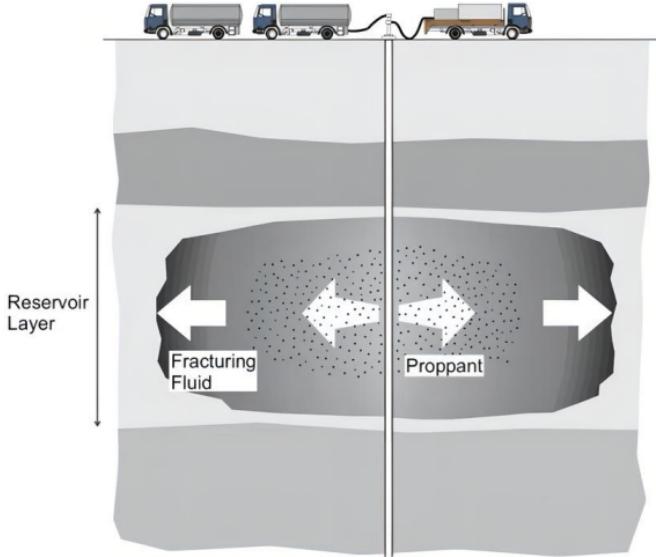


# Hydraulic fracturing (HF)

- ▶ Essential components of HF modeling
- ▶ Governing equations
- ▶ Basic HF model geometries



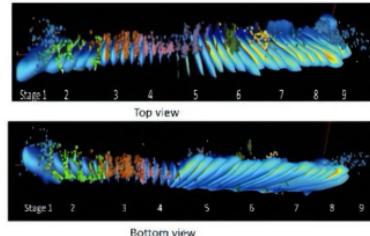
Illustrations by Don Foley

**Crack it:** Drillers bore down to a shale layer that can be 5,000 feet deep or more, then turn and continue horizontally as much as another 5,000 feet. The drill bit is retracted (bottom diagram, left); water, sand and chemicals are pumped down the well to fracture the rock (center), releasing gas that flows back up with the fluid (right). The treated wastewater is held in surface ponds or tanks.

# Essential pieces of a hydraulic fracture model

## 1. Volume balance of the injected fluid (incompressible):

$$\text{Volume injected} = \text{Fracture volume} + \text{leak-off}$$

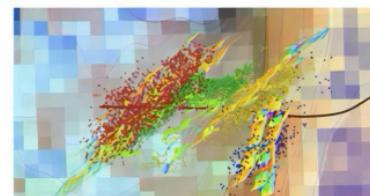


## 2. Fluid flow equations:

Viscous pressure drop inside the fracture

## 3. Rock equilibrium (elasticity):

$$\text{Fluid pressure} = \text{Stress} + \text{Stiffness} * \text{FracWidth}$$

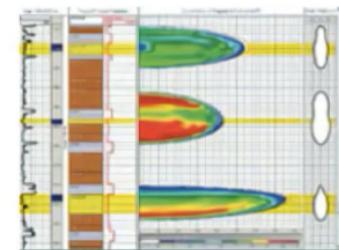


## 4. Propagation condition:

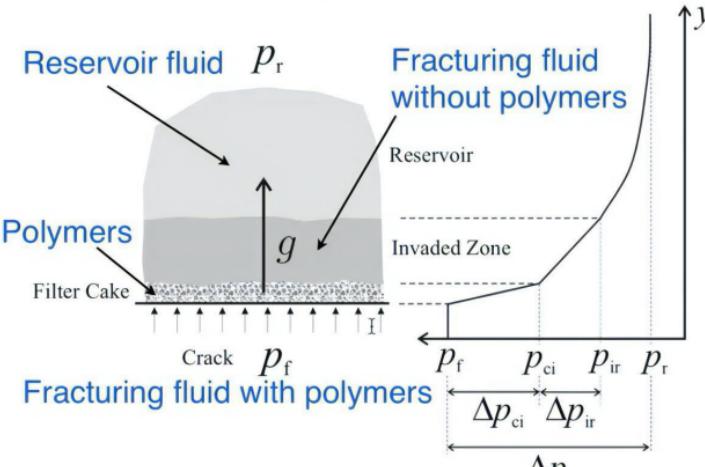
Some parameter reaches a critical value near the front

## 5. Proppant transport (not covered):

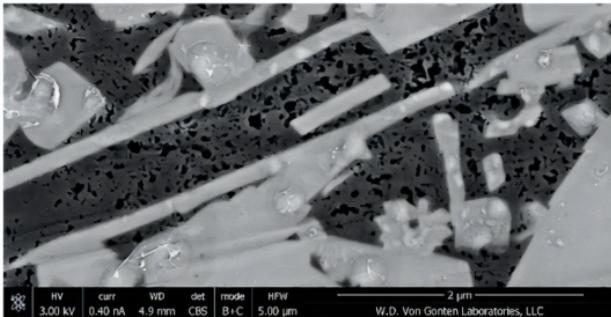
Particles flow with fluid + gravitational settling



# Carter's leak-off model



J. Adachi, PhD thesis



## Flow through filter cake:

$$g_c = \alpha \frac{dh_c}{dt}$$

$$g_c = \frac{\kappa_c}{\mu} \frac{\Delta p_{ci}}{h_c}$$

Solution:

$$g_c = \frac{C_c}{\sqrt{t}} \quad C_c = \sqrt{\alpha \frac{\kappa_c}{\mu} \frac{\Delta p_{ci}}{h_c}}$$

## Flow through invaded zone:

$$g_i = \frac{\kappa}{\mu_{filt}} \frac{\Delta p_{ir}}{h_i}$$

$$g_i = \phi \frac{dh_i}{dt}$$

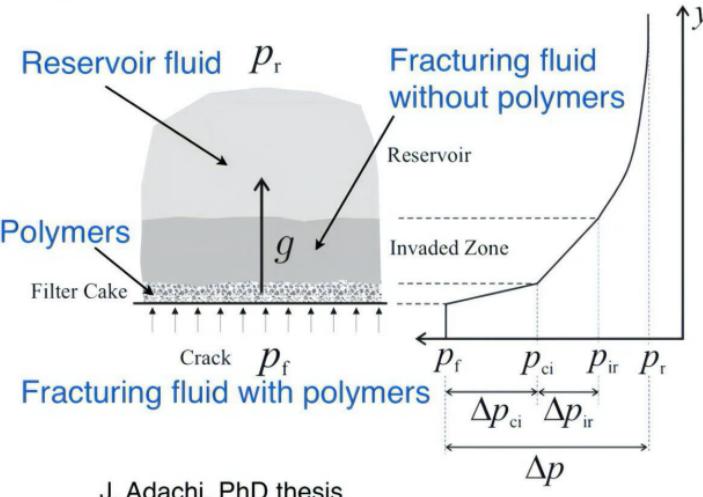
this is Darcy's law (quasi-static flow)

this is volume balance that states that the volume of fluid leaked into the formation determines the size of the invasion zone

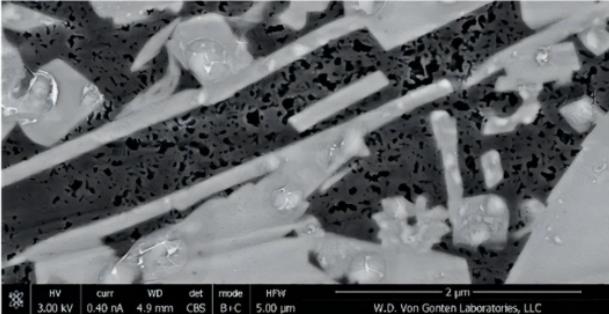
Solution:

$$g_i = \frac{C_i}{\sqrt{t}} \quad C_i = \sqrt{\phi \frac{\kappa}{\mu_{filt}} \frac{\Delta p_{ir}}{h_i}}$$

# Carter's leak-off model



J. Adachi, PhD thesis



**Flow in reservoir:**

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial y^2} \quad \text{1D diffusion equation (volume balance + Darcy)}$$

$$p|_{t=0} = p_r \quad \text{initial condition}$$

$$p|_{y=0} = p_{ir} \quad \text{boundary condition}$$

To solve this equation, introduce new variable:

$$\xi = \frac{y}{\sqrt{4Dt}} \implies -\frac{y}{2t\sqrt{4Dt}} p' = \frac{D}{4Dt} p'' \implies -2\xi p' = p''$$

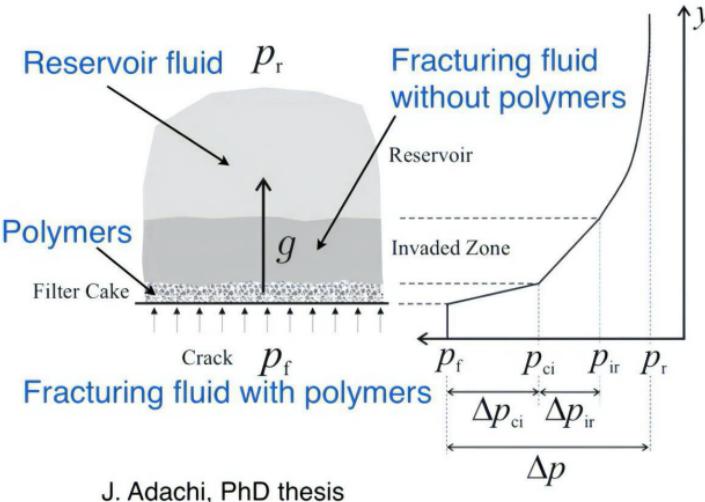
Solution of the above differential equation is:

$$p = p_r + (p_{ir} - p_r) \operatorname{erfc}(\xi^2) \quad \operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

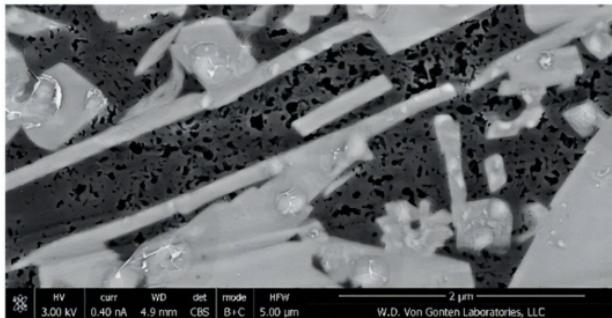
The leak-off flux is then (from Darcy):

$$g_r = -\frac{k_r}{\mu_r} \frac{\partial p}{\partial y} \Big|_{y=0} = \frac{C_r}{\sqrt{t}} \quad C_r = \sqrt{\frac{k_r c_t \phi}{\pi \mu_r}} (p_{ir} - p_r)$$

## Carter's leak-off model



J. Adachi, PhD thesis



**Combined result if all the mechanisms are present:**

$$\Delta p = \Delta p_{ci} + \Delta p_{ir} + (p_{ir} - p_r)$$

$$C_l = \frac{2C_c C_i C_r}{C_c C_i + \sqrt{C_c^2 C_i^2 + 4C_r^2(C_c^2 + C_i^2)}}$$

In the above result, the individual leak-off coefficients are computed by using the total pressure drop, i.e. the reservoir part is given by

$$C_r = \sqrt{\frac{k_r c_t \phi}{\pi \mu_r}} \Delta p \quad I$$

**Recall the main assumptions of the model:**

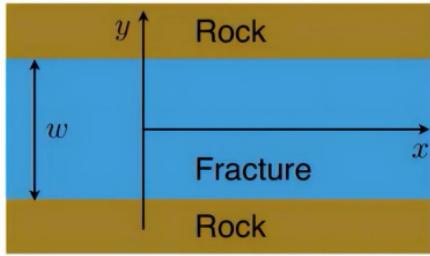
1D diffusion, i.e. the diffusion length scale should be much smaller than the fracture size

The net pressure is often neglected, whereby  $\Delta p = \sigma_0 - p_r$

It is implicitly assumed that one type of fracturing fluid is used

More reading: Economides & Nolte 2000, section 6-4.

# Fluid flow



$$v = v_x(y)$$

given the geometry, we have only one component of the velocity vector that varies only across the channel

$$\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y}$$

this comes from Navier-Stokes equations or equilibrium equations

$$\tau = \mu \frac{\partial v}{\partial y}$$

this states that the rheology is Newtonian

Herschel-Bulkley  $\tau = \tau_0 + k\dot{\gamma}^n$

$$v|_{y=\pm w/2} = 0 \quad \text{this is no-slip boundary condition at the fracture walls}$$

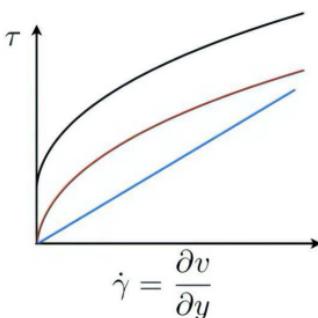
Power-law  $\tau = k\dot{\gamma}^n$

General solution:

Newtonian  $\tau = \mu \dot{\gamma}$

$$v = \frac{\partial p}{\partial x} \frac{y^2}{2} + Ay + B$$

Actual solution:

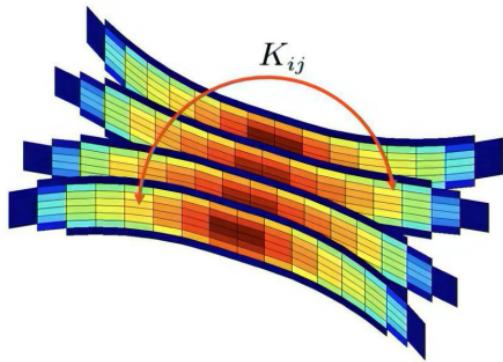


$$\dot{\gamma} = \frac{\partial v}{\partial y}$$

Total flux is:

$$q = \int_{-w/2}^{w/2} v(y) dy = -\frac{w^3}{12\mu} \frac{\partial p}{\partial x}$$

# Elasticity



Elasticity equation ensures that rock surrounding open fracture(s) is in equilibrium

Every open element induces a stress change (all components) in the whole space

The interaction coefficient (induced stress divided by aperture) depends on the elastic properties and the distance from the element and generally decays quickly  $\sim 1/r^3$  for 3D geometry

For a plane strain fracture, the elasticity equation reads:

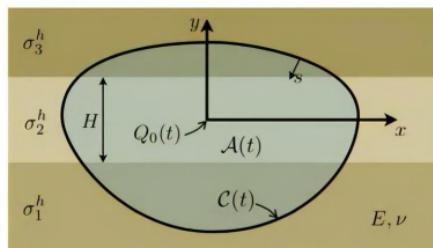
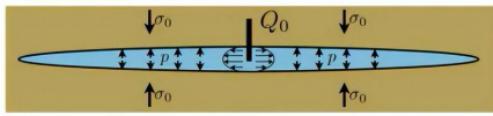
$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^l \frac{w(s) ds}{(x-s)^2} \quad \text{and} \quad E' = \frac{E}{1-\nu^2}$$

For a planar fracture, the elasticity equation reads:

$$p(x, y, t) = \sigma^h(y) - \frac{E'}{8\pi} \int_{\mathcal{A}(t)} \frac{w(x', y', t) dx' dy'}{[(x' - x)^2 + (y' - y)^2]^{3/2}},$$

For general expressions in 3D, see Crouch and Starfield, 1983

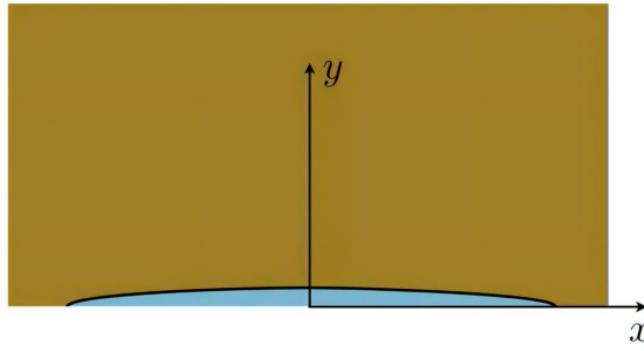
For expressions in layered materials, see Peirce and Siebrits, 2000



## **Lecture 2: Essential pieces of hydraulic fracturing: part 2**

Egor Dontsov

# Derivation of elasticity equation (plane strain)



Governing equations in terms of displacements

$$(2\mu + \lambda) \frac{\partial^2 u_x}{\partial x^2} + (\lambda + \mu) \frac{\partial^2 u_y}{\partial x \partial y} + \mu \frac{\partial^2 u_x}{\partial y^2} = 0,$$

$$(2\mu + \lambda) \frac{\partial^2 u_y}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u_x}{\partial x \partial y} + \mu \frac{\partial^2 u_y}{\partial x^2} = 0$$

Hooke's law

$$\begin{aligned}\sigma_{xx} &= 2\mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy}), \\ \sigma_{yy} &= 2\mu\epsilon_{yy} + \lambda(\epsilon_{xx} + \epsilon_{yy}), \\ \sigma_{xy} &= 2\mu\epsilon_{xy}\end{aligned}$$

Boundary conditions

$$u_y|_{y=0} = \frac{w}{2} \quad \sigma_{xy}|_{y=0} = 0$$

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Equilibrium equations

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} &= 0\end{aligned}$$

Need to solve for

$$\sigma_{yy}|_{y=0} - ?$$

System of ODEs

$$\frac{\partial \hat{u}_x}{\partial y} = \hat{d}_x,$$

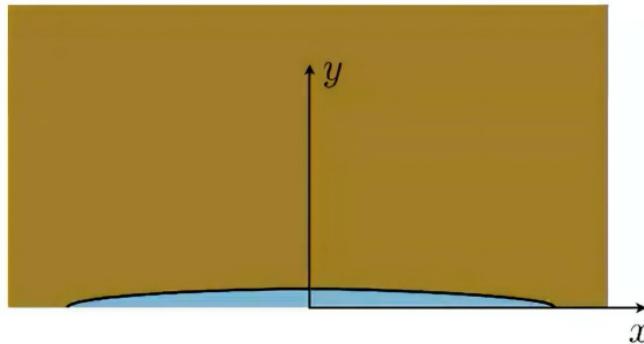
$$\frac{\partial \hat{u}_y}{\partial y} = \hat{d}_y,$$

$$\frac{\partial \hat{d}_x}{\partial y} = \frac{2\mu + \lambda}{\mu} k^2 \hat{u}_x + \frac{\lambda + \mu}{\mu} i k d_x + \frac{\mu}{2\mu + \lambda} k^2 \hat{u}_y,$$

$$\frac{\partial \hat{d}_y}{\partial y} = \frac{\mu + \lambda}{2\mu + \lambda} i k \hat{d}_x + \frac{\mu}{2\mu + \lambda} k^2 \hat{d}_x + \frac{\mu}{2\mu + \lambda} k^2 \hat{u}_y,$$

$$\boxed{\mathbf{Y}' = \mathbf{A}\mathbf{Y}}$$

## Derivation of elasticity equation (plane strain)



Boundary conditions

$$\hat{\sigma}_{xy}|_{y=0} = 0, \\ \hat{u}_y|_{y=0} = \hat{w}(k)/2,$$

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y}$$

Eigenvalues of  $A$ :  $k, k, -k, -k$ .

Solution (resonance)

$$\mathbf{Y} = c_1 \mathbf{v}_1 e^{-|k|y} + c_2 (\mathbf{v}_1 y + \mathbf{v}_2) e^{-|k|y}$$

Solution in frequency domain

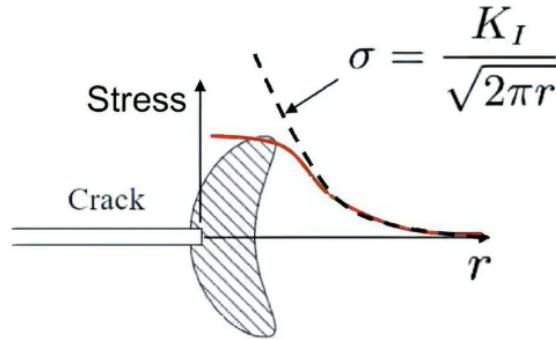
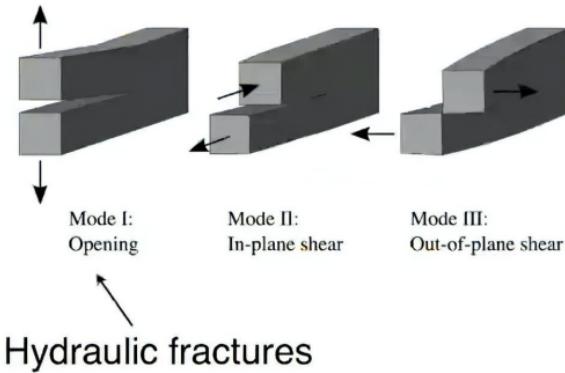
$$\hat{p} = \frac{\hat{w}|k|}{4} E' = -\frac{ik\hat{w}E'}{4} \frac{|k|}{(-ik)} = -\frac{1}{i} \text{sgn}k \frac{d\hat{w}}{dx}.$$

[https://en.wikipedia.org/wiki/Fourier\\_transform](https://en.wikipedia.org/wiki/Fourier_transform)

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$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^l \frac{w(s) ds}{(x-s)^2}$$

## Propagation condition



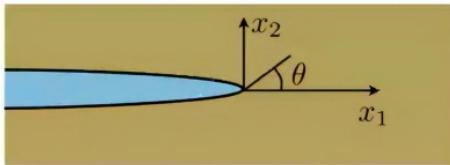
$K_I$  - Stress Intensity Factor (SIF)

Propagation condition:  $K_I = K_{Ic}$

$K_{Ic}$  - fracture toughness



# Mode I solution near the tip



$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \left[ \cos\left(\frac{1}{2}\theta\right) \left\{ 1 - \sin\left(\frac{1}{2}\theta\right) \sin\left(\frac{3}{2}\theta\right) \right\} \right]$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \left[ \cos\left(\frac{1}{2}\theta\right) \left\{ 1 + \sin\left(\frac{1}{2}\theta\right) \sin\left(\frac{3}{2}\theta\right) \right\} \right]$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \left[ \cos\left(\frac{1}{2}\theta\right) \sin\left(\frac{1}{2}\theta\right) \cos\left(\frac{3}{2}\theta\right) \right]$$

$$u_1 = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \left[ \cos\left(\frac{1}{2}\theta\right) \left\{ \kappa - 1 + 2 \sin^2\left(\frac{1}{2}\theta\right) \right\} \right]$$

$$u_2 = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \left[ \sin\left(\frac{1}{2}\theta\right) \left\{ \kappa + 1 - 2 \cos^2\left(\frac{1}{2}\theta\right) \right\} \right]$$

$$\kappa = 3 - 4\nu \quad \mu = \frac{E}{2(1+\nu)}$$

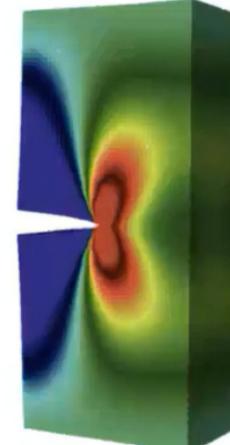
## Solution methodology:

- Write elasticity equations via Airy stress function
- Solve the equations assuming stress-free crack and finite displacement at the tip
- See lecture notes on fracture mechanics for more info: <http://www.mate.tue.nl/~piet/edu/frm/pdf/frmsyl1213.pdf>

Fracture width around  
the crack tip:

$$w = \sqrt{\frac{32}{\pi}} \frac{K_I(1-\nu^2)}{E} \sqrt{r}$$

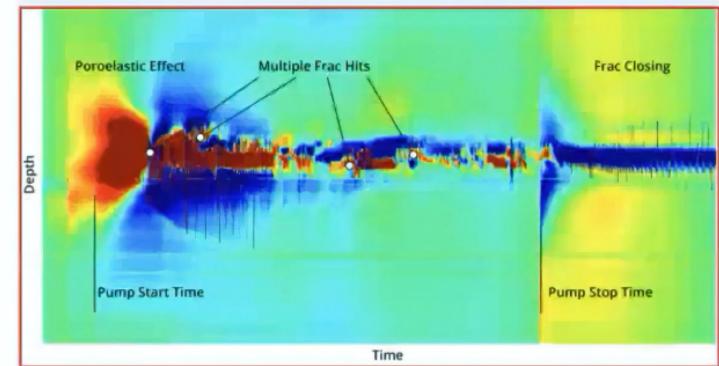
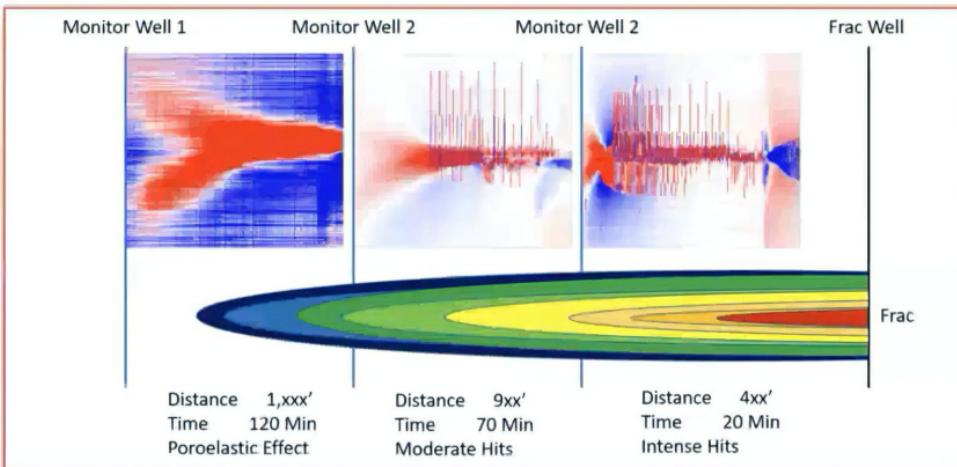
Stress field around the crack tip:



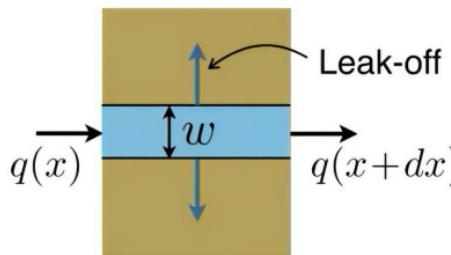
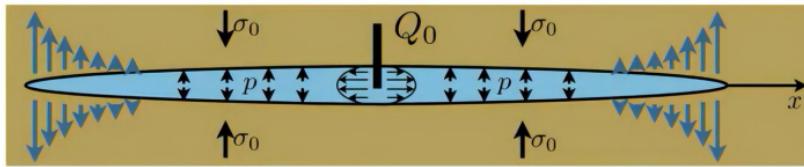
# Observation of crack tip stress in the field



- Fiber optics cables are used to measure stretch versus time along the cable length
- A cable is often placed in the neighboring horizontal well, while the primary well is being fractured
- The characteristic “ears” of the approaching crack are clearly visible



## Volume balance for a plane strain HF

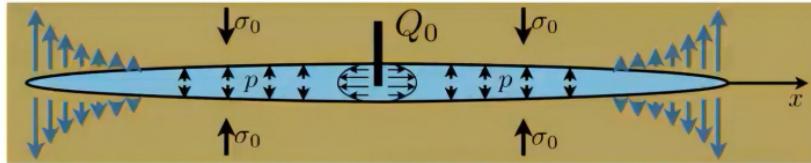


	New volume	Flux in	Leak-off
	$w(t + dt)dx = w(t)dx + q(x) - q(x + dx) - g_l dt dx + Q_0 dt \int \delta(x) dx$		
	Previous volume	Flux out	Source

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Fracture width      I  
 $\frac{\partial w}{\partial t}$       Leak-off       $\frac{C'}{\sqrt{t - t_0(x)}}$       Source

# Mathematical model for a plane strain HF



Volume balance of fluid

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Fracture width  
 Leak-off  
 Source

Laminar fluid flow flux

$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$

Scaled quantities

$$C' = 2C_L \quad \mu' = 12\mu \quad E' = \frac{E}{1-\nu^2} \quad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

Elasticity

$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^l \frac{w(s) ds}{(x-s)^2}$$

Fracture length  
 Fluid pressure

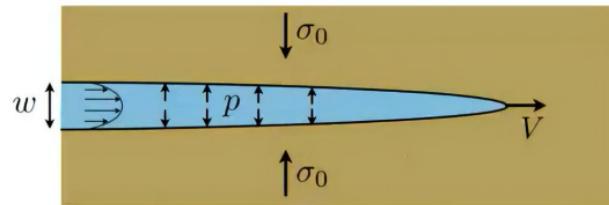
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Propagation condition (LEFM)

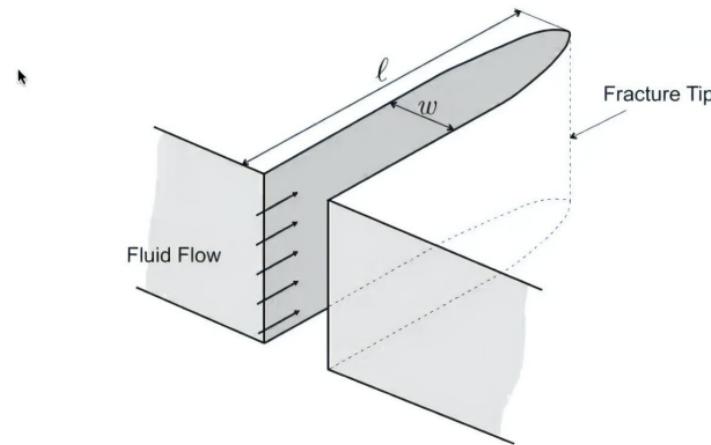
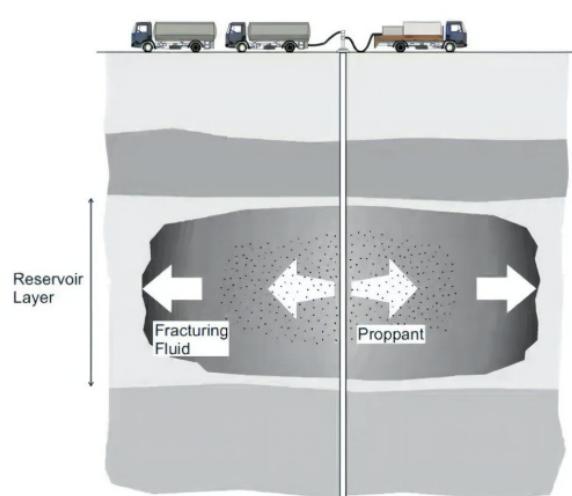
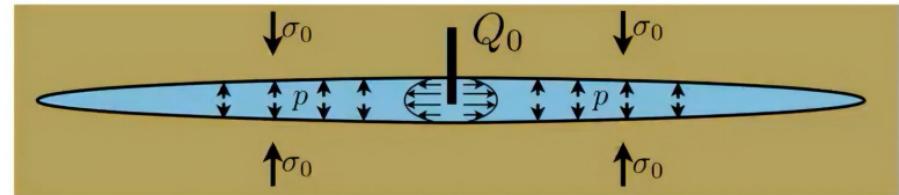
$$w \rightarrow \frac{K'}{E'} \sqrt{l-x} \quad (K_I = K_{Ic})$$

## HF geometries - the simplest

Semi-infinite (tip region)



Khristianovich–Zheltov–Geertsma–De Klerk (KGD)

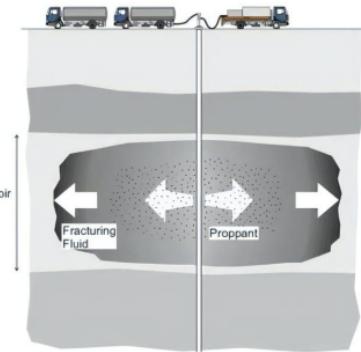
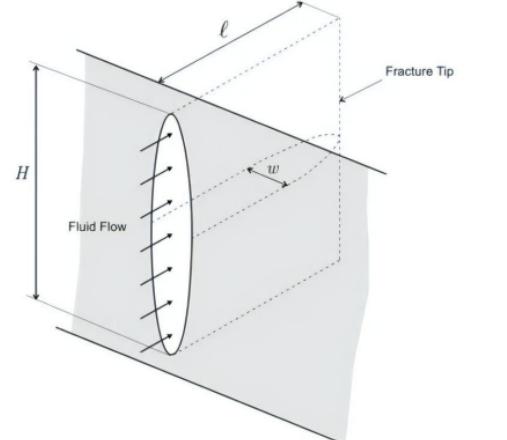
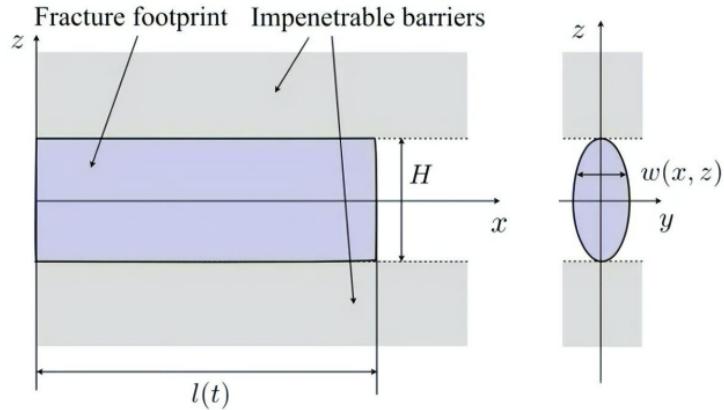


Khristianovic SA, Zheltov YP. 1955 Formation of vertical fractures by means of highly viscous fluids. In Proc. 4th World Petroleum Congress, Rome, Italy, 6–16 June, vol. 2, pp. 579–586.

# HF geometries

## Perkins–Kern–Nordgren (PKN)

T.K. Perkins, L.R. Kern, Widths of hydraulic fractures, J. Pet. Tech. Trans. AIME (1961) 937–949.  
R.P. Nordgren, Propagation of vertical hydraulic fractures, Soc. Petrol. Eng. J. (1972) 306–314.

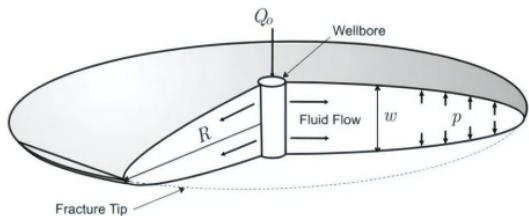


In conventional field applications, solution evolves from KGD geometry at early times to PKN geometry for late times

KGD (early time) -> PKN (developed fracture)

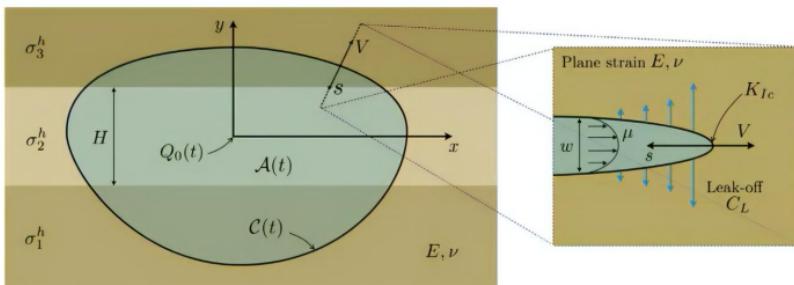
# HF geometries

## Radial

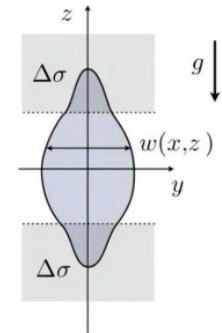
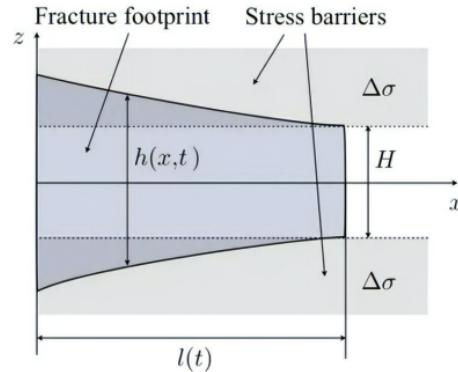


## Planar-3D

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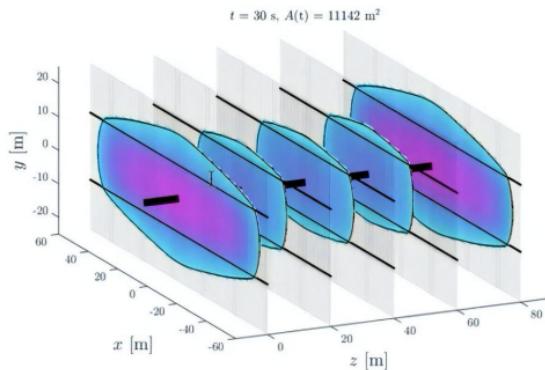
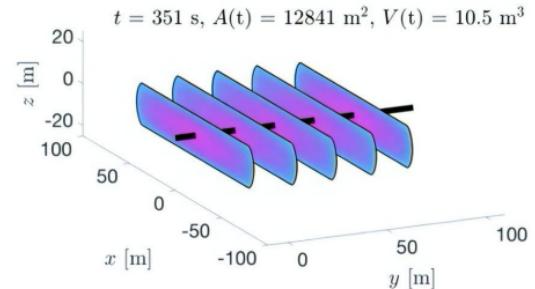
## Pseudo-3D



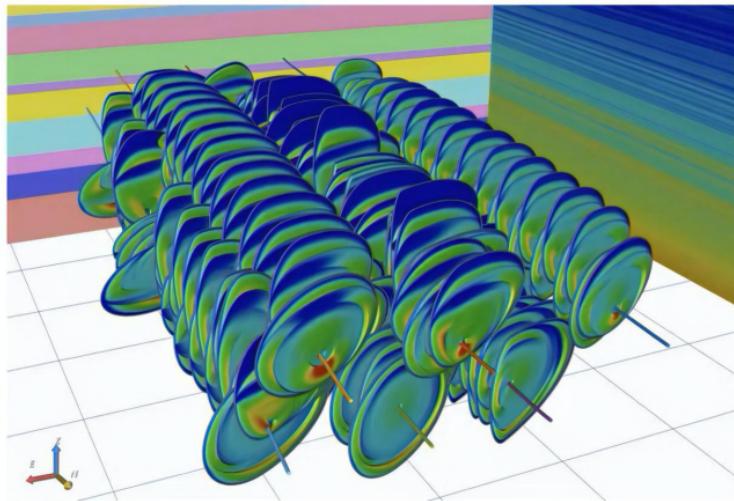
Settari A, Cleary M. Development and testing of a pseudo-three-dimensional model of hydraulic fracture geometry (P3DH). SPE 10505; 1982. p. 185–214.

# HF geometries

Multi-fracture

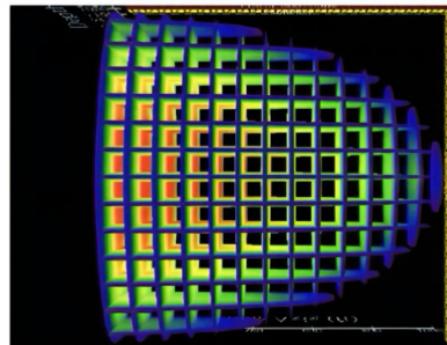
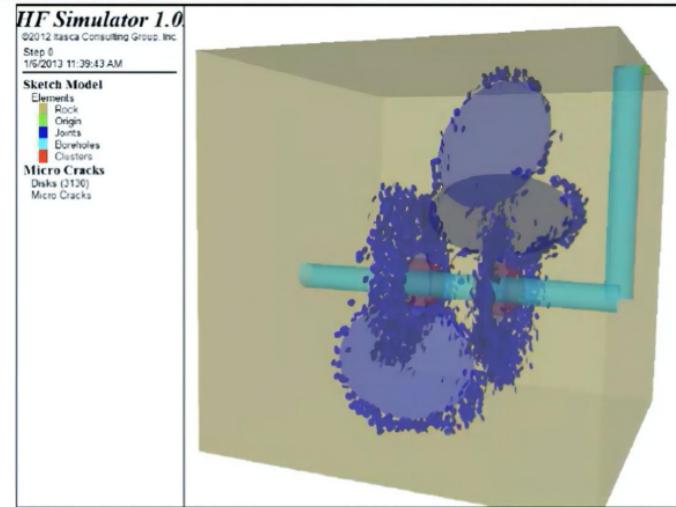
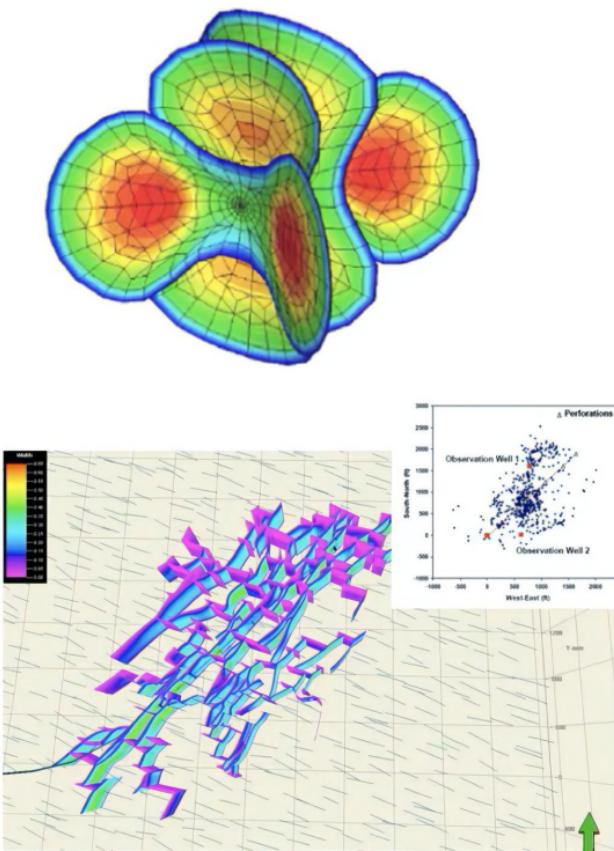


Multi-well



Natural fractures, curved fractures, etc.

## HF geometries - other complex



## **Lecture 3: Semi-infinite hydraulic fracture**

Egor Dontsov

## Recall from lecture 1

- Essential pieces of HF model

- Volume balance and leak-off
- Fluid flow
- Elasticity
- Propagation condition
- Proppant transport

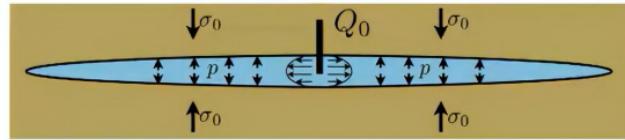
- Various fracture geometries

- Semi-infinite
- KGD (plane strain)
- PKN
- Radial
- Pseudo-3D
- Planar 3D
- Complex

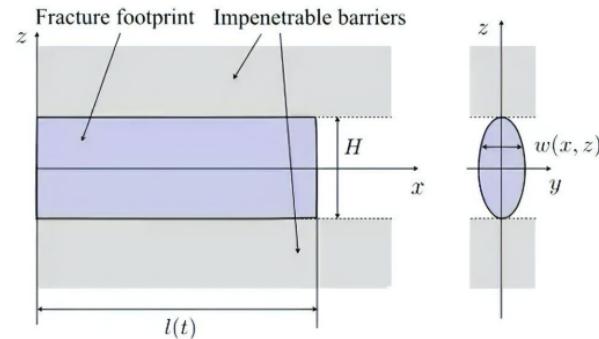
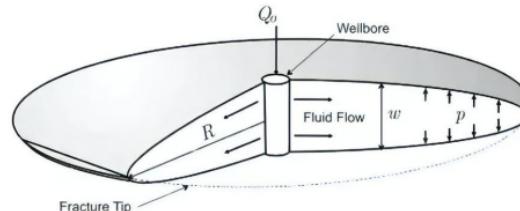
- Governing equations

- KGD (plane strain)

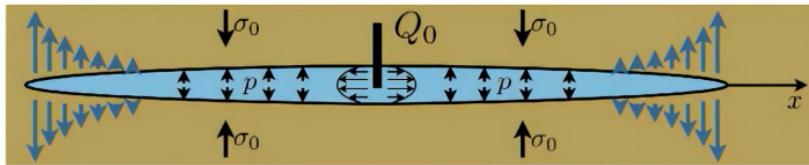
- Derivation of elasticity equation



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# Mathematical model for plane strain HF



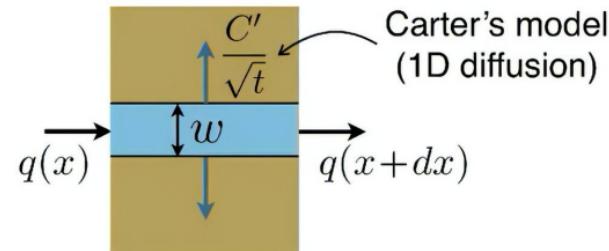
Scaled quantities

$$C' = 2C_L \quad \mu' = 12\mu$$

Volume balance of fluid

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Fracture width  
Leak-off  
Source



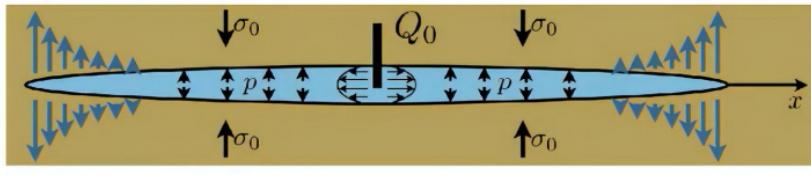
Laminar fluid flow flux

$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$

$$v = -(\frac{1}{4}w^2 - y^2) \frac{1}{2\mu} \frac{\partial p}{\partial x}$$

$$q = \int_{-w/2}^{w/2} v \, dy = -\frac{w^3}{12\mu} \frac{\partial p}{\partial x}$$

# Mathematical model for plane strain HF



Scaled quantities

$$C' = 2C_L \quad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2} \quad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

Volume balance of fluid

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Fracture width  
 Leak-off      Source

I      Elasticity      Fracture length  
 $p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^l \frac{w(s) ds}{(x-s)^2}$   
 Fluid pressure

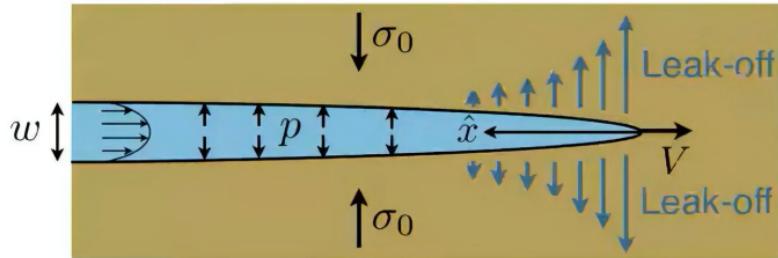
Laminar fluid flow flux

$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$

Propagation condition (LEFM)

$$w \rightarrow \frac{K'}{E'} \sqrt{l-x} \quad (K_I = K_{Ic})$$

## Tip asymptotics: semi-infinite hydraulic fracture



Material parameters

$$C' = 2C_L \quad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2} \quad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

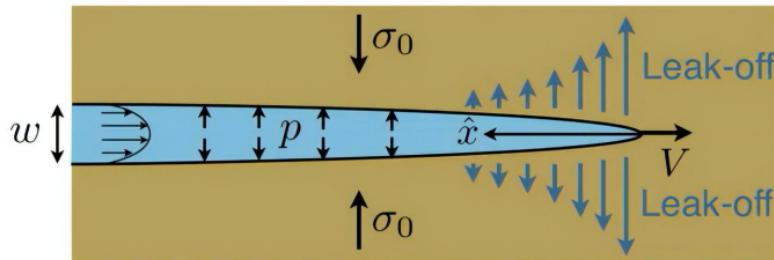
Fluid volume balance

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = 0 \quad \xrightarrow{w(Vt-x)} \quad \frac{q}{w} = V + 2C'V^{1/2}\frac{\hat{x}^{1/2}}{w}$$

Elasticity

$$p = \sigma(\hat{x}) + \frac{E'}{4\pi} \int_0^\infty \frac{dw(\hat{s})}{d\hat{s}} \frac{d\hat{s}}{\hat{x} - \hat{s}}$$

# Tip asymptotics: semi-infinite hydraulic fracture



Material parameters

$$C' = 2C_L \quad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2} \quad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

Fluid volume balance

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = 0$$

Traveling wave

$$w(Vt-x)$$

$$\frac{q}{w} = V + 2C'V^{1/2}\frac{\hat{x}^{1/2}}{w}$$

Elasticity

$$w = \frac{K'}{E'}\hat{x}^{1/2} + \frac{4}{\pi E'} \int_0^\infty K(\hat{x}, \hat{s})(p(\hat{s}) - \sigma(\hat{s})) d\hat{s}$$

$$K(\hat{x}, \hat{s}) = \ln \left| \frac{\hat{x}^{1/2} + \hat{s}^{1/2}}{\hat{x}^{1/2} - \hat{s}^{1/2}} \right| - 2 \frac{\hat{x}^{1/2}}{\hat{s}^{1/2}}$$

Fluid flow

$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x} \longrightarrow q = \frac{w^3}{\mu'} \frac{dp}{d\hat{x}}$$

LEFM propagation condition

$$w = \frac{K'}{E'}\hat{x}^{1/2}, \quad \hat{x} \rightarrow 0$$

## Non-singular formulation

### 1. Integrate elasticity equation by parts

$$w = \frac{K'}{E'} x^{1/2} - \frac{4}{\pi E'} \int_0^\infty F(x, s) \frac{dp}{ds} ds \quad F(x, s) = (s - x) \ln \left| \frac{x^{1/2} + s^{1/2}}{x^{1/2} - s^{1/2}} \right| - 2x^{1/2}s^{1/2}$$

### 2. Substitute pressure gradient into the result

$$w(x) = \frac{K'}{E'} x^{1/2} - \frac{4}{\pi E'} \int_0^\infty F(x, s) \frac{\mu'}{w(s)^2} \left[ V + 2C'V^{1/2} \frac{s^{1/2}}{w(s)} \right] ds$$

### 3. Apply scaling

$$\tilde{w} = \frac{E' w}{K' x^{1/2}}, \quad \chi = \frac{2C' E'}{V^{1/2} K'}, \quad \tilde{x} = (x/l)^{1/2}, \quad \tilde{s} = (s/l)^{1/2}, \quad l = \left( \frac{K'^3}{\mu' E'^2 V} \right)^2$$

### 4. Final result

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi} \int_0^\infty G(\tilde{s}/\tilde{x}) \left[ \frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3} \right] d\tilde{s}$$

w = “toughness” + “viscosity” + “leak-off”

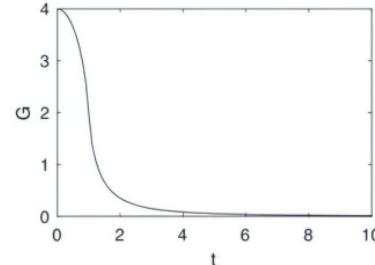
$$G(t) = \frac{1-t^2}{t} \ln \left| \frac{1+t}{1-t} \right| + 2$$

Non-singular

# Limiting vertex solutions

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi} \int_0^\infty G(\tilde{s}/\tilde{x}) \left[ \frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3} \right] d\tilde{s}$$

w = “toughness” + “viscosity” + “leak-off”



Toughness dominates

$$\tilde{w}_k = 1, \quad \rightarrow \quad w_k = \frac{K'}{E'} x^{1/2}$$

Viscosity dominates

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi} \int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2} d\tilde{s} \quad \rightarrow \quad \begin{aligned} \tilde{w}_m &= \beta_m \tilde{x}^{1/3} \\ \beta_m &= 2^{1/3} 3^{5/6} \end{aligned} \quad \rightarrow \quad w_m = \beta_m \left( \frac{\mu' V}{E'} \right)^{1/3} x^{2/3}$$

Desroches et al 1994

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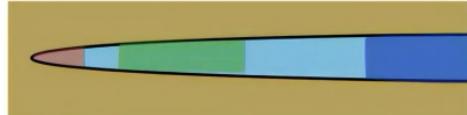
Leak-off dominates

$$\tilde{w}(\tilde{x}) = \frac{8\chi}{\pi} \int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^3} d\tilde{s} \quad \rightarrow \quad \begin{aligned} \tilde{w}_{\tilde{m}} &= \beta_{\tilde{m}} \chi^{1/4} \tilde{x}^{1/4} \\ \beta_{\tilde{m}} &= \frac{4}{15^{1/4} (\sqrt{2}-1)^{1/4}} \end{aligned} \quad \rightarrow \quad w_{\tilde{m}} = \beta_{\tilde{m}} \left( \frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} x^{5/8}$$

Lenoah 1995

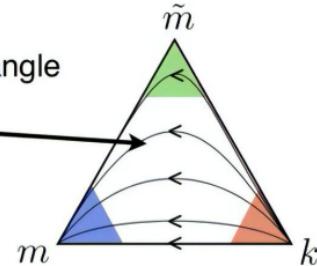
# Order of limiting solutions

$$w \propto x^{1/2} \quad w \propto x^{5/8} \quad w \propto x^{2/3}$$



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Parametric triangle



## Toughness

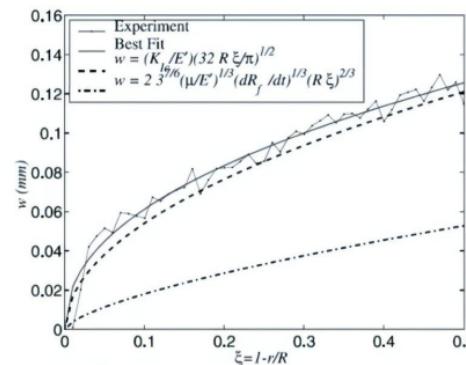
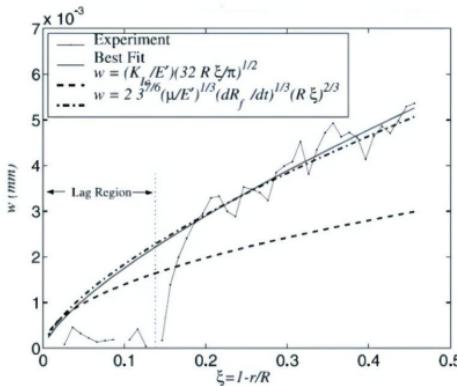
$$w_k = \frac{K'}{E'} x^{1/2},$$

## Leak-off

$$w_{\tilde{m}} = \beta_{\tilde{m}} \left( \frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} x^{5/8},$$

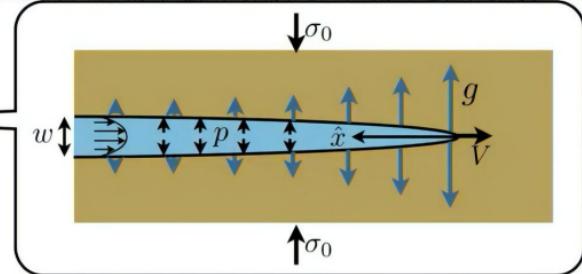
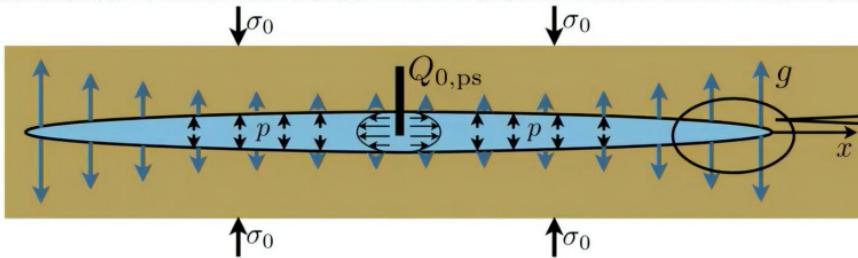
## Viscosity

$$w_m = \beta_m \left( \frac{\mu' V}{E'} \right)^{1/3} x^{2/3}$$



Bunger&Jeffrey

# Governing equations for a semi-infinite hydraulic fracture



Volume balance and flow

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = Q_{0,ps}(t)\delta(x),$$

$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x},$$

Use moving coordinate

$$\hat{x} = Vt - x$$



$$V \frac{dw}{d\hat{x}} - \frac{dq}{d\hat{x}} + \frac{C'}{\sqrt{\hat{x}/V}} = 0$$

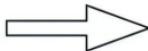
$$\frac{w^2}{\mu'} \frac{dp}{d\hat{x}} = V + 2C' \frac{\sqrt{V\hat{x}}}{w}$$

Substitute flux and integrate

Elasticity and propagation

$$p(x, t) = \sigma_0 - \frac{E'}{4\pi} \int_{-l_1}^{l_2} \frac{w(s)ds}{(x-s)^2},$$

$$\lim_{s \rightarrow 0} \frac{w}{s^{1/2}} = \begin{cases} \frac{K'}{E'}, & \text{if } V > 0, \\ \frac{K'_L}{E'}, & \text{if } V = 0. \end{cases}$$



$$w(\hat{x}) = \frac{K'}{E'} \hat{x}^{1/2} - \frac{4}{\pi E'} \int_0^\infty F(\hat{x}, \hat{s}) \frac{d(p - \sigma_0)}{d\hat{s}} d\hat{s},$$

$$F(\hat{x}, \hat{s}) = (\hat{s} - \hat{x}) \ln \left| \frac{\hat{x}^{1/2} + \hat{s}^{1/2}}{\hat{x}^{1/2} - \hat{s}^{1/2}} \right| - 2\hat{x}^{1/2} \hat{s}^{1/2}.$$

# Non-singular formulation

Substitute pressure gradient

$$w(\hat{x}) = \frac{K'}{E'} \hat{x}^{1/2} - \frac{4}{\pi E'} \int_0^\infty F(\hat{x}, \hat{s}) \frac{d(p - \sigma_0)}{d\hat{s}} d\hat{s},$$

$$\frac{w^2}{\mu'} \frac{dp}{d\hat{x}} = V + 2C' \frac{\sqrt{V\hat{x}}}{w}$$



$$w(\hat{x}) = \frac{K'}{E'} \hat{x}^{1/2} - \frac{4}{\pi E'} \int_0^\infty F(\hat{x}, \hat{s}) \frac{\mu'}{w(\hat{s})^2} \left[ V + 2C' V^{1/2} \frac{\hat{s}^{1/2}}{w(\hat{s})} \right] d\hat{s}.$$



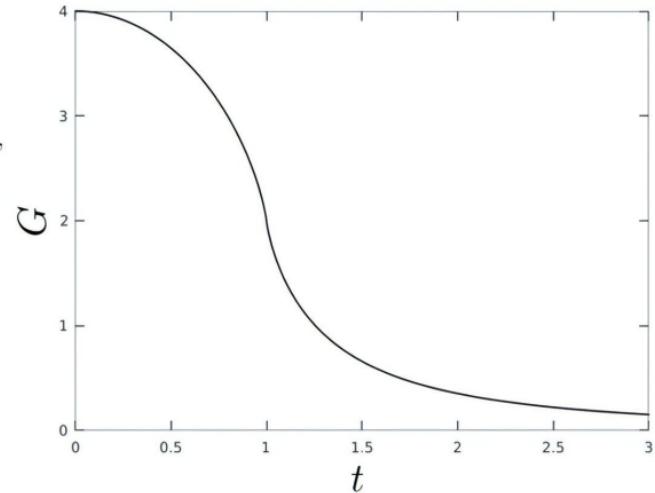
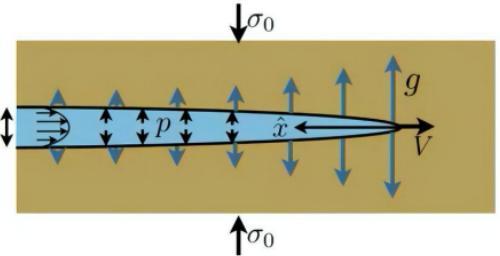
Scaling

$$\begin{aligned} \tilde{w} &= \frac{E' w}{K' \hat{x}^{1/2}}, & \chi &= \frac{2C' E'}{V^{1/2} K'}, & \tilde{x} &= (\hat{x}/l)^{1/2}, \\ \tilde{s} &= (\hat{s}/l)^{1/2}, & l &= \left( \frac{K'^3}{\mu' E'^2 V} \right)^2, \end{aligned}$$

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi} \int_0^\infty G(\tilde{s}/\tilde{x}) \left[ \frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3} \right] d\tilde{s}, \quad \text{Non-singular formulation}$$

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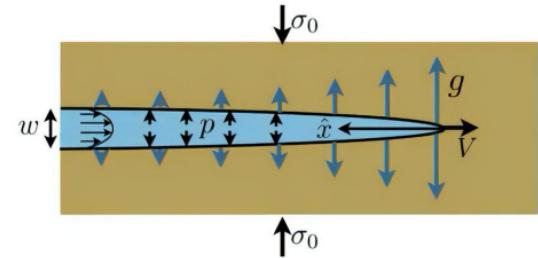
$$G(t) = \frac{1-t^2}{t} \ln \left| \frac{1+t}{1-t} \right| + 2. \quad \text{Non-singular kernel}$$



## Non-singular formulation

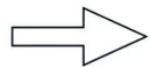
$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi} \int_0^\infty G(\tilde{s}/\tilde{x}) \left[ \frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3} \right] d\tilde{s},$$

w = “toughness” + “viscosity” + “leak-off”



Toughness dominates

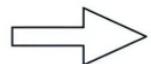
$$\tilde{w}_k = 1,$$



$$w_k = \frac{K'}{E'} \hat{x}^{1/2},$$

Viscosity dominates

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi} \int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2} d\tilde{s}$$



$$\begin{aligned}\tilde{w}_m &= \beta_m \tilde{x}^{1/3} \\ \beta_m &= 2^{1/3} 3^{5/6}\end{aligned}$$



$$w_m = \beta_m \left( \frac{\mu' V}{E'} \right)^{1/3} \hat{x}^{2/3}$$

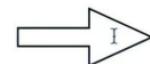
Desroches et al 1994

Leak-off dominates

$$\tilde{w}(\tilde{x}) = \frac{8\chi}{\pi} \int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^3} d\tilde{s}$$



$$\begin{aligned}\tilde{w}_{\tilde{m}} &= \beta_{\tilde{m}} \chi^{1/4} \tilde{x}^{1/4} \\ \beta_{\tilde{m}} &= \frac{4}{15^{1/4} (\sqrt{2}-1)^{1/4}}\end{aligned}$$



$$w_{\tilde{m}} = \beta_{\tilde{m}} \left( \frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} \hat{x}^{5/8},$$

Lenoah 1995

# Derivation of the viscosity solution

Governing integral equation

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi} \int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2} d\tilde{s}$$

Form of the solution

$$\tilde{w} = \beta_m \tilde{x}^{\alpha_m}$$



$$\beta_m \tilde{x}^{\alpha_m} = \frac{8\tilde{x}^{2\alpha_m-1}}{\pi\beta_m^2} \int_0^\infty \frac{G(t)}{t^{2\alpha_m}} dt$$

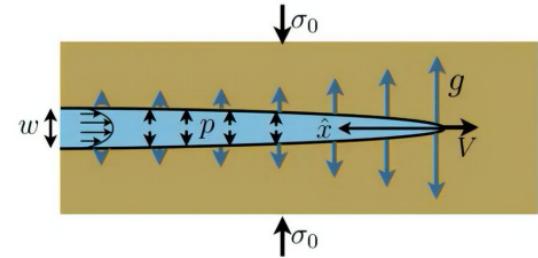


$$\int_0^\infty \frac{G(t)}{t^\alpha} dt = \frac{2\pi}{\alpha(2-\alpha)} \tan\left(\frac{\pi}{2}\alpha\right) \quad (\text{see derivation in the notes})$$

$$\alpha_m = 1/3 \quad \beta_m = 2^{1/3} 3^{5/6},$$

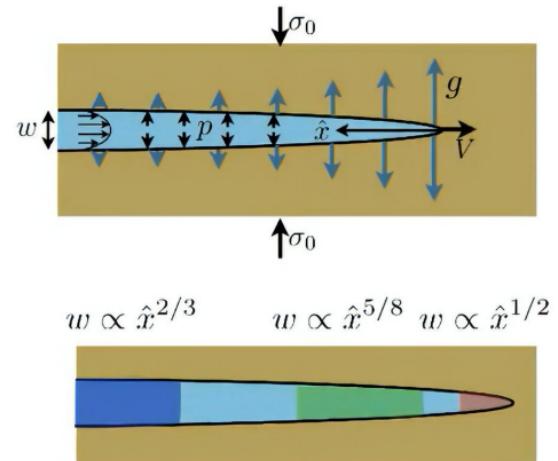
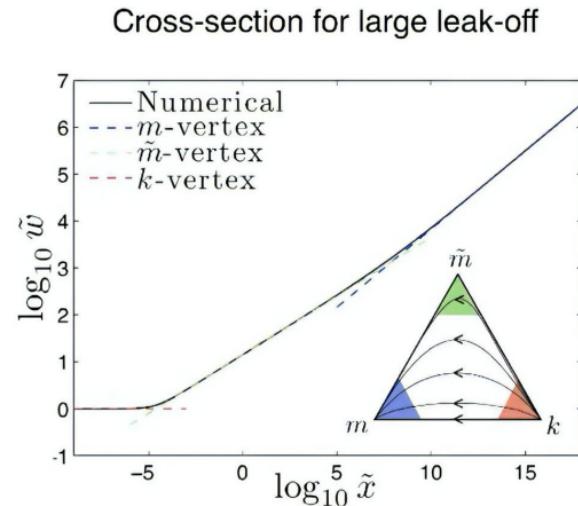
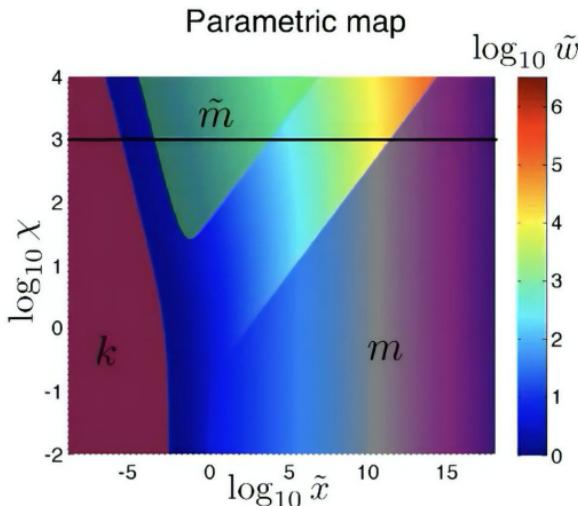


$$\tilde{w}_m = \beta_m \tilde{x}^{1/3}$$



Leak-off dominated solution can be derived in a similar way.

## Structure of the solution



Vertex solutions:

$$\tilde{w}_k = 1, \quad \tilde{w}_{\tilde{m}} = \beta_{\tilde{m}} \chi^{1/4} \tilde{x}^{1/4}, \quad \tilde{w}_m = \beta_m \tilde{x}^{1/3},$$

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Solution transitions gradually from one limiting solution to another starting from toughness, then passing through leak-off (for large leak-off), and then reaching viscosity vertex.

## Approximate solution

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi} \int_0^\infty G(\tilde{s}/\tilde{x}) \left[ \frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3} \right] d\tilde{s},$$

Numerical solution of this equation is time consuming and can be a limiting factor for some applications, such as using it as a propagation condition for a planar fracture. So, need to construct an efficient approximation.



Differentiate

$$\frac{d\tilde{w}(\tilde{x})}{d\tilde{x}} = -\frac{8}{\pi} \int_0^\infty G'(\tilde{s}/\tilde{x}) \frac{\tilde{s}}{\tilde{x}^2} \left[ \frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3} \right] d\tilde{s}, \quad \tilde{w}(0) = 1.$$



Assume that  $\tilde{w} \propto \tilde{x}^\delta$

$$\frac{d\tilde{w}(\tilde{x})}{d\tilde{x}} = -\frac{8}{\pi} \int_0^\infty G'(\tilde{s}/\tilde{x}) \frac{\tilde{x}^{2\delta-1}}{\tilde{s}^{2\delta-1}} \frac{d\tilde{s}}{\tilde{x}} \left[ \frac{\tilde{s}^{2\delta}}{\tilde{w}(\tilde{s})^2 \tilde{x}^{2\delta}} \right] - \frac{8}{\pi} \int_0^\infty G'(\tilde{s}/\tilde{x}) \frac{\tilde{x}^{3\delta-1}}{\tilde{s}^{3\delta-1}} \frac{d\tilde{s}}{\tilde{x}} \left[ \frac{\chi \tilde{s}^{3\delta}}{\tilde{w}(\tilde{s})^3 \tilde{x}^{3\delta}} \right].$$

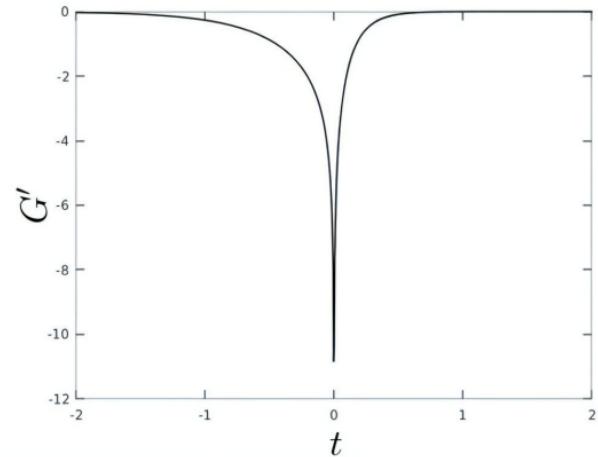


Replace G' with Delta function

$$\tilde{w}' = \frac{C_1(\delta)}{\tilde{w}^2} + \frac{\chi C_2(\delta)}{\tilde{w}^3}, \quad \delta = \tilde{x} \frac{\tilde{w}'}{\tilde{w}}, \quad \tilde{w}(0) = 1,$$

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$$C_1(\delta) = \frac{4(1-2\delta)}{\delta(1-\delta)} \tan(\pi\delta), \quad C_2(\delta) = \frac{16(1-3\delta)}{3\delta(2-3\delta)} \tan\left(\frac{3\pi}{2}\delta\right).$$



# Approximate solution

$$\tilde{w}' = \frac{C_1(\delta)}{\tilde{w}^2} + \frac{\chi C_2(\delta)}{\tilde{w}^3}, \quad \delta = \tilde{x} \frac{\tilde{w}'}{\tilde{w}}, \quad \tilde{w}(0) = 1,$$



Solve assuming constant C1 and C2

$$\tilde{w}^3 - 1 - \frac{3}{2}b(\tilde{w}^2 - 1) + 3b^2(\tilde{w} - 1) - 3b^3 \ln\left(\frac{b + \tilde{w}}{b + 1}\right) = 3C_1(\delta)\tilde{x}, \quad b = \frac{C_2(\delta)}{C_1(\delta)}\chi.$$



Take  $C_1(1/3) = \beta_m^3/3$  and  $C_2(1/4) = \beta_m^4/4$   
to match viscosity and leak-off limits exactly

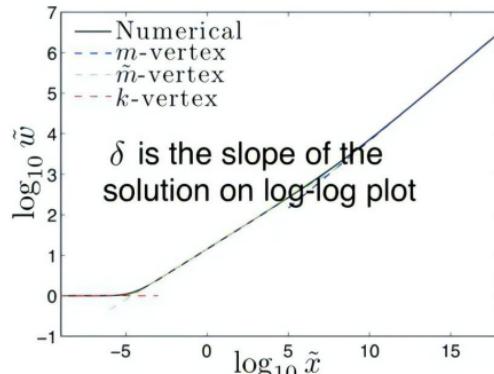
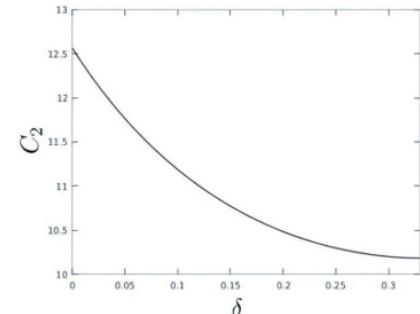
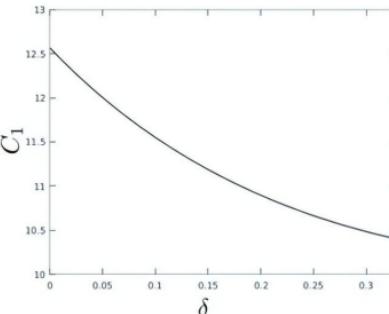
$$\tilde{w}_0^3 - 1 - \frac{3}{2}b_0(\tilde{w}_0^2 - 1) + 3b_0^2(\tilde{w}_0 - 1) - 3b_0^3 \ln\left(\frac{b_0 + \tilde{w}_0}{b_0 + 1}\right) = \beta_m^3 \tilde{x}, \quad b_0 = \frac{3\beta_m^4}{4\beta_m^3}\chi \approx 0.9912\chi.$$

The above is an implicit zeroth order solution for the problem, to get a better approximation, compute

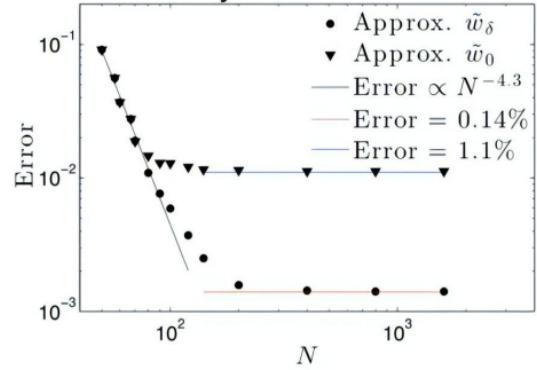
$$\delta = \frac{\beta_m^3 \tilde{x}(\tilde{w}_0)}{3\tilde{w}_0^3} \left(1 + \frac{b_0}{\tilde{w}_0}\right),$$

and re-evaluate the solution.

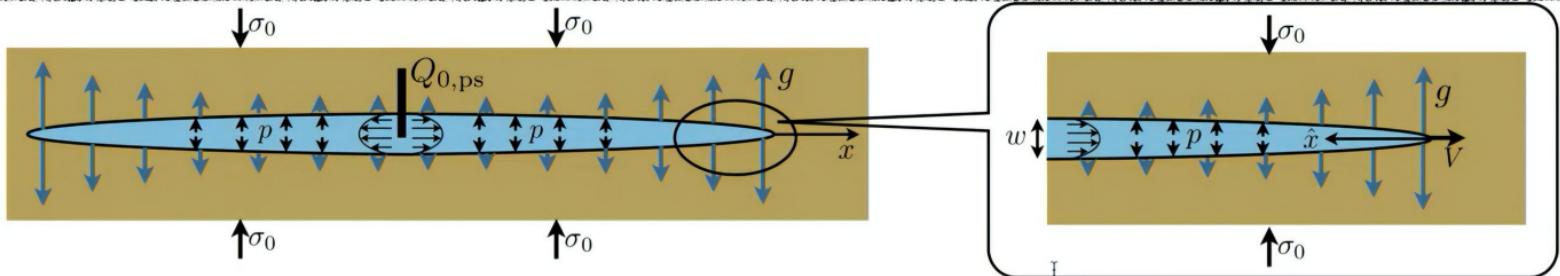
$$C_1(\delta) = \frac{4(1-2\delta)}{\delta(1-\delta)} \tan(\pi\delta), \quad C_2(\delta) = \frac{16(1-3\delta)}{3\delta(2-3\delta)} \tan\left(\frac{3\pi}{2}\delta\right).$$



## Error analysis

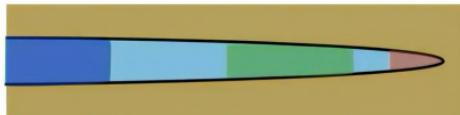


## Things to remember for the semi-infinite hydraulic fracture

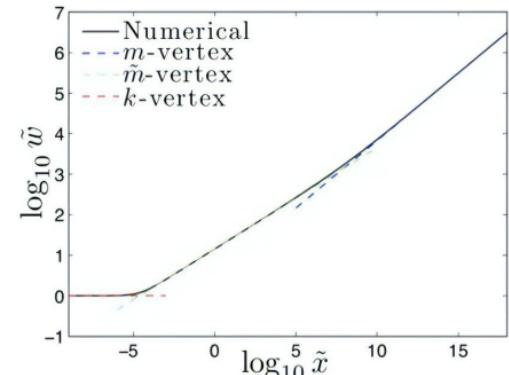


- Semi-infinite geometry describes tip region of a finite hydraulic fracture
- There are three limiting analytic solutions: toughness, viscosity, and leak-off
- The global solution gradually transitions from one limiting case to another
- There is computationally efficient approximate solution for the problem that can be used as a propagation condition for finite fractures

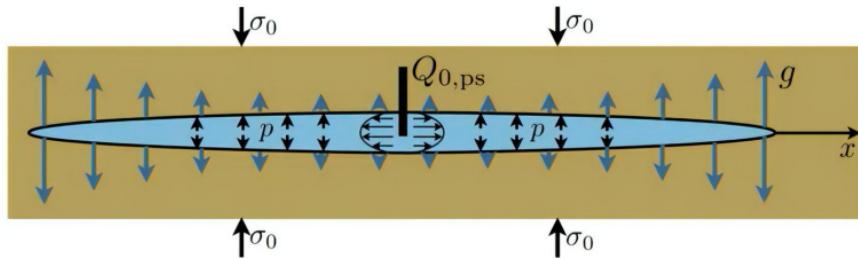
$$w \propto \hat{x}^{2/3} \quad w \propto \hat{x}^{5/8} \quad w \propto \hat{x}^{1/2}$$



$$w_k = \frac{K'}{E'} \hat{x}^{1/2}, \quad w_{\tilde{m}} = \beta_{\tilde{m}} \left( \frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} \hat{x}^{5/8}, \quad w_m = \beta_m \left( \frac{\mu' V}{E'} \right)^{1/3} \hat{x}^{2/3}.$$



# Plane strain hydraulic fracture



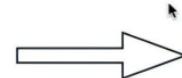
Governing equations

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = Q_{0,ps}(t)\delta(x),$$

$$q = -\frac{w^3}{\mu'} \frac{\partial p_n}{\partial x},$$

$$p_n(x) = -\frac{E'}{4\pi} \int_{-l}^l \frac{w(s)ds}{(x-s)^2},$$

$$w \rightarrow \frac{K'}{E'} \sqrt{l-x}, \quad x \rightarrow l.$$



Scales

$$\frac{w_*}{t} = \frac{q_*}{l_*} = \frac{C'}{t^{1/2}} = \frac{Q_{0,ps}}{l_*},$$

$$q_* = \frac{w_*^3 p_*}{\mu' l_*},$$

$$p_* = \frac{E' w_*}{l_*},$$

$$w_* = \frac{K'}{E'} l_*^{1/2}.$$

6 equations, 4 unknowns

# Scaling for viscosity-storage solution

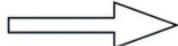
Viscosity-storage => no toughness, no leak-off

$$\frac{w_*}{t} = \frac{q_*}{l_*} = \frac{\cancel{C'}}{\cancel{t^{1/2}}} = \frac{Q_{0,\text{ps}}}{l_*},$$

$$q_* = \frac{w_*^3 p_*}{\mu' l_*},$$

$$p_* = \frac{E' w_*}{l_*},$$

$$\cancel{w_*} = \cancel{\frac{K'}{E'}} l_*^{1/2}.$$



Scaling solution

$$l_* = \left( \frac{Q_{0,\text{ps}}^3 E' t^4}{\mu'} \right)^{1/6}, \quad w_* = \left( \frac{\mu' Q_{0,\text{ps}}^3 t^2}{E'} \right)^{1/6}, \quad p_* = \left( \frac{\mu' E'^2}{t} \right)^{1/3}$$

M-vertex solution

$$w_m(\xi, t) = 1.1265 \left( \frac{\mu' Q_0^3 t^2}{E'} \right)^{1/6} (1+\xi)^{0.588} (1-\xi)^{2/3},$$

$$p_m(\xi, t) = 2.7495 \left( \frac{\mu' E'^2}{t} \right)^{1/3} \mathcal{F}\left(\xi, 0.588, \frac{2}{3}\right),$$

$$l_m(t) = 0.6159 \left( \frac{Q_0^3 E' t^4}{\mu'} \right)^{1/6},$$

# Scaling for toughness-storage solution

Toughness-storage => no viscosity, no leak-off

$$\frac{w_*}{t} = \frac{q_*}{l_*} = \frac{\cancel{C'}}{\cancel{t^{1/2}}} = \frac{Q_{0,\text{ps}}}{l_*},$$

$$q_* = \cancel{\frac{w_*^3 p_*}{\mu' l_*}},$$

$$p_* = \frac{E' w_*}{l_*},$$

$$w_* = \frac{K'}{E'} l_*^{1/2}.$$

Scaling solution



$$l_* = \left( \frac{E' Q_{0,\text{ps}} t}{K'} \right)^{2/3}, \quad w_* = \left( \frac{K'^2 Q_{0,\text{ps}} t}{E'^2} \right)^{1/3}, \quad p_* = \left( \frac{K'^4}{E'^2 Q_{0,\text{ps}} t} \right)^{1/3}$$

I

K-vertex solution

$$w_k(\xi, t) = 0.6828 \left( \frac{K'^2 Q_0 t}{E'^2} \right)^{1/3} (1 - \xi^2)^{1/2},$$

$$p_k(\xi, t) = 0.1831 \left( \frac{K'^4}{E' Q_0 t} \right)^{1/3},$$

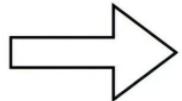
$$l_k(t) = 0.9324 \left( \frac{E' Q_0 t}{K'} \right)^{2/3}.$$

## Toughness-viscosity transition

---

Use either

$$\begin{aligned} l_m &\sim l_k \\ w_m &\sim w_k \end{aligned}$$



Dimensionless toughness

$$K_m = \left( \frac{K'^4}{\mu' E'^3 Q_{0,\text{ps}}} \right)^{1/4}$$

$$p_m \sim p_k$$

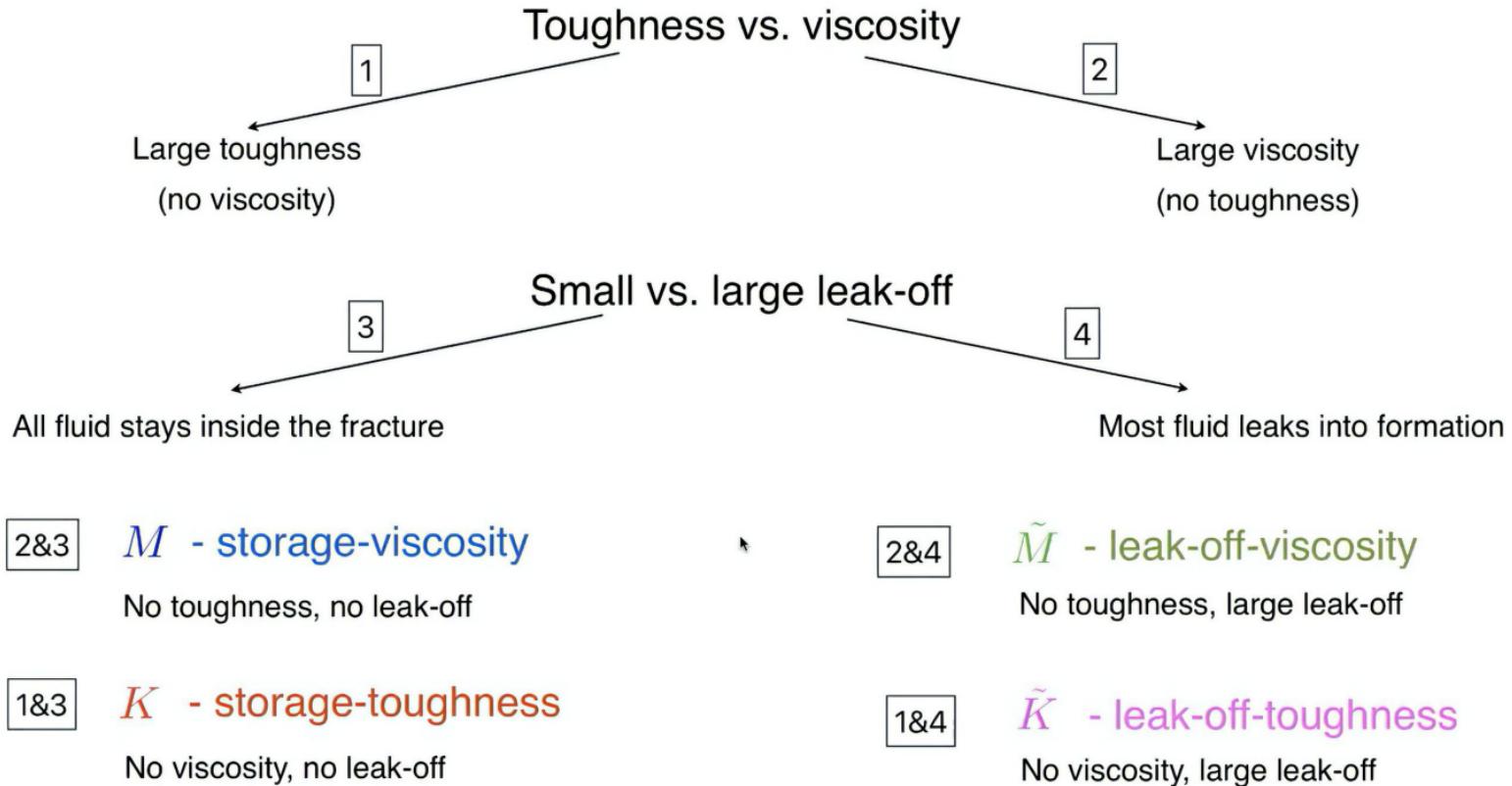
$K_m \ll 1$  M vertex (viscosity dominated)

$K_m \gg 1$  K vertex (toughness dominated)

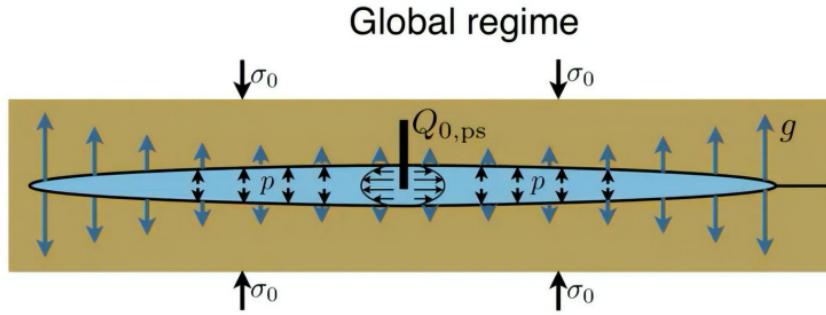
$K_m \sim 1$  M-K transition

**This is only part of the story, how about leak-off?**

# Hydraulic fracture regimes of propagation



# Tip asymptotics vs. regime



*M* - storage-viscosity

No toughness, no leak-off

*K* - storage-toughness

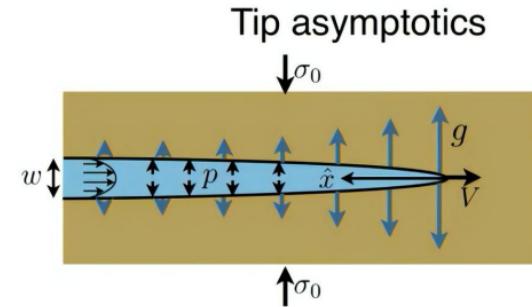
No viscosity, no leak-off

*M̃* - leak-off-viscosity

No toughness, large leak-off

*K̃* - leak-off-toughness

No viscosity, large leak-off



Viscosity

$$w_m = \beta_m \left( \frac{\mu' V}{E'} \right)^{1/3} \hat{x}^{2/3}$$

Toughness

$$w_k = \frac{K'}{E'} \hat{x}^{1/2}$$

Leak-off

$$w_{\tilde{m}} = \beta_{\tilde{m}} \left( \frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} \hat{x}^{5/8}$$

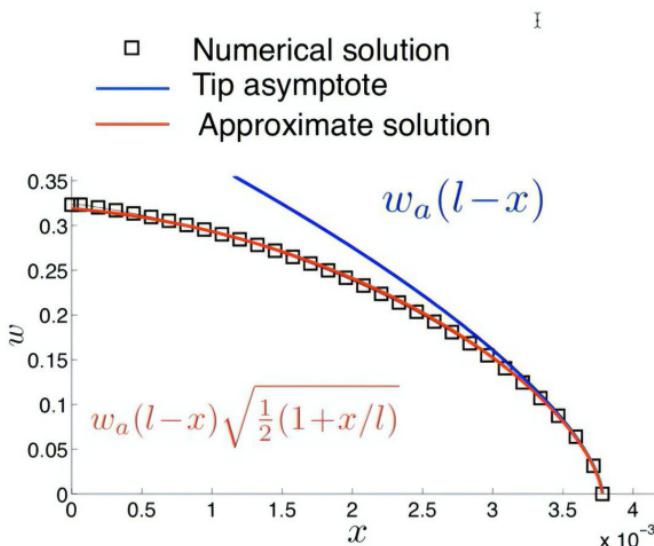
# Solution for the problem

- **Numerical solution**

- Discretize governing equations using finite differences, see Dontsov, 2017

- **Approximate solution**

- Global volume balance
- Approximate solution for width based on the tip asymptote



Global volume balance

$$\int_0^l \left( w(x, t) + 2C' \sqrt{t - t_0(x)} \right) dx = \frac{Q_{0,ps} t}{2},$$
$$l(t) \propto t^\alpha \implies x/l = (t_0/t)^\alpha$$

Approximate solution for width

$$w(x, t) = \left( \frac{l+x}{2l} \right)^\lambda w_a(l-x),$$

$\lambda$  - fitting parameter

# Regimes of propagation for a plane strain hydraulic fracture

$M$  - storage-viscosity

No toughness, no leak-off

$K$  - storage-toughness

No viscosity, no leak-off

$\tilde{M}$  - leak-off-viscosity

No toughness, large leak-off

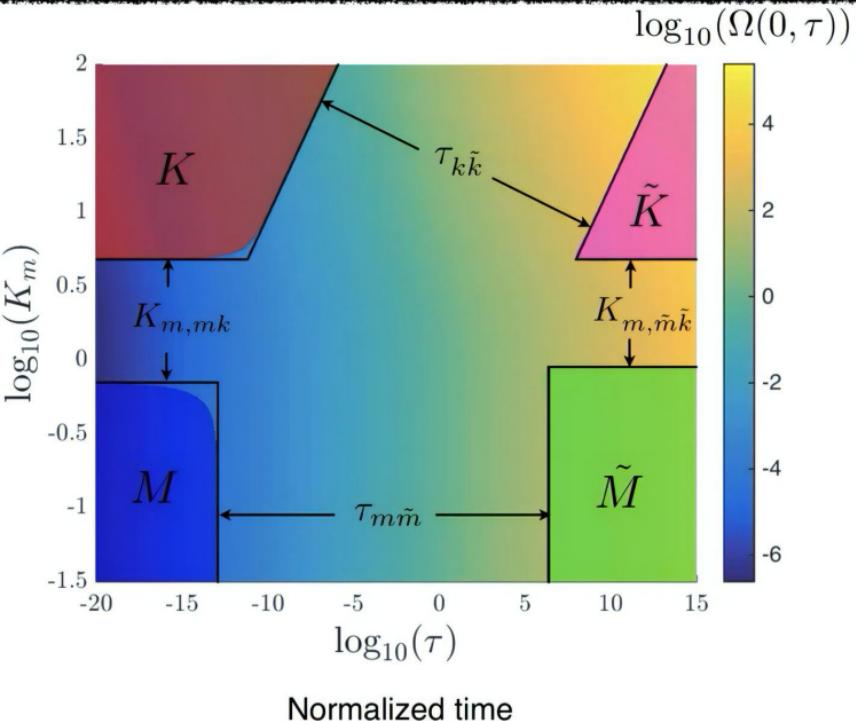
$\tilde{K}$  - leak-off-toughness

No viscosity, large leak-off

Zones of applicability of limiting solutions  
are defined to within 1% of error

Bounds for these zones (solid black  
lines) are known

Normalized toughness



$$K_m = \left( \frac{K'^4}{\mu' E'^3 Q_{0,ps}} \right)^{1/4} \quad \tau = \frac{t}{t_{m\bar{m}}} \quad t_{m\bar{m}} = \frac{\mu' Q_{0,ps}^3}{E' C'^6}$$

# Limiting solutions for a plane strain hydraulic fracture

## $M$ - storage-viscosity

$$\begin{aligned} w_m(\xi, t) &= 1.1265 \left( \frac{\mu' Q_{0,ps}^3 t^2}{E'} \right)^{1/6} (1+\xi)^{0.588} (1-\xi)^{2/3}, \\ p_m(\xi, t) &= 2.7495 \left( \frac{\mu' E'^2}{t} \right)^{1/3} \mathcal{F}(\xi, 0.588, \frac{2}{3}), \\ l_m(t) &= 0.6159 \left( \frac{Q_{0,ps}^3 E' t^4}{\mu'} \right)^{1/6}, \end{aligned}$$

## $K$ - storage-toughness

$$\begin{aligned} w_k(\xi, t) &= 0.6828 \left( \frac{K'^2 Q_{0,ps} t}{E'^2} \right)^{1/3} (1-\xi^2)^{1/2}, \\ p_k(\xi, t) &= 0.1831 \left( \frac{K'^4}{E' Q_{0,ps} t} \right)^{1/3}, \\ l_k(t) &= 0.9324 \left( \frac{E' Q_{0,ps} t}{K'} \right)^{2/3}. \end{aligned}$$

Elasticity function:

## $\tilde{M}$ - leak-off-viscosity

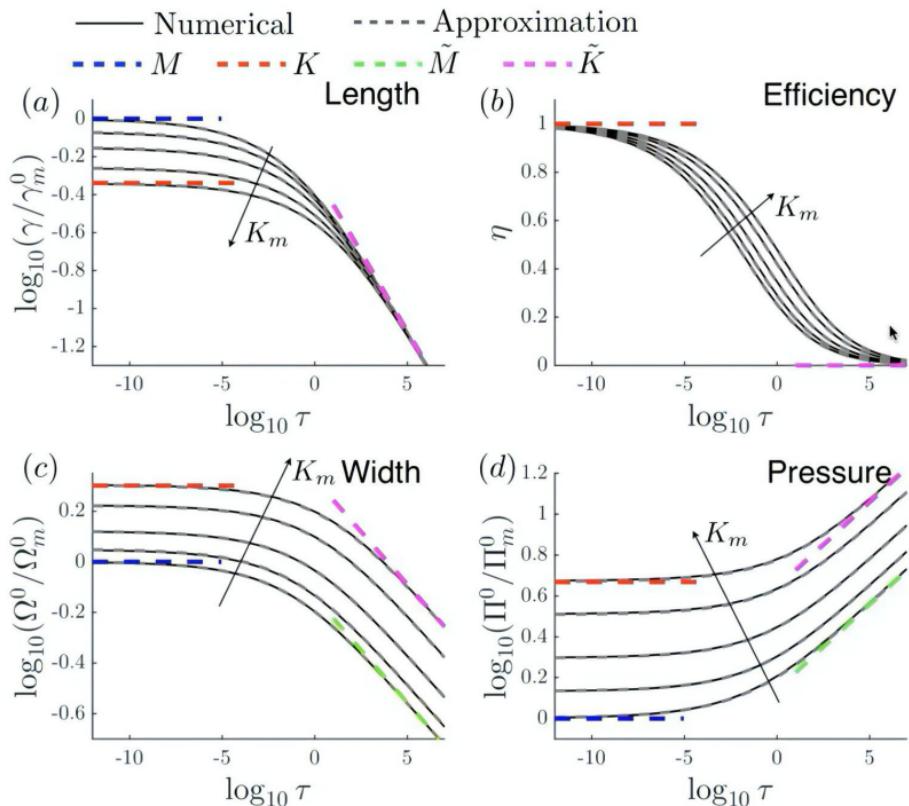
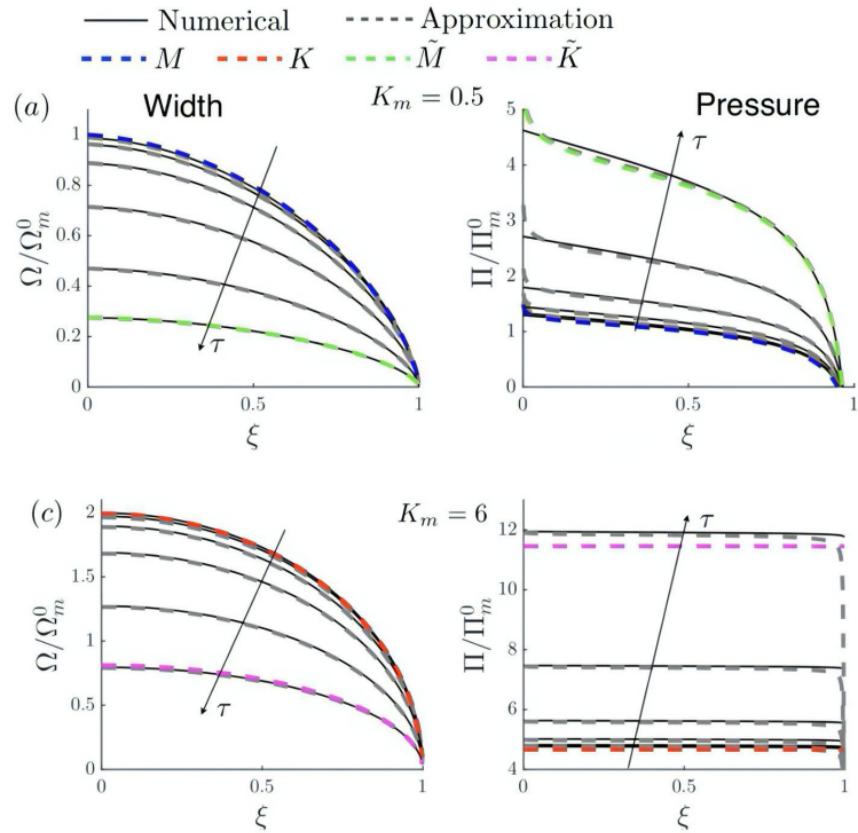
$$\begin{aligned} w_{\tilde{m}}(\xi, t) &= 0.8165 \left( \frac{\mu' Q_{0,ps}^3 t}{E' C'^2} \right)^{1/4} (1+\xi)^{0.520} (1-\xi)^{5/8}, \\ p_{\tilde{m}}(\xi, t) &= 3.6783 \left( \frac{C'^2 \mu' E'^3}{Q_{0,ps} t} \right)^{1/4} \mathcal{F}(\xi, 0.520, \frac{5}{8}), \\ l_{\tilde{m}}(t) &= 0.3183 \frac{Q_{0,ps} t^{1/2}}{C'}, \end{aligned}$$

## $\tilde{K}$ - leak-off-toughness

$$\begin{aligned} w_{\tilde{k}}(\xi, t) &= 0.3989 \left( \frac{K'^4 Q_{0,ps}^2 t}{E'^4 C'^2} \right)^{1/4} (1-\xi^2)^{1/2}, \\ p_{\tilde{k}}(\xi, t) &= 0.3133 \left( \frac{K'^4 C'^2}{Q_{0,ps}^2 t} \right)^{1/4}, \\ l_{\tilde{k}}(t) &= 0.3183 \frac{Q_{0,ps} t^{1/2}}{C'}, \end{aligned}$$

$$\begin{aligned} \mathcal{F}(\xi, \lambda, \bar{\delta}) &= \frac{1}{2^{1+\lambda}\pi} \int_0^1 \frac{\partial M(\xi, s)}{\partial s} (1+s)^\lambda (1-s)^{\bar{\delta}} ds, \\ M(\xi, s) &= \frac{\xi}{\xi^2 - s^2}, \end{aligned}$$

# Numerical vs. approximate solutions



# Things to remember for the plane strain hydraulic fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote
- The existence of approximate solution constructed using global volume balance and tip asymptote
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless toughness and dimensionless time

$M$  - storage-viscosity

No toughness, no leak-off

Viscosity

$$w_m = \beta_m \left( \frac{\mu' V}{E'} \right)^{1/3} \hat{x}^{2/3}$$

$K$  - storage-toughness

No viscosity, no leak-off

Toughness

$$w_k = \frac{K'}{E'} \hat{x}^{1/2}$$

$\tilde{M}$  - leak-off-viscosity

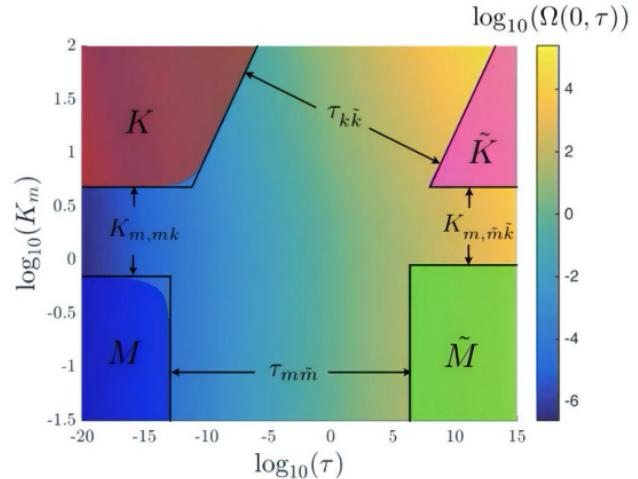
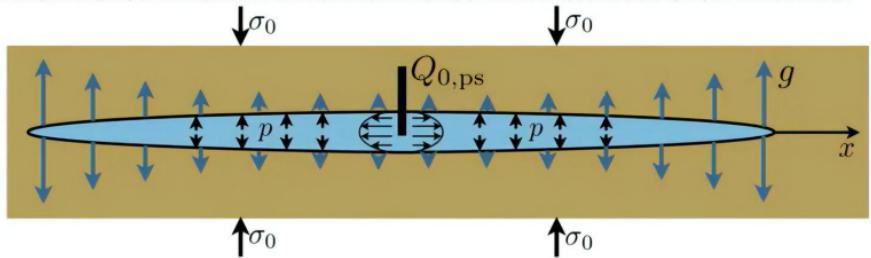
No toughness, large leak-off

Leak-off

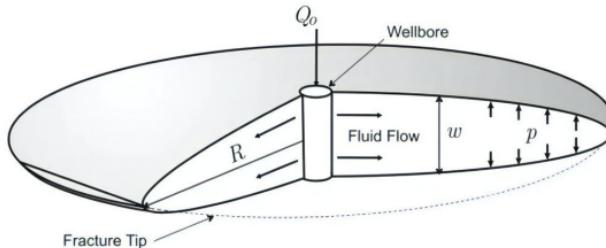
$$w_{\tilde{m}} = \beta_{\tilde{m}} \left( \frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} \hat{x}^{5/8}$$

$\tilde{K}$  - leak-off-toughness

No viscosity, large leak-off



# Radial hydraulic fracture



Governing equations

$$\frac{\partial w}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (rq) + \frac{C'}{\sqrt{t - t_0(r)}} = Q_0 \delta(r),$$

$$q = -\frac{w^3}{\mu'} \frac{\partial p_n}{\partial r},$$

$$p_n(r, t) = -\frac{E'}{2\pi R} \int_0^R M\left(\frac{r}{R}, \frac{r'}{R}\right) \frac{\partial w(r', t)}{\partial r'} dr',$$

$$w \rightarrow \frac{K'}{E'}(R-r)^{1/2}, \quad r \rightarrow R.$$

Scales

$$\frac{w_*}{t} = \frac{q}{R_*} = \frac{C'}{t^{1/2}} = \frac{Q_0}{R_*^2},$$

$$q = \frac{w_*^3 p_*}{\mu' R_*},$$

$$p_* = \frac{E' w_*}{R_*},$$

$$w = \frac{K' R_*^{1/2}}{E'}.$$

6 equations, 4 unknowns

## Scaling for viscosity-storage solution

Viscosity-storage => no toughness, no leak-off

$$\frac{w_*}{t} = \frac{q}{R_*} = \cancel{\frac{C'}{t^{1/2}}} = \frac{Q_0}{R_*^2},$$

$$q = \frac{w_*^3 p_*}{\mu' R_*},$$

$$p_* = \frac{E' w_*}{R_*},$$

$$w = \cancel{\frac{K' R_*^{1/2}}{E'}}.$$



Scaling solution

$$R_* = \left( \frac{Q_0^3 E' t^4}{\mu'} \right)^{1/9}, \quad w_* = \left( \frac{\mu'^2 Q_0^3 t}{E'^2} \right)^{1/9}, \quad p_* = \left( \frac{\mu' E'^2}{t} \right)^{1/3}$$

M-vertex solution<sup>I</sup>

$$w_m(\rho, t) = 1.1901 \left( \frac{\mu'^2 Q_0^3 t}{E'^2} \right)^{1/9} (1+\rho)^{0.487} (1-\rho)^{2/3},$$

$$p_m(\rho, t) = 2.4019 \left( \frac{\mu' E'^2}{t} \right)^{1/3} \mathcal{F}(\rho, 0.487, \frac{2}{3}),$$

$$R_m(t) = 0.6944 \left( \frac{Q_0^3 E' t^4}{\mu'} \right)^{1/9},$$

# Regimes of propagation for a radial hydraulic fracture

$M$  - storage-viscosity

No toughness, no leak-off

$K$  - storage-toughness

No viscosity, no leak-off

$\tilde{M}$  - leak-off-viscosity

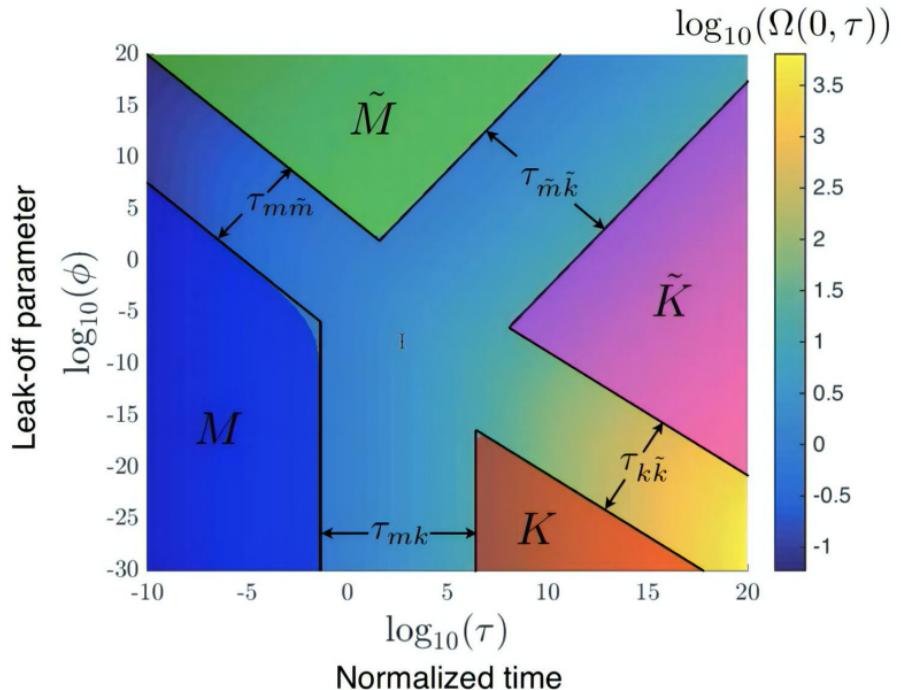
No toughness, large leak-off

$\tilde{K}$  - leak-off-toughness

No viscosity, large leak-off

Zones of applicability of limiting solutions  
are defined to within 1% of error

Bounds for these zones (solid black  
lines) are known



$$\phi = \frac{\mu'^3 E'^{11} C'^4 Q_0}{K'^{14}} \quad \tau = \frac{t}{t_{mk}} \quad t_{mk} = \left( \frac{\mu'^5 E'^{13} Q_0^3}{K'^{18}} \right)^{1/2}$$

# Regimes of propagation for radial HF

## $M$ - storage-viscosity

$$\begin{aligned} w_m(\rho, t) &= 1.1901 \left( \frac{\mu'^2 Q_0^3 t}{E'^2} \right)^{1/9} (1+\rho)^{0.487} (1-\rho)^{2/3}, \\ p_m(\rho, t) &= 2.4019 \left( \frac{\mu' E'^2}{t} \right)^{1/3} \mathcal{F}(\rho, 0.487, \frac{2}{3}), \\ R_m(t) &= 0.6944 \left( \frac{Q_0^3 E' t^4}{\mu'} \right)^{1/9}. \end{aligned}$$

## $\tilde{M}$ - leak-off-viscosity

$$\begin{aligned} w_{\tilde{m}}(\rho, t) &= 1.0574 \left( \frac{\mu'^4 Q_0^6 t}{E'^4 C'^2} \right)^{1/16} (1+\rho)^{0.397} (1-\rho)^{5/8}, \\ p_{\tilde{m}}(\rho, t) &= 3.0931 \left( \frac{\mu'^4 E'^{12} C'^6}{Q_0^2 t^3} \right)^{1/16} \mathcal{F}(\rho, 0.397, \frac{5}{8}), \\ R_{\tilde{m}}(t) &= 0.4502 \left( \frac{Q_0^2 t}{C'^2} \right)^{1/4}, \end{aligned}$$

## $K$ - storage-toughness

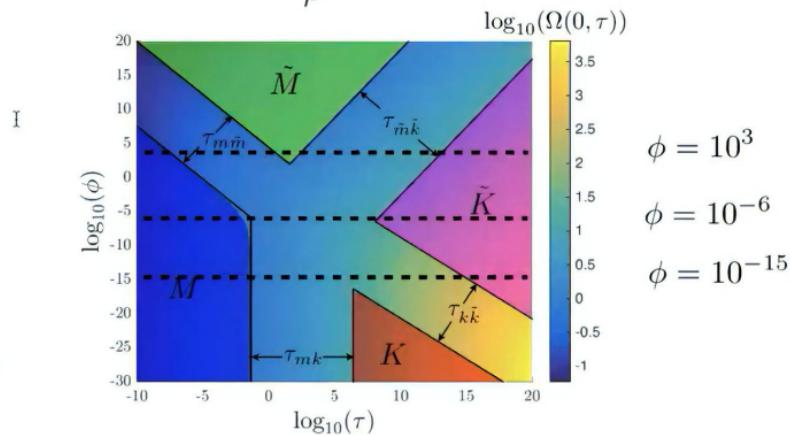
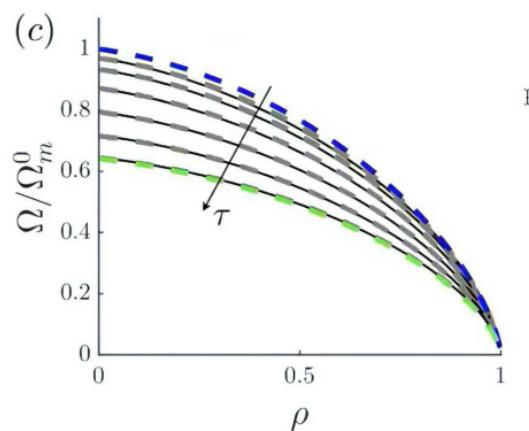
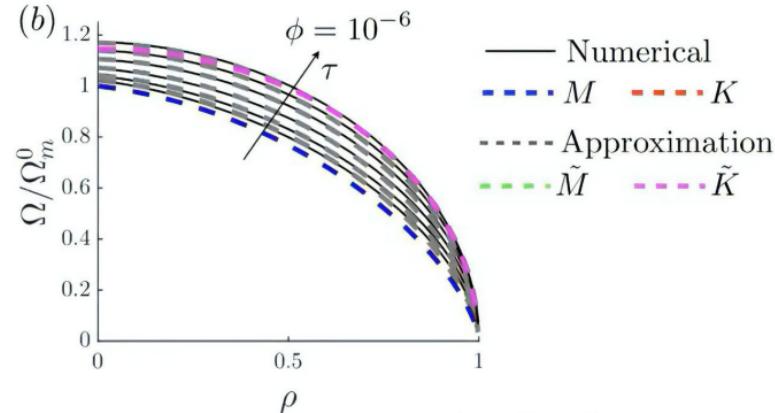
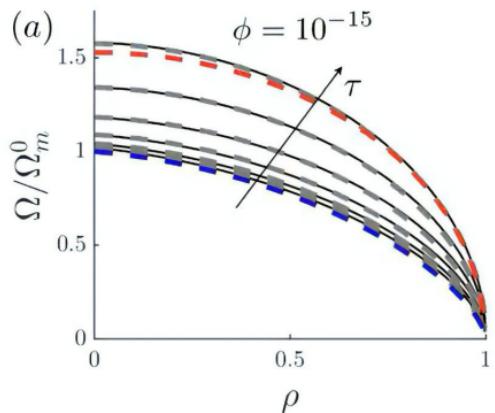
$$\begin{aligned} w_k(\rho, t) &= 0.6537 \left( \frac{K'^4 Q_0 t}{E'^4} \right)^{1/5} (1-\rho^2)^{1/2}, \\ p_k(\rho, t) &= 0.3004 \left( \frac{K'^6}{E' Q_0 t} \right)^{1/5}, \\ R_k(t) &= 0.8546 \left( \frac{E' Q_0 t}{K'} \right)^{2/5}. \end{aligned}$$

## $\tilde{K}$ - leak-off-toughness

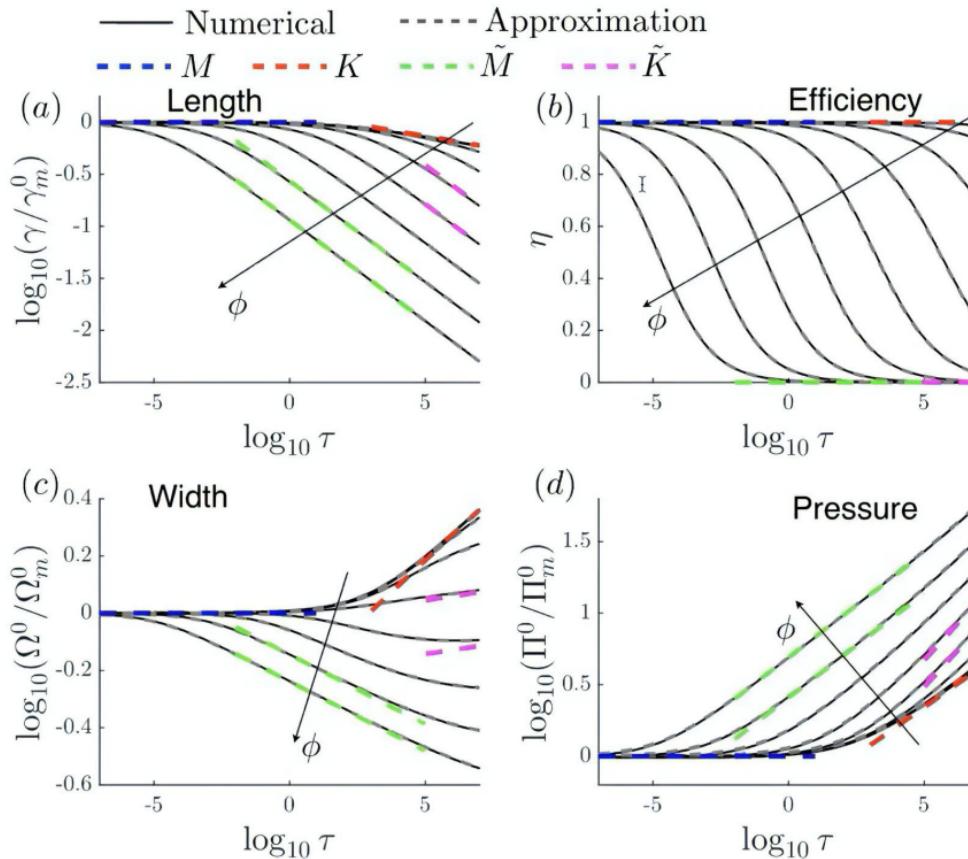
$$\begin{aligned} w_{\tilde{k}}(\rho, t) &= 0.4744 \left( \frac{K'^8 Q_0^2 t}{E'^8 C'^2} \right)^{1/8} (1-\rho^2)^{1/2}, \\ p_{\tilde{k}}(\rho, t) &= 0.4139 \left( \frac{K'^8 C'^2}{Q_0^2 t} \right)^{1/8}, \\ R_{\tilde{k}}(t) &= 0.4502 \left( \frac{Q_0^2 t}{C'^2} \right)^{1/4}. \end{aligned}$$

Elasticity function:  $\mathcal{F}(\rho, \lambda, \bar{\delta}) = \frac{1}{2^{1+\lambda}\pi} \int_0^1 \frac{\partial M(\rho, s)}{\partial s} (1+s)^\lambda (1-s)^{\bar{\delta}} ds,$

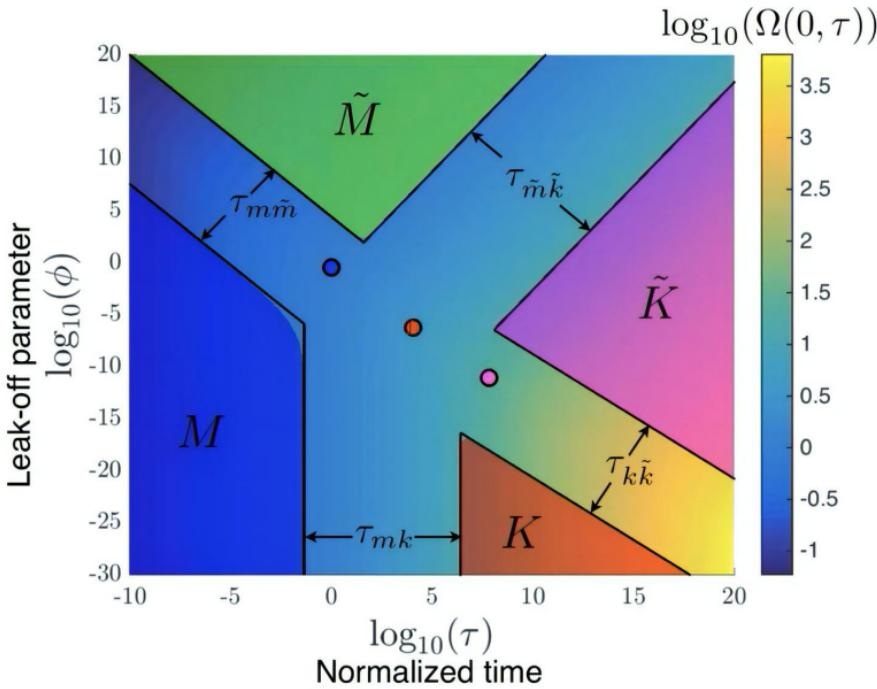
# Comparison with numerical solution



## Comparison with numerical solution



# Radial hydraulic fracture: examples



$$\phi = \frac{\mu'^3 E'^{11} C'^4 Q_0}{K'^{14}} \quad \tau = \frac{t}{t_{mk}} \quad t_{mk} = \left( \frac{\mu'^5 E'^{13} Q_0^3}{K'^{18}} \right)^{1/2}$$

## Field values

$$E' = 36 \text{ GPa}$$

$$K_{Ic} = 2.6 \text{ MPa} \cdot \text{m}^{1/2}$$

$$Q_0 = 18 \text{ bbl/min} = 0.05 \text{ m}^3/\text{s}$$

$$t = 100 \text{ min}$$

$$C' = 4.3 \times 10^{-5} \text{ m/s}^{1/2}$$

## Slick water

$$\mu = 1.5 \text{ cP} = 1.5 \times 10^{-3} \text{ Pa} \cdot \text{s}$$

## Linear gel

$$\mu = 100 \text{ cP} = 0.1 \text{ Pa} \cdot \text{s}$$

## Lab values (rock)

$$E' = 36 \text{ GPa}$$

$$K_{Ic} = 2.6 \text{ MPa} \cdot \text{m}^{1/2}$$

$$C' = 4.3 \times 10^{-5} \text{ m/s}^{1/2}$$

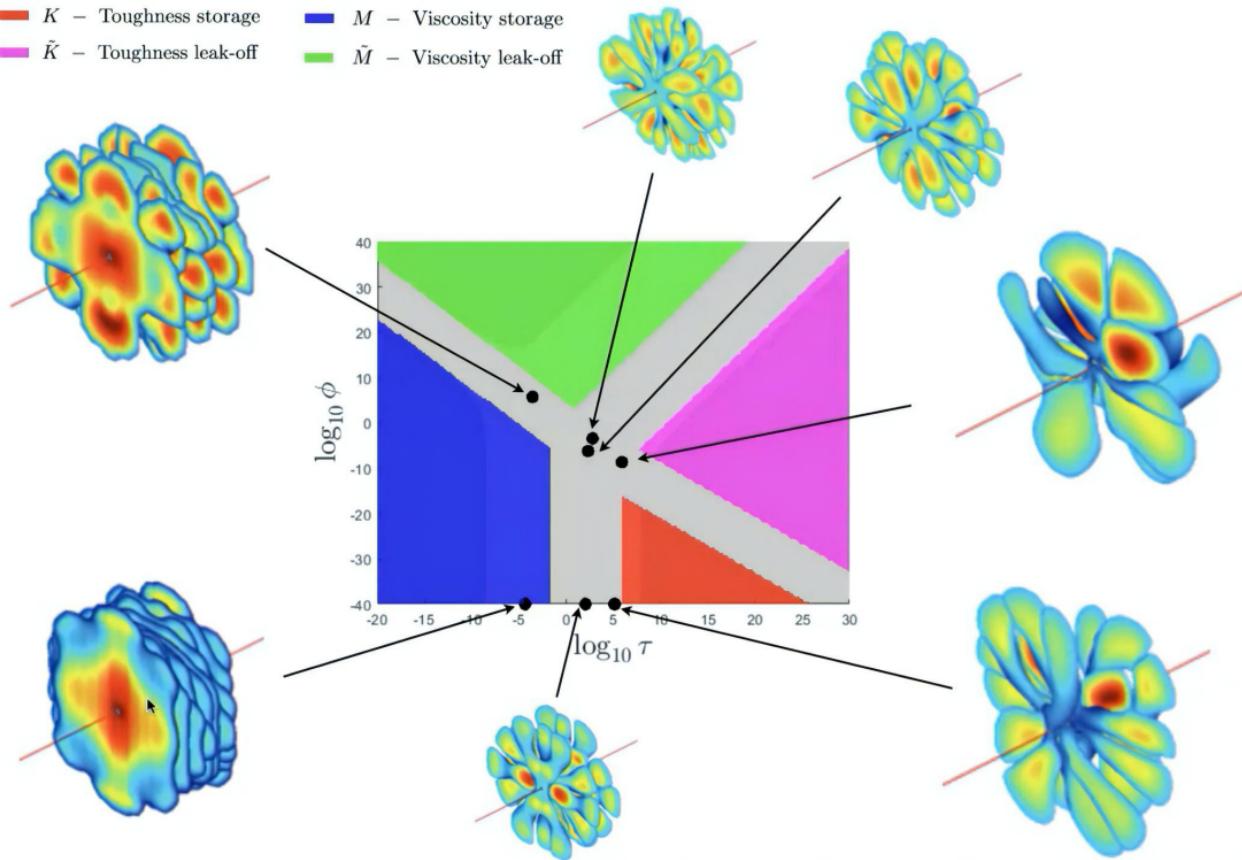
$$\mu = 1.5 \text{ cP} = 1.5 \times 10^{-3} \text{ Pa} \cdot \text{s}$$

$$t = 1 \text{ min} \quad Q_0 = 2.6 \times 10^{-6} \text{ m}^3/\text{s}$$

## Application to multiple fractures

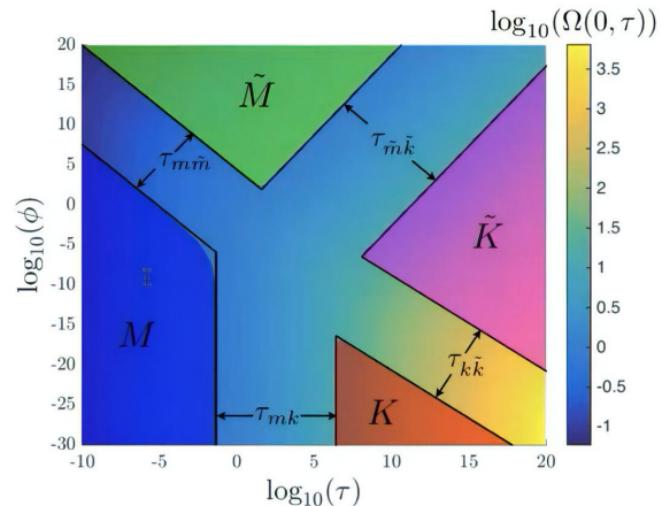
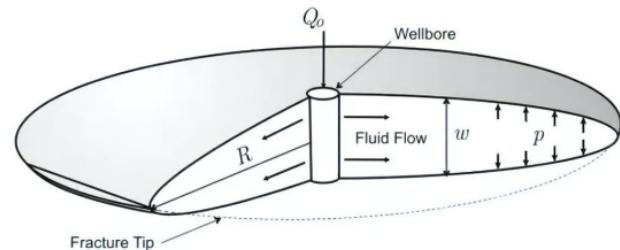
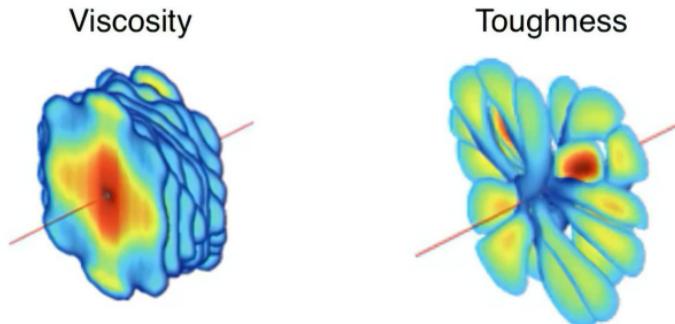
—  $K$  — Toughness storage  
—  $\tilde{K}$  — Toughness leak-off

—  $M$  — Viscosity storage  
—  $\tilde{M}$  — Viscosity leak-off



# Things to remember for radial hydraulic fracture

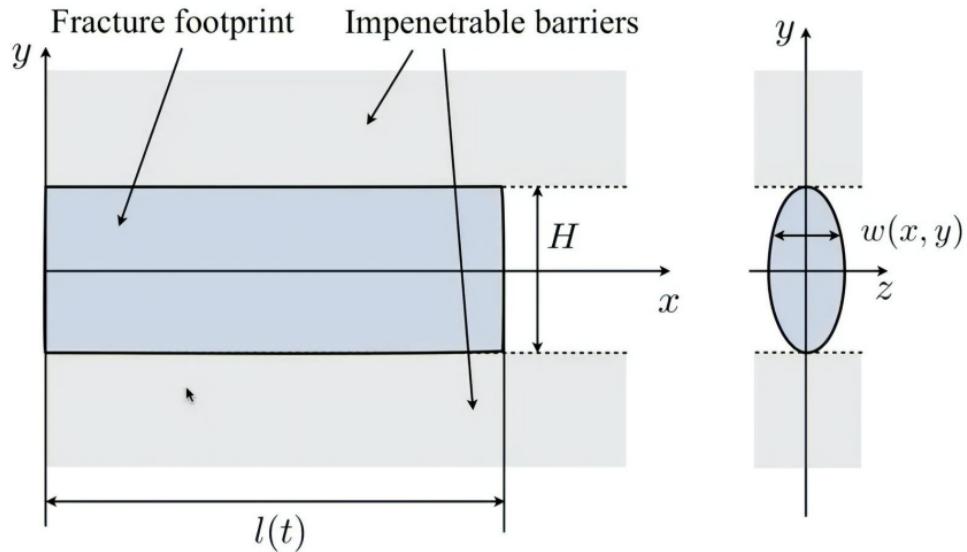
- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote (the same as for plane strain)
- The existence of approximate solution constructed using global volume balance and tip asymptote (similar to plane strain)
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless leak-off and dimensionless time
- Fracture regimes affect morphology of multiple hydraulic fractures



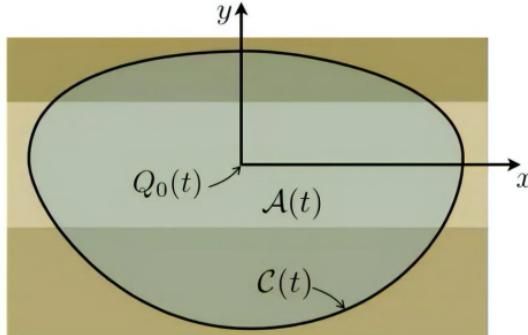
E.V. Dontsov. An approximate solution for a penny-shaped hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *R. Soc. open sci.*, 3:160737, 2016.

## PKN or constant height hydraulic fracture

- Fracture height is assumed to be constant
- Length  $\gg$  height (long time solution)
- Flux is predominantly horizontal
- Pressure is assumed to be constant in the vertical cross-section
- Far away from the tip, local elasticity applies. That is the pressure is determined by the width at the particular point (as opposed to non-local relation for which the pressure depends on the integral of the width with some kernel)
- Fracture width is elliptical in each cross-section
- These assumptions allow to reformulate the two-dimensional fracture problem as one-dimensional, effectively solving the solution in the vertical direction analytically



# Governing equations for PKN geometry



Scaling       $E' = \frac{E}{1-\nu^2}, \quad \mu' = 12\mu, \quad K' = \sqrt{\frac{32}{\pi}} K_{Ic}, \quad C' = 2C_l,$

Apply vertical averaging

$$w(x, y) = \frac{4}{\pi} \bar{w}(x) \sqrt{1 - \left(\frac{2y}{H}\right)^2}, \quad \bar{w}(x) = \frac{1}{H} \int_{-H/2}^{H/2} w(x, y) dy,$$

Volume balance       $\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{q} + \frac{C'}{\sqrt{t-t_0(x,y)}} = Q_0(t) \delta(x, y),$

Fluid flow       $\mathbf{q} = -\frac{w^3}{\mu'} \nabla p,$

Elasticity       $p(x, y, t) = \sigma_0(y) - \frac{E'}{8\pi} \int_{\mathcal{A}(t)} \frac{w(x', y', t) dx' dy'}{[(x' - x)^2 + (y' - y)^2]^{3/2}}, \quad \xrightarrow{\text{I}}$

Propagation       $\lim_{s \rightarrow 0} \frac{w}{s^{1/2}} = \begin{cases} \frac{K'}{E'}, & \text{if } V > 0, \\ \frac{K'_l}{E'}, & \text{if } V = 0. \end{cases}$

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}_x}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H} \delta(x),$$

$$\bar{q}_x = -\frac{1}{H\mu'} \frac{\partial p}{\partial x} \int_{-H/2}^{H/2} w^3 dy = -\frac{\bar{w}^3}{\pi^2 \mu} \frac{\partial p}{\partial x},$$

$$p(x) = -\frac{2E'}{\pi^2 H} \int_{-l(t)}^{l(t)} \bar{w}(x') \frac{dG(2(x'-x)/H)}{dx'} dx'$$

$$\lim_{s \rightarrow 0} \frac{w}{s^{1/2}} = \frac{K'}{E'}$$

# Elasticity equation for PKN fracture

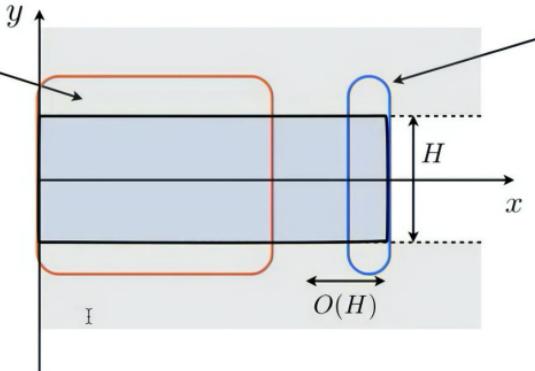
$$p(x) = -\frac{2E'}{\pi^2 H} \int_{-l(t)}^{l(t)} \bar{w}(x') \frac{dG(2(x'-x)/H)}{dx'} dx'$$

$$G(s) = \frac{\sqrt{1+s^2}}{s} E\left(\frac{1}{1+s^2}\right), \quad G(s) \approx \frac{\pi}{2} \text{sign}(s), \quad |s| \gg 1, \quad G(s) \approx \frac{1}{s}, \quad s \ll 1,$$

Far away from  
the tip ( $x \gg H$ )

$$p(x) = \frac{2E' \bar{w}(x)}{\pi H},$$

Local elasticity equation



Near the tip  
( $x \sim H$ )

$$p(x, t) = \sigma_0 - \frac{E'}{4\pi} \int_{-l_1}^{l_2} \frac{w(s)ds}{(x-s)^2},$$

Plane strain elasticity equation

## Two options for the solution

Local elasticity + special toughness boundary condition

$$p(x) = \frac{2E' \bar{w}(x)}{\pi H},$$

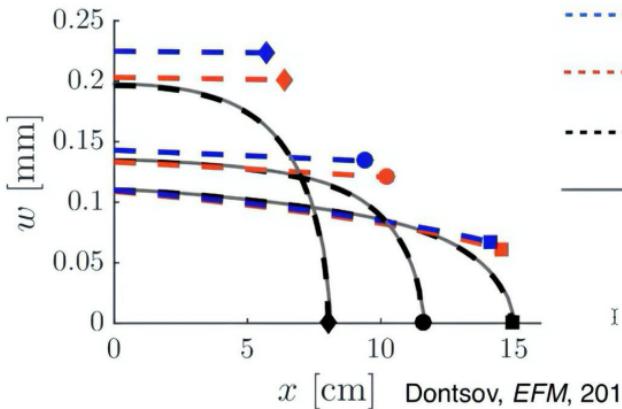
$$\bar{w}(l) = \frac{\sqrt{\pi H} K_{Ic}}{E'}.$$

Non-elasticity + standard toughness boundary condition

$$p(x) = -\frac{2E'}{\pi^2 H} \int_{-l(t)}^{l(t)} \bar{w}(x') \frac{dG(2(x'-x)/H)}{dx'} dx'$$

$$\lim_{s \rightarrow 0} \frac{w}{s^{1/2}} = \frac{K'}{E'}$$

■  $K_{Ic} = 0.47 \text{ MPa}\cdot\text{m}^{1/2}$  ●  $K_{Ic} = 0.94 \text{ MPa}\cdot\text{m}^{1/2}$  ♦  $K_{Ic} = 1.57 \text{ MPa}\cdot\text{m}^{1/2}$



..... PKN + local elasticity

..... PKN + local elasticity

..... PKN + non-local elasticity

— Fully planar ILSA

$$\bar{w}_R(l) = \left(\frac{\pi}{2}\right)^{3/2} \frac{\sqrt{H} K_{Ic}}{E'}. \quad \text{Nolte, 1991}$$

$$\bar{w}_E(l) = \frac{\sqrt{\pi H} K_{Ic}}{E'}, \quad \text{Sarvaramini & Garagash, 2015}$$

- Non-local elasticity has superior accuracy
- Formulation with local elasticity can be used for analysis

## Tip region of PKN fracture

Equations for finite fracture

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}_x}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H} \delta(x),$$

$$\bar{q}_x = -\frac{1}{H\mu'} \frac{\partial p}{\partial x} \int_{-H/2}^{H/2} w^3 dy = -\frac{\bar{w}^3}{\pi^2 \mu} \frac{\partial p}{\partial x},$$

$$p(x) = \frac{2E' \bar{w}(x)}{\pi H},$$

$$\bar{w}(l) = \frac{\sqrt{\pi H} K_{Ic}}{E'}.$$

Semi-infinite PKN fracture

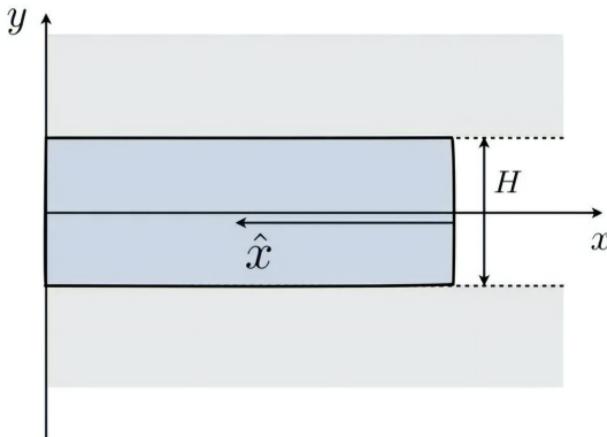
Use moving coordinate

$$\hat{x} = Vt - x$$



$$\frac{E'}{2\pi^3 \mu H} \frac{d\bar{w}^4}{d\hat{x}} = V\bar{w} + 2C' \sqrt{V\hat{x}},$$

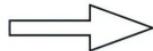
$$\bar{w}(0) = \frac{\sqrt{\pi H} K_{Ic}}{E'}.$$



## Known solutions for the tip region

$$\frac{E'}{2\pi^3 \mu H} \frac{d\bar{w}^4}{d\hat{x}} = V\bar{w} + 2C' \sqrt{V\hat{x}},$$

$$\bar{w}(0) = \frac{\sqrt{\pi H} K_{Ic}}{E'}.$$



### Limiting solutions

$$\bar{w}_k = \frac{\sqrt{\pi H} K_{Ic}}{E'}, \quad \bar{w}_m = \left( \frac{3\pi^3 \mu H V}{2E'} \right)^{1/3} \hat{x}^{1/3}, \quad \bar{w}_{\tilde{m}} = \left( \frac{8\pi^3 \mu H C' V^{1/2}}{3E'} \right)^{1/4} \hat{x}^{3/8},$$

Toughness

Viscosity

Leak-off

### Exact edge solutions

$$\bar{w}_{km} = (\bar{w}_k^3 + \bar{w}_m^3)^{1/3}, \quad \bar{w}_{k\tilde{m}} = (\bar{w}_k^4 + \bar{w}_{\tilde{m}}^4)^{1/4}.$$

I

Less accurate, but simple approximation

$$\bar{w}_{m\tilde{m}k} = (\bar{w}_k^p + \bar{w}_m^p + \bar{w}_{\tilde{m}}^p)^{1/p}, \quad p = 3.4,$$

More accurate approximation

$$\bar{w}_{m\tilde{m}k} = \left[ w_{km} (\bar{w}_{km}^4 + \bar{w}_{\tilde{m}}^4)^{1/4} + w_{k\tilde{m}} (\bar{w}_{k\tilde{m}}^3 + \bar{w}_m^3)^{1/3} \right] [w_{km} + w_{k\tilde{m}}]^{-1},$$

# Parametric space

Original equation

$$\frac{E'}{2\pi^3 \mu H} \frac{d\bar{w}^4}{d\hat{x}} = V\bar{w} + 2C'\sqrt{V\hat{x}},$$

$$\bar{w}(0) = \frac{\sqrt{\pi H} K_{Ic}}{E'}.$$

$$\Omega = \frac{\bar{w}}{\bar{w}_k} = \frac{E' \bar{w}}{(\pi H)^{1/2} K_{Ic}}, \quad \xi = \frac{\pi^{3/2} \mu V E'^2 \hat{x}}{2 K_{Ic}^3 H^{1/2}}, \quad \chi = \left( \frac{8 C'^2 K_{Ic}}{\pi^{5/2} \mu H^{1/2} V^2} \right)^{1/2}.$$

Scaling



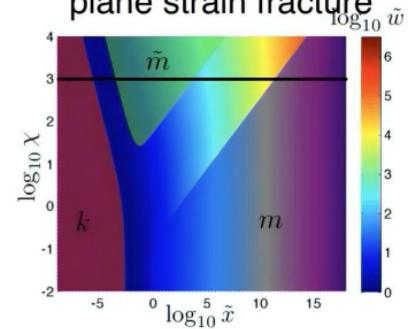
Dimensionless equation

$$\frac{d\Omega}{d\xi} = \frac{1}{\Omega^2} + \frac{\chi \xi^{1/2}}{\Omega^3}, \quad \Omega(0) = 1.$$

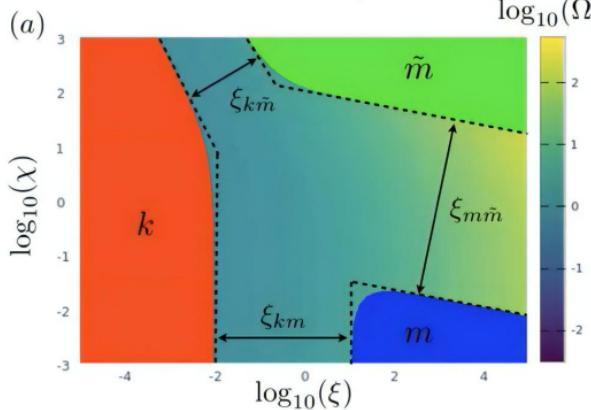
Vertex solutions

$$\Omega_k = 1, \quad \Omega_m = (3\xi)^{1/3}, \quad \Omega_{\tilde{m}} = \left( \frac{8\chi}{3} \right)^{1/4} \xi^{3/8},$$

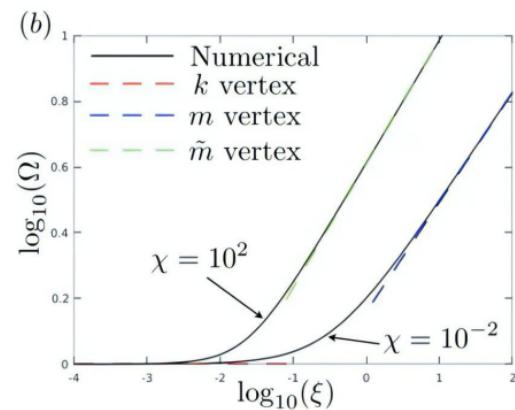
Recall the map for semi-infinite plane strain fracture



Parametric space



Solution cross-section



Transition zones

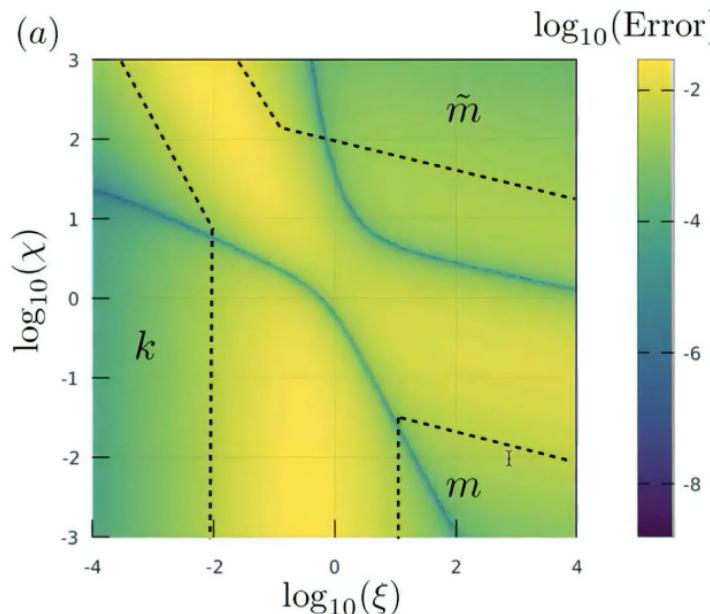
$$\begin{aligned} \xi_{km} &= \xi, & \xi_{km,1} &\approx 0.010, & \xi_{km,2} &\approx 11, \\ \xi_{k\tilde{m}} &= \xi \chi^{2/3}, & \xi_{k\tilde{m},1} &\approx 0.062, & \xi_{k\tilde{m},2} &\approx 4.4, \\ \xi_{m\tilde{m}} &= \xi \chi^6, & \xi_{m\tilde{m},1} &\approx 1.85 \times 10^{-8}, & \xi_{m\tilde{m},2} &\approx 2.1 \times 10^{12}. \end{aligned}$$

# Accuracy of the approximations

Less accurate, but simple approximation

$$\bar{w}_{m\tilde{m}k} = (\bar{w}_k^p + \bar{w}_m^p + \bar{w}_{\tilde{m}}^p)^{1/p}, \quad p = 3.4,$$

(a)

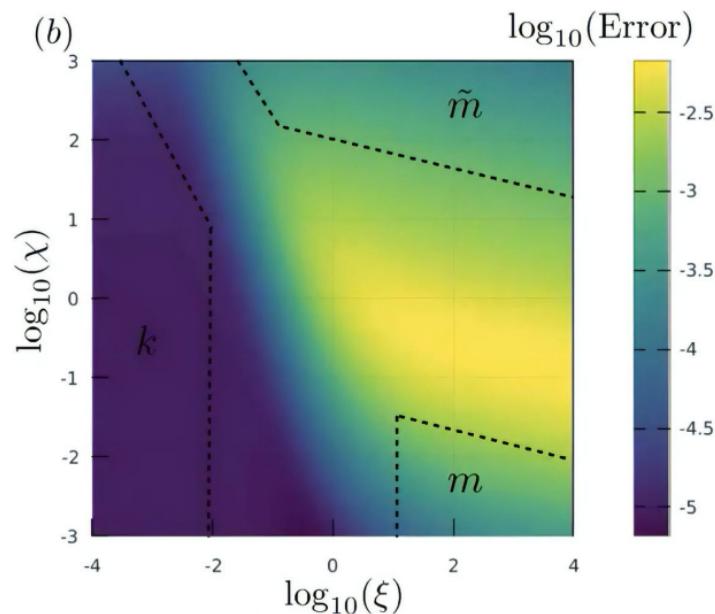


Max error ~3%

More accurate approximation

$$\bar{w}_{m\tilde{m}k} = [w_{km}(\bar{w}_{km}^4 + \bar{w}_{\tilde{m}}^4)^{1/4} + w_{k\tilde{m}}(\bar{w}_{k\tilde{m}}^3 + \bar{w}_m^3)^{1/3}][w_{km} + w_{k\tilde{m}}]^{-1},$$

(b)



Max error <1%

# Vertex solutions for the finite fracture: storage viscosity

## Governing equations

$$\frac{\partial \bar{w}}{\partial t} - \frac{E'}{2\pi^3 \mu H} \frac{\partial^2 \bar{w}^4}{\partial x^2} = \frac{Q_0}{H} \delta(x), \quad \int_0^l \bar{w}(x) dx = \frac{Q_0 t}{2H}, \quad \bar{w}(l) = 0.$$

$$\frac{1}{4}(1-\xi)^{1/3}f(\xi) - \xi \frac{d(1-\xi)^{1/3}f(\xi)}{d\xi} - \frac{3}{4} \frac{d^2(1-\xi)^{4/3}(f(\xi))^4}{d\xi^2} = 0,$$

$$f|_{\xi=1} = 1.$$

$$f(\xi) \approx 1 - \frac{1-\xi}{96}. \quad \text{Correction is very small!}$$

## Explicitly incorporate tip solution

$$\bar{w} = \underbrace{\left( \frac{3\pi^3 \mu H}{2E'} \right)^{1/3} (\dot{l})^{1/3} l^{1/3} (1-\xi)^{1/3} f(\xi)}_{\text{Tip solution}}, \quad \xi = \frac{x}{l(t)},$$

Correction

$$\left( \frac{3\pi^3 \mu H}{2E'} \right)^{1/3} (\dot{l})^{1/3} l^{4/3} \int_0^1 (1-\xi)^{1/3} f(\xi) d\xi = \frac{Q_0 t}{2H}. \longrightarrow l = c \left( \frac{5E' Q_0^3}{48\pi^3 \mu H^4} \right)^{1/5} t^{4/5},$$

$$c^{5/3} \int_0^1 (1-\xi)^{1/3} f(\xi) d\xi = 1.$$

## Summary of the solution

$$\begin{aligned} \bar{w}_M &= 1.76 \left( \frac{\mu Q_0^2}{E' H} \right)^{1/5} t^{1/5} (1-\xi)^{1/3} \left( 1 - \frac{1-\xi}{96} \right), \\ p_M &= 1.12 \left( \frac{\mu E'^4 Q_0^2}{H^6} \right)^{1/5} t^{1/5} (1-\xi)^{1/3} \left( 1 - \frac{1-\xi}{96} \right), \\ l_M &= 0.38 \left( \frac{E' Q_0^3}{\mu H^4} \right)^{1/5} t^{4/5}. \end{aligned}$$

# Vertex solutions for the finite fracture: leak-off viscosity

## Governing equations

$$-\frac{E'}{2\pi^3\mu H} \frac{\partial^2 \bar{w}^4}{\partial x^2} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H} \delta(x), \quad 2C' \int_0^l \sqrt{t-t_0(x)} dx = \frac{Q_0 t}{2H}, \quad \bar{w}(l) = 0.$$



$$l \propto t^\alpha \quad t_0(x) = t(x/l)^{1/\alpha}$$

$$2C't^{1/2}l \int_0^1 \sqrt{1-\xi^{1/\alpha}} d\xi = \frac{Q_0 t}{2H}. \xrightarrow{\alpha=1/2} l = \frac{Q_0 t^{1/2}}{\pi C' H}.$$

$$-\frac{E'}{2\pi^3\mu H} \frac{\partial^2 \bar{w}^4}{\partial x^2} + \frac{C'}{\sqrt{t}\sqrt{1-(x/l)^2}} = \frac{Q_0}{H} \delta(x), \quad \bar{w}(l) = 0,$$



$$\bar{w} = \left( \frac{2\pi\mu Q_0^2}{E'C'H} \right)^{1/4} t^{1/8} g(\xi), \quad g(\xi) = \left[ \xi \left( \sin^{-1}(\xi) - \frac{\pi}{2} \right) + \sqrt{1-\xi^2} \right]^{1/4}, \quad \xi = \frac{x}{l}.$$



$$g(\xi) \approx \left( \frac{8}{9} \right)^{1/8} \underline{(1-\xi)^{3/8}} \left[ 1 - \left( 1 - \left( \frac{9}{8} \right)^{1/8} \right) (1-\xi) \right].$$

**Tip solution**

**Correction**

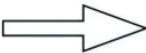
Correction is very small!

## I Summary of the solution

$$\begin{aligned} \bar{w}_{\tilde{M}} &= \left( \frac{2\pi\mu Q_0^2}{E'C'H} \right)^{1/4} t^{1/8} g(\xi), \\ p_{\tilde{M}} &= \left( \frac{32\mu E'^3 Q_0^2}{\pi^3 C' H^5} \right)^{1/4} t^{1/8} g(\xi), \\ l_{\tilde{M}} &= \frac{Q_0 t^{1/2}}{\pi C' H}. \end{aligned}$$

# Vertex solutions for the finite fracture: storage and leak-off toughness

No viscosity implies no pressure gradient



Width is spatially constant and is determined by the boundary condition at the tip

Storage toughness

$$\int_0^l \bar{w}(x) dx = \frac{Q_0 t}{2H}, \quad \bar{w}(l) = \frac{\sqrt{\pi H} K_{Ic}}{E'}.$$

Summary of the solution

$$\bar{w}_K = \frac{K_{Ic} \sqrt{\pi H}}{E'},$$

$$p_K = \frac{2K_{Ic}}{\sqrt{\pi H}},$$

$$l_K = \frac{E' Q_0 t}{\sqrt{4\pi} K_{Ic} H^{3/2}}.$$

Leak-off toughness

$$\int_0^l 2C' \sqrt{t - t_0(x)} dx = \frac{Q_0 t}{2H}, \quad \bar{w}(l) = \frac{\sqrt{\pi H} K_{Ic}}{E'}.$$

$$\downarrow \quad l \propto t^\alpha \quad t_0(x) = t(x/l)^{1/\alpha}$$

$$2C' t^{1/2} l \int_0^1 \sqrt{1 - \xi^{1/\alpha}} d\xi = \frac{Q_0 t}{2H}. \xrightarrow{\alpha = 1/2} l = \frac{Q_0 t^{1/2}}{\pi C' H}.$$

Summary of the solution

$$\bar{w}_{\tilde{K}} = \frac{K_{Ic} \sqrt{\pi H}}{E'},$$

$$p_{\tilde{K}} = \frac{2K_{Ic}}{\sqrt{\pi H}},$$

$$l_{\tilde{K}} = \frac{Q_0 t^{1/2}}{\pi C' H}.$$

I

## Fast approximate solution for a finite PKN fracture

We saw earlier that all the vertex solutions are approximated accurately by the tip solution. Therefore, the solution is sought in the following form:

$$\bar{w} = \bar{w}_a(1-\xi)^\delta, \quad \bar{w}_a = \bar{w}_{m\tilde{m}k}(x=l, V=\dot{l}) \quad \delta = \frac{d \log(\bar{w}_{m\tilde{m}k})}{d \log(\hat{x})}, \quad \xi = \frac{x}{l},$$

Full solution      Tip asymptote      Spatial behavior from asymptote

I

Global volume balance

$$\int_0^l [\bar{w}(x) + 2C' \sqrt{t - t_0(x)}] dx = \frac{Q_0 t}{2H}.$$

$$l \propto t^\alpha \quad t_0(x) = t(x/l)^{1/\alpha}$$

$$\frac{\bar{w}_a(l, \alpha l/t)l}{1+\delta} + \sqrt{\pi} C' t^{1/2} l \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})} = \frac{Q_0 t}{2H},$$

$$\delta = \frac{d \log(\bar{w}_{m\tilde{m}k})}{d \log(\hat{x})}, \quad \alpha = \frac{d \log(l)}{d \log(t)},$$

This system of equations can be solved quickly

# Scaling

Original system of equations

$$\frac{\partial \bar{w}}{\partial t} - \frac{E'}{2\pi^3 \mu H} \frac{\partial^2 \bar{w}^4}{\partial x^2} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H} \delta(x),$$

$$\bar{w}(l) = \frac{\sqrt{\pi H} K_{Ic}}{E'}.$$

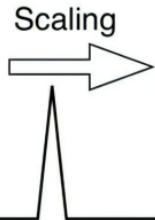
$$\frac{dl}{dt} = \frac{\bar{q}_x(l)}{\bar{w}(l)} = -\frac{2E'}{3\pi^3 \mu H} \frac{\partial \bar{w}^3}{\partial x} \Big|_{x=l},$$

Dimensionless system of equations

$$\frac{\partial \Omega}{\partial \tau} - \frac{\xi \dot{\lambda}}{\lambda} \frac{\partial \Omega}{\partial \xi} - \frac{1}{\lambda^2} \frac{\partial^2 \Omega^4}{\partial \xi^2} + \frac{\phi}{\sqrt{\tau-\tau_0(\xi)}} = \delta(\xi),$$

$$\Omega(1) = 1, \quad \text{I}$$

$$\frac{d\lambda}{d\tau} = -\frac{4}{3\lambda} \frac{\partial \Omega^3}{\partial \xi} \Big|_{\xi=1}.$$

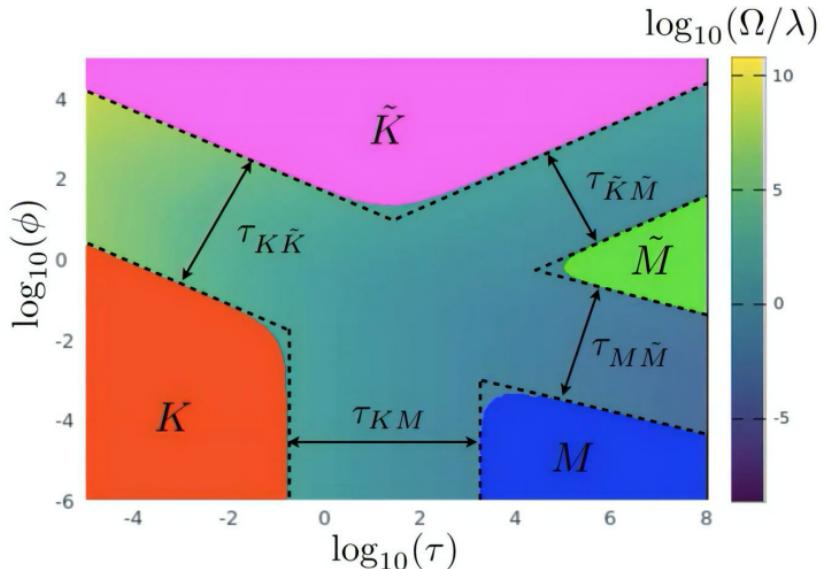


$\Omega = \frac{\bar{w}}{w_*},$	$\lambda = \frac{l}{l_*},$	$\tau = \frac{t}{t_*},$	$\xi = x/l(t)$	$\phi = \left( \frac{H^5 K_{Ic}^6 C'^4}{4\pi^3 E'^4 \mu^2 Q_0^4} \right)^{1/4}.$
$w_* = \frac{(\pi H)^{1/2} K_{Ic}}{E'},$	$l_* = \frac{H^2 K_{Ic}^4}{2\pi E'^3 \mu Q_0},$	$t_* = \frac{H^{7/2} K_{Ic}^5}{2\pi^{1/2} E'^4 \mu Q_0^2}.$		

The dimensionless system of equations can be solved numerically using standard finite difference technique.

# Parametric space

- Parametric space is computed using the fast solution
- Zones of applicability of the vertex solutions are indicated
- The zone boundaries are quantified



$$\tau = \frac{2\pi^{1/2} E'^4 \mu Q_0^2 t}{H^{7/2} K_{Ic}^5} \quad \phi = \left( \frac{H^5 K_{Ic}^6 C'^4}{4\pi^3 E'^4 \mu^2 Q_0^4} \right)^{1/4}.$$

I

$M$  - storage-viscosity

No toughness, no leak-off

$K$  - storage-toughness

No viscosity, no leak-off

$\tilde{M}$  - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$  - leak-off-toughness

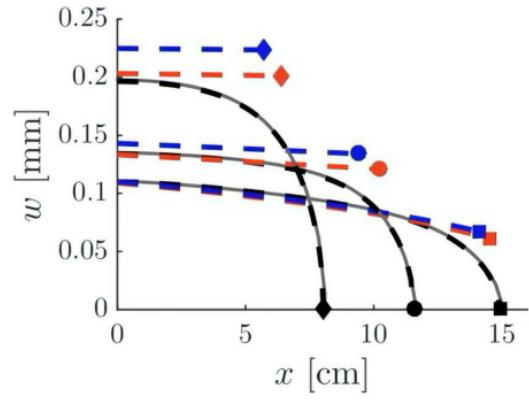
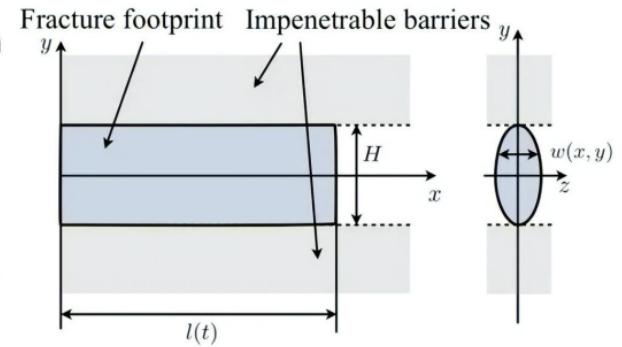
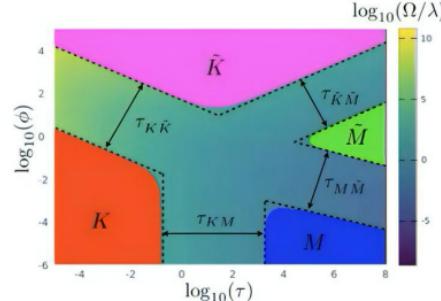
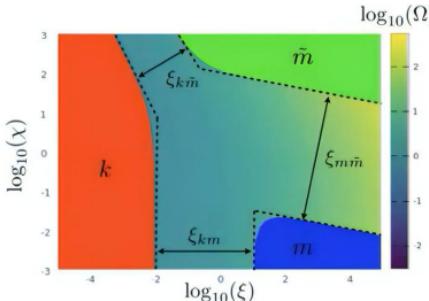
No viscosity, large leak-off

## Transitions

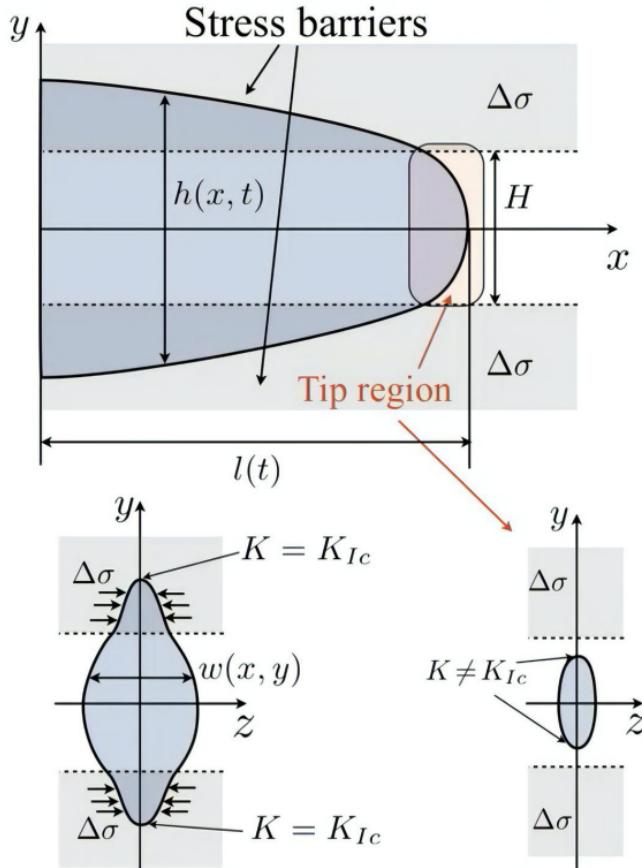
$$\begin{aligned}
 \tau_{MK} &= \tau, & \tau_{MK,1} &= 0.16, & \tau_{MK,2} &= 1.6 \times 10^3, \\
 \tau_{K\tilde{K}} &= \tau\phi^2, & \tau_{K\tilde{K},1} &= 5.6 \times 10^{-5}, & \tau_{K\tilde{K},2} &= 2.8 \times 10^3, \\
 \tau_{\tilde{K}\tilde{M}} &= \tau\phi^{-2}, & \tau_{\tilde{K}\tilde{M},1} &= 0.18, & \tau_{\tilde{K}\tilde{M},2} &= 6.6 \times 10^4, \\
 \tau_{M\tilde{M}} &= \tau\phi^{10/3}, & \tau_{M\tilde{M},1} &= 2.3 \times 10^{-7}, & \tau_{M\tilde{M},2} &= 2.8 \times 10^3.
 \end{aligned}$$

# Things to remember

- Assumptions of PKN model, including constant height, length  $\gg$  height, horizontal flux, constant pressure in each vertical cross-section, elliptical width in each vertical cross-section
- There are two approaches to solve the problem:
  - Use non-local elasticity, which is good for numerical scheme and leads to superior accuracy
  - Use local elasticity with a specific boundary condition at the tip, which is less accurate, but easier for analysis
- PKN fracture has its own formulation for the tip region, which is different from that for a semi-infinite plane strain fracture, approximate solution exists
- Parametric space for the finite fracture is evaluated
- Limiting vertex solutions exist, as well as the global approximate solution

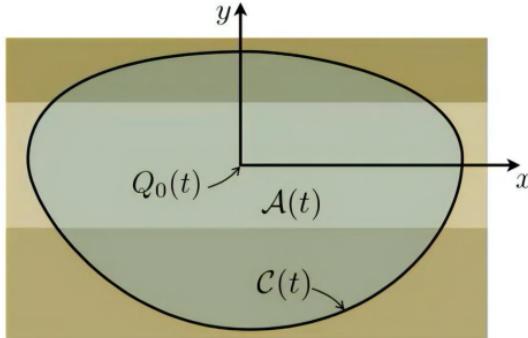


# Enhanced pseudo-3D (EP3D) model



- Extension of PKN model that allows height growth
- Symmetric stress barriers
- Fluid pressure is constant in each vertical cross-section, i.e.  $p=p(x)$  (toughness dominated propagation with toughness at the vertical tips controlling height growth)
- Fracture opening and height in  $(y, z)$  cross-sections:
  - Plane strain analytic solution outside of the tip region  $h>H$
  - **Radial solution in the tip region**
- **Non-local elasticity**
$$p(x) = \int_{-l(t)}^{l(t)} G(x, x') \bar{w}(x') dx'$$
- **Asymptotic solution at the tip**
- **Viscous height growth (see paper Dontsov&Peirce, 2015)**
- Solution with local elasticity, without viscous height growth, without radial solution at the tip, and without using the tip asymptote corresponds to classical P3D model

# Governing equations



Scaling       $E' = \frac{E}{1-\nu^2}, \quad \mu' = 12\mu, \quad K' = \sqrt{\frac{32}{\pi}} K_{Ic}, \quad C' = 2C_l,$

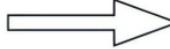
Apply vertical averaging

$$\bar{w}(x) = \frac{1}{H} \int_{-H/2}^{H/2} w(x, y) dy,$$

We also need  $h(\bar{w})$

Volume balance       $\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{q} + \frac{C'}{\sqrt{t-t_0(x,y)}} = Q_0(t)\delta(x,y),$

Fluid flow       $\mathbf{q} = -\frac{w^3}{\mu'} \nabla p,$

Elasticity       $p(x, y, t) = \sigma_0(y) - \frac{E'}{8\pi} \int_{A(t)} \frac{w(x', y', t) dx' dy'}{[(x' - x)^2 + (y' - y)^2]^{3/2}},$  

Propagation       $\lim_{s \rightarrow 0} \frac{w}{s^{1/2}} = \begin{cases} \frac{K'}{E'}, & \text{if } V > 0, \\ \frac{K'_l}{E'}, & \text{if } V = 0. \end{cases}$

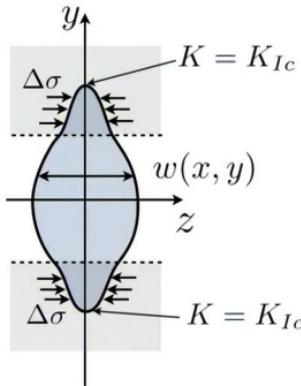
$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}_x}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H} \delta(x),$$

$$\bar{q}_x = -\frac{1}{H\mu'} \frac{\partial p}{\partial x} \int_{-h/2}^{h/2} w^3 dy,$$

$$p(x) = \int_{-l(t)}^{l(t)} G(x, x') \bar{w}(x') dx'$$

$$\lim_{s \rightarrow 0} \frac{w}{s^{1/2}} = \frac{K'}{E'}$$

# Width solution



Interior (plane strain solution for a uniformly pressurized fracture)

$$w(x, y) = \frac{2}{E'} \sqrt{\frac{2}{\pi h}} K_{Ic} \chi + \frac{4\Delta\sigma}{\pi E'} \left\{ -y \log \left| \frac{H\chi + 2y\psi}{H\chi - 2y\psi} \right| + \frac{H}{2} \log \left| \frac{\chi + \psi}{\chi - \psi} \right| \right\},$$

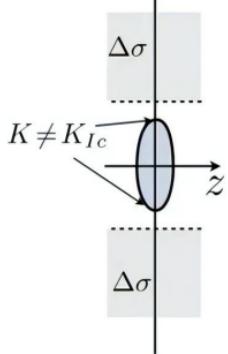
Averaging

$$\bar{w}(x) = \frac{1}{H} \int_{-H/2}^{H/2} w(x, y) dy,$$

$$\chi = \sqrt{h^2 - 4y^2}, \psi = \sqrt{h^2 - H^2},$$

$$\bar{w} = \frac{H}{E'} \left( \sqrt{\frac{\pi}{2H}} K_{Ic} \left( \frac{h}{H} \right)^{3/2} + \Delta\sigma \sqrt{\frac{h^2}{H^2} - 1} \right), \quad h > H,$$

I



Tip (radial solution in toughness regime)

$$w_{\text{rad}} = \frac{4K_{Ic}}{\sqrt{\pi l} E'} \sqrt{l^2 - x^2 - y^2},$$

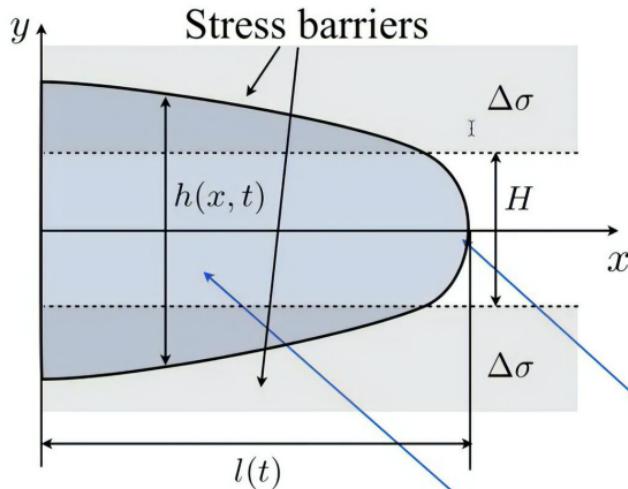
Averaging

$$\bar{w}_{\text{rad}} = \frac{1}{H} \int_{-h_{\text{rad}}/2}^{h_{\text{rad}}/2} w_{\text{rad}} dy = \frac{\pi K_{Ic}}{2H E' \sqrt{\pi l}} h_{\text{rad}}^2,$$

$$h_{\text{rad}} = 2\sqrt{l^2 - x^2}.$$

$$h_{\text{rad}} = \bar{w}_{\text{rad}}^{1/2} (2l)^{1/4} \sqrt{\frac{\sqrt{2} H E'}{\sqrt{\pi} K_{Ic}}}. \quad w_{\text{rad}} = \frac{4H}{\pi h_{\text{rad}}} \bar{w}_{\text{rad}} \sqrt{1 - \left( \frac{2y}{h_{\text{rad}}} \right)^2},$$

# Non-local elasticity



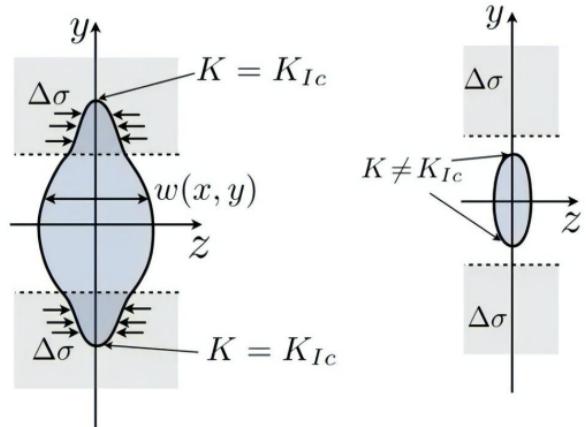
Plane strain (local) elasticity

$$p_{P3D} = \sqrt{\frac{2}{\pi h}} K_{IC} + \Delta\sigma \left( 1 - \frac{2}{\pi} \arcsin\left(\frac{H}{h}\right) \right)$$

KGD

Non-local elasticity

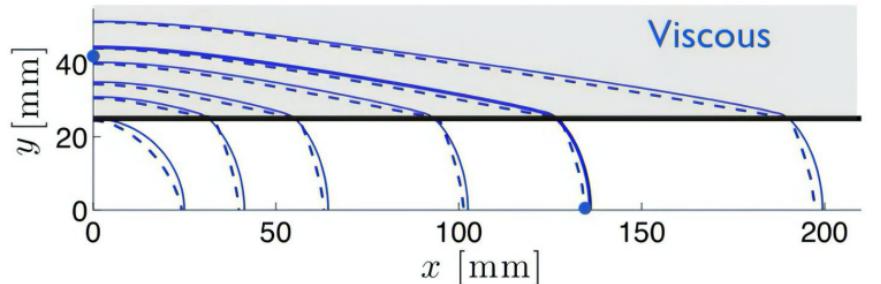
$$p(x) = -\frac{E'}{8\pi} \int_{-l(t)}^{l(t)} \int_{-\frac{1}{2}h(x',t)}^{\frac{1}{2}h(x',t)} \frac{w(x', y') dy' dx'}{((x'-x)^2 + y'^2)^{3/2}} = \int_{-l(t)}^{l(t)} G(x, x') \bar{w}(x') dx'$$



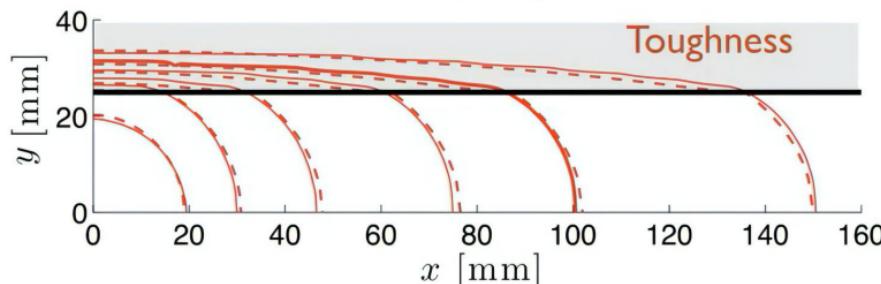
$$w(x, y) = \frac{2}{E'} \sqrt{\frac{2}{\pi h}} K_{Ic} \chi + \frac{4\Delta\sigma}{\pi E'} \left\{ -y \log \left| \frac{H\chi + 2y\psi}{H\chi - 2y\psi} \right| + \frac{H}{2} \log \left| \frac{\chi + \psi}{\chi - \psi} \right| \right\},$$

For faster evaluation, the width is approximated by two ellipses and the integral is computed analytically

# Enhanced P3D (EP3D) vs. ILSA

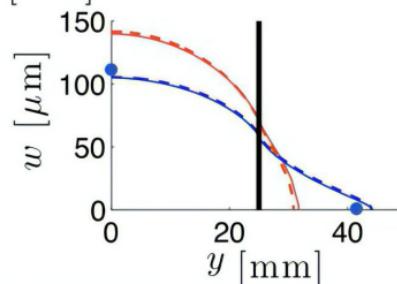
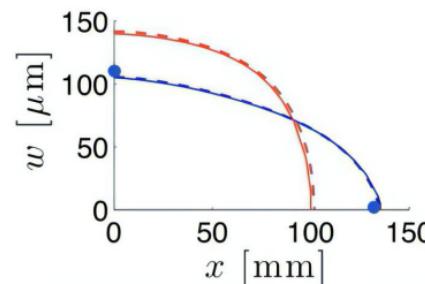


—  $K' = 0$  ILSA  
 .....  $K' = 0$  EP3D

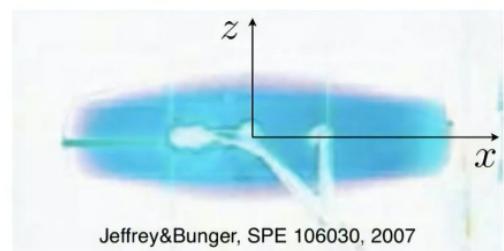


— ILSA  
 $K' = 3.0 \text{ MPa}\cdot\text{m}^{1/2}$

..... EP3D  
 $K' = 3.0 \text{ MPa}\cdot\text{m}^{1/2}$

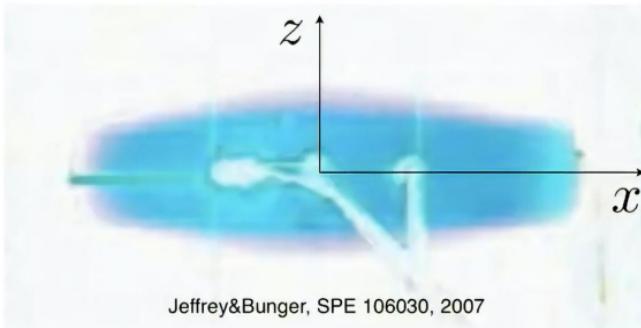


● Experiment  $K' = 0$

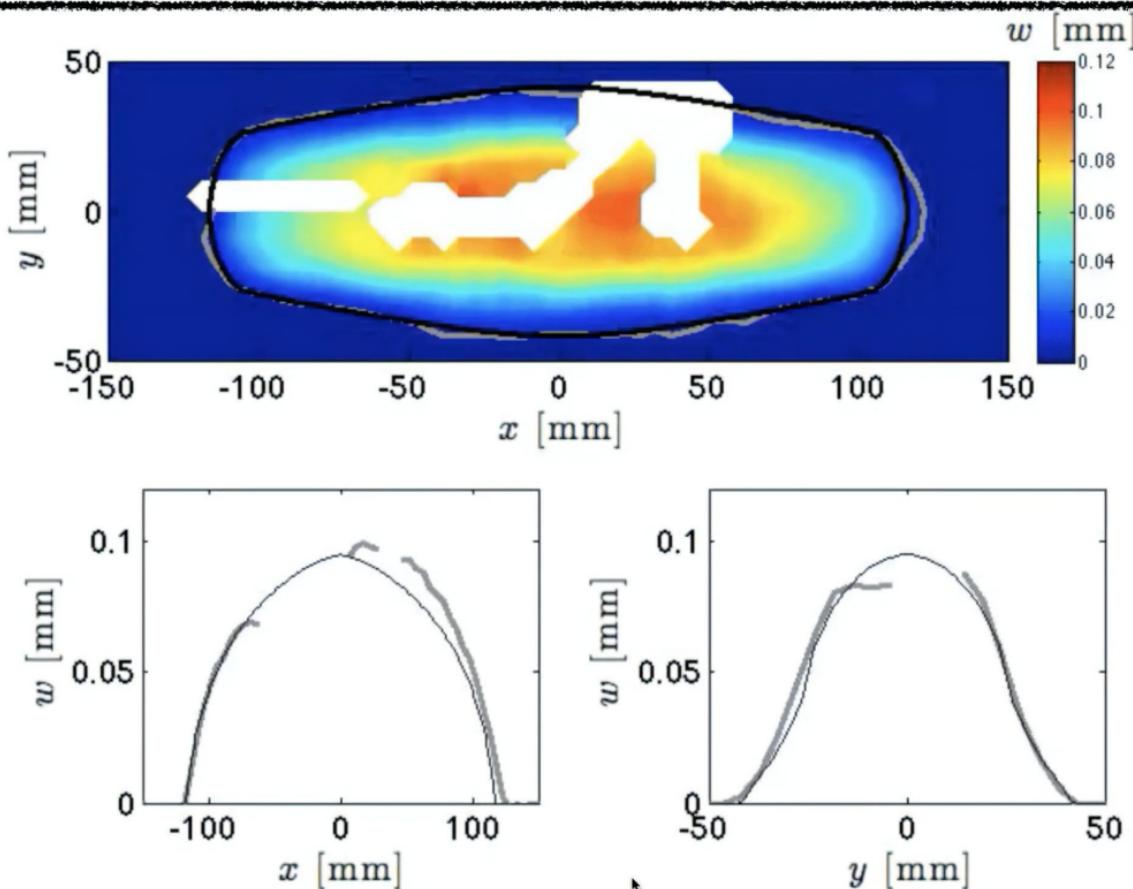


## Comparison with experiment

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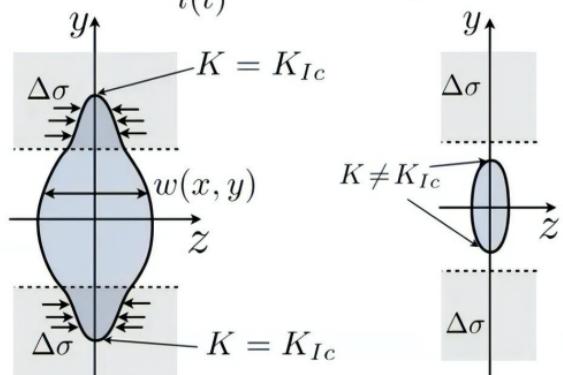
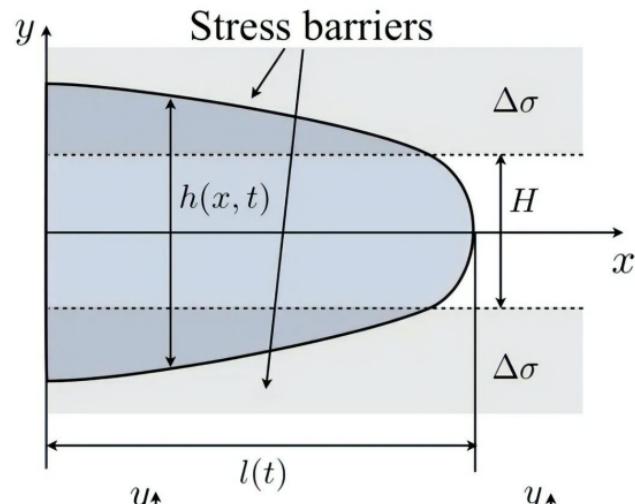
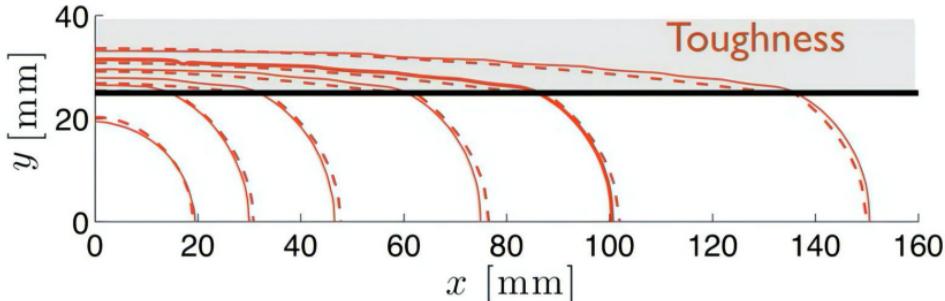


## Comparison with experiment



## Things to remember

- Assumptions of P3D model, including equilibrium height growth model, length  $\gg$  height, horizontal flux, constant pressure in each vertical cross-section, plane strain solution for width in each vertical cross-section
- Non-local elasticity and other corrections allow to significantly improve accuracy of the model
- There are extensions for asymmetric stress layers as well as multiple layers
- This model is suited more for numerical calculations, rather than analysis. However, analysis of the classical P3D model can be found in Adachi et. al, 2010.



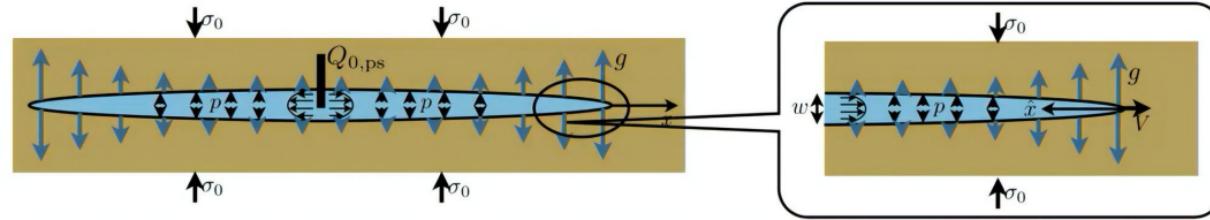
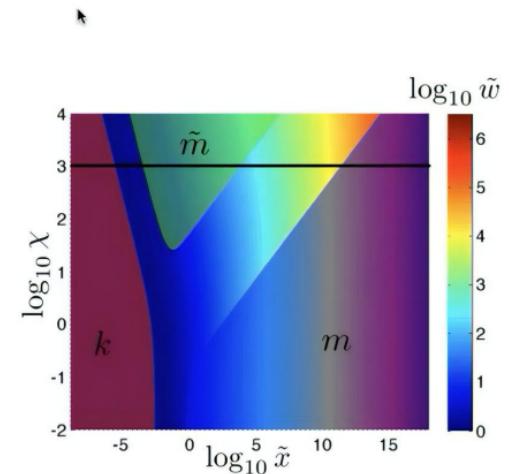
# Recall main learnings: fundamentals and semi-infinite fracture

- Work towards a balanced life and invest in your knowledge!

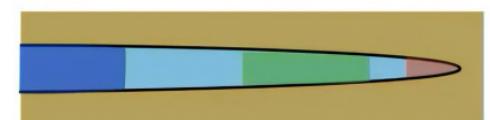
An investment in knowledge pays the best interest.

-Benjamin Franklin

- Essential pieces of a hydraulic fracture model: volume balance, fluid flow, elasticity, propagation, proppant transport
- Derivation of the governing equations for planar and plane strain fracture geometries
- Semi-infinite hydraulic fracture as a model for the tip region
  - There are three limiting analytic solutions: toughness, viscosity, and leak-off
  - The global solution gradually transitions from one limiting case to another
  - There is computationally efficient approximate solution for the problem that can be used as a propagation condition for finite fractures



$$w \propto \hat{x}^{2/3} \quad w \propto \hat{x}^{5/8} \quad w \propto \hat{x}^{1/2}$$



## Recall main learnings: plane strain fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote
- The existence of approximate solution constructed using global volume balance and tip asymptote
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless toughness and dimensionless time

$M$  - storage-viscosity

No toughness, no leak-off

$K$  - storage-toughness

No viscosity, no leak-off

$\tilde{M}$  - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$  - leak-off-toughness

No viscosity, large leak-off

Viscosity

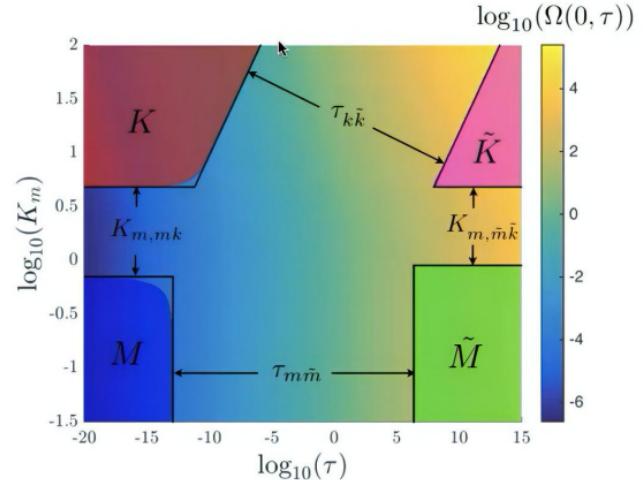
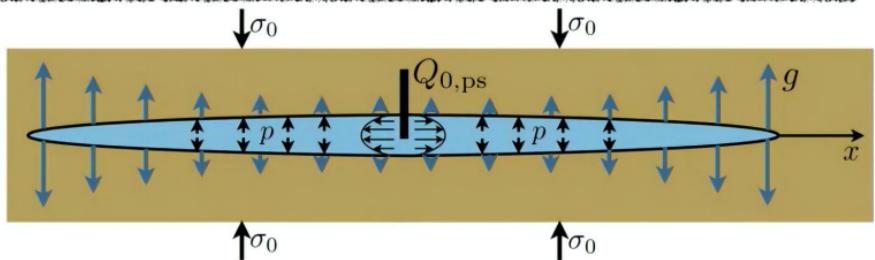
$$w_m = \beta_m \left( \frac{\mu' V}{E'} \right)^{1/3} \hat{x}^{2/3}$$

Toughness

$$w_k = \frac{K'}{E'} \hat{x}^{1/2}$$

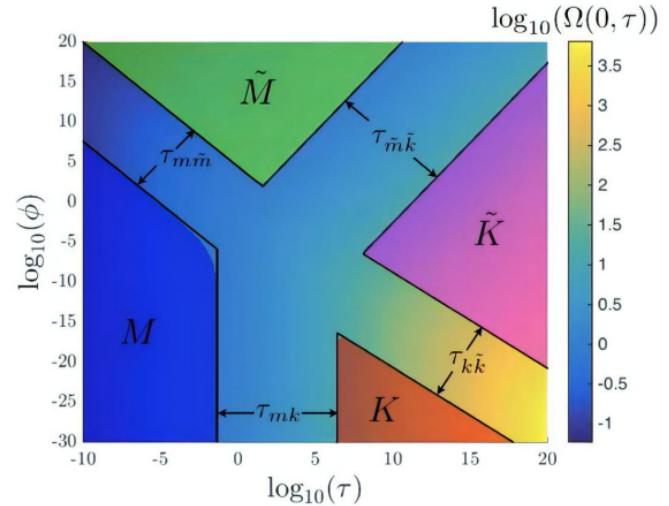
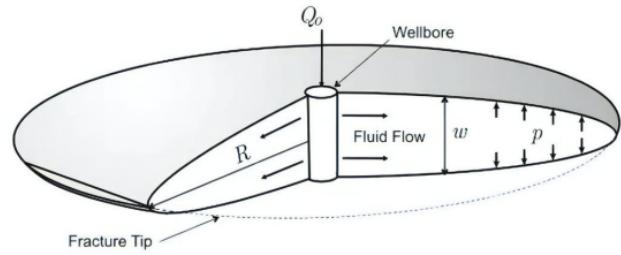
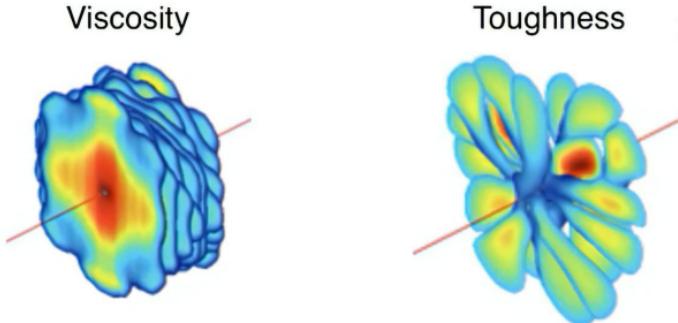
Leak-off

$$w_{\bar{m}} = \beta_{\bar{m}} \left( \frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} \hat{x}^{5/8}$$



## Recall main learnings: radial fracture

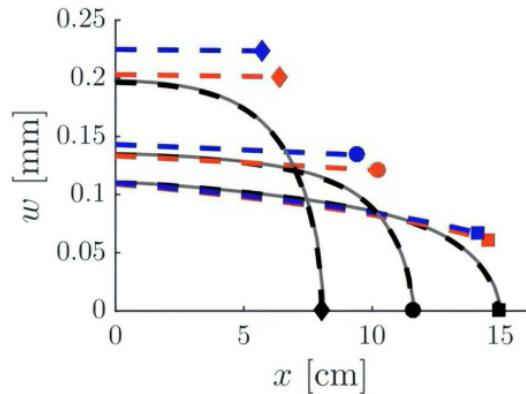
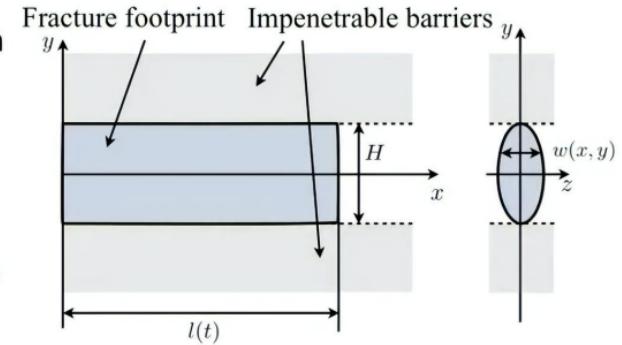
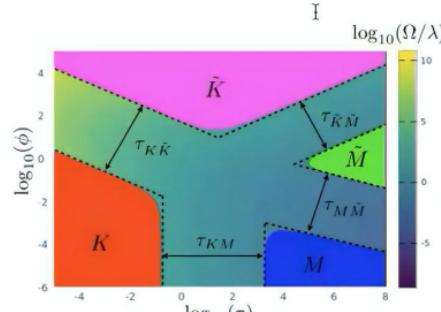
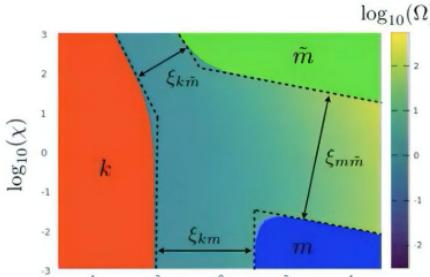
- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote (the same as for plane strain)
- The existence of approximate solution constructed using global volume balance and tip asymptote (similar to plane strain)
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless leak-off and dimensionless time
- Fracture regimes affect morphology of multiple hydraulic fractures



E.V. Dontsov. An approximate solution for a penny-shaped hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *R. Soc. open sci.*, 3:160737, 2016.

## Recall main learnings: PKN fracture

- Assumptions of PKN model, including constant height, length  $\gg$  height, horizontal flux, constant pressure in each vertical cross-section, elliptical width in each vertical cross-section
- There are two approaches to solve the problem:
  - Use non-local elasticity, which is good for numerical scheme and leads to superior accuracy
  - Use local elasticity with a specific boundary condition at the tip, which is less accurate, but easier for analysis
- PKN fracture has its own formulation for the tip region, which is different from that for a semi-infinite plane strain fracture, approximate solution exists
- Parametric space for the finite fracture is evaluated
- Limiting vertex solutions exist, as well as the global approximate solution



## Recall main learnings: pseudo-3D fracture

- Assumptions of P3D model, including equilibrium height growth model, length  $\gg$  height, horizontal flux, constant pressure in each vertical cross-section, plane strain solution for width in each vertical cross-section
- Non-local elasticity and other corrections allow to significantly improve accuracy of the model
- There are extensions for asymmetric stress layers as well as multiple layers
- This model is suited more for numerical calculations, rather than analysis. However, analysis of the classical P3D model can be found in Adachi et. al, 2010.

