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Simplified Productivity Equations for Horizontal Wells Producing at Constant Rate and Constant Pressure

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Abstract

This paper presents a simple model to calculate the productivity of horizontal wells producing at constant flowing bottomhole pressure, p_{wf}, or constant rate from bounded reservoirs. We propose new, easy-to-use correlations to calculate the shape factor and partial penetration skin for both cases. The shape factor is calculated using an analytical solution of a fully penetrating well; and the partial penetration skin is calculated using a numerical solution. The correlations are developed using nonlinear regression (with R^2 of 99%) of more than 800 numerical simulation runs for different reservoir aspect ratios, well locations and well penetration ratios. The productivity of wells producing at constant p_{wf} can be quite different from that of wells producing at constant rate. The models proposed in this paper are only approximate. In many cases they proved to be more accurate than other methods currently available in the industry. However, errors could be generated when the proposed models are used outside the range of applicability of the correlations.

Introduction

Several analytical models have been proposed to calculate the productivity of horizontal wells. In 1988, Joshi[1] proposed a steady state model using potential-fluid-flow theory. This model is only useful for first approximations and comparisons with vertical well. However, it does not apply for wells producing at pseudo-steady state. In the same year, Mutalik *et al.*[2] studied the effect of drainage area shape on the productivity of horizontal wells. They gave tables of shape and skin factors for different drainage area shapes and well penetration ratios. In 1989, Babu and Odeh[3] developed a pseudo-steady state model using Green's functions. They reduced their original infinite series solution into equations for shape factor and partial penetration skin. Although their expression for the shape factor is quite simple, the expression for the partial penetration skin is complicated. In 1991, Thompson *et al.*[4] proposed an algorithm to compute the horizontal well pressure response in bounded reservoirs. Their algorithm switches between two infinite series solutions (with different rate of convergence) to improve the overall convergence rate. In 1994, Economides *et al.*[5] proposed a pseudo-steady state computer model using the continuous point source solution. Their model also accounts for well orientation.

All of the above models are either complicated to use or require computer programming. None of them provides for calculating the productivity of horizontal wells operating at constant p_{wf}. This study provides a simple model to calculate the productivity of horizontal wells producing at constant p_{wf} or constant rate.

Constant Rate Model

As shown in Fig. 1, we consider a homogeneous, anisotropic, rectangular reservoir with a horizontal well running parallel to one of the reservoir sides. Similar to the approach taken by Babu and Odeh[3], we use the well-known productivity equation of a vertical well to describe the productivity of a horizontal well producing at constant rate:

$$J_{CR} = \frac{k_{eq} b_{eq}}{141.2 B\mu \left(\frac{1}{2}\ln \frac{4A_{eq}}{\gamma r_{weq}^2} - \frac{1}{2}\ln C_A + s_P \right)} \quad \text{.....(1)}$$

C_A is Dietz's[6] shape factor, and s_P is partial penetration skin factor. Other symbols are defined at the end of the paper. Correlations for C_A and s_P are given in Table 1. Note that C_A is based on the area $(a_{eq}h_{eq})$, Fig.1.

The permeability anisotropy is accounted for in this equation by using coordinate transformation[7], which is described in Appendix A. Therefore, only transformed well and reservoir dimensions are used in Eq. (1) and Eq. (2).

Constant p_{wf} Model

The productivity equation for wells producing at constant p_{wf} is obtained by replacing C_A with C_{ACP} and s_P with s_{PCP}:

$$J_{CP} = \frac{k_{eq} b_{eq}}{141.2 B\mu \left(\frac{1}{2} \ln \frac{4A_{eq}}{\gamma r_{weq}^2} - \frac{1}{2} \ln C_{ACP} + s_{PCP} \right)} \quad (2)$$

Correlations for C_{ACP} and s_{PCP} are given in Table 1. Helmy and Wattenbarger[8] showed that shape factors, (and well productivity) differ for wells producing at constant p_{wf} versus constant rate. They indicated that the difference is larger as the area ($a_{eq}h_{eq}$) gets more elongated, which is the common case for horizontal wells.

Shape Factor Correlations

A horizontal well could be thought of as a vertical well rotated 90 degrees. In doing so, the drainage area of the vertical well becomes the side of the reservoir for a horizontal well (area $a_{eq}h_{eq}$). Analytical solution for a fully penetrating well is used to calculate the shape factors for different area ($a_{eq}h_{eq}$) and well locations. The solution is generated using the method of images.[6,8,9] The exact values of shape factors calculated from the analytical solution are correlated with the equations of Table 1. The equation for the constant rate shape factor, C_A, is identical to Babu and Odeh's[3], which was introduced in 1989. This equation is valid (with less than 1% error) when $a_{eq} \geq 0.75h_{eq}$ and $\min(x_{weq}, a_{eq}-x_{weq}) \geq 0.75h_{eq}$; a condition that is satisfied for all cases of interest. Table 2 shows some examples of the insignificant error in calculating the shape factor using this equation.

Constant p_{wf} shape factors, C_{ACP}, are different from constant rate shape factors[8]. The constant p_{wf} shape factors are correlated with a different equation as shown in Table 1. The error in calculating the shape factor using this equation is less than 4% when $a_{eq} \geq 2.0h_{eq}$. Again, Table 2 shows some examples of the insignificant error in calculating the constant p_{wf} shape factor using this equation.

Partial Penetration Skin Correlations

The partial penetration skin is a slightly more complicated issue than the shape factors. Analytical solutions for partially penetrating wells usually suffer from slow convergence. Considering the large number of cases needed to be studied to obtain a reasonable correlation for the partial penetration skin, analytical solutions were not an attractive approach. Therefore, we decided to use numerical simulation.[15]

About 400 simulation runs were made with different reservoir dimensions and well penetration ratios for each case: constant p_{wf} and constant rate. The following list gives the cases considered:

1) z_{weq}/h_{eq}: 0.1, 0.3 and 0.5,
2) x_{weq}/a_{eq}: 0.1, 0.3 and 0.5,
3) h_{eq}/a_{eq}: 0.5, 0.1 and 0.01,
4) L_{eq}/b_{eq}: 0.2, 0.47 and 0.73,
5) y_{weq}/b_{eq}: 0.1, 0.3 and 0.5,
6) a_{eq}/b_{eq}: 0.2, 1.0 and 5.0.

All possible combinations were simulated and the results were correlated with the equations shown in Table 1 (s_P and s_{PCP}). The above list represents the boundaries of the range of applicability for the proposed model of this study.

To show the adequacy of the correlation of the partial penetration skin, the dimensionless productivity index, Eqs. (A-5) and (A-8), calculated from the correlated values of skin is plotted against the dimensionless productivity index obtained from the numerical simulation, Figs. 2 and 3. The constant rate correlation gives a productivity index with an R^2 value of 0.9975. And the constant pressure correlation gives a productivity index with an R^2 value of 0.9977.

As we will show later in the examples, for most practical cases the correlation is fairly accurate and matches well with the analytical solutions. However, when the reservoir and well parameters lie on the boundaries of the range of applicability of the correlations (when well penetration is closed to 0.2, the ratio a_{eq}/b_{eq} is nearly 1.0 and the aspect ratio h_{eq}/a_{eq} is close to 0.5), the correlation is not as accurate. The maximum error (difference between analytical solution and correlation) we have seen in the cases studied is 20%.

Penetration Skin: Constant Rate vs. Constant p_{wf}

An interesting point that deserves a special attention here, is the dependence of the partial penetration skin factor on the inner boundary condition of the system, i.e. constant rate versus constant p_{wf}. The partial penetration skin is calculated by subtracting the analytical solution of a fully penetrating well from the analytical solution of a partially penetrating well. Fig. 4 shows the results of one of the cases we studied. The analytical solution used here is that developed by Gringarten and Ramey[10] for partially penetrating vertical wells in infinite reservoirs. We created boundaries using the method of images, and rotated the system 90 degrees to simulate a horizontal well in a bounded reservoir.

In Fig. 4, the skin factor is plotted versus the dimensionless time, t_D, for the constant rate case, and versus the dimensionless material balance time, N_{pD}/q_D, for the constant p_{wf} case (dimensionless variables are defined at the end of the paper). The constant p_{wf} skin factor during boundary dominated flow is lower than the constant rate skin during pseudo-steady state. This causes an increase in the productivity index of wells producing at constant p_{wf} (as compared to those producing at constant rate). However, this effect is opposite to the effect of the shape factors. In a previous paper[8], we showed that the effect of shape factors is to reduce the productivity index of wells producing at constant p_{wf}. As we will show later in the examples, for some cases the effect of the shape factors offsets the effect of the partial penetration skin, and the result is that we get the same productivity index for constant rate and constant p_{wf} cases.

Example Applications

We consider two examples here to show the application of the proposed models. In both examples, we calculate the productivity of a horizontal well as a function of well penetration and compare with an analytical solution, Babu and Odeh approximation and Joshi steady state solution. The analytical solution used here was developed by Economides, *et al.*[5,14] using integration of the continuous point source solution. The solution assumes a line source well with infinite wellbore conductivity.

Example 1. Reservoir and well parameters are listed in the insert of Fig. 5. It is clear from the plot, that the proposed correlation traces the analytical solution. Also, it is important to note that the constant p_{wf} productivity is lower than the constant rate productivity for all well penetrations. However this may not be true for other cases, as will be shown in Example 2. Also from Fig. 5, it is shown that Babu and Odeh approximation falls below the analytical solution. Although Joshi's solution is for steady state case, it is included here only for comparison purposes.

Example 2. Reservoir and well parameters are listed in the insert of Fig. 6. This example shows that the constant rate and constant p_{wf} productivity indexes could be very close. As indicated earlier in the paper, in some cases the effect of partial penetration skin offsets the effect of shape factor resulting in the same productivity index for constant p_{wf} and constant rate. Fig. 6 also shows that Babu and Odeh approximation matches very well with our correlation and the analytical solution. It has to be noted that each of the models available in the industry is only valid within the range of assumptions made during the development of the model.

Conclusions

From the preceding discussion, the following are our main conclusions:

1) a new simple model was developed to calculate the productivity of wells producing at constant rate or constant p_{wf}.
2) We used an analytical solution to develop a correlation for shape factors for both cases: constant rate or constant p_{wf}. The error in calculating then shape factor is less than 1% for the constant rate case and less than 4% for the constant p_{wf} case. However, we used numerical simulation to calculate the partial penetration skin factor.
3) The model has a n overall R^2 correlation factor of 0.9975 for the constant rate case and 0.9977 for the constant p_{wf} case.
4) The effect of partial penetration skin on the productivity index is opposite to the effect of shape factors. Because of this fact, constant p_{wf} productivity may be the same as constant rate productivity.
5) Example application shows that the proposed model matches fairly well with analytical solutions.
6) The model (and any other model for that matter) will generate errors if used outside the range of applicability. The errors at the boundaries of that range could be as high as 20%.

Transformed Coordinates

The following are the transformed coordinates and dimensions referred to in this paper:

$$x_{eq} = \sqrt{\frac{k_{eq}}{k_x}}\,x,\ y_{eq} = \sqrt{\frac{k_{eq}}{k_y}}\,y,\ z_{eq} = \sqrt{\frac{k_{eq}}{k_z}}\,z,$$

$$x_{weq} = \sqrt{\frac{k_{eq}}{k_x}}\,x_w,\ y_{weq} = \sqrt{\frac{k_{eq}}{k_y}}\,y_w,\ z_{weq} = \sqrt{\frac{k_{eq}}{k_z}}\,z_w,$$

$$a_{eq} = \sqrt{\frac{k_{eq}}{k_x}}\,a,\ b_{eq} = \sqrt{\frac{k_{eq}}{k_y}}\,b,\ h_{eq} = \sqrt{\frac{k_{eq}}{k_z}}\,h,$$

$$L_{eq} = \sqrt{\frac{k_{eq}}{k_y}}\,L,\ r_{weq} = \tfrac{1}{2} r_w\left(\sqrt[4]{\frac{k_x}{k_z}} + \sqrt[4]{\frac{k_z}{k_x}}\right),$$

$$A_{eq} = a_{eq} h_{eq},\ \text{and}\ k_{eq} = \sqrt[3]{k_x k_y k_z}\,.$$

Dimensionless Variables

The following are the dimensionless variables referred to in this paper:

$$t_D = \frac{0.00633 k_{eq} t}{\phi \mu c_t r_{weq}^2},\ t_{DAeq} = \frac{0.00633 k_{eq} t}{\phi \mu c_t A_{eq}}$$

and $J_D = \dfrac{141.2 B\mu}{kh} J$.

Constant rate case: $p_D = \dfrac{k_{eq} b_{eq} (p_i - p)}{141.2 qB\mu}$.

Constant pressure case: $p_D = \dfrac{(p_i - p)}{(p_i - p_{wf})}$,

$$q_D = \frac{141.2 qB\mu}{k_{eq} b_{eq} (p_i - p_{wf})},\ N_{pD} = \frac{0.8938 BN_p}{\phi c_t r_{weq}^2 b_{eq} (p_i - p_{wf})}$$

and $\dfrac{N_{pD}}{q_D}$ = dimensionless material balance time.

Nomenclature

A = well drainage area, L^2, ft^2
a = reservoir width, L, ft
B = formation volume factor, rB/STB
b = reservoir length, L, ft
C = shape factor, dimensionless
c = compressibility, Lt^2/m, $psia^{-1}$
h = net formation thickness, L, ft
J = Productivity index, L^4t/m, STB/D/psia
k = reservoir permeability, L^2, md
N_p = cumulative production, L^3, STB
p = pressure, m/Lt^2, psia
$\overline{p}$ = average reservoir pressure, m/Lt^2, psia
q = production rate, L^3/t, STB/D
r = radius, L, ft
s = skin factor, dimensionless
t = time, t, days
ϕ = porosity, fraction
μ = viscosity, m/Lt, cp

Subscripts

A = based on area
CP = constant pressure
CR = constant rate
D = dimensionless
eq = equivalent
f = flowing
h = horizontal
i = initial
o = oil
P = partial penetration
t = total
v = vertical
w = well
x = x-direction
y = y-direction
z = z-direction

Constants

γ = 1.781 (exponential of Euler's constant)

Acknowledgment

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Appendix A

The diffusivity equation for a homogeneous and anisotropic system in rectangular coordinates is given by:

$$k_x \frac{\partial^2 p}{\partial x^2} + k_y \frac{\partial^2 p}{\partial y^2} + k_z \frac{\partial^2 p}{\partial z^2} = \phi \mu c_t \frac{\partial p}{\partial t} \quad \text{........ (A-1)}$$

This equation is converted into an *equivalent* isotropic equation using coordinate transformation (list of transformed coordinates is given at the end of the paper):

$$\frac{\partial^2 p}{\partial x_{eq}^2}+\frac{\partial^2 p}{\partial y_{eq}^2}+\frac{\partial^2 p}{\partial z_{eq}^2}=\frac{\phi\mu c_t}{k_{eq}}\frac{\partial p}{\partial t} \quad \text{(A-2)}$$

Constant Rate Case: We choose to use a productivity index equation similar to that of a vertical well:

$$J_{CR}=\frac{k_{eq}b_{eq}}{141.2B\mu\left(\frac{1}{2}\ln\frac{4A_{eq}}{\gamma r_{weq}^2}-\frac{1}{2}\ln C_A+s_P\right)}, \quad \text{(A-3)}$$

In this equation, C_A is Dietz's[6] shape factor for wells operating at constant rate; and s_P is a skin factor that allows for the partial penetration of the well. Thus, the long-term solution of the isotropic diffusivity equation with constant rate boundary condition could be written as:

$$p_D=\frac{1}{2}\ln\frac{4A_{eq}}{\gamma r_{weq}^2}-\frac{1}{2}\ln C_A+s_P+2\pi t_{DAeq}. \quad \text{(A-4)}$$

And the dimensionless productivity index:

$$J_{DCR}=\frac{1}{\frac{1}{2}\ln\frac{4A}{\gamma C_A r_w^2}+s_P}. \quad \text{(A-5)}$$

Constant Pressure Case: The productivity index equation for a well producing at constant p_{wf} is given by:

$$J_{CP}=\frac{k_{eq}b_{eq}}{141.2B\mu\left(\frac{1}{2}\ln\frac{4A_{eq}}{\gamma r_{weq}^2}-\frac{1}{2}\ln C_{ACP}+s_{PCP}\right)}. \quad \text{(A-6)}$$

C_{ACP} is constant pressure shape factor[7] for wells operating at constant p_{wf}. s_{PCP} is a skin factor that allows for the partial penetration of the well. As outlined by Helmy and Wattenbarger[7], the long-term solution of the isotropic diffusivity equation with constant p_{wf} boundary condition is:

$$\frac{1}{q_D}=\frac{1}{2}\ln\frac{4A_{eq}}{\gamma r_{weq}^2}-\frac{1}{2}\ln C_{ACP}+s_{PCP}+2\pi\frac{r_{weq}^2}{A_{eq}}\frac{N_{pD}}{q_D}. \quad \text{(A-7)}$$

And the dimensionless productivity index in this case is:

$$J_{DCP}=\frac{1}{\frac{1}{2}\ln\frac{4A}{\gamma C_{ACP} r_w^2}+s_{PCP}}. \quad \text{(A-8)}$$

SI Metric Conversion Factors

cp ×1.0	E-03	=	Pa • s
ft ×3.048	E-01	=	m
ft² ×9.290 304	E-02	=	m²
ft³ ×2.831 685	E-02	=	m³
in. ×2.54	E+00	=	cm
lbf ×4.448 222	E+00	=	N
md ×9.869 233	E-04	=	μm²
psi ×6.894 757	E+00	=	kPa

TABLE 1 – EQUATIONS FOR SHAPE AND SKIN FACTORS

Constant Rate	$\ln C_A = 4.485 - \left(4.187 - 12.56\left(\frac{x_{weq}}{a_{eq}}\right) + 12.56\left(\frac{x_{weq}}{a_{eq}}\right)^2\right)\left(\frac{a_{eq}}{h_{eq}}\right) + 2.0\ln\left(\sin\left(\frac{\pi z_{weq}}{h_{eq}}\right)\right) + \ln\left(\frac{a_{eq}}{h_{eq}}\right)$
	$s_P = \left(\left(\frac{b_{eq}}{L_{eq}}\right)^{0.858} - 1\right) \times \left\{-0.025 + 0.022\ln C_A - 3.781\ln\left(\frac{h_{eq}}{a_{eq}}\right) + \left[\frac{1.289 - 4.751\left(\frac{y_{weq}}{b_{eq}}\right) + 4.652\left(\frac{y_{weq}}{b_{eq}}\right)^2 + 1.654\left(\frac{L_{eq}}{b_{eq}}\right) - 1.718\left(\frac{L_{eq}}{b_{eq}}\right)^2}{\left(\frac{h_{eq}}{a_{eq}}\right)\left(\frac{a_{eq}}{b_{eq}}\right)^{1.472}}\right]\right\}$
Constant Pressure	$\ln C_{ACP} = 2.607 - \left(4.74 - 10.353\left(\frac{x_{weq}}{a_{eq}}\right)^{1.115} + 9.165\left(\frac{x_{weq}}{a_{eq}}\right)^{2.838}\right)\left(\frac{a_{eq}}{h_{eq}}\right)^{1.011} + 1.810\ln\left(\sin\left(\frac{\pi z_{weq}}{h_{eq}}\right)\right) + 2.056\ln\left(\frac{a_{eq}}{h_{eq}}\right)$
	$s_{PCP} = \left(\left(\frac{b_{eq}}{L_{eq}}\right)^{1.233} - 1\right) \times \left\{2.894 + 0.003\ln C_A - 0.453\ln\left(\frac{h_{eq}}{a_{eq}}\right) + \left[\frac{0.388 - 1.278\left(\frac{y_{weq}}{b_{eq}}\right) + 0.715\left(\frac{y_{weq}}{b_{eq}}\right)^2 + 1.278\left(\frac{L_{eq}}{b_{eq}}\right) - 1.215\left(\frac{L_{eq}}{b_{eq}}\right)^2}{\left(\frac{h_{eq}}{a_{eq}}\right)\left(\frac{a_{eq}}{b_{eq}}\right)^{1.711}}\right]\right\}$

TABLE 2 – COMPARISON OF VALUES OF SHAPE FACTORS

z_{weq}/h_{eq}	x_{weq}/a_{eq}	h_{eq}/a_{eq}	Constant Rate			Constant p_{wf}		
			This study exact $\ln C_A$	Odeh approx. $\ln C_A$	Error	This study exact $\ln C_{ACP}$	This study approx. $\ln C_{ACP}$	Error
0.1	0.1	0.1	-26.146	-26.124	0.1%	-35.353	-35.400	0.1%
0.1	0.1	0.01	-299.040	-298.885	0.1%	-405.238	-406.539	0.3%
0.1	0.3	0.1	-11.060	-11.052	0.1%	-18.294	-18.736	2.4%
0.1	0.3	0.01	-148.244	-148.165	0.1%	-242.309	-235.627	2.8%
0.1	0.5	0.1	-6.033	-6.028	0.1%	-7.511	-7.511	0.0%
0.1	0.5	0.01	-97.978	-97.925	0.1%	-120.280	-120.495	0.2%
0.3	0.1	0.1	-24.217	-24.199	0.1%	-33.846	-33.654	0.6%
0.3	0.1	0.01	-297.115	-296.960	0.1%	-404.016	-404.793	0.2%
0.3	0.3	0.1	-9.135	-9.127	0.1%	-16.961	-16.990	0.2%
0.3	0.3	0.01	-146.319	-146.240	0.1%	-241.225	-233.880	3.0%
0.3	0.5	0.1	-4.108	-4.103	0.1%	-5.765	-5.765	0.0%
0.3	0.5	0.01	-96.530	-96.000	0.5%	-118.586	-118.748	0.1%
0.5	0.1	0.1	-23.791	-23.775	0.1%	-33.420	-33.279	0.4%
0.5	0.1	0.01	-296.692	-296.536	0.1%	-403.554	-404.418	0.2%
0.5	0.3	0.1	-8.711	-8.703	0.1%	-16.602	-16.615	0.1%
0.5	0.3	0.01	-145.895	-145.816	0.1%	-240.000	-233.505	2.7%
0.5	0.5	0.1	-3.685	-3.679	0.2%	-5.389	-5.390	0.0%
0.5	0.5	0.01	-95.630	-95.576	0.1%	-118.395	-118.373	0.0%

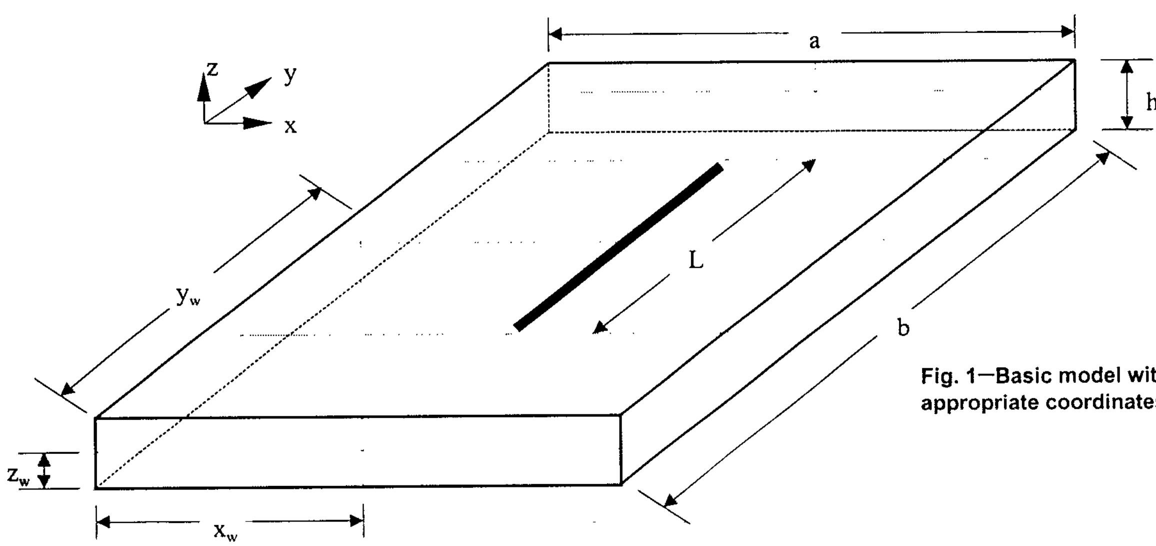

Fig. 1—Basic model with appropriate coordinates.

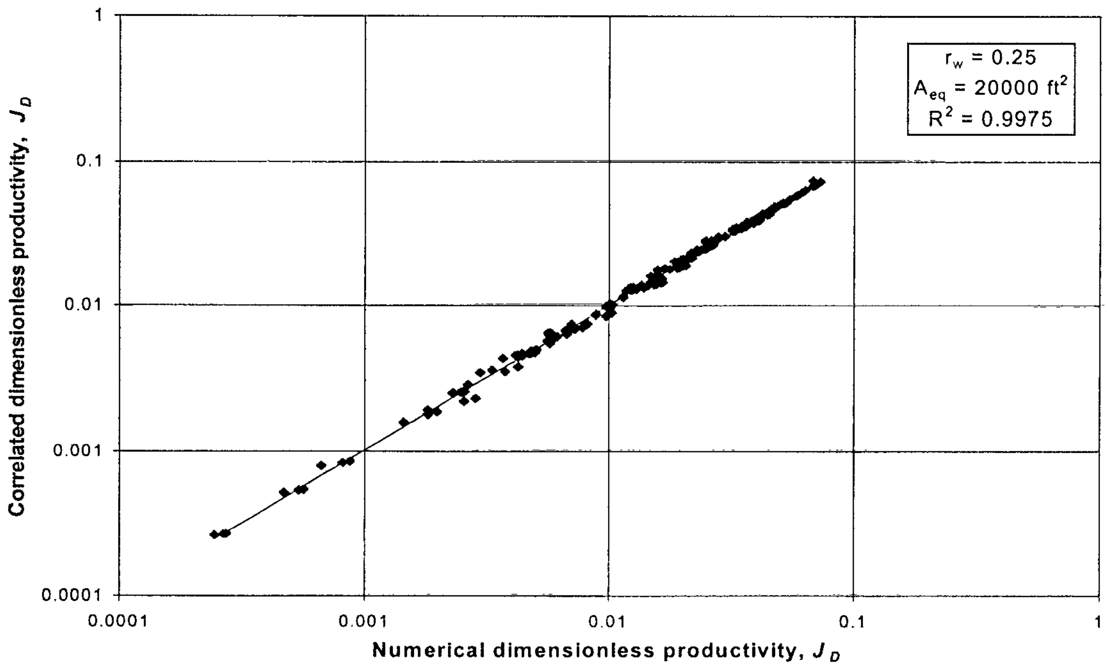

Fig. 2–Results of the correlation of the constant rate partial penetration skin for the cases studied.

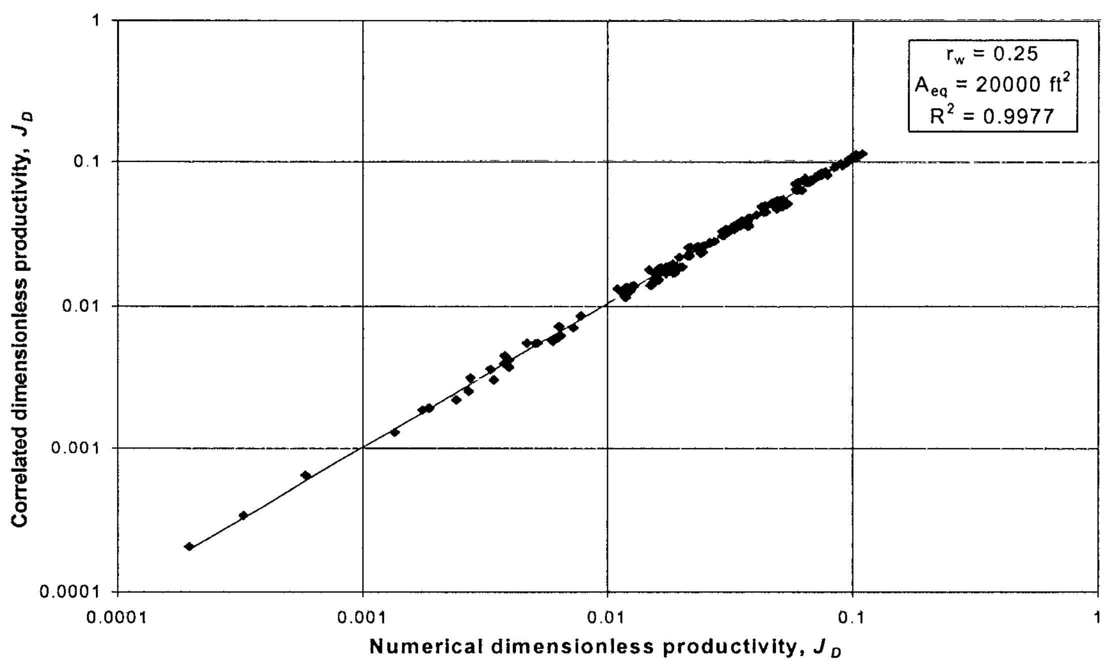

Fig. 3–Results of the correlation of the constant p_{wf} partial penetration skin for the cases studied.

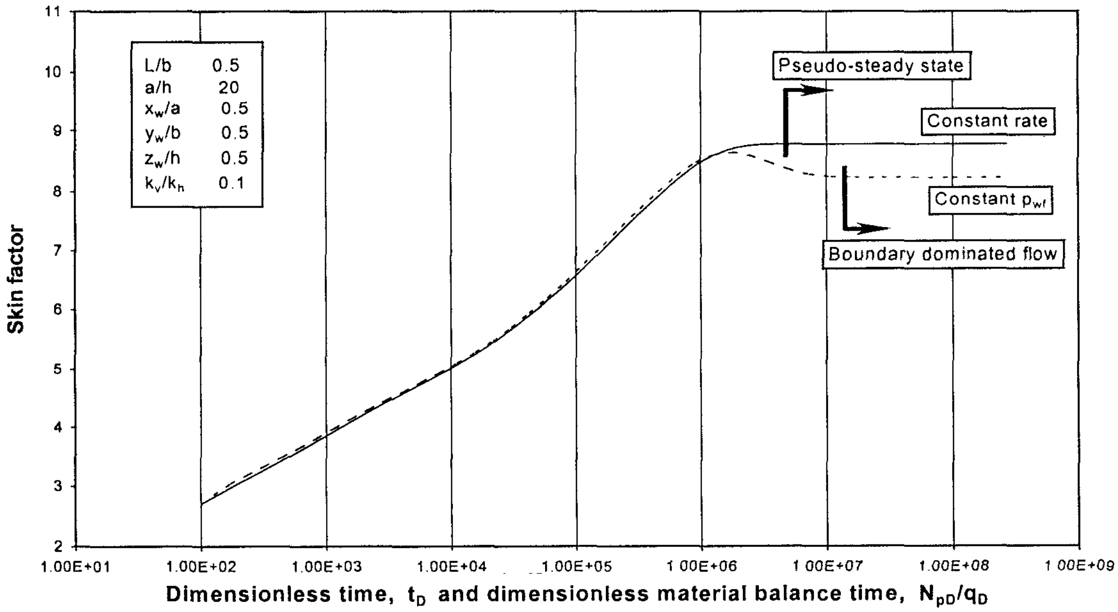

Fig. 4–Partial penetration skin is different for constant rate and constant p_{wf}.

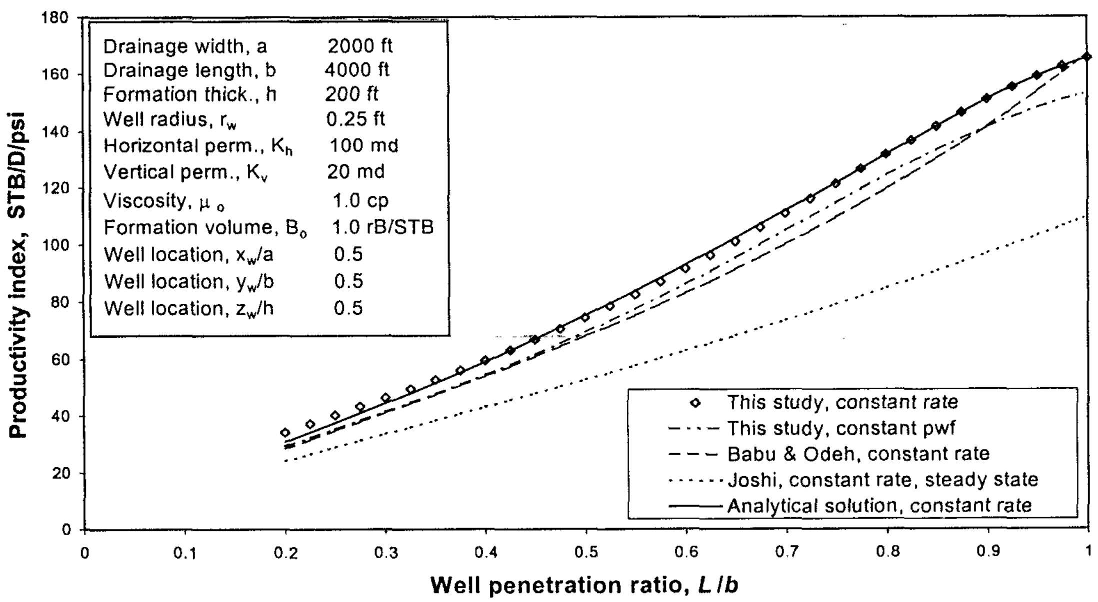

Fig. 5–Example 1, application of the proposed model, a good match with analytical solution.

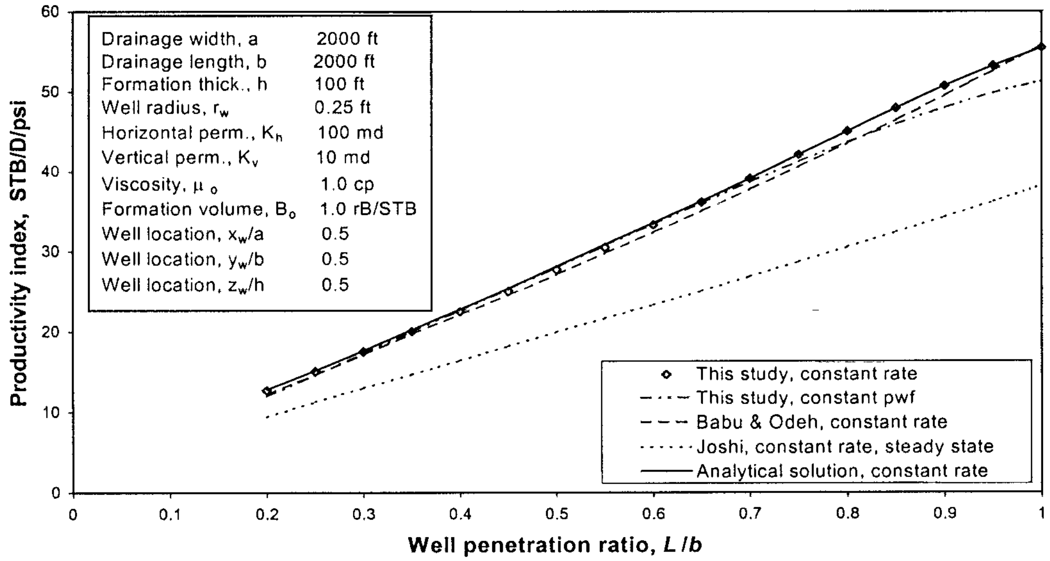

Fig. 6– Example 2, application of the proposed model, a good match with analytical solution. Constant p_{wf} and constant rate solutions are close.