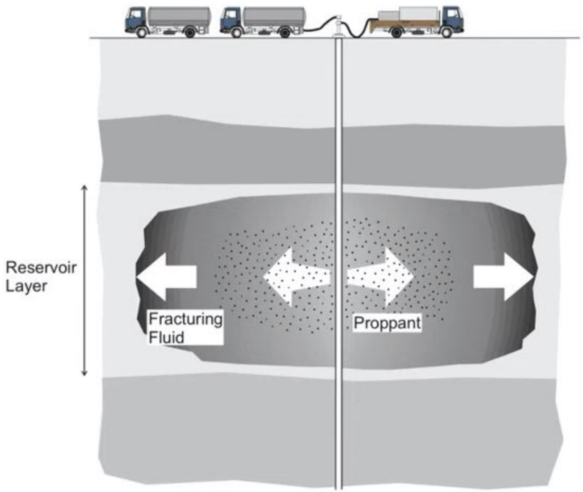
Hydraulic fracturing (HF)

- **Essential components of HF modeling**
- **Governing equations**
- **Basic HF model geometries**



Essential pieces of a hydraulic fracture model

1. Volume balance of the injected fluid (incompressible):

Volume injected = Fracture volume + leak-off

2. Fluid flow equations:

Viscous pressure drop inside the fracture

3. Rock equilibrium (elasticity):

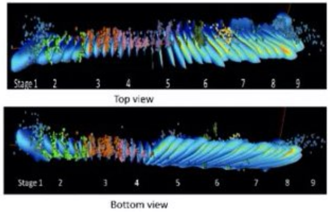
Fluid pressure = Stress + Stiffness*FracWidth

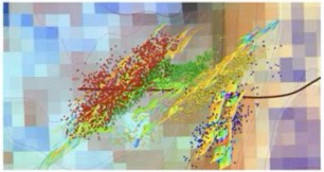
4. Propagation condition:

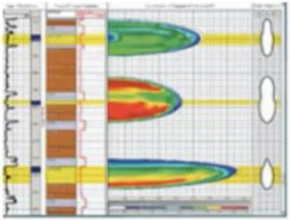
Some parameter reaches a critica value near the front

5. Proppant transport (not covered):

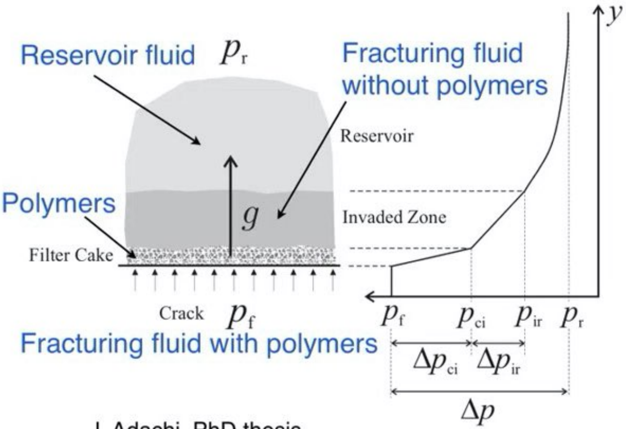
Particles flow with fluid + gravitational settling



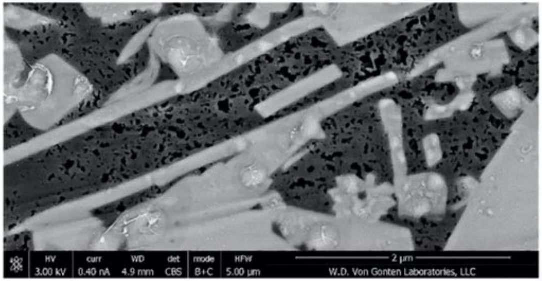




Carter's leak-off model



J. Adachi, PhD thesis



Flow through filter cake:

$$g_c = \alpha \frac{\mathrm{d}h_c}{\mathrm{d}t}$$

this equation says that the growth rate of the filter cake is linearly proportional to the amount of fluid leaked from the fracture (the constant of proporitonality is measured experimentally)

$$g_c = \frac{\kappa_c}{\mu}\frac{\Delta p_{ci}}{h_c}$$

this is Darcy's law (quasi-static flow)

Solution: $$g_c = \frac{C_c}{\sqrt{t}} \qquad C_c = \sqrt{\alpha \frac{\kappa_c}{\mu}\frac{\Delta p_{ci}}{h_c}}$$

Flow through invaded zone:

$$g_i = \frac{\kappa}{\mu_{filt}}\frac{\Delta p_{ir}}{h_i}$$

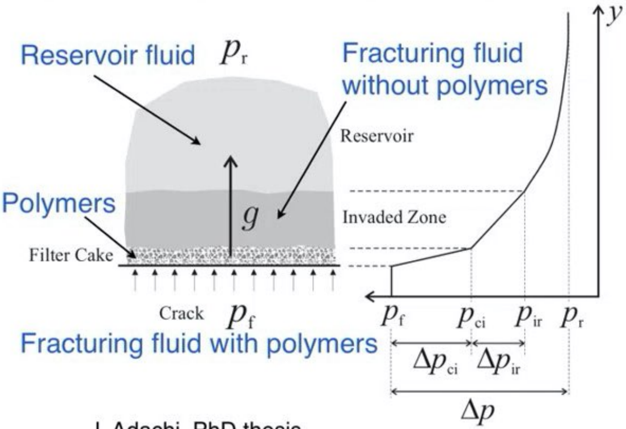
this is Darcy's law (quasi-static flow)

$$g_i = \phi \frac{\mathrm{d}h_i}{\mathrm{d}t}$$

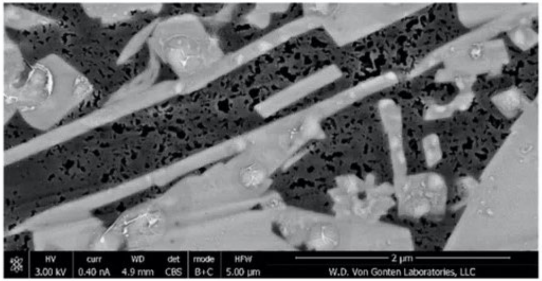
this is volume balance that states that the volume of fluid leaked into the formation determines the size of the invasion zone

Solution: $$g_i = \frac{C_i}{\sqrt{t}} \qquad C_i = \sqrt{\phi \frac{\kappa}{\mu_{filt}}\frac{\Delta p_{ir}}{h_i}}$$

Carter's leak-off model



J. Adachi, PhD thesis



Flow in reservoir:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial y^2}$$ 1D diffusion equation (volume balance + Darcy)

$$p|_{t=0} = p_r$$ initial condition

$$p|_{y=0} = p_{ir}$$ boundary condition

To solve this equation, introduce new variable:

$$\xi = \frac{y}{\sqrt{4Dt}} \Longrightarrow -\frac{y}{2t\sqrt{4Dt}}p' = \frac{D}{4Dt}p'' \Longrightarrow -2\xi p' = p''$$

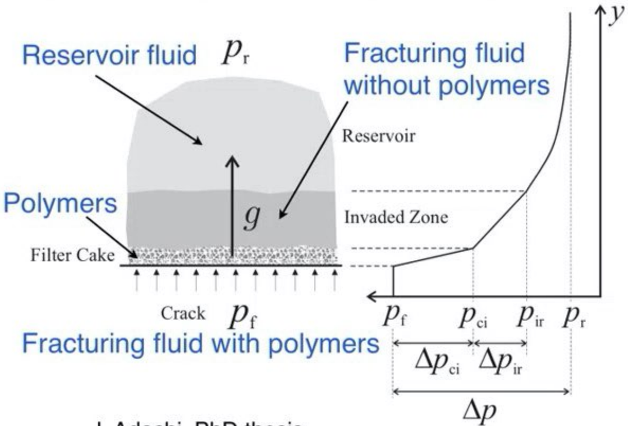
Solution of the above differential equation is:

$$p = p_r + (p_{ir} - p_r)\mathrm{erfc}(\xi^2) \qquad \mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

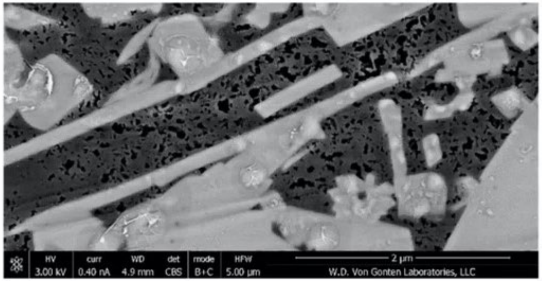
The leak-off flux is then (from Darcy):

$$g_r = -\frac{k_r}{\mu_r}\frac{\partial p}{\partial y}\bigg|_{y=0} = \frac{C_r}{\sqrt{t}} \qquad C_r = \sqrt{\frac{k_r c_t \phi}{\pi \mu_r}}(p_{ir} - p_r)$$

Carter's leak-off model



J. Adachi, PhD thesis



Combined result if all the mechanisms are present:

$$\Delta p = \Delta p_{ci} + \Delta p_{ir} + (p_{ir} - p_r)$$

$$C_l = \frac{2C_c C_i C_r}{C_c C_i + \sqrt{C_c^2 C_i^2 + 4C_r^2(C_c^2 + C_i^2)}}$$

In the above result, the individual leak-off coefficients are computed by using the total pressure drop, i.e. the reservoir part id given by

$$C_r = \sqrt{\frac{k_r c_t \phi}{\pi \mu_r}} \Delta p$$

Recall the main assumptions of the model:

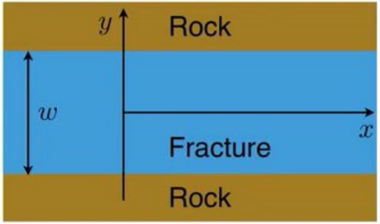
1D diffusion, i.e. the diffusion length scale should be much smaller than the fracture size

The net pressure is often neglected, whereby $\Delta p = \sigma_0 - p_r$

It is implicitly assumed that one type of fracturing fluid is used

More reading: Economides&Nolte 2000, section 6-4.

Fluid flow



$$v = v_x(y)$$ given the geometry, we have only one component of the velocity vector that varies only across the channel

$$\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y}$$ this comes from Navier-Stokes equations or equilibrium equations

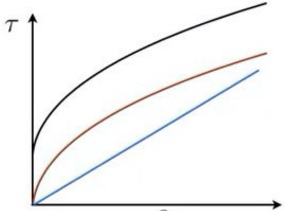
$$\tau = \mu \frac{\partial v}{\partial y}$$ this states that the rheology is Newtonian

$$v|_{y=\pm w/2} = 0$$ this is no-slip boundary condition at the fracture walls

Herschel-Bulkley $\tau = \tau_0 + k\dot{\gamma}^n$

Power-law $\tau = k\dot{\gamma}^n$

Newtonian $\tau = \mu\dot{\gamma}$



$$\dot{\gamma} = \frac{\partial v}{\partial y}$$

General solution:

$$v = \frac{\partial p}{\partial x}\frac{y^2}{2} + Ay + B$$

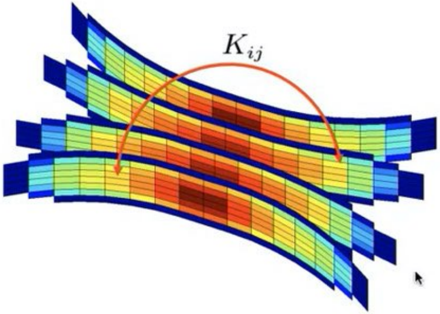
Actual solution:

$$v = -\frac{\partial p}{\partial x}\frac{w^2 - 4y^2}{8\mu}$$

Total flux is:

$$q = \int_{-w/2}^{w/2} v(y)\, dy = -\frac{w^3}{12\mu}\frac{\partial p}{\partial x}$$

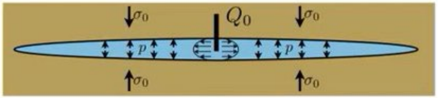
Elasticity



Elasticity equation ensures that rock surrounding open fracture(s) is in equilibrium

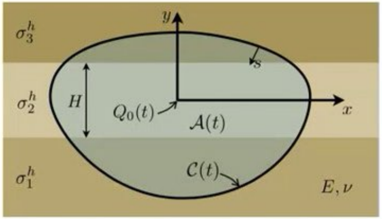
Every open element induces a stress change (all components) in the whole space

The interaction coefficient (induced stress divided by aperture) depends on the elastic properties and the distance from the element and generally decays quickly ~1/r^3 for 3D geometry



For a plane strain fracture, the elasticity equation reads:

$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^{l} \frac{w(s)\,ds}{(x-s)^2} \qquad E' = \frac{E}{1-\nu^2}$$



For a planar fracture, the elasticity equation reads:

$$p(x,y,t) = \sigma^h(y) - \frac{E'}{8\pi} \int_{\mathcal{A}(t)} \frac{w(x',y',t)\mathrm{d}x'\mathrm{d}y'}{\left[(x'-x)^2+(y'-y)^2\right]^{3/2}},$$

For general expressions in 3D, see Crouch and Starfield, 1983

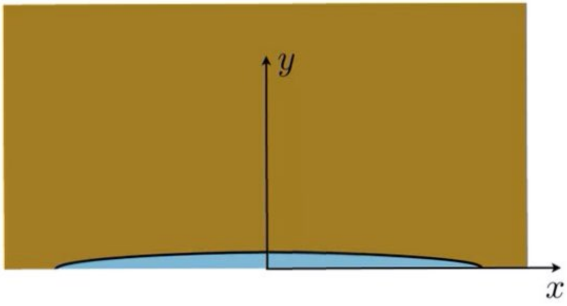
For expressions in layered materials, see Peirce and Siebrits, 2000

Lecture 2: Essential pieces of hydraulic fracturing: part 2

Egor Dontsov

Novosibirsk State University, March 2, 2021

Derivation of elasticity equation (plane strain)



Hooke's law

$$\sigma_{xx} = 2\mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy}),$$
$$\sigma_{yy} = 2\mu\epsilon_{yy} + \lambda(\epsilon_{xx} + \epsilon_{yy}),$$
$$\sigma_{xy} = 2\mu\epsilon_{xy}$$

Equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

Boundary conditions

$$u_y|_{y=0} = \frac{w}{2} \qquad \sigma_{xy}|_{y=0} = 0$$

Need to solve for

$$\sigma_{yy}|_{y=0} - ?$$

Governing equations in terms of displacements

$$(2\mu+\lambda)\frac{\partial^2 u_x}{\partial x^2} + (\lambda+\mu)\frac{\partial^2 u_y}{\partial x \partial y} + \mu\frac{\partial^2 u_x}{\partial y^2} = 0,$$
$$(2\mu+\lambda)\frac{\partial^2 u_y}{\partial y^2} + (\lambda+\mu)\frac{\partial^2 u_x}{\partial x \partial y} + \mu\frac{\partial^2 u_y}{\partial x^2} = 0$$

Apply Fourier transform

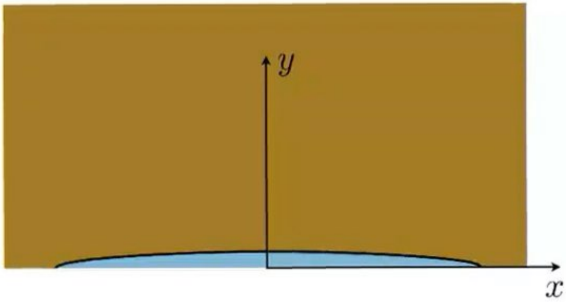
$$\hat{u}_x(k) = \int_{\infty}^{\infty} u_x(x) e^{ikx}\mathrm{d}x,$$
$$u_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}_x(k) e^{-ikx}\mathrm{d}k,$$

System of ODEs

$$\frac{\partial \hat{u}_x}{\partial y} = \hat{d}_x,$$
$$\frac{\partial \hat{u}_y}{\partial y} = \hat{d}_y,$$
$$\frac{\partial \hat{d}_x}{\partial y} = \frac{2\mu+\lambda}{\mu}k^2\hat{u}_x + \frac{\lambda+\mu}{\mu}ik\hat{d}_x + \frac{\mu}{2\mu+\lambda}k^2\hat{u}_y,$$
$$\frac{\partial \hat{d}_y}{\partial y} = \frac{\mu+\lambda}{2\mu+\lambda}ik\hat{d}_x + \frac{\mu}{2\mu+\lambda}k^2\hat{d}_x + \frac{\mu}{2\mu+\lambda}k^2\hat{u}_y,$$

$$\boxed{\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}}$$

Derivation of elasticity equation (plane strain)



Boundary conditions

$$\hat{\sigma}_{xy}|_{y=0} = 0,$$
$$\hat{u}_y|_{y=0} = \hat{w}(k)/2,$$

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$$

Eigenvalues of A: $k, k, -k, -k$.

Solution (resonance)

$$\boldsymbol{Y} = c_1 \boldsymbol{v_1} e^{-|k|y} + c_2(\boldsymbol{v}_1 y + \boldsymbol{v_2})e^{-|k|y}$$

Solution in frequency domain

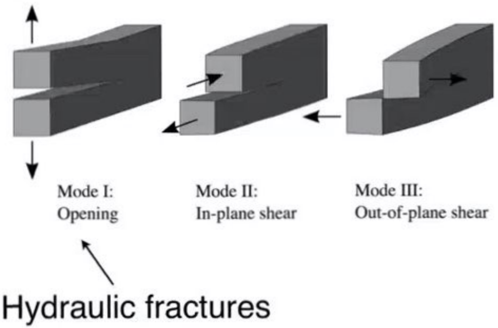
$$\hat{p} = \frac{\hat{w}|k|}{4}E' = -\frac{ik\hat{w}E'}{4}\frac{|k|}{(-ik)} = -\frac{1}{i}\mathrm{sgn}k\frac{\mathrm{d}\hat{w}}{\mathrm{d}x}.$$

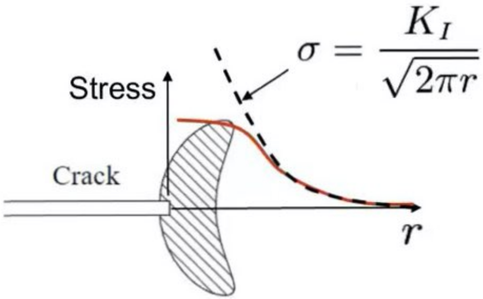
$$\Downarrow$$

https://en.wikipedia.org/wiki/Fourier_transform

$$p = \sigma_0 - \frac{E'}{4\pi}\int_{-l}^{l}\frac{w(s)\,ds}{(x-s)^2}$$

Propagation condition





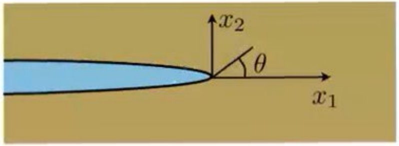
K_I - Stress Intensity Factor (SIF)

Propagation condition: $K_I = K_{Ic}$

K_{Ic} - fracture toughness

https://en.wikipedia.org/wiki/Fracture_mechanics

Mode I solution near the tip



$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1 - \sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1 + \sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\sin(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right]$$

$$u_1 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\cos(\tfrac{1}{2}\theta)\left\{\kappa - 1 + 2\sin^2(\tfrac{1}{2}\theta)\right\}\right]$$

$$u_2 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\sin(\tfrac{1}{2}\theta)\left\{\kappa + 1 - 2\cos^2(\tfrac{1}{2}\theta)\right\}\right]$$

$$\kappa = 3 - 4\nu \qquad \mu = \frac{E}{2(1+\nu)}$$

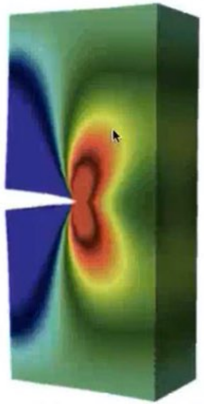
Solution methodology:

- Write elasticity equations via Airy stress function
- Solve the equations assuming stress-free crack and finite displacement at the tip
- See lecture notes on fracture mechanics for more info: http://www.mate.tue.nl/~piet/edu/frm/pdf/frmsyl1213.pdf

Fracture width around the crack tip:

$$w = \sqrt{\frac{32}{\pi}}\frac{K_I(1-\nu^2)}{E}\sqrt{r}$$

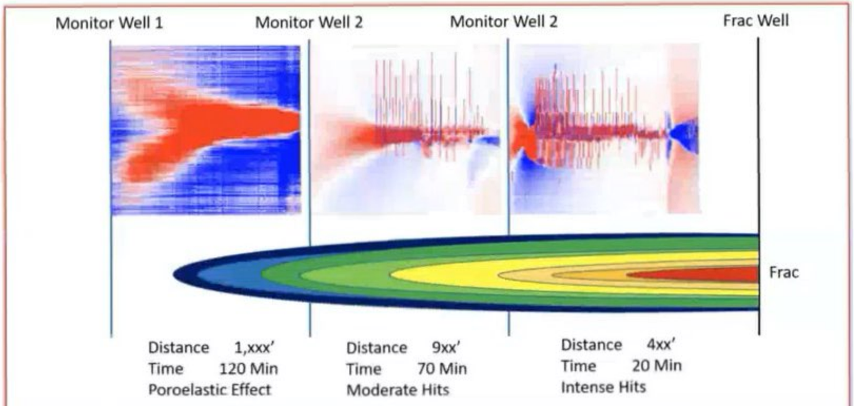
Stress field around the crack tip:

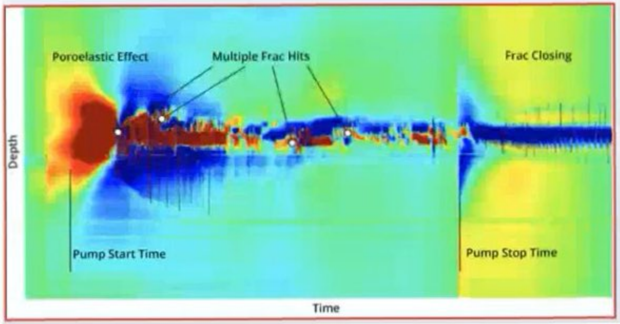


http://umich.edu/~compphys/gradientElasticity.html

Observation of crack tip stress in the field

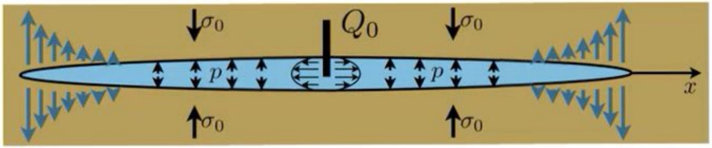
- Fiber optics cables are used to measure stretch versus time along the cable length
- A cable is often placed in the neighboring horizontal well, while the primary well is being fractured
- The characteristic "ears" of the approaching crack are clearly visible

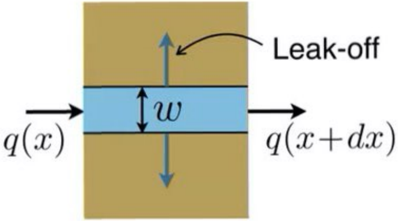




Silixa.com Richter et al. 2020

Volume balance for a plane strain HF





New volume Flux in Leak-off

$$w(t+dt)dx = w(t)dx + q(x) - q(x+dx) - g_l dt dx + Q_0 dt \int \delta(x)dx$$

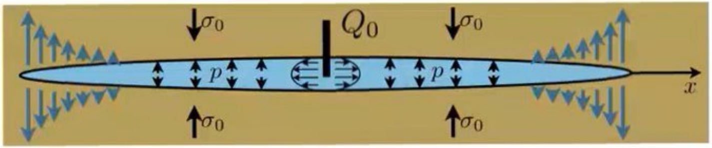
Previous volume Flux out Source

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Leak-off Source

Mathematical model for a plane strain HF



Scaled quantities

$$C' = 2C_L \qquad \mu' = 12\mu \qquad E' = \frac{E}{1-\nu^2} \qquad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

Volume balance of fluid

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Leak-off

Source

Elasticity

Fracture length

$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^{l} \frac{w(s)\, ds}{(x-s)^2}$$

Fluid pressure

Laminar fluid flow flux

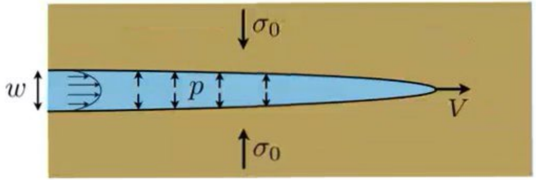
$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$

Propagation condition (LEFM)

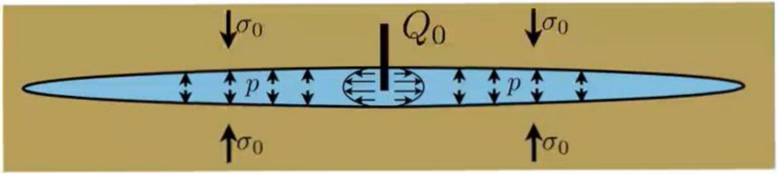
$$w \to \frac{K'}{E'} \sqrt{l - x} \qquad (K_I = K_{Ic})$$

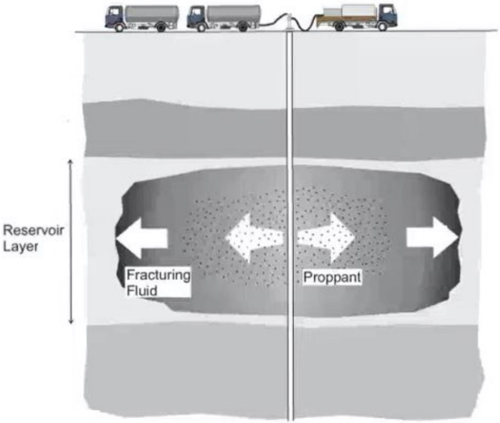
HF geometries - the simplest

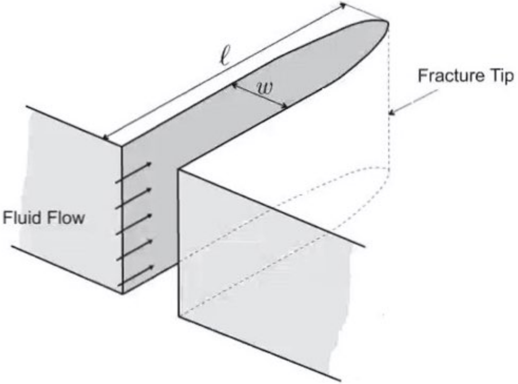
Semi-infinite (tip region)



Khristianovich–Zheltov–Geertsma–De Klerk (KGD)





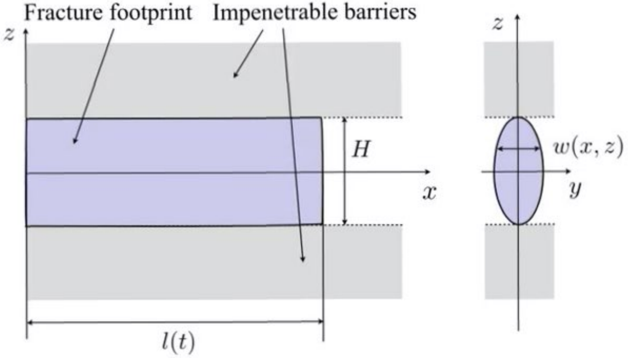


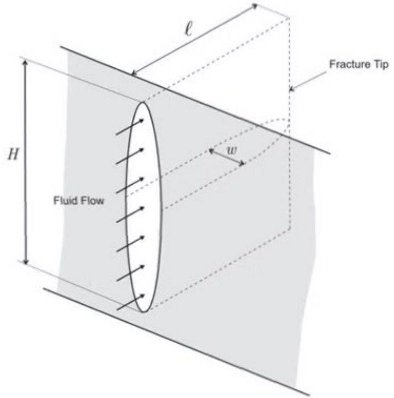
Khristianovic SA, Zheltov YP. 1955 Formation of vertical fractures by means of highly viscous fluids. In Proc. 4th World Petroleum Congress, Rome, Italy, 6–16 June, vol. 2, pp. 579–586.

HF geometries

Perkins–Kern–Nordgren (PKN)

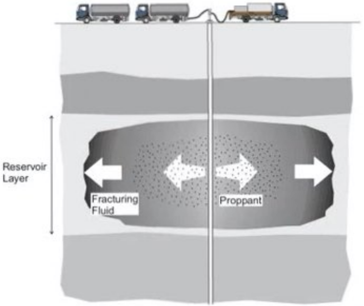
T.K. Perkins, L.R. Kern, Widths of hydraulic fractures, J. Pet. Tech. Trans. AIME (1961) 937–949.
R.P. Nordgren, Propagation of vertical hydraulic fractures, Soc. Petrol. Eng. J. (1972) 306–314.





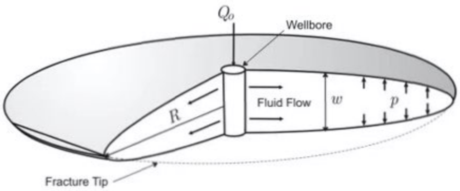
In conventional field applications, solution evolves from KGD geometry at early times to PKN geometry for late times

KGD (early time) -> PKN (developed fracture)

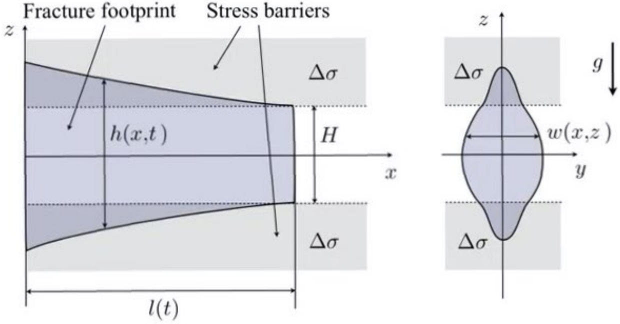


HF geometries

Radial

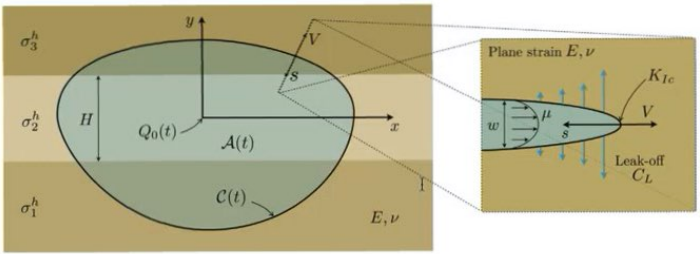


Pseudo-3D



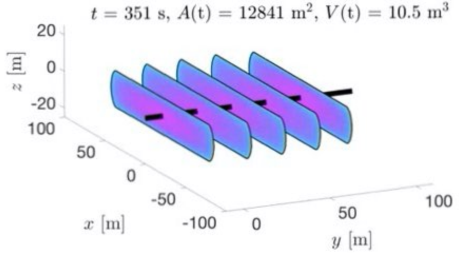
Settari A, Cleary M. Development and testing of a pseudo-three-dimensional model of hydraulic fracture geometry (P3DH). SPE 10505; 1982. p. 185–214.

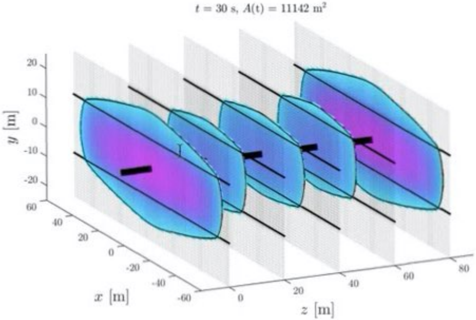
Planar-3D



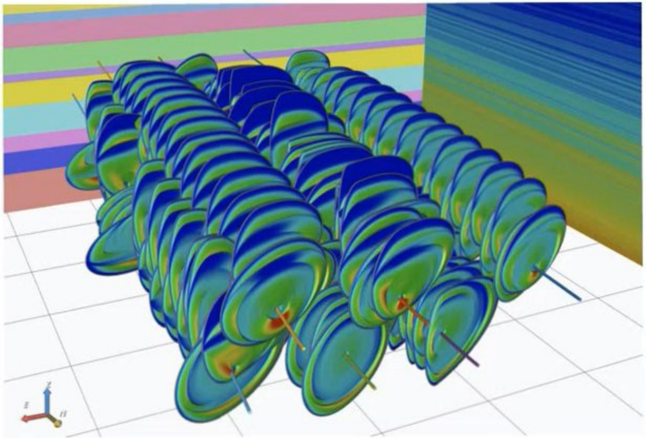
HF geometries

Multi-fracture



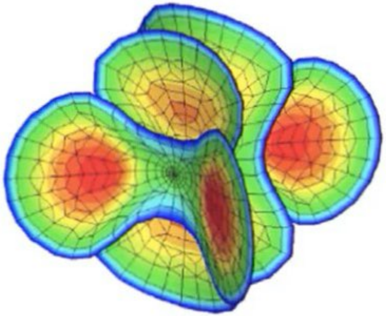


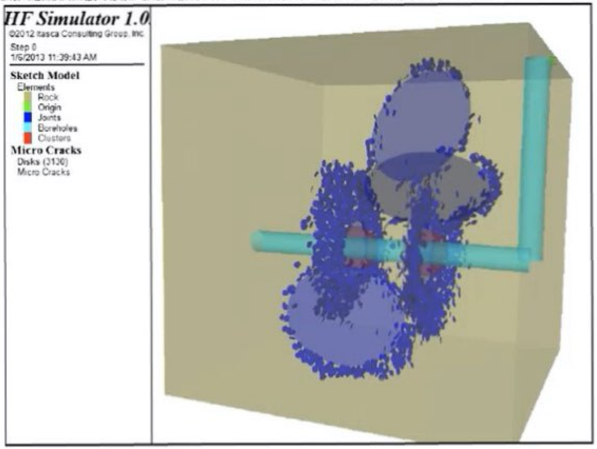
Multi-well

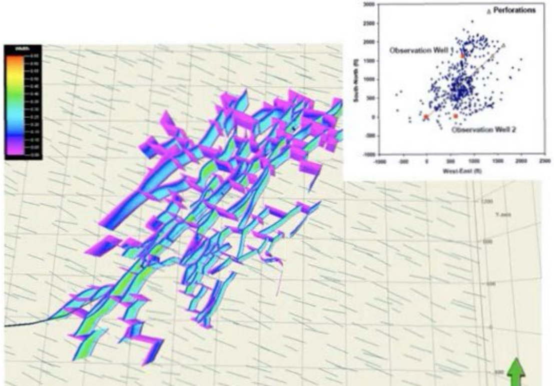


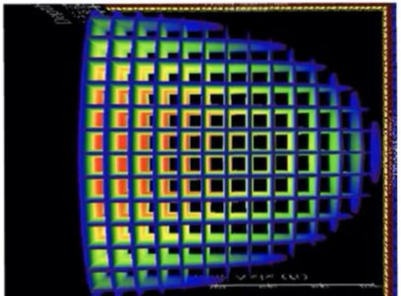
Natural fractures, curved fractures, etc.

HF geometries - other complex









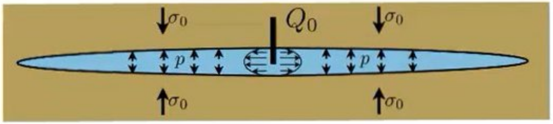
Lecture 3: Semi-infinite hydraulic fracture

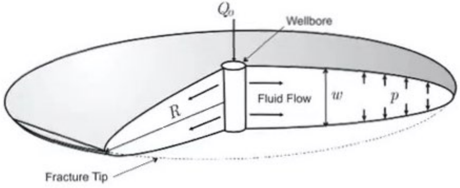
Egor Dontsov

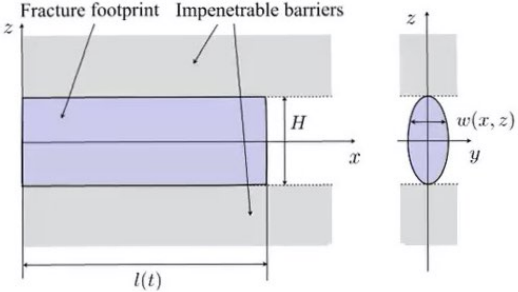
Novosibirsk State University, March 2, 2021

Recall from lecture 1

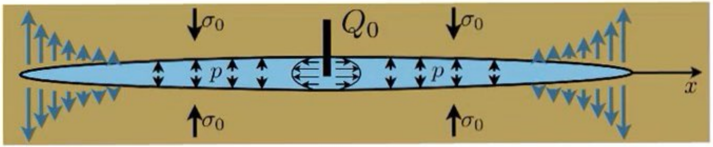
- **Essential pieces of HF model**
  - Volume balance and leak-off
  - Fluid flow
  - Elasticity
  - Propagation condition
  - Proppant transport
- **Various fracture geometries**
  - Semi-infinite
  - KGD (plane strain)
  - PKN
  - Radial
  - Pseudo-3D
  - Planar 3D
  - Complex
- **Governing equations**
  - KGD (plane strain)
- **Derivation of elasticity equation**







Mathematical model for plane strain HF



Scaled quantities

$$C' = 2C_L \qquad \mu' = 12\mu$$

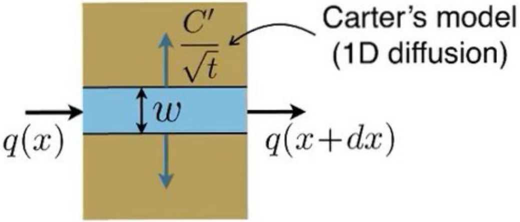
Volume balance of fluid

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

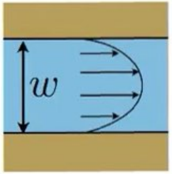
Leak-off

Source



Laminar fluid flow flux

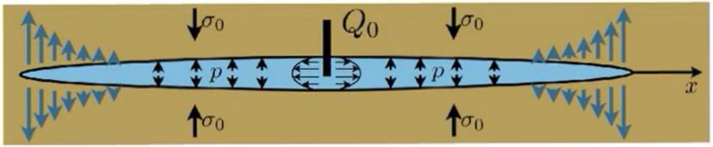
$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$



$$v = -\left(\tfrac{1}{4}w^2 - y^2\right) \frac{1}{2\mu} \frac{\partial p}{\partial x}$$

$$q = \int_{-w/2}^{w/2} v \, dy = -\frac{w^3}{12\mu} \frac{\partial p}{\partial x}$$

Mathematical model for plane strain HF



Scaled quantities

$$C' = 2C_L \qquad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2}$$

Volume balance of fluid

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Leak-off

Source

Elasticity

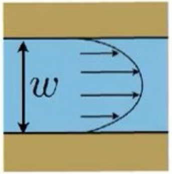
Fracture length

$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^{l} \frac{w(s)\, ds}{(x-s)^2}$$

Fluid pressure

Laminar fluid flow flux

$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$



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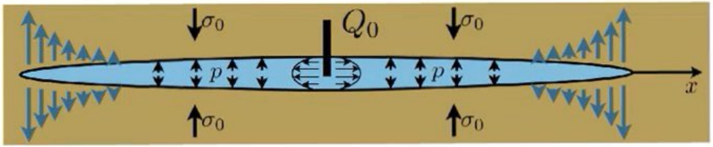
Mathematical model for plane strain HF



Scaled quantities

$$C' = 2C_L \qquad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2} \quad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

Volume balance of fluid

Fracture width

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = Q_0 \delta(x)$$

Leak-off

Source

Elasticity

Fracture length

$$p = \sigma_0 - \frac{E'}{4\pi} \int_{-l}^{l} \frac{w(s)\, ds}{(x-s)^2}$$

Fluid pressure

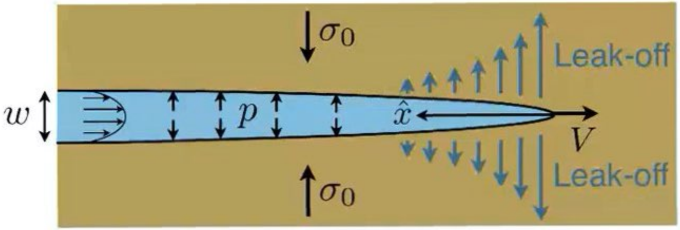
Laminar fluid flow flux

$$q = -\frac{w^3}{\mu'} \frac{\partial p}{\partial x}$$

Propagation condition (LEFM)

$$w \to \frac{K'}{E'} \sqrt{l - x} \qquad (K_I = K_{Ic})$$

Tip asymptotics: semi-infinite hydraulic fracture



Material parameters

$$C' = 2C_L \qquad \mu' = 12\mu$$

$$E' = \frac{E}{1-\nu^2} \qquad K' = \frac{8K_{Ic}}{\sqrt{2\pi}}$$

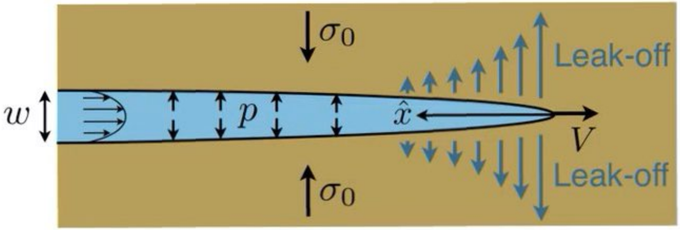
Fluid volume balance

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = 0 \quad \xrightarrow[\;]{\text{Traveling wave} \; w(Vt-x)} \quad \frac{q}{w} = V + 2C'V^{1/2}\frac{\hat{x}^{1/2}}{w}$$

Elasticity

$$w = \frac{K'}{E'}\hat{x}^{1/2} + \frac{4}{\pi E'}\int_0^\infty K(\hat{x},\hat{s})(p(\hat{s})-\sigma(\hat{s}))\, d\hat{s} \qquad K(\hat{x},\hat{s}) = \ln\left|\frac{\hat{x}^{1/2}+\hat{s}^{1/2}}{\hat{x}^{1/2}-\hat{s}^{1/2}}\right| - 2\frac{\hat{x}^{1/2}}{\hat{s}^{1/2}}$$

Tip asymptotics: semi-infinite hydraulic fracture



Material parameters

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Fluid volume balance

$$\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} + \frac{C'}{\sqrt{t - t_0(x)}} = 0 \xrightarrow[]{\substack{\text{Traveling wave} \\ w(Vt-x)}} \frac{q}{w} = V + 2C'V^{1/2}\frac{\hat{x}^{1/2}}{w}$$

Elasticity

$$w = \frac{K'}{E'}\hat{x}^{1/2} + \frac{4}{\pi E'}\int_0^\infty K(\hat{x},\hat{s})(p(\hat{s})-\sigma(\hat{s}))\,d\hat{s} \qquad K(\hat{x},\hat{s}) = \ln\left|\frac{\hat{x}^{1/2}+\hat{s}^{1/2}}{\hat{x}^{1/2}-\hat{s}^{1/2}}\right| - 2\frac{\hat{x}^{1/2}}{\hat{s}^{1/2}}$$

Fluid flow

$$q = -\frac{w^3}{\mu'}\frac{\partial p}{\partial x} \longrightarrow q = \frac{w^3}{\mu'}\frac{dp}{d\hat{x}}$$

LEFM propagation condition

$$w = \frac{K'}{E'}\hat{x}^{1/2}, \qquad \hat{x} \to 0$$

Non-singular formulation

1. Integrate elasticity equation by parts

$$w = \frac{K'}{E'} x^{1/2} - \frac{4}{\pi E'} \int_0^\infty F(x,s) \frac{dp}{ds}\, ds \qquad F(x,s) = (s-x)\ln\left|\frac{x^{1/2}+s^{1/2}}{x^{1/2}-s^{1/2}}\right| - 2x^{1/2}s^{1/2}$$

2. Substitute pressure gradient into the result

$$w(x) = \frac{K'}{E'} x^{1/2} - \frac{4}{\pi E'} \int_0^\infty F(x,s) \frac{\mu'}{w(s)^2}\left[V + 2C'V^{1/2}\frac{s^{1/2}}{w(s)}\right] ds$$

3. Apply scaling

$$\tilde{w} = \frac{E'w}{K'x^{1/2}}, \qquad \chi = \frac{2C'E'}{V^{1/2}K'}, \qquad \tilde{x} = (x/l)^{1/2}, \qquad \tilde{s} = (s/l)^{1/2}, \qquad l = \left(\frac{K'^3}{\mu' E'^2 V}\right)^2$$

4. Final result

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right] d\tilde{s} \qquad G(t) = \frac{1-t^2}{t}\ln\left|\frac{1+t}{1-t}\right| + 2$$

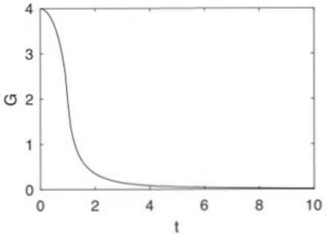
Non-singular

w = “toughness” + “viscosity” + “leak-off”

Limiting vertex solutions

$$\tilde{w}(\tilde{x}) = 1 + \frac{8}{\pi}\int_0^\infty G(\tilde{s}/\tilde{x})\left[\frac{1}{\tilde{w}(\tilde{s})^2} + \frac{\chi}{\tilde{w}(\tilde{s})^3}\right] d\tilde{s}$$

w = “toughness” + “viscosity” + “leak-off”



Toughness dominates

$$\tilde{w}_k = 1, \quad \Longrightarrow \quad w_k = \frac{K'}{E'} x^{1/2}$$

Viscosity dominates

$$\tilde{w}(\tilde{x}) = \frac{8}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^2} d\tilde{s} \quad \Longrightarrow \quad \begin{aligned} \tilde{w}_m &= \beta_m \tilde{x}^{1/3} \\ \beta_m &= 2^{1/3} 3^{5/6} \end{aligned} \quad \Longrightarrow \quad w_m = \beta_m \left(\frac{\mu' V}{E'}\right)^{1/3} x^{2/3}$$

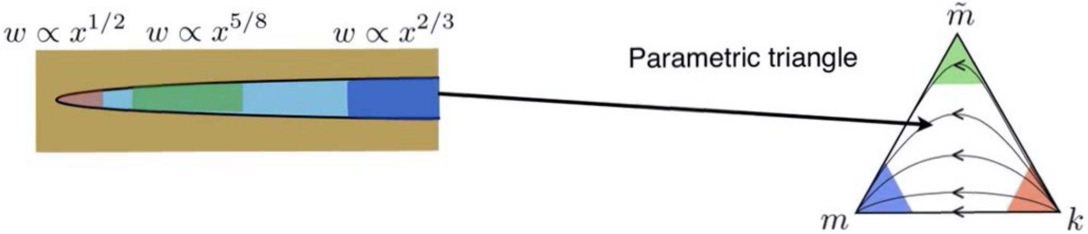
Desroches et al 1994

Leak-off dominates

$$\tilde{w}(\tilde{x}) = \frac{8\chi}{\pi}\int_0^\infty \frac{G(\tilde{s}/\tilde{x})}{\tilde{w}(\tilde{s})^3} d\tilde{s} \quad \Longrightarrow \quad \begin{aligned} \tilde{w}_{\tilde{m}} &= \beta_{\tilde{m}} \chi^{1/4} \tilde{x}^{1/4} \\ \beta_{\tilde{m}} &= \frac{4}{15^{1/4}(\sqrt{2}-1)^{1/4}} \end{aligned} \quad \Longrightarrow \quad w_{\tilde{m}} = \beta_{\tilde{m}} \left(\frac{4\mu'^2 V C'^2}{E'^2}\right)^{1/8} x^{5/8}$$

Lenoah 1995

Order of limiting solutions



Toughness

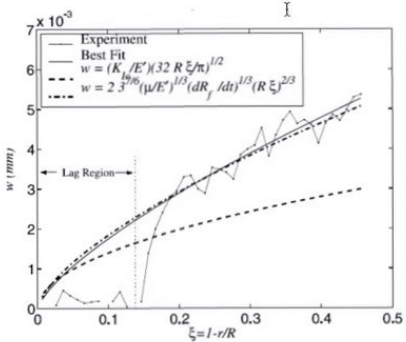
$$w_k = \frac{K'}{E'} x^{1/2},$$

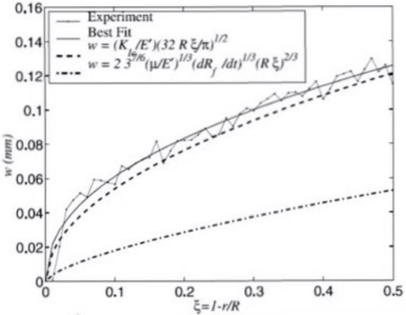
Leak-off

$$w_{\tilde{m}} = \beta_{\tilde{m}} \left(\frac{4\mu'^2 V C'^2}{E'^2} \right)^{1/8} x^{5/8},$$

Viscosity

$$w_m = \beta_m \left(\frac{\mu' V}{E'} \right)^{1/3} x^{2/3}$$





Bunger&Jeffrey