Solution of Linear System of Equations and Matrix Inversion

The solution of the system of equations gives n unknown values $x1, x2, \ldots, x$n, which satisfy the system simultaneously. If $m > n$, the system usually will have an infinite number of solutions.

If $|A| \neq 0$ then the system will have a unique solution.

If $|A| = 0$,

Then there exists no solution.

Numerical methods for finding the solution of the system of equations are classified as direct and iterative methods. In direct methods, we get the solution of the system after performing all the steps involved in the procedure. The direct methods consist of elimination methods and decomposition methods.

Under elimination methods, we consider, Gaussian elimination and Gauss-Jordan elimination methods

Crout's reduction also known as Cholesky's reduction is considered under decomposition methods.

Under iterative methods, the initial approximate solution is assumed to be known and is improved towards the exact solution in an iterative way. We consider Jacobi, Gauss-Seidel and relaxation methods under iterative methods.

Gaussian Elimination Method

In this method, the solution to the system of equations is obtained in two stages.

i) the given system of equations is reduced to an equivalent upper triangular form using elementary transformations
ii) the upper triangular system is solved using back substitution procedure

This method is explained by considering a system of n equations in n unknowns in the form as follows

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\ \ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\}$$

Stage I: Substituting the value of $x1$ from first equation into the rest

$$\left.\begin{aligned} x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n &= b'_1 \\ a'_{22}x_2 + \cdots + a'_{2n}x_n &= b'_2 \\ &\ \ \vdots \\ a'_{n2}x_2 + \cdots + a'_{nn}x_n &= b'_n \end{aligned}\right\}$$

Now, the last ($n-1$) equations are independent of $x1$, that is, $x1$ is eliminated from the last ($n-1$) equations.
This procedure is repeated and $x2$ is eliminated from 3rd, 4th, …, n-th equations
The same procedure is repeated till the given system assumes the upper triangular form:

$$\left.\begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \cdots + c_{2n}x_n &= d_2 \\ &\vdots \\ c_{nn}x_n &= d_n \end{aligned}\right\}$$

Stage II: The values of the unknowns are determined by backward substitution.
First xn is found from the last equation and then substitution this value of xn in the preceding equation will give the value of xn-1. Continuing this way, we can find the values of all other unknowns

Example

Solve the following system of equations using Gaussian elimination method

$$\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 4y - 3z &= 3 \\ -2x + 3y - z &= 1 \end{aligned}$$

Solution

Stage I *(Reduction to upper-triangular form):*
Divide first equation by 2 and then subtract the resulting equation (multiplied by 4 and –2) from the 2nd and 3rd equations respectively. Thus, we eliminate x from the 2nd and 3rd equations.
The resulting new system is given by

$$\left.\begin{aligned} x + \frac{3}{2}y - \frac{z}{2} &= \frac{5}{2} \\ -2y - z &= -7 \\ 6y - 2z &= 6 \end{aligned}\right\}$$

Now, we divide the 2nd equation by –2 and eliminate y from the last equation and the modified system is given by

$$\left.\begin{aligned} x+\frac{3}{2}y-\frac{z}{2}&=\frac{5}{2} \\ y+\frac{z}{2}&=\frac{7}{2} \\ -5z&=-15 \end{aligned}\right\}$$

Stage II (Backward substitution):
From the last equation, we get

$z = 3$

Using this value of z, the second equation gives

$$y=\frac{7}{2}-\frac{3}{2}=2$$

Putting these values of y and z in the first equation, we get

$$x=\frac{5}{2}+\frac{3}{2}-3=1$$

Thus, the solution of the given system is given by
x = 1, y = 2, z = 3

Partial and Full Pivoting

The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewrite the equations in a different order to avoid zero pivots. Changing the order of equations is called pivoting.

Partial pivoting

If the pivot happens to be zero, then the i-*th* column elements are searched for the numerically largest element. Let the *j-th* row ($j > i$) contains this element, then we interchange the *i-th* equation with the *j-th* equation and proceed for elimination. This process is continued whenever pivots become zero during elimination.
For example, let us examine the solution of the following simple system

$$\begin{aligned} 10^{-5}x_1+x_2&=1 \\ x_1+x_2&=2 \end{aligned}$$

Using Gaussian elimination method with and without partial pivoting, assuming that we require the solution accurate to only four decimal places. The solution by Gaussian elimination gives $x1 = 0$, $x2 = 1$.

If we use partial pivoting, the system takes the form

$$x_1 + x_2 = 2$$

$$10^{-5} x_1 + x_2 = 1$$

Using Gaussian elimination method, the solution is found to be $x1 = 1$, $x2 = 1$, which is a meaningful and perfect result.

In full pivoting (or complete pivoting), we interchange rows *and* columns, such that the largest element in the matrix of the variables also get changed. Full pivoting, in fact, is more complicated than the partial pivoting. Partial pivoting is preferred for hand computation.

The general form of a system of m linear equations in n unknowns $x1, x2, x3, \ldots, xn$ can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Using matrix notation, the above system can be written in compact form as

$$[A](X) = (B)$$

Note:

1. This method fails if any of he pivots become zero in such cases, by interchanging the rows we can get the non-zero pivots.

Example

Solve the system of equations by Gaussian elimination method with partial pivoting**.**

$$x + y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \\ 16 \end{pmatrix}$$

To start with, we observe that the pivot element

$a_{11} = 1 (\neq 0)$.

However, a glance at the first column reveals that the numerically largest element is 3 which is in second row. Hence *R12*

Thus the given equation takes the form after partial pivoting

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 \\ 7 \\ 16 \end{pmatrix}$$

Stage I (Reduction to upper triangular form):

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}$$

Stage II (Back substitution):

$z = 3$

$-y + 1 = 0$ or $y = 1$

$x + 1 + 4 = 8$ or $x = 3$

Example

Solve the following system of equations by Gaussian elimination method with partial pivoting

$$\begin{aligned} 0x_1 + 4x_2 + 2x_3 + 8x_4 &= 24 \\ 4x_1 + 10x_2 + 5x_3 + 4x_4 &= 32 \\ 4x_1 + 5x_2 + 6.5x_3 + 2x_4 &= 26 \\ 9x_1 + 4x_2 + 4x_3 + 0x_4 &= 21 \end{aligned}$$

Solution

In matrix notation, the given system can be written as

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 26 \\ 21 \end{bmatrix}$$

To start with, we observe that the pivot row, that is, the first row has a zero pivot element ($a11 = 0$). This row should be interchanged with any row following it, which on becoming a pivot row should not have a zero pivot element.

While interchanging rows it is better to interchange the first and fourth rows, which is called partial pivoting and get,

$$\begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix}$$

Stage I (Reduction to upper-triangular form):

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 8.2222 & 3.2222 & 4 \\ 0 & 3.2222 & 4.7222 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.333 \\ 22.6666 \\ 16.6666 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 3.4594 & 0.4324 \\ 0 & 0 & 0.4324 & 6.0540 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.333 \\ 2.7568 \\ 7.78836 \\ 12.9728 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 1 & 0.1250 \\ 0 & 0 & 0 & 5.999 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 2.7568 \\ 2.2500 \\ 11.9999 \end{bmatrix}$$

Stage II
Back substitution

$$x_1 = 1.0, \qquad x_2 = 1.0, \qquad x_3 = 2.0, \qquad x_4 = 2.0$$

Example

Solve the system of equations $\begin{array}{c} 3x + y - z = 3 \\ 2x - 8y + z = -5 \\ x - 2y + 9z = 8 \end{array}$ using Gauss elimination method.

Solution
The given system is equivalent to

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$$

$$\mathrm{A} \qquad X = B$$

the argumented matrix is

$$[A \mid B] = \begin{bmatrix} 3 & 1 & -1 & & 3 \\ 2 & -8 & 1 & | & -5 \\ 1 & -2 & 9 & & 8 \end{bmatrix}$$

now making A *as an upper triangular matrix*

$$\sim \begin{bmatrix} 3 & 1 & -1 & & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & | & -7 \\ 0 & \frac{-7}{3} & \frac{28}{3} & & 7 \end{bmatrix} R_2 - \frac{2}{3}R_1, R_3 - \frac{1}{3}R_1$$

now choosing $\frac{-26}{3}$ *as the pivot from the second column,*

$$\sim \begin{bmatrix} 3 & 1 & -1 & & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & | & -7 \\ 0 & 0 & \frac{693}{78} & & \frac{231}{26} \end{bmatrix} R_2 - \frac{2}{3}R_1, R_3 - \frac{1}{3}R_1$$

from this we get

$$3x + y - z = 3$$

$$\frac{-26}{3}y + \frac{5}{3}z = -7$$

$$\frac{693}{78}z = \frac{231}{26}$$

now by back substitution $z = 1$

$$\frac{-26}{3}y = -7 - \frac{5}{3}(z) = -7 - \frac{5}{3}(1) = \frac{-26}{3} \Rightarrow y = 1$$

$$now\ x = \frac{1}{3}[3 - y + z] = \frac{1}{3}[3 - 1 + 1] = 1$$

so the solution is $x = y = z = 1$

Example

Using Gauss elimination method, solve the system.

$$3.15x - 1.96y + 3.85z = 12.95$$

$$2.13x + 5.12y - 2.89z = -8.61$$

$$5.92x + 3.05y + 2.15z = 6.88$$

Solution

The given system is equivalent to

$$\underbrace{\begin{bmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}}_{\mathrm{B}}$$

$$[A \mid B] = \begin{bmatrix} 3.15 & -1.96 & 3.85 & & 12.95 \\ 2.13 & 5.12 & -2.89 & | & -8.61 \\ 5.92 & 3.05 & 2.15 & & 6.88 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3.15 & -1.96 & 3.85 & & 12.95 \\ 0 & 6.4453 & -5.4933 & | & -17.3667 \\ 0 & 6.7335 & -5.0855 & & -17.4578 \end{bmatrix} \quad R_2 - \frac{2.13}{3.15}R_1, R_3 - \frac{5.92}{3.15}R_1$$

$$choo\sin g \;\; 6.4453 \;\; as \;\; pivot$$

$$\sim \begin{bmatrix} 3.15 & -1.96 & 3.85 & & 12.95 \\ 0 & 6.4453 & -5.4933 & | & -17.3667 \\ 0 & 0 & 0.6534 & & 0.6853 \end{bmatrix} \quad R_3 - \frac{6.7335}{6.4453}R_1$$

$$form\ this\ ,we\ get$$

$$3.15x - 1.96y + 3.85z = 12.95$$

$$6.4453y - 5.4933z = -17.3667$$

$$0.6534z = 0.6853$$

$$by\ backward\ substitution$$

$$z = \frac{0.6853}{0.6534} = 1.0488215$$

$$y = \frac{5.4933 - 17.3667}{6.4453} = -1.8005692$$

$$x = \frac{1.96y - 3.85z + 12.95}{3.15} = 1.708864$$

Example

$$x_1+x_2+x_3+x_4=2$$
$$x_1+x_2+3x_3-2x_4=-6$$
$$2x_1+3x_2-x_3+2x_4=7$$
$$x_1+2x_2+x_3-x_4=-2$$

the given system in matrix form is

$$\underbrace{\begin{bmatrix}1&1&1&1\\1&1&3&-2\\2&3&-1&2\\1&2&1&-1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}_{X}=\underbrace{\begin{bmatrix}2\\-6\\7\\-2\end{bmatrix}}_{B}$$

$$[A|B]=\begin{bmatrix}1&1&1&1&&2\\1&1&3&-2&|&-6\\2&3&-1&2&&7\\1&2&1&-1&&-2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1&1&&2\\0&0&2&-3&|&-8\\0&1&-3&0&&3\\0&1&0&-2&&-4\end{bmatrix}R_2-R_1,R_3-2R_1,R_4-R_1$$

since the element in the second column is zero so interchanging the rows

$$\sim\begin{bmatrix}1&1&1&1&&2\\0&1&-3&0&|&3\\0&0&2&-3&&-8\\0&1&0&-2&&-4\end{bmatrix}R_{23}$$

$$\sim\begin{bmatrix}1&1&1&1&&2\\0&1&-3&0&|&3\\0&0&2&-3&&-8\\0&1&3&-2&&-4\end{bmatrix}R_4-R_2$$

now the pivot is 2, *therefore*

$$\sim\begin{bmatrix}1&1&1&1&&2\\0&1&-3&0&&3\\0&0&2&-3&|&-8\\0&0&0&\frac{5}{2}&&5\end{bmatrix}R_4-\frac{3}{2}R_3$$

from this we get

$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_2 - 3x_3 = 3$$

$$2x_3 - 3x_4 = -8$$

$$(\frac{5}{2})x_4 = 5$$

now $x_4 = 2$

$$x_3 = \frac{1}{2}(-8 + 3x_4) = \frac{1}{2}(-8 + 6) = -1$$

now

$$x_2 = 3 + 3x_3 = 3 - 3 = 0$$

now from equation 1

$$x_1 = 2 - x_2 - x_3 - x_4 = 2 - 0 + 1 - 2 = 1$$

so $x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 2$