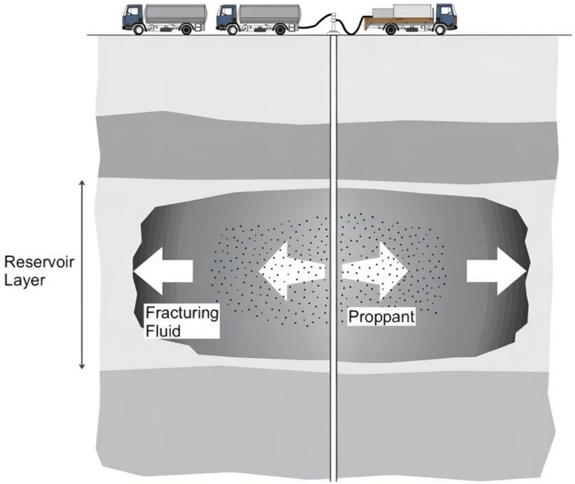
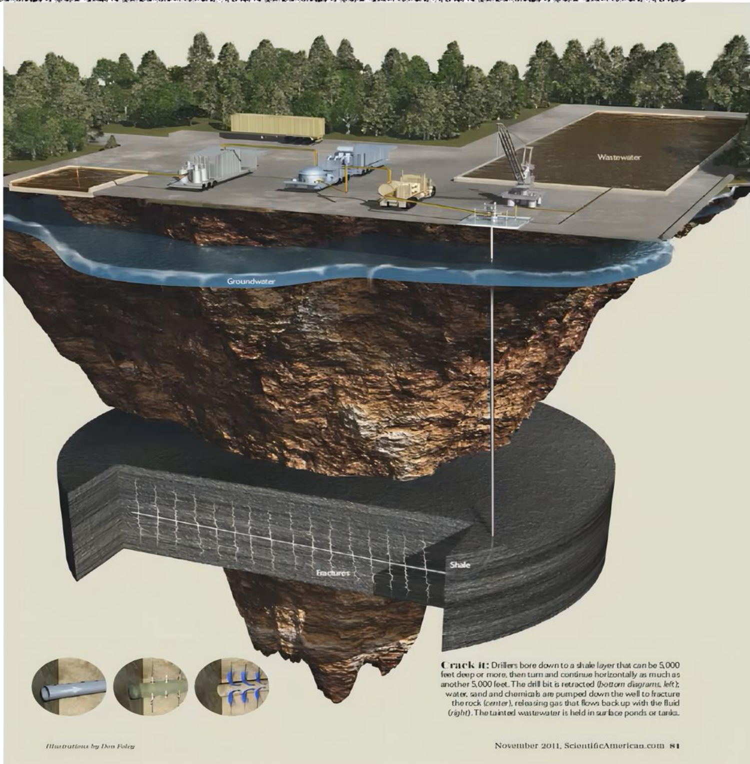
Hydraulic fracturing (HF)

- **Essential components of HF modeling**
- **Governing equations**
- **Basic HF model geometries**





Essential pieces of a hydraulic fracture model

1. Volume balance of the injected fluid (incompressible):

Volume injected = Fracture volume + leak-off

2. Fluid flow equations:

Viscous pressure drop inside the fracture

3. Rock equilibrium (elasticity):

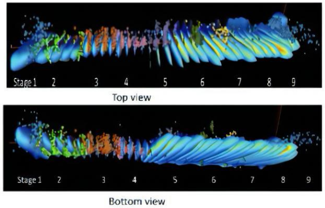
Fluid pressure = Stress + Stiffness*FracWidth

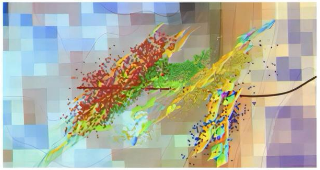
4. Propagation condition:

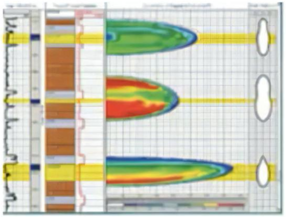
Some parameter reaches a critica value near the front

5. Proppant transport (not covered):

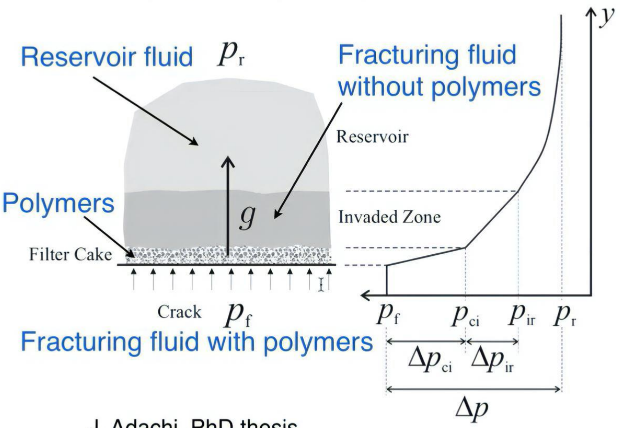
Particles flow with fluid + gravitational settling



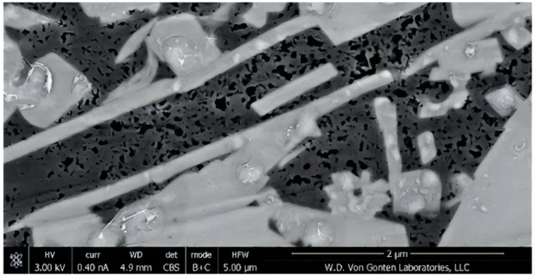




Carter's leak-off model



J. Adachi, PhD thesis



Flow through filter cake:

$$g_c = \alpha \frac{dh_c}{dt}$$

this equation says that the growth rate of the filter cake is linearly proportional to the amount of fluid leaked from the fracture (the constant of proporitonality is measured experimentally)

$$g_c = \frac{\kappa_c}{\mu} \frac{\Delta p_{ci}}{h_c}$$

this is Darcy's law (quasi-static flow)

Solution: $$g_c = \frac{C_c}{\sqrt{t}} \qquad C_c = \sqrt{\alpha \frac{\kappa_c}{\mu} \frac{\Delta p_{ci}}{h_c}}$$

Flow through invaded zone:

$$g_i = \frac{\kappa}{\mu_{filt}} \frac{\Delta p_{ir}}{h_i}$$

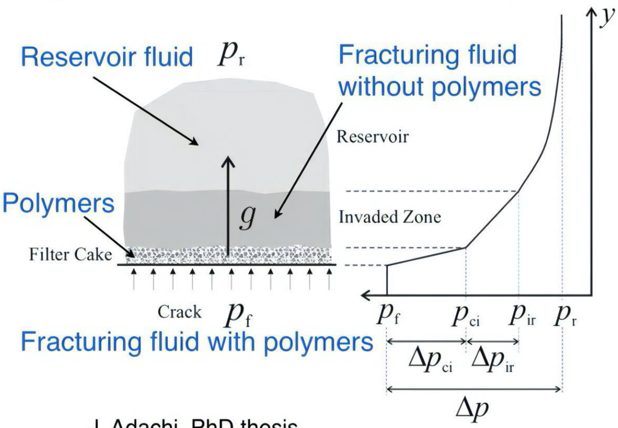
this is Darcy's law (quasi-static flow)

$$g_i = \phi \frac{dh_i}{dt}$$

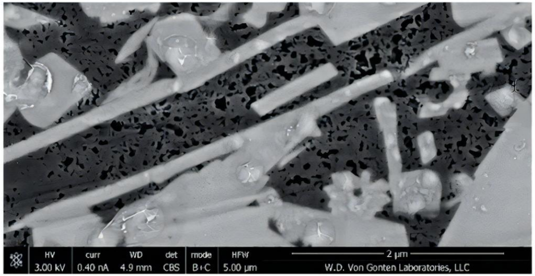
this is volume balance that states that the volume of fluid leaked into the formation determines the size of the invasion zone

Solution: $$g_i = \frac{C_i}{\sqrt{t}} \qquad C_i = \sqrt{\phi \frac{\kappa}{\mu_{filt}} \frac{\Delta p_{ir}}{h_i}}$$

Carter’s leak-off model



J. Adachi, PhD thesis



Flow in reservoir:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial y^2}$$ 1D diffusion equation (volume balance + Darcy)

$$p|_{t=0} = p_r$$ initial condition

$$p|_{y=0} = p_{ir}$$ boundary condition

To solve this equation, introduce new variable:

$$\xi = \frac{y}{\sqrt{4Dt}} \Rightarrow -\frac{y}{2t\sqrt{4Dt}}p' = \frac{D}{4Dt}p'' \Rightarrow -2\xi p' = p''$$

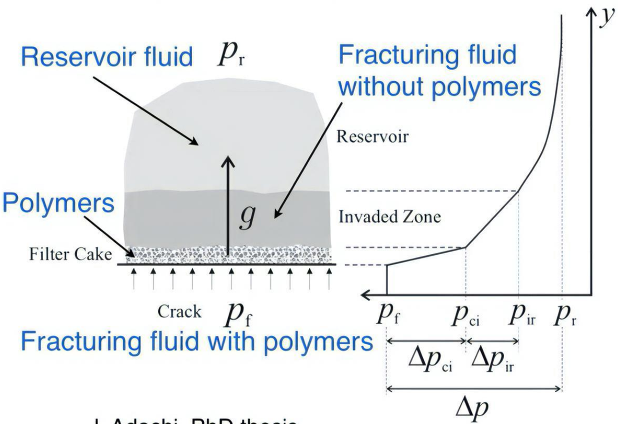
Solution of the above differential equation is:

$$p = p_r + (p_{ir} - p_r)\mathrm{erfc}(\xi^2) \qquad \mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

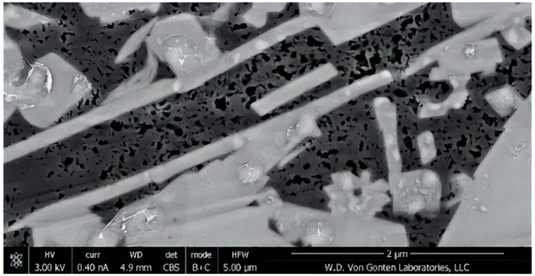
The leak-off flux is then (from Darcy):

$$g_r = -\frac{k_r}{\mu_r}\frac{\partial p}{\partial y}\bigg|_{y=0} = \frac{C_r}{\sqrt{t}} \qquad C_r = \sqrt{\frac{k_r c_t \phi}{\pi \mu_r}}(p_{ir} - p_r)$$

Carter's leak-off model



J. Adachi, PhD thesis



Combined result if all the mechanisms are present:

$$\Delta p = \Delta p_{ci} + \Delta p_{ir} + (p_{ir} - p_r)$$

$$C_l = \frac{2C_cC_iC_r}{C_cC_i + \sqrt{C_c^2C_i^2 + 4C_r^2(C_c^2 + C_i^2)}}$$

In the above result, the individual leak-off coefficients are computed by using the total pressure drop, i.e. the reservoir part id given by

$$C_r = \sqrt{\frac{k_r c_t \phi}{\pi \mu_r}} \Delta p$$

Recall the main assumptions of the model:

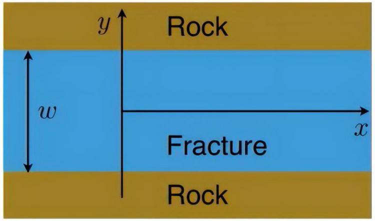
1D diffusion, i.e. the diffusion length scale should be much smaller than the fracture size

The net pressure is often neglected, whereby $\Delta p = \sigma_0 - p_r$

It is implicitly assumed that one type of fracturing fluid is used

More reading: Economides&Nolte 2000, section 6-4.

Fluid flow



$v = v_x(y)$ given the geometry, we have only one component of the velocity vector that varies only across the channel

$\dfrac{\partial p}{\partial x} = \dfrac{\partial \tau}{\partial y}$ this comes from Navier-Stokes equations or equilibrium equations

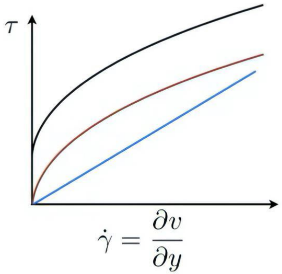
$\tau = \mu \dfrac{\partial v}{\partial y}$ this states that the rheology is Newtonian

$v|_{y=\pm w/2} = 0$ this is no-slip boundary condition at the fracture walls

Herschel-Bulkley $\tau = \tau_0 + k\dot{\gamma}^n$

Power-law $\tau = k\dot{\gamma}^n$

Newtonian $\tau = \mu\dot{\gamma}$



General solution:

$$v = \frac{\partial p}{\partial x}\frac{y^2}{2} + Ay + B$$

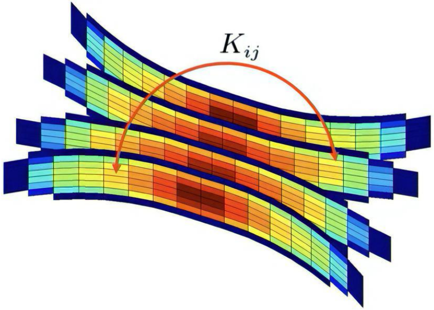
Actual solution:

$$v = -\frac{\partial p}{\partial x}\frac{w^2 - 4y^2}{8\mu}$$

Total flux is:

$$q = \int_{-w/2}^{w/2} v(y)\, dy = -\frac{w^3}{12\mu}\frac{\partial p}{\partial x}$$

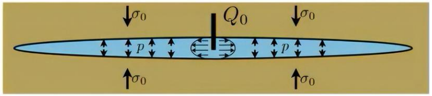
Elasticity



Elasticity equation ensures that rock surrounding open fracture(s) is in equilibrium

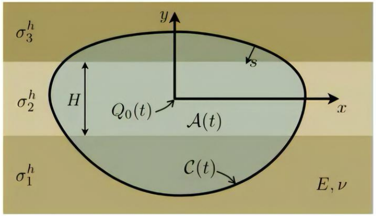
Every open element induces a stress change (all components) in the whole space

The interaction coefficient (induced stress divided by aperture) depends on the elastic properties and the distance from the element and generally decays quickly ~1/r^3 for 3D geometry



For a plane strain fracture, the elasticity equation reads:

$$p = \sigma_0 - \frac{E'}{4\pi}\int_{-l}^{l} \frac{w(s)\,ds}{(x-s)^2} \qquad E' = \frac{E}{1-\nu^2}$$



For a planar fracture, the elasticity equation reads:

$$p(x,y,t) = \sigma^h(y) - \frac{E'}{8\pi}\int_{\mathcal{A}(t)} \frac{w(x',y',t)\mathrm{d}x'\mathrm{d}y'}{\left[(x'-x)^2+(y'-y)^2\right]^{3/2}},$$

For general expressions in 3D, see Crouch and Starfield, 1983

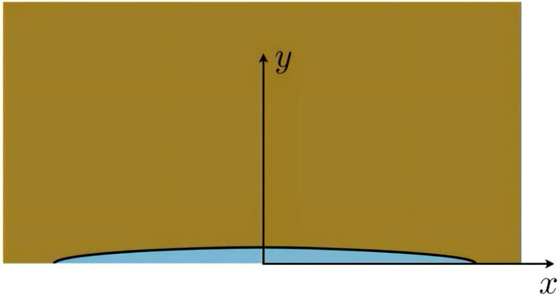
For expressions in layered materials, see Peirce and Siebrits, 2000

Lecture 2: Essential pieces of hydraulic fracturing: part 2

Egor Dontsov

Novosibirsk State University, March 2, 2021

Derivation of elasticity equation (plane strain)



Hooke's law

$$\sigma_{xx} = 2\mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy}),$$
$$\sigma_{yy} = 2\mu\epsilon_{yy} + \lambda(\epsilon_{xx} + \epsilon_{yy}),$$
$$\sigma_{xy} = 2\mu\epsilon_{xy}$$

Equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

Boundary conditions

$$u_y|_{y=0} = \frac{w}{2} \qquad \sigma_{xy}|_{y=0} = 0$$

Need to solve for

$$\sigma_{yy}|_{y=0} - ?$$

Governing equations in terms of displacements

$$(2\mu + \lambda)\frac{\partial^2 u_x}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 u_y}{\partial x \partial y} + \mu\frac{\partial^2 u_x}{\partial y^2} = 0,$$
$$(2\mu + \lambda)\frac{\partial^2 u_y}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u_x}{\partial x \partial y} + \mu\frac{\partial^2 u_y}{\partial x^2} = 0$$

Apply Fourier transform

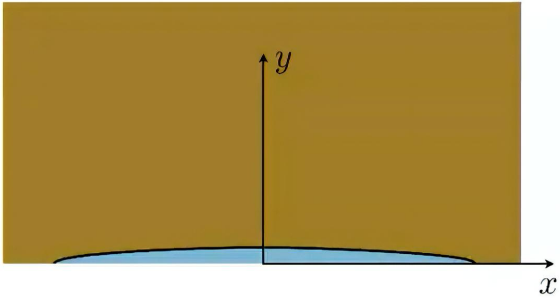
$$\hat{u}_x(k) = \int_{\infty}^{\infty} u_x(x) e^{ikx} \mathrm{d}x,$$
$$u_x(x) = \frac{1}{2\pi}\int_{\infty}^{\infty} \hat{u}_x(k) e^{-ikx} \mathrm{d}k,$$

System of ODEs

$$\frac{\partial \hat{u}_x}{\partial y} = \hat{d}_x,$$
$$\frac{\partial \hat{u}_y}{\partial y} = \hat{d}_y,$$
$$\frac{\partial \hat{d}_x}{\partial y} = \frac{2\mu + \lambda}{\mu} k^2 \hat{u}_x + \frac{\lambda + \mu}{\mu} ik\hat{d}_x + \frac{\mu}{2\mu + \lambda} k^2 \hat{u}_y,$$
$$\frac{\partial \hat{d}_y}{\partial y} = \frac{\mu + \lambda}{2\mu + \lambda} ik\hat{d}_x + \frac{\mu}{2\mu + \lambda} k^2 \hat{d}_x + \frac{\mu}{2\mu + \lambda} k^2 \hat{u}_y,$$

$$\boxed{\boldsymbol{Y' = AY}}$$

Derivation of elasticity equation (plane strain)



Boundary conditions

$$\hat{\sigma}_{xy}|_{y=0} = 0,$$
$$\hat{u}_y|_{y=0} = \hat{w}(k)/2,$$

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$$

Eigenvalues of A: $k, k, -k, -k.$

Solution (resonance)

$$\boldsymbol{Y} = c_1 \boldsymbol{v_1} e^{-|k|y} + c_2(\boldsymbol{v}_1 y + \boldsymbol{v_2})e^{-|k|y}$$

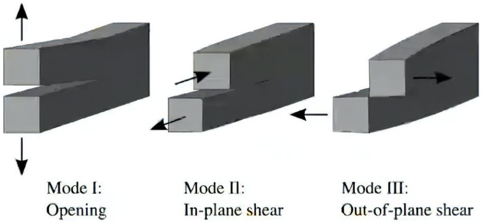
Solution in frequency domain

$$\hat{p} = \frac{\hat{w}|k|}{4}E' = -\frac{ik\hat{w}E'}{4}\frac{|k|}{(-ik)} = -\frac{1}{i}\mathrm{sgn}k\frac{\mathrm{d}\hat{w}}{\mathrm{d}x}.$$

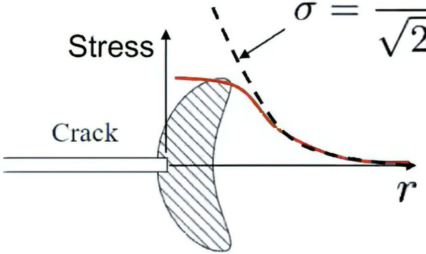
https://en.wikipedia.org/wiki/Fourier_transform

$$p = \sigma_0 - \frac{E'}{4\pi}\int_{-l}^{l}\frac{w(s)\,ds}{(x-s)^2}$$

Propagation condition



Hydraulic fractures



$$\sigma = \frac{K_I}{\sqrt{2\pi r}}$$

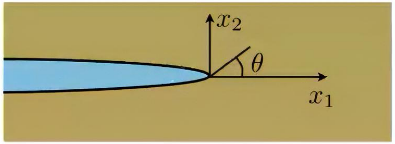
K_I - Stress Intensity Factor (SIF)

Propagation condition: $K_I = K_{Ic}$

K_{Ic} - fracture toughness

https://en.wikipedia.org/wiki/Fracture_mechanics

Mode I solution near the tip



$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1 - \sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1 + \sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\sin(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right]$$

$$u_1 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\cos(\tfrac{1}{2}\theta)\left\{\kappa - 1 + 2\sin^2(\tfrac{1}{2}\theta)\right\}\right]$$

$$u_2 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\sin(\tfrac{1}{2}\theta)\left\{\kappa + 1 - 2\cos^2(\tfrac{1}{2}\theta)\right\}\right]$$

$$\kappa = 3 - 4\nu \qquad \mu = \frac{E}{2(1+\nu)}$$

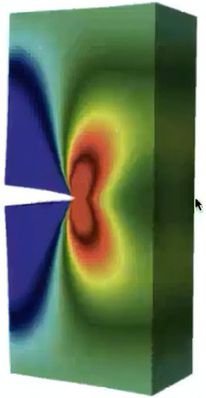
Solution methodology:

- Write elasticity equations via Airy stress function
- Solve the equations assuming stress-free crack and finite displacement at the tip
- See lecture notes on fracture mechanics for more info: http://www.mate.tue.nl/~piet/edu/frm/pdf/frmsyl1213.pdf

Fracture width around the crack tip:

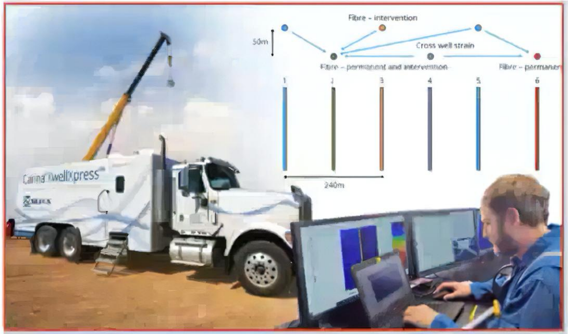
$$w = \sqrt{\frac{32}{\pi}}\frac{K_I(1-\nu^2)}{E}\sqrt{r}$$

Stress field around the crack tip:

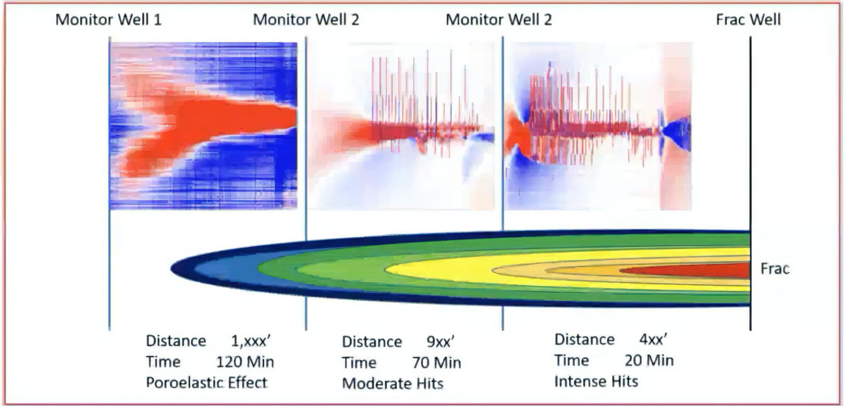


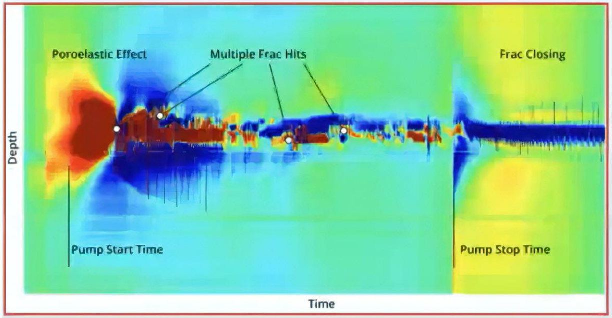
http://umich.edu/~compphys/gradientElasticity.html

Observation of crack tip stress in the field



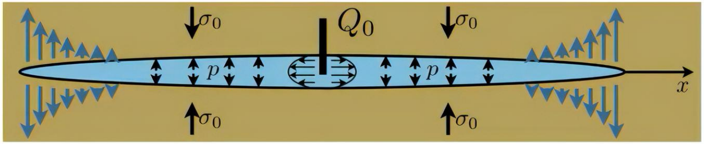
- Fiber optics cables are used to measure stretch versus time along the cable length
- A cable is often placed in the neighboring horizontal well, while the primary well is being fractured
- The characteristic "ears" of the approaching crack are clearly visible

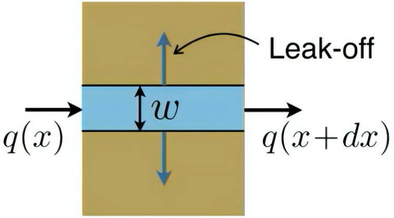




Silixa.com Richter et al. 2020

Volume balance for a plane strain HF





New volume   Flux in   Leak-off

$$w(t+dt)dx = w(t)dx + q(x) - q(x+dx) - g_l dt dx + Q_0 dt \int \delta(x)dx$$

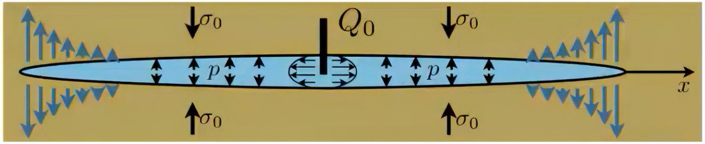
Previous volume   Flux out   Source

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Leak-off   Source

Mathematical model for a plane strain HF



Scaled quantities

$$C' = 2C_L \qquad \mu' = 12\mu \qquad E' = \frac{E}{1-\nu^2} \qquad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

Volume balance of fluid

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Leak-off

Source

Elasticity

Fracture length

$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^{l} \frac{w(s)\, ds}{(x-s)^2}$$

Fluid pressure

Laminar fluid flow flux

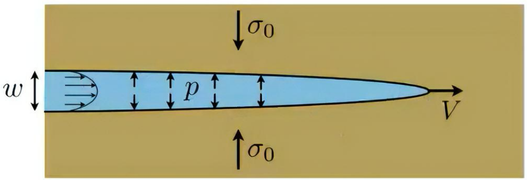
$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$

Propagation condition (LEFM)

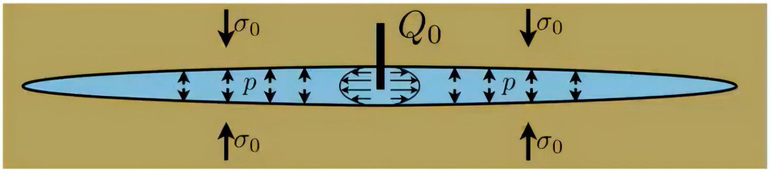
$$w \to \frac{K'}{E'} \sqrt{l - x} \qquad (K_I = K_{Ic})$$

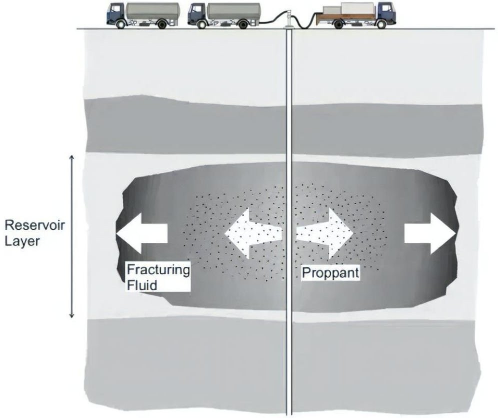
HF geometries - the simplest

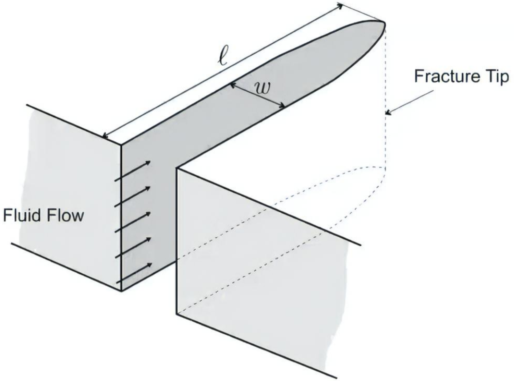
Semi-infinite (tip region)



Khristianovich–Zheltov–Geertsma–De Klerk (KGD)





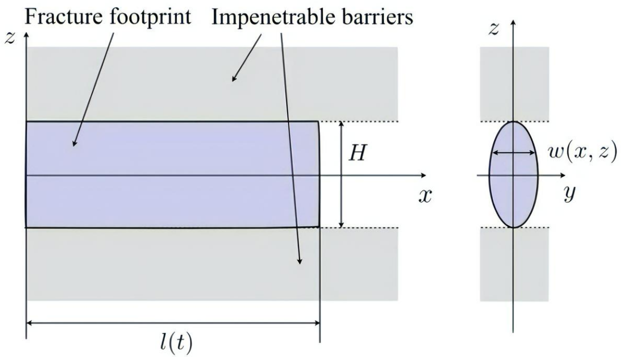


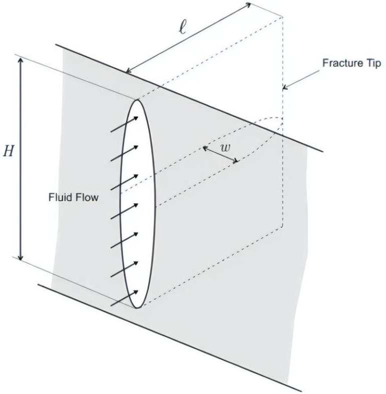
Khristianovic SA, Zheltov YP. 1955 Formation of vertical fractures by means of highly viscous fluids. In Proc. 4th World Petroleum Congress, Rome, Italy, 6–16 June, vol. 2, pp. 579–586.

HF geometries

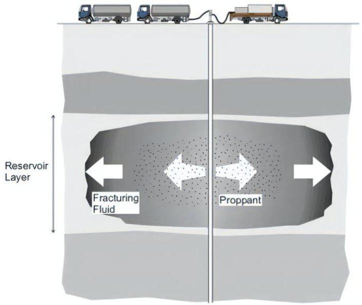
Perkins–Kern–Nordgren (PKN)

T.K. Perkins, L.R. Kern, Widths of hydraulic fractures, J. Pet. Tech. Trans. AIME (1961) 937–949.
R.P. Nordgren, Propagation of vertical hydraulic fractures, Soc. Petrol. Eng. J. (1972) 306–314.





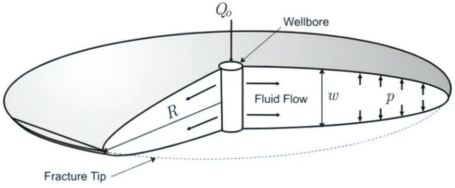
In conventional field applications, solution evolves from KGD geometry at early times to PKN geometry for late times



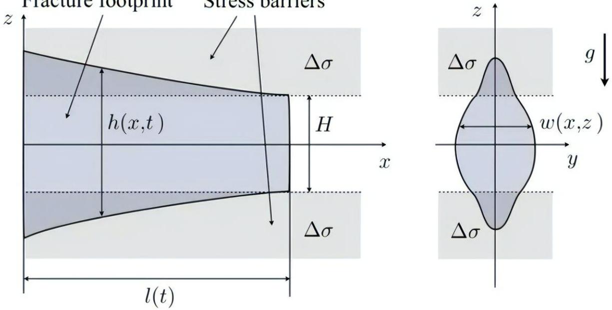
KGD (early time) -> PKN (developed fracture)

HF geometries

Radial

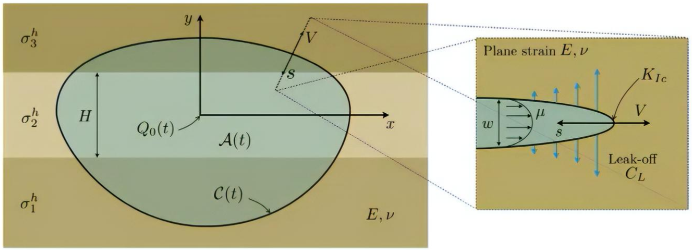


Pseudo-3D



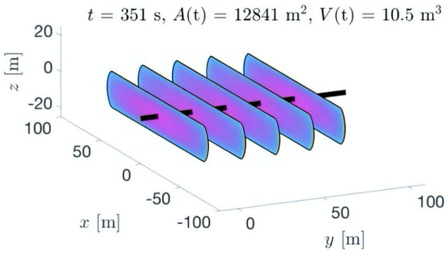
Settari A, Cleary M. Development and testing of a pseudo-three-dimensional model of hydraulic fracture geometry (P3DH). SPE 10505; 1982. p. 185–214.

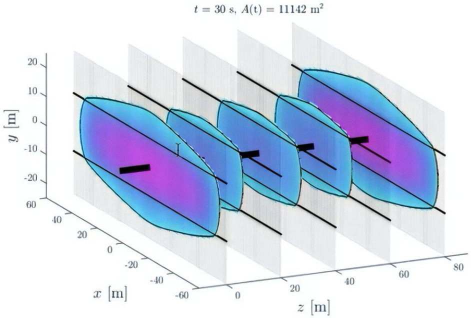
Planar-3D



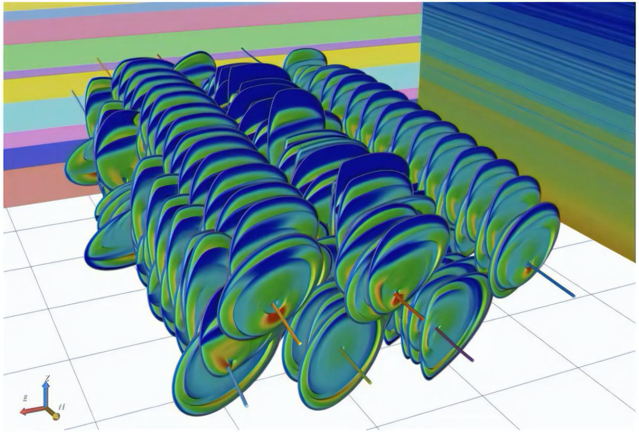
HF geometries

Multi-fracture



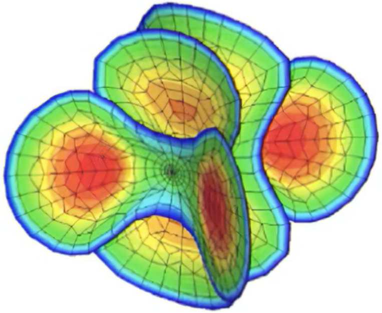


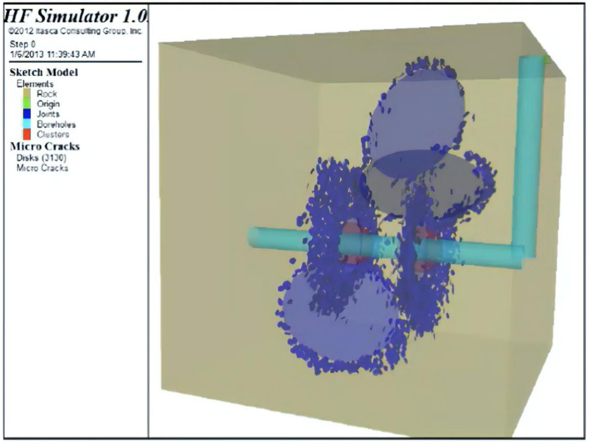
Multi-well

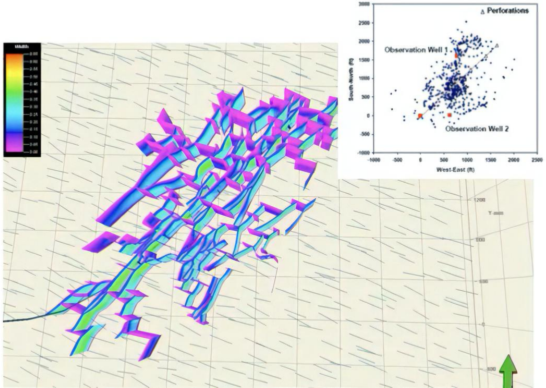


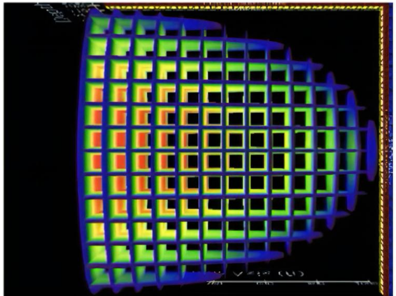
Natural fractures, curved fractures, etc.

HF geometries - other complex









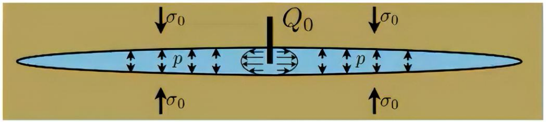
Lecture 3: Semi-infinite hydraulic fracture

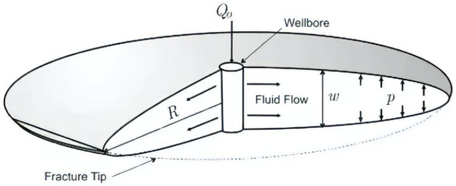
Egor Dontsov

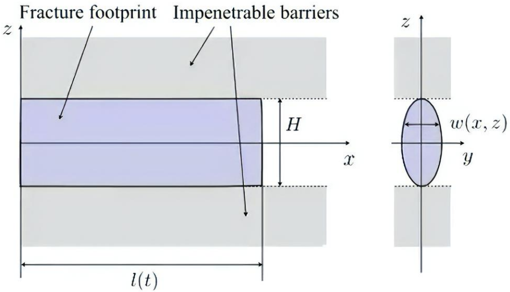
Novosibirsk State University, March 2, 2021

Recall from lecture 1

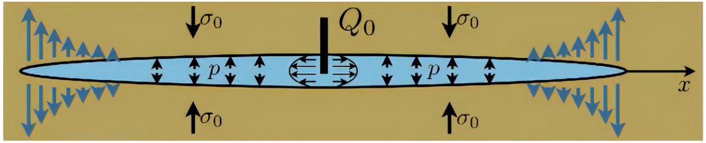
- **Essential pieces of HF model**
  - Volume balance and leak-off
  - Fluid flow
  - Elasticity
  - Propagation condition
  - Proppant transport
- **Various fracture geometries**
  - Semi-infinite
  - KGD (plane strain)
  - PKN
  - Radial
  - Pseudo-3D
  - Planar 3D
  - Complex
- **Governing equations**
  - KGD (plane strain)
- **Derivation of elasticity equation**







Mathematical model for plane strain HF



Scaled quantities

$$C' = 2C_L \qquad \mu' = 12\mu$$

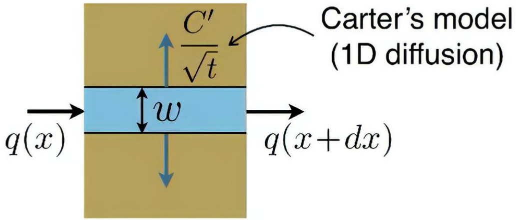
Volume balance of fluid

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

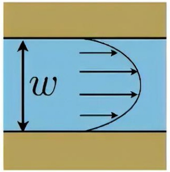
Leak-off

Source



Laminar fluid flow flux

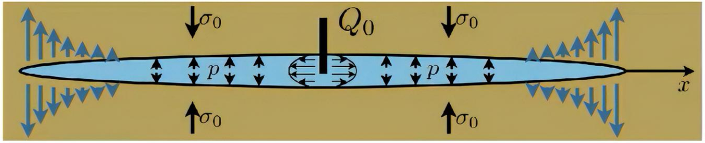
$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$



$$v = -\left(\tfrac{1}{4}w^2 - y^2\right) \frac{1}{2\mu} \frac{\partial p}{\partial x}$$

$$q = \int_{-w/2}^{w/2} v \, dy = -\frac{w^3}{12\mu} \frac{\partial p}{\partial x}$$

Mathematical model for plane strain HF



Scaled quantities

$$C' = 2C_L \qquad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2} \qquad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

Volume balance of fluid

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Leak-off

Source

Elasticity

Fracture length

$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^{l} \frac{w(s)\, ds}{(x-s)^2}$$

Fluid pressure

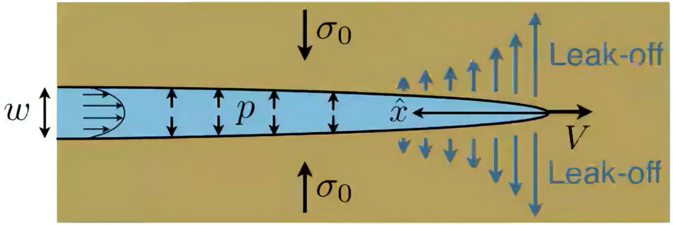
Laminar fluid flow flux

$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$

Propagation condition (LEFM)

$$w \to \frac{K'}{E'} \sqrt{l - x} \qquad (K_I = K_{Ic})$$

Tip asymptotics: semi-infinite hydraulic fracture



Material parameters

$$C' = 2C_L \qquad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2} \qquad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

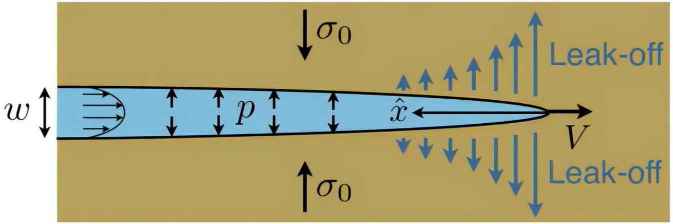
Fluid volume balance

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = 0 \xrightarrow[]{\text{Traveling wave } w(Vt-x)} \frac{q}{w} = V + 2C'V^{1/2}\frac{\hat{x}^{1/2}}{w}$$

Elasticity

$$p = \sigma(\hat{x}) + \frac{E'}{4\pi}\int_0^{\infty} \frac{dw(\hat{s})}{d\hat{s}} \frac{d\hat{s}}{\hat{x} - \hat{s}}$$

Tip asymptotics: semi-infinite hydraulic fracture



Material parameters

$$C' = 2C_L \qquad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2} \qquad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

Fluid volume balance

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = 0 \xrightarrow[\;]{\text{Traveling wave} \atop w(Vt-x)} \frac{q}{w} = V + 2C'V^{1/2}\frac{\hat{x}^{1/2}}{w}$$

Elasticity

$$w = \frac{K'}{E'}\hat{x}^{1/2} + \frac{4}{\pi E'}\int_0^\infty K(\hat{x},\hat{s})(p(\hat{s}) - \sigma(\hat{s}))\,d\hat{s} \qquad K(\hat{x},\hat{s}) = \ln\left|\frac{\hat{x}^{1/2}+\hat{s}^{1/2}}{\hat{x}^{1/2}-\hat{s}^{1/2}}\right| - 2\frac{\hat{x}^{1/2}}{\hat{s}^{1/2}}$$

Fluid flow

$$q = -\frac{w^3}{\mu'}\frac{\partial p}{\partial x} \longrightarrow q = \frac{w^3}{\mu'}\frac{dp}{d\hat{x}}$$

LEFM propagation condition

$$w = \frac{K'}{E'}\hat{x}^{1/2}, \qquad \hat{x} \to 0$$

Non-singular formulation

1. Integrate elasticity equation by parts

$$w = \frac{K'}{E'}x^{1/2} - \frac{4}{\pi E'}\int_0^\infty F(x,s)\frac{dp}{ds}\,ds \qquad F(x,s) = (s-x)\ln\left|\frac{x^{1/2}+s^{1/2}}{x^{1/2}-s^{1/2}}\right| - 2x^{1/2}s^{1/2}$$

2. Substitute pressure gradient into the result

$$w(x) = \frac{K'}{E'}x^{1/2} - \frac{4}{\pi E'}\int_0^\infty F(x,s)\frac{\mu'}{w(s)^2}\left[V + 2C'V^{1/2}\frac{s^{1/2}}{w(s)}\right]ds$$

3. Apply scaling

$$\tilde{w} = \frac{E'w}{K'x^{1/2}}, \qquad \chi = \frac{2C'E'}{V^{1/2}K'}, \qquad \tilde{x} = (x/l)^{1/2}, \qquad \tilde{s} = (s/l)^{1/2}, \qquad l = \left(\frac{K'^3}{\mu' E'^2 V}\right)^2$$

4. Final result

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right]d\tilde{s} \qquad G(t) = \frac{1-t^2}{t}\ln\left|\frac{1+t}{1-t}\right| + 2$$

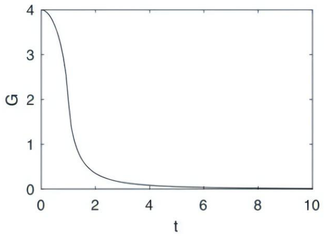
Non-singular

w = “toughness” + “viscosity” + “leak-off”

Limiting vertex solutions

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right] d\tilde{s}$$

w = “toughness” + “viscosity” + “leak-off”



Toughness dominates

$$\tilde{w}_k = 1, \quad \Longrightarrow \quad w_k = \frac{K'}{E'}x^{1/2}$$

Viscosity dominates

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2}\, d\tilde{s} \quad \Longrightarrow \quad \begin{aligned} \tilde{w}_m &= \beta_m \tilde{x}^{1/3} \\ \beta_m &= 2^{1/3}3^{5/6} \end{aligned} \quad \Longrightarrow \quad w_m = \beta_m\left(\frac{\mu' V}{E'}\right)^{1/3} x^{2/3}$$

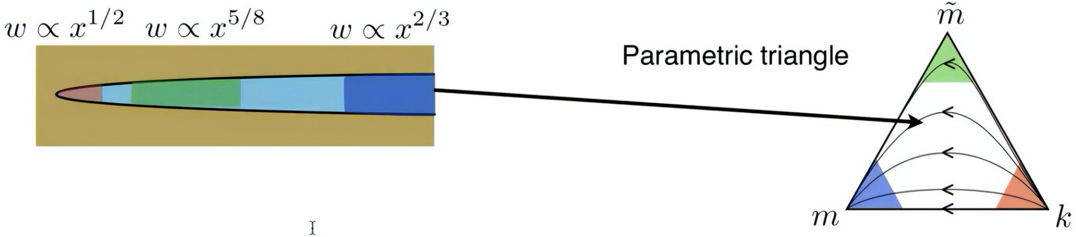
Desroches et al 1994

Leak-off dominates

$$\tilde{w}(\tilde{x}) = \frac{8\chi}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^3}\, d\tilde{s} \quad \Longrightarrow \quad \begin{aligned} \tilde{w}_{\tilde{m}} &= \beta_{\tilde{m}} \chi^{1/4}\tilde{x}^{1/4} \\ \beta_{\tilde{m}} &= \frac{4}{15^{1/4}(\sqrt{2}-1)^{1/4}} \end{aligned} \quad \Longrightarrow \quad w_{\tilde{m}} = \beta_{\tilde{m}}\left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8} x^{5/8}$$

Lenoah 1995

Order of limiting solutions



Toughness

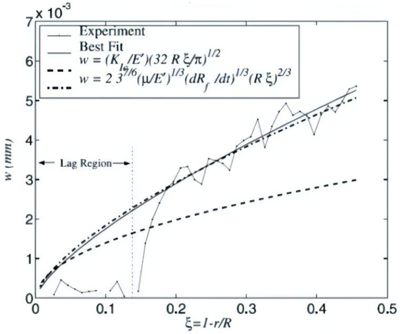
$$w_k = \frac{K'}{E'} x^{1/2},$$

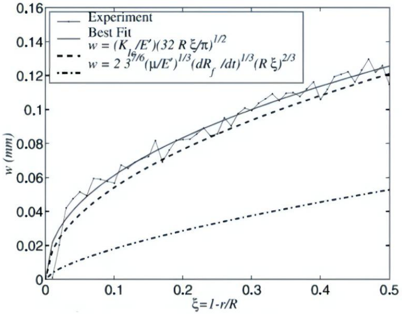
Leak-off

$$w_{\tilde{m}} = \beta_{\tilde{m}} \left(\frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} x^{5/8},$$

Viscosity

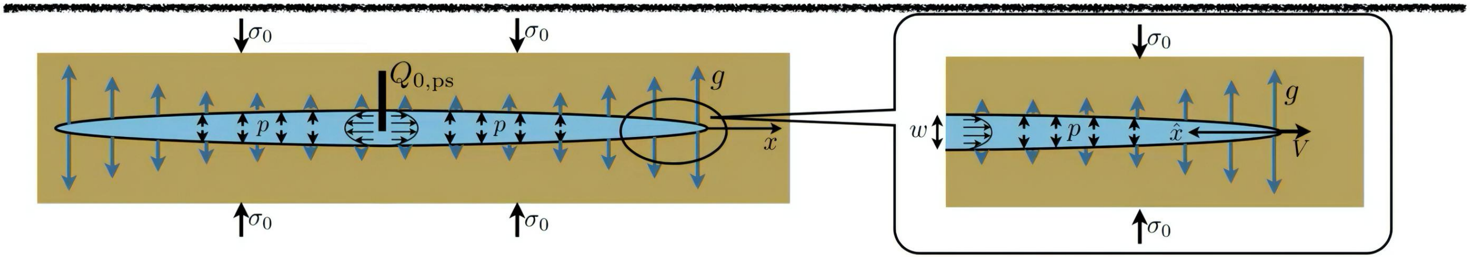
$$w_m = \beta_m \left(\frac{\mu' V}{E'} \right)^{1/3} x^{2/3}$$





Bunger&Jeffrey

Governing equations for a semi-infinite hydraulic fracture



Use moving coordinate

$$\hat{x} = Vt - x$$

Volume balance and flow

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_{0,\mathrm{ps}}(t)\delta(x),$$

$$q = -\frac{w^3}{\mu'}\frac{\partial p}{\partial x},$$

$$\Longrightarrow \quad V\frac{dw}{d\hat{x}} - \frac{dq}{d\hat{x}} + \frac{C'}{\sqrt{\hat{x}/V}} = 0$$

Substitute flux and integrate

$$\Longrightarrow \quad \frac{w^2}{\mu'}\frac{dp}{d\hat{x}} = V + 2C'\frac{\sqrt{V\hat{x}}}{w}$$

Elasticity and propagation

$$p(x,t) = \sigma_0 - \frac{E'}{4\pi}\int_{-l_1}^{l_2} \frac{w(s)ds}{(x-s)^2},$$

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \begin{cases} \dfrac{K'}{E'}, & \text{if } V > 0, \\ \dfrac{K'_I}{E'}, & \text{if } V = 0. \end{cases}$$

$$\Longrightarrow \quad w(\hat{x}) = \frac{K'}{E'}\hat{x}^{1/2} - \frac{4}{\pi E'}\int_0^\infty F(\hat{x},\hat{s})\frac{d(p-\sigma_0)}{d\hat{s}}d\hat{s},$$

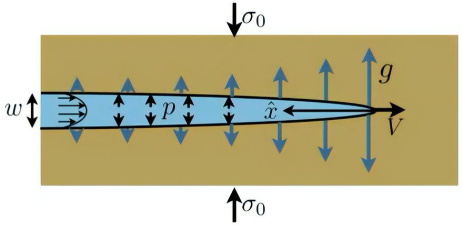
$$F(\hat{x},\hat{s}) = (\hat{s}-\hat{x})\ln\left|\frac{\hat{x}^{1/2}+\hat{s}^{1/2}}{\hat{x}^{1/2}-\hat{s}^{1/2}}\right| - 2\hat{x}^{1/2}\hat{s}^{1/2}.$$

Non-singular formulation

Substitute pressure gradient

$$w(\hat{x}) = \frac{K'}{E'}\hat{x}^{1/2} - \frac{4}{\pi E'}\int_0^\infty F(\hat{x},\hat{s})\frac{d(p-\sigma_0)}{d\hat{s}}d\hat{s},$$

$$\frac{w^2}{\mu'}\frac{dp}{d\hat{x}} = V + 2C'\frac{\sqrt{V\hat{x}}}{w}$$



$$w(\hat{x}) = \frac{K'}{E'}\hat{x}^{1/2} - \frac{4}{\pi E'}\int_0^\infty F(\hat{x},\hat{s})\frac{\mu'}{w(\hat{s})^2}\left[V + 2C'V^{1/2}\frac{\hat{s}^{1/2}}{w(\hat{s})}\right]d\hat{s}.$$

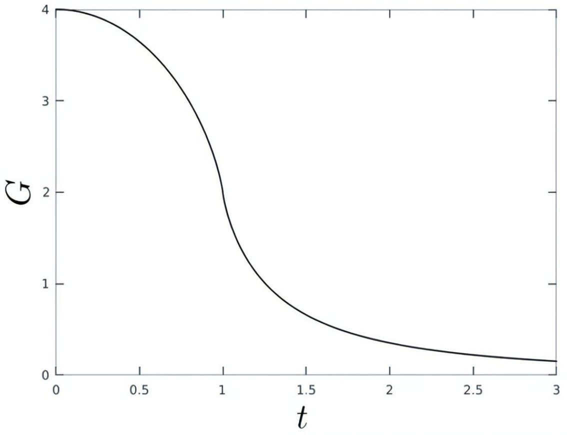
Scaling

$$\tilde{w} = \frac{E'w}{K'\hat{x}^{1/2}}, \qquad \chi = \frac{2C'E'}{V^{1/2}K'}, \qquad \tilde{x} = (\hat{x}/l)^{1/2},$$

$$\tilde{s} = (\hat{s}/l)^{1/2}, \qquad l = \left(\frac{K'^3}{\mu' E'^2 V}\right)^2,$$

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right]d\tilde{s},$$ Non-singular formulation

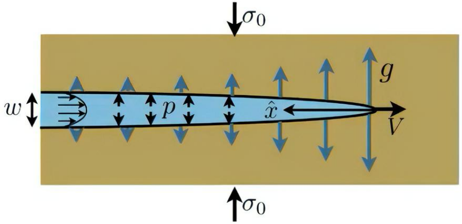
$$G(t) = \frac{1-t^2}{t}\ln\left|\frac{1+t}{1-t}\right| + 2.$$ Non-singular kernel



Non-singular formulation

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right] d\tilde{s},$$

w = “toughness” + “viscosity” + “leak-off”



Toughness dominates

$$\tilde{w}_k = 1, \quad \Longrightarrow \quad w_k = \frac{K'}{E'}\hat{x}^{1/2},$$

Viscosity dominates

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2}\, d\tilde{s} \quad \Longrightarrow \quad \tilde{w}_m = \beta_m \tilde{x}^{1/3}, \quad \beta_m = 2^{1/3}3^{5/6} \quad \Longrightarrow \quad w_m = \beta_m\left(\frac{\mu' V}{E'}\right)^{1/3}\hat{x}^{2/3}$$

Desroches et al 1994

Leak-off dominates

$$\tilde{w}(\tilde{x}) = \frac{8\chi}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^3}\, d\tilde{s} \quad \Longrightarrow \quad \tilde{w}_{\tilde{m}} = \beta_{\tilde{m}}\chi^{1/4}\tilde{x}^{1/4}, \quad \beta_{\tilde{m}} = \frac{4}{15^{1/4}(\sqrt{2}-1)^{1/4}} \quad \Longrightarrow \quad w_{\tilde{m}} = \beta_{\tilde{m}}\left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8}\hat{x}^{5/8},$$

Lenoah 1995

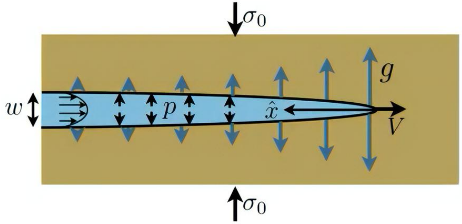
Derivation of the viscosity solution

Governing integral equation

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2}\, d\tilde{s}$$

Form of the solution

$$\tilde{w} = \beta_m \tilde{x}^{\alpha_m}$$



$$\Downarrow$$

$$\beta_m \tilde{x}^{\alpha_m} = \frac{8\tilde{x}^{2\alpha_m - 1}}{\pi \beta_m^2}\int_0^\infty \frac{G(t)}{t^{2\alpha_m}}\, dt$$

$$\Downarrow \quad \int_0^\infty \frac{G(t)}{t^\alpha}\, dt = \frac{2\pi}{\alpha(2-\alpha)}\tan\left(\frac{\pi}{2}\alpha\right)$$

(see derivation in the notes)

$$\alpha_m = 1/3 \qquad \beta_m = 2^{1/3}3^{5/6},$$

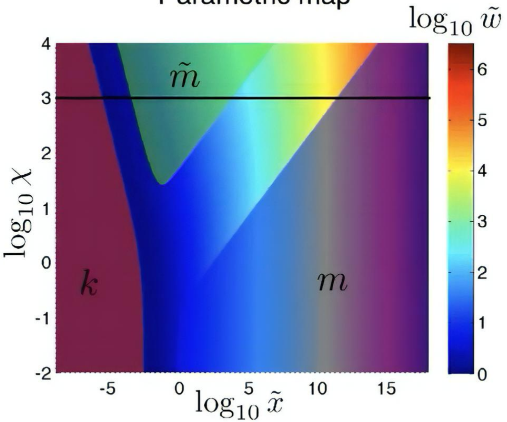
$$\Downarrow$$

$$\tilde{w}_m = \beta_m \tilde{x}^{1/3}$$

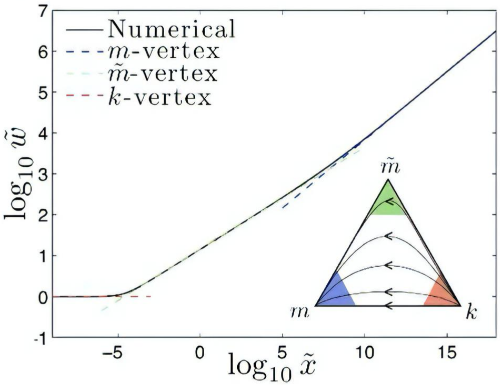
Leak-off dominated solution can be derived in a similar way.

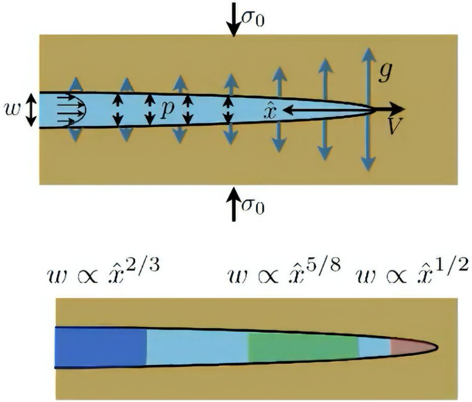
Structure of the solution

Parametric map



Cross-section for large leak-off





Vertex solutions:

$$\tilde{w}_k = 1, \qquad \tilde{w}_{\tilde{m}} = \beta_{\tilde{m}}\chi^{1/4}\tilde{x}^{1/4}, \qquad \tilde{w}_m = \beta_m \tilde{x}^{1/3},$$

Solution transitions gradually from one limiting solution to another starting from toughness, then passing through leak-off (for large leak-off), and then reaching viscosity vertex.

Approximate solution

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right]d\tilde{s},$$

Numerical solution of this equation is time consuming and can be a limiting factor for some applications, such as using it as a propagation condition for a planar fracture. So, need to construct an efficient approximation.

Differentiate

$$\frac{d\tilde{w}(\tilde{x})}{d\tilde{x}} = -\frac{8}{\pi}\int_0^\infty G'(\tilde{s}/\tilde{x})\frac{\tilde{s}}{\tilde{x}^2}\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right]d\tilde{s}, \qquad \tilde{w}(0) = 1.$$

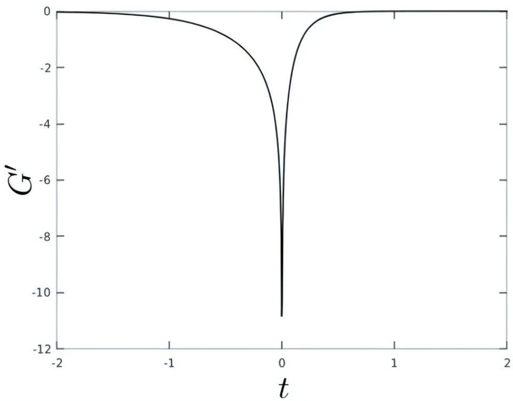
Assume that $\tilde{w} \propto \tilde{x}^\delta$

$$\frac{d\tilde{w}(\tilde{x})}{d\tilde{x}} = -\frac{8}{\pi}\int_0^\infty G'(\tilde{s}/\tilde{x})\frac{\tilde{x}^{2\delta-1}}{\tilde{s}^{2\delta-1}}\frac{d\tilde{s}}{\tilde{x}}\left[\frac{\tilde{s}^{2\delta}}{\tilde{w}(\tilde{s})^2\tilde{x}^{2\delta}}\right] - \frac{8}{\pi}\int_0^\infty G'(\tilde{s}/\tilde{x})\frac{\tilde{x}^{3\delta-1}}{\tilde{s}^{3\delta-1}}\frac{d\tilde{s}}{\tilde{x}}\left[\frac{\chi\tilde{s}^{3\delta}}{\tilde{w}(\tilde{s})^3\tilde{x}^{3\delta}}\right].$$

Replace G’ with Delta function

$$\tilde{w}' = \frac{C_1(\delta)}{\tilde{w}^2} + \frac{\chi C_2(\delta)}{\tilde{w}^3}, \qquad \delta = \tilde{x}\frac{\tilde{w}'}{\tilde{w}}, \qquad \tilde{w}(0) = 1,$$

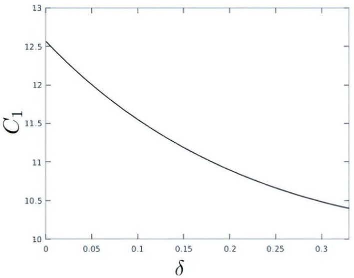
$$C_1(\delta) = \frac{4(1-2\delta)}{\delta(1-\delta)}\tan(\pi\delta), \qquad C_2(\delta) = \frac{16(1-3\delta)}{3\delta(2-3\delta)}\tan\left(\frac{3\pi}{2}\delta\right).$$

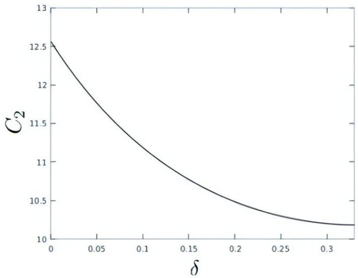


Approximate solution

$$\tilde{w}' = \frac{C_1(\delta)}{\tilde{w}^2} + \frac{\chi C_2(\delta)}{\tilde{w}^3}, \qquad \delta = \tilde{x}\frac{\tilde{w}'}{\tilde{w}}, \qquad \tilde{w}(0) = 1,$$

$$C_1(\delta) = \frac{4(1-2\delta)}{\delta(1-\delta)}\tan(\pi\delta), \qquad C_2(\delta) = \frac{16(1-3\delta)}{3\delta(2-3\delta)}\tan\Big(\frac{3\pi}{2}\delta\Big).$$





Solve assuming constant C1 and C2

$$\tilde{w}^3 - 1 - \frac{3}{2}b(\tilde{w}^2-1) + 3b^2(\tilde{w}-1) - 3b^3\ln\Big(\frac{b+\tilde{w}}{b+1}\Big) = 3C_1(\delta)\tilde{x}, \qquad b = \frac{C_2(\delta)}{C_1(\delta)}\chi.$$

Take $C_1(1/3) = \beta_m^3/3$ and $C_2(1/4) = \beta_{\tilde{m}}^4/4$

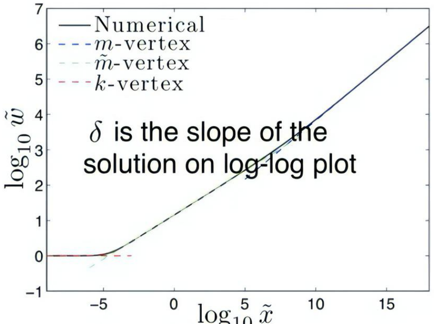
to match viscosity and leak-off limits exactly

$$\tilde{w}_0^3 - 1 - \frac{3}{2}b_0(\tilde{w}_0^2-1) + 3b_0^2(\tilde{w}_0-1) - 3b_0^3\ln\Big(\frac{b_0+\tilde{w}_0}{b_0+1}\Big) = \beta_m^3\tilde{x}, \qquad b_0 = \frac{3\beta_{\tilde{m}}^4}{4\beta_m^3}\chi \approx 0.9912\,\chi.$$

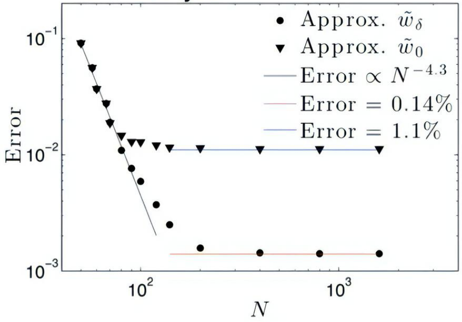
The above is an implicit zeroth order solution for the problem, to get a better approximation, compute

$$\delta = \frac{\beta_m^3\tilde{x}(\tilde{w}_0)}{3\tilde{w}_0^3}\Big(1 + \frac{b_0}{\tilde{w}_0}\Big),$$

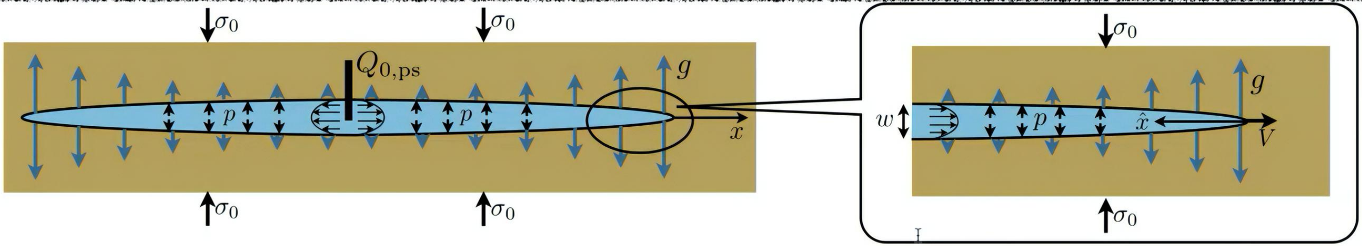
and re-evaluate the solution.



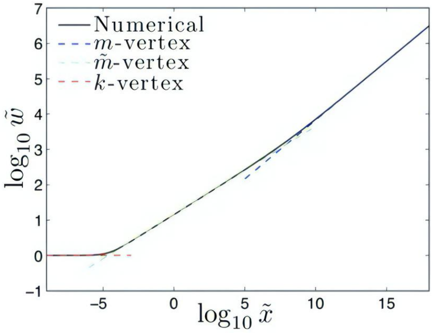
Error analysis

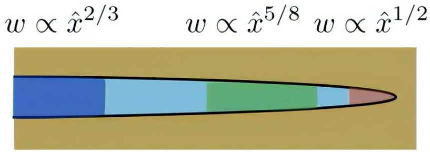


Things to remember for the semi-infinite hydraulic fracture



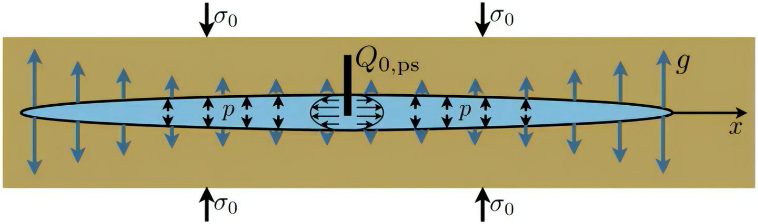
- Semi-infinite geometry describes tip region of a finite hydraulic fracture
- There are three limiting analytic solutions: toughness, viscosity, and leak-off
- The global solution gradually transitions from one limiting case to another
- There is computationally efficient approximate solution for the problem that can be used as a propagation condition for finite fractures





$$w_k = \frac{K'}{E'}\hat{x}^{1/2}, \qquad w_{\tilde{m}} = \beta_{\tilde{m}}\left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8}\hat{x}^{5/8}, \qquad w_m = \beta_m\left(\frac{\mu' V}{E'}\right)^{1/3}\hat{x}^{2/3}.$$

Plane strain hydraulic fracture



Governing equations

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = Q_{0,\mathrm{ps}}(t)\delta(x),$$

$$q = -\frac{w^3}{\mu'}\frac{\partial p_n}{\partial x},$$

$$p_n(x) = -\frac{E'}{4\pi}\int_{-l}^{l}\frac{w(s)ds}{(x-s)^2},$$

$$w \to \frac{K'}{E'}\sqrt{l-x}, \qquad x \to l.$$

$\Longrightarrow$

Scales

$$\frac{w_*}{t} = \frac{q_*}{l_*} = \frac{C'}{t^{1/2}} = \frac{Q_{0,\mathrm{ps}}}{l_*},$$

$$q_* = \frac{w_*^3 p_*}{\mu' l_*},$$

$$p_* = \frac{E' w_*}{l_*},$$

$$w_* = \frac{K'}{E'} l_*^{1/2}.$$

6 equations, 4 unknowns

Scaling for viscosity-storage solution

Viscosity-storage => no toughness, no leak-off

$$\frac{w_*}{t} = \frac{q_*}{l_*} = \cancel{\frac{C'}{t^{1/2}}} = \frac{Q_{0,\mathrm{ps}}}{l_*},$$

$$q_* = \frac{w_*^3 p_*}{\mu' l_*},$$

$$p_* = \frac{E' w_*}{l_*},$$

$$\cancel{w_* = \frac{K'}{E'} l_*^{1/2}}.$$

$\Rightarrow$

Scaling solution

$$l_* = \left(\frac{Q_{0,\mathrm{ps}}^3 E' t^4}{\mu'}\right)^{1/6}, \qquad w_* = \left(\frac{\mu' Q_{0,\mathrm{ps}}^3 t^2}{E'}\right)^{1/6}, \qquad p_* = \left(\frac{\mu' E'^2}{t}\right)^{1/3}$$

M-vertex solution

$$w_m(\xi, t) = 1.1265 \left(\frac{\mu' Q_0^3 t^2}{E'}\right)^{1/6} (1+\xi)^{0.588}(1-\xi)^{2/3},$$

$$p_m(\xi, t) = 2.7495 \left(\frac{\mu' E'^2}{t}\right)^{1/3} \mathcal{F}\left(\xi, 0.588, \tfrac{2}{3}\right),$$

$$l_m(t) = 0.6159 \left(\frac{Q_0^3 E' t^4}{\mu'}\right)^{1/6},$$

Scaling for toughness-storage solution

Toughness-storage => no viscosity, no leak-off

$$\frac{w_*}{t} = \frac{q_*}{l_*} = \cancel{\frac{C'}{t^{1/2}}} = \frac{Q_{0,\mathrm{ps}}}{l_*},$$

$$\cancel{q_* = \frac{w_*^3 p_*}{\mu' l_*}},$$

$$p_* = \frac{E' w_*}{l_*},$$

$$w_* = \frac{K'}{E'} l_*^{1/2}.$$

$\Longrightarrow$

Scaling solution

$$l_* = \left(\frac{E' Q_{0,\mathrm{ps}} t}{K'}\right)^{2/3}, \qquad w_* = \left(\frac{K'^2 Q_{0,\mathrm{ps}} t}{E'^2}\right)^{1/3}, \qquad p_* = \left(\frac{K'^4}{E'^2 Q_{0,\mathrm{ps}} t}\right)^{1/3}$$

K-vertex solution

$$w_k(\xi, t) = 0.6828 \left(\frac{K'^2 Q_0 t}{E'^2}\right)^{1/3} (1-\xi^2)^{1/2},$$

$$p_k(\xi, t) = 0.1831 \left(\frac{K'^4}{E' Q_0 t}\right)^{1/3},$$

$$l_k(t) = 0.9324 \left(\frac{E' Q_0 t}{K'}\right)^{2/3}.$$

Toughness-viscosity transition

Use either

$$l_m \sim l_k$$

$$w_m \sim w_k$$

$$p_m \sim p_k$$

$\Rightarrow$

Dimensionless toughness

$$K_m = \left(\frac{K'^4}{\mu' E'^3 Q_{0,\mathrm{ps}}} \right)^{1/4}$$

$K_m \ll 1$ M vertex (viscosity dominated)

$K_m \gg 1$ K vertex (toughness dominated)

$K_m \sim 1$ M-K transition

This is only part of the story, how about leak-off?

Hydraulic fracture regimes of propagation

Toughness vs. viscosity

1 → Large toughness (no viscosity)

2 → Large viscosity (no toughness)

Small vs. large leak-off

3 → All fluid stays inside the fracture

4 → Most fluid leaks into formation

2&3 M - storage-viscosity

No toughness, no leak-off

2&4 $\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

1&3 K - storage-toughness

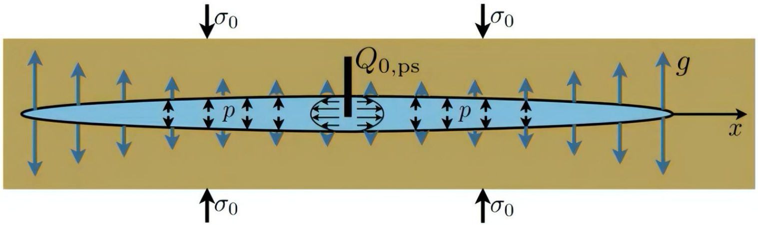
No viscosity, no leak-off

1&4 $\tilde{K}$ - leak-off-toughness

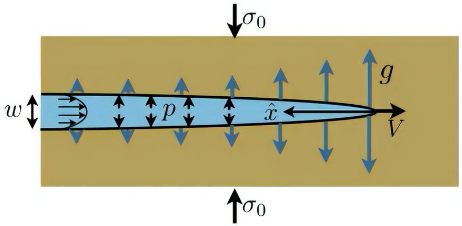
No viscosity, large leak-off

Tip asymptotics vs. regime

Global regime



Tip asymptotics



M - storage-viscosity

No toughness, no leak-off

K - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off

Viscosity

$$w_m = \beta_m \left(\frac{\mu' V}{E'}\right)^{1/3} \hat{x}^{2/3}$$

Toughness

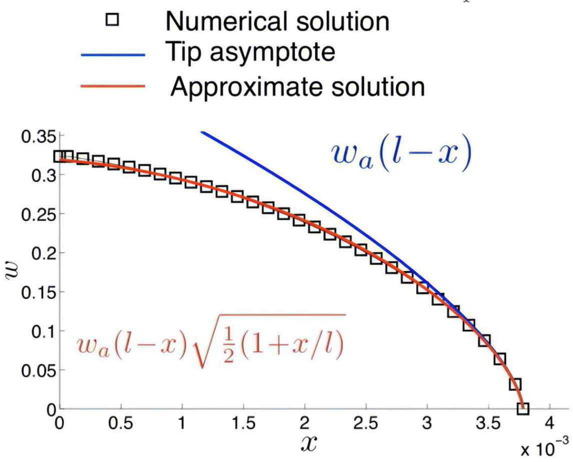
$$w_k = \frac{K'}{E'} \hat{x}^{1/2}$$

Leak-off

$$w_{\tilde{m}} = \beta_{\tilde{m}} \left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8} \hat{x}^{5/8}$$

Solution for the problem

- **Numerical solution**
  - Discretize governing equations using finite differences, see Dontsov, 2017
- **Approximate solution**
  - Global volume balance
  - Approximate solution for width based on the tip asymptote



Global volume balance

$$\int_0^l \left(w(x,t) + 2C'\sqrt{t-t_0(x)} \right) dx = \frac{Q_{0,\mathrm{ps}} t}{2},$$

$$l(t) \propto t^\alpha \implies x/l = (t_0/t)^\alpha$$

Approximate solution for width

$$w(x,t) = \left(\frac{l+x}{2l} \right)^\lambda w_a(l-x),$$

λ - fitting parameter

Regimes of propagation for a plane strain hydraulic fracture

M - storage-viscosity

No toughness, no leak-off

K - storage-toughness

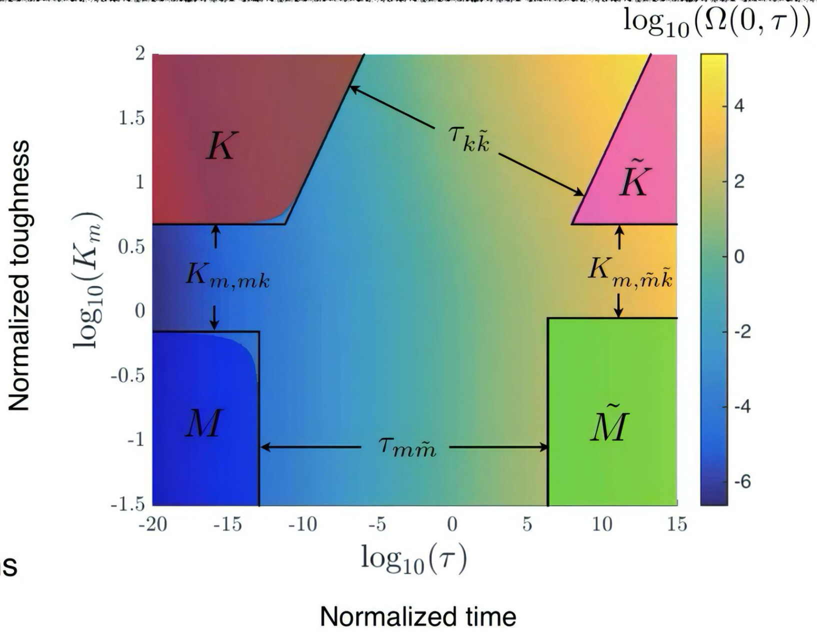
No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off



Zones of applicability of limiting solutions are defined to within 1% of error

Bounds for these zones (solid black lines) are known

$$K_m = \left(\frac{K'^4}{\mu' E'^3 Q_{0,\mathrm{ps}}}\right)^{1/4} \qquad \tau = \frac{t}{t_{m\tilde{m}}} \qquad t_{m\tilde{m}} = \frac{\mu' Q_{0,\mathrm{ps}}^3}{E' C'^6}$$

Limiting solutions for a plane strain hydraulic fracture

M - storage-viscosity

$$\begin{aligned} w_m(\xi,t) &= 1.1265\left(\frac{\mu' Q_{0,\mathrm{ps}}^3 t^2}{E'}\right)^{1/6}(1+\xi)^{0.588}(1-\xi)^{2/3}, \\ p_m(\xi,t) &= 2.7495\left(\frac{\mu' E'^2}{t}\right)^{1/3}\mathcal{F}(\xi, 0.588, \tfrac{2}{3}), \\ l_m(t) &= 0.6159\left(\frac{Q_{0,\mathrm{ps}}^3 E' t^4}{\mu'}\right)^{1/6}, \end{aligned}$$

$\tilde{M}$ - leak-off-viscosity

$$\begin{aligned} w_{\tilde{m}}(\xi,t) &= 0.8165\left(\frac{\mu' Q_{0,\mathrm{ps}}^3 t}{E' C'^2}\right)^{1/4}(1+\xi)^{0.520}(1-\xi)^{5/8}, \\ p_{\tilde{m}}(\xi,t) &= 3.6783\left(\frac{C'^2 \mu' E'^3}{Q_{0,\mathrm{ps}} t}\right)^{1/4}\mathcal{F}(\xi, 0.520, \tfrac{5}{8}), \\ l_{\tilde{m}}(t) &= 0.3183\,\frac{Q_{0,\mathrm{ps}} t^{1/2}}{C'}, \end{aligned}$$

K - storage-toughness

$$\begin{aligned} w_k(\xi,t) &= 0.6828\left(\frac{K'^2 Q_{0,\mathrm{ps}} t}{E'^2}\right)^{1/3}(1-\xi^2)^{1/2}, \\ p_k(\xi,t) &= 0.1831\left(\frac{K'^4}{E' Q_{0,\mathrm{ps}} t}\right)^{1/3}, \\ l_k(t) &= 0.9324\left(\frac{E' Q_{0,\mathrm{ps}} t}{K'}\right)^{2/3}. \end{aligned}$$

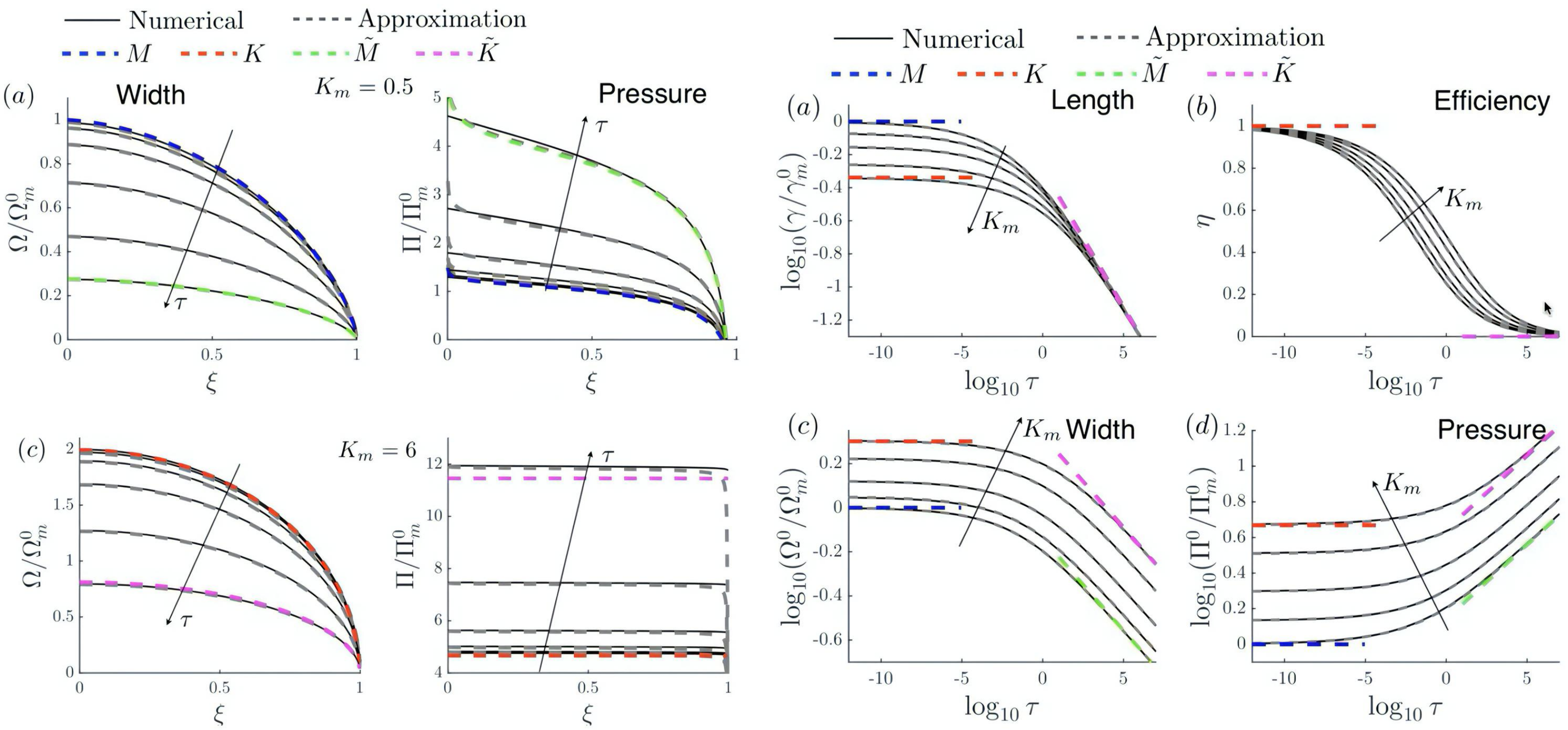
$\tilde{K}$ - leak-off-toughness

$$\begin{aligned} w_{\tilde{k}}(\xi,t) &= 0.3989\left(\frac{K'^4 Q_{0,\mathrm{ps}}^2 t}{E'^4 C'^2}\right)^{1/4}(1-\xi^2)^{1/2}, \\ p_{\tilde{k}}(\xi,t) &= 0.3133\left(\frac{K'^4 C'^2}{Q_{0,\mathrm{ps}}^2 t}\right)^{1/4}, \\ l_{\tilde{k}}(t) &= 0.3183\,\frac{Q_{0,\mathrm{ps}} t^{1/2}}{C'}, \end{aligned}$$

Elasticity function:

$$\begin{aligned} \mathcal{F}(\xi,\lambda,\bar{\delta}) &= \frac{1}{2^{1+\lambda}\pi}\int_0^1 \frac{\partial M(\xi,s)}{\partial s}(1+s)^{\lambda}(1-s)^{\bar{\delta}}\,ds, \\ M(\xi,s) &= \frac{\xi}{\xi^2 - s^2}, \end{aligned}$$

Numerical vs. approximate solutions



Things to remember for the plane strain hydraulic fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote
- The existence of approximate solution constructed using global volume balance and tip asymptote
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless toughness and dimensionless time

M - storage-viscosity

No toughness, no leak-off

K - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off

Viscosity

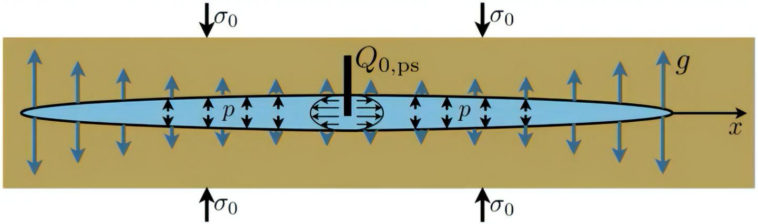
$$w_m = \beta_m \left(\frac{\mu' V}{E'}\right)^{1/3} \hat{x}^{2/3}$$

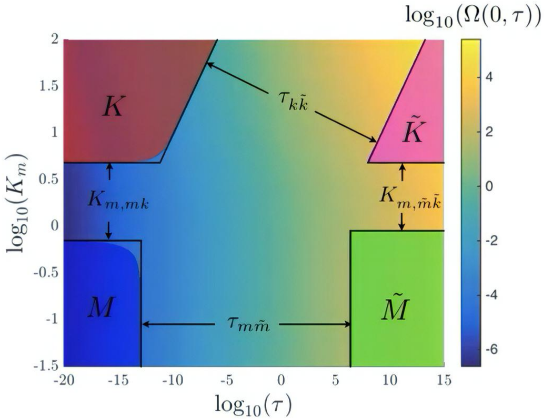
Toughness

$$w_k = \frac{K'}{E'} \hat{x}^{1/2}$$

Leak-off

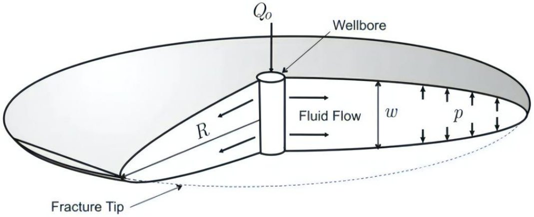
$$w_{\tilde{m}} = \beta_{\tilde{m}} \left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8} \hat{x}^{5/8}$$





E.V. Dontsov. An approximate solution for a plane strain hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *Int. J. Fract.*, 205:221–237, 2017.

Radial hydraulic fracture



Governing equations

$$\frac{\partial w}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rq) + \frac{C'}{\sqrt{t - t_0(r)}} = Q_0 \delta(r),$$

$$q = -\frac{w^3}{\mu'}\frac{\partial p_n}{\partial r},$$

$$p_n(r,t) = -\frac{E'}{2\pi R}\int_0^R M\left(\frac{r}{R}, \frac{r'}{R}\right)\frac{\partial w(r',t)}{\partial r'}\, dr',$$

$$w \to \frac{K'}{E'}(R-r)^{1/2}, \qquad r \to R.$$

Scales

$$\frac{w_*}{t} = \frac{q_*}{R_*} = \frac{C'}{t^{1/2}} = \frac{Q_0}{R_*^2},$$

$$q_* = \frac{w_*^3 p_*}{\mu' R_*},$$

$$p_* = \frac{E' w_*}{R_*},$$

$$w_* = \frac{K' R_*^{1/2}}{E'}.$$

6 equations, 4 unknowns

Scaling for viscosity-storage solution

Viscosity-storage => no toughness, no leak-off

$$\frac{w_*}{t} = \frac{q}{R_*} \cancel{= \frac{C'}{t^{1/2}}} = \frac{Q_0}{R_*^2},$$

$$q = \frac{w_*^3 p_*}{\mu' R_*},$$

$$p_* = \frac{E' w_*}{R_*},$$

$$\cancel{w = \frac{K' R_*^{1/2}}{E'}}.$$

Scaling solution

$$R_* = \left(\frac{Q_0^3 E' t^4}{\mu'}\right)^{1/9}, \qquad w_* = \left(\frac{\mu'^2 Q_0^3 t}{E'^2}\right)^{1/9}, \qquad p_* = \left(\frac{\mu' E'^2}{t}\right)^{1/3}$$

M-vertex solution

$$\begin{aligned}
w_m(\rho, t) &= 1.1901 \left(\frac{\mu'^2 Q_0^3 t}{E'^2}\right)^{1/9} (1+\rho)^{0.487} (1-\rho)^{2/3}, \\
p_m(\rho, t) &= 2.4019 \left(\frac{\mu' E'^2}{t}\right)^{1/3} \mathcal{F}(\rho, 0.487, \tfrac{2}{3}), \\
R_m(t) &= 0.6944 \left(\frac{Q_0^3 E' t^4}{\mu'}\right)^{1/9},
\end{aligned}$$

Regimes of propagation for a radial hydraulic fracture

M - storage-viscosity

No toughness, no leak-off

K - storage-toughness

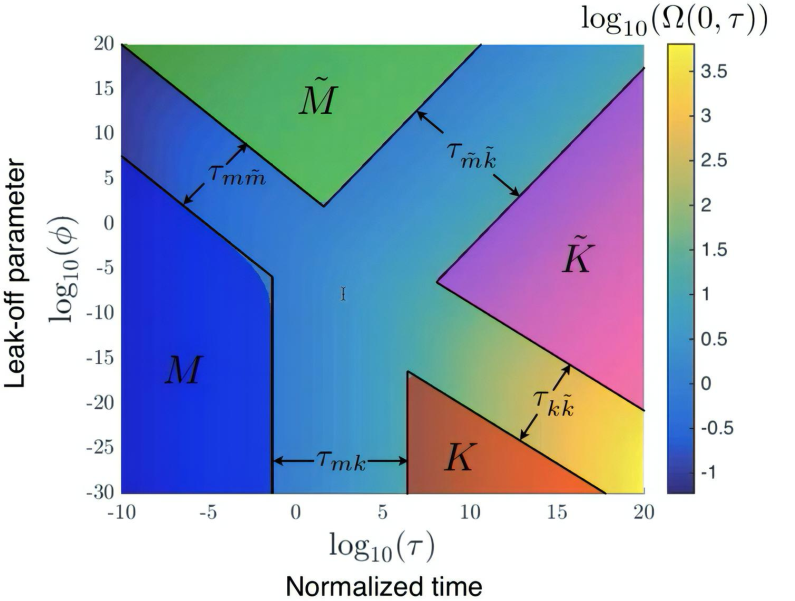
No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off



Zones of applicability of limiting solutions are defined to within 1% of error

Bounds for these zones (solid black lines) are known

$$\phi = \frac{\mu'^3 E'^{11} C'^4 Q_0}{K'^{14}} \qquad \tau = \frac{t}{t_{mk}} \qquad t_{mk} = \left(\frac{\mu'^5 E'^{13} Q_0^3}{K'^{18}}\right)^{1/2}$$

Regimes of propagation for radial HF

M - storage-viscosity

$$
\begin{aligned}
w_m(\rho,t) &= 1.1901\left(\frac{\mu'^2 Q_0^3 t}{E'^2}\right)^{1/9}(1+\rho)^{0.487}(1-\rho)^{2/3},\\
p_m(\rho,t) &= 2.4019\left(\frac{\mu' E'^2}{t}\right)^{1/3}\mathcal{F}(\rho,0.487,\tfrac{2}{3}),\\
R_m(t) &= 0.6944\left(\frac{Q_0^3 E' t^4}{\mu'}\right)^{1/9},
\end{aligned}
$$

$\tilde{M}$ - leak-off-viscosity

$$
\begin{aligned}
w_{\tilde{m}}(\rho,t) &= 1.0574\left(\frac{\mu'^4 Q_0^6 t}{E'^4 C'^2}\right)^{1/16}(1+\rho)^{0.397}(1-\rho)^{5/8},\\
p_{\tilde{m}}(\rho,t) &= 3.0931\left(\frac{\mu'^4 E'^{12} C'^6}{Q_0^2 t^3}\right)^{1/16}\mathcal{F}(\rho,0.397,\tfrac{5}{8}),\\
R_{\tilde{m}}(t) &= 0.4502\left(\frac{Q_0^2 t}{C'^2}\right)^{1/4},
\end{aligned}
$$

K - storage-toughness

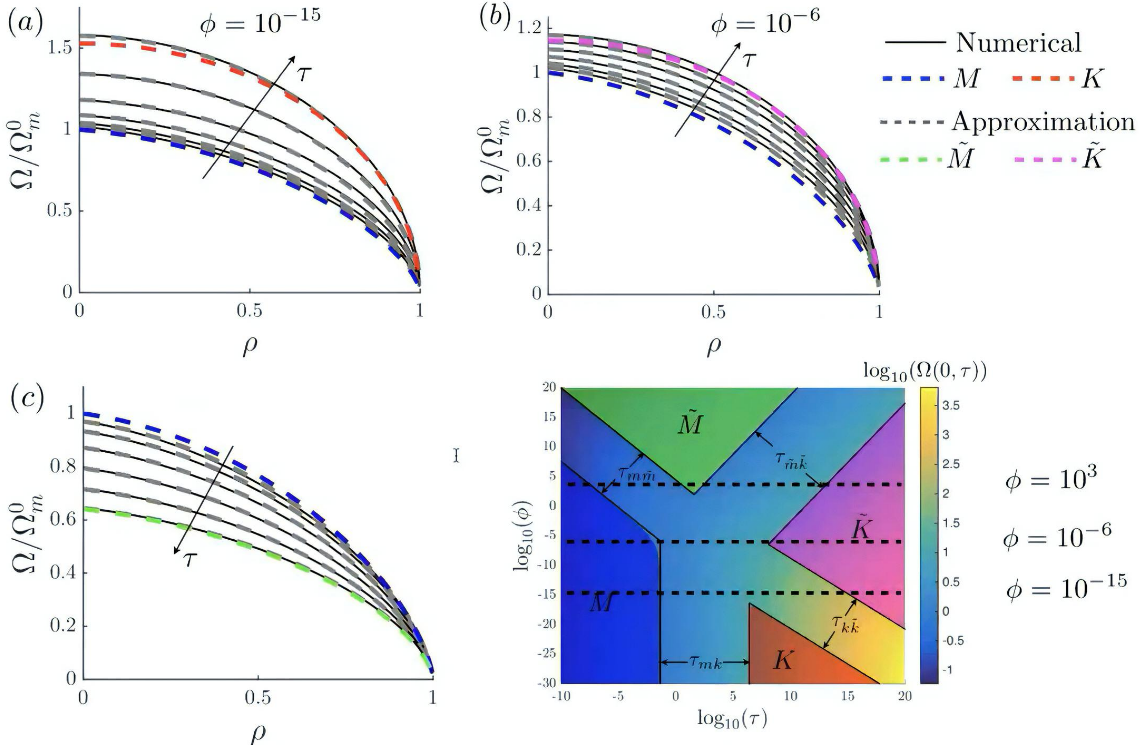
$$
\begin{aligned}
w_k(\rho,t) &= 0.6537\left(\frac{K'^4 Q_0 t}{E'^4}\right)^{1/5}(1-\rho^2)^{1/2},\\
p_k(\rho,t) &= 0.3004\left(\frac{K'^6}{E' Q_0 t}\right)^{1/5},\\
R_k(t) &= 0.8546\left(\frac{E' Q_0 t}{K'}\right)^{2/5}.
\end{aligned}
$$

$\tilde{K}$ - leak-off-toughness

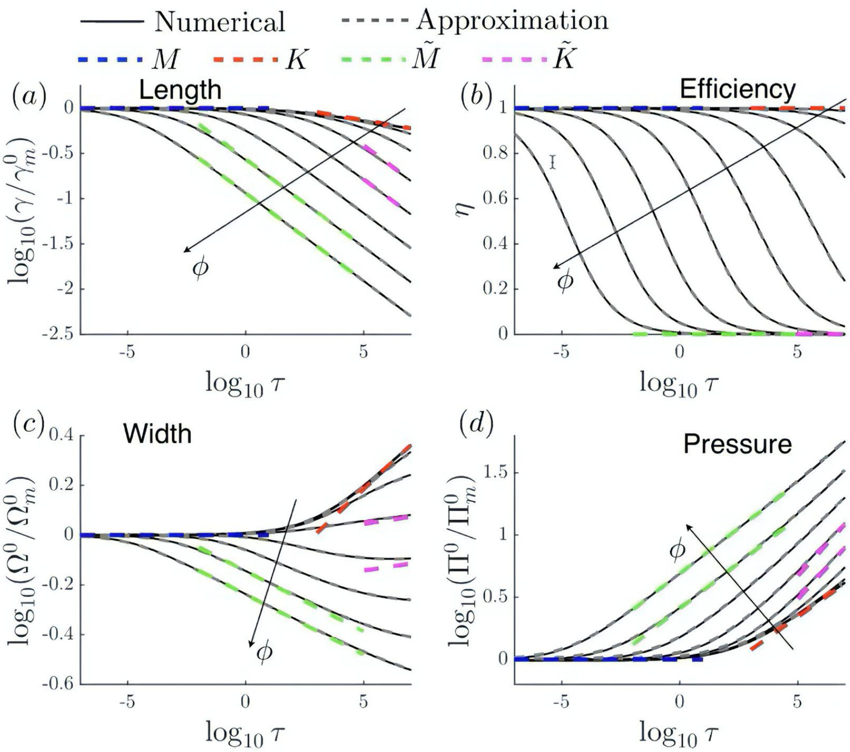
$$
\begin{aligned}
w_{\tilde{k}}(\rho,t) &= 0.4744\left(\frac{K'^8 Q_0^2 t}{E'^8 C'^2}\right)^{1/8}(1-\rho^2)^{1/2},\\
p_{\tilde{k}}(\rho,t) &= 0.4139\left(\frac{K'^8 C'^2}{Q_0^2 t}\right)^{1/8},\\
R_{\tilde{k}}(t) &= 0.4502\left(\frac{Q_0^2 t}{C'^2}\right)^{1/4}.
\end{aligned}
$$

Elasticity function: $\mathcal{F}(\rho,\lambda,\bar{\delta}) = \dfrac{1}{2^{1+\lambda}\pi}\displaystyle\int_0^1 \frac{\partial M(\rho,s)}{\partial s}(1+s)^{\lambda}(1-s)^{\bar{\delta}}\,ds,$

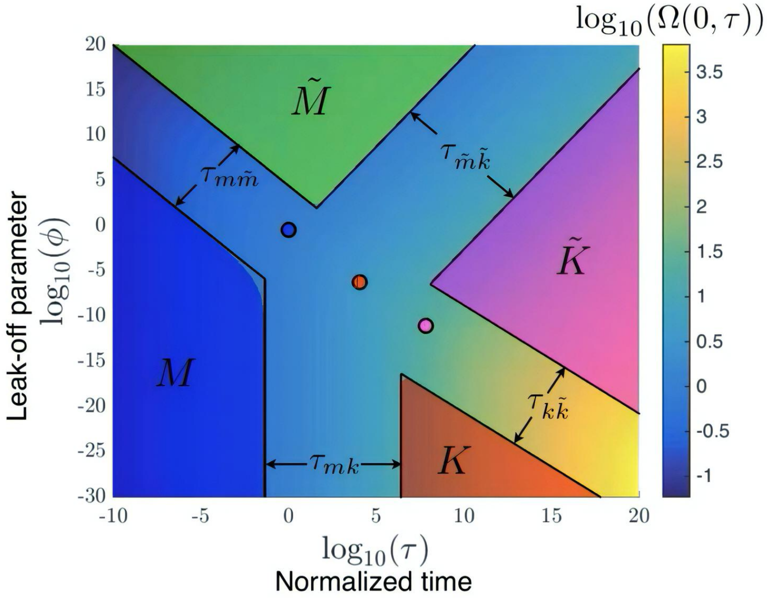
Comparison with numerical solution



Comparison with numerical solution



Radial hydraulic fracture: examples



$$\phi = \frac{\mu'^3 E'^{11} C'^4 Q_0}{K'^{14}} \qquad \tau = \frac{t}{t_{mk}} \qquad t_{mk} = \left(\frac{\mu'^5 E'^{13} Q_0^3}{K'^{18}}\right)^{1/2}$$

Field values

$E' = 36$ GPa

$K_{Ic} = 2.6$ MPa $\cdot$ m$^{1/2}$

$Q_0 = 18$ bbl/min $= 0.05$ m^3/s

$t = 100$ min

$C' = 4.3 \times 10^{-5}$ m/s$^{1/2}$

Slick water

$\mu = 1.5$ cP $= 1.5 \times 10^{-3}$ Pa $\cdot$ s

Linear gel

$\mu = 100$cP $= 0.1$ Pa $\cdot$ s

Lab values (rock)

$E' = 36$ GPa

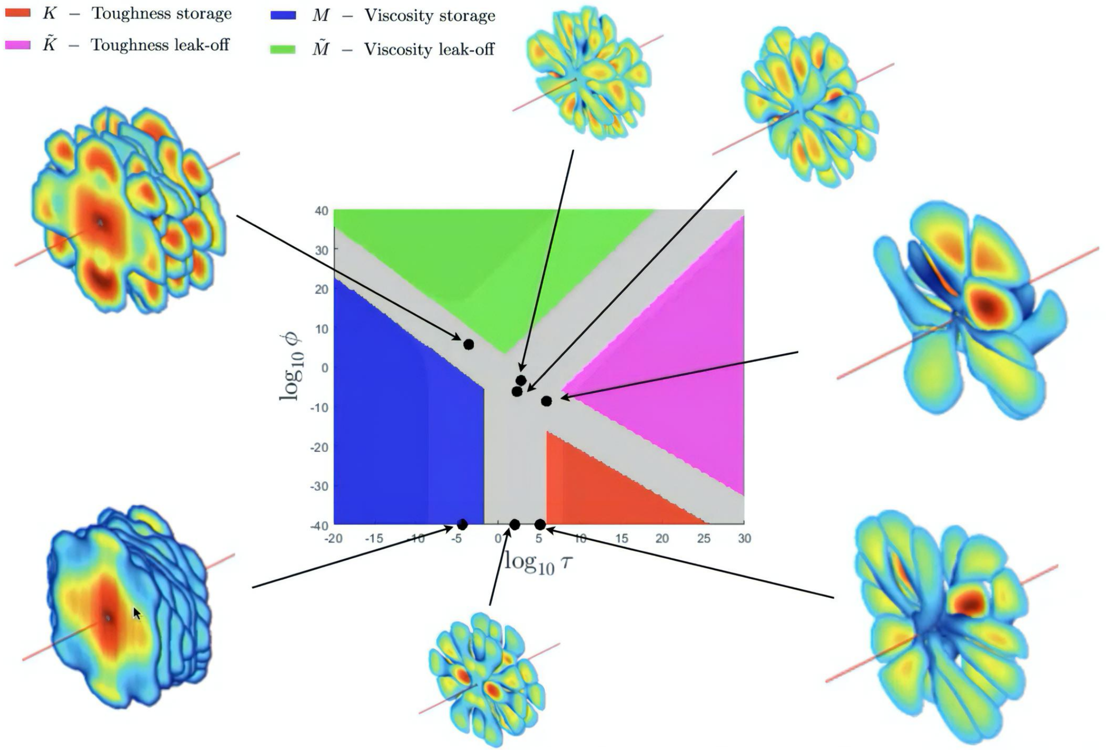
$K_{Ic} = 2.6$ MPa $\cdot$ m$^{1/2}$

$C' = 4.3 \times 10^{-5}$ m/s$^{1/2}$

$\mu = 1.5$ cP $= 1.5 \times 10^{-3}$ Pa $\cdot$ s

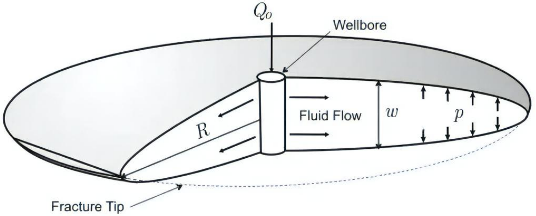
$t = 1$ min $\quad Q_0 = 2.6 \times 10^{-6}$ m^3/s

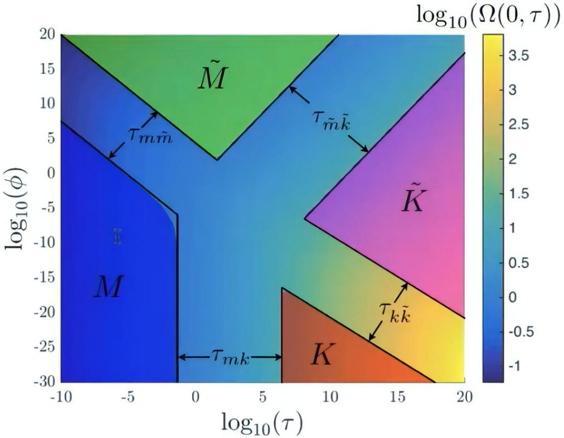
Application to multiple fractures



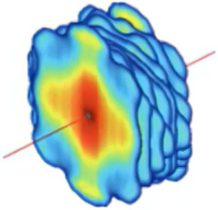
Things to remember for radial hydraulic fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote (the same as for plane strain)
- The existence of approximate solution constructed using global volume balance and tip asymptote (similar to plane strain)
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless leak-off and dimensionless time
- Fracture regimes affect morphology of multiple hydraulic fractures

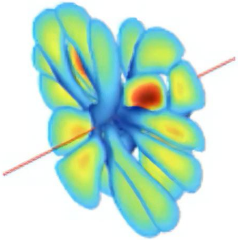




Viscosity



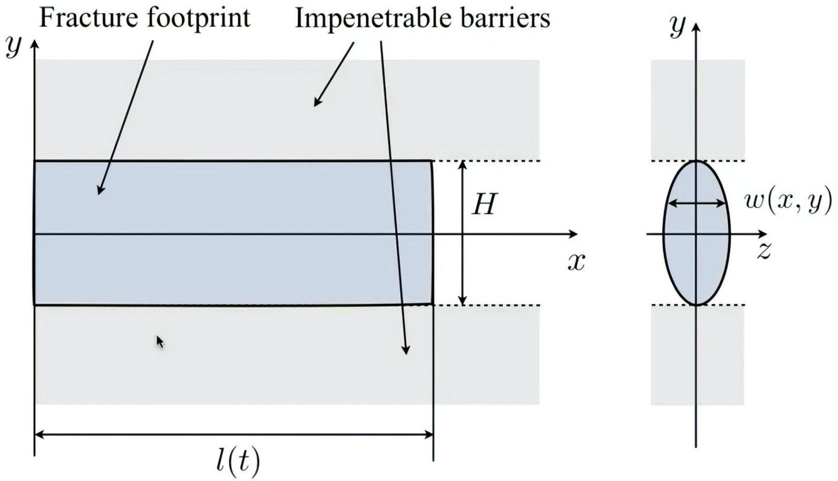
Toughness



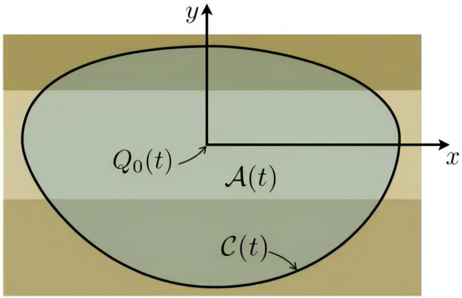
E.V. Dontsov. An approximate solution for a penny-shaped hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *R. Soc. open sci.*, 3:160737, 2016.

PKN or constant height hydraulic fracture

- Fracture height is assumed to be constant
- Length >> height (long time solution)
- Flux is predominantly horizontal
- Pressure is assumed to be constant in the vertical cross-section
- Far away from the tip, local elasticity applies. That is the pressure is determined by the width at the particular point (as opposed to non-local relation for which the pressure depends on the integral of the width with some kernel)
- Fracture width is elliptical in each cross-section
- These assumptions allow to reformulate the two-dimensional fracture problem as one-dimensional, effectively solving the solution in the vertical direction analytically



Governing equations for PKN geometry



Scaling $\quad E' = \dfrac{E}{1-\nu^2}, \qquad \mu' = 12\mu, \qquad K' = \sqrt{\dfrac{32}{\pi}}K_{Ic}, \qquad C' = 2C_l,$

Apply vertical averaging

$$w(x,y) \;=\; \frac{4}{\pi}\bar{w}(x)\sqrt{1-\left(\frac{2y}{H}\right)^2}, \qquad \bar{w}(x) = \frac{1}{H}\int_{-H/2}^{H/2} w(x,y)\,dy,$$

Volume balance $\quad \dfrac{\partial w}{\partial t} + \nabla\cdot\mathbf{q} + \dfrac{C'}{\sqrt{t-t_0(x,y)}} = Q_0(t)\delta(x,y),$

Fluid flow $\quad \mathbf{q} = -\dfrac{w^3}{\mu'}\nabla p,$

Elasticity $\quad p(x,y,t) = \sigma_0(y) - \dfrac{E'}{8\pi}\displaystyle\int_{\mathcal{A}(t)} \dfrac{w(x',y',t)\mathrm{d}x'\mathrm{d}y'}{[(x'-x)^2+(y'-y)^2]^{3/2}},$

Propagation $\quad \displaystyle\lim_{s\to 0}\frac{w}{s^{1/2}} = \begin{cases} \dfrac{K'}{E'}, & \text{if } V>0,\\ \dfrac{K_I'}{E'}, & \text{if } V=0. \end{cases}$

$\Longrightarrow$

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}_x}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} \;=\; \frac{Q_0}{H}\delta(x),$$

$$\bar{q}_x \;=\; -\frac{1}{H\mu'}\frac{\partial p}{\partial x}\int_{-H/2}^{H/2} w^3\,dy \;=\; -\frac{\bar{w}^3}{\pi^2\mu}\frac{\partial p}{\partial x},$$

$$p(x) \;=\; -\frac{2E'}{\pi^2 H}\int_{-l(t)}^{l(t)} \bar{w}(x')\frac{dG(2(x'-x)/H)}{dx'}\,dx'$$

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}$$

Elasticity equation for PKN fracture

$$p(x) \;=\; -\frac{2E'}{\pi^2 H}\int_{-l(t)}^{l(t)} \bar{w}(x')\frac{dG(2(x'-x)/H)}{dx'}\,dx'$$

$$G(s) \;=\; \frac{\sqrt{1+s^2}}{s}\mathrm{E}\Big(\frac{1}{1+s^2}\Big), \qquad G(s) \approx \frac{\pi}{2}\mathrm{sign}(s), \quad |s| \gg 1, \qquad G(s) \approx \frac{1}{s}, \quad s \ll 1,$$

Far away from the tip (x>>H)

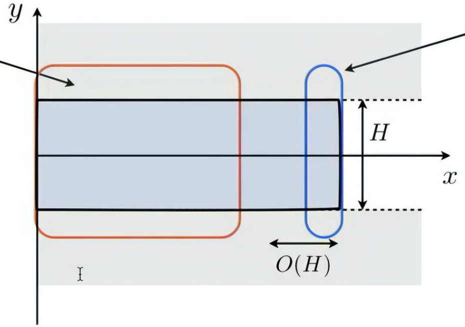
$$p(x) \;=\; \frac{2E'\bar{w}(x)}{\pi H},$$

Local elasticity equation

Near the tip (x~H)

$$p(x,t) = \sigma_0 - \frac{E'}{4\pi}\int_{-l_1}^{l_2} \frac{w(s)ds}{(x-s)^2},$$

Plane strain elasticity equation



Two options for the solution

Local elasticity + special toughness boundary condition

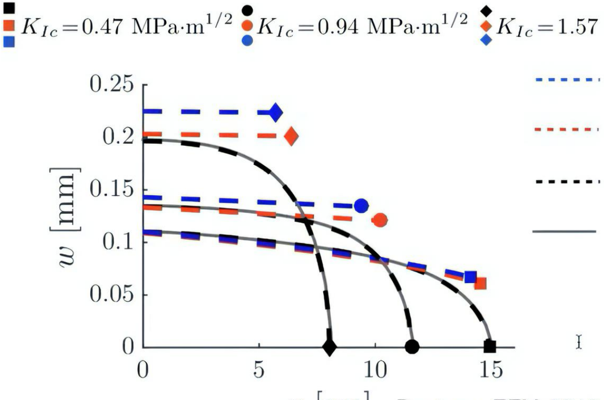
$$p(x) \;=\; \frac{2E'\bar{w}(x)}{\pi H},$$

$$\bar{w}(l) \;=\; \frac{\sqrt{\pi H}\,K_{Ic}}{E'}.$$

Non-elasticity + standard toughness boundary condition

$$p(x) \;=\; -\frac{2E'}{\pi^2 H}\int_{-l(t)}^{l(t)} \bar{w}(x')\frac{dG(2(x'-x)/H)}{dx'}\,dx'$$

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}$$



Dontsov, *EFM*, 2016

$$\bar{w}_{\mathrm{R}}(l) \;=\; \left(\frac{\pi}{2}\right)^{3/2}\frac{\sqrt{H}\,K_{Ic}}{E'}.$$ Nolte,1991

$$\bar{w}_{\mathrm{E}}(l) \;=\; \frac{\sqrt{\pi H}\,K_{Ic}}{E'},$$ Sarvaramini &Garagash, 2015

- Non-local elasticity has superior accuracy
- Formulation with local elasticity can be used for analysis

Tip region of PKN fracture

Equations for finite fracture

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}_x}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H}\delta(x),$$

$$\bar{q}_x = -\frac{1}{H\mu'}\frac{\partial p}{\partial x}\int_{-H/2}^{H/2} w^3\,dy = -\frac{\bar{w}^3}{\pi^2\mu}\frac{\partial p}{\partial x},$$

$$p(x) = \frac{2E'\bar{w}(x)}{\pi H},$$

$$\bar{w}(l) = \frac{\sqrt{\pi H}\,K_{Ic}}{E'}.$$

Use moving coordinate

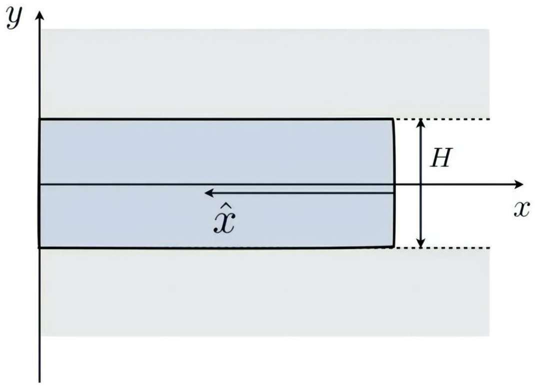
$$\hat{x} = Vt - x$$

$\Rightarrow$

Semi-infinite PKN fracture

$$\frac{E'}{2\pi^3\mu H}\frac{d\bar{w}^4}{d\hat{x}} = V\bar{w} + 2C'\sqrt{V\hat{x}},$$

$$\bar{w}(0) = \frac{\sqrt{\pi H}\,K_{Ic}}{E'}.$$



Known solutions for the tip region

$$\frac{E'}{2\pi^3\mu H}\frac{d\bar{w}^4}{d\hat{x}} = V\bar{w} + 2C'\sqrt{V\hat{x}},$$

$$\bar{w}(0) = \frac{\sqrt{\pi H}\,K_{Ic}}{E'}.$$

Limiting solutions

$$\bar{w}_k = \frac{\sqrt{\pi H}\,K_{Ic}}{E'}, \qquad \bar{w}_m = \left(\frac{3\pi^3\mu H V}{2E'}\right)^{1/3}\hat{x}^{1/3}, \qquad \bar{w}_{\tilde{m}} = \left(\frac{8\pi^3\mu H C' V^{1/2}}{3E'}\right)^{1/4}\hat{x}^{3/8},$$

Toughness — Viscosity — Leak-off

Exact edge solutions

$$\bar{w}_{km} = \left(\bar{w}_k^3 + \bar{w}_m^3\right)^{1/3}, \qquad \bar{w}_{k\tilde{m}} = \left(\bar{w}_k^4 + \bar{w}_{\tilde{m}}^4\right)^{1/4}.$$

Less accurate, but simple approximation

$$\bar{w}_{m\tilde{m}k} = \left(\bar{w}_k^p + \bar{w}_m^p + \bar{w}_{\tilde{m}}^p\right)^{1/p}, \qquad p = 3.4,$$

More accurate approximation

$$\bar{w}_{m\tilde{m}k} = \left[w_{km}\left(\bar{w}_{km}^4 + \bar{w}_{\tilde{m}}^4\right)^{1/4} + w_{k\tilde{m}}\left(\bar{w}_{k\tilde{m}}^3 + \bar{w}_m^3\right)^{1/3}\right]\left[w_{km} + w_{k\tilde{m}}\right]^{-1},$$

Parametric space

Original equation

$$\frac{E'}{2\pi^3\mu H}\frac{d\bar{w}^4}{d\hat{x}} = V\bar{w} + 2C'\sqrt{V\hat{x}},$$

$$\bar{w}(0) = \frac{\sqrt{\pi H}\,K_{Ic}}{E'}.$$

Scaling

$$\Omega = \frac{\bar{w}}{\bar{w}_k} = \frac{E'\bar{w}}{(\pi H)^{1/2}K_{Ic}}, \qquad \xi = \frac{\pi^{3/2}\mu V E'^2\hat{x}}{2K_{Ic}^3 H^{1/2}}, \qquad \chi = \left(\frac{8C'^2K_{Ic}}{\pi^{5/2}\mu H^{1/2}V^2}\right)^{1/2}.$$

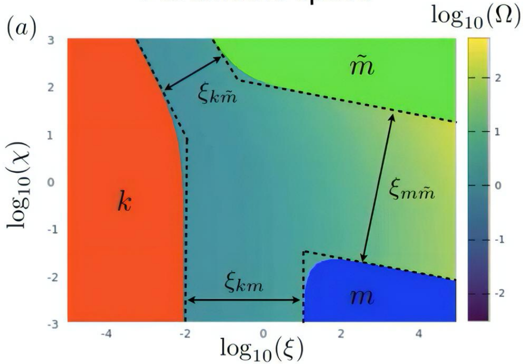
Dimensionless equation

$$\frac{d\Omega}{d\xi} = \frac{1}{\Omega^2} + \frac{\chi\xi^{1/2}}{\Omega^3}, \qquad \Omega(0) = 1.$$

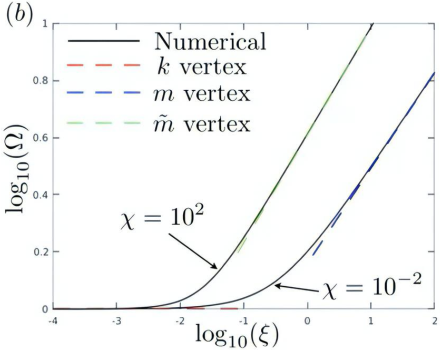
Vertex solutions

$$\Omega_k = 1, \qquad \Omega_m = (3\xi)^{1/3}, \qquad \Omega_{\tilde{m}} = \left(\frac{8\chi}{3}\right)^{1/4}\xi^{3/8},$$

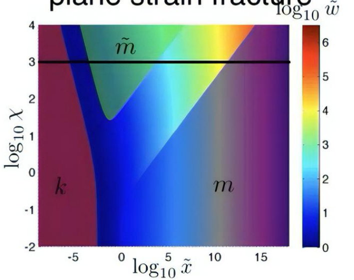
Parametric space



Solution cross-section



Recall the map for semi-infinite plane strain fracture



Transition zones

$$\xi_{km} = \xi, \qquad \xi_{km,1} \approx 0.010, \qquad \xi_{km,2} \approx 11,$$

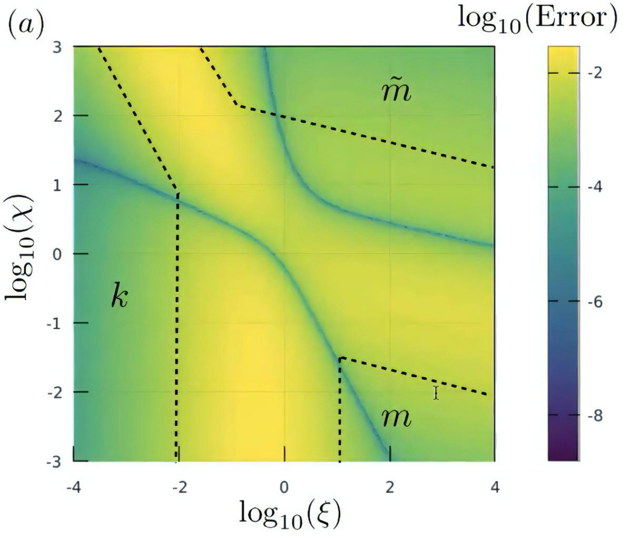
$$\xi_{k\tilde{m}} = \xi\chi^{2/3}, \qquad \xi_{k\tilde{m},1} \approx 0.062, \qquad \xi_{k\tilde{m},2} \approx 4.4,$$

$$\xi_{m\tilde{m}} = \xi\chi^6, \qquad \xi_{m\tilde{m},1} \approx 1.85\times10^{-8}, \qquad \xi_{m\tilde{m},2} \approx 2.1\times10^{12}.$$

Accuracy of the approximations

Less accurate, but simple approximation

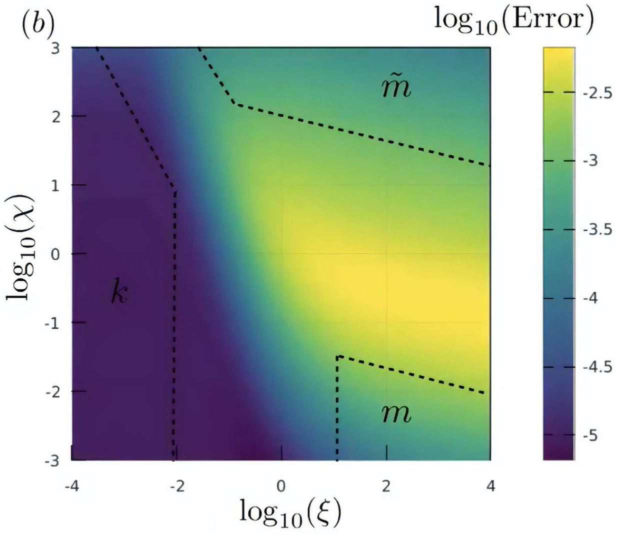
$$\bar{w}_{m\tilde{m}k} = \left(\bar{w}_k^p + \bar{w}_m^p + \bar{w}_{\tilde{m}}^p\right)^{1/p}, \qquad p = 3.4,$$



Max error ~3%

More accurate approximation

$$\bar{w}_{m\tilde{m}k} = \left[w_{km}\left(\bar{w}_{km}^4 + \bar{w}_{\tilde{m}}^4\right)^{1/4} + w_{k\tilde{m}}\left(\bar{w}_{k\tilde{m}}^3 + \bar{w}_m^3\right)^{1/3}\right]\left[w_{km} + w_{k\tilde{m}}\right]^{-1},$$



Max error <1%

Vertex solutions for the finite fracture: storage viscosity

Governing equations

$$\frac{\partial \bar{w}}{\partial t} - \frac{E'}{2\pi^3 \mu H}\frac{\partial^2 \bar{w}^4}{\partial x^2} = \frac{Q_0}{H}\delta(x), \qquad \int_0^l \bar{w}(x)\,dx = \frac{Q_0 t}{2H}, \qquad \bar{w}(l) = 0.$$

Explicitly incorporate tip solution

$$\bar{w} = \underbrace{\left(\frac{3\pi^3\mu H}{2E'}\right)^{1/3}(\dot{l})^{1/3} l^{1/3}(1-\xi)^{1/3}}_{\text{Tip solution}} \overbrace{f(\xi)}^{\text{Correction}}, \qquad \xi = \frac{x}{l(t)},$$

$$\left(\frac{3\pi^3\mu H}{2E'}\right)^{1/3}(\dot{l})^{1/3} l^{4/3}\int_0^1 (1-\xi)^{1/3} f(\xi)\,d\xi = \frac{Q_0 t}{2H}. \longrightarrow l = c\left(\frac{5E'Q_0^3}{48\pi^3\mu H^4}\right)^{1/5} t^{4/5},$$

$$c^{5/3}\int_0^1 (1-\xi)^{1/3} f(\xi)\,d\xi = 1.$$

$$\frac{1}{4}(1-\xi)^{1/3} f(\xi) - \xi\frac{d(1-\xi)^{1/3} f(\xi)}{d\xi} - \frac{3}{4}\frac{d^2(1-\xi)^{4/3}(f(\xi))^4}{d\xi^2} = 0,$$

$$f\big|_{\xi=1} = 1.$$

$$c = \left(\frac{224}{167}\right)^{3/5} \approx 1.193.$$

$$f(\xi) \approx 1 - \frac{1-\xi}{96}.$$

Correction is very small!

Summary of the solution

$$\bar{w}_M = 1.76\left(\frac{\mu Q_0^2}{E'H}\right)^{1/5} t^{1/5}(1-\xi)^{1/3}\left(1 - \frac{1-\xi}{96}\right),$$

$$p_M = 1.12\left(\frac{\mu E'^4 Q_0^2}{H^6}\right)^{1/5} t^{1/5}(1-\xi)^{1/3}\left(1 - \frac{1-\xi}{96}\right),$$

$$l_M = 0.38\left(\frac{E'Q_0^3}{\mu H^4}\right)^{1/5} t^{4/5}.$$

Vertex solutions for the finite fracture: leak-off viscosity

Governing equations

$$-\frac{E'}{2\pi^3\mu H}\frac{\partial^2 \bar{w}^4}{\partial x^2} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H}\delta(x), \qquad 2C'\int_0^l \sqrt{t-t_0(x)}\,dx = \frac{Q_0 t}{2H}, \qquad \bar{w}(l) = 0.$$

$$l \propto t^\alpha \qquad t_0(x) = t(x/l)^{1/\alpha}$$

$$2C't^{1/2}l\int_0^1 \sqrt{1-\xi^{1/\alpha}}\,d\xi = \frac{Q_0 t}{2H}. \xrightarrow{\alpha = 1/2} l = \frac{Q_0 t^{1/2}}{\pi C' H}.$$

$$-\frac{E'}{2\pi^3\mu H}\frac{\partial^2 \bar{w}^4}{\partial x^2} + \frac{C'}{\sqrt{t}\sqrt{1-(x/l)^2}} = \frac{Q_0}{H}\delta(x), \qquad \bar{w}(l) = 0,$$

$$\bar{w} = \left(\frac{2\pi\mu Q_0^2}{E'C'H}\right)^{1/4} t^{1/8} g(\xi), \qquad g(\xi) = \left[\xi\left(\sin^{-1}(\xi) - \frac{\pi}{2}\right) + \sqrt{1-\xi^2}\right]^{1/4}, \qquad \xi = \frac{x}{l}.$$

$$g(\xi) \approx \left(\frac{8}{9}\right)^{1/8} \underline{(1-\xi)^{3/8}}\left[1 - \left(1 - \left(\frac{9}{8}\right)^{1/8}\right)(1-\xi)\right].$$

Tip solution

Correction

Correction is very small!

Summary of the solution

$$\bar{w}_{\tilde{M}} = \left(\frac{2\pi\mu Q_0^2}{E'C'H}\right)^{1/4} t^{1/8} g(\xi),$$

$$p_{\tilde{M}} = \left(\frac{32\mu E'^3 Q_0^2}{\pi^3 C' H^5}\right)^{1/4} t^{1/8} g(\xi),$$

$$l_{\tilde{M}} = \frac{Q_0 t^{1/2}}{\pi C' H}.$$

Vertex solutions for the finite fracture: storage and leak-off toughness

No viscosity implies no pressure gradient $\Longrightarrow$ Width is spatially constant and is determined by the boundary condition at the tip

Storage toughness

$$\int_0^l \bar{w}(x)\,dx = \frac{Q_0 t}{2H}, \qquad \bar{w}(l) = \frac{\sqrt{\pi H}\,K_{Ic}}{E'}.$$

Summary of the solution

$$\begin{aligned}
\bar{w}_K &= \frac{K_{Ic}\sqrt{\pi H}}{E'},\\
p_K &= \frac{2K_{Ic}}{\sqrt{\pi H}},\\
l_K &= \frac{E' Q_0 t}{\sqrt{4\pi} K_{Ic} H^{3/2}}.
\end{aligned}$$

Leak-off toughness

$$\int_0^l 2C'\sqrt{t-t_0(x)}\,dx = \frac{Q_0 t}{2H}. \qquad \bar{w}(l) = \frac{\sqrt{\pi H}\,K_{Ic}}{E'}.$$

$$\Big\downarrow \; l \propto t^\alpha \quad t_0(x) = t(x/l)^{1/\alpha}$$

$$2C' t^{1/2} l \int_0^1 \sqrt{1-\xi^{1/\alpha}}\,d\xi = \frac{Q_0 t}{2H}. \xrightarrow{\alpha = 1/2} l = \frac{Q_0 t^{1/2}}{\pi C' H}.$$

Summary of the solution

$$\begin{aligned}
\bar{w}_{\tilde{K}} &= \frac{K_{Ic}\sqrt{\pi H}}{E'},\\
p_{\tilde{K}} &= \frac{2K_{Ic}}{\sqrt{\pi H}},\\
l_{\tilde{K}} &= \frac{Q_0 t^{1/2}}{\pi C' H}.
\end{aligned}$$

Fast approximate solution for a finite PKN fracture

We saw earlier that all the vertex solutions are approximated accurately by the tip solution. Therefore, the solution is sought in the following form:

$$\bar{w} = \bar{w}_a(1-\xi)^{\delta}, \qquad \bar{w}_a = \bar{w}_{m\tilde{m}k}(x=l, V=\dot{l}) \qquad \delta = \frac{d\log(\bar{w}_{m\tilde{m}k})}{d\log(\hat{x})}, \qquad \xi = \frac{x}{l},$$

Full solution

Tip asymptote

Spatial behavior from asymptote

Global volume balance

$$\int_0^l \left[\bar{w}(x) + 2C'\sqrt{t-t_0(x)}\right] dx \;=\; \frac{Q_0 t}{2H}.$$

$$\Big\downarrow \; l \propto t^{\alpha} \quad t_0(x) = t(x/l)^{1/\alpha}$$

$$\frac{\bar{w}_a(l, \alpha l/t)l}{1+\delta} + \sqrt{\pi}C' t^{1/2} l \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})} \;=\; \frac{Q_0 t}{2H},$$

$$\delta = \frac{d\log(\bar{w}_{m\tilde{m}k})}{d\log(\hat{x})}, \qquad \alpha = \frac{d\log(l)}{d\log(t)},$$

This system of equations can be solved quickly

Scaling

Original system of equations

$$\frac{\partial \bar{w}}{\partial t} - \frac{E'}{2\pi^3 \mu H}\frac{\partial^2 \bar{w}^4}{\partial x^2} + \frac{C'}{\sqrt{t - t_0(x)}} = \frac{Q_0}{H}\delta(x),$$

$$\bar{w}(l) = \frac{\sqrt{\pi H}\, K_{Ic}}{E'}.$$

$$\frac{dl}{dt} = \frac{\bar{q}_x(l)}{\bar{w}(l)} = -\frac{2E'}{3\pi^3 \mu H}\frac{\partial \bar{w}^3}{\partial x}\bigg|_{x=l},$$

Dimensionless system of equations

$$\frac{\partial \Omega}{\partial \tau} - \frac{\xi \dot{\lambda}}{\lambda}\frac{\partial \Omega}{\partial \xi} - \frac{1}{\lambda^2}\frac{\partial^2 \Omega^4}{\partial \xi^2} + \frac{\phi}{\sqrt{\tau - \tau_0(\xi)}} = \delta(\xi),$$

$$\Omega(1) = 1,$$

$$\frac{d\lambda}{d\tau} = -\frac{4}{3\lambda}\frac{\partial \Omega^3}{\partial \xi}\bigg|_{\xi=1}.$$

Scaling

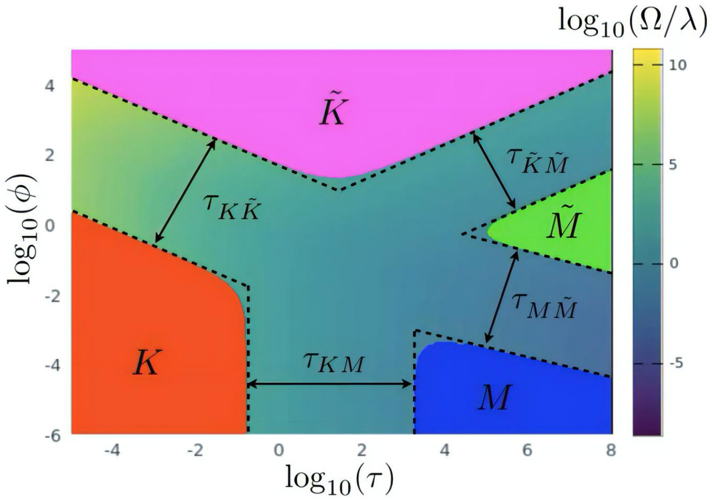
$$\Omega = \frac{\bar{w}}{w_*}, \quad \lambda = \frac{l}{l_*}, \quad \tau = \frac{t}{t_*}, \quad \xi = x/l(t) \quad \phi = \left(\frac{H^5 K_{Ic}^6 C'^4}{4\pi^3 E'^4 \mu^2 Q_0^4}\right)^{1/4}.$$

$$w_* = \frac{(\pi H)^{1/2} K_{Ic}}{E'}, \quad l_* = \frac{H^2 K_{Ic}^4}{2\pi E'^3 \mu Q_0}, \quad t_* = \frac{H^{7/2} K_{Ic}^5}{2\pi^{1/2} E'^4 \mu Q_0^2}.$$

The dimensionless system of equations can be solved numerically using standard finite difference technique.

Parametric space

- Parametric space is computed using the fast solution
- Zones of applicability of the vertex solutions are indicated
- The zone boundaries are quantified



$$\tau = \frac{2\pi^{1/2} E'^4 \mu Q_0^2 t}{H^{7/2} K_{Ic}^5} \qquad \phi = \left(\frac{H^5 K_{Ic}^6 C'^4}{4\pi^3 E'^4 \mu^2 Q_0^4}\right)^{1/4}.$$

M - storage-viscosity

No toughness, no leak-off

K - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

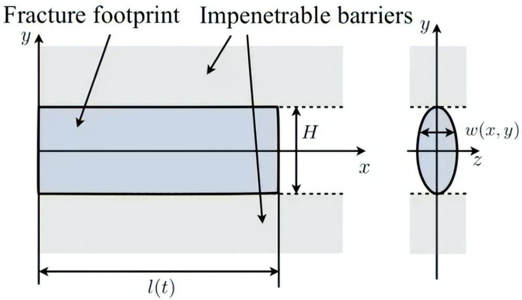
No viscosity, large leak-off

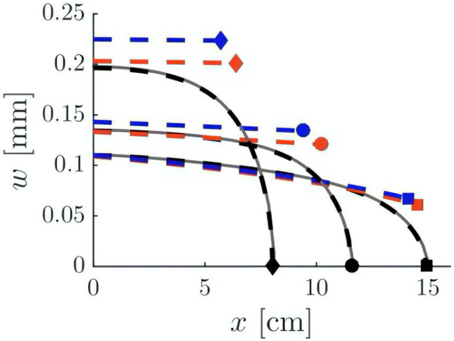
Transitions

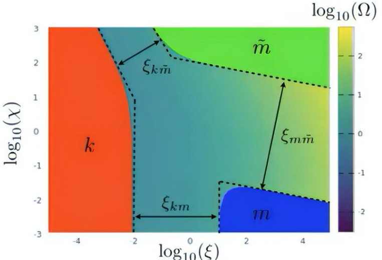
$$\begin{aligned}
\tau_{MK} &= \tau, & \tau_{MK,1} &= 0.16, & \tau_{MK,2} &= 1.6 \times 10^3, \\
\tau_{K\tilde{K}} &= \tau\phi^2, & \tau_{K\tilde{K},1} &= 5.6 \times 10^{-5}, & \tau_{K\tilde{K},2} &= 2.8 \times 10^3, \\
\tau_{\tilde{K}\tilde{M}} &= \tau\phi^{-2}, & \tau_{\tilde{K}\tilde{M},1} &= 0.18, & \tau_{\tilde{K}\tilde{M},2} &= 6.6 \times 10^4, \\
\tau_{M\tilde{M}} &= \tau\phi^{10/3}, & \tau_{M\tilde{M},1} &= 2.3 \times 10^{-7}, & \tau_{M\tilde{M},2} &= 2.8 \times 10^3.
\end{aligned}$$

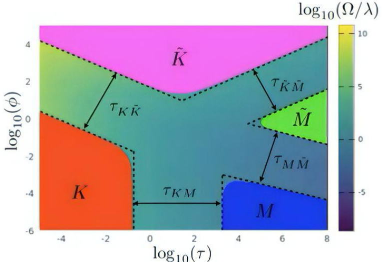
Things to remember

- Assumptions of PKN model, including constant height, length >> height, horizontal flux, constant pressure in each vertical cross-section, elliptical width in each vertical cross-section
- There are two approaches to solve the problem:
  - Use non-local elasticity, which is good for numerical scheme and leads to superior accuracy
  - Use local elasticity with a specific boundary condition at the tip, which is less accurate, but easier for analysis
- PKN fracture has its own formulation for the tip region, which is different from that for a semi-infinite plane strain fracture, approximate solution exists
- Parametric space for the finite fracture is evaluated
- Limiting vertex solutions exist, as well as the global approximate solution

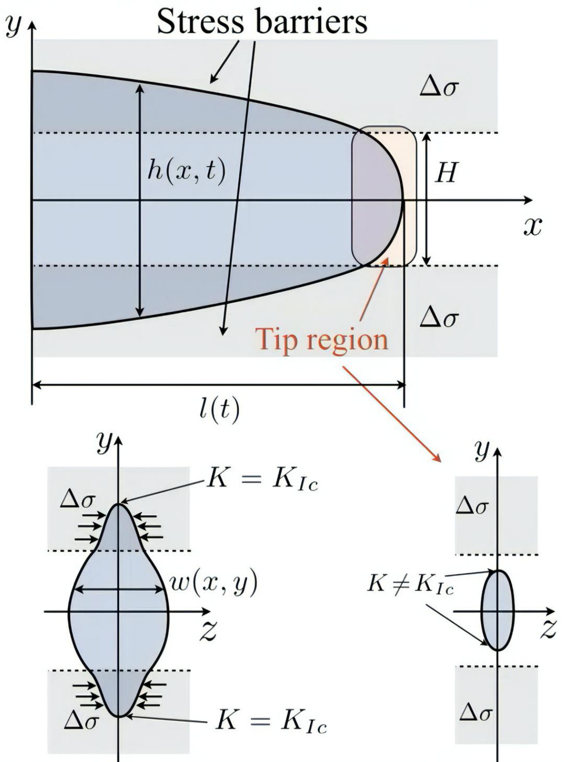








Enhanced pseudo-3D (EP3D) model

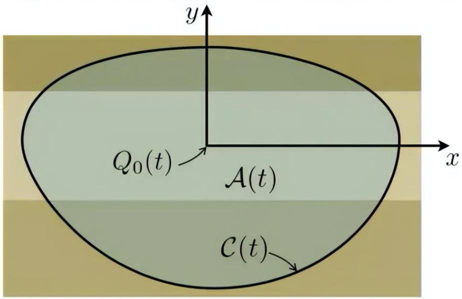


- Extension of PKN model that allows height growth
- Symmetric stress barriers
- Fluid pressure is constant in each vertical cross-section, i.e. *p=p(x)* (toughness dominated propagation with toughness at the vertical tips controlling height growth)
- Fracture opening and height in *(y,z)* cross-sections:
  - Plane strain analytic solution outside of the tip region h>H
  - **Radial solution in the tip region**
- **Non-local elasticity**

$$p(x) = \int_{-l(t)}^{l(t)} G(x,x')\bar{w}(x')\,dx'$$

- **Asymptotic solution at the tip**
- **Viscous height growth (see paper Dontsov&Peirce, 2015)**
- Solution with local elasticity, without viscous height growth, without radial solution at the tip, and without using the tip asymptote corresponds to classical P3D model

Governing equations



Scaling $E' = \dfrac{E}{1-\nu^2}, \qquad \mu' = 12\mu, \qquad K' = \sqrt{\dfrac{32}{\pi}} K_{Ic}, \qquad C' = 2C_l,$

Apply vertical averaging

$$\bar{w}(x) = \frac{1}{H}\int_{-h/2}^{h/2} w(x,y)\,dy,$$

We also need $h(\bar{w})$

Volume balance $$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{q} + \frac{C'}{\sqrt{t-t_0(x,y)}} = Q_0(t)\delta(x,y),$$

Fluid flow $$\mathbf{q} = -\frac{w^3}{\mu'}\nabla p,$$

Elasticity $$p(x,y,t) = \sigma_0(y) - \frac{E'}{8\pi}\int_{\mathcal{A}(t)} \frac{w(x',y',t)\mathrm{d}x'\mathrm{d}y'}{[(x'-x)^2+(y'-y)^2]^{3/2}},$$

Propagation $$\lim_{s\to 0}\frac{w}{s^{1/2}} = \begin{cases} \dfrac{K'}{E'}, & \text{if } V>0, \\ \dfrac{K_I'}{E'}, & \text{if } V=0. \end{cases}$$

$\Rightarrow$

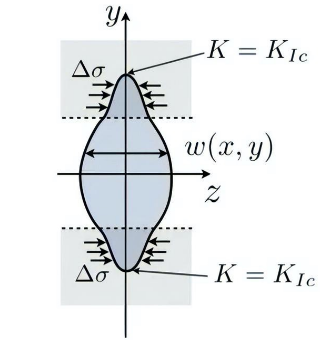
$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}_x}{\partial x} + \frac{C'}{\sqrt{t-t_0(x)}} = \frac{Q_0}{H}\delta(x),$$

$$\bar{q}_x = -\frac{1}{H\mu'}\frac{\partial p}{\partial x}\int_{-h/2}^{h/2} w^3\,dy,$$

$$p(x) = \int_{-l(t)}^{l(t)} G(x,x')\bar{w}(x')\,dx'$$

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}$$

Width solution



Interior (plane strain solution for a uniformly pressurized fracture)

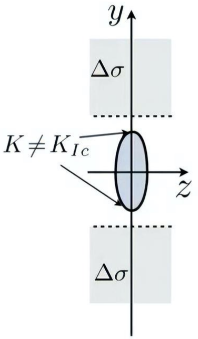
$$w(x,y) \;=\; \frac{2}{E'}\sqrt{\frac{2}{\pi h}}K_{Ic}\chi + \frac{4\Delta\sigma}{\pi E'}\left\{-y\log\left|\frac{H\chi+2y\psi}{H\chi-2y\psi}\right| + \frac{H}{2}\log\left|\frac{\chi+\psi}{\chi-\psi}\right|\right\},$$

$$\chi = \sqrt{h^2-4y^2},\, \psi = \sqrt{h^2-H^2},$$

Averaging

$$\bar{w}(x) = \frac{1}{H}\int_{-h/2}^{h/2} w(x,y)\,dy,$$

$$\bar{w} \;=\; \frac{H}{E'}\left(\sqrt{\frac{\pi}{2H}}K_{Ic}\left(\frac{h}{H}\right)^{3/2} + \Delta\sigma\sqrt{\frac{h^2}{H^2}-1}\right), \qquad h > H,$$



Tip (radial solution in toughness regime)

$$w_{\rm rad} = \frac{4K_{Ic}}{\sqrt{\pi l}E'}\sqrt{l^2-x^2-y^2},$$

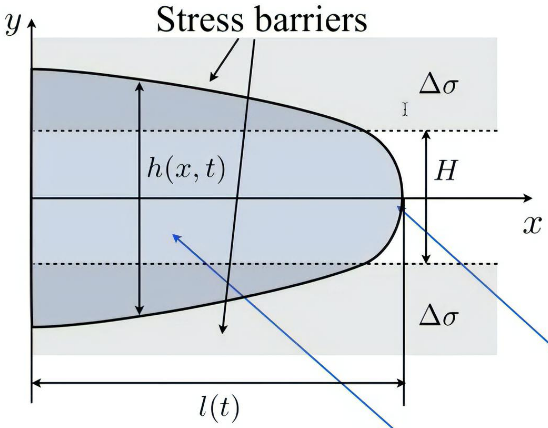
$$h_{\rm rad} = 2\sqrt{l^2-x^2}.$$

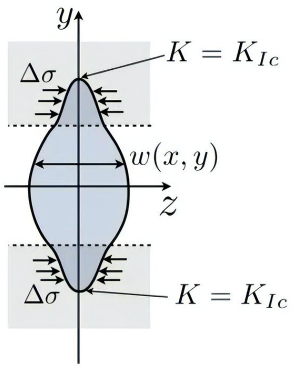
Averaging

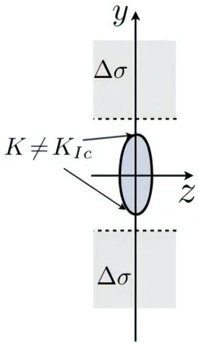
$$\bar{w}_{\rm rad} = \frac{1}{H}\int_{-h_{\rm rad}/2}^{h_{\rm rad}/2} w_{\rm rad}\,dy = \frac{\pi K_{Ic}}{2HE'\sqrt{\pi l}}h_{\rm rad}^2,$$

$$h_{\rm rad} = \bar{w}_{\rm rad}^{1/2}(2l)^{1/4}\sqrt{\frac{\sqrt{2}HE'}{\sqrt{\pi}K_{Ic}}}. \qquad w_{\rm rad} = \frac{4H}{\pi h_{\rm rad}}\bar{w}_{\rm rad}\sqrt{1-\left(\frac{2y}{h_{\rm rad}}\right)^2},$$

Non-local elasticity







Plane strain (local) elasticity

$$p_{P3D} \;=\; \sqrt{\frac{2}{\pi h}}K_{IC} + \Delta\sigma\Big(1 - \frac{2}{\pi}\arcsin\Big(\frac{H}{h}\Big)\Big)$$

KGD

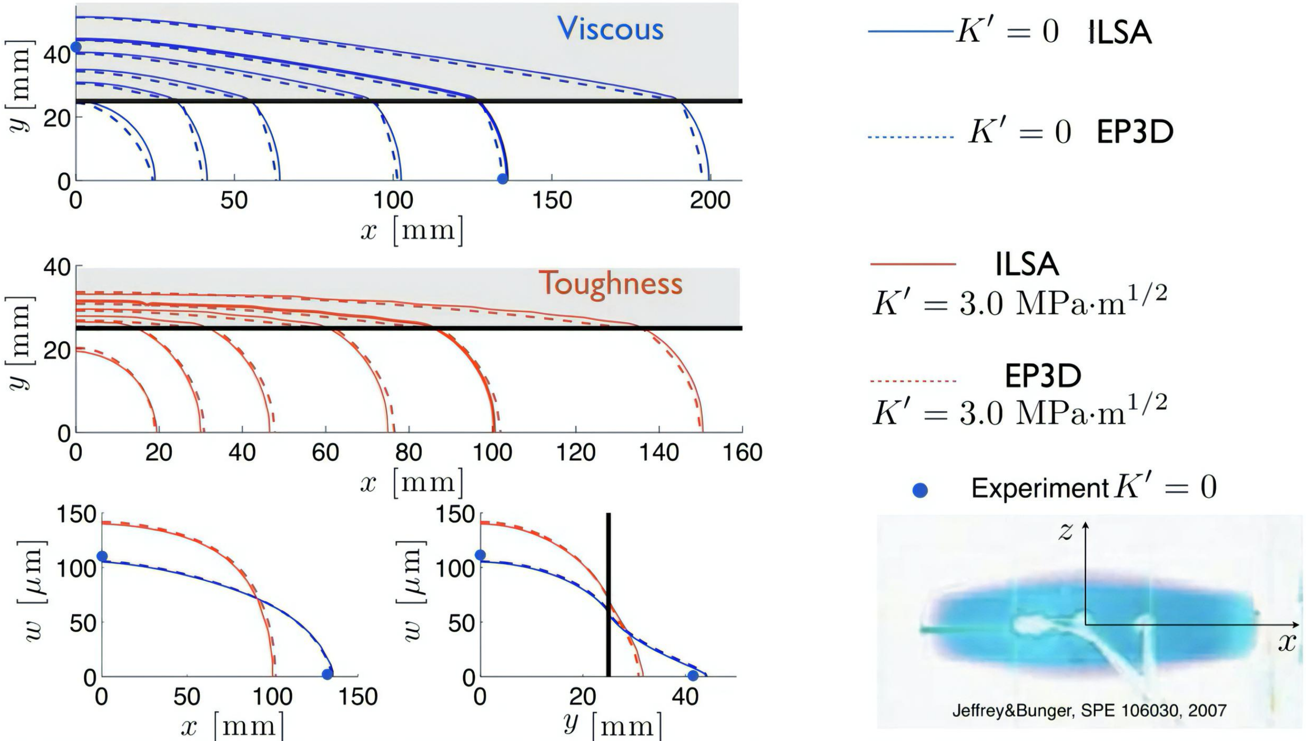
$$w(x,y) \;=\; \frac{2}{E'}\sqrt{\frac{2}{\pi h}}K_{Ic}\chi + \frac{4\Delta\sigma}{\pi E'}\left\{-y\log\left|\frac{H\chi+2y\psi}{H\chi-2y\psi}\right| + \frac{H}{2}\log\left|\frac{\chi+\psi}{\chi-\psi}\right|\right\},$$

For faster evaluation, the width is approximated by two ellipses and the integral is computed analytically

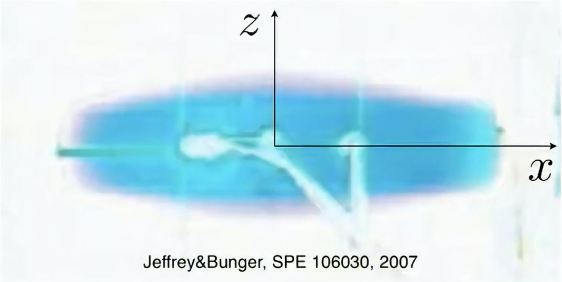
Non-local elasticity

$$p(x) \;=\; -\frac{E'}{8\pi}\int_{-l(t)}^{l(t)}\int_{-\frac{1}{2}h(x',t)}^{\frac{1}{2}h(x',t)}\frac{w(x',y')\,dy'\,dx'}{((x'-x)^2+y'^2)^{3/2}} = \int_{-l(t)}^{l(t)} G(x,x')\bar{w}(x')\,dx'$$

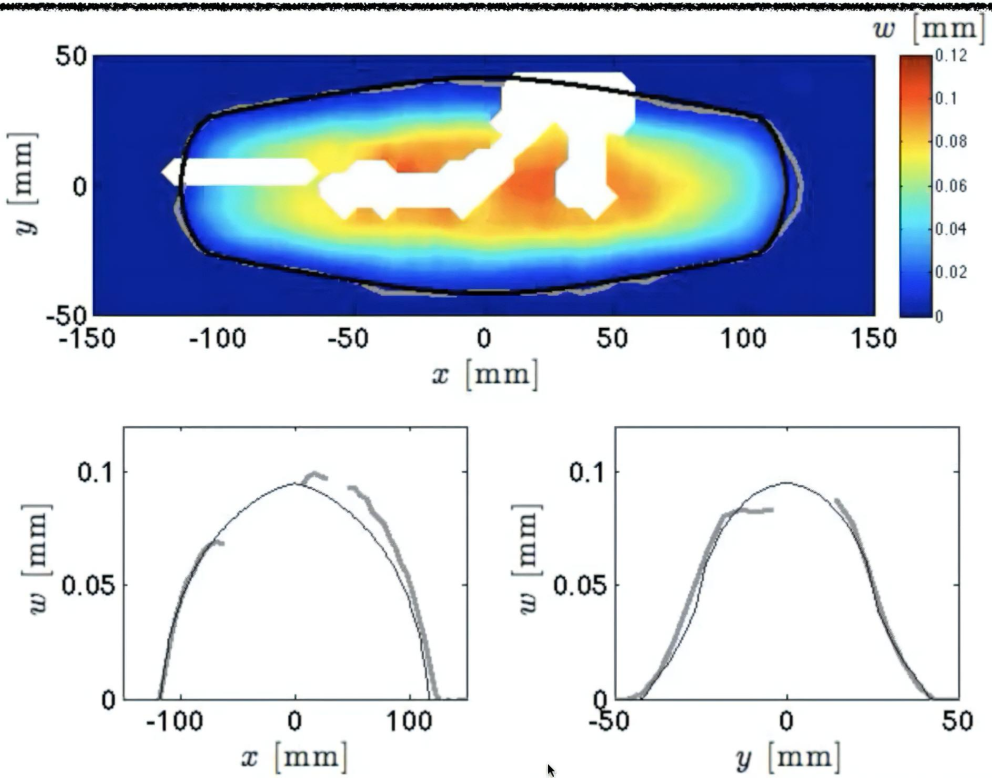
Enhanced P3D (EP3D) vs. ILSA



Comparison with experiment

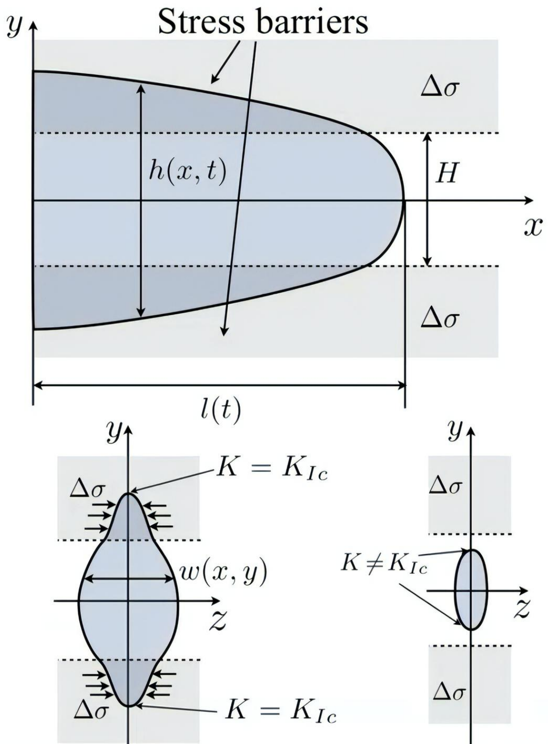


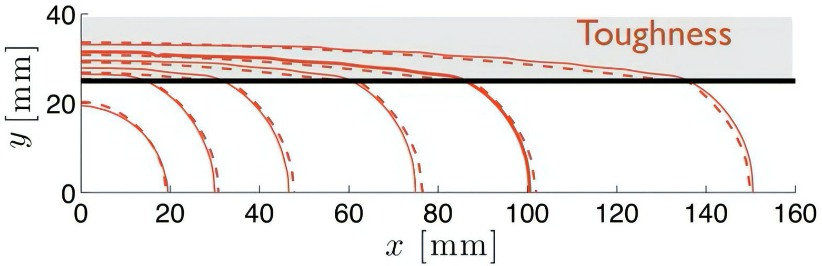
Comparison with experiment



Things to remember

- Assumptions of P3D model, including equilibrium height growth model, length >> height, horizontal flux, constant pressure in each vertical cross-section, plane strain solution for width in each vertical cross-section
- Non-local elasticity and other corrections allow to significantly improve accuracy of the model
- There are extensions for asymmetric stress layers as well as multiple layers
- This model is suited more for numerical calculations, rather than analysis. However, analysis of the classical P3D model can be found in Adachi et. al, 2010.





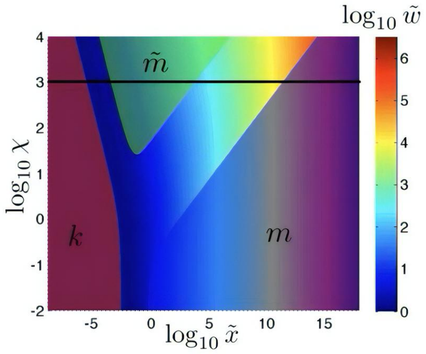
Recall main learnings: fundamentals and semi-infinite fracture

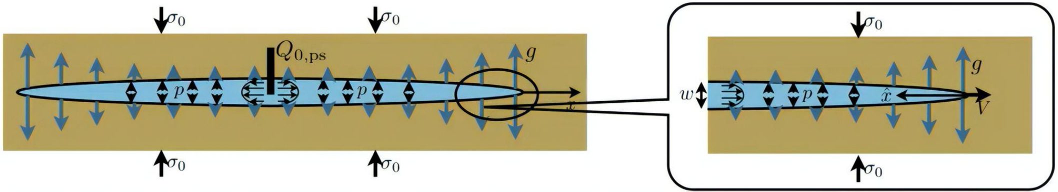
- Work towards a balanced life and invest in your knowledge!

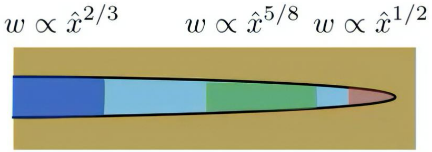
  An investment in knowledge pays the best interest.
  -Benjamin Franklin

- Essential pieces of a hydraulic fracture model: volume balance, fluid flow, elasticity, propagation, proppant transport
- Derivation of the governing equations for planar and plane strain fracture geometries

- Semi-infinite hydraulic fracture as a model for the tip region
  - There are three limiting analytic solutions: toughness, viscosity, and leak-off
  - The global solution gradually transitions from one limiting case to another
  - There is computationally efficient approximate solution for the problem that can be used as a propagation condition for finite fractures

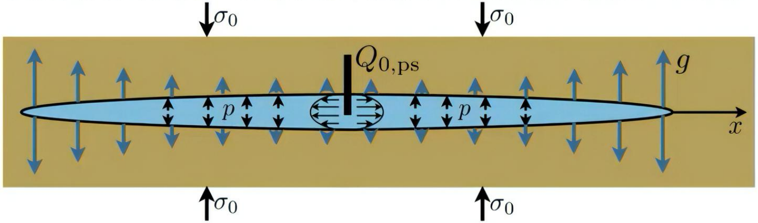






Recall main learnings: plane strain fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote
- The existence of approximate solution constructed using global volume balance and tip asymptote
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless toughness and dimensionless time



M - storage-viscosity

No toughness, no leak-off

K - storage-toughness

No viscosity, no leak-off

$\tilde{M}$ - leak-off-viscosity

No toughness, large leak-off

$\tilde{K}$ - leak-off-toughness

No viscosity, large leak-off

Viscosity

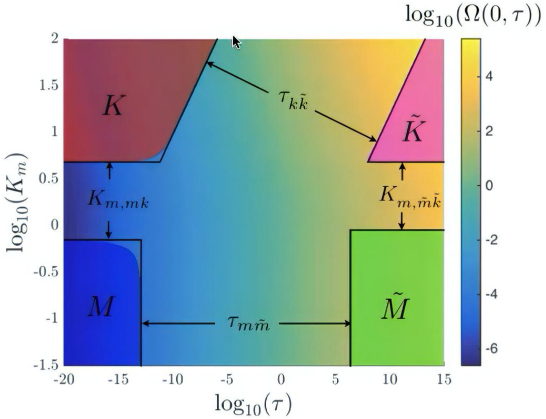
$w_m = \beta_m \left(\frac{\mu' V}{E'}\right)^{1/3} \hat{x}^{2/3}$

Toughness

$w_k = \frac{K'}{E'} \hat{x}^{1/2}$

Leak-off

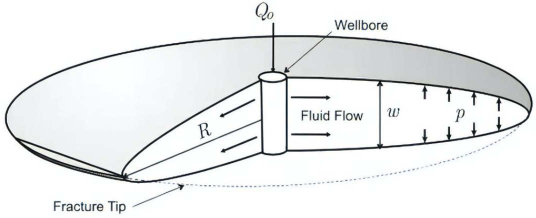
$w_{\tilde{m}} = \beta_{\tilde{m}} \left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8} \hat{x}^{5/8}$

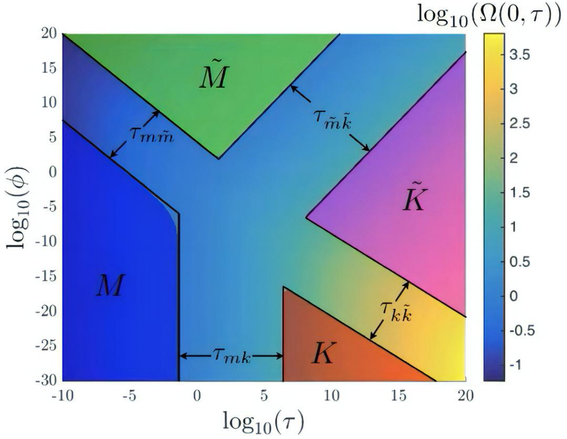


E.V. Dontsov. An approximate solution for a plane strain hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *Int. J. Fract.*, 205:221–237, 2017.

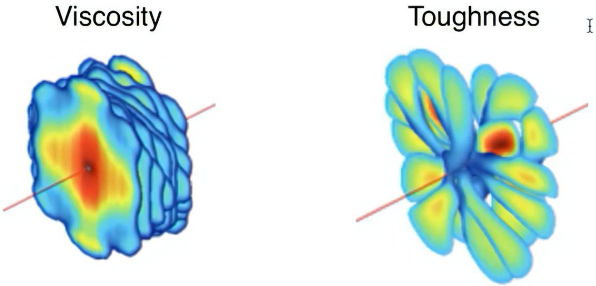
Recall main learnings: radial fracture

- Estimation of the solution based on scaling
- Definition of fracture regimes
- The relationship between the regimes for a finite fracture and tip asymptote (the same as for plane strain)
- The existence of approximate solution constructed using global volume balance and tip asymptote (similar to plane strain)
- The existence of explicit expressions for limiting or vertex cases
- Parametric space for the problem, two dimensionless parameters, dimensionless leak-off and dimensionless time
- Fracture regimes affect morphology of multiple hydraulic fractures



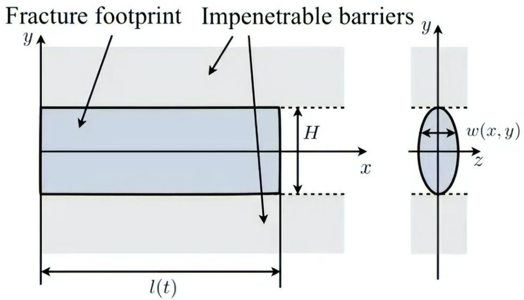


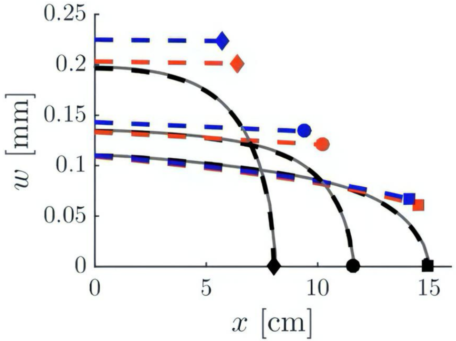
E.V. Dontsov. An approximate solution for a penny-shaped hydraulic fracture that accounts for fracture toughness, fluid viscosity, and leak-off. *R. Soc. open sci.*, 3:160737, 2016.

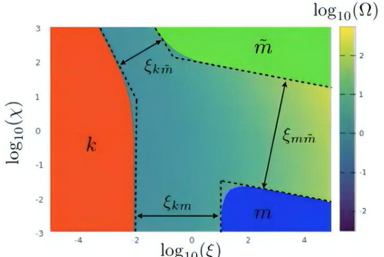


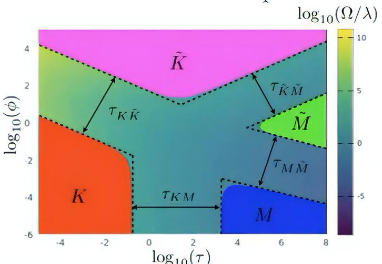
Recall main learnings: PKN fracture

- Assumptions of PKN model, including constant height, length >> height, horizontal flux, constant pressure in each vertical cross-section, elliptical width in each vertical cross-section
- There are two approaches to solve the problem:
  - Use non-local elasticity, which is good for numerical scheme and leads to superior accuracy
  - Use local elasticity with a specific boundary condition at the tip, which is less accurate, but easier for analysis
- PKN fracture has its own formulation for the tip region, which is different from that for a semi-infinite plane strain fracture, approximate solution exists
- Parametric space for the finite fracture is evaluated
- Limiting vertex solutions exist, as well as the global approximate solution









Recall main learnings: pseudo-3D fracture

- Assumptions of P3D model, including equilibrium height growth model, length >> height, horizontal flux, constant pressure in each vertical cross-section, plane strain solution for width in each vertical cross-section
- Non-local elasticity and other corrections allow to significantly improve accuracy of the model
- There are extensions for asymmetric stress layers as well as multiple layers
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