$x + 12y + 3z = 39.66$$

$$3x + 4y + 15z = 54.8$$

$$x = \frac{1}{10}\left[7.74 - y + 2z\right]$$

$$y = \frac{1}{12}\left[39.66 - x - 3z\right]$$

$$z = \frac{1}{15}\left[54.8 - 3x - 4y\right]$$

we start with an initial aproximation $x_0 = y_0 = z_0 = 0$

substituting these

first iteration

$$x_1 = \frac{1}{10}\left[7.74 - (0) + 2(0)\right] = 0.774$$

$$y_1 = \frac{1}{12}\left[39.66 - (0) - 3(0)\right] = 1.1383333$$

$$z_1 = \frac{1}{15}\left[54.8 - 3(0) - 4(0)\right] = 3.6533333$$

Second iteration

$$x_2 = \frac{1}{10}\left[7.74 - 1.1383333 + 2(3.6533333)\right] = 1.3908333$$

$$y_2 = \frac{1}{12}\left[39.66 - 0.774 - 3(3.6533333)\right] = 2.3271667$$

$$z_2 = \frac{1}{15}\left[54.8 - 3(0.774) - 4(1.1383333)\right] = 3.1949778$$

third iteration

$$x_3 = \frac{1}{10}\left[7.74 - 2.3271667 + 2(3.1949778)\right] = 1.1802789$$

$$y_3 = \frac{1}{12}\left[39.66 - 1.3908333 - 3(3.1949778)\right] = 2.3903528$$

$$z_3 = \frac{1}{15}\left[54.8 - 3(1.3908333) - 4(2.3271667)\right] = 2.7545889$$

fourth iteration

$$x_4 = \frac{1}{10}\left[7.74 - 2.5179962 + 2(2.7798501)\right] = 1.0781704$$

$$y_4 = \frac{1}{12}\left[39.66 - 1.1802789 - 3(2.7545889)\right] = 2.51779962$$

$$z_4 = \frac{1}{15}\left[54.8 - 3(1.1802789) - 4(2.3903528)\right] = 2.7798501$$