Numerical Analysis

Course Contents

Solution of Non Linear Equations
Solution of Linear System of Equations
Approximation of Eigen Values
Interpolation and Polynomial Approximation
Numerical Differentiation
Numerical Integration
Numerical Solution of Ordinary Differential Equations

Introduction

We begin this chapter with some of the basic concept of representation of numbers on computers and errors introduced during computation. Problem solving using computers and the steps involved are also discussed in brief.

Number (s) System (s)

In our daily life, we use numbers based on the decimal system. In this system, we use ten symbols 0, 1,…,9 and the number 10 is called the base of the system.

Thus, when a base *N* is given, we need *N* different symbols 0, 1, 2, …,($N-1$) to represent an arbitrary number.

The number systems commonly used in computers are

Base, N	Number
2	Binary
8	Octal
10	Decimal
16	Hexadecimal

An arbitrary real number, *a* can be written as

$$a = a_m N^m + a_{m-1} N^{m-1} + \cdots + a_1 N^1 + a_0 + a_{-1} N^{-1} + \cdots + a_{-m} N^{-m}$$

In binary system, it has the form,

$$a = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0 + a_{-1} 2^{-1} + \cdots + a_{-m} 2^{-m}$$

The decimal number 1729 is represented and calculated

$$(1729)_{10} = 1\times10^3 + 7\times10^2 + 2\times10^1 + 9\times10^0$$

While the decimal equivalent of binary number 10011001 is

$$1\times2^0 + 0\times2^{-1} + 0\times2^{-2} + 1\times2^{-3} + 1\times2^{-4} + 0\times2^{-5} + 0\times2^{-6} + 1\times2^{-7}$$

$$= 1 + \frac{1}{8} + \frac{1}{16} + \frac{1}{128}$$

$$= (1.1953125)_{10}$$

Electronic computers use binary system whose base is 2. The two symbols used in this system are 0 and 1, which are called *binary digits* or simply *bits*.

The internal representation of any data within a computer is in binary form. However, we prefer data input and output of numerical results in decimal system. Within the computer, the arithmetic is carried out in binary form.

<u>Conversion of decimal number 47 into its binary equivalent</u>

Sol.

2	47	Remainder
2	23	1
2	11	1
2	5	1
2	2	1
2	1	0
	0	1 Most significant bit

$(47)_{10} = (101111)_2$

Binary equivalent of the decimal fraction 0.7625

Sol.

		Product	Integer
0.7625	x2	1.5250	1
0.5250	x2	1.0500	1
0.05	x2	0.1	0
0.1	x2	0.2	0
0.2	x2	0.4	0
0.4	x2	0.8	0
0.8	x2	1.6	1
0.6	x2	1.2	1
0.2	x2	0.4	0

$(0.7625)_{10} = (0.11....11(0011))_2$

Conversion $(59)_{10}$ into binary and then into octal.

Sol.

2	29	1
2	14	1
2	7	0
2	3	1
2	1	1
	0	1

$$(59)_{10} = (11011)_2$$

$$(111011)_2 = 111011 = (73)_8$$

Errors in Computations

Numerically, computed solutions are subject to certain errors. It may be fruitful to identify the error sources and their growth while classifying the errors in numerical computation. These are

Inherent errors,

Local round-off errors

Local truncation errors

Inherent errors

It is that quantity of error which is present in the statement of the problem itself, before finding its solution. It arises due to the simplified assumptions made in the mathematical modeling of a problem. It can also arise when the data is obtained from certain physical measurements of the parameters of the problem.

Local round-off errors

Every computer has a finite word length and therefore it is possible to store only a fixed number of digits of a given input number. Since computers store information in binary form, storing an exact decimal number in its binary form into the computer memory gives an error. This error is computer dependent.

At the end of computation of a particular problem, the final results in the computer, which is obviously in binary form, should be converted into decimal form-a form understandable to the user-before their print out. Therefore, an additional error is committed at this stage too.

This error is called *local round-off* error.

$$(0.7625)_{10} = (0.110000110011)_2$$

If a particular computer system has a word length of 12 bits only, then the decimal number 0.7625 is stored in the computer memory in binary form as 0.110000110011. However, it is equivalent to 0.76245.

Thus, in storing the number 0.7625, we have committed an error equal to 0.00005, which is the round-off error; inherent with the computer system considered.

Thus, we define the *error* as

$$\text{Error} = \text{True value} - \text{Computed value}$$

Absolute error, denoted by |Error|,

While, the *relative error* is defined as

$$\text{Relative error } = \frac{|\text{Error}|}{|\text{True value}|}$$

Local truncation error

It is generally easier to expand a function into a power series using Taylor series expansion and evaluate it by retaining the first few terms. For example, we may approximate the function $f(x) = \cos x$ by the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

If we use only the first three terms to compute *cos x* for a given *x*, we get an approximate answer. Here, the error is due to truncating the series. Suppose, we retain the first *n* terms, the truncation error (TE) is given by

$$\text{TE} \le \frac{x^{2n+2}}{(2n+2)!}$$

The TE is independent of the computer used.
If we wish to compute *cos x* for accurate with five significant digits, the question is, how many terms in the expansion are to be included? In this situation

$$\frac{x^{2n+2}}{(2n+2)!} < .5\times 10^{-5} = 5\times 10^{-6}$$

Taking logarithm on both sides, we get

$$(2n+2)\log x - \log[(2n+2)!]$$
$$< \log_{10} 5 - 6\log_{10} 10 = 0.699 - 6 = -5.3$$

or

$$\log[(2n+2)!] - (2n+2)\log x > 5.3$$

We can observe that, the above inequality is satisfied for $n = 7$. Hence, seven terms in the expansion are required to get the value of *cos x*, with the prescribed accuracy
The truncation error is given by

$$\text{TE} \le \frac{x^{16}}{16!}$$

Polynomial

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ where n is a positive integer and $a_0, a_1, a_2 + ...a_n$ are real constants, such type of expression is called an nth degree polynomial in x if $a_0 \neq 0$

Algebraic equation:

An equation f(x)=0 is said to be the algebraic equation in x if it is purely a polynomial in x.

For example

$x^5 + x^4 + 3x^2 + x - 6 = 0$ It is a fifth order polynomial and so this equation is an algebraic equation.

$x^3 - 6 = 0$

$x^6 - 7x = 0$

$y^4 - 4y^3 + 3y^2 - y - 2 = 0$ *polynomial in y*

$t^4 - 6t^2 - 21 = 0$ *polynomail in t*

These all are the examples of the polynomial or algebraic equations.

Some facts

1. Every equation of the form f(x)=0 has at least one root ,it may be real or complex.
2. Every polynomial of nth degree has n and only n roots.
3. If f(x) =0 is an equation of odd degree, then it has at least one real root whose sign is opposite to that of last term.
4. If f(x)=0 is an equation of even degree whose last term is negative then it has at least one positive and at least one negative root .

Transcendental equation

An equation is said to be transcendental equation if it has logarithmic, trigonometric and exponential function or combination of all these three.

For example

$e^x - 5x - 3 = 0$ it is a transcendental equation as it has an exponential function

$e^x - \sin x = 0$

$\ln x - \sin x = 0$

$2\sec^2 x - \tan x - e^x = 0$

These all are the examples of transcendental equation.

Root of an equation

For an equation f(x) =0 to find the solution we find such value which satisfy the equation f(x)=0,these values are known as the roots of the equation .

A value a is known as the root of an equation f(x) =0 if and only if f (a) =0.

Properties of an Algebraic equation

1. Complex roots occur in the pairs. That is ,If (a+ib) is a root of f(x)=0 then (a-ib) is also a root of the equation
2. if x=a is a root of the equation f(x)=0 a polynomial of nth degree ,then (x-a) is a factor of f(x) and by dividing f(x) by (x-a) we get a polynomial of degree n-1.

Descartes rule of signs

This rule shows the relation ship between the signs of coefficients of an equation and its roots.

"The number of positive roots of an algebraic equation f(x) =0 with real coefficients can not exceed the number of changes in the signs of the coefficients in the polynomial f(x) =0.similarly the number of negative roots of the equation can not exceed the number of changes in the sign of coefficients of f (-x) =0"

Consider the equation $x^3-3x^2+4x-5=0$ here it is an equation of degree three and there are three changes in the signs
First +ve to –ve second –ve to +ve and third +ve to –ve so the tree roots will be positive
Now $f(-x)=-x^3-3x^2-4x-5$ so there is no change of sign so there will be no negative root of this equation.

Intermediate value property

If f(x) is a real valued continuous function in the closed interval $a \le x \le b$ if f(a) and f(b) have opposite signs once; that is f(x)=0 has at least one root β such that $a \le \beta \le b$
Simply
If f(x)=0 is a polynomial equation and if f(a) and f(b) are of different signs ,then f(x)=0 must have at least one real root between a and b.

Numerical methods for solving either algebraic or transcendental equation are classified into two groups

Direct methods

Those methods which do not require any information about the initial approximation of root to start the solution are known as direct methods.
The examples of direct methods are Graefee root squaring method, Gauss elimination method and Gauss Jordan method. All these methods do not require any type of initial approximation.

Iterative methods

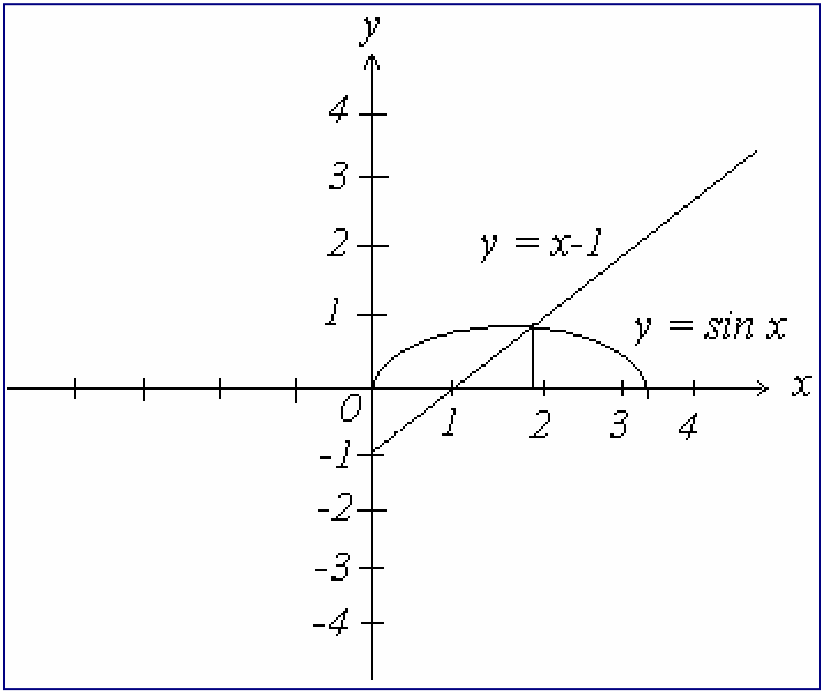
These methods require an initial approximation to start.

Bisection method, Newton raphson method, secant method, jacobie method are all examples of iterative methods.

How to get an initial approximation?

The initial approximation may be found by two methods either by graphical method or analytical method
Graphical method

The equation f(x)=0 can be rewritten as $f_1(x)=f_2(x)$ and initial approximation of f(x) may be taken as the abscissa of the point of intersection of graphs of
$y=f_1(x)\ and\ y=f_2(x)$
for example $f(x)=x-\sin x-1=0$
so this may be written as $x-1=\sin x$ Now we shall draw the graphs of
$y=x-1\ and\ y=\sin x$



Here both the graphs cut each other at 1.9 so the initial approximation should be taken as 1.9

Analytical method

This method is based on the intermediate value property in this we locate two values a and b such that f(a) and f(b) have opposite signs then we use the fact that the root lies

between both these points ,this method can be used for both transcendental and algebraic equations.
Consider the equation

$$f(x) = 3x - \sqrt{1+\sin x} = 0$$
$$f(0) = -1$$
$$f(1) = 3 - \sqrt{1+\sin(1\times\frac{180}{\pi})} = 3 - \sqrt{1+0.84147} = 1.64299$$

Here f(0) and f(1) are of opposite signs making use of intermediate value property we infer that one root lies between 0 and 1 .
So in analytical method we must always start with an initial interval (a,b) so that f(a) and f(b) have opposite signs.

Bisection method (Bolzano)

Suppose you have to locate the root of the equation f(x)=0 in an interval say (x_0, x_1) ,let $f(x_0)$ and $f(x_1)$ are of opposite signs such that $f(x_0)f(x_1) < 0$

Then the graph of the function crossed the x-axis between x_0 *and* x_1 which exists the existence of at least one root in the interval (x_0, x_1).

The desired root is approximately defined by the mid point $x_2 = \frac{x_0 + x_1}{2}$ if $f(x_2) = 0$ then x_2 is the root of the equation otherwise the root lies either between x_0 *and* x_2 or x_1 *and* x_2

Now we define the next approximation by $x_3 = \frac{x_0 + x_2}{2}$ provided $f(x_0)f(x_2) < 0$ then root may be found between x_0 *and* x_2

If provided $f(x_1)f(x_2) < 0$ then root may be found between x_1 *and* x_2 by $x_3 = \frac{x_1 + x_2}{2}$

Thus at each step we find the desired root to the required accuracy or narrow the range to half the previous interval.
This process of halving the intervals is continued in order to get smaller and smaller interval within which the desired root lies. Continuation of this process eventually gives us the required root.

NOTE: In numerical analysis all the calculation are carried out in radians mode and the value of pi is supposed to be 3.14

Example
Solve $x^3 - 9x + 1 = 0$ for the root between x=2 and x=4 by bisection method

Solution:
Here we are given the interval (2,4) so we need not to carry out intermediate value property to locate initial approximation.
Here

$f(x) = x^3 - 9x + 1 = 0$

$now\ f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = 9$

$f(4) = 4^3 - 9(4) + 1 = 64 - 36 + 1 = 29$

$here\ f(2)\ \ f(3) < 0\ so\ root\ lies\ between\ 2\ and\ 4$

$x_0 = 2 \quad x_1 = 4$

$$x_2 = \frac{2+4}{2} = 3$$

$f(3) = 3^3 - 9(3) + 1 = 27 - 27 + 1 = 1$

$here\ f(2)f(3) < 0\ so\ the\ root\ lies\ between\ 2\ nad\ 3$

$$x_3 = \frac{2+3}{2} = 2.5$$

$f(2.5) = 2.5^3 - 9(2.5) + 1 = 15.625 - 22.5 + 1 = -5.875 < 0$

$so\ the\ root\ lies\ between\ 2.5\ and\ 3\ as\ f(2.5)f(3) < 0$

$$x_4 = \frac{2.5+3}{2} = 2.75$$

now

$similarly\ x_5 = 2.875\ and\ x_6 = 2.9375\ and\ the\ process\ is\ continued$

$untill\ the\ desired\ accuracy\ is\ obtained.$

n	xn	$f(xn)$
2	3	1.0
3	2.5	-5.875
4	2.75	-2.9531
5	2.875	-1.1113
6	2.9375	-0.0901

When to stop the process of iteration?

Here in the solved example neither accuracy is mentioned and nor number of iteration is mentioned by when you are provided with the number of iteration then you will carry those number of iteration and will stop but second case is accuracy like you are asked to find root with an accuracy of 10^{-3} then you will check the accuracy by subtracting two consecutive root like 2.135648 and 2.135769

2.135769-2.135648=0.000121

So the accuracy of 10^{-3} is achieved so you will stop here.

Example:

Carry out the five iterations for the function $f(x) = 2x\cos(2x) - (x+1)^2$

Note: All the calculations should be done in the radians.

Solution:

$$f(x) = 2x\cos(2x) - (x+1)^2$$

$$f(-1) = 2(-1)\cos(-2) - (-1+1)^2 = -2(-0.4161) = +0.8322 > 0$$

$$f(0) = 2(0)\cos(0) - (0+1)^2 = -1 = -1 < 0$$

$$\textit{so the root lies between } 0 \textit{ and } -1 \textit{ as } f(0)f(-1) < 0$$

$$x_2 = \frac{0-1}{2} = -0.5$$

$$f(-0.5) = 2(-0.5)\cos(-1) - (-0.5+1)^2 = -0.5403 - 0.25 = -0.7903 < 0$$

$$\textit{so root lies between } -1 \textit{ and } -0.5 \textit{ as } f(-1)f(-0.5)$$

$$x_3 = \frac{-0.5-1}{2} = -0.75$$

$$f(-0.75) = 2(-0.75)\cos(-1.5) - (-0.75+1)^2 = -0.106 - 0.0625 = -0.1686 < 0$$

$$\textit{so root lies between } -1 \textit{ and } -0.75 \textit{ as } f(-1)f(-0.75)$$

$$x_4 = \frac{-0.75-1}{2} = -0.875$$

$$f(-0.875) = 2(-0.875)\cos(-1.75) - (-0.875+1)^2 = 0.3119 - 0.015625 = 0.296275 > 0$$

$$\textit{so root lies between } -0.875 \textit{ and } -0.75 \textit{ as } f(-0.75)f(-0.875)$$

$$x_5 = \frac{-0.75-0.875}{2} = -0.8125$$

$$f(-0.8125) = 2(-0.8125)\cos(-1.625) - (-0.8125+1)^2 = 0.0880 - 0.0351 = 0.052970 > 0$$

$$\textit{so root lies between } -0.8125 \textit{ and } -0.75 \textit{ as } f(-0.75)f(-0.8125)$$

$$x_5 = \frac{-0.75-0.8125}{2} = -0.78125$$

Example :

Carry out the first five iterations $f(x) = x\cos x - 2x^2 + 3x - 1 \quad , \quad 1.2 \le x \le 1.3$

Note: All the calculations should be done in the radians.

Solution:

$$f(x) = x\cos x - 2x^2 + 3x - 1$$

$$f(1.2) = 1.2\cos 1.2 - 2(1.2)^2 + 3(1.2) - 1$$

$$= 1.2(0.3623) - 2(1.44) + 3.6 - 1 = 0.1548 > 0$$

$$f(1.3) = 1.3\cos 1.3 - 2(1.3)^2 + 3(1.3) - 1$$

$$= 1.3(0.2674) - 2(1.69) + 9.3 - 1 = -0.1322 < 0$$

as $f(1.2)f(1.3) < 0$ *so the root lies between both*

$$x_2 = \frac{1.2 + 1.3}{2} = 1.25$$

$$f(1.25) = 1.25\cos 1.25 - 2(1.25)^2 + 3(1.25) - 1$$

$$= 1.25(0.3153) - 2(1.5625) + 3.75 - 1 = 0.0191 > 0$$

as $f(1.25)f(1.3) < 0$ *so the root lies between both*

$$x_3 = \frac{1.25 + 1.3}{2} = 1.275$$

$$f(1.275) = 1.275\cos 1.275 - 2(1.275)^2 + 3(1.275) - 1$$

$$= 1.275(0.2915) - 2(1.6256) + 3.825 - 1 = -0.0545 < 0$$

as $f(1.25)f(1.275) < 0$ *so the root lies between both*

$$x_4 = \frac{1.25 + 1.275}{2} = 1.2625$$

$$f(1.2625) = 1.2625\cos 1.2625 - 2(1.2625)^2 + 3(1.2625) - 1$$

$$= 1.275(0.3034) - 2(1.5939) + 3.7875 - 1 = -0.0172 < 0$$

as $f(1.25)f(1.265) < 0$ *so the root lies between both*

$$x_5 = \frac{1.25 + 1.2625}{2} = 1.25625$$

$$f(1.25625) = 1.25625\cos 1.25625 - 2(1.25625)^2 + 3(1.25625) - 1$$

$$= 1.25625(0.3093) - 2(1.5781) + 3(1.25625) - 1 = 0.00108 > 0$$

as $f(1.25625)f(1.265) < 0$ *so the root lies between both*

$$x_6 = \frac{1.25625 + 1.2625}{2} = 1.259375$$