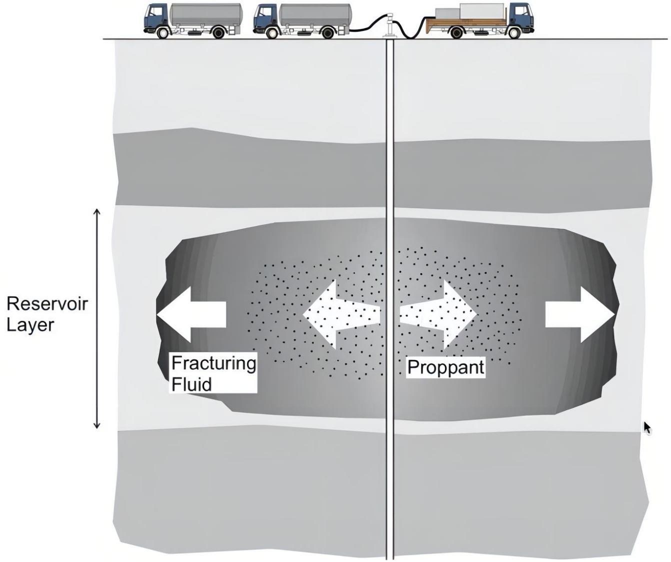
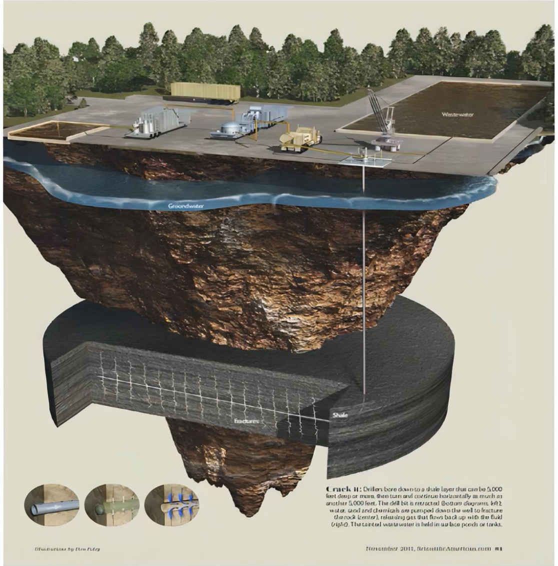
# Hydraulic fracturing (HF)

**Conventional:** vertical well, single fracture




J. Adachi, PhD thesis

**Unconventional:** horizontal well(s), multiple fractures, up to 1000s per group of wells




Scientific American, November 2011

# Essential pieces of a hydraulic fracture model

**1. Volume balance of the injected fluid (incompressible):**

Volume injected = Fracture volume + leak-off

**2. Fluid flow equations:**

Viscous pressure drop inside the fracture

**3. Rock equilibrium (elasticity):**

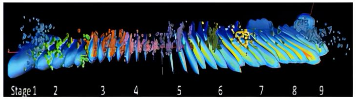
Fluid pressure = Stress + Stiffness*FracWidth

**4. Propagation condition:**

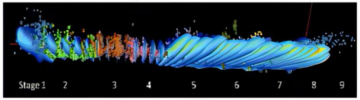
Some parameter reaches a critical value near the front

**5. Proppant transport (not covered):**

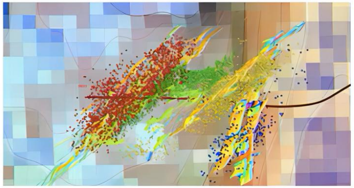
Particles flow with fluid + gravitational settling

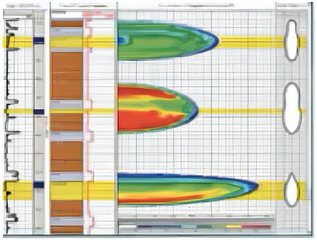



Top view

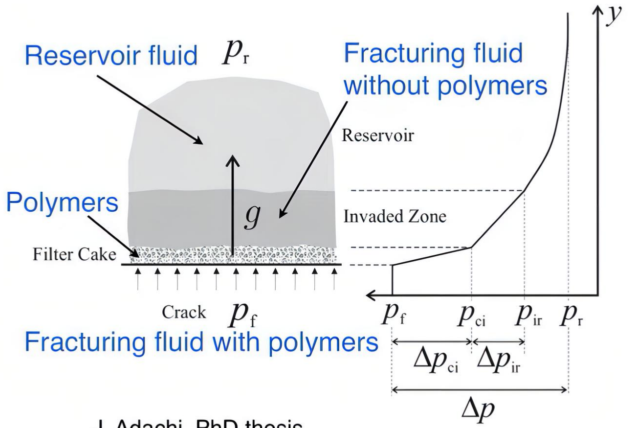



Bottom view





# Carter's leak-off model

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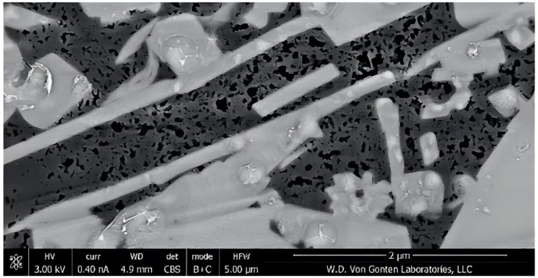



**Flow through filter cake:**

$$g_c = \alpha \frac{dh_c}{dt}$$

this equation says that the growth rate of the filter cake is linearly proportional to the amount of fluid leaked from the fracture (the constant of proportionality is related to the content of polymers)

$$g_c = \frac{\kappa_c}{\mu} \frac{\Delta p_{ci}}{h_c}$$

this is Darcy's law (quasi-static flow)

Solution: $$g_c = \frac{C_c}{\sqrt{t}} \qquad C_c = \sqrt{\frac{\alpha \kappa_c \Delta p_{ci}}{2\mu_b}}$$

**Flow through invaded zone:**

$$g_i = \frac{\kappa}{\mu_{filt}} \frac{\Delta p_{ir}}{h_i}$$

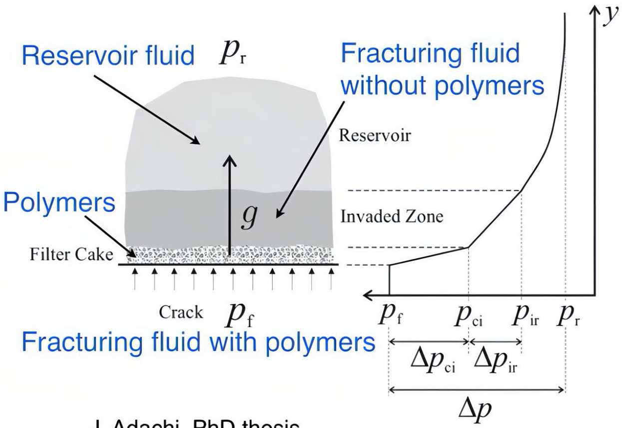
this is Darcy's law (quasi-static flow)

$$g_i = \phi \frac{dh_i}{dt}$$

this is volume balance that states that the volume of fluid leaked into the formation determines the size of the invasion zone

Solution: $$g_i = \frac{C_i}{\sqrt{t}} \qquad C_i = \sqrt{\frac{\phi \kappa_r \Delta p_{ir}}{2\mu_b}}$$

# Carter's leak-off model

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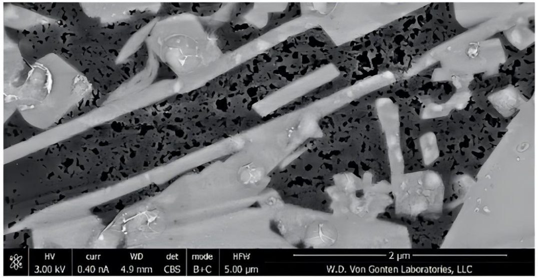



**Flow in reservoir:**

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial y^2}$$ 1D diffusion equation (volume balance + Darcy)

$$p|_{t=0} = p_r$$ initial condition

$$p|_{y=0} = p_{ir}$$ boundary condition

$$D = \frac{\kappa_r}{\mu_r c_t \phi}$$ diffusion coefficient

To solve this equation, introduce new variable:

$$\xi = \frac{y}{\sqrt{4Dt}} \Rightarrow -\frac{y}{2t\sqrt{4Dt}}p' = \frac{D}{4Dt}p'' \Rightarrow -2\xi p' = p''$$

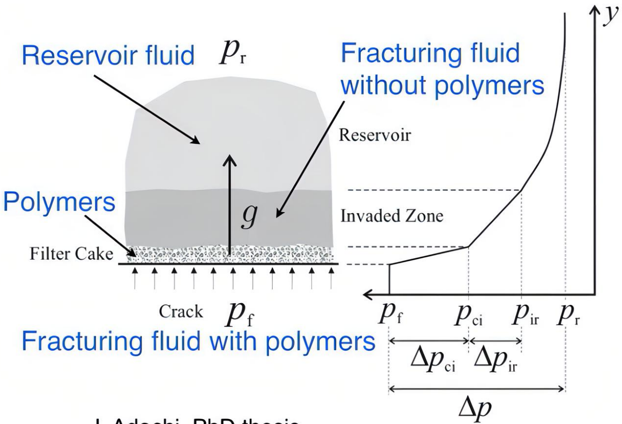
Solution of the above differential equation is:

$$p = p_r + (p_{ir} - p_r)\mathrm{erfc}(\xi) \qquad \mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

The leak-off flux is then (from Darcy):

$$g_r = -\frac{k_r}{\mu_r}\frac{\partial p}{\partial y}\bigg|_{y=0} = \frac{C_r}{\sqrt{t}} \qquad C_r = \sqrt{\frac{k_r c_t \phi}{\pi \mu_r}}(p_{ir} - p_r)$$

# Carter's leak-off model

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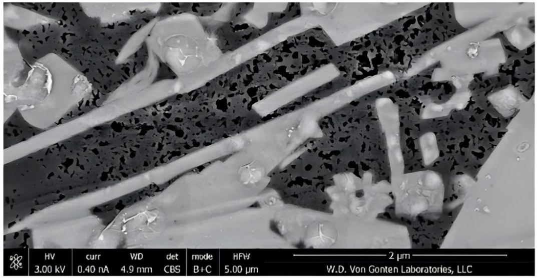



**Combined result if all the mechanisms are present:**

$$g = \frac{C_l}{\sqrt{t}}, \qquad C_l = \frac{2\bar{C}_c\bar{C}_i\bar{C}_r}{\bar{C}_c\bar{C}_i + \sqrt{\bar{C}_c^2\bar{C}_i^2 + 4\bar{C}_r^2(\bar{C}_c^2 + \bar{C}_i^2)}},$$

In the above result, the individual leak-off coefficients are computed by using the total pressure drop, i.e.

$$\bar{C}_c = \sqrt{\frac{\alpha\kappa_c\Delta p}{2\mu_b}}, \qquad \bar{C}_i = \sqrt{\frac{\phi\kappa_r\Delta p}{2\mu_b}}, \qquad \bar{C}_r = \sqrt{\frac{\kappa_r c_t\phi}{\pi\mu_r}}\Delta p.$$

**Recall the main assumptions of the model:**

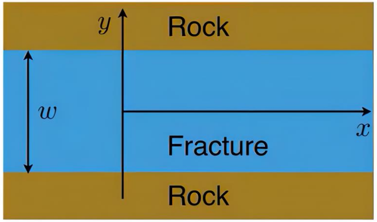
1D diffusion, i.e. the diffusion length scale should be much smaller than the fracture size

The net pressure is often neglected, whereby $\Delta p = \sigma_0 - p_r$

It is implicitly assumed that one type of fracturing fluid is used

More reading: Economides&Nolte 2000, section 6-4.

# Fluid flow




$v = v_x(y)$ given the geometry, we have only one component of the velocity vector that varies only across the channel

$\dfrac{\partial p}{\partial x} = \dfrac{\partial \tau}{\partial y}$ this comes from Navier-Stokes equations or equilibrium equations

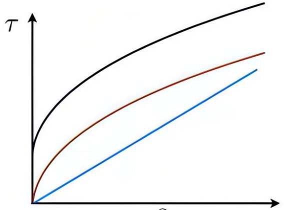
$\tau = \mu \dfrac{\partial v}{\partial y}$ this states that the rheology is Newtonian

$v|_{y=\pm w/2} = 0$ this is no-slip boundary condition at the fracture walls

Herschel-Bulkley $\tau = \tau_0 + k\dot{\gamma}^n$

Power-law $\tau = k\dot{\gamma}^n$

Newtonian $\tau = \mu\dot{\gamma}$




$$\dot{\gamma} = \frac{\partial v}{\partial y}$$

General solution:

$$v = \frac{\partial p}{\partial x}\frac{y^2}{2} + Ay + B$$

Actual solution:

$$v = -\frac{\partial p}{\partial x}\frac{w^2 - 4y^2}{8\mu}$$

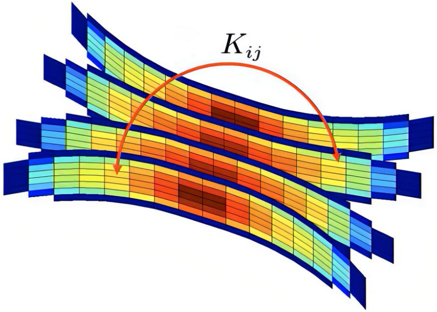
Total flux is: $q = \displaystyle\int_{-w/2}^{w/2} v(y)\,dy = -\frac{w^3}{12\mu}\frac{\partial p}{\partial x}$

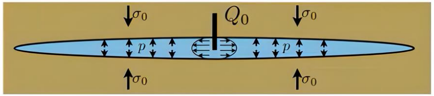
HB fluid:

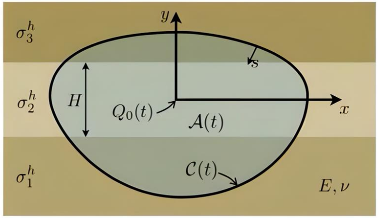
$$q_{\mathrm{HB}} = -\frac{w^{2+1/n}}{M'^{1/n}}\frac{\partial p}{\partial x}\left|\frac{\partial p}{\partial x}\right|^{1/n-1}\left(1 - \frac{2\tau_0}{w}\left|\frac{\partial p}{\partial x}\right|^{-1}\right)^{1+1/n}\left(1 + \frac{2\tau_0}{w}\left|\frac{\partial p}{\partial x}\right|^{-1}\frac{n}{n+1}\right)$$

$$M' = \frac{2^{n+1}(2n+1)^n}{n^n}k,$$

# Elasticity




Elasticity equation ensures that rock surrounding open fracture(s) is in equilibrium.

Every open element induces a stress change (all components) in the whole space.

The interaction coefficient (induced stress divided by aperture) depends on the elastic properties and the distance from the element and generally decays quickly ~1/r^3 for 3D geometry.

For a plane strain fracture, the elasticity equation reads:

$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^{l} \frac{w(s)\, ds}{(x-s)^2} \qquad E' = \frac{E}{1-\nu^2}$$

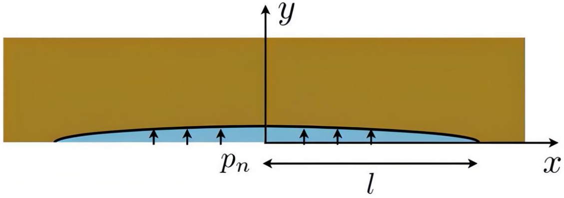
For a planar fracture, the elasticity equation reads:

$$p(x,y,t) = \sigma^h(y) - \frac{E'}{8\pi} \int_{\mathcal{A}(t)} \frac{w(x',y',t)\mathrm{d}x'\mathrm{d}y'}{\left[(x'-x)^2 + (y'-y)^2\right]^{3/2}},$$

For general expressions in 3D, see Crouch and Starfield, 1983

For expressions in layered materials, see Peirce and Siebrits, 2000

# Derivation of elasticity equation (plane strain)




## Hooke's law

$$\sigma_{xx} = 2\mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy}),$$
$$\sigma_{yy} = 2\mu\epsilon_{yy} + \lambda(\epsilon_{xx} + \epsilon_{yy}),$$
$$\sigma_{xy} = 2\mu\epsilon_{xy}$$

## Equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

## Normal stress continuity

$$\sigma_{yy}|_{y=0} = -p_n = -(p - \sigma_0)$$

## Boundary conditions

$$u_y|_{y=0} = \frac{w}{2} \qquad \sigma_{xy}|_{y=0} = 0$$

## Need to solve for

$$\sigma_{yy}|_{y=0} - ?$$

## Governing equations in terms of displacements

$$(2\mu + \lambda)\frac{\partial^2 u_x}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 u_y}{\partial x \partial y} + \mu\frac{\partial^2 u_x}{\partial y^2} = 0,$$
$$(2\mu + \lambda)\frac{\partial^2 u_y}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u_x}{\partial x \partial y} + \mu\frac{\partial^2 u_y}{\partial x^2} = 0$$

Apply Fourier transform

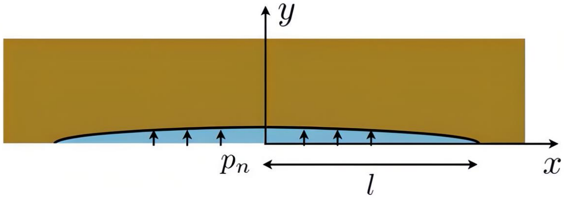
$$\mathcal{F}(g) = \hat{g}(k) = \int_{-\infty}^{+\infty} g(x)e^{-ikx}\mathrm{d}x,$$
$$\mathcal{F}\left(\frac{\partial g}{\partial x}\right) = ik\mathcal{F}(g), \quad \mathcal{F}\left(\frac{\partial^2 g}{\partial x^2}\right) = -k^2\mathcal{F}(g),$$

## System of ODEs

$$\frac{\partial \hat{u}_x}{\partial y} = \hat{d}_x,$$
$$\frac{\partial \hat{u}_y}{\partial y} = \hat{d}_y,$$
$$\frac{\partial \hat{d}_x}{\partial y} = \frac{2\mu + \lambda}{\mu}k^2\hat{u}_x - \frac{\lambda + \mu}{\mu}ik\hat{d}_x + \frac{\mu}{2\mu + \lambda}k^2\hat{u}_y,$$
$$\frac{\partial \hat{d}_y}{\partial y} = -\frac{\mu + \lambda}{2\mu + \lambda}ik\hat{d}_x + \frac{\mu}{2\mu + \lambda}k^2\hat{d}_x + \frac{\mu}{2\mu + \lambda}k^2\hat{u}_y,$$

$$\boxed{\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}}$$

# Derivation of elasticity equation (plane strain)




$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$$

Eigenvalues of $A$: $k, k, -k, -k$.

Solution (resonance)

$$\boldsymbol{Y} = c_1 \boldsymbol{v_1} e^{-|k|y} + c_2(\boldsymbol{v}_1 y + \boldsymbol{v_2}) e^{-|k|y}$$

Boundary conditions

$$\hat{\sigma}_{xy}|_{y=0} = 0,$$
$$\hat{u}_y|_{y=0} = \hat{w}(k)/2,$$

Solution in frequency domain

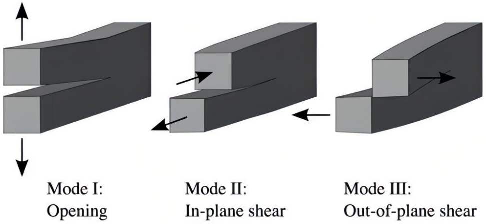
$$\hat{p}_n = \frac{\hat{w}|k|}{4}E' = \frac{ik\hat{w}E'}{4}\frac{|k|}{ik} = -\frac{iE'}{4}\operatorname{sgn}(k)\widehat{\frac{\mathrm{d}w}{\mathrm{d}x}},$$

$$\widehat{\frac{\mathrm{d}w}{\mathrm{d}x}} = ik\hat{w}$$

$$p_n(x) = -\frac{1}{2\pi}\frac{iE'}{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\operatorname{sgn}(k)\frac{\mathrm{d}w}{\mathrm{d}s}e^{ik(x-s)}\mathrm{d}s\mathrm{d}k.$$

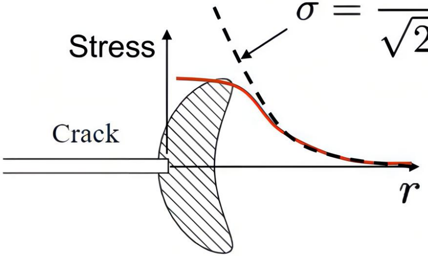
$$p = \sigma_0 - \frac{E'}{4\pi}\int_{-l}^{l}\frac{w(s)\,ds}{(x-s)^2}$$

# Propagation condition




Hydraulic fractures

$$\sigma = \frac{K_I}{\sqrt{2\pi r}}$$



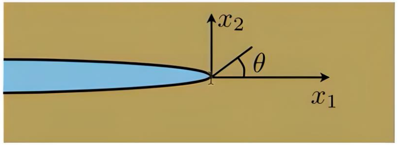

$K_I$ - Stress Intensity Factor (SIF)

Propagation condition: $K_I = K_{Ic}$

$K_{Ic}$ - fracture toughness

https://en.wikipedia.org/wiki/Fracture_mechanics

# Mode I solution near the tip




$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1 - \sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1 + \sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\sin(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right]$$

$$u_1 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\cos(\tfrac{1}{2}\theta)\left\{\kappa - 1 + 2\sin^2(\tfrac{1}{2}\theta)\right\}\right]$$

$$u_2 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\sin(\tfrac{1}{2}\theta)\left\{\kappa + 1 - 2\cos^2(\tfrac{1}{2}\theta)\right\}\right]$$

$$\kappa = 3 - 4\nu \qquad \mu = \frac{E}{2(1+\nu)}$$

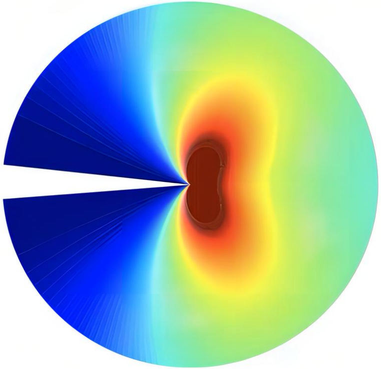
## Solution methodology:

- Write elasticity equations via Airy stress function
- Solve the equations assuming stress-free crack and finite displacement at the tip
- See lecture notes on fracture mechanics for more info: http://www.mate.tue.nl/~piet/edu/frm/pdf/frmsyl1213.pdf

Fracture width around the crack tip:

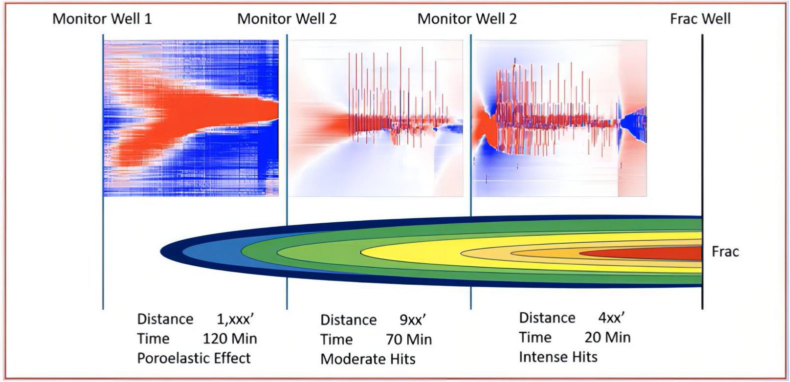
$$w = \sqrt{\frac{32}{\pi}}\frac{K_I(1-\nu^2)}{E}\sqrt{r}$$

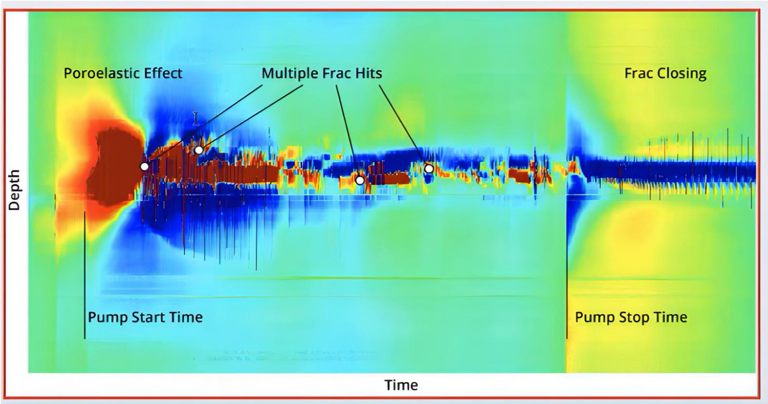
Stress field around the crack tip:



# Observation of the crack tip stress in the field

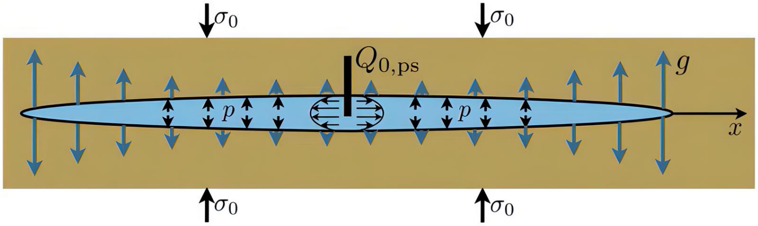
- Fiber optic cables are used to measure stretch versus time along the cable length
- A cable is often placed in a neighboring horizontal well, while the primary well is being fractured
- The characteristic lobes of the approaching crack are clearly visible

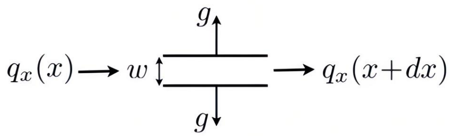



Silixa.com Richter et al. 2020

# Volume balance for a plane strain hydraulic fracture




New volume — Flux in — Leak-off

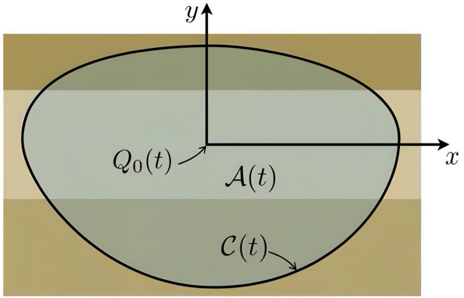
$$w(t+dt)dx = w(t)dx + q_x(x)dt - q_x(x+dx)dt - 2g\,dx\,dt$$

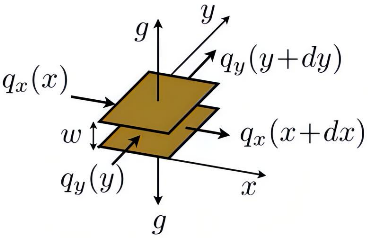
Previous volume — Flux out

Source term

$$\frac{\partial w}{\partial t} + \frac{\partial q_x}{\partial x} + 2g = Q_{0,\mathrm{ps}}\delta(x)$$

# Volume balance for a planar hydraulic fracture




New volume | Flux in | Flux in | Leak-off

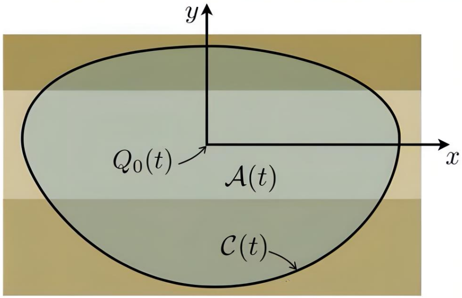
$$w(t+dt)dxdy = w(t)dxdy + q_x(x)dtdy - q_x(x+dx)dtdy + q_y(y)dtdx - q_y(y+dy)dtdx - 2gdxdydt$$

Previous volume | Flux out | Flux out

Source term

$$\frac{\partial w}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + 2g = Q_0\delta(x, y)$$

# Governing equations for a planar hydraulic fracture




**Scaling** $$E' = \frac{E}{1-\nu^2}, \qquad \mu' = 12\mu, \qquad K' = \sqrt{\frac{32}{\pi}} K_{Ic}, \qquad C' = 2C_l,$$

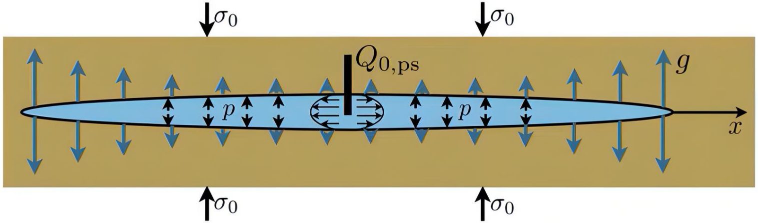
**Volume balance** $$\frac{\partial w}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{q} + \frac{C'}{\sqrt{t-t_0(x,y)}} = Q_0(t)\delta(x,y),$$

**Fluid flow** $$\mathbf{q} = -\frac{w^3}{\mu'}\boldsymbol{\nabla} p,$$

**Elasticity** $$p(x,y,t) = \sigma_0(y) - \frac{E'}{8\pi}\int_{\mathcal{A}(t)} \frac{w(x',y',t)\mathrm{d}x'\mathrm{d}y'}{[(x'-x)^2+(y'-y)^2]^{3/2}},$$

**Propagation** $$\lim_{s\to 0}\frac{w}{s^{1/2}} = \begin{cases} \dfrac{K'}{E'}, & \text{if } V>0,\\[1ex] \dfrac{K'_I}{E'}, & \text{if } V=0.\end{cases}$$

# Governing equations for a plane strain hydraulic fracture




Scaling $\quad E' = \dfrac{E}{1-\nu^2}, \qquad \mu' = 12\mu, \qquad K' = \sqrt{\dfrac{32}{\pi}} K_{Ic}, \qquad C' = 2C_l,$

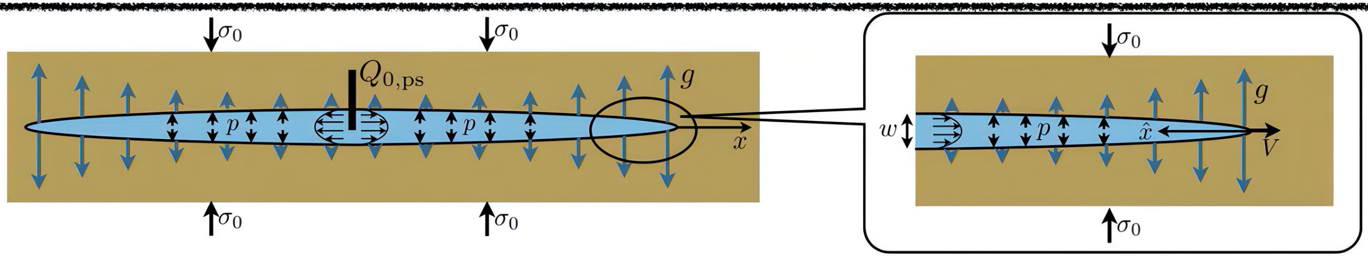
Volume balance $\quad \dfrac{\partial w}{\partial t} + \dfrac{\partial q}{\partial x} + \dfrac{C'}{\sqrt{t - t_0(x)}} = Q_{0,\mathrm{ps}}(t)\delta(x),$

Fluid flow $\quad q = -\dfrac{w^3}{\mu'}\dfrac{\partial p}{\partial x},$

Elasticity $\quad p(x,t) = \sigma_0 - \dfrac{E'}{4\pi}\displaystyle\int_{-l_1}^{l_2} \dfrac{w(s)ds}{(x-s)^2},$

Propagation $\quad \displaystyle\lim_{s\to 0} \dfrac{w}{s^{1/2}} = \begin{cases} \dfrac{K'}{E'}, & \text{if } V > 0, \\ \dfrac{K'_I}{E'}, & \text{if } V = 0. \end{cases}$

# Governing equations for a semi-infinite hydraulic fracture




Use moving coordinate

$$\hat{x} = Vt - x$$

**Volume balance and flow**

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_{0,\mathrm{ps}}(t)\delta(x),$$

$$q = -\frac{w^3}{\mu'}\frac{\partial p}{\partial x},$$

$$\Longrightarrow \quad V\frac{dw}{d\hat{x}} - \frac{dq}{d\hat{x}} + \frac{C'}{\sqrt{\hat{x}/V}} = 0$$

Substitute flux and integrate

$$\Longrightarrow \quad \frac{w^2}{\mu'}\frac{dp}{d\hat{x}} = V + 2C'\frac{\sqrt{V\hat{x}}}{w}$$

**Elasticity and propagation**

$$p(x,t) = \sigma_0 - \frac{E'}{4\pi}\int_{-l_1}^{l_2} \frac{w(s)ds}{(x-s)^2},$$

$$\lim_{s \to 0} \frac{w}{s^{1/2}} = \begin{cases} \dfrac{K'}{E'}, & \text{if } V > 0, \\ \dfrac{K'_I}{E'}, & \text{if } V = 0. \end{cases}$$

$$\Longrightarrow \quad w(\hat{x}) = \frac{K'}{E'}\hat{x}^{1/2} - \frac{4}{\pi E'}\int_0^\infty F(\hat{x},\hat{s})\frac{d(p-\sigma_0)}{d\hat{s}}d\hat{s},$$

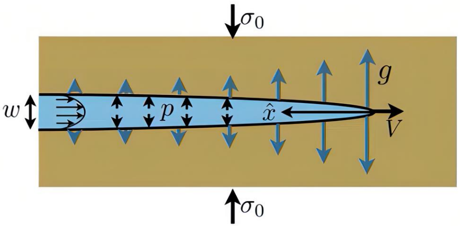
$$F(\hat{x},\hat{s}) = (\hat{s}-\hat{x})\ln\left|\frac{\hat{x}^{1/2}+\hat{s}^{1/2}}{\hat{x}^{1/2}-\hat{s}^{1/2}}\right| - 2\hat{x}^{1/2}\hat{s}^{1/2}.$$

# Non-singular formulation

Substitute pressure gradient

$$w(\hat{x}) = \frac{K'}{E'}\hat{x}^{1/2} - \frac{4}{\pi E'}\int_0^\infty F(\hat{x},\hat{s})\frac{d(p-\sigma_0)}{d\hat{s}}d\hat{s},$$

$$\frac{w^2}{\mu'}\frac{dp}{d\hat{x}} = V + 2C'\frac{\sqrt{V\hat{x}}}{w}$$




$$w(\hat{x}) = \frac{K'}{E'}\hat{x}^{1/2} - \frac{4}{\pi E'}\int_0^\infty F(\hat{x},\hat{s})\frac{\mu'}{w(\hat{s})^2}\left[V + 2C'V^{1/2}\frac{\hat{s}^{1/2}}{w(\hat{s})}\right]d\hat{s}.$$

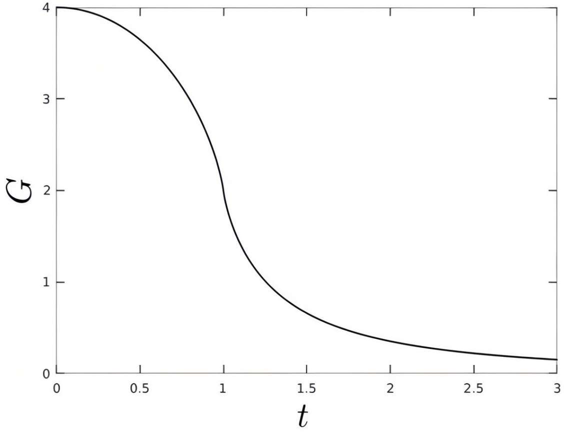
Scaling

$$\tilde{w} = \frac{E'w}{K'\hat{x}^{1/2}}, \qquad \chi = \frac{2C'E'}{V^{1/2}K'}, \qquad \tilde{x} = (\hat{x}/l)^{1/2},$$

$$\tilde{s} = (\hat{s}/l)^{1/2}, \qquad l = \left(\frac{K'^3}{\mu' E'^2 V}\right)^2,$$

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right]d\tilde{s},$$ Non-singular formulation

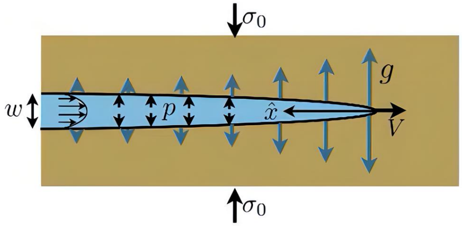
$$G(t) = \frac{1-t^2}{t}\ln\left|\frac{1+t}{1-t}\right| + 2.$$ Non-singular kernel



# Non-singular formulation

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right] d\tilde{s},$$

w = “toughness” + “viscosity” + “leak-off”




## Toughness dominates

$$\tilde{w}_k = 1, \quad \Longrightarrow \quad w_k = \frac{K'}{E'}\hat{x}^{1/2},$$

## Viscosity dominates

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2}\, d\tilde{s} \quad \Longrightarrow \quad \begin{aligned}\tilde{w}_m &= \beta_m \tilde{x}^{1/3}\\ \beta_m &= 2^{1/3}3^{5/6}\end{aligned} \quad \Longrightarrow \quad w_m = \beta_m\left(\frac{\mu' V}{E'}\right)^{1/3}\hat{x}^{2/3}$$

Desroches et al 1994

## Leak-off dominates

$$\tilde{w}(\tilde{x}) = \frac{8\chi}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^3}\, d\tilde{s} \quad \Longrightarrow \quad \begin{aligned}\tilde{w}_{\tilde{m}} &= \beta_{\tilde{m}}\chi^{1/4}\tilde{x}^{1/4}\\ \beta_{\tilde{m}} &= \frac{4}{15^{1/4}(\sqrt{2}-1)^{1/4}}\end{aligned} \quad \Longrightarrow \quad w_{\tilde{m}} = \beta_{\tilde{m}}\left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8}\hat{x}^{5/8},$$

Lenoah 1995

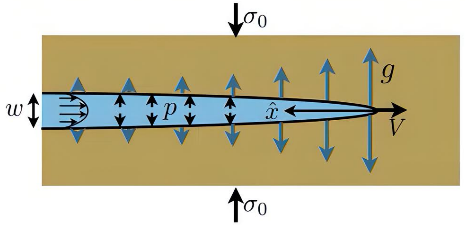
# Derivation of the viscosity solution

Governing integral equation

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2}\,d\tilde{s}$$

Form of the solution

$$\tilde{w} = \beta_m \tilde{x}^{\alpha_m}$$




$$\beta_m \tilde{x}^{\alpha_m} = \frac{8\tilde{x}^{2\alpha_m - 1}}{\pi\beta_m^2}\int_0^\infty \frac{G(t)}{t^{2\alpha_m}}\,dt$$

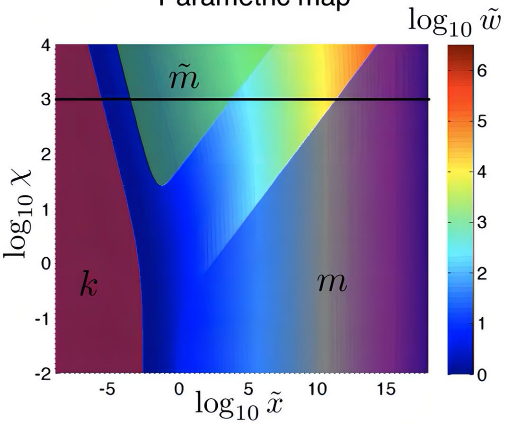
$$\int_0^\infty \frac{G(t)}{t^\alpha}\,dt = \frac{2\pi}{\alpha(2-\alpha)}\tan\left(\frac{\pi}{2}\alpha\right)$$ (see derivation in the notes)

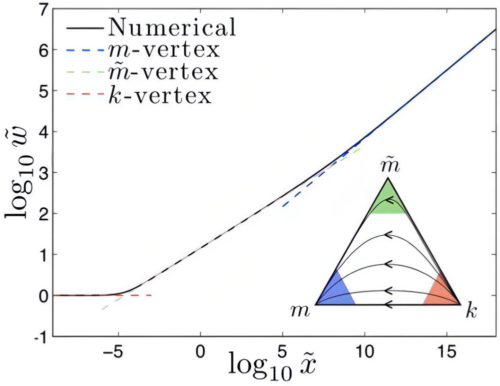
$$\alpha_m = 1/3 \qquad \beta_m = 2^{1/3}3^{5/6},$$

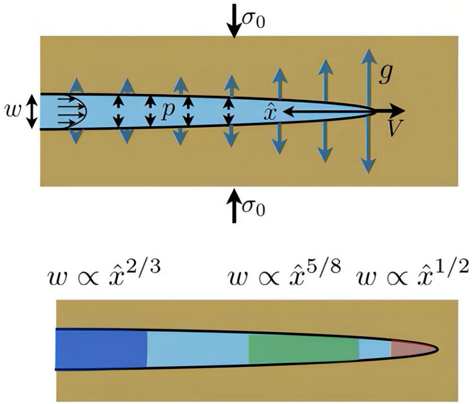
$$\tilde{w}_m = \beta_m \tilde{x}^{1/3}$$

Leak-off dominated solution can be derived in a similar way.

# Structure of the solution




Vertex solutions:

$$\tilde{w}_k = 1, \qquad \tilde{w}_{\tilde{m}} = \beta_{\tilde{m}}\chi^{1/4}\tilde{x}^{1/4}, \qquad \tilde{w}_m = \beta_m \tilde{x}^{1/3},$$

Solution transitions gradually from one limiting solution to another starting from toughness, then passing through leak-off (for large leak-off), and then reaching viscosity vertex.

# Approximate solution

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right]d\tilde{s},$$

Numerical solution of this equation is time consuming and can be a limiting factor for some applications, such as using it as a propagation condition for a planar fracture. So, need to construct an efficient approximation.

Differentiate

$$\frac{d\tilde{w}(\tilde{x})}{d\tilde{x}} = -\frac{8}{\pi}\int_0^\infty G'(\tilde{s}/\tilde{x})\frac{\tilde{s}}{\tilde{x}^2}\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right]d\tilde{s}, \qquad \tilde{w}(0) = 1.$$

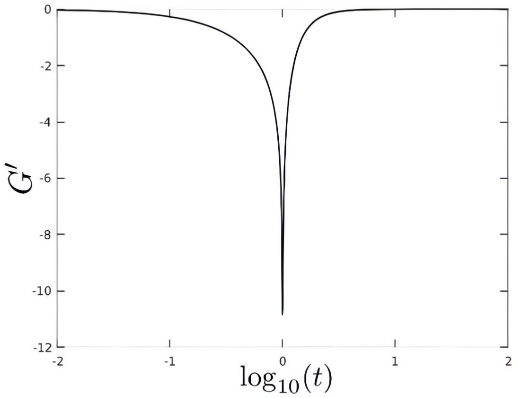
Assume that $\tilde{w} \propto \tilde{x}^\delta$

$$\frac{d\tilde{w}(\tilde{x})}{d\tilde{x}} = -\frac{8}{\pi}\int_0^\infty G'(\tilde{s}/\tilde{x})\frac{\tilde{x}^{2\delta-1}}{\tilde{s}^{2\delta-1}}\frac{d\tilde{s}}{\tilde{x}}\left[\frac{\tilde{s}^{2\delta}}{\tilde{w}(\tilde{s})^2\tilde{x}^{2\delta}}\right] - \frac{8}{\pi}\int_0^\infty G'(\tilde{s}/\tilde{x})\frac{\tilde{x}^{3\delta-1}}{\tilde{s}^{3\delta-1}}\frac{d\tilde{s}}{\tilde{x}}\left[\frac{\chi\tilde{s}^{3\delta}}{\tilde{w}(\tilde{s})^3\tilde{x}^{3\delta}}\right].$$

Replace G’ with Delta function (set $\tilde{s} = \tilde{x}$ in [ ] and compute integrals)

$$\tilde{w}' = \frac{C_1(\delta)}{\tilde{w}^2} + \frac{\chi C_2(\delta)}{\tilde{w}^3}, \qquad \delta = \tilde{x}\frac{\tilde{w}'}{\tilde{w}}, \qquad \tilde{w}(0) = 1,$$

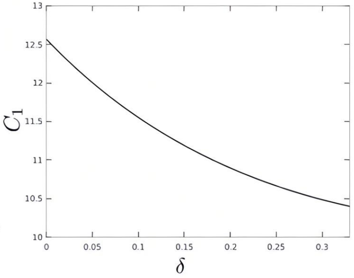
$$C_1(\delta) = \frac{4(1-2\delta)}{\delta(1-\delta)}\tan(\pi\delta), \qquad C_2(\delta) = \frac{16(1-3\delta)}{3\delta(2-3\delta)}\tan\left(\frac{3\pi}{2}\delta\right).$$

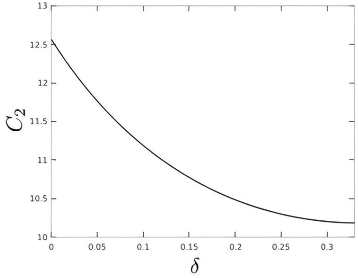


# Approximate solution

$$\tilde{w}' = \frac{C_1(\delta)}{\tilde{w}^2} + \frac{\chi C_2(\delta)}{\tilde{w}^3}, \qquad \delta = \tilde{x}\frac{\tilde{w}'}{\tilde{w}}, \qquad \tilde{w}(0) = 1,$$

$$C_1(\delta) = \frac{4(1-2\delta)}{\delta(1-\delta)}\tan(\pi\delta), \qquad C_2(\delta) = \frac{16(1-3\delta)}{3\delta(2-3\delta)}\tan\Big(\frac{3\pi}{2}\delta\Big).$$




Solve assuming constant C1 and C2

$$\tilde{w}^3 - 1 - \frac{3}{2}b(\tilde{w}^2 - 1) + 3b^2(\tilde{w} - 1) - 3b^3\ln\Big(\frac{b+\tilde{w}}{b+1}\Big) = 3C_1(\delta)\tilde{x}, \qquad b = \frac{C_2(\delta)}{C_1(\delta)}\chi.$$

Take $C_1(1/3) = \beta_m^3/3$ and $C_2(1/4) = \beta_{\tilde{m}}^4/4$

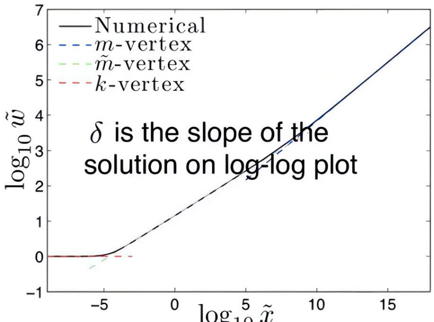
to match viscosity and leak-off limits exactly

$$\tilde{w}_0^3 - 1 - \frac{3}{2}b_0(\tilde{w}_0^2 - 1) + 3b_0^2(\tilde{w}_0 - 1) - 3b_0^3\ln\Big(\frac{b_0+\tilde{w}_0}{b_0+1}\Big) = \beta_m^3\tilde{x}, \qquad b_0 = \frac{3\beta_{\tilde{m}}^4}{4\beta_m^3}\chi \approx 0.9912\,\chi.$$

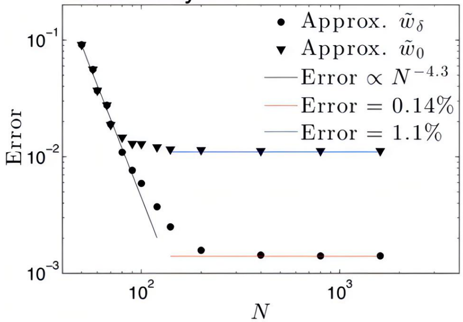
The above is an implicit zeroth order solution for the problem, to get a better approximation, compute

$$\delta = \frac{\beta_m^3\tilde{x}(\tilde{w}_0)}{3\tilde{w}_0^3}\Big(1 + \frac{b_0}{\tilde{w}_0}\Big),$$

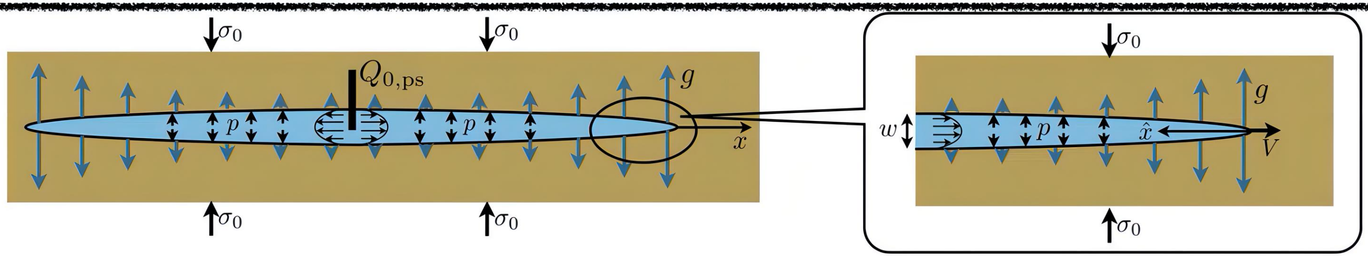
and re-evaluate the solution.



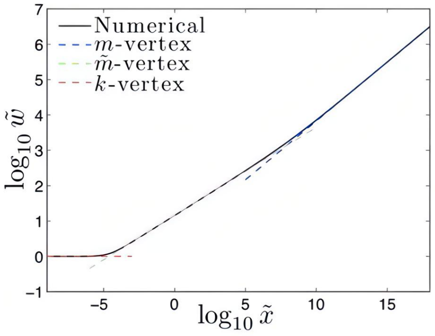

Error analysis

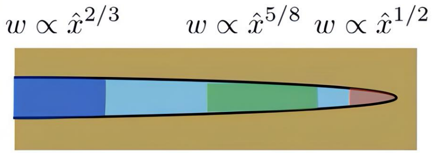


# Things to remember for the semi-infinite hydraulic fracture



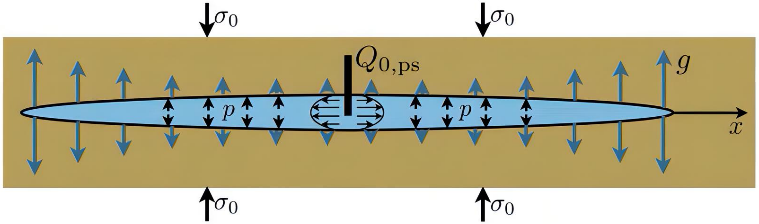

- Semi-infinite geometry describes tip region of a finite hydraulic fracture
- There are three limiting analytic solutions: toughness, viscosity, and leak-off
- The global solution gradually transitions from one limiting case to another
- There is computationally efficient approximate solution for the problem that can be used as a propagation condition for finite fractures




$$w_k = \frac{K'}{E'}\hat{x}^{1/2}, \qquad w_{\tilde{m}} = \beta_{\tilde{m}}\left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8}\hat{x}^{5/8}, \qquad w_m = \beta_m\left(\frac{\mu' V}{E'}\right)^{1/3}\hat{x}^{2/3}.$$

# Plane strain hydraulic fracture




Governing equations

$$\frac{\partial w}{\partial t}+\frac{\partial q}{\partial x}+\frac{C'}{\sqrt{t-t_0(x)}}=Q_{0,\mathrm{ps}}(t)\delta(x),$$

$$q=-\frac{w^3}{\mu'}\frac{\partial p_n}{\partial x},$$

$$p_n(x)=-\frac{E'}{4\pi}\int_{-l}^{l}\frac{w(s)ds}{(x-s)^2},$$

$$w\to\frac{K'}{E'}\sqrt{l-x},\qquad x\to l.$$

$\Longrightarrow$

Scales

$$\frac{w_*}{t}=\frac{q_*}{l_*}=\frac{C'}{t^{1/2}}=\frac{Q_{0,\mathrm{ps}}}{l_*},$$

$$q_*=\frac{w_*^3p_*}{\mu' l_*},$$

$$p_*=\frac{E'w_*}{l_*},$$

$$w_*=\frac{K'}{E'}l_*^{1/2}.$$

6 equations, 4 unknowns

# Scaling for viscosity-storage solution

Viscosity-storage => no toughness, no leak-off

$$\frac{w_*}{t} = \frac{q_*}{l_*} = \cancel{\frac{C'}{t^{1/2}}} = \frac{Q_{0,\mathrm{ps}}}{l_*},$$

$$q_* = \frac{w_*^3 p_*}{\mu' l_*},$$

$$p_* = \frac{E' w_*}{l_*},$$

$$\cancel{w_* = \frac{K'}{E'} l_*^{1/2}}.$$

$\Rightarrow$

Scaling solution

$$l_* = \left(\frac{Q_{0,\mathrm{ps}}^3 E' t^4}{\mu'}\right)^{1/6}, \qquad w_* = \left(\frac{\mu' Q_{0,\mathrm{ps}}^3 t^2}{E'}\right)^{1/6}, \qquad p_* = \left(\frac{\mu' E'^2}{t}\right)^{1/3}$$

## M-vertex solution

$$w_m(\xi, t) = 1.1265 \left(\frac{\mu' Q_0^3 t^2}{E'}\right)^{1/6} (1+\xi)^{0.588}(1-\xi)^{2/3},$$

$$p_m(\xi, t) = 2.7495 \left(\frac{\mu' E'^2}{t}\right)^{1/3} \mathcal{F}\left(\xi, 0.588, \tfrac{2}{3}\right),$$

$$l_m(t) = 0.6159 \left(\frac{Q_0^3 E' t^4}{\mu'}\right)^{1/6},$$

# Scaling for toughness-storage solution

Toughness-storage => no viscosity, no leak-off

$$\frac{w_*}{t} = \frac{q_*}{l_*} = \cancel{\frac{C'}{t^{1/2}}} = \frac{Q_{0,\mathrm{ps}}}{l_*},$$

$$\cancel{q_* = \frac{w_*^3 p_*}{\mu' l_*}},$$

$$p_* = \frac{E' w_*}{l_*},$$

$$w_* = \frac{K'}{E'} l_*^{1/2}.$$

Scaling solution

$$l_* = \left(\frac{E' Q_{0,\mathrm{ps}} t}{K'}\right)^{2/3}, \qquad w_* = \left(\frac{K'^2 Q_{0,\mathrm{ps}} t}{E'^2}\right)^{1/3}, \qquad p_* = \left(\frac{K'^4}{E'^2 Q_{0,\mathrm{ps}} t}\right)^{1/3}$$

## K-vertex solution

$$w_k(\xi, t) = 0.6828 \left(\frac{K'^2 Q_0 t}{E'^2}\right)^{1/3} (1-\xi^2)^{1/2},$$

$$p_k(\xi, t) = 0.1831 \left(\frac{K'^4}{E' Q_0 t}\right)^{1/3},$$

$$l_k(t) = 0.9324 \left(\frac{E' Q_0 t}{K'}\right)^{2/3}.$$

# Toughness-viscosity transition

Use either

$$l_m \sim l_k$$

$$w_m \sim w_k$$

$$p_m \sim p_k$$

$\Rightarrow$

Dimensionless toughness

$$K_m = \left(\frac{K'^4}{\mu' E'^3 Q_{0,\mathrm{ps}}}\right)^{1/4}$$

$K_m \ll 1$ M vertex (viscosity dominated)

$K_m \gg 1$ K vertex (toughness dominated)

$K_m \sim 1$ M-K transition

**This is only part of the story, how about leak-off?**

# Hydraulic fracture regimes of propagation

## Toughness vs. viscosity

$\boxed{1}$ Large toughness (no viscosity)

$\boxed{2}$ Large viscosity (no toughness)

## Small vs. large leak-off

$\boxed{3}$ All fluid stays inside the fracture

$\boxed{4}$ Most fluid leaks into formation

$\boxed{2\&3}$ $M$ - storage-viscosity

No toughness, no leak-off

$\boxed{1\&3}$ $K$ - storage-toughness

No viscosity, no leak-off

$\boxed{2\&4}$ $\tilde{M}$ - leak-off-viscosity

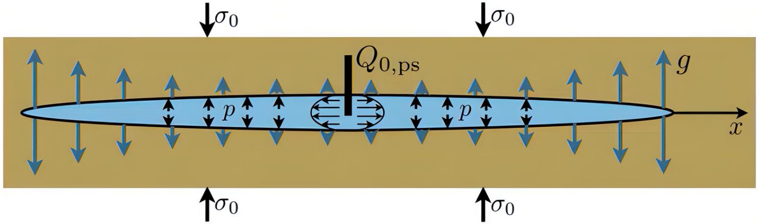
No toughness, large leak-off

$\boxed{1\&4}$ $\tilde{K}$ - leak-off-toughness

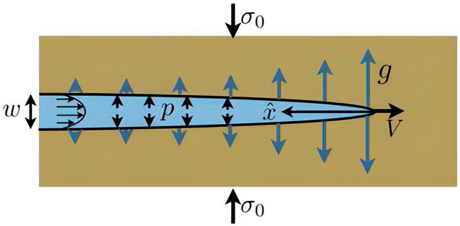
No viscosity, large leak-off

# Tip asymptotics vs. regime

## Global regime




## Tip asymptotics




$M$ - storage-viscosity

No toughness, no leak-off

$K$ - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off

Viscosity

$$w_m = \beta_m \left( \frac{\mu' V}{E'} \right)^{1/3} \hat{x}^{2/3}$$

Toughness

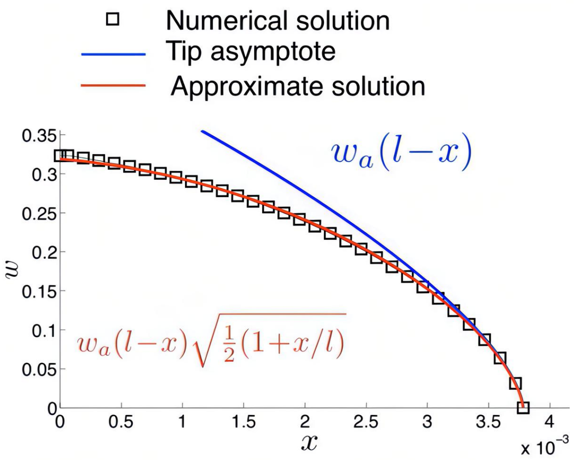
$$w_k = \frac{K'}{E'} \hat{x}^{1/2}$$

Leak-off

$$w_{\tilde{m}} = \beta_{\tilde{m}} \left( \frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} \hat{x}^{5/8}$$

# Solution for the problem

- **Numerical solution**
  - Discretize governing equations using finite differences, see Dontsov, 2017
- **Approximate solution**
  - Global volume balance
  - Approximate solution for width based on the tip asymptote




Global volume balance

$$\int_0^l \left( w(x,t) + 2C'\sqrt{t-t_0(x)} \right) dx = \frac{Q_{0,\mathrm{ps}} t}{2},$$

$$l(t) \propto t^\alpha \implies x/l = (t_0/t)^\alpha$$

Approximate solution for width

$$w(x,t) = \left(\frac{l+x}{2l}\right)^\lambda w_a(l-x),$$

$\lambda$ - fitting parameter

# Regimes of propagation for a plane strain hydraulic fracture

$M$ - storage-viscosity

No toughness, no leak-off

$K$ - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

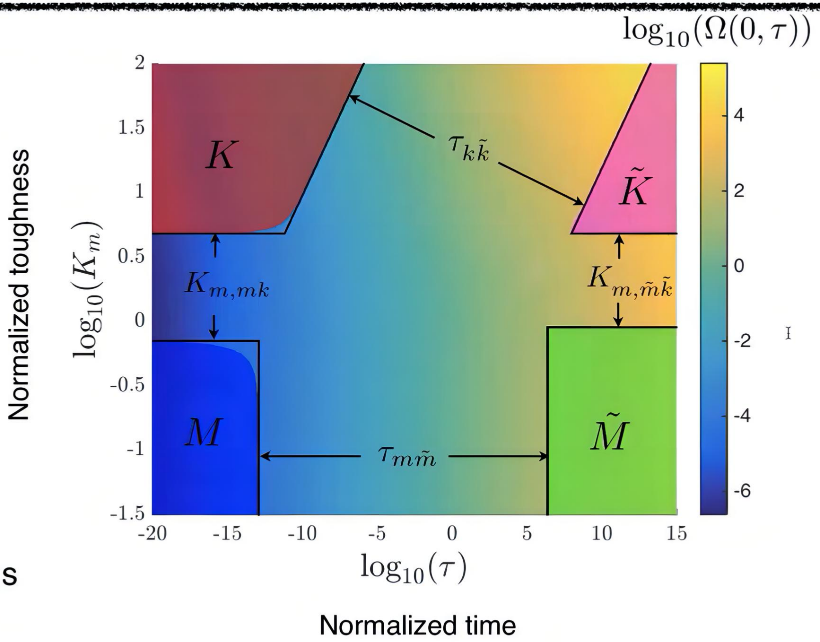
No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off

Zones of applicability of limiting solutions are defined to within 1% of error

Bounds for these zones (solid black lines) are known




$$K_m = \left(\frac{K'^4}{\mu' E'^3 Q_{0,\mathrm{ps}}}\right)^{1/4} \qquad \tau = \frac{t}{t_{m\tilde{m}}} \qquad t_{m\tilde{m}} = \frac{\mu' Q_{0,\mathrm{ps}}^3}{E' C'^6}$$

# Limiting solutions for a plane strain hydraulic fracture

## $M$ - storage-viscosity

$$
\begin{aligned}
w_m(\xi,t) &= 1.1265\left(\frac{\mu' Q_{0,\mathrm{ps}}^3\, t^2}{E'}\right)^{1/6}(1+\xi)^{0.588}(1-\xi)^{2/3},\\
p_m(\xi,t) &= 2.7495\left(\frac{\mu' E'^2}{t}\right)^{1/3}\mathcal{F}(\xi, 0.588, \tfrac{2}{3}),\\
l_m(t) &= 0.6159\left(\frac{Q_{0,\mathrm{ps}}^3 E'\, t^4}{\mu'}\right)^{1/6},
\end{aligned}
$$

## $\tilde{M}$ - leak-off-viscosity

$$
\begin{aligned}
w_{\tilde{m}}(\xi,t) &= 0.8165\left(\frac{\mu' Q_{0,\mathrm{ps}}^3\, t}{E' C'^2}\right)^{1/4}(1+\xi)^{0.520}(1-\xi)^{5/8},\\
p_{\tilde{m}}(\xi,t) &= 3.6783\left(\frac{C'^2 \mu' E'^3}{Q_{0,\mathrm{ps}} t}\right)^{1/4}\mathcal{F}(\xi, 0.520, \tfrac{5}{8}),\\
l_{\tilde{m}}(t) &= 0.3183\,\frac{Q_{0,\mathrm{ps}} t^{1/2}}{C'},
\end{aligned}
$$

## $K$ - storage-toughness

$$
\begin{aligned}
w_k(\xi,t) &= 0.6828\left(\frac{K'^2 Q_{0,\mathrm{ps}} t}{E'^2}\right)^{1/3}(1-\xi^2)^{1/2},\\
p_k(\xi,t) &= 0.1831\left(\frac{K'^4}{E' Q_{0,\mathrm{ps}} t}\right)^{1/3},\\
l_k(t) &= 0.9324\left(\frac{E' Q_{0,\mathrm{ps}} t}{K'}\right)^{2/3}.
\end{aligned}
$$

## $\tilde{K}$ - leak-off-toughness

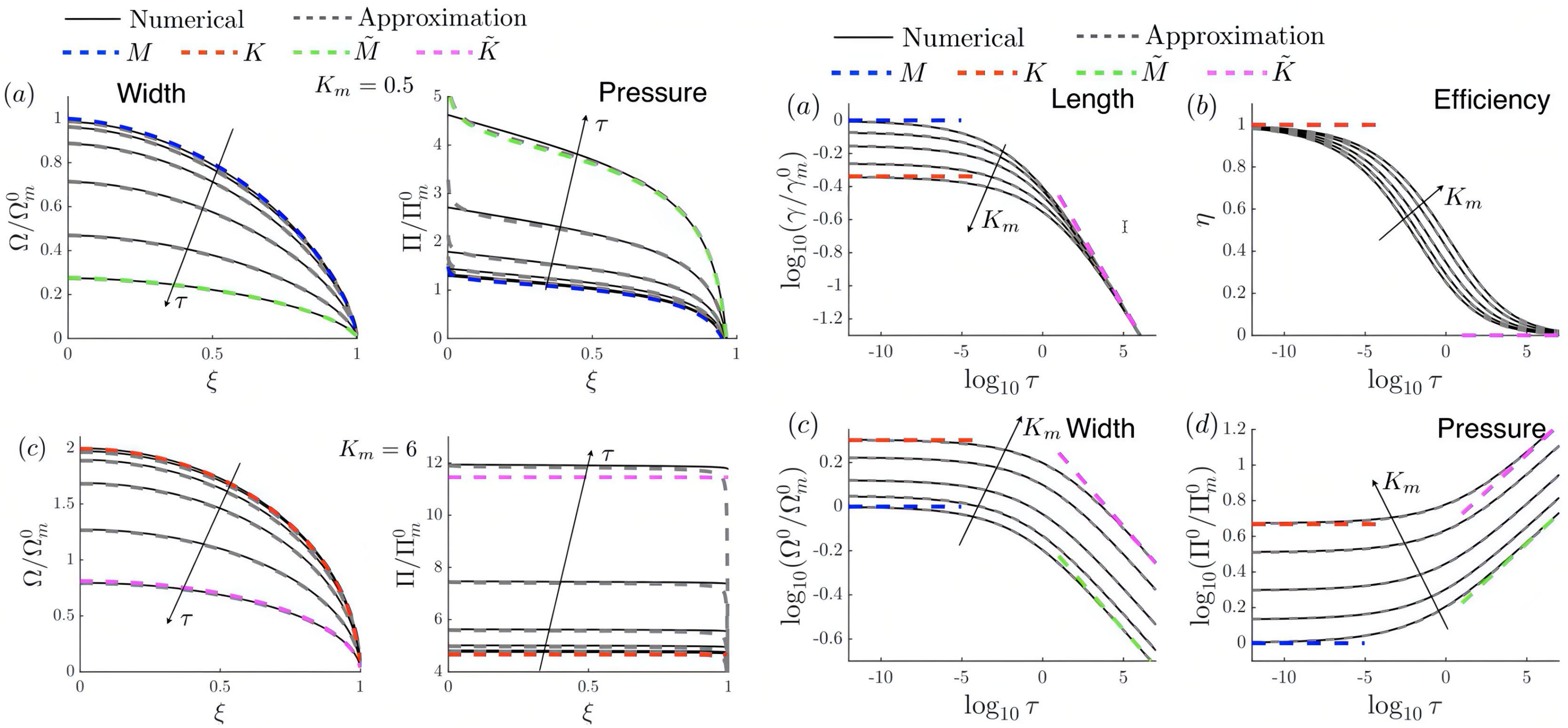
$$
\begin{aligned}
w_{\tilde{k}}(\xi,t) &= 0.3989\left(\frac{K'^4 Q_{0,\mathrm{ps}}^2 t}{E'^4 C'^2}\right)^{1/4}(1-\xi^2)^{1/2},\\
p_{\tilde{k}}(\xi,t) &= 0.3133\left(\frac{K'^4 C'^2}{Q_{0,\mathrm{ps}}^2 t}\right)^{1/4},\\
l_{\tilde{k}}(t) &= 0.3183\,\frac{Q_{0,\mathrm{ps}} t^{1/2}}{C'},
\end{aligned}
$$

Elasticity function:

$$
\mathcal{F}(\xi,\lambda,\bar{\delta}) = \frac{1}{2^{1+\lambda}\pi}\int_0^1 \frac{\partial M(\xi,s)}{\partial s}(1+s)^{\lambda}(1-s)^{\bar{\delta}}\,ds,
$$

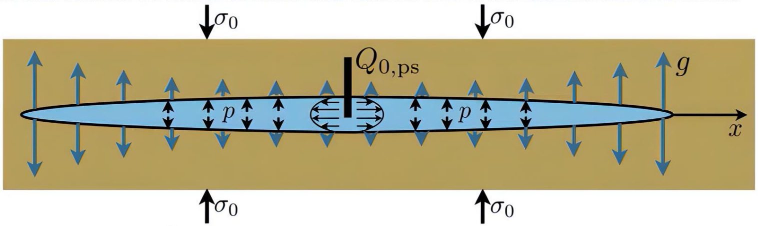
$$
M(\xi,s) = \frac{\xi}{\xi^2 - s^2},
$$

# Numerical vs. approximate solutions



# Things to remember for the plane strain hydraulic fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote
- The existence of approximate solution constructed using global volume balance and tip asymptote
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless toughness and dimensionless time




$M$ - storage-viscosity

No toughness, no leak-off

$K$ - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off

Viscosity

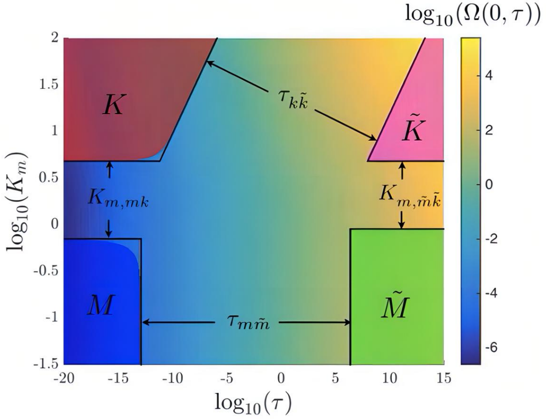
$$w_m = \beta_m \left(\frac{\mu' V}{E'}\right)^{1/3} \hat{x}^{2/3}$$

Toughness

$$w_k = \frac{K'}{E'} \hat{x}^{1/2}$$

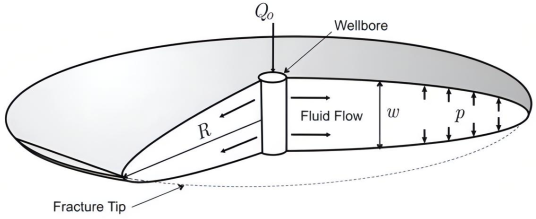
Leak-off

$$w_{\tilde{m}} = \beta_{\tilde{m}} \left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8} \hat{x}^{5/8}$$




E.V. Dontsov. An approximate solution for a plane strain hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *Int. J. Fract.*, 205:221–237, 2017.

# Radial hydraulic fracture




Governing equations

$$\frac{\partial w}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rq) + \frac{C'}{\sqrt{t - t_0(r)}} = Q_0 \delta(r),$$

$$q = -\frac{w^3}{\mu'}\frac{\partial p_n}{\partial r},$$

$$p_n(r,t) = -\frac{E'}{2\pi R}\int_0^R M\left(\frac{r}{R}, \frac{r'}{R}\right)\frac{\partial w(r',t)}{\partial r'}\,dr',$$

$$w \to \frac{K'}{E'}(R-r)^{1/2}, \qquad r \to R.$$

$$\Longrightarrow$$

Scales

$$\frac{w_*}{t} = \frac{q}{R_*} = \frac{C'}{t^{1/2}} = \frac{Q_0}{R_*^2},$$

$$q = \frac{w_*^3 p_*}{\mu' R_*},$$

$$p_* = \frac{E' w_*}{R_*},$$

$$w = \frac{K' R_*^{1/2}}{E'}.$$

6 equations, 4 unknowns

# Scaling for viscosity-storage solution

Viscosity-storage => no toughness, no leak-off

$$\frac{w_*}{t} = \frac{q}{R_*} = \cancel{\frac{C'}{t^{1/2}}} = \frac{Q_0}{R_*^2},$$

$$q = \frac{w_*^3 p_*}{\mu' R_*},$$

$$p_* = \frac{E' w_*}{R_*},$$

$$\cancel{w = \frac{K' R_*^{1/2}}{E'}}.$$

$\Rightarrow$

Scaling solution

$$R_* = \left(\frac{Q_0^3 E' t^4}{\mu'}\right)^{1/9}, \qquad w_* = \left(\frac{\mu'^2 Q_0^3 t}{E'^2}\right)^{1/9}, \qquad p_* = \left(\frac{\mu' E'^2}{t}\right)^{1/3}$$

## M-vertex solution

$$\begin{aligned} w_m(\rho,t) &= 1.1901\left(\frac{\mu'^2 Q_0^3\, t}{E'^2}\right)^{1/9}(1+\rho)^{0.487}(1-\rho)^{2/3}, \\ p_m(\rho,t) &= 2.4019\left(\frac{\mu' E'^2}{t}\right)^{1/3}\mathcal{F}(\rho, 0.487, \tfrac{2}{3}), \\ R_m(t) &= 0.6944\left(\frac{Q_0^3 E'\, t^4}{\mu'}\right)^{1/9}, \end{aligned}$$

# Regimes of propagation for a radial hydraulic fracture

$M$ - storage-viscosity

No toughness, no leak-off

$K$ - storage-toughness

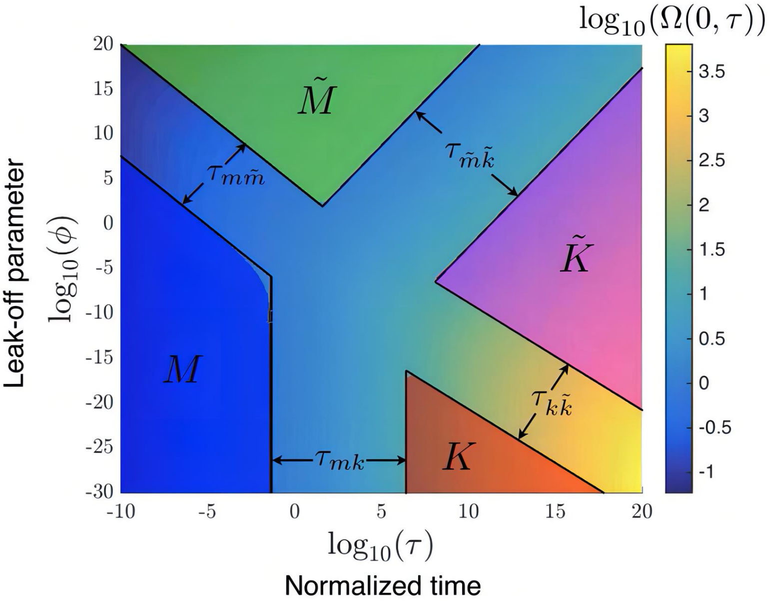
No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off




Zones of applicability of limiting solutions are defined to within 1% of error

Bounds for these zones (solid black lines) are known

$$\phi = \frac{\mu'^3 E'^{11} C'^4 Q_0}{K'^{14}} \qquad \tau = \frac{t}{t_{mk}} \qquad t_{mk} = \left(\frac{\mu'^5 E'^{13} Q_0^3}{K'^{18}}\right)^{1/2}$$

# Regimes of propagation for radial HF

## $M$ - storage-viscosity

$$
\begin{aligned}
w_m(\rho,t) &= 1.1901\left(\frac{\mu'^2 Q_0^3 t}{E'^2}\right)^{1/9}(1+\rho)^{0.487}(1-\rho)^{2/3},\\
p_m(\rho,t) &= 2.4019\left(\frac{\mu' E'^2}{t}\right)^{1/3}\mathcal{F}(\rho,0.487,\tfrac{2}{3}),\\
R_m(t) &= 0.6944\left(\frac{Q_0^3 E' t^4}{\mu'}\right)^{1/9},
\end{aligned}
$$

## $\tilde{M}$ - leak-off-viscosity

$$
\begin{aligned}
w_{\tilde{m}}(\rho,t) &= 1.0574\left(\frac{\mu'^4 Q_0^6 t}{E'^4 C'^2}\right)^{1/16}(1+\rho)^{0.397}(1-\rho)^{5/8},\\
p_{\tilde{m}}(\rho,t) &= 3.0931\left(\frac{\mu'^4 E'^{12} C'^6}{Q_0^2 t^3}\right)^{1/16}\mathcal{F}(\rho,0.397,\tfrac{5}{8}),\\
R_{\tilde{m}}(t) &= 0.4502\left(\frac{Q_0^2 t}{C'^2}\right)^{1/4},
\end{aligned}
$$

## $K$ - storage-toughness

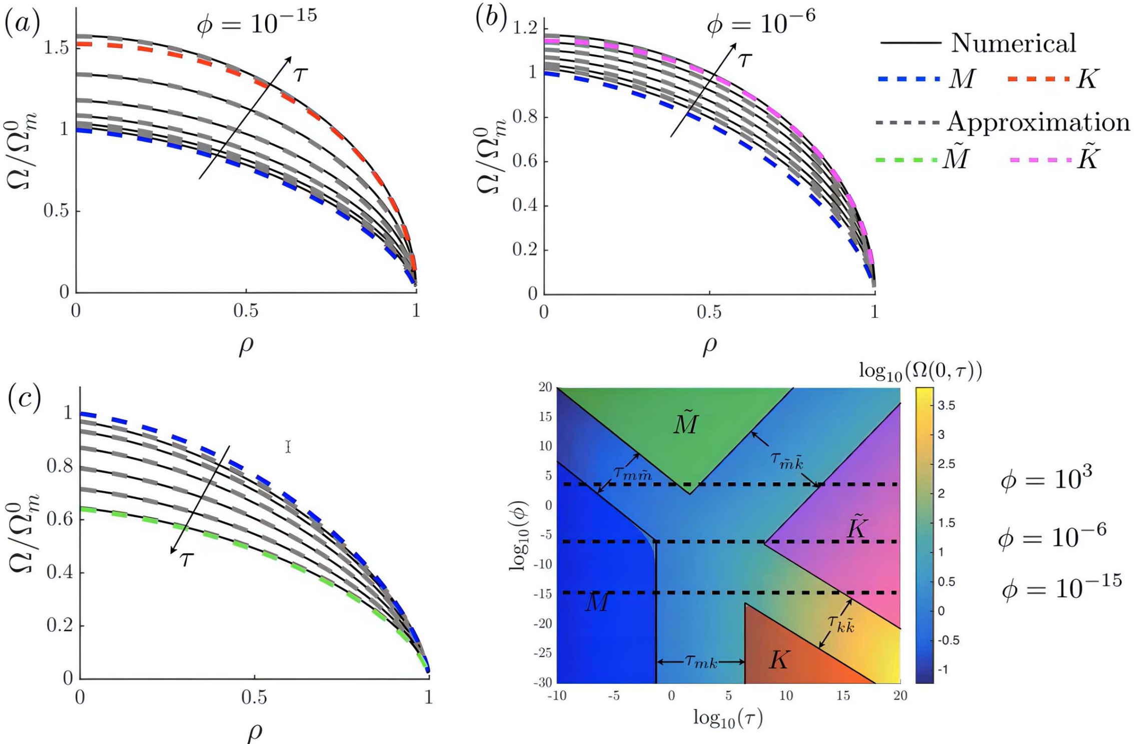
$$
\begin{aligned}
w_k(\rho,t) &= 0.6537\left(\frac{K'^4 Q_0 t}{E'^4}\right)^{1/5}(1-\rho^2)^{1/2},\\
p_k(\rho,t) &= 0.3004\left(\frac{K'^6}{E' Q_0 t}\right)^{1/5},\\
R_k(t) &= 0.8546\left(\frac{E' Q_0 t}{K'}\right)^{2/5}.
\end{aligned}
$$

## $\tilde{K}$ - leak-off-toughness

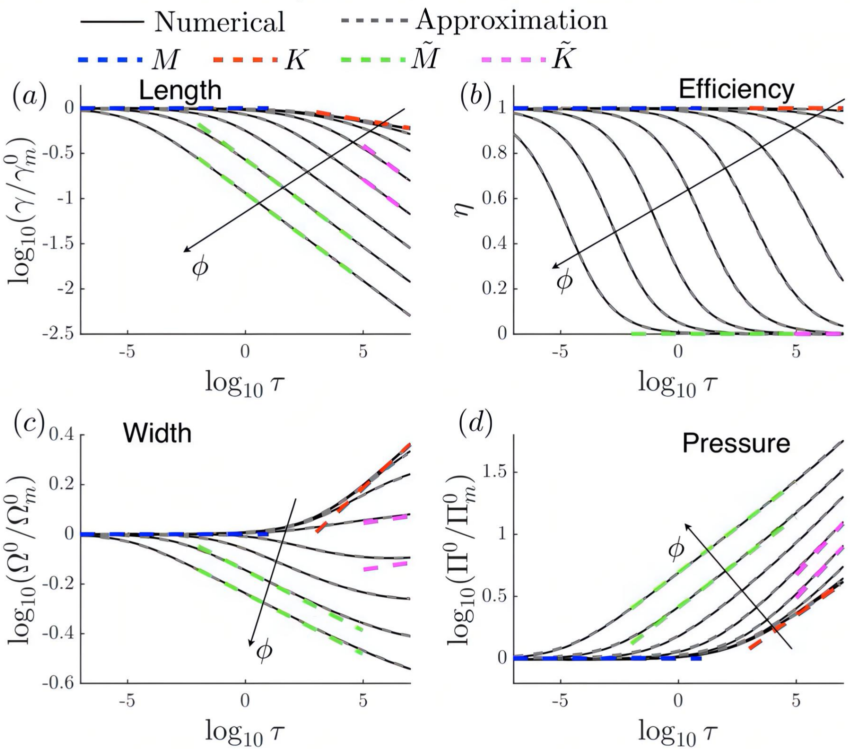
$$
\begin{aligned}
w_{\tilde{k}}(\rho,t) &= 0.4744\left(\frac{K'^8 Q_0^2 t}{E'^8 C'^2}\right)^{1/8}(1-\rho^2)^{1/2},\\
p_{\tilde{k}}(\rho,t) &= 0.4139\left(\frac{K'^8 C'^2}{Q_0^2 t}\right)^{1/8},\\
R_{\tilde{k}}(t) &= 0.4502\left(\frac{Q_0^2 t}{C'^2}\right)^{1/4}.
\end{aligned}
$$

Elasticity function: $\mathcal{F}(\rho,\lambda,\bar{\delta}) = \frac{1}{2^{1+\lambda}\pi}\int_0^1 \frac{\partial M(\rho,s)}{\partial s}(1+s)^{\lambda}(1-s)^{\bar{\delta}}\,ds,$

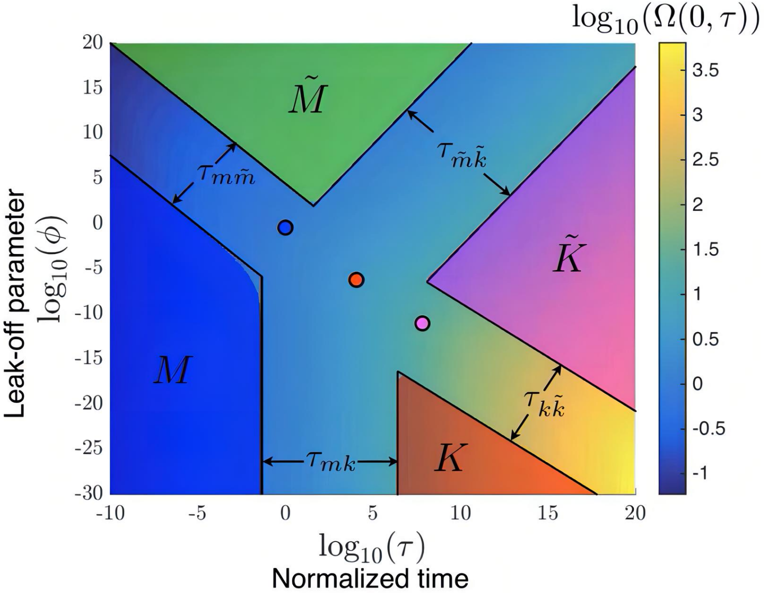
# Comparison with numerical solution



# Comparison with numerical solution



# Radial hydraulic fracture: examples




$$\phi = \frac{\mu'^3 E'^{11} C'^4 Q_0}{K'^{14}} \qquad \tau = \frac{t}{t_{mk}} \qquad t_{mk} = \left(\frac{\mu'^5 E'^{13} Q_0^3}{K'^{18}}\right)^{1/2}$$

**Field values**

$E' = 36$ GPa

$K_{Ic} = 2.6$ MPa $\cdot$ m$^{1/2}$

$Q_0 = 18$ bbl/min $= 0.05$ m$^3$/s

$t = 100$ min

$C' = 4.3 \times 10^{-5}$ m/s$^{1/2}$

**Slick water**

$\mu = 1.5$ cP $= 1.5 \times 10^{-3}$ Pa $\cdot$ s

**Linear gel**

$\mu = 100$cP $= 0.1$ Pa $\cdot$ s

**Lab values (rock)**

$E' = 36$ GPa

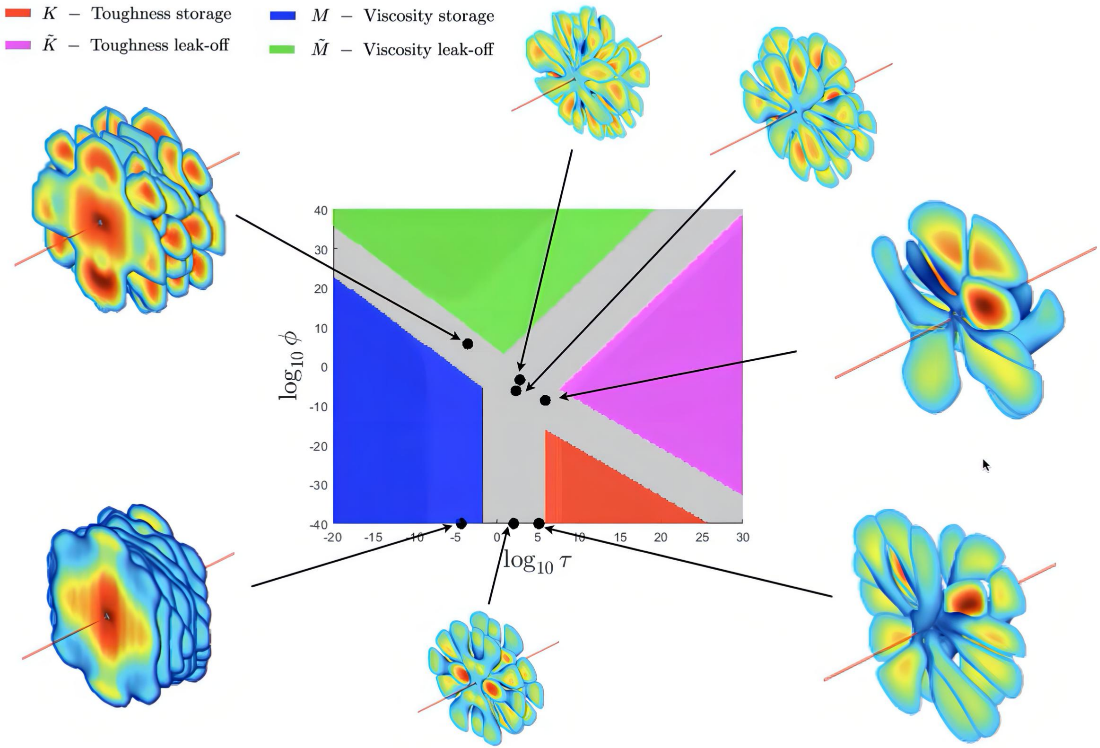
$K_{Ic} = 2.6$ MPa $\cdot$ m$^{1/2}$

$C' = 4.3 \times 10^{-5}$ m/s$^{1/2}$

$\mu = 1.5$ cP $= 1.5 \times 10^{-3}$ Pa $\cdot$ s

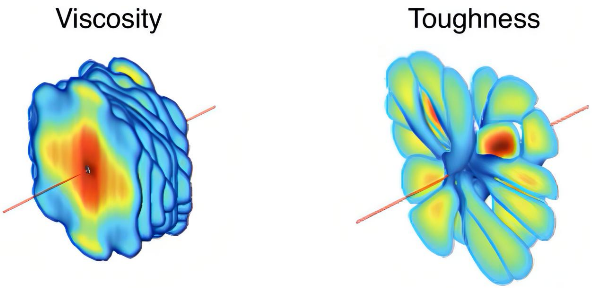
$t = 1$ min $\quad Q_0 = 2.6 \times 10^{-6}$ m$^3$/s

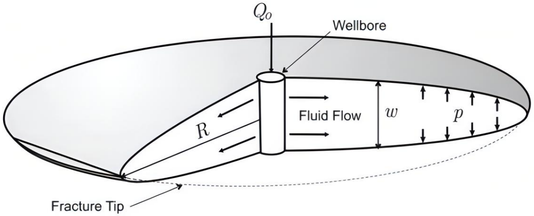
# Application to multiple fractures

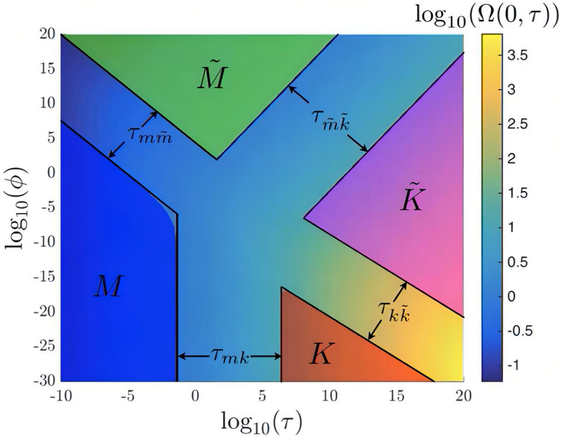


# Things to remember for radial hydraulic fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote (the same as for plane strain)
- The existence of approximate solution constructed using global volume balance and tip asymptote (similar to plane strain)
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless leak-off and dimensionless time
- Fracture regimes affect morphology of multiple hydraulic fractures



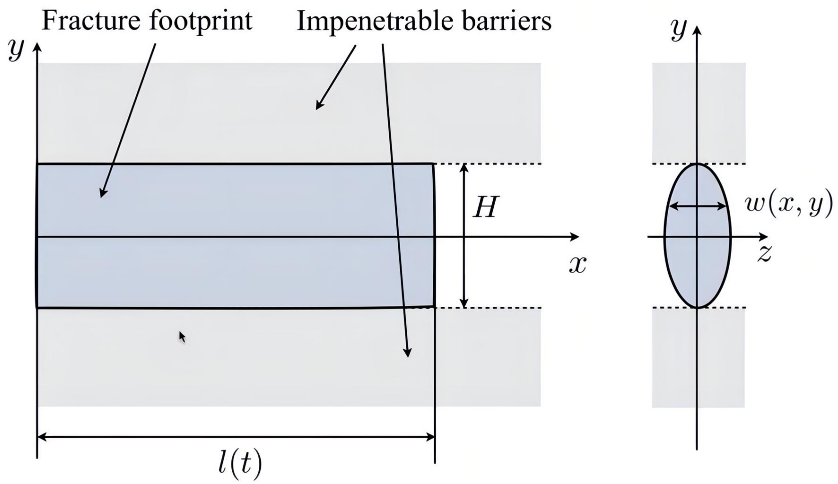



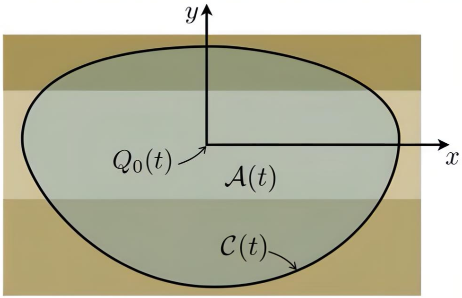

E.V. Dontsov. An approximate solution for a penny-shaped hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *R. Soc. open sci.*, 3:160737, 2016.

# PKN or constant height hydraulic fracture

- Fracture height is assumed to be constant
- Length >> height (long time solution)
- Flux is predominantly horizontal
- Pressure is assumed to be constant in the vertical cross-section
- Far away from the tip, local elasticity applies. That is the pressure is determined by the width at the particular point (as opposed to non-local relation for which the pressure depends on the integral of the width with some kernel)
- Fracture width is elliptical in each cross-section
- These assumptions allow to reformulate the two-dimensional fracture problem as one-dimensional, effectively solving the solution in the vertical direction analytically



# Governing equations for PKN geometry




Scaling $E' = \dfrac{E}{1-\nu^2}$, $\mu' = 12\mu$, $K' = \sqrt{\dfrac{32}{\pi}} K_{Ic}$, $C' = 2C_l$,

Apply vertical averaging

$$w(x,y) = \frac{4}{\pi}\bar{w}(x)\sqrt{1-\left(\frac{2y}{H}\right)^2}, \qquad \bar{w}(x) = \frac{1}{H}\int_{-H/2}^{H/2} w(x,y)\,dy,$$

Volume balance $\dfrac{\partial w}{\partial t} + \nabla\cdot\mathbf{q} + \dfrac{C'}{\sqrt{t-t_0(x,y)}} = Q_0(t)\delta(x,y),$

Fluid flow $\mathbf{q} = -\dfrac{w^3}{\mu'}\nabla p,$

Elasticity $p(x,y,t) = \sigma_0(y) - \dfrac{E'}{8\pi}\displaystyle\int_{\mathcal{A}(t)} \frac{w(x',y',t)\mathrm{d}x'\mathrm{d}y'}{[(x'-x)^2+(y'-y)^2]^{3/2}},$

Propagation $\displaystyle\lim_{s\to 0}\frac{w}{s^{1/2}} = \begin{cases} \dfrac{K'}{E'}, & \text{if } V>0,\\[2mm] \dfrac{K_I'}{E'}, & \text{if } V=0. \end{cases}$

$\Longrightarrow$

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}_x}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H}\delta(x),$$

$$\bar{q}_x = -\frac{1}{H\mu'}\frac{\partial p}{\partial x}\int_{-H/2}^{H/2} w^3\,dy = -\frac{\bar{w}^3}{\pi^2\mu}\frac{\partial p}{\partial x},$$

$$p(x) = -\frac{2E'}{\pi^2 H}\int_{-l(t)}^{l(t)} \bar{w}(x')\frac{dG(2(x'-x)/H)}{dx'}\,dx'$$

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}$$

# Elasticity equation for PKN fracture

$$p(x) = -\frac{2E'}{\pi^2 H}\int_{-l(t)}^{l(t)} \bar{w}(x')\frac{dG(2(x'-x)/H)}{dx'}\,dx'$$

$$G(s) = \frac{\sqrt{1+s^2}}{s}\mathrm{E}\left(\frac{1}{1+s^2}\right), \qquad G(s) \approx \frac{\pi}{2}\mathrm{sign}(s), \quad |s| \gg 1, \qquad G(s) \approx \frac{1}{s}, \quad s \ll 1,$$

Far away from the tip (x>>H)

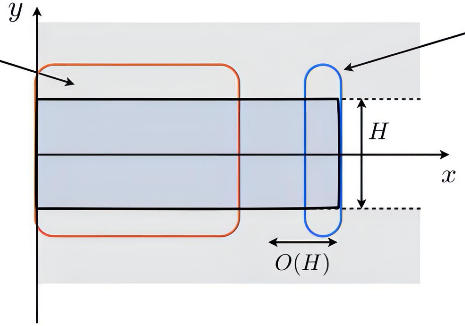
$$p(x) = \frac{2E'\bar{w}(x)}{\pi H},$$

Local elasticity equation

Near the tip (x~H)

$$p(x,t) = \sigma_0 - \frac{E'}{4\pi}\int_{-l_1}^{l_2}\frac{w(s)ds}{(x-s)^2},$$

Plane strain elasticity equation



# Two options for the solution

## Local elasticity + special toughness boundary condition

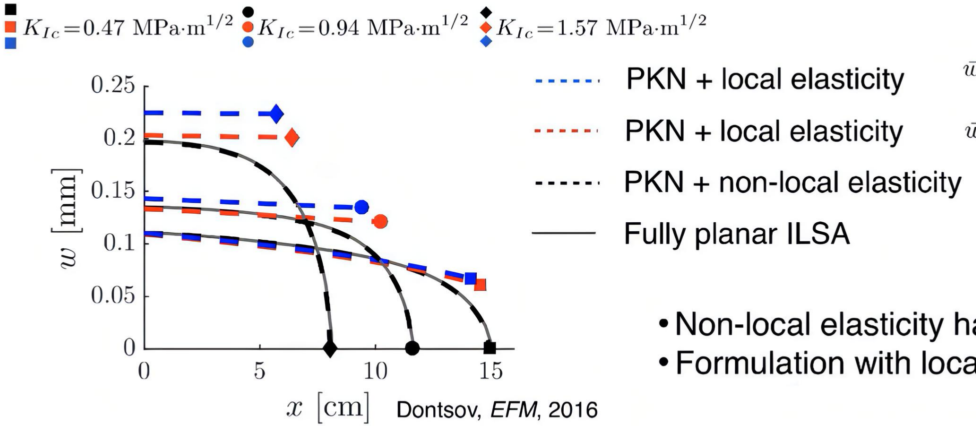
$$p(x) = \frac{2E'\bar{w}(x)}{\pi H},$$

$$\bar{w}(l) = \frac{\sqrt{\pi H}\,K_{Ic}}{E'}.$$

## Non-elasticity + standard toughness boundary condition

$$p(x) = -\frac{2E'}{\pi^2 H}\int_{-l(t)}^{l(t)} \bar{w}(x')\frac{dG(2(x'-x)/H)}{dx'}\,dx'$$

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}$$




Dontsov, *EFM*, 2016

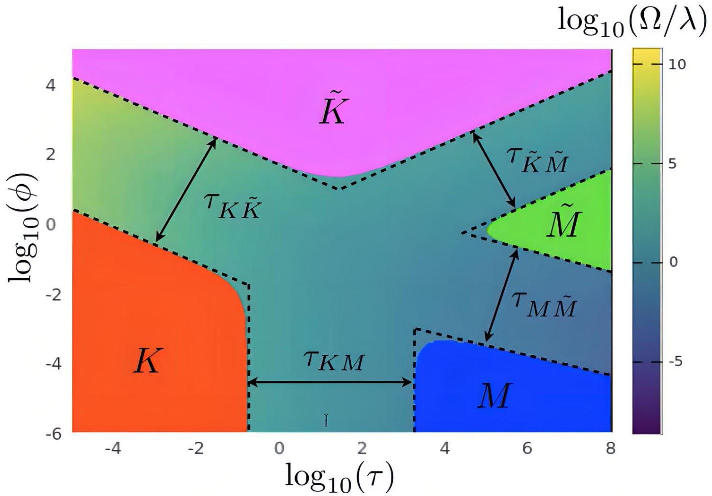
$$\bar{w}_{\mathrm{R}}(l) = \left(\frac{\pi}{2}\right)^{3/2}\frac{\sqrt{H}\,K_{Ic}}{E'}.$$ Nolte,1991

$$\bar{w}_{\mathrm{E}}(l) = \frac{\sqrt{\pi H}\,K_{Ic}}{E'},$$ Sarvaramini &Garagash, 2015

- Non-local elasticity has superior accuracy
- Formulation with local elasticity can be used for analysis

# Parametric space

- Parametric space is computed using the fast solution
- Zones of applicability of the vertex solutions are indicated
- The zone boundaries are quantified




$$\tau = \frac{2\pi^{1/2}E'^4\mu Q_0^2 t}{H^{7/2}K_{Ic}^5} \qquad \phi = \left(\frac{H^5 K_{Ic}^6 C'^4}{4\pi^3 E'^4 \mu^2 Q_0^4}\right)^{1/4}.$$

$M$ - storage-viscosity

No toughness, no leak-off

$K$ - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

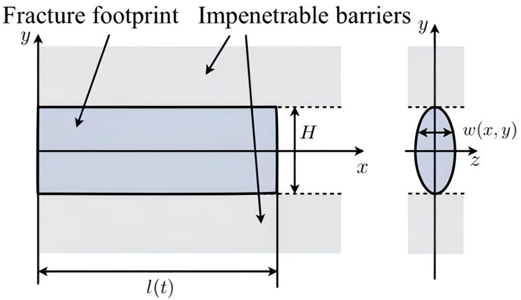
No viscosity, large leak-off

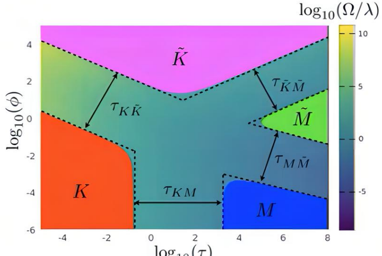
Transitions

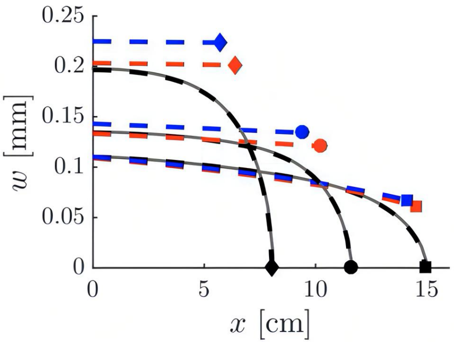
$$\begin{aligned}
\tau_{MK} &= \tau, & \tau_{MK,1} &= 0.16, & \tau_{MK,2} &= 1.6\times 10^3,\\
\tau_{K\tilde{K}} &= \tau\phi^2, & \tau_{K\tilde{K},1} &= 5.6\times 10^{-5}, & \tau_{K\tilde{K},2} &= 2.8\times 10^3,\\
\tau_{\tilde{K}\tilde{M}} &= \tau\phi^{-2}, & \tau_{\tilde{K}\tilde{M},1} &= 0.18, & \tau_{\tilde{K}\tilde{M},2} &= 6.6\times 10^4,\\
\tau_{M\tilde{M}} &= \tau\phi^{10/3}, & \tau_{M\tilde{M},1} &= 2.3\times 10^{-7}, & \tau_{M\tilde{M},2} &= 2.8\times 10^3.
\end{aligned}$$

# Things to remember

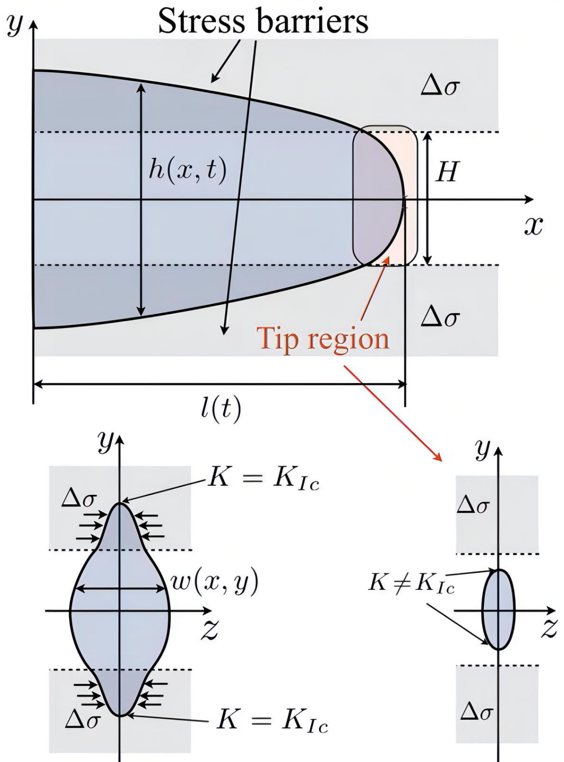
- Assumptions of PKN model, including constant height, length >> height, horizontal flux, constant pressure in each vertical cross-section, elliptical width in each vertical cross-section
- There are two approaches to solve the problem:
  - Use non-local elasticity, which is good for numerical scheme and leads to superior accuracy
  - Use local elasticity with a specific boundary condition at the tip, which is less accurate, but easier for analysis
- Parametric space for the finite fracture is evaluated
- Limiting vertex solutions exist, as well as the global approximate solution https://arxiv.org/abs/2110.13088



# Enhanced pseudo-3D (EP3D) model

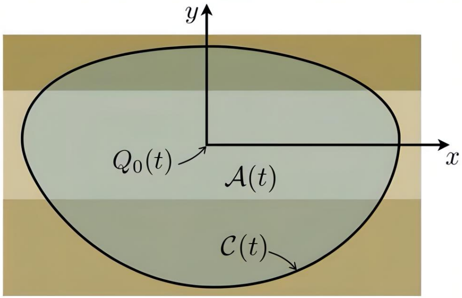



- Extension of PKN model that allows height growth
- Symmetric stress barriers
- Fluid pressure is constant in each vertical cross-section, i.e. *p=p(x)* (toughness dominated propagation with toughness at the vertical tips controlling height growth)
- Fracture opening and height in *(y,z)* cross-sections:
  - Plane strain analytic solution outside of the tip region h>H
  - **Radial solution in the tip region**
- **Non-local elasticity**

$$p(x) = \int_{-l(t)}^{l(t)} G(x,x')\bar{w}(x')\,dx'$$

- **Asymptotic solution at the tip**
- **Viscous height growth (see paper Dontsov&Peirce, 2015)**
- Solution with local elasticity, without viscous height growth, without radial solution at the tip, and without using the tip asymptote corresponds to classical P3D model

# Governing equations




Scaling $E' = \dfrac{E}{1-\nu^2}, \qquad \mu' = 12\mu, \qquad K' = \sqrt{\dfrac{32}{\pi}} K_{Ic}, \qquad C' = 2C_l,$

Apply vertical averaging

$$\bar{w}(x) = \frac{1}{H}\int_{-H/2}^{H/2} w(x,y)\,dy,$$

We also need $h(\bar{w})$

Volume balance $\dfrac{\partial w}{\partial t} + \nabla \cdot \mathbf{q} + \dfrac{C'}{\sqrt{t - t_0(x,y)}} = Q_0(t)\delta(x,y),$

Fluid flow $\mathbf{q} = -\dfrac{w^3}{\mu'}\nabla p,$

Elasticity $p(x,y,t) = \sigma_0(y) - \dfrac{E'}{8\pi}\displaystyle\int_{\mathcal{A}(t)} \frac{w(x',y',t)\mathrm{d}x'\mathrm{d}y'}{[(x'-x)^2+(y'-y)^2]^{3/2}},$

Propagation $\displaystyle\lim_{s\to 0}\frac{w}{s^{1/2}} = \begin{cases} \dfrac{K'}{E'}, & \text{if } V > 0, \\ \dfrac{K_I'}{E'}, & \text{if } V = 0. \end{cases}$

$\Rightarrow$

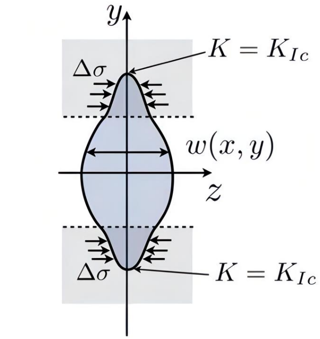
$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}_x}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = \frac{Q_0}{H}\delta(x),$$

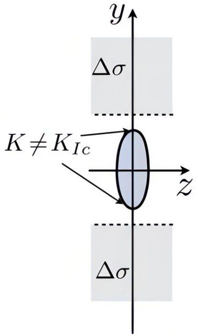
$$\bar{q}_x = -\frac{1}{H\mu'}\frac{\partial p}{\partial x}\int_{-h/2}^{h/2} w^3\,dy,$$

$$p(x) = \int_{-l(t)}^{l(t)} G(x,x')\bar{w}(x')\,dx'$$

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}$$

# Width solution




## Interior (plane strain solution for a uniformly pressurized fracture)

$$w(x,y) = \frac{2}{E'}\sqrt{\frac{2}{\pi h}}K_{Ic}\chi + \frac{4\Delta\sigma}{\pi E'}\left\{-y\log\left|\frac{H\chi+2y\psi}{H\chi-2y\psi}\right| + \frac{H}{2}\log\left|\frac{\chi+\psi}{\chi-\psi}\right|\right\},$$

$$\chi = \sqrt{h^2-4y^2},\ \psi = \sqrt{h^2-H^2},$$

Averaging

$$\bar{w}(x) = \frac{1}{H}\int_{-H/2}^{H/2} w(x,y)\,dy,$$

$$\bar{w} = \frac{H}{E'}\left(\sqrt{\frac{\pi}{2H}}K_{Ic}\left(\frac{h}{H}\right)^{3/2} + \Delta\sigma\sqrt{\frac{h^2}{H^2}-1}\right), \qquad h > H,$$

## Tip (radial solution in toughness regime)

$$w_{\text{rad}} = \frac{4K_{Ic}}{\sqrt{\pi l}E'}\sqrt{l^2-x^2-y^2},$$

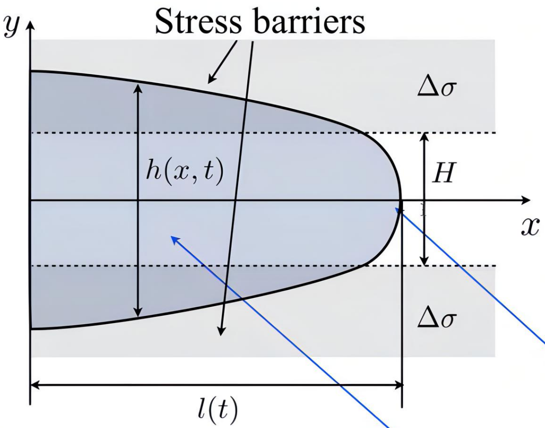
$$h_{\text{rad}} = 2\sqrt{l^2-x^2}.$$

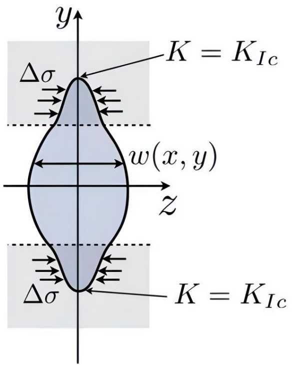
Averaging

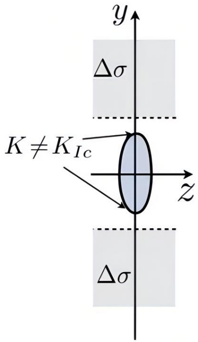
$$\bar{w}_{\text{rad}} = \frac{1}{H}\int_{-h_{\text{rad}}/2}^{h_{\text{rad}}/2} w_{\text{rad}}\,dy = \frac{\pi K_{Ic}}{2HE'\sqrt{\pi l}}h_{\text{rad}}^2,$$

$$h_{\text{rad}} = \bar{w}_{\text{rad}}^{1/2}(2l)^{1/4}\sqrt{\frac{\sqrt{2}HE'}{\sqrt{\pi}K_{Ic}}}. \qquad w_{\text{rad}} = \frac{4H}{\pi h_{\text{rad}}}\bar{w}_{\text{rad}}\sqrt{1-\left(\frac{2y}{h_{\text{rad}}}\right)^2},$$

# Non-local elasticity




Plane strain (local) elasticity

$$p_{P3D} \;=\; \sqrt{\frac{2}{\pi h}}K_{IC} + \Delta\sigma\Big(1 - \frac{2}{\pi}\arcsin\Big(\frac{H}{h}\Big)\Big)$$

KGD

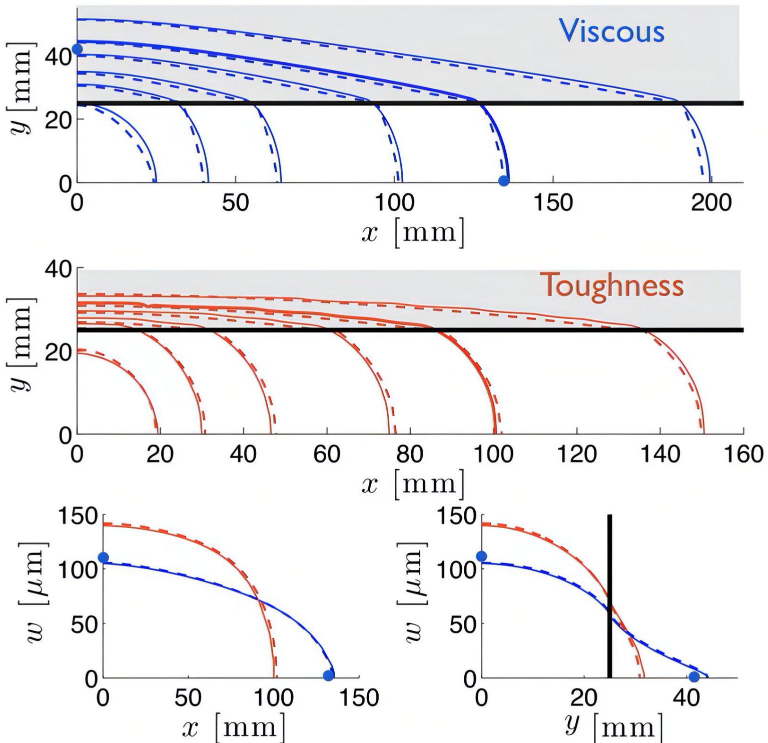
$$w(x,y) \;=\; \frac{2}{E'}\sqrt{\frac{2}{\pi h}}K_{Ic}\chi + \frac{4\Delta\sigma}{\pi E'}\left\{-y\log\left|\frac{H\chi+2y\psi}{H\chi-2y\psi}\right| + \frac{H}{2}\log\left|\frac{\chi+\psi}{\chi-\psi}\right|\right\},$$

For faster evaluation, the width is approximated by two ellipses and the integral is computed analytically

Non-local elasticity

$$p(x) \;=\; -\frac{E'}{8\pi}\int_{-l(t)}^{l(t)}\int_{-\frac{1}{2}h(x',t)}^{\frac{1}{2}h(x',t)}\frac{w(x',y')\,dy'\,dx'}{((x'-x)^2+y'^2)^{3/2}} = \int_{-l(t)}^{l(t)} G(x,x')\bar{w}(x')\,dx'$$

# Enhanced P3D (EP3D) vs. ILSA



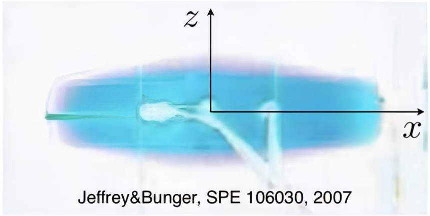

$K' = 0$ ILSA

$K' = 0$ EP3D

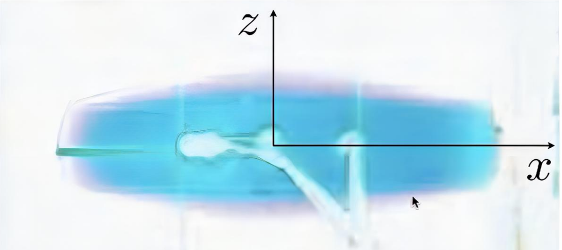
ILSA
$K' = 3.0\ \mathrm{MPa{\cdot}m^{1/2}}$

EP3D
$K' = 3.0\ \mathrm{MPa{\cdot}m^{1/2}}$

Experiment $K' = 0$

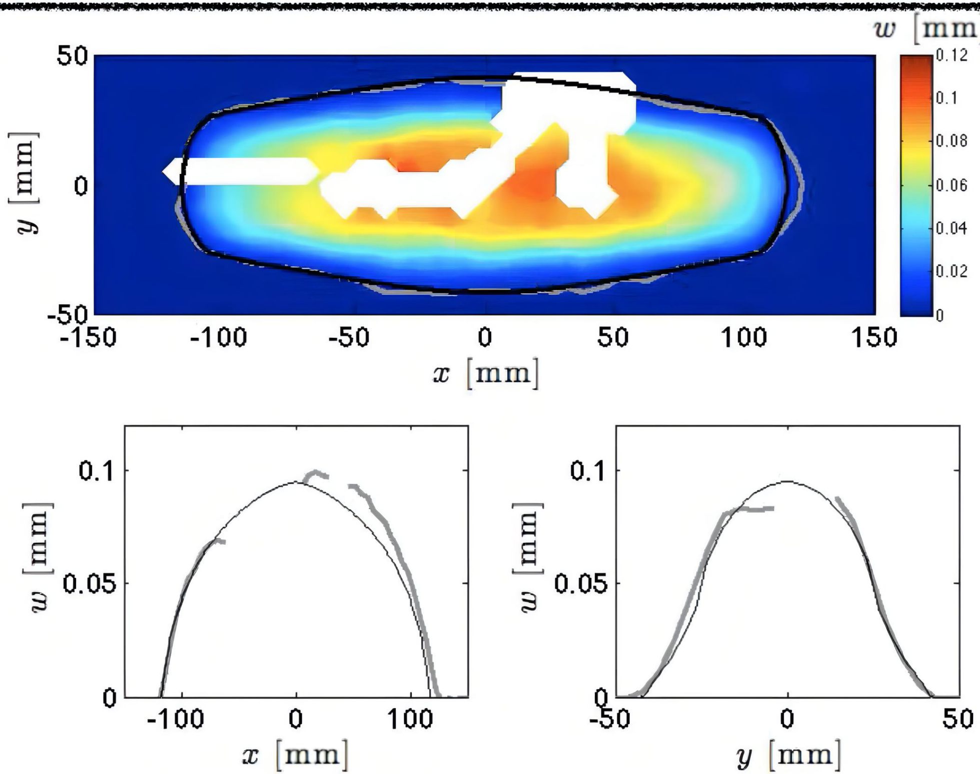


# Comparison with experiment



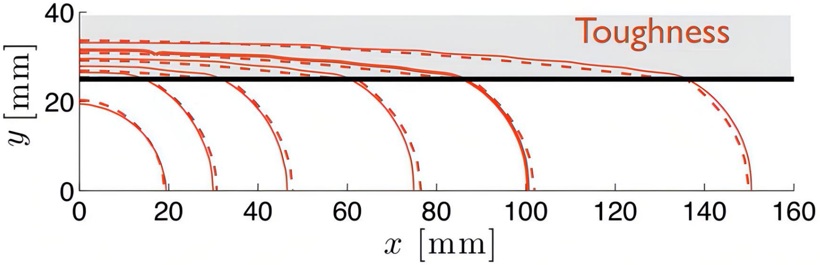
Jeffrey&Bunger, SPE 106030, 2007

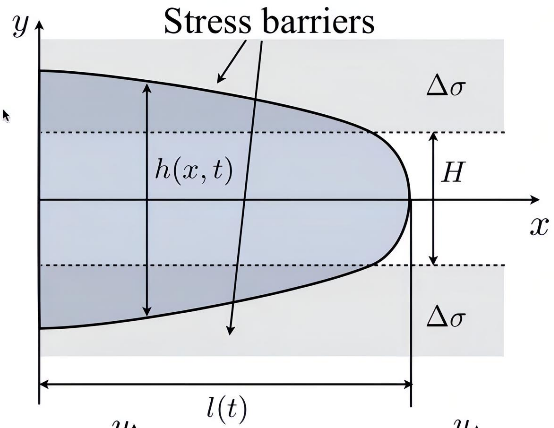
# Comparison with experiment

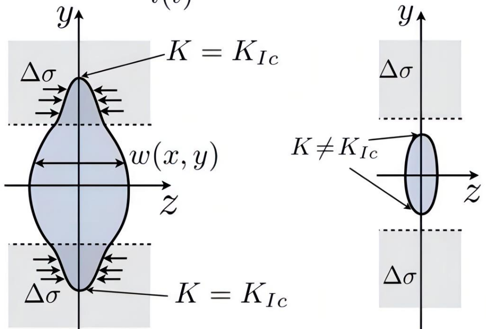


# Things to remember

- Assumptions of P3D model, including equilibrium height growth model, length >> height, horizontal flux, constant pressure in each vertical cross-section, plane strain solution for width in each vertical cross-section
- Non-local elasticity and other corrections allow to significantly improve accuracy of the model
- There are extensions for asymmetric stress layers as well as multiple layers
- This model is suited more for numerical calculations, rather than analysis. However, analysis of the classical P3D model can be found in Adachi et. al, 2010.



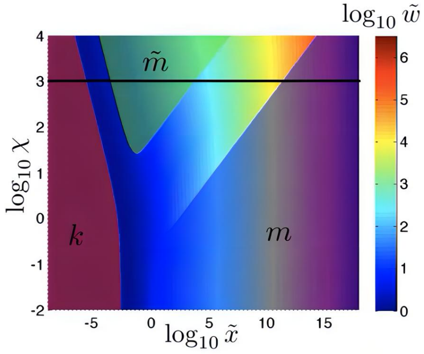
# Recall main learnings: fundamentals and semi-infinite fracture

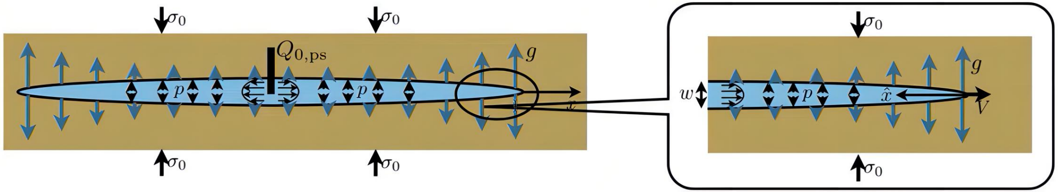
- Work towards a balanced life and invest in your knowledge!

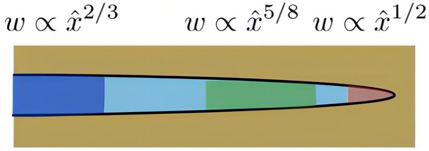
  An investment in knowledge pays the best interest.
  -Benjamin Franklin

- Essential pieces of a hydraulic fracture model: volume balance, fluid flow, elasticity, propagation, proppant transport
- Derivation of the governing equations for planar and plane strain fracture geometries

- Semi-infinite hydraulic fracture as a model for the tip region
  - There are three limiting analytic solutions: toughness, viscosity, and leak-off
  - The global solution gradually transitions from one limiting case to another
  - There is computationally efficient approximate solution for the problem that can be used as a propagation condition for finite fractures

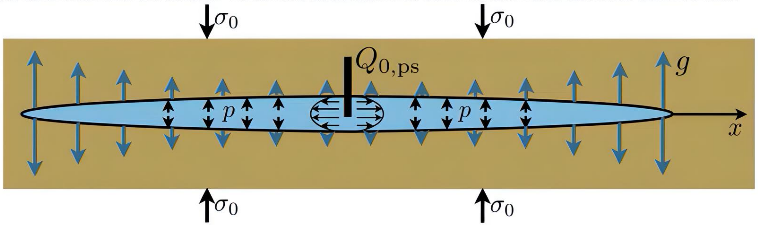


# Recall main learnings: plane strain fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote
- The existence of approximate solution constructed using global volume balance and tip asymptote
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless toughness and dimensionless time




$M$ - storage-viscosity

No toughness, no leak-off

$K$ - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off

Viscosity

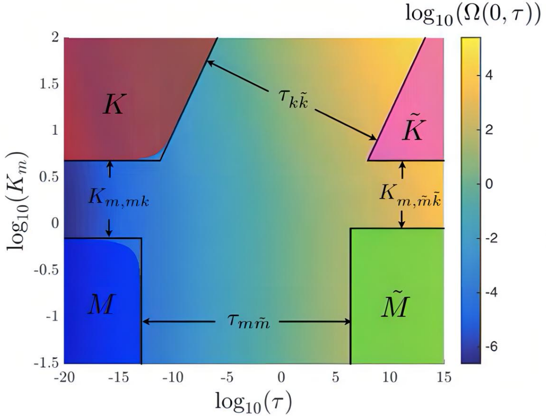
$$w_m = \beta_m \left(\frac{\mu' V}{E'}\right)^{1/3} \hat{x}^{2/3}$$

Toughness

$$w_k = \frac{K'}{E'} \hat{x}^{1/2}$$

Leak-off

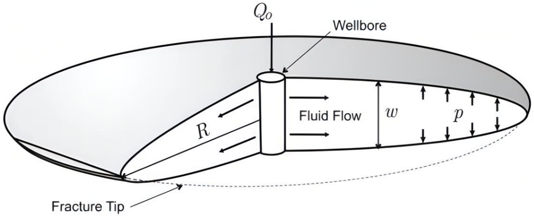
$$w_{\tilde{m}} = \beta_{\tilde{m}} \left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8} \hat{x}^{5/8}$$

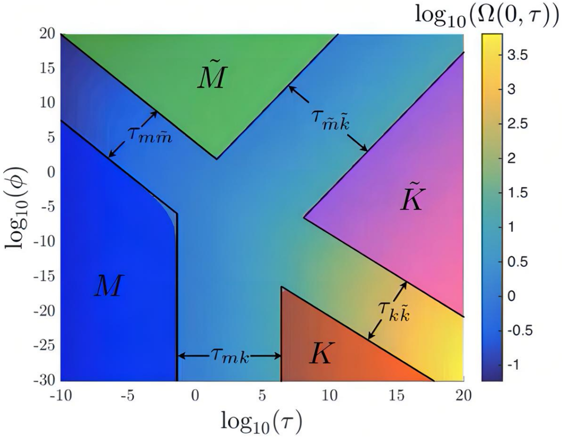



E.V. Dontsov. An approximate solution for a plane strain hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *Int. J. Fract.*, 205:221–237, 2017.

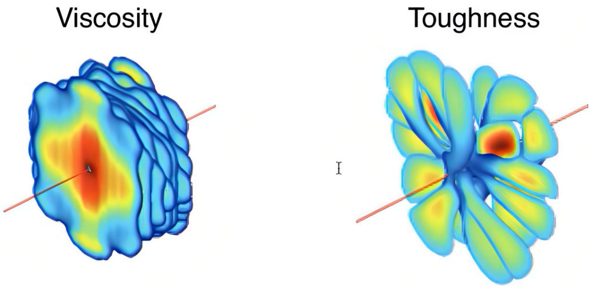
# Recall main learnings: radial fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote (the same as for plane strain)
- The existence of approximate solution constructed using global volume balance and tip asymptote (similar to plane strain)
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless leak-off and dimensionless time
- Fracture regimes affect morphology of multiple hydraulic fractures



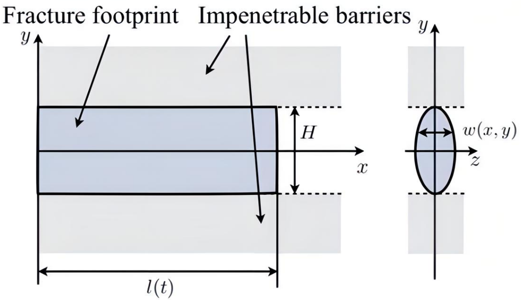



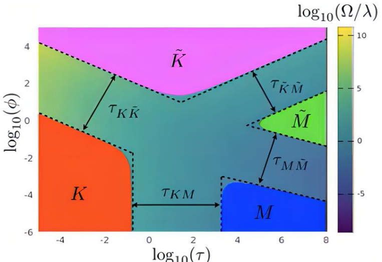

E.V. Dontsov. An approximate solution for a penny-shaped hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *R. Soc. open sci.*, 3:160737, 2016.

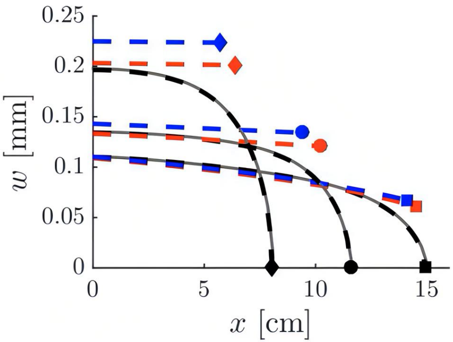


# Recall main learnings: PKN fracture

- Assumptions of PKN model, including constant height, length >> height, horizontal flux, constant pressure in each vertical cross-section, elliptical width in each vertical cross-section
- There are two approaches to solve the problem:
  - Use non-local elasticity, which is good for numerical scheme and leads to superior accuracy
  - Use local elasticity with a specific boundary condition at the tip, which is less accurate, but easier for analysis
- Parametric space for the finite fracture is evaluated
- Limiting vertex solutions exist, as well as the global approximate solution

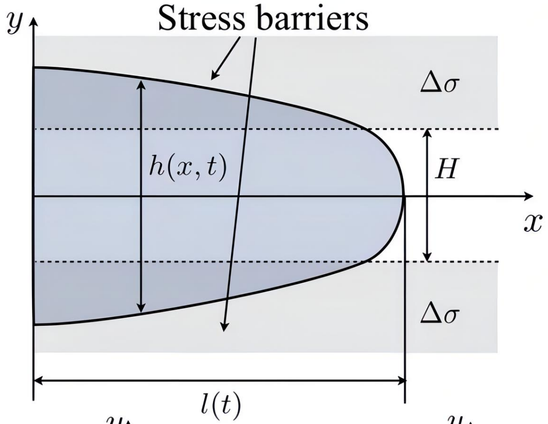


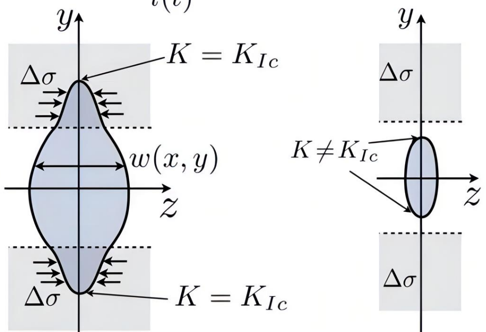



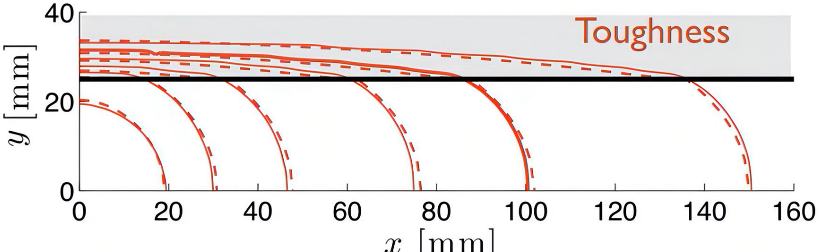



# Recall main learnings: pseudo-3D fracture

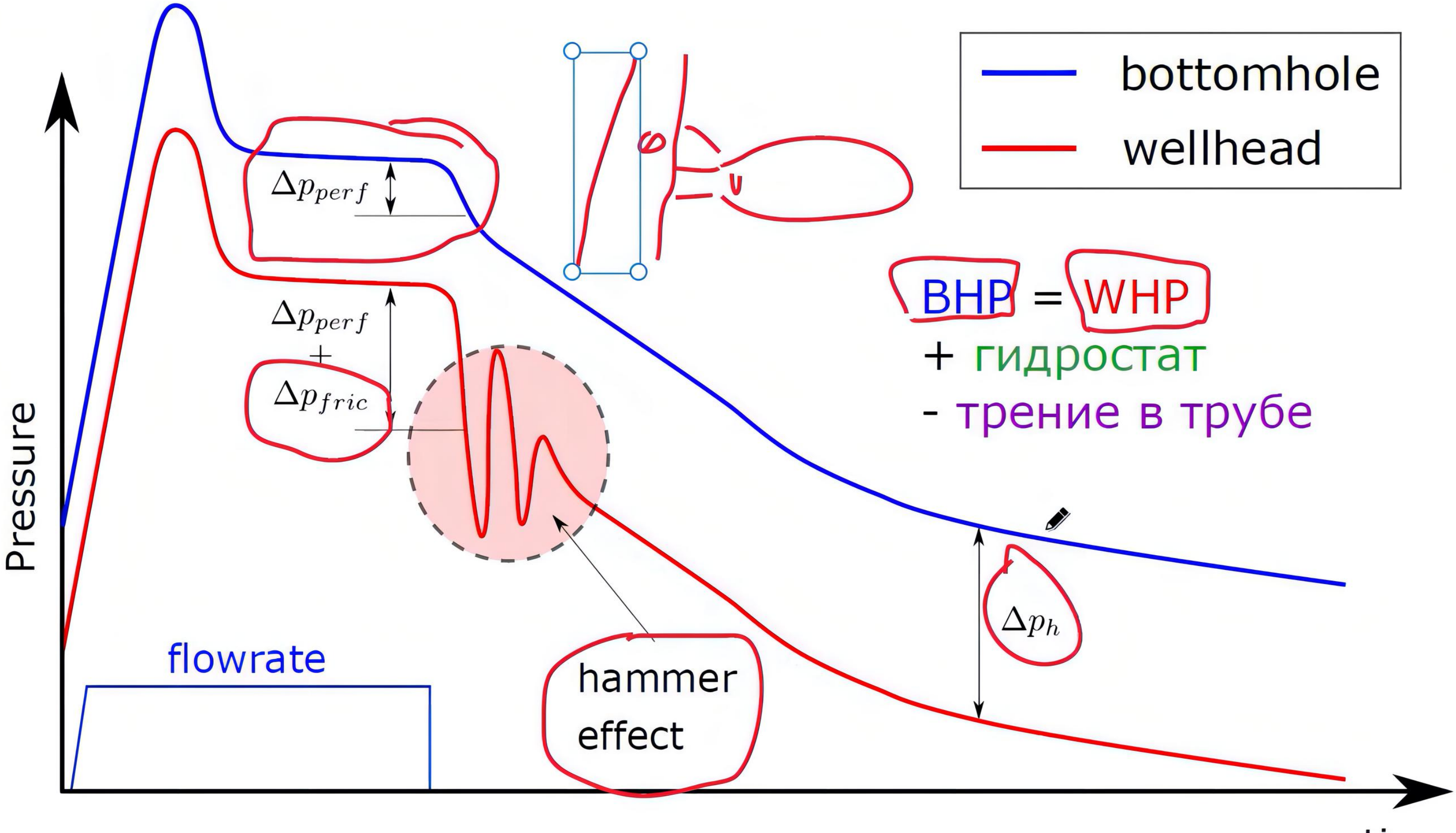
- Assumptions of P3D model, including equilibrium height growth model, length >> height, horizontal flux, constant pressure in each vertical cross-section, plane strain solution for width in each vertical cross-section
- Non-local elasticity and other corrections allow to significantly improve accuracy of the model
- There are extensions for asymmetric stress layers as well as multiple layers
- This model is suited more for numerical calculations, rather than analysis. However, analysis of the classical P3D model can be found in Adachi et. al, 2010.



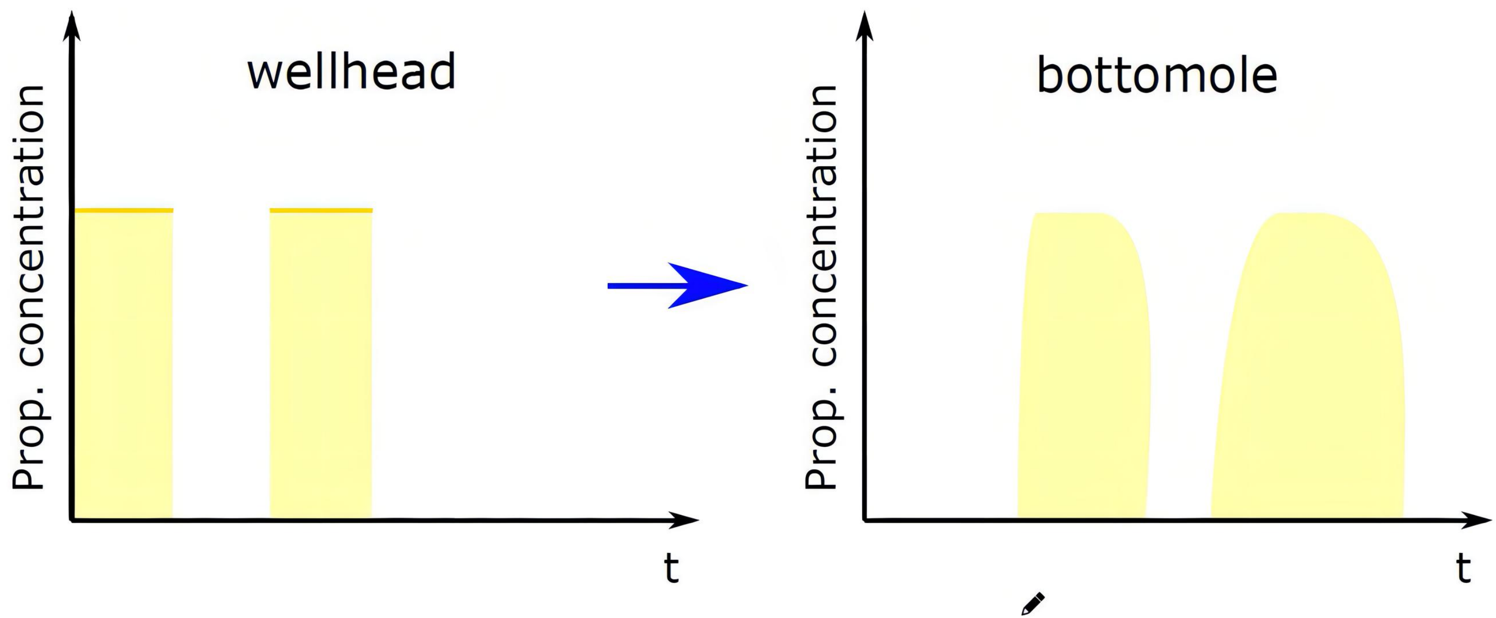



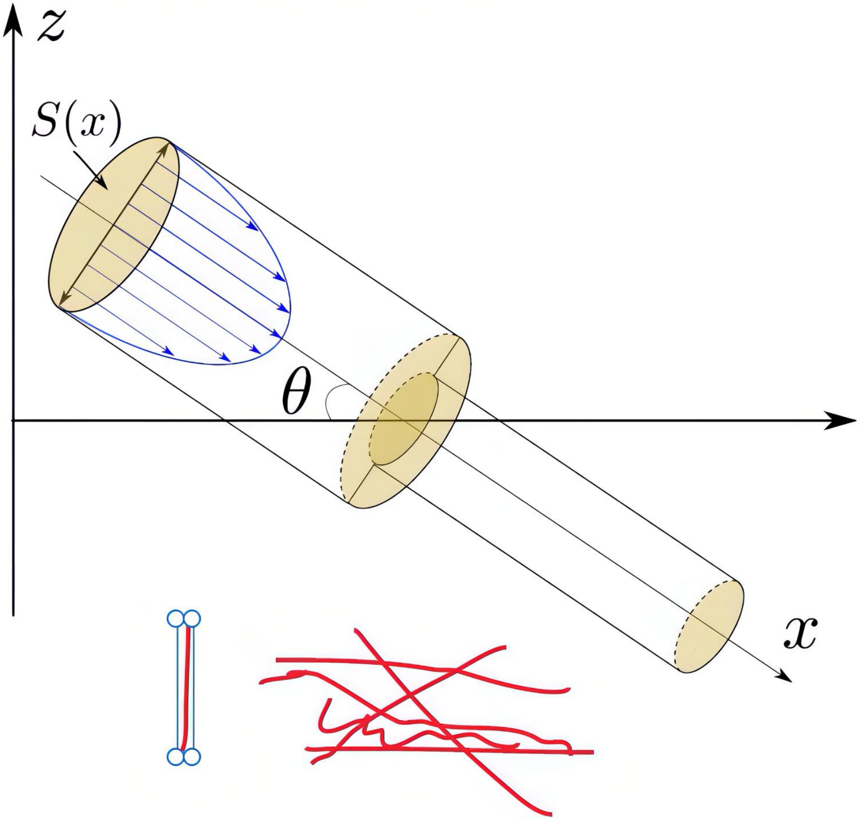
# Зачем моделировать скважину?



# Зачем моделировать скважину?



# Предположения




- Наклонная скважина, переменного радиуса $R$
- Односкоростная модель $\vec{u}_p = \vec{u}_f = \vec{u}_m$
- Жидкость степенная
- Течение не расслаивается
- Ламинарный, переходный, турбулентный режимы
- Сжимаемостью пренебрегаем

# Определяющие уравнения

**Законы сохранения** объема жидкости и проппанта:

$$\frac{\partial(cS(x))}{\partial t} + \frac{\partial(cS(x)u_p)}{\partial x} = 0, \quad c_p \equiv c \tag{1}$$

$$\frac{\partial((1-c)S(x))}{\partial t} + \frac{\partial((1-c)S(x)u_f)}{\partial x} = 0, \quad c_f \equiv 1 - c \tag{2}$$

Односкоростная модель $u_p = u_f = u_m$, где $u_m$ — **среднеобъемная** усредненная скорость смеси по сечению $S(x)$

$$\frac{\partial(S(x)u_m)}{\partial x} = \frac{\partial(Q(t,x))}{\partial x} = 0 \tag{3}$$

$$Q(t,x) = Su_m = const(t) = Q_{inlet}(t), \quad S = \pi R^2 \tag{4}$$

Граничное условие на устье

$$c|_{x=0} = c_{inlet}(t). \tag{5}$$

# Численный алгоритма

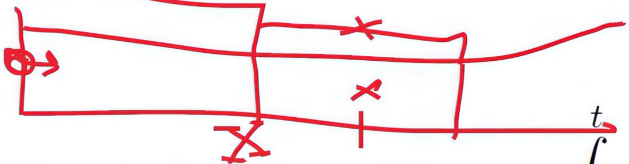
**Упр.** Доказать, что уравнение переноса для проппанта можно переписать в виде

$$\boxed{\frac{\partial c}{\partial t} + u_m \frac{\partial c}{\partial x} = 0.} \tag{6}$$

Разобьем суммарное время закачки на $k-1$ временных интервалов

$$\Delta t_k = t_k - t_{k-1}, \quad t_0 = 0. \tag{7}$$

$F_k$ значение величины $F$ в момент времени $t_k$.
В лагранжевых координатах $(t, X)$ на интервале $[t_k, t_{k+1}]$ имеем решение вида



$$c(t, X(t)) = c(t, X|_{t=t_k}), \quad X(t) = X|_{t=t_k} + \int\limits_{t_k}^{t} u_m(X(s))ds. \tag{8}$$

# Численный алгоритма

$$\frac{dx}{dt} = u_m$$

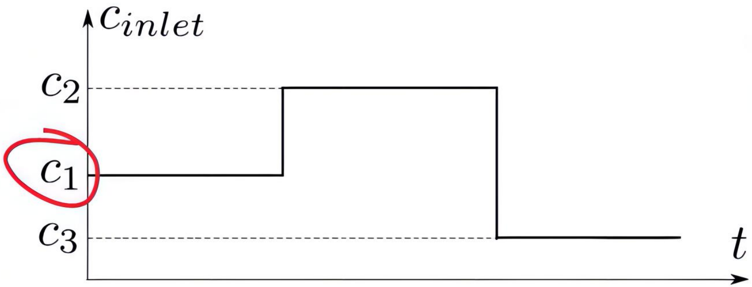



Figure: Кусочно-постоянное по времени граничное условие на концентрацию проппанта

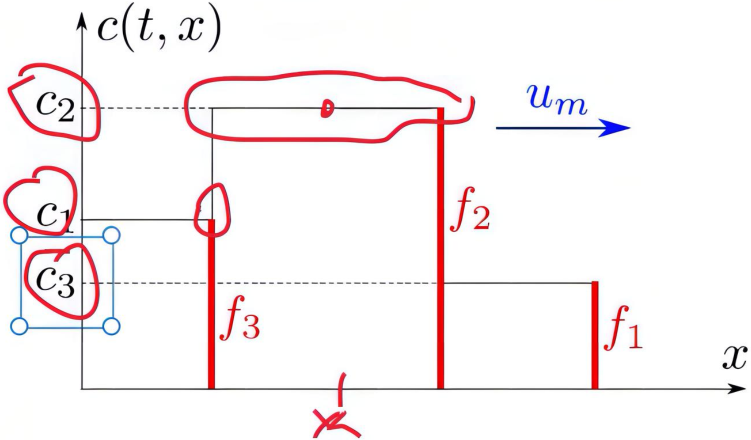



Figure: Фронты концентрации проппанта, распространяющиеся в скважине

# Численный алгоритма

$\varepsilon = \frac{2R}{L}$

Аналогично выводу транпорта проппанта для трещины из ур-ий НС. Вдоль оси $Ox$:

$$0 = -\frac{\mathrm{d}p(x)}{\mathrm{d}x} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rx}) + \rho g \sin\theta.$$

$$\tau_{ij} = \boxed{2\mu_s D_{ij}}, \quad D = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right), \quad \boxed{\tau_{rx} = \mu_s \frac{\partial u_x}{\partial r}}$$

**Формула Нолти:**

вязкость

$K$, $n$

$$\mu_s(c) = \mu_f \left(1 - \frac{c}{c_{max}}\right)^{-2.5 \boxed{n_{clean}}},$$

$c_{max} = 0.65$ — максимальная концентрация упаковки.

$$0 = -\frac{\mathrm{d}p(x)}{\mathrm{d}x} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\mu_s\frac{\partial u_x}{\partial r}\right) + \rho g \sin\theta \tag{9}$$

$\times r$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\mu_s\frac{\partial u_x}{\partial r}\right) = \left(\frac{\mathrm{d}p(x)}{\mathrm{d}x} - \rho g\sin\theta\right) r \tag{10}$$

С учетом $\tau_{rx}|_{r=0} = 0$: ← условие рег-ти $u_x$

$$\mu_s\frac{\partial u_x}{\partial r} = \left(\frac{\mathrm{d}p}{\mathrm{d}x} - \rho g\sin\theta\right)\frac{r}{2} \tag{11}$$

$\cancel{\ln r} \to \infty$

Гран. условие $u_x|_{r=R} = 0$:

$$u_x(r) = \frac{1}{4\mu_s}\left(\frac{\mathrm{d}p}{\mathrm{d}x} - \rho g\sin\theta\right) r^2 + C =$$

$$\underbrace{-\frac{R^2}{4\mu_s}\left(\frac{\mathrm{d}p}{\mathrm{d}x} - \rho g\sin\theta\right)}_{u_{max}}\left(1 - \left(\frac{r^2}{R^2}\right)\right) = u_{max}\left(1 - \left(\frac{r^2}{R^2}\right)\right) \tag{12}$$

# Вычисление средней скорости

$$
u_m = \frac{1}{|S|}\int\limits_S u_x dS = \frac{1}{\pi R^2}\int\limits_0^{2\pi}\int\limits_0^{R} u_{max} r\left(1-\left(\frac{r^2}{R^2}\right)\right) dr d\varphi =
$$

$$
= u_{max}\frac{2\cancel{\pi}\cancel{R^2}}{\cancel{\pi}\cancel{R^2}}\int\limits_0^{R/R}\frac{r}{R}\left(1-\left(\frac{r^2}{R^2}\right)\right) d\left(\frac{r}{R}\right) = \frac{u_{max}}{2} \quad (13)
$$

$$
\boxed{u_m = -\frac{R^2}{8\mu_s}\left(\frac{\mathrm{d}p}{\mathrm{d}x} - \rho g \sin\theta\right)} \quad (14)
$$

В целом отсюда можно найти давление для ламинарного течения!

# Средняя скорость для степенной жидкости

**Упр.** Найти профиль и среднюю скорость для степенной жидкости

$$\tau_{ij} = K_s \dot{\gamma}^{n-1} D_{ij}, \quad \dot{\gamma} = \sqrt{\frac{1}{2} \sum_{i,j=1}^{3} D_{ij}^2},$$

**Ответ:**

$$u_x = u_{max} \left(1 - \left(\frac{r}{R}\right)^{(n+1)/n}\right)$$

Выражение для $u_{max}$ и $u_m$ найдите сами.

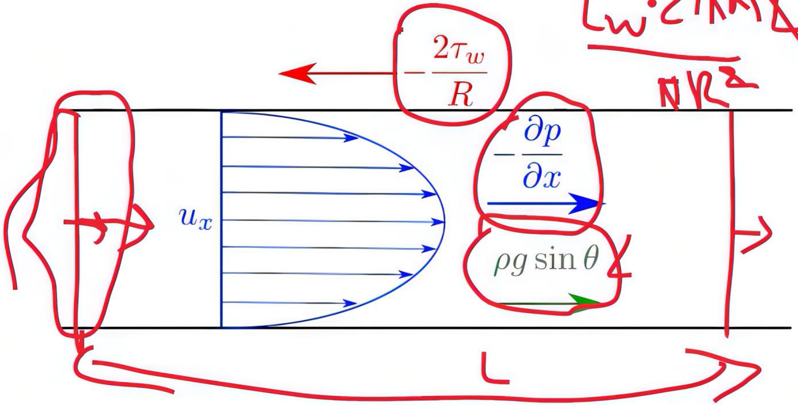
$$0 = -\frac{\mathrm{d}p(x)}{\mathrm{d}x} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rx}) + \rho g \sin\theta \quad \Bigg| \quad \frac{1}{\pi R^2}\int\limits_S (\cdot) dS \tag{15}$$

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}x} = -\frac{2\tau_w}{R} + \bar{\rho} g \sin\theta, \tag{16}$$

где $\tau_w = -\tau_{rx}\big|_{r=R}$ — напряжение сдвига (трения) на стенке трубы.

**Его можно измерить!**

$\frac{\tau_w \cdot 2\pi R L}{\pi R^2} = -(p_2 - p_1)S + S\rho g \sin\theta$




**Коэффициент трения Фаннинга**

$$f_s = \frac{\tau_w}{\rho u_m^2/2}$$

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{\rho u_m^2}{R} f_s + \rho g \sin\theta$$

$$u_x = 2u_m\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\tau_w = -\mu_s \frac{\partial u_x}{\partial r}\bigg|_{r=R} = \frac{4\mu_s u_m}{R} \tag{17}$$

$$f_s = \frac{\tau_w}{\rho u_m^2/2} = \frac{4\mu_s \cancel{u_m}}{R\rho u_m^{\cancel{2}}/2} = \frac{8 \cdot 2}{\rho u_m (2R)\mu_s} = \boxed{\frac{16}{Re}}$$

$$Re = \frac{\rho u_m (2R)}{\mu_s}$$

**Упр.** Показать, что для степенной жидкости

$$f_s = \frac{16}{Re'}, \quad Re' = \frac{\rho u_m^{2-n}(2R)^n}{K_s\left(\frac{3n+1}{4n}\right)^n 8^{n-1}}$$ — обобщ. число Рейнольдса

*Metzner, A. B. & Reed, J. C.* **Flow of Non-Newtonian Fluids - Correlation of the Laminar, Transition, and Turbulent-Flow Regions** // Aiche Journal, 1, 434–440, 1955

# Турбулентное течение

Экспериментальная корреляция

$$1/\sqrt{f_{s,turb}} = \frac{4.0}{n^{0.75}} \log_{10}\left(Re' f_{s,turb}^{1-n/2}\right) - \frac{0.4}{n^{1.2}}. \qquad (18)$$

*Dodge, D. W. & Metzner, A. B.* **Turbulent Flow of Non-Newtonian Systems** // AICHE Journal, 5, 189–204, 1959

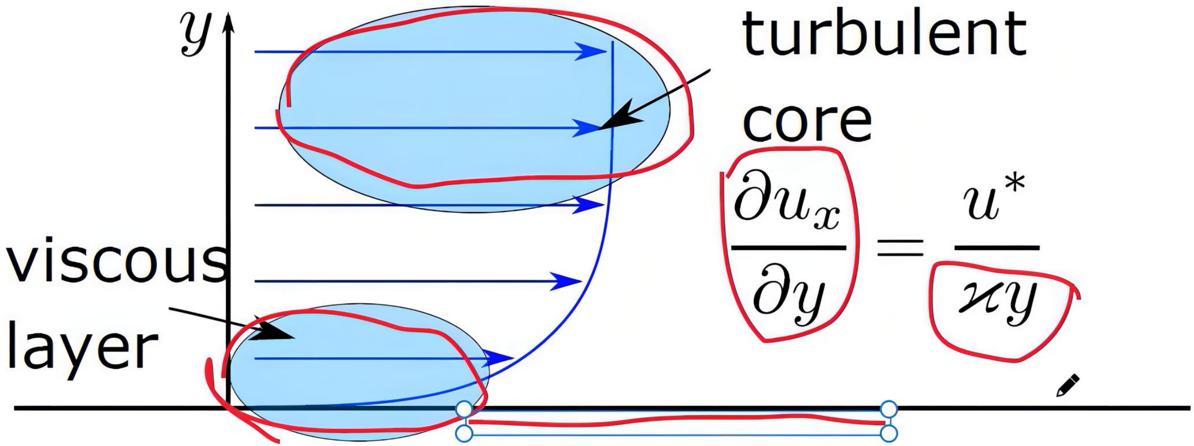
Ограничение $Pr\,(Re')^2 f_{s,turb} > 5 \times 10^5$

Хорошая аппроксимация

$$\boxed{f_{s,turb} = 0.079 n^{0.675} (Re')^{-0.25}} \qquad (19)$$

*Rohsenow, W.M., Hartnett, J.P., Cho, Y.I.* **Handbook of Heat Transfer** (Third Edition)// The McGraw-Hill Companies, Inc., New York, NY (USA), 1998

# Two layered mixing model

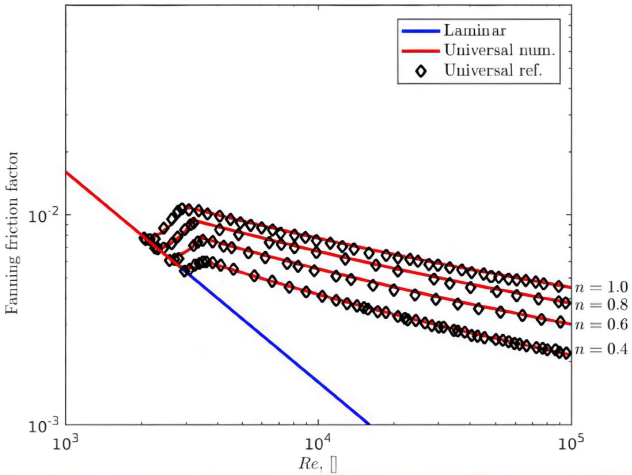



$$\frac{\partial u_x}{\partial y} = \frac{u^*}{\varkappa y}$$

**Турбулентная диффузия пропорциональна $y$ (фон Карман)**

$$u_x \sim A \log_{10} y + B \qquad (20)$$

$$f_s = \begin{cases} f_{s,\,lam}, & Re' \leqslant Re'_{lam}, \\ 10^{\log_{10} f_{s,\,lam} + \log_{10}\left(\frac{f_{s,\,turb}}{f_{s,\,lam}}\right) \frac{\log_{10}\left(Re'/Re'_{lam}\right)}{\log_{10}\left(Re'_{turb}/Re'_{lam}\right)}}, & Re'_{lam} \leqslant Re' \leqslant Re'_{turb} \\ f_{s,\,turb}, & Re' \geqslant Re'_{turb}, \end{cases}$$




$Re'_{lam} = 3250 - 1150n$, $Re'_{turb} = 4150 - 1150n$ — критические значения числа Рейнольдса при переходе в разные режимы

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}x} = -\frac{2\tau_w}{R} + \bar{\rho} g \sin\theta, \tag{21}$$

$$p_{bh}(t,x) = p_{wh}(t) + \Delta p_h(t,x) - \Delta p_{fric}(t,x), \tag{22}$$

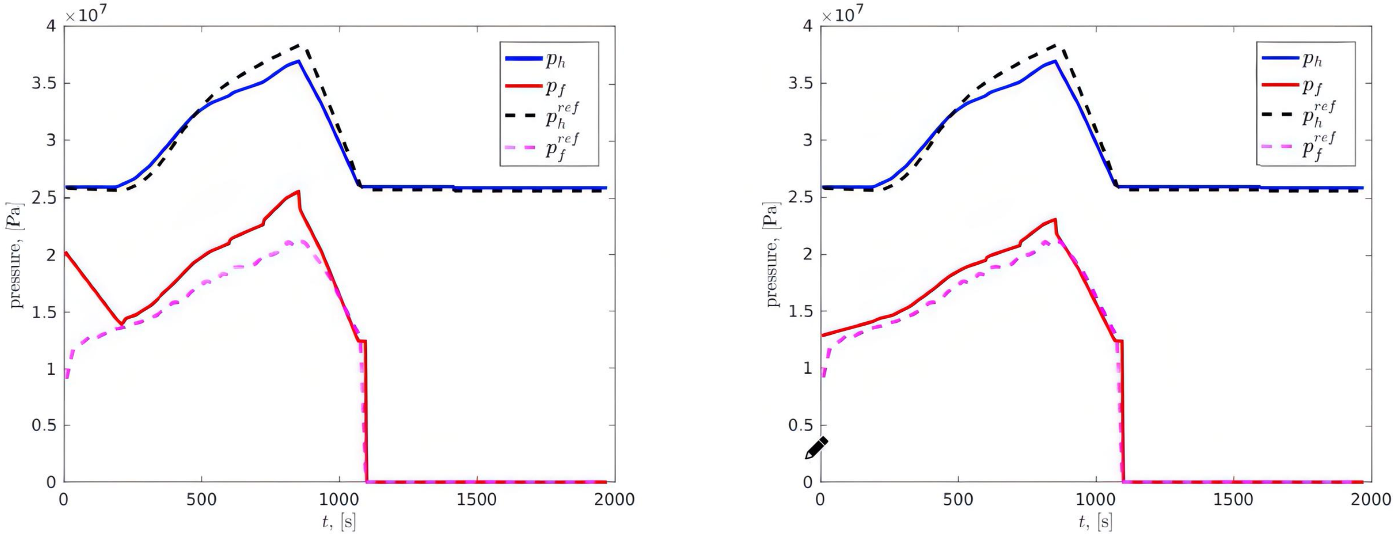
**Гидростатика:**

$$\Delta p_h(t,x) = \int_0^x \rho_s(c(t,s)) g \sin(\theta(s))\, ds \tag{23}$$

$$\rho_s(c) = \rho_p c + \rho_f (1-c) \tag{24}$$

**Трение:**

$$\Delta p_{fric}(t,x) = \int_0^x \frac{2\tau_w(t,s)}{R(s)}\, ds \text{ — давление трения,} \tag{25}$$

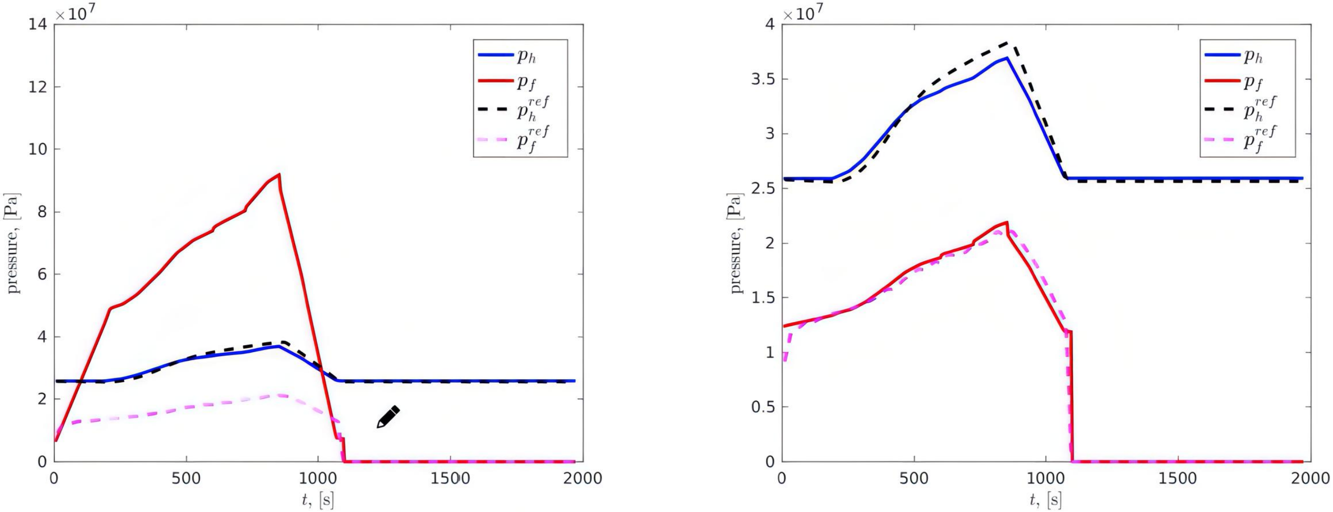



(a) (б)

Figure: Гидростатическое давление и давление трения (модель VS MFRAC). Для базового давления трения используются таблицы

(a): формула Nolte, нач. жидкость №1

(б): формула Keck, нач. жидкость №2

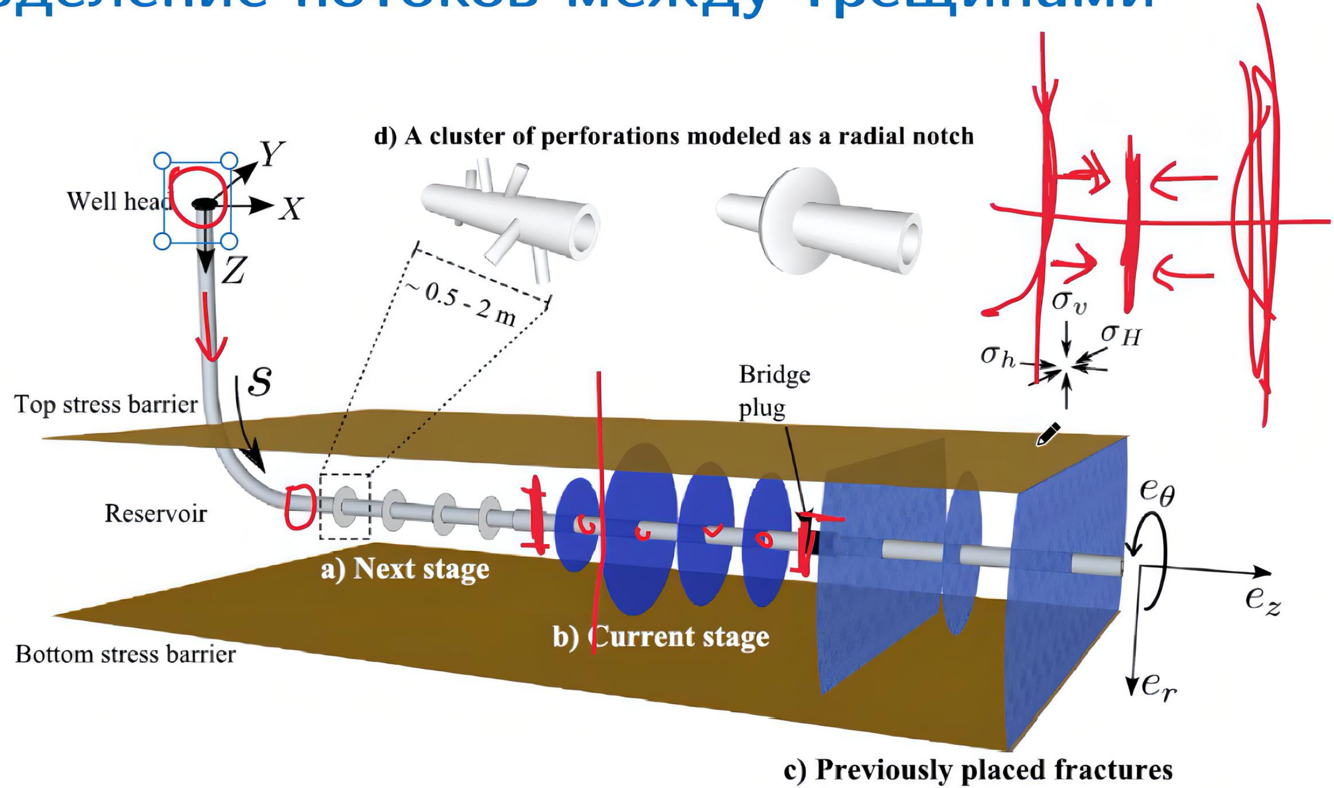


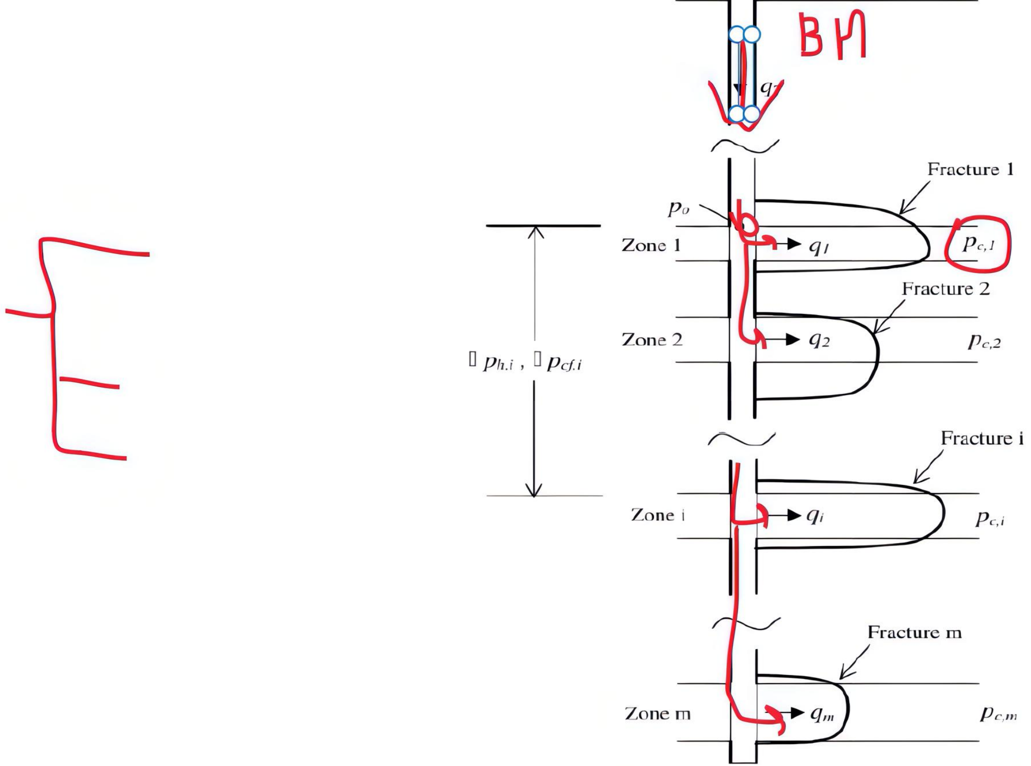

(а) (б)

Figure: Гидростатическое давление и давление трения (модель VS MFRAC). Давления трения расчитывается по формуле Virk

(а): формула Nolte, нач. жидкость №1
(б): формула Keck, нач. жидкость №2, $n = 1.0, 0.88, 0.47$ и $K = 10^{-3}, 40 \times 10^{-3}, 0.85$ Па$\cdot$с$^n$

# Разделение потоков между трещинами






Необходимо расчитывать при одновременной закачке в несколько портов

## Two Kirchhoff's laws

1. Slurry is incompressible $\Longrightarrow$ volume conservation:

$$\boxed{Q_0 = \sum_{i=1}^{N} Q_i} \tag{26}$$


2. Pressure continuity:

$$p_0 = \underbrace{\sigma_{min,i} + p_{net,i}}_{p_{frac}} + \Delta p_{perf,i} - \sum_{j=1}^{i} \Delta p_{h,j} + \sum_{j=1}^{i} \Delta p_{fric,j}, \tag{27}$$

where

- $\sigma_{min,i}$ — closure pressure (min in-situ stress) in $i^{\text{th}}$ port
- $p_{net,i} = p_{frac,i} - \sigma_{min,i}$ — net pressure in $i^{\text{th}}$ port( from frac model)
- $\Delta p_{perf,i}$ — perforation pressure drop at $i^{\text{th}}$ port
- $\Delta p_{h,i}$ — hydrostatic pressure drop between $i^{\text{th}}$ and $(i-1)^{\text{th}}$ ports
- $\Delta p_{fric,i}$ — friction pressure drop between $i^{\text{th}}$ and $(i-1)^{\text{th}}$ ports

## Analytical approach (PKN model)

- $p_{net,i}$ — net pressure

$$\boxed{p_{net,i}(Q_i) = a_i \, Q_i^{\frac{n}{2n+3}} \, V_i^{\frac{1}{2n+3}},} \tag{28}$$

$$a_i = \left( \frac{(n+3)(2n+1)^n \, K \, (E_i')^{2n+2}}{\pi \, 2^{2n} \, n^n \, \phi^n \, h_i^{3n+3}} \right)^{\frac{1}{2n+3}}, \tag{29}$$

where

- $a_i$ — stiffness parameter
- $Q_i, V_i$ — injection rate and fracture volume in i'th layer
- $K, n$ — power-law fluid rheology parameters
- $E_i'$ — plane strain modulus
- $\phi$ — geometric parameter
- $h_i$ — pay zone thickness

# Fracture net pressure

## Numerical approach

- $p_{net,i}$ — net pressure

$$p_{net,i} = p_{net,i}(Q_i) \tag{30}$$

## Preparation for Newton method

$$p_{net,i}(Q_i + \Delta Q_i) \approx p_{net,i}(Q_i) + \frac{\partial p_{net,i}}{\partial Q_i}(Q_i)\Delta Q_i \tag{31}$$

## Calculate derivatives

$$\frac{\partial p_{net,i}}{\partial Q_i}(Q_i) \approx \frac{p_{net,i}(Q_i + \delta Q_i) - p_{net,i}(Q_i)}{\delta Q_i}, \quad \delta Q_i = 10^{-3} \times Q_i \tag{32}$$

# Perforation friction

- $\Delta p_{perf,i}(Q_i)$ — perforation friction pressure drop

$$\Delta p_{perf,i} = \frac{8\rho_s}{\pi^2\, C_{d,i}^2\, n_{p,i}^2\, d_{p,i}^4} Q_i|Q_i|, \tag{33}$$

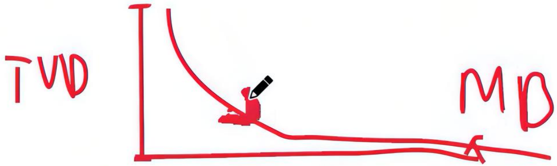
where

- $\rho_s$ — (averaged) density of the slurry
- $n_{p,i}, d_{p,i}$ — number and diameter of perforations
- $C_{d,i} = \frac{min(d_{jet})}{d_p}$ — dimensionless discharge coefficient:
  - $C_{d,i} \in [0.5, 0.6]$ without solids
  - $C_{d,i} \in [0.6, 0.95]$ with solids due to perforations erosion

# Hydrostatic pressure drop

- $\Delta p_{h,i}$ is hydrostatic pressure change in each casing interval according to the weight of the fluid column:

$$\Delta p_{h,i}(t,x) = \int_{x_{i-1}}^{x_i} \rho(c(t,s)) \cdot g \cdot \sin(\theta(s))ds, \tag{34}$$




where

  - $x_i$ — $i^{\text{th}}$ port MD
  - $\rho(c(t,s))$ — density of the slurry, that depends on dynamic proppant concentration
  - $g$ — gravitational acceleration
  - $\theta(s)$ — angle between wellbore and land surface in given point (cross-section)

# Tube friction pressure drop

- $\Delta p_{fric,i}$ is friction pressure drop in each casing interval

$$\Delta p_{fric,i} = \int_{x_{i-1}}^{x_i} f\,\frac{\rho u_{m,i}^2}{R_i} = \int_{x_{i-1}}^{x_i} \frac{\rho(c(t,s))\, f(Re)\left(Q_0 - \sum_{j=1}^{i-1} Q_j\right)^2}{R_i(s)\, S_i^2(s)}\,ds \tag{35}$$

where

- $f = \frac{\tau}{\rho u_{m,i}^2/2}$ — Fanning friction factor: ratio between the local shear stress and the local flow kinetic energy density
- $\rho(c(t,s))$ — density of the slurry
- $u_{m,i} = \frac{Q_0 - \sum_{j=1}^{i-1} Q_j}{S_i}$ — average flow velocity in the pipe (**important:** no $Q_i$ term since we mean pressure drop along wellbore till given i'th port)
- $S_i$ — cross-section pipe area for i'th port
- $R_i$ — radius of given circular pipe
- $Re$ — Reynold's number

Equations (26)-(27) generate (n+1)x(n+1) system of nonlinear equations in (n+1) unknowns, which can be expressed in vector form

$$Q^T = [Q_1, Q_2, ..., Q_N, p_0], \tag{36}$$

and residual vector

$$F^T = [F_1, F_2, ..., F_N, F_{N+1}], \tag{37}$$

$= 0$

$\to \min$

where

$$F_i = \begin{cases} \sigma_{min,i} + p_{net,i} + \Delta p_{perf,i} - \\ \sum_{j=1}^{i} \Delta p_{h,j} + \sum_{j=1}^{i} \Delta p_{fric,j} - p_0, & \text{for } i \leq N \\ Q_0 - \sum_{i=1}^{N} Q_i, & \text{for } i = N+1 \end{cases}$$

# Iterative procedure

then Jacobian matrix can be defined as

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial Q_1} & \cdots & \frac{\partial F_1}{\partial Q_N} & \frac{\partial F_1}{\partial p_0} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial F_{N+1}}{\partial Q_1} & \cdots & \frac{\partial F_{N+1}}{\partial Q_N} & \frac{\partial F_{N+1}}{\partial p_0} \end{bmatrix} \tag{38}$$

so the iterative procedure for estimating the vector $\overline{Q}$ is that via

$$\overline{Q}^{k+1} = \overline{Q}^{k} - J^{-1}\,\overline{F}^{k}, \tag{39}$$

the convergence condition $|\overline{Q}^{k+1} - \overline{Q}^{k}|^2 \leq \tau \sim 10^{-4}$ can be checked. If no, the next iteration in (39) goes until this scheme converges.

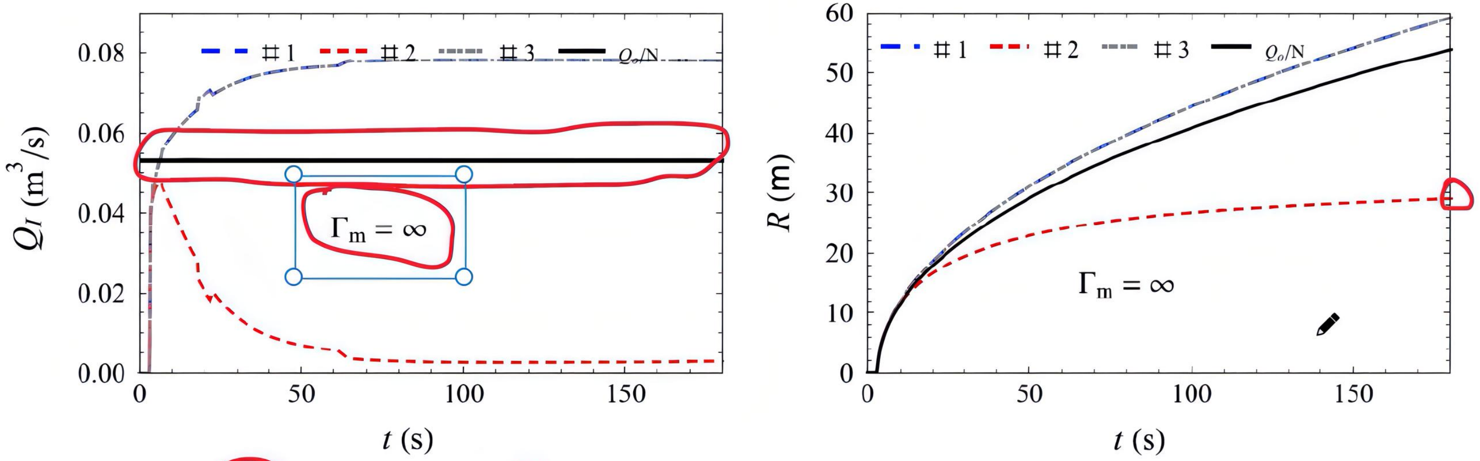
# Initial guess

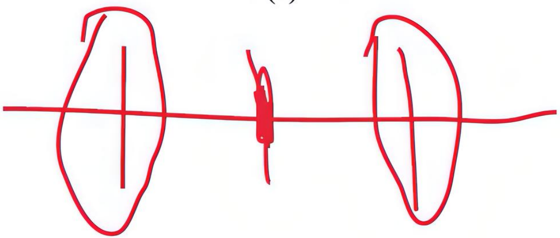
During the computation the initial guess for the iterative procedure will be the values from the previous timestep, however, there are no such values at the beginning.
The impact of the initial solution on the hydraulic fracturing problem decreases as volume of injected fluid increases above volume of the initial fracture, then for heuristic reasons all the injection rates considered to be uniform, while the bottomhole pressure equals to the pressure of closed fracture — closure pressure in the layer:

$$Q_i = Q_0/N \quad \forall i \in [1, N], \qquad p_0 = \sigma_i \tag{40}$$

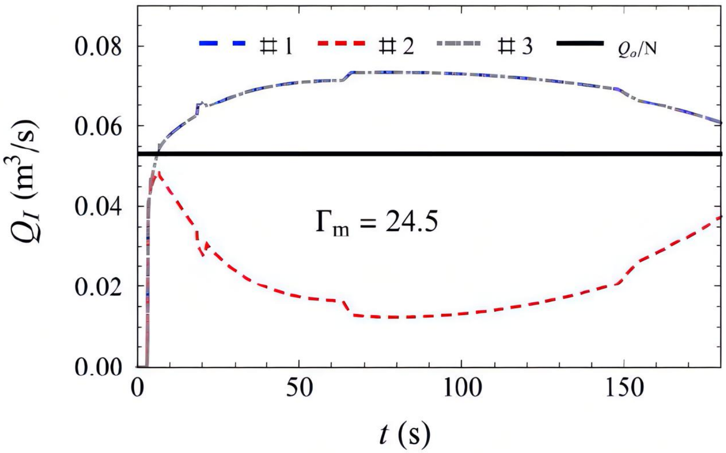
# Разделение потоков между трещинами

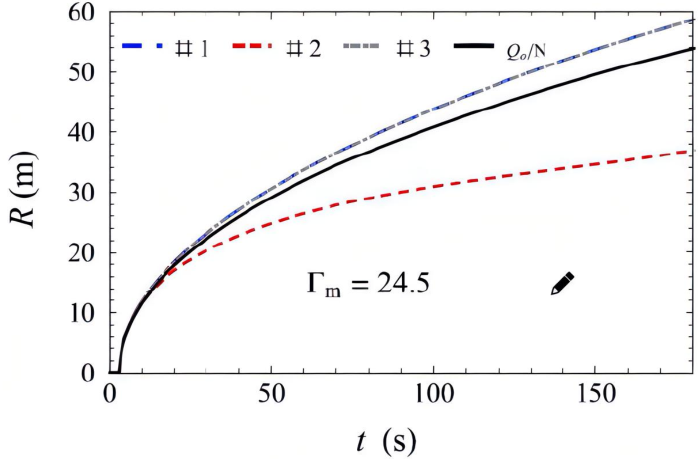
$$\Gamma_m = \frac{\sim \text{stress shadow}}{\Delta p_{ref}}$$



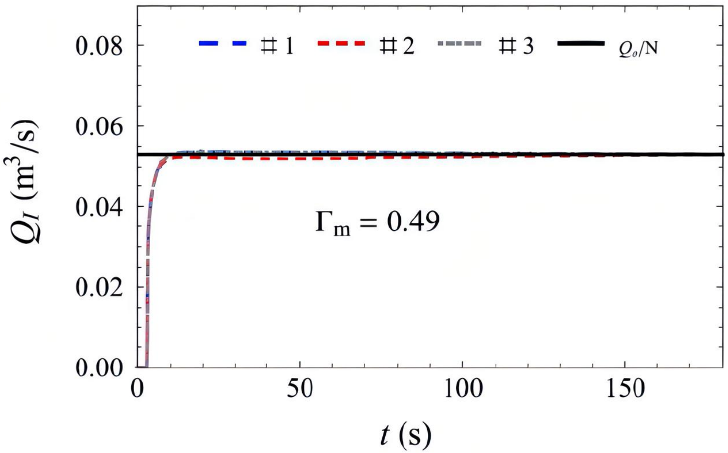



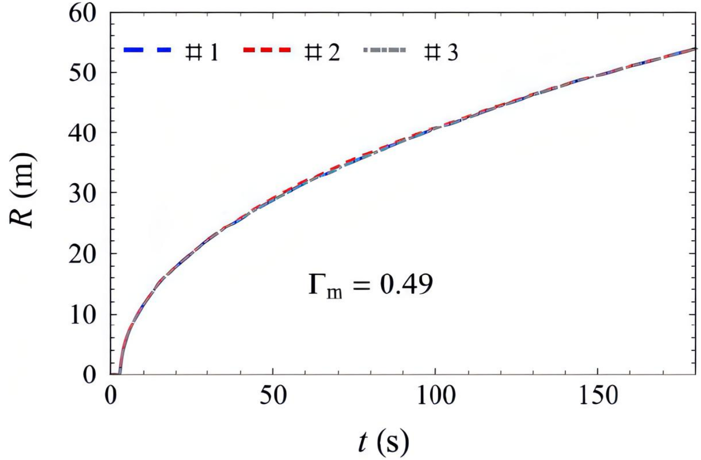
# Разделение потоков между трещинами



# Разделение потоков между трещинами



# Введение

Во время гидравлического разрыва пласта (ГРП) необходимо знать информацию о давлении в нижней части скважины и параметрах созданной трещины.

Одним из способов быстрой оценки параметров трещины является анализ волнового пакета, возникающего при резкой остановке закачки.

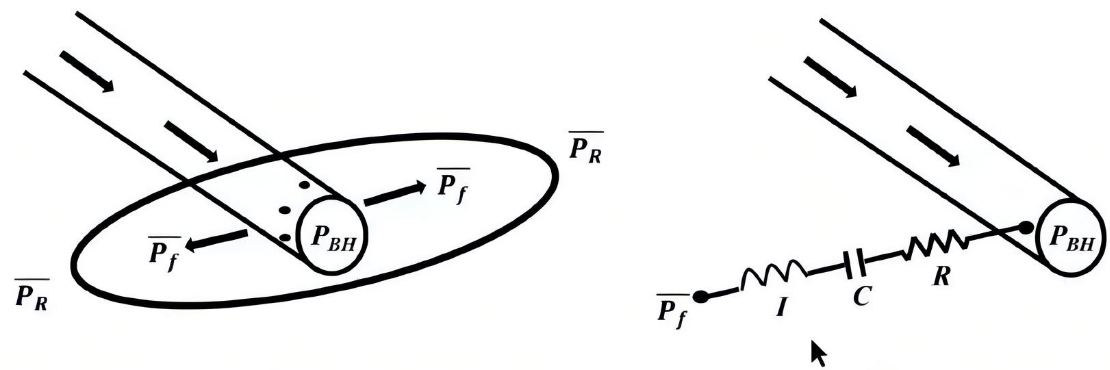
Работы по изучению гидроудара и влияния трещины на вид волнового пакета начались в 80-е годы прошлого века [1], [3] и ведутся до сих пор [2], [5], [6]

В работах [2], [6] была предложена модель расчета гидроудара в скважине и нахождения характеристик трещины.

Целью данной работы является определение оптимального набора параметров, необходимого для расчета волнового пакета.

# Введение

Рассматривается течение слабосжимаемой жидкости в скважине, соединенной с трещиной через тонкие каналы.




Течение в скважине описывается уравнениями акустики, влияние трещины ГРП учитывается через условие на нижней границе.

Характеристики трещины находятся из параметров R, C, I [8]

# Введение

Приведем модель, рассмотренную в работе Carey, Mondal, Sharma 2015 [2]:

- **уравнения акустики**:

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{c^2}{\rho_0} \frac{\partial \rho}{\partial x} = g \sin\theta - f \frac{u|u|}{R}, \tag{2}$$

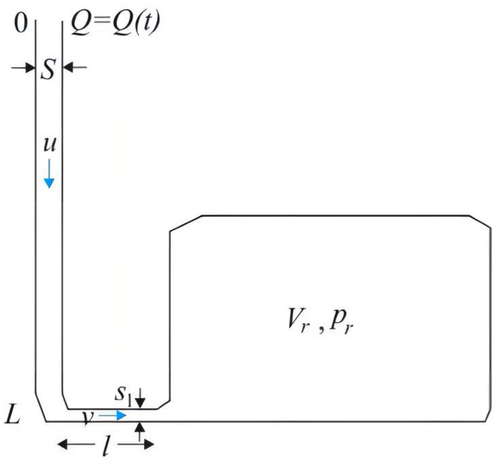
  где $u$ — скорость, $\rho$ — плотность, $p = c^2 (\rho - \rho_0) + p_0$, $c^2 = \mathrm{const}$, $\theta$ — угол наклона, $f$ — коэффициент трения Фаннинга

- **граничные условия**:
  - при $x = 0$: $Q = S_0 u = Q(t)$ — задан расход от времени
  - при $x = L$:

$$p_L - p_r = RQ + \frac{1}{C} \int Q dt + I \frac{dQ}{dt}, \tag{3}$$

    где $p_L$ — давление на дне скважины, $p_r$ — давление в резервуаре (трещине)

# Математическая модель: законы сохранения




Рассматривается скважина переменного сечения $S(x)$, которая через перфорации площадью $S_p$ и длиной $l_p$ сопрягается с резервуаром (трещиной).

Обозначим $u$ — скорость жидкости в скважине, $v$ — скорость жидкости в перфорациях, $p_r$ — равновесное давление в трещине.

Предположим, что объем трещины меняется несущественно $V_r \equiv \mathrm{const}$ и давление внутри трещины распределено равномерно $p_r = p_r(t)$

Течение в скважине описывается уравнениями акустики в каналовом приближении

$$(\rho S)_t + (\rho S u)_x = 0 \tag{4}$$

$$u_t + \left(\frac{u^2}{2} + c_0^2 \ln\frac{\rho}{\rho_0}\right)_x = g\sin\theta - f\frac{u|u|}{R} \tag{5}$$

# Математическая модель: граничные условия

- **При** $x = 0$ :

  На верхней границе задается расход жидкости
  $Q = Q(t) = u(t) \cdot S(0)$

- **При** $x = L$ :

  Закон сохранения массы для резервуара

$$\rho_t = Mv, \quad M = \frac{\rho_0 S_p}{V_r}, \tag{6}$$

Уравнение импульса для перфораций

$$v_t = \frac{K}{\rho_0}(p_L - p_r) - \varphi, \quad K = \frac{1}{l_p}, \tag{7}$$

для ламинарного случая $\varphi = \frac{8\pi\nu v}{S_p}$, для турбулентного течения $\varphi = f\frac{v|v|}{r_p}$,

$$S(L) \cdot u(L, t) = S_p \cdot v(t) \tag{8}$$

# Математическая модель

Выпишем условие Carey, Mondal на нижней границе в виде:

$$\Delta p = IQ_{tt} + RQ_t + \frac{Q}{C}, \tag{9}$$

В ламинарном случае из уравнений (6), (7) получаем условие

$$\Delta p = \frac{\rho_0}{K} v_{tt} + \frac{8\pi\mu}{S_p K} v_t + c^2 M v, \tag{10}$$

Таким образом, при выборе коэффициентов

$$I = \frac{\rho_0}{K S_p}, \quad R = \frac{8\pi\mu}{S_p^2 K}, \quad C = \frac{1}{c^2 S_p M}$$

граничные условия в обоих моделях совпадают

# Математическая модель

Условие (10) упрощается в случае, когда можно пренебречь $v_t$ в левой части (7),

$$\rho_t = K_1 \left(p_L - p_r\right), \quad K_1 = \frac{KMS_p}{8\rho_0 \pi \nu}, \quad M = \frac{\rho_0 S_p}{V_r}, \qquad (11)$$

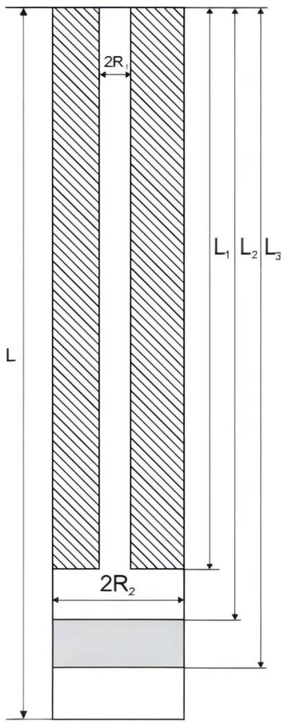
приходим к условию, в которое входит давление в резервуаре.

Расчет волнового пакета в скважинах Patzek & De, 2000 и Hwang, 2017 проводился по двум моделям:

- модель с условием (10) на нижней границе (пересчет коэффициентов R, C, I)
- упрощенная модель с условием (11) на нижней границе

Было показано, что обе модели дают хорошее совпадение с экспериментальными данными, поэтому для рассматриваемых скважин достаточно использовать более простую равновесную модель.

# Примеры расчетов: Wang




В работе Wang, 2008 приведено описание гидроудара в экспериментальной скважине, конструкция которой приведена на рисунке слева.

Было замечено, что для данной скважины давление в нижней части за время существования волнового пакета практически не менялось.

Это позволило расщепить полную задачу на задачи в верхней и нижней частях скважины, т.к. в точке $x = L_1$ условия (6), (7) заменяются на условие $p(L_1) = \mathrm{const}$

# Примеры расчетов

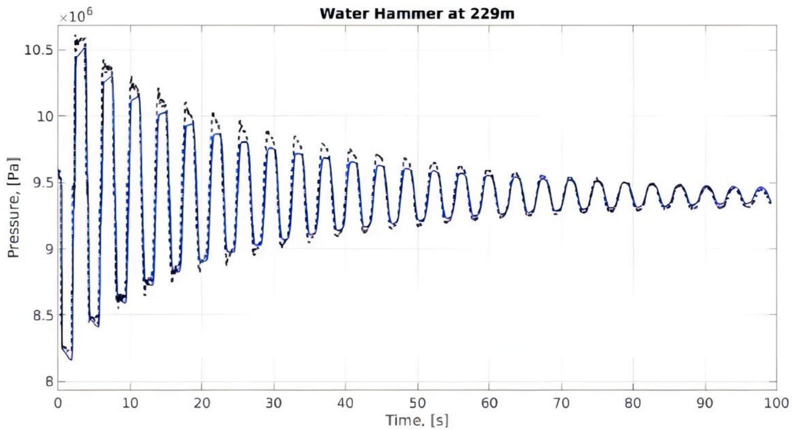
В процессе эксперимента (Wang) резко прекращалась подача воды в скважину и проводились замеры давления на глубинах 229, 762 и 1356 метров.

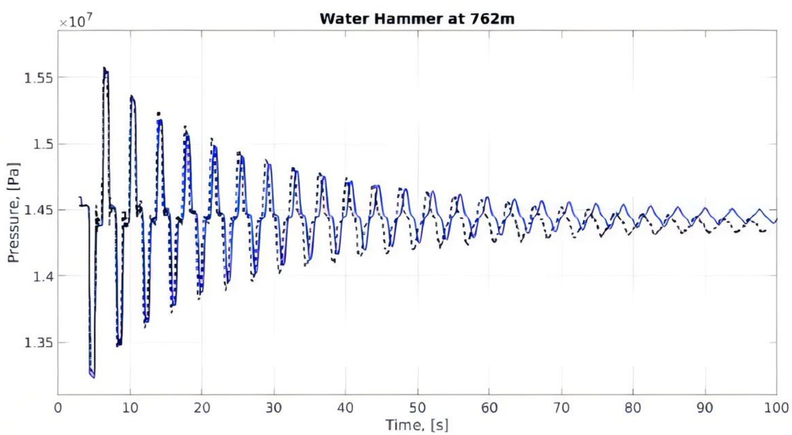
Параметры скважины и жидкости, использующиеся в расчетах, были выбраны в соответствии с экспериментальными данными

$$L = 1433 \text{ м},\ L_1 = 1301 \text{ м},\ L_2 = 1379 \text{ м},\ L = 1402 \text{ м}, \qquad (12)$$

$R_1 = 0.03711$ м, $R_2 = 0.0889$ м, $c_0 = 1370$ м/с, $p_a = 7.5{\cdot}10^6$ Па, $u_0 = 0.887$ м/с, $\theta = 7.3°$, $g = 9.8{\cdot}\cos\varphi$ м/$c^2$, $K_0' = K_0 c_0^2 = 200$ 1/с.

# Примеры расчетов: течение в верхней части



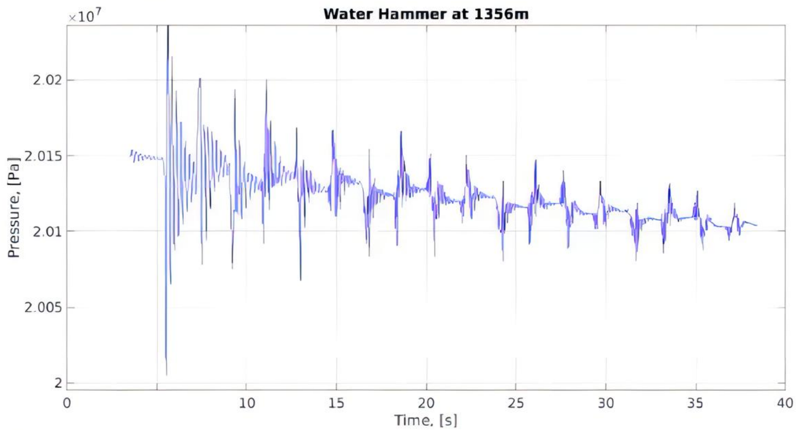




На верхнем рисунке — реальное давление и результаты расчетов на глубине 229 м

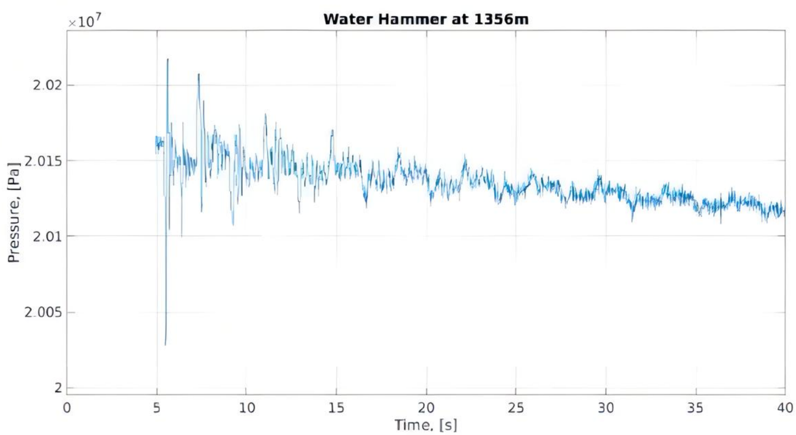
На нижнем рисунке — на глубине 762 м.

Данный результат был получен без привлечения дополнительной информации о течении в нижней части скважины.

# Примеры расчетов: течение в нижней части

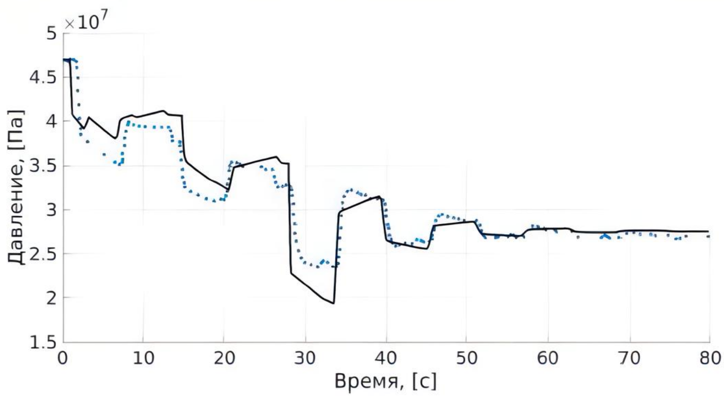


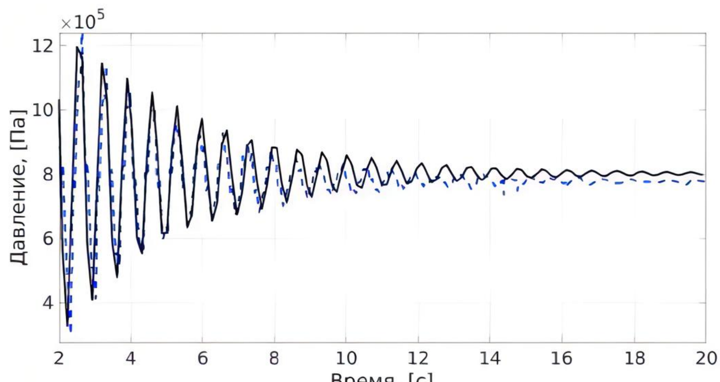

На верхнем рисунке показаны результаты расчетов давления на глубине 1356 м .




На нижнем рисунке — реальное значение давления на глубине 1356 м.

Рассмотрим скважины с более сложной конструкцией, описанные в работах Patzek & De, 2000 и Hwang, 2017




На верхнем рисунке показано сравнение реального давления (син.) и расчета по равновесной модели. Экспериментальные данные из работы [6].

На нижнем рисунке — расчет одной из скважин [4].

# Заключение

- В работе была получена модель, описывающая развитие волнового пакета в скважине, соединенной с резервуаром. Данная модель при выборе коэффициентов может быть сведена к [2], [6].
- Показано, что в данную модель можно упростить и снизить количество неизвестных параметров.
- Проведены расчеты по полной и упрощенной моделям и показано, что обе модели обеспечивают хорошее совпадение с экспериментом
- Получен набор параметров, необходимых для расчета волнового пакета:
  - конструкция скважины
  - реология жидкости
  - скачок давления на устье при остановке закачки

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[2] M. Carey, S. Mondal, M. Sharma, Analysis of water hammer signatures for fracture diagnostics, Paper SPE 174866-MS presented at the SPE Annual Technical Conference and Exhibition, 2015.

[3] C. Holzhaugen, H. Egan, Fracture diagnostics in East Texas and Western Colorado using hydraulic-impedance method, Paper SPE 15215 presented at the SPE Unconventional Gas technology Symposium, 1986.

[4] T. Patzek, A. De, Lossy transmission line model of hydrofractured well dynamics, Journal of Petroleum Science and Engineering, 2000, vol. 25, no. 12, P. 59–77.

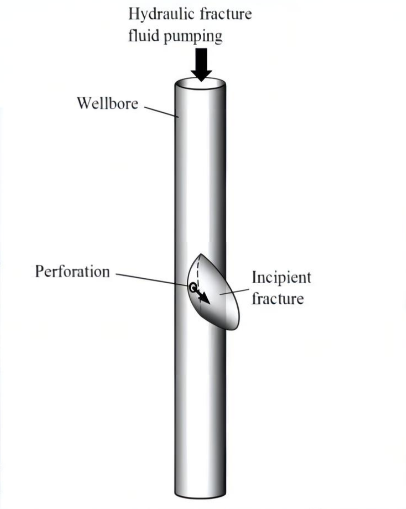
[5] X. Wang, K. Hovem, D. Moos, Y. Quan, Water hammer effects on water injection well performance and longevity, Paper SPE 112282 presented at the SPE International Symposium and Exhibition on Formation Damage Control, 2008.

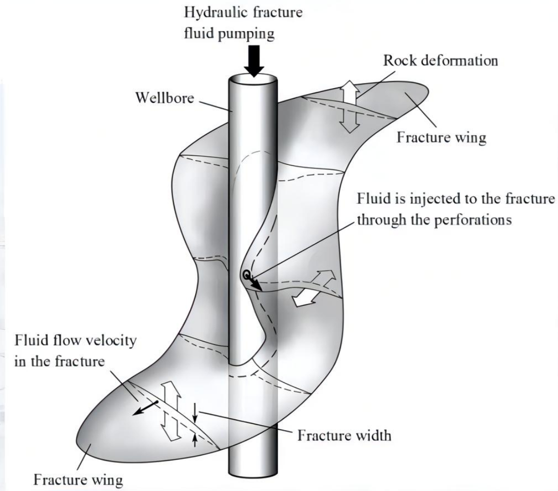
[6] J. Hwang, J. Szabian, M. Sharma, Hydraulic Fracture Diagnostics and Stress Interference Analysis by Water Hammer Signatures in Multi-Stage Pumping Data, URTeC 2687423.

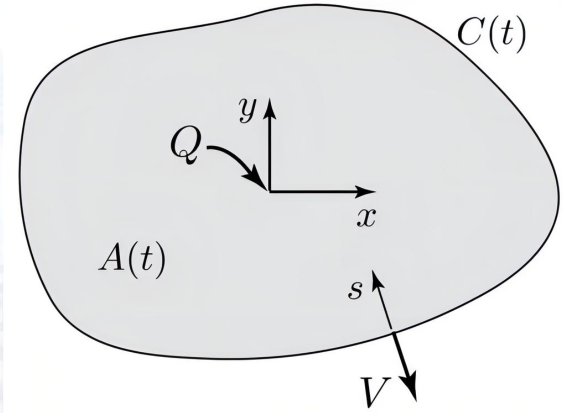
[7] S. Choi, W. Huang, Impact of water hammer in deep sea water injection wells, Paper SPE 146300 presented at the SPE Annual Technical Conference and Exhibition, 2011.

[8] J. Shlyapoberdsky, J.K. Wang, W.W. Walhaug, Overpressure calibrated design of hydraulic fracture stimulations, Paper SPE 18194, 1988.

# Моделирование гидроразрыва




## Инициация трещины

Деформация
Разрушение

## Начальное распространение

Деформация породы
Разрушение породы
Движение жидкости
Химические процессы

## Развитая трещина

Плоская деформация
Плоское разрушение
Движение жидкости
Фильтрация в породу
Перенос проппанта
Химические процессы

# Инициация трещины

## Этапы расчета :

- Напряженно-деформированное состояние
  - Численное решение уравнений упругости
  - Приближенные аналитические решения
  - Учет дополнительных эффектов
  - Серийные численные расчеты НДС
- Критерий инициации
  - Классические
  - Учет эффекта размера
- Начальное распространение трещины

# Расчет напряженно-деформированного состояния

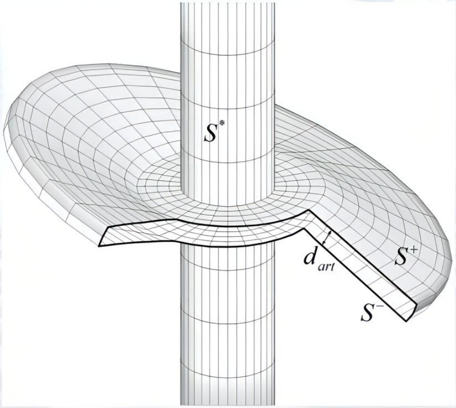
## Постановка задачи

$\sigma_{ij,j}(x) = 0, \quad x \in V$ уравнения равновесия

$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{kk}$ закон Гука

$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ определение деформаций

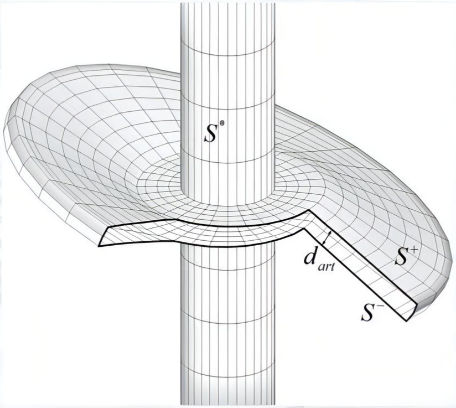
$t(x)|_{S_t} = g(x)$ граничные условия



# Численные решения уравнений упругого равновесия

## Метод граничных элементов

Искусственный разрез для классического МГЭ




Интегральное уравнение для точки границы

$$\boxed{-0.5u_i\left(x'\right)=\int\limits_S U_{ij}\left(x',x\right)t_j\left(x\right)dS\left(x\right)-\int\limits_S T_{ij}\left(x',x\right)u_j\left(x\right)dS\left(x\right)}$$

Выражение координат и неизвестных функций через базисные функции

$$\mathbf{x}\left(\xi_1,\xi_2\right)=\sum_{m=1}^{M}\mathbf{x}^m\varphi_m\left(\xi_1,\xi_2\right),\ \mathbf{u}\left(\xi_1,\xi_2\right)=\sum_{m=1}^{M}\mathbf{u}^m\varphi_m\left(\xi_1,\xi_2\right),\ \mathbf{t}\left(\xi_1,\xi_2\right)=\sum_{m=1}^{M}\mathbf{t}^m\varphi_m\left(\xi_1,\xi_2\right.$$

Система линейных уравнений для коэффициентов разложения

$$-0.5u_j\left(x'\right)+\sum_{n=1}^{N}\sum_{m=1}^{M}U_{ij}^{nm}t_j^{nm}=\sum_{n=1}^{N}\sum_{m=1}^{M}T_{ij}^{nm}u_j^{nm}$$

$$\boxed{\left(T^*-\tfrac{1}{2}I\right)\mathbf{u}=U^*\mathbf{t}}$$

# Приближенные решения (2D)

## Задача Кирша

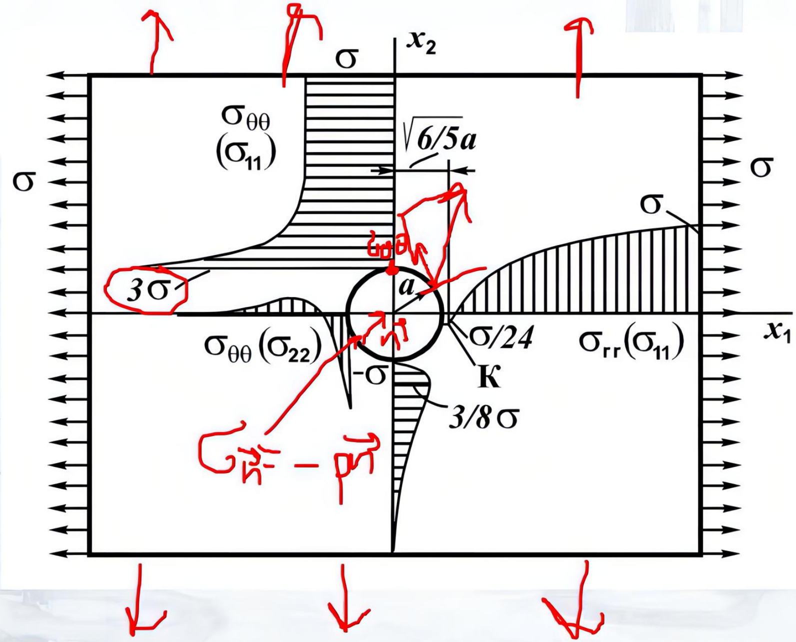
Одностороннее растяжение пластины с круговым вырезом

$$\sigma_{\theta\theta}(r = a) = \sigma - 2\,\sigma\cos(2\theta)$$

Растяжение с 2х сторон:

$$\sigma_{\theta\theta}(r = a) = \sigma_x^\infty - 2\,\sigma_x^\infty\cos(2\theta)$$
$$+\ \sigma_y^\infty - 2\,\sigma_y^\infty\cos(2\theta - \pi) =$$
$$(\sigma_x^\infty + \sigma_y^\infty) - 2\,(\sigma_x^\infty - \sigma_y^\infty)\cos(2\theta)$$

$+p$



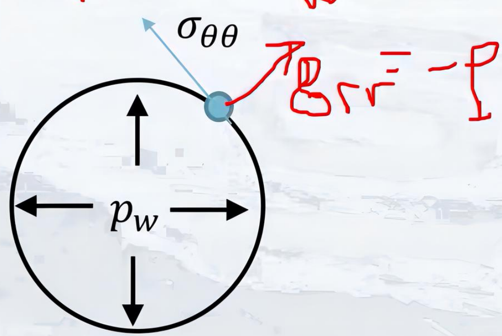

Тимошенко, Гудиер Теория упругости //Гл.4, стр.107

## **Упр.** Растяжение внутренним давлением

Одностороннее растяжение пластины с круговым вырезом

$u_r = u_r(r)$

$$\sigma_{\theta\theta} = \frac{a^2}{r^2}p_w$$




Нулевые напряжения на бесконечности

# Приближенные решения (2D)

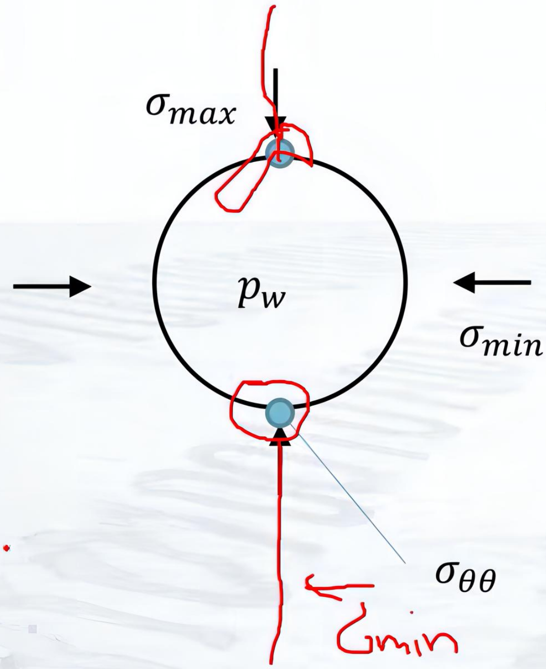
## Полное решение

Вводя переобозначения

$$\sigma_x^{\infty} = -\sigma_{min}$$
$$\sigma_y^{\infty} = -\sigma_{max}$$

$$\sigma_{\theta\theta}(r = a) = p_w - (\sigma_{max} + \sigma_{min}) - 2\,(\sigma_{max} - \sigma_{min})\cos(2\theta) \rightarrow max$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta}(\theta) = 0$$




## Давление инициации

Растягивающие напряжения

$$\sigma_{\theta\theta} = -3\sigma_{min} + \sigma_{max} + p_w$$

Условие разрушения

$$\max\{\sigma_{\theta\theta}\} = \sigma_t$$

## Необходимое давление в скважине

$$\boxed{p_w = 3\sigma_{min} - \sigma_{max} + \cancel{\sigma_t}}$$

# Приближенные решения (3D)

## Полное решение

$$\sigma_{rr} = -p$$

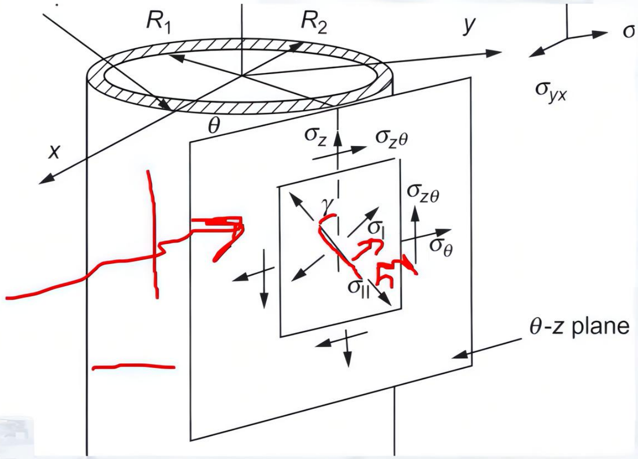
$$\sigma_{\theta\theta} = p + \sigma_{xx}(1 - 2\cos 2\theta) + \sigma_{yy}(1 + 2\cos 2\theta) - 4\sigma_{xy}\sin 2\theta$$

$$\sigma_{zz} = \sigma_{zz}^{\infty} - \nu\left[2\left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta + 4\sigma_{xy}\sin 2\theta\right]$$

$$\sigma_{r\theta} = 0$$

$$\sigma_{rz} = 0$$

$$\sigma_{\theta z} = -2\sigma_{xz}\sin\theta + 2\sigma_{yz}\cos\theta$$




Удовлетворяют условиям совместности
(Уравнения Бельтрами-Митчела)!

*$\sigma_{zz}^{\infty}$ - $\sigma_{zz}$ вдали (для ясности)
$\gamma$ – угол между трещиной и осью z

$$\boxed{\sigma_{\max}(\theta) = \frac{\sigma_{zz} + \sigma_{\theta\theta}}{2} + \sqrt{\left(\frac{\sigma_{zz} - \sigma_{\theta\theta}}{2}\right)^2 + \sigma_{\theta z}^2}}$$

При $\sigma_{\theta z} = 0$
$\sigma_{max} = \max(\sigma_{\theta\theta}, \sigma_{zz})$



Ching H. Yew, Xiaowei Weng **Mechanics of Hydraulic Fracturing 2nd Edition**

# Пример (3D)

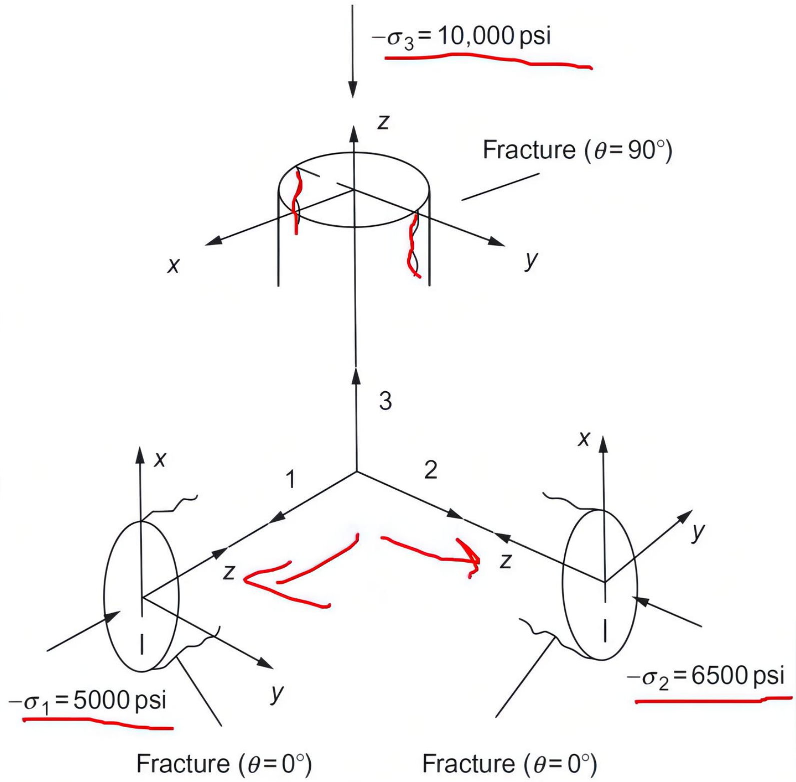
## Вертикальная скважина

$$\sigma_{\theta\theta} - p_w$$
$$= (\sigma_x^{\infty} + \sigma_y^{\infty}) - 2\,(\sigma_x^{\infty} - \sigma_y^{\infty})\cos(2\theta) =$$
$$-(5000 + 6500) + 2\,(5000-6500)\cos(2\theta) =$$
$$-11500-3000\cos(2\theta)$$

$$\mathbf{max}(\boldsymbol{\sigma_{\theta\theta}}) = -11500 - 3000 * (-1) =$$
$$= -8500 + p_w$$

$$\boldsymbol{\theta} = \pm\pi/2$$

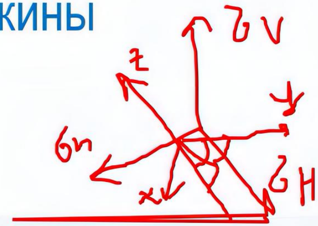
$$\sigma_{\theta\theta} = -3\sigma_{min} + \sigma_{max} + p_w =$$
$$-15000 + 6500 + \mathrm{p_w} = -8500 + \mathrm{p_w}$$

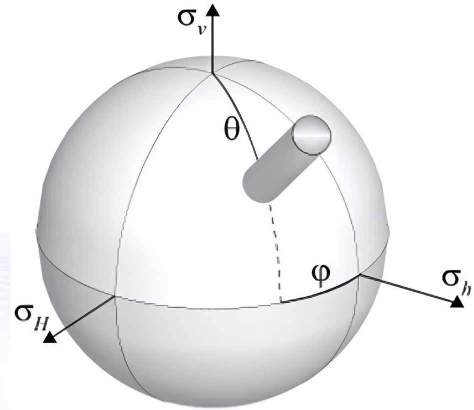



## Горизонтальная скважина (упр.)

$$\gamma = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{\theta z}}{\sigma_{\theta\theta} - \sigma_{zz}}\right)$$

# НДС для перфорированной скважины




1. Напряжения в СК, связанной со скважиной

$$\boxed{\sigma_H, \sigma_h, \sigma_v} \to \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \ldots$$

2. Напряжения в цилиндрической СК

$$\sigma_{xx}, \sigma_{yy}, \sigma_{xy} \to \sigma_{\theta\theta}, \sigma_{rr}, \sigma_{r\theta}, \ldots$$

3. Добавка давления в скважине

$$\sigma_{rr} + p_w$$

4. Плоская задача для перфорации

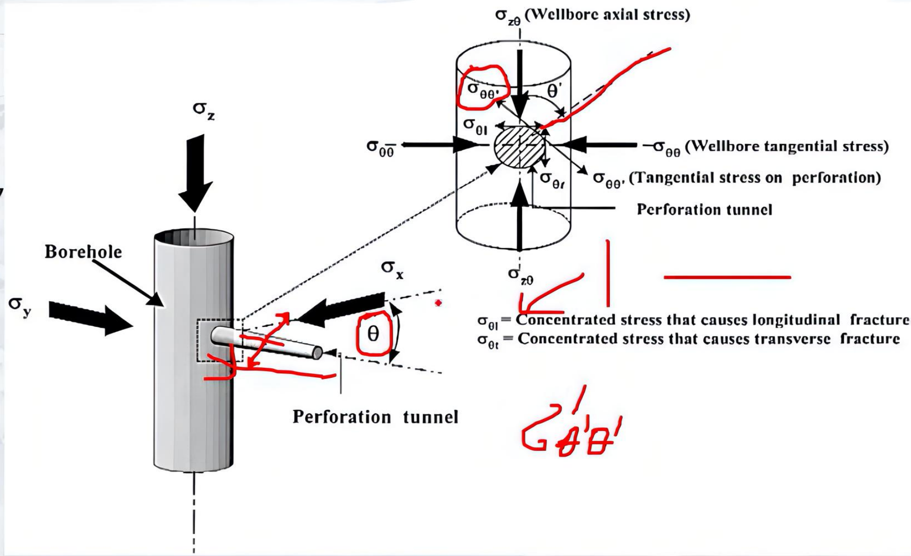
$$\sigma_{\theta\theta}, \sigma_{rr}, \sigma_{r\theta}, \ldots \to \sigma_{xx}^p, \sigma_{yy}^p, \sigma_{xy}^p$$

5. Напряжения на границе перфорации в цилиндрической СК

$$\sigma_{xx}^p, \sigma_{yy}^p, \sigma_{xy}^p \to \sigma_{\theta\theta}^p, \sigma_{rr}^p, \sigma_{r\theta}^p$$

6. Давление инициации для перфорации

$$\sigma_{\theta\theta}^p, \sigma_{rr}^p, \sigma_{r\theta}^p \to p_{init}$$




***θ*** – угол ориентации перфорации!

***θ'*** - окружной угол в сечении перфорации, отсчит. от $\sigma_{zz}$



M.M. Hossain, M.K. Rahman, S.S. Rahman **Hydraulic fracture initiation and propagation: roles of wellbore trajectory, perforation and stress regimes** // Journal of Petroleum Science and Engineering 27 2000 129–149 (Сжимающие напряжения – положительные в этой статье! Поэтому напряжения с точностью до знака!)

# Примеры для перфорированной скважины

## Вертикальная скважина

$\sigma_z = \sigma_v \; \sigma_x = \sigma_H \; \sigma_y = \sigma_h$

$$\sigma_{\theta\theta'} = (\sigma_x + \sigma_y + \sigma_{z\theta}) + 2(\sigma_x + \sigma_y - \sigma_{z\theta})\cos 2\theta' - 2(\sigma_x - \sigma_y)(\cos 2\theta + 2\cos 2\theta \cos 2\theta') - 4\tau_{xy}(1 + 2\cos 2\theta)\sin 2\theta - 4\tau_{z\theta}\sin 2\theta' - P_w(2\cos 2\theta' + 2) \qquad (29)$$

$\boldsymbol{\theta}$ = 0 – ориентация вдоль макс. горизонт. напряжения

$\boldsymbol{\theta}'$ = 0

$\sigma_{\theta 1} = 9\sigma_h - 3\sigma_H - \sigma_{z\theta} - 4P_w$ ~ -bt

$\sigma_{z\theta} = \sigma_v - 2\nu(\sigma_H - \sigma_h)$

С минусом, т.к. сжимающие стрессы положительны

Поперечная трещина $\boldsymbol{\theta}' = \pi/2$

$$\boxed{\sigma_H \geq \sigma_h + \frac{\sigma_v}{2\nu}}$$

## Горизонтальная скважина $\boldsymbol{\theta}$ = 0 и $\boldsymbol{\theta}$ = 90

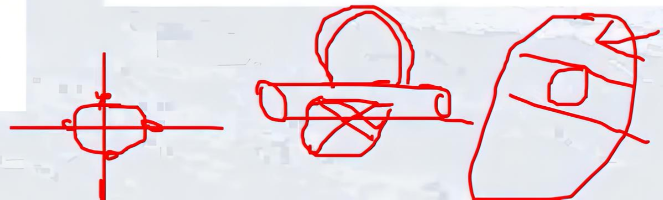
$$P_{w0} = \frac{1}{4}\{9\sigma_H - \sigma_h - 3\sigma_v + 2\nu(\sigma_v - \sigma_H)\}$$

$$P_{w90} = \frac{1}{4}\{9\sigma_v - \sigma_h - 3\sigma_H + 2\nu(\sigma_v - \sigma_H)\}$$

Поперечная трещина

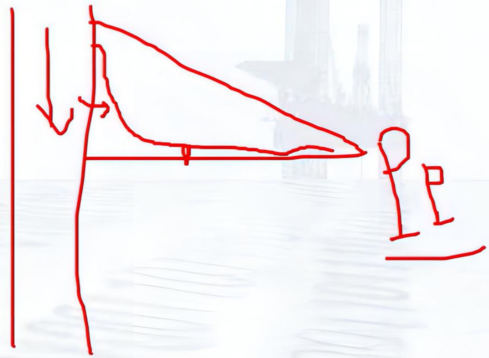
$\boldsymbol{\theta}' = \pi/2$

$\sigma_{\theta t} = 3\sigma_h - (\sigma_v - \sigma_H)(2 - 6\nu) - \sigma_v - \sigma_H$



# Дополнительные эффекты

## Пористость




$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - \alpha\delta_{ij}p$$

закон Гука

$$P_{\mathrm{wf}} = 3\sigma_{\mathrm{h}} - \sigma_{\mathrm{H}} - P_{\mathrm{p}} + \sigma_{\mathrm{t}}$$

Верхняя оценка

$$P_{\mathrm{wf,\,lower}} = \frac{3\sigma_{\mathrm{h}} - \sigma_{\mathrm{H}} - 2\eta P_{\mathrm{p}} + \sigma_{\mathrm{t}}}{2(1-\upsilon)}$$

$\boxed{\text{Нижняя оценка}}$

Valko Economides **Hydraulic fracture mechanics**, 1995

## Температура

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - \beta(3\lambda + 2\mu)(T - T_0)\delta_{ij}$$

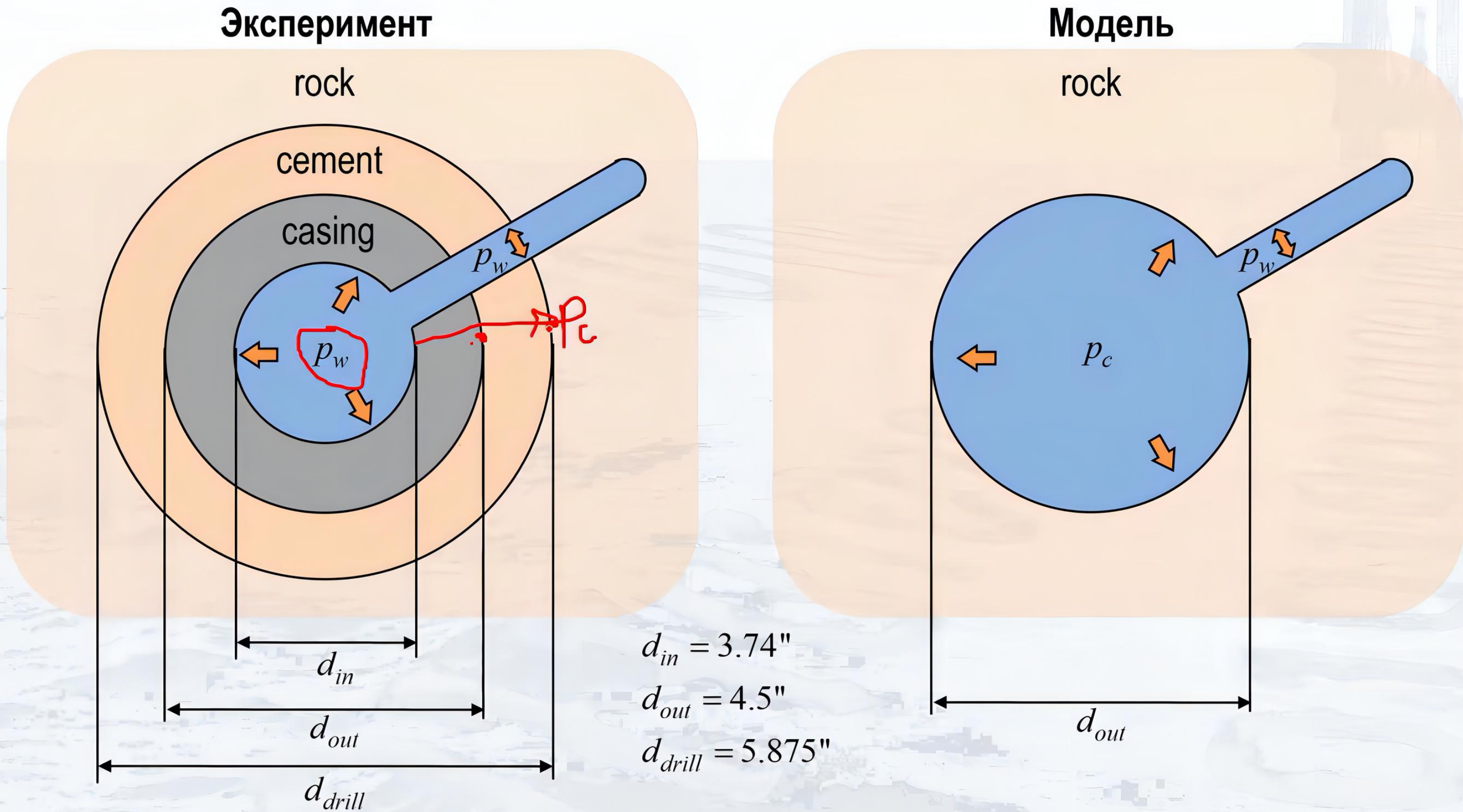
закон Гука

## Обсаженная колонна

… 3. Добавка давления в скважине

$\sigma_{rr} + Cp_w,\qquad C \sim 0.2 \div 0.5$

# Обсаженная колонна




$d_{in} = 3.74''$

$d_{out} = 4.5''$

$d_{drill} = 5.875''$

$$\boxed{p_c = p_0 + k \cdot (p_w - p_0)}$$

Условие непрерывности смещений и напряжений

$p_w$ – давление жидкости в скважине

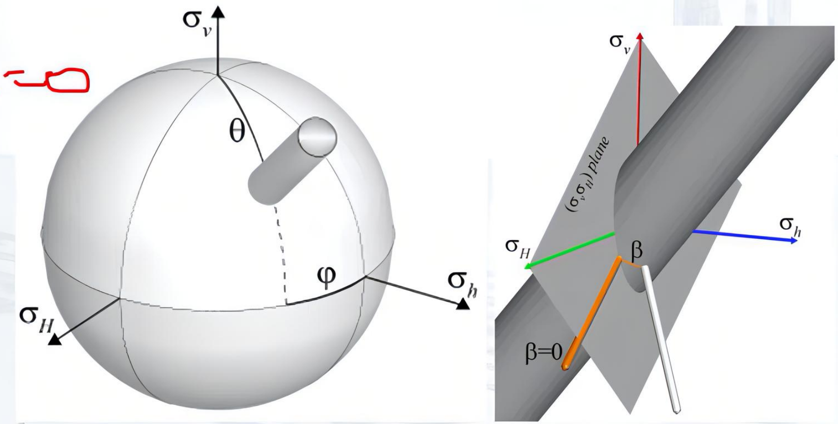
$p_c$ – давление, ослабленное casing

$p_0$ – давление цементирования

$k$ – коэффициент ослабления давления за счет обсадной колонны

# Серийные численные расчеты

Ориентация скважины и перфорации




Вспомогательные задачи

1. $\sigma_{xx} = 1, \sigma_{xy} = 0, \sigma_{yy} = 0, \ldots$ $p_w = 0$

2. $\sigma_{xx} = 0, \sigma_{xy} = 1, \sigma_{yy} = 0, \ldots$

…

6. $\sigma_{xx} = 0, \sigma_{xy} = 0, \ldots, \sigma_{zz} = 1,$

7. $\boldsymbol{\sigma} = 0, p_w = 1$

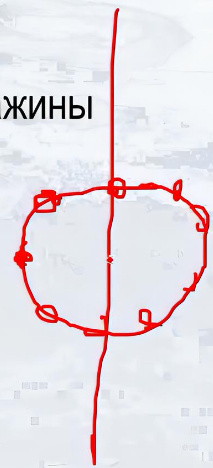
Решая находим $\sigma_3^k(\boldsymbol{x})$ для каждой задачи k=1…7

$\boldsymbol{\sigma}$ – тензор напряжений на бесконечности

Для каждой ориентации

1. Находим $\sigma_k = \sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ ... в СК скважины

2. $\sigma_3\ (\boldsymbol{x}) = \sum_k \sigma_3^k(\boldsymbol{x})\,\sigma_k$

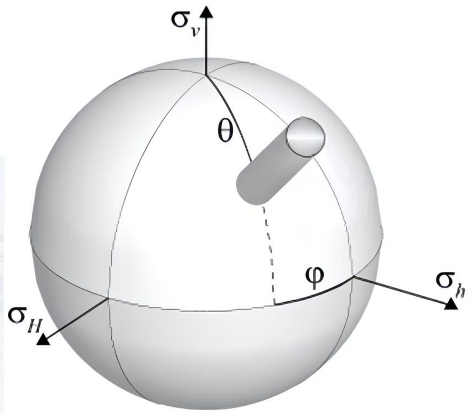


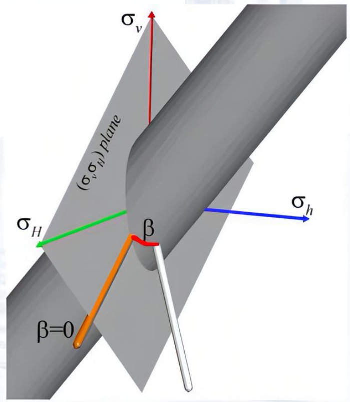
# Серийные численные расчеты

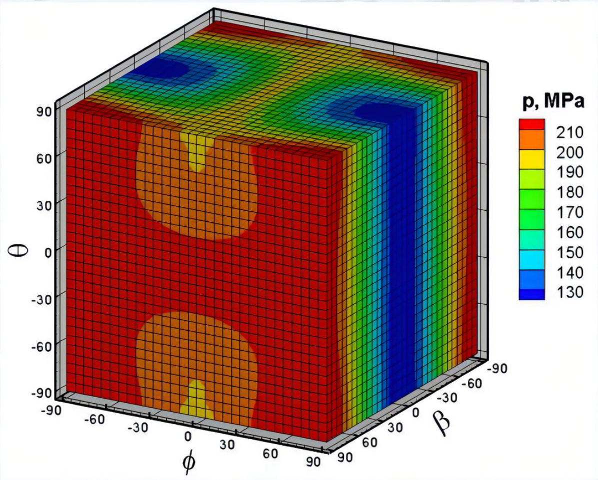
Для каждой ориентации получаем давление инициации $p_{init}(\theta,\phi,\beta)$

Как анализировать три параметра?

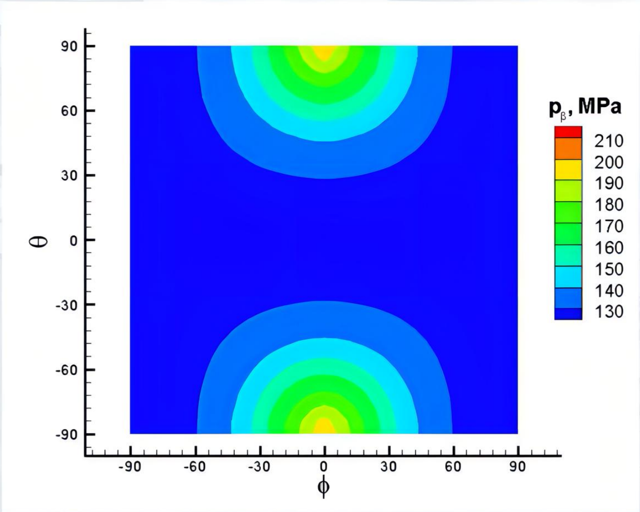
Ориентация скважины и перфорации




$$p_\beta(\theta,\phi) = \min_\beta\left(p_{init}(\theta,\phi,\beta)\right)$$



# Влияние ориентации скважины и перфорации на давление инициации

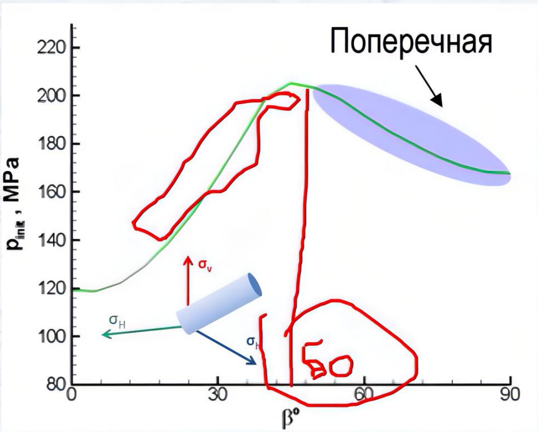

## Месторождение Амин (Оман )

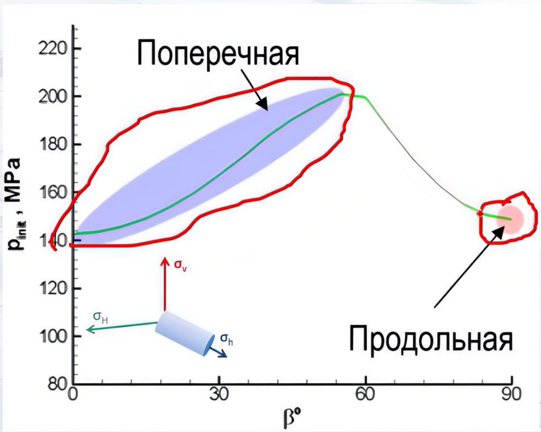
$E = 50\,GPa \quad \sigma_H = 116\,MPa$

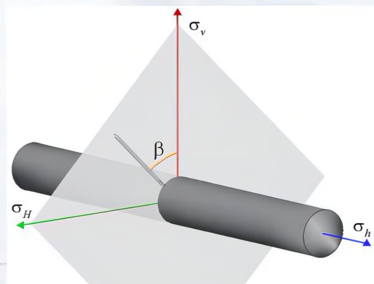
$\nu = 0.13 \quad \sigma_h = 96\,MPa$

$\sigma_t = 20\,MPa \quad \sigma_v = 118\,MPa$

## Зависимость давления от ориентации перфорации



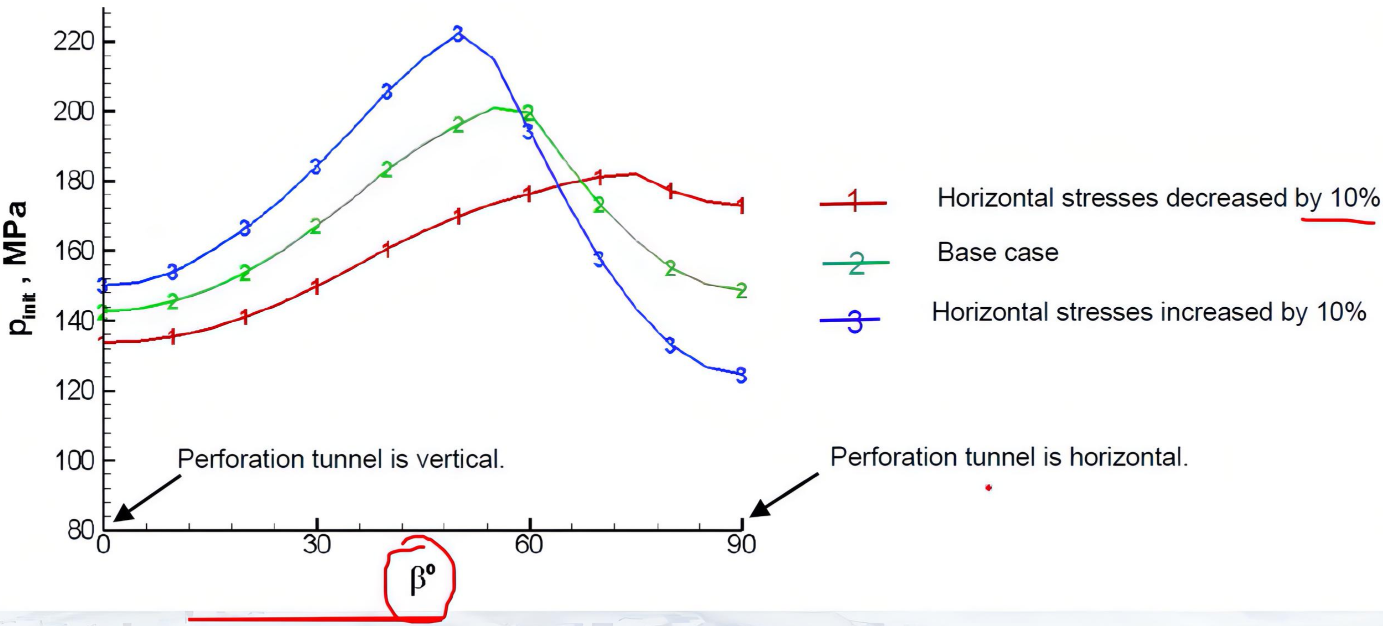

Наклонная скважина

Горизонтальна скважина

Другие вариант см. в

С.Г. **Черный** В.Н. **Лапин**. Д.В. **Есипов** Д.С. **Куранаков. Методы моделирования** зарождения и распространения трещин, 2016

# Влияние точности определения средних напряжений $\sigma_h$ H



# Критерии инициации трещины

$$F_s(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \dots) \geq F_m(\sigma_t, K_{Ic}, \dots)$$

$F_s(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \dots)$ – зависит от НДС

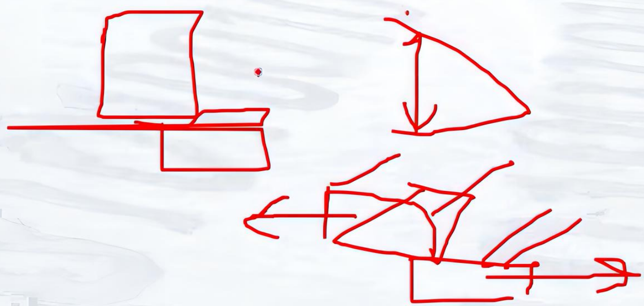
$F_m(\sigma_t, K_{Ic}, \dots)$ – определяется материалом и геометрией задачи

## Классические критерии

### Растягивающих напряжений

$\sigma_3 \geq \sigma_t$,

$\sigma_3 = \max \sigma_k$, $\sigma_k$ - собственные числа $\boldsymbol{\sigma}$



*Разрушится ли материал, если напряжения только сжимающие ?*

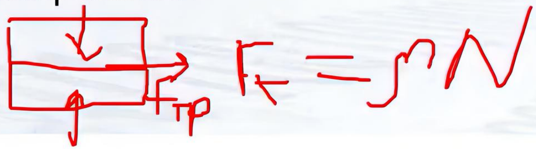
# Критерии для сжимающих напряжений

## Треска

$\tau(\boldsymbol{\sigma}) \geq \tau_c$

$\tau(\boldsymbol{\sigma})$ – максимальные сдвиговые напряжения

$\tau_c$ - параметр материала



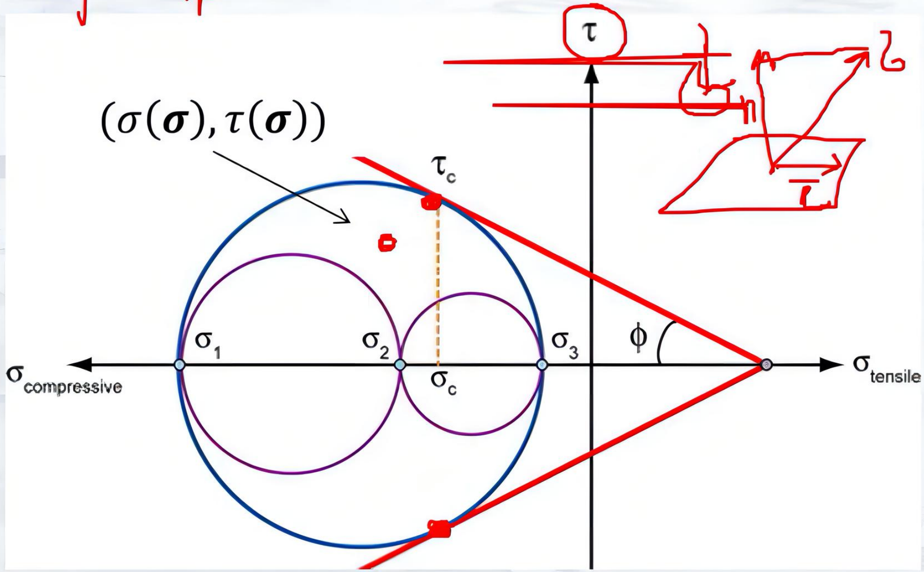

## Мора-Кулона

$\tau(\boldsymbol{\sigma}) \geq \sigma(\boldsymbol{\sigma}) \tan\phi + c$

c – сцепление

$\phi$ – угол внутреннего трения

Разрушение возникает,
если прямая касается внешнего
Круга Мора




https://ru.qwe.wiki/wiki/Mohr%E2%80%93Coulomb_theory

https://www.youtube.com/watch?v=TcHXLJCPx68

# Учет эффекта размера

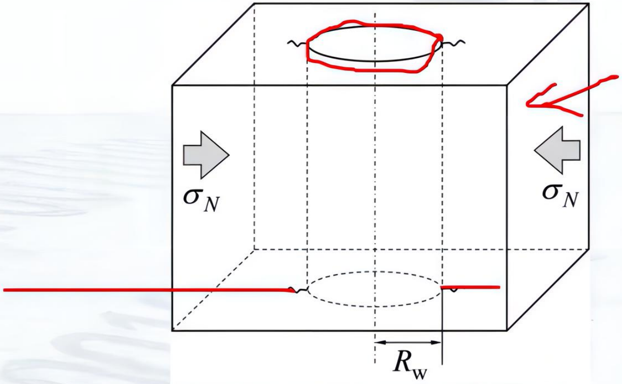
## Эффект размера

Эксперимент о сжатии куба с отверстием

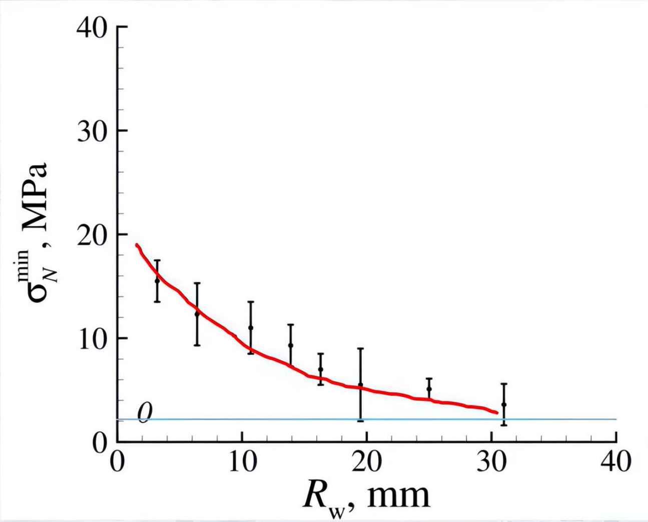
Точка разрушения известна

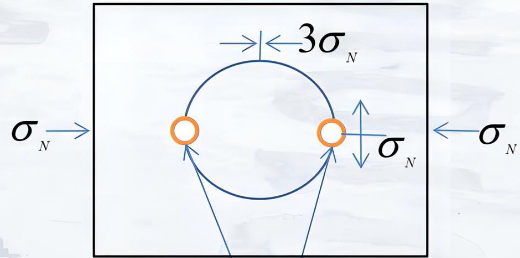
Напряжения в точках разрушения $\sigma_3 = \sigma_N$

Разрушение наступит при $\sigma_3 = \sigma_N = \sigma_t$




Зависимость давления разрушения от размера отверстия

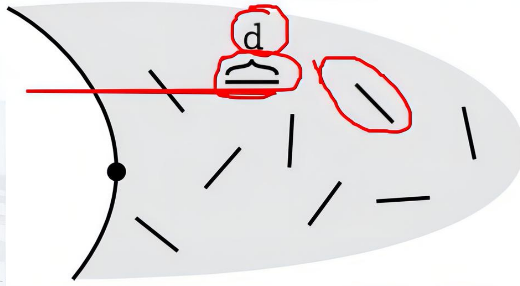


# Учет трещин в материале

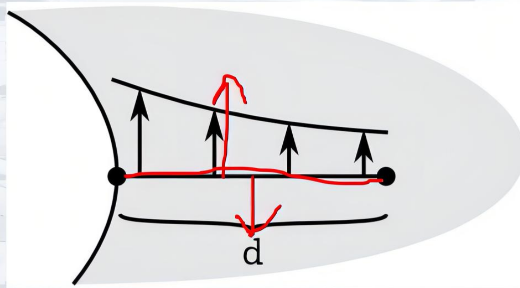
$K_I(\boldsymbol{\sigma}) \geq K_{Ic}$

$K_{Ic}$ - трещиностойкость материала




## Осреднения

$$\frac{1}{d}\int_0^d \sigma_y dx \geq \sigma_t$$




$$\frac{1}{V}\int_V \sigma_{\theta\theta} dV \geq \sigma_t$$

$$\frac{1}{V}\int_V S \quad dV \geq S_c$$

$S = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ - плотность энергии деформации

$S_c = \frac{\sigma_t^2}{2E}$ - критическое значение

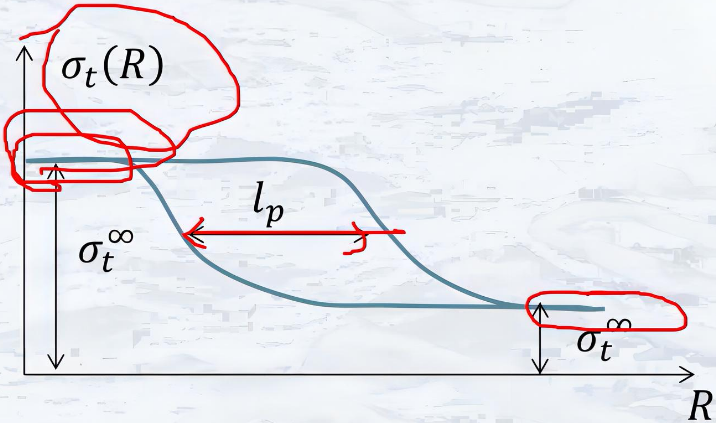
$d, V$ – размер области осреднения параметры материала

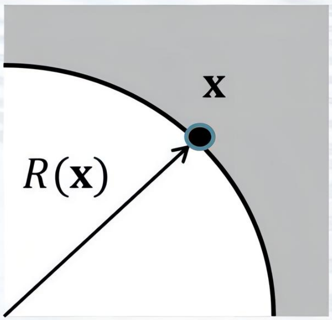
# Кривизна поверхности

$$\sigma_3 \geq \sigma_t(R)$$

$$\sigma_t(R) = \frac{\sigma_t^0 + R_l(R)\sigma_t^\infty}{1 + R_l(R)}, \qquad R_l(R) = \frac{R}{l_p}$$

$l_p, \sigma_t^0, \sigma_t^\infty$ - параметры материала






Bazant Z. P., Planas J. Fracture and size effect in concrete and other quasi-brittle materials. Boca Raton, Florida: CRC Press LLC, 1998. 640 p.

# Влияние эффекта размера на давление инициации