MCLA Concice Review

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Contents

Part I

Multivariable Calculus

Chapter 11

Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

x and y are given as functions of a third variable t, a **parameter**, by the equations

$$x = f(t) \qquad y = g(t)$$

Each value of t determines a point $(x,\, y)$. As t varies, the point $(x,\, y) = (f(t),\, g(t))$ varies and traces out a curve C, a **parametric curve**.

Tangents

f and g are differentiable functions and we want to find the tangent line at a point $(f(t),\, g(t))$. The Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \bullet \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad if\ \frac{dx}{dt} \neq 0$$

d^2y/dx^2 can be found by replacing y with dy/dx in the equation above

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$$

Example

Find the tangent to the cycloid $x = r(\theta - sin\,\theta)$, $y = r(1 - cos\,\theta)$ at the point where $\theta = \pi/3$.

Solution

The slope of the tangent is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\ sin\ \theta}{r(1 - cos\ \theta)} = \frac{sin\ \theta}{1 - cos\ \theta}$$

When $\theta = \pi/3$, we have

$$x = r\left(\frac{\pi}{3} - sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y = r\left(1 - cos\frac{\pi}{3}\right) = \frac{r}{2}$$

and

$$\frac{dy}{dx} = \frac{sin(\pi/3)}{1 - cos(\pi/3)} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right)$$

Areas

Area under a curve $y = F(x)$ form a to b is $A = \int_a^b F(x)dx$. If the curve is traced out with the parametric equations $f(t)$ and $g(t)$, we can calculate the area by using the Substitution Rule for Definite Integrals:

$$A = \int_a^b ydx = \int_\alpha^\beta g(t)f'(t)dt$$

Example

Find the area under one arch of the cycloid

$$x = r(\theta - sin\ \theta) \qquad y = r(1 - cos\ \theta)$$

Solution

One arch of the cycloid is given by $0 \le \theta \le 2\pi$. Using the substitution rule with $y = r(1 - cos\ \theta)$ and $dx = r(1 - cos\ \theta)d\theta$, we have

$$A = \int_0^{2\pi r} y\ dx = \int_0^{2\pi r} r(1 - cos\ \theta)r(1 - cos\ \theta)d\theta$$

$$= r^2(\frac{3}{2} \cdot 2\pi) = 3\pi r^2$$

Arc Length

To find the length L of a curve C in the from $y = F(x)$

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

C can also be described with parametric equations, and we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_\alpha^\beta \sqrt{1 + (\frac{dy/dt}{dx/dt})^2} \frac{dx}{dt} dt$$

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha,\beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Surface Area

Suppose a curve C is rotated about the x-axis. If C is traversed exactly once as t increases from α to β, then the area of the surface is given by

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The general symbolic formulas $S = \int 2\pi y ds$ and $S = \int 2\pi x ds$ are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example

Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution

The sphere is obtained by rotating the semicircle

$$x = r cos\, t \qquad y = r sin\, t \qquad 0 \leq t \leq \pi$$

about the x-axis. Then, we get

$$S = \int_0^\pi 2\pi r\ sin\, t \sqrt{(-r sin\, t)^2 + (r cos\, t)^2} dt$$

$$= 2\pi r^2 \int_0^\pi sin\, t\, dt = 2\pi r^2 (-cos\, t)]_0^\pi = 4\pi r^2$$

11.2 Polar Coordinates

r is the ditsance from O to P and θ is the angle between the polar axis and the line OP. The **polar coordinates** of P are in the form $(r,\, \theta)$,

If a point P has Cartesian coordinates $(x,\, y)$ and polar coordinates $(r,\, \theta)$

$$cos\, \theta = \frac{x}{r} \qquad y = rsin\, \theta$$

and so

$$x = rcos\, \theta \qquad y = rsin\, \theta$$

To find r and θ when x and y are known, we use

$$r^2 = x^2 + y^2 \qquad tan\, \theta = \frac{y}{x}$$

Example

What curve is represented by the polar equation θ=1?

Solution

This curve consists of all points $(r,\, \theta)$ such that θ is 1 radian. It is a straight line that passes through O and makes an angle of 1 radian with the polar axis.

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we write its parametric equations as

$$x = rcos\, \theta = f(\theta)\, cos\, \theta \qquad y = rsin\, \theta = f(\theta)\, sin\, \theta$$

Then, using the method of finding slopes of parametric curves and the Product Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} sin\, \theta + r\, cos\, \theta}{\frac{dr}{d\theta} cos\, \theta - r\, sin\, \theta}$$

11.3 Areas and Lengths in Polar Coordinates

Using the formula for the area of a sector of a circle

$$A = \frac{1}{2} r^2 \theta$$

the formula for the area A of the polar region R is

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2\, d\theta = \int_a^b \frac{1}{2} r^2\, d\theta$$

Arc Length

To find the length of a polar curve r=$f(\theta)$, we write the parametric equations as

$$x = r cos\,\theta = f(\theta)\,cos\,\theta \qquad y = r sin\,\theta = f(\theta)\,sin\,\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\,cos\,\theta - r\,sin\,\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\,sin\,\theta + r\,cos\,\theta$$

so, using $cos^2\theta + sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{dr}{d\theta})^2 + r^2$$

Therefore the length of a curve is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

11.4 Conic Sections

Parabolas

An equation of the parabola with focus (0, p) and directrix $y = -p$ is

$$x^2 = 4py$$

Ellipses

The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \leq b > 0$$

has focus ($\pm c$, 0), where $c^2 = a^2 - b^2$, and vertices ($\pm a$, 0)
The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \leq b > 0$$

has focus (0, $\pm c$), where $c^2 = a^2 - b^2$, and vertices (0, $\pm a$)

Hyerpbolas

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci($\pm c$, 0), where $c^2 = a^2 + b^2$, vertices ($\pm a$, 0), and asymptomes $y = \pm(b/a)x$.
The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci(0, $\pm c$), where $c^2 = a^2 + b^2$, vertices (0, $\pm a$), and asymptomes $y = \pm(a/b)x$.

11.5 Conic Sections in Polar Coordinates

11.5.1 Elasticity Theorem

Let P be a fixed point (the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

. The conic is

1. an ellipse if $e < 1$.
2. a parabola if $e = 1$.
3. a hyperbola if $e > 1$.

11.5.2 Polar Equation

The polar equation of an ellipse with focus at the origin, emimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 + e\, cos\, \theta}$$

The **perihelion distance** from a planet to the sun is $a(1 - e)$ and the **aphelion distance** is $a(1 + e)$.

Chapter 12

Infinite Sequences and Series

12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L$$

If $\lim_{n\to\infty} a_n$ exists, we say that the sequence **converges**, otherwise we say that the sequence **diverges**.

Properties of convergent sequences

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$
4. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$

Anither useful fact about limits of sequences is given by the following theorem

$$If \lim_{n\to\infty} |a_n| = 0, then \lim_{n\to\infty} a_n = 0$$

Example

Find $\lim_{n\to\infty} \frac{n}{n+1}$

Solution

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$
$$= \frac{1}{1+0} = 1$$

In the previous example, the constant r is defined as 1. The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all values of r.

$$\lim_{n\to\infty} r^n = \left\{ \begin{cases} [11]0 & \text{if} - 1 < r < 1 \\ 1 & \text{if} r = 1 \end{cases} \right. \tag{12.1}$$

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M \qquad$ for all $n \geq 1$
It is **bounded below** if there is a number m such that

$$m \leq a_n \qquad \text{for all} \quad n \geq 1$$

12.2 Series

An infinite sequence $\{a^n\}_{n=1}^{\infty}$ can be described as a **series** which is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n$$

Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote its nth partial sum:

$$s_n = sum_{i=1}^{n} a_i = a_{\cdot} + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

$$\sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. Otheriwse, it's called **divergent**. The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

Example

Find the sum of the series $\sum_{n=1}^{\infty}(\frac{3}{n(n+1)} + \frac{1}{2^n})$ The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} - \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

We know that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So, the given series is convergent and

$$\sum_{n=1}^{\infty}(\frac{3}{n(n+1)} + \frac{1}{2^n}) = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 3 \cdot 1 + 1 = 4$$

A **harmonic series** follows the following formula (and is divergent):

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

If the series a_n is convergent, then $\lim_{n\to\infty} a_n = 0$

Test for Divergence

If $\lim_{n\to\infty}$ does not exist or if $\lim_{n\to\infty} \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution

$$\lim_{n=\infty} a_n = \lim_{n=\infty} \frac{n^2}{5n^2+4} = \lim_{n=\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

Properties of convergent series

i $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

ii $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

iii $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

12.3 The Integral Test and Estimates of Sums

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words, (i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution

The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1,\infty)$ so we use the integral test:

$$\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_1^{t} \frac{1}{x^2+1}dx = \lim_{t\to\infty} \tan^{-1} x\Big)_1^t$$

$$= \lim_{t\to\infty}(\tan^{-1} t - \frac{\pi}{4}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$$

The p-series $\sum_{n=1}^{\infty}$ is convergent if $p > 1$ and divergent if $p \leq 1$. **Remainder estimate for the Integral Test**: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_N = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

12.4 Comparison Tests

The Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

i If $\sum b_n$ is convergent and $a_n \leq b_n$, for all n, then $\sum a_n$ is also convergent.

ii If $\sum b_n$ is divergent and $a_n \geq b_n$, for all n, then $\sum a_n$ is also divergent.

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.
For large n the dominant term in the denominator is $2n^2$ so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

is convergent by part (i) of the Comparison Test.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both series diverge.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence. We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n-1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 > 0$$

Since this limi t exists, and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

12.5 Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative.

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots \qquad (b_n > 0)$$

satisfies

1. $b_{n+1} \leq b_n \qquad$ for all n
2. $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Example

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

satisfies

1. $b_{n+1} \leq b_n \qquad$ because $\frac{1}{n+1} < \frac{1}{n}$
2. $\lim_{n\to\infty} b_n = \lim_n n \to \infty \frac{1}{n} = 0$

then the series is convergent by the Akternating Series Test.

12.6 Absolute Convergence and the Ratio and Root Tests

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent. A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

The Ratio Test

1. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.
2. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$, or $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, the ratio test is inconclusive.

Example

Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Solution

We use the Ratio test with $a_n = (-1)^n n^3/3^n$:

$$|\frac{a_{n+1}}{a_N}| = |\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}| = \frac{(n+1)^3}{3^{n+1} \cdot \frac{3^n}{n^3}}$$
$$= \frac{1}{3}(\frac{n+1}{n})^3 = \frac{1}{3}(1+\frac{1}{n})^3 \to \frac{1}{3} < 1$$

Root Test

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Example

Test the convergence of the series $\sum_{n=1}^{\infty}(\frac{2n+3}{3n+2})^n$.

Solution

$$a_n = (\frac{2n+3}{3n+2})^n$$
$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

12.7 Strategy for Testing Series

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, we converges if $|r| < 1$ and diverges if $|r| \geq 1$
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tets should be considered. In particular, if a_n is a rational/algebraic function of n, then the series should be compared with a p-series. The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the conmparison test to $\sum a_n$ and test for absolute convergence.

4. If you see that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products are often conveniently tested using the Ratio Test.

7. If a_n is of of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the Integral Test is effective

12.8 Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is a variable and the c_n's are constants called the **coeffeicients** of the series.
For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilites:

1. The series converges only when $x = a$.

2. The sereis converges for all x.

3. There is a positive number R such that the sereis converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R is called the **radius of convergence** of the power series. The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. Below is a summarization of the radius and interval of convergence for each of the examples considered in this section.

12.9 Representations of Functions as Power Series

Example

Find a power series representation $1/(x+2)$.

Solution

In order to put this function in the form of the left side, we first factor a 2 from the denominator:

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2[1-(-\frac{x}{2})]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-fracx2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

The series converges when $|-x/2| < 1$, that is $|x| < 2$. So the interval of convergence is $(-2, 2)$.

We would like to be able to differentiate and integrate such functions, and the following theorem will help. This is called **term-by-term differentiation and integration**. If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and (i) $f(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$

(ii)$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

12.10 Taylor and Maclaurin Series

If f has a power series expansion at a, then it must be of the following form.

$$f(x) = \sum_{n=0}^{\infty}\frac{f^(n)(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \tag{12.2}$$

For the special case $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty}\frac{f^(n)(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

Below are some important Maclaurin series that have been derived in this section and the preceding one.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Solution

We have $f^(n)(2) = e^2$ and so, putting $a = 2$ in the definition of a Taylor series, we get

$$\sum_{n=0}^{\infty} \frac{f^(n)(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

We see that radius of convergence is $R = \infty$.

Chapter 13

Vectors and the Geometry of Space

13.1 Three-Dimensional Coordinate System

The three xy, yz, and xz axes are called **octants**.

The distance $|P_1P_2|$ between the points $P_1(x, y, z)$ and $P_2(x, y, z)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center of the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

13.2 Vectors

If $\vec{u}$ and $\vec{v}$ are vectors positioned so the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then the **sum** $u + v$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.

Scalar Multiplication: If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\vec{v}$ if $c < 0$, If $c = 0$ or $v = 0$, then $c\mathbf{v} = 0$.

Components

The components of a vector $\vec{a}$ and you can write

$$a = \langle a_1, a_2 \rangle \qquad ora = \langle a_1, a_2, a_3 \rangle$$

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{a}$ with representation $\vec{AB}$ is

$$a = \langle x_2 - x_1, y_2, -y_1, z_2 - z_1 \rangle$$

The length of the three-dimensional vector $a = \langle a_1, a_2, a_3 \rangle$ is

$$|a| = a_1^2 + a_2^2 + a_3^2$$

Properties of Vectors If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in V_n and c and d are scalars, then

1. $a + b = b + a$
2. $a + (b + c) = (a + b) + c$
3. $a + 0 = a$
4. $a + (-a) = 0$
5. $c(a + b) = c\vec{a} + c\vec{b}$
6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(\vec{a})$
8. $1\vec{a} = \vec{a}$

Any vector in V_3 can be expressed in terms of the **standard basis vectors i, j**, and **k**. For instance,

$$\langle 1, -2, 6 \rangle = \hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}$$

13.3 The Dot Product

If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\vec{a}$ and $\vec{b}$ is the number $a \cdot b$ given by

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties of the Dot Product If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in V_3 and c is a scalar, then

1. $a \cdot a = |a|^2$
2. $a \cdot b = b \cdot a$
3. $a \cdot (b + c) = a \cdot b + a \cdot c$
4. $(c\vec{a}) \cdot b = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$
5. $0 \cdot a = 0$

If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $a \cdot b = 0$

Projections

Scalar projection of $\vec{b}$ onto $\vec{a}$: $comp_a b = \frac{a \cdot b}{|a|}$ Vector projection of $\vec{b}$ onto $\vec{a}$: $proj_a b = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|}\right)$

Example

Find the scalar projection and vector projection of $b = \langle 1, 1, 2 \rangle$ onto $a = \langle -2, 3, 1 \rangle$. Since $|a| = \sqrt{-2^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of $\vec{b}$ onto $\vec{a}$ is

$$comp_a b = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is

$$proj_a b = \frac{3}{\sqrt{14}} \frac{\vec{a}}{|\vec{a}|} = \frac{3}{14} a = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

13.4 The Cross Product

The **cross product** $a \times b$ of two vectors $\vec{a}$ and $\vec{b}$ is a vector. If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\vec{a}$ and $\vec{b}$ is the vector

$$a \times b = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

If $a = \langle 1, 3, 4 \rangle$ and $b = \langle 2, 7, -5 \rangle$, then

$$a \times b = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \hat{\mathbf{k}}$$

$$= (-15 - 28)\,\hat{\mathbf{i}} - (-5 - 8)\,\hat{\mathbf{j}} + (7 - 6)khat = -43\,\hat{\mathbf{i}} + 13\,\hat{\mathbf{j}} + \,\hat{\mathbf{k}}$$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

$$a \times b = 0$$

Properties of cross product

1. $a \times b = -b \times a$
2. $(c\vec{a})\vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $a \times (b + c) = a \times b + a \times c$
4. $(a + b) \times c = a \times c + b \times c$
5. $a \cdot (b \times c) = (a \times b) \cdot c$
6. $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

13.5 Equations of Lines and Planes

There is a scalar t such that

$$r = r_0 + tv$$

. Therefore, the **parametric equations** we have

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

. Another way of describing a line L; if a, b, or c is 0, we can solve each of these equations for t and obtain

$$\frac{x - x_0}{a} = \frac{y - y_0)}{b} = \frac{z - z_0}{c}$$

The line segment from r_0 to r_1 is given by the vector equation

$$r(t) = (1 - t)r_0 + tr_1 \qquad 0 \leq t \leq 1$$

Planes

The normal vector $\vec{n}$ is orthogonal to every vector in the given plane. In particular, n is orthogonal to $r - r_0$ and so we have

$$n \cdot (r - r_0) = 0$$

The **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $n = \langle a, b, c \rangle$ Another way we can rewrite the equation:

$$ax + by + cz + d = 0$$

Thus the formula for distance D:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example

Find the distance between the parallel planes $10x+2y-2z = 5$ and $5x+y-z = 1$

Solution

Note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel.

Chapter 14

Vector Functions

14.1 Vector Functions and Space Curves

A **vector-valued function**, or **vector function** is simply a function whose domain is a set of real numbers and whose range is a set of vectors. If $f(t)$. $g(t)$, $r(t)$ are the components of the vector $r(t)$, then f, g, and h, are real valued functions called the **component function** of **r** and we can write

$$r(t) = \langle f(t), g(t), h(t) \rangle = f(t)' \,\hat{\mathbf{i}} + g(t)' \,\hat{\mathbf{j}} + h(t)' \,\hat{\mathbf{k}}$$

If $r(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} r(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Example

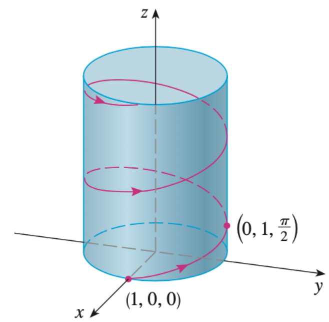
Sketch the curve whose vector equation is

$$r(t) = \cos t \,\hat{\mathbf{i}} + \sin t \,\hat{\mathbf{j}} + t \,\hat{\mathbf{k}}$$

Solution

The parametric equations for this curve are

$$x = \cos t \qquad y = \sin t \qquad z = t$$



The figure shown below is known as a **helix**.

14.2 Derivatives and Integrals of Vector Functions

The **derivative** r' of a vector function r is defined as

$$\frac{dx}{dt} = r'(t) = \lim_{h \to t} \frac{r(t+h) - r(t)}{h}$$

The **tangent line** to C at P is defined to be the line throguh P parallel to the tangent vector $r'(t)$. It is defined as

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

The following theorem gives us a conveninet method for computing the derivative vector function of r; just differentiate each component of r.
If $r(t) = \langle f(t), g(t), h(t) \rangle$ where f, g, and h are differentiable functions, then

$$r'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h'(t)\,\hat{\mathbf{k}}$$

Example

Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$

Solution

The vector equation of the helix is $r(t) = \langle 2cost, sint, t \rangle$, so

$$r'(t) = \langle -2sint, cost, 1 \rangle$$

The parameter value corresponding to the point is $\pi/2$, so the tangent vector there is $r'(\pi/2) = \langle -2, 0, 1 \rangle$. As a result, its parametric equations are

$$x = -2t \qquad y = 1 \qquad z = \pi/2 + t$$

The **second derivative** of a vector function r is defined as $r'\prime = (r')'$. A curve given by a vector function $r(t)$ on an interv al I is called **smooth** if r' is continuous and $r'(t) \neq 0$.

Differentiation Rules

Suppose u and v are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$
2. $\frac{d}{dt}[cu(t)] = cu'(t)$
3. $\frac{d}{dt}[u(t)[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$
4. $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$
5. $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$
6. $\frac{d}{dt}[uf(t)] = f'(t)u'(f(t))$ (Chain Rule)

Integrals

The **definite integral** of a continuous function $r(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. We can express the integral of r in terms of the integrals of its component functions f, g, and h.

$$\int_a^b r(t)dt = \left(\int_a^b f(t)dt\,\hat{\mathbf{i}}\right) + \left(\int_a^b g(t)dt\,\hat{\mathbf{j}}\right) + \left(\int_a^b h(t)dt\,\hat{\mathbf{k}}\right)$$

We can extedn the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b r(t)dt = R(t)\Big)_a^b = R(b) - R(a)$$

where R is an antiderivative of r.

Example

If $r'(t) = 2\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$, then find the antiderivative.

Solution

$$\int r(t)dt = \left(\int 2\cos t dt\right) \hat{\mathbf{i}} + \left(\int \sin t dt\right) \hat{\mathbf{j}} + (2tdt)\ \hat{\mathbf{k}}$$

$$= 2\sin t\, \hat{\mathbf{i}} - \cos t\, \hat{\mathbf{j}} + t^2\, \hat{\mathbf{k}} + C$$

where C is a vector constant of intergration, and

$$\int_0^{\pi/2} = [2\sin t\, \hat{\mathbf{i}} - \cos t\, \hat{\mathbf{j}} + t^2\ \hat{\mathbf{k}}]_0^{\pi/2} = 2\, \hat{\mathbf{i}} +\ \hat{\mathbf{j}} + \frac{\pi^2}{4}\ \hat{\mathbf{k}}$$

14.3 Arc Length and Curvature

The length of a space curve can be defined if the curve is traversed exactly once as t iincreases from a to b, then it can be shown that its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

=

$$\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$$

A more compact form is shown below

$$L = \int_a^b |r'(t)| dt$$

Example

Find the length of the arc of the circular helix with vector equation $r(t) = \cos t\, \hat{\mathbf{i}} + \sin t\, \hat{\mathbf{j}} + t\, \hat{\mathbf{k}}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Solution

Since $r'(t) = -\sin t\, \hat{\mathbf{i}} + \cos t\, \hat{\mathbf{j}} +\ \hat{\mathbf{k}}$, we have

$$|r'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$ is described by the parameter interval $0 \leq t \leq 2\pi$, and so, we have

$$L = \int_0^{2\pi} |r'(t)| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$$

Parametrization

A single curve C can be represented by more than one vector function. For instance, the twisted cubic

$$r_1(t) = \left\langle t, t^2, t^3 \right\rangle$$

could also be represented by the function

$$r_2(u) = \left\langle e^u, e^{2u}, e^{3u} \right\rangle$$

We say that these previous equations are parametrizations of the curve C. It also shows that the arc length is independent of the parametrization used.
Now we suppose that C is a piecewise-smooth curve given by a vector function $r(t) = \left\langle f(t)\,\hat{\mathbf{i}}, g(t)\,\hat{\mathbf{j}}, h(t)\,\hat{\mathbf{k}} \right\rangle$, $a \leq t \leq b$, and C is tranversed exactly once as t increases from a to b. We define its **arc length function** s by

$$s(t) = \int_a^t |r'(t)| du = \int_a^t \sqrt{(\frac{dx}{du})^2 + (\frac{dy}{du})^2 + (\frac{dz}{du})^2}$$

When both sides are differentiated,

$$\frac{ds}{dt} = |r'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises natrually from the shape of the curve and does not depend on a particular coordinate system. The reparametrization can be done in terms of s by substituting for t: $r = r(t(s))$.

Example

Reparametrize the helix $r(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

Solution

The initial point $(1, 0, 0)$ corresponds to the parameter value $t = 0$. From the previous example, we have

$$\frac{ds}{dt} = |r'(t)| = \sqrt{2}$$

and so $\quad s = s(t) = \int_0^t |r'(u)| du = \int_0^t \sqrt{2} du = \sqrt{2} t$
Therefore, $t = s/\sqrt{2}$ and the required reparametrization is obtained by substituting for t:

$$r(t(s)) = \cos s/\sqrt{2}\,\hat{\mathbf{i}} + \sin s/\sqrt{2} + (s/\sqrt{2})\,\hat{\mathbf{k}}$$

Curvature

The **curvature** of a curve is

$$\kappa = |\frac{dT}{ds}|$$

where **T** is the tangent vector. Another way to define the curvature is

$$\kappa = \frac{|T'(t)|}{|r'(t)|}$$

The curvature of the curve given by the vector function **r** is

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

The Normal and Binomial Vectors

The **principal unit normal vector N(t)** or simply **unit normal** as

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

The vector $B(t) = T(t) \times N(t)$ is called the **binormal vector**. It is perpendicular to both T and N and is also a unit vector.

Example

Find the unit normal and binormal vectors for the circular helix

$$r(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$$

Solutiom

We first compute the ingredients needed for the unit normal vector:

$$r'(t) = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \hat{\mathbf{k}} \qquad |r'(t)| = \sqrt{2}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$T'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}}) \qquad |T'(t)| = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = (-\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}}) = \langle -\cos t, -\sin t, 0\rangle$$

$$B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1\rangle$$

14.4 Motion in Space: Velocity and Acceleration

: Suppose that a particle moves through space so its position vector at time t is $r(t)$. For such small values of h, the vector

$$\frac{r(t+h)-r(t)}{h}$$

gives the average velocity over a time interval of length h and its limit is the **velocoty vector v(t)** at time t:

$$v(t) = \lim_{h} \to 0 \frac{r(t+h)-r(t)}{h} = r'(t)$$

The **speed** of the particle at time t is the magnitude of the velocity vector, that is, $|v(t)|$.
As in the case of one-dimensional motion, the **acceleration** of the particle is defined as the derivative of the velocity:

$$a(t) = v'(t) = r''(t)$$

Example

The position vector of an object moving in a plane is given by $r(t) = t^3\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$. Find its velocity, speed, and acceleration when $t = 1$.

Solution

The velocity and acceleration at time t are

$$v(t) = r'(t) = 3t^2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}$$

$$a(t) = r'' = 6t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}$$

and the speed is

$$|v(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

When $t = 1$, we have

$$v(1) = 3\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} \qquad a(1) = 6\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} \qquad |v(1)| = \sqrt{13}$$

The parametric equations of a trajectory are

$$x = (v_0 \cos\alpha) \qquad y = (v_0 sin\alpha)t - \frac{1}{2}gt^2$$

Tangential and Normal Components of Acceleration

Proof for equation to relate both tangential and normal acceleration:

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{v(t)}{|v(t)|} = \frac{\mathbf{v}}{v}$$

and so

$$\mathbf{v} = vT$$

If we differentiate both sides of this equation with respect to t, we get

$$a = v'$$

If we use the expression for the curvature, we have

$$\kappa = \frac{|T'|}{|r'|} = \frac{|T'|}{v}$$

Therefore, the equatopn formed is

$$a = v'T + \kappa v^2 N$$

where

$$a_T = v' \qquad a_N = \kappa v^2$$

Kepler's Laws of Planetary Motion

Kepler's Laws:

1. A planet revolves around the sun in an ellipitical orbit with the sun in one focus.
2. The line joining the Sun to a planet sweeps out equal areas in equal times.
3. The square of th period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Chapter 15

Partial Derivatives

15.1 Functions of Several Variables

A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values f takes on. Sometimes it is written as $z = f(x, y)$, where x and y are independent variables and z is the dependent variable.

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a cosntant (in the range of f).

Example

Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

Solution

The level curves are

$$6 - 3x - 2y = k \qquad or \qquad 3x + 2y + (k - 6) = 0$$

. The curves are drawn below, with the graph showing a family of lines with slope $-\frac{3}{2}$.

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) which can also be written as $T = f(x, y, z)$.

15.2 Limits and Continuity

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b) along any path that stays within the domain of f. Then we say that the **limit of** $f(x,y)$ **as** (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $lim_{(x,y)\to(a,b)}$ does not exist.

Example

15.3 Partial Derivatives

If g has a derivative at a, then we call it the **partial derivative of** f **with respect to** x **at** (a,b) and denoite it by $f_x(a,b)$. Thus

$$f_x(a,b) = g'(a) \qquad where \qquad g(x) = f(x,b)$$

.

Notations for Partial Derivatives

If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Example

If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution

Holdying y constant and differentiating with respect to x, we get

$$f_x(x,y) = 3x^2 + 2xy^3$$

and so

$$f_x(2,1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiatng with respect to y, we get

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Higher Derivatives

If f is a function of two varaibles, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$ which are called the **second partial derivatives**

Example

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

In the previous example, we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x = 2y^3 \qquad f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2 \qquad f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Suppose f is defined on a disk D that contains the point (a, b). If the functions $f_{xy}(a, b) = f_{yx}(a, b)$ are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial Differential Equations

Laplace's equation plays a role in problems of heat conduction, fluid flow, and electric potential. Its solutions are called **harmonic functions**.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The **wave equation** describes the motion of a waveform, which could be an ocean wave, sound wave, a light wave, or a wave traveling along a vibrating string.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The Cobb-Douglas production function models the total production P of an economic system as a function of the amount of labor L and the capital investment K.

$$P(L, K) = bL^\alpha K^{1-\alpha}$$

15.4 Tangent Planes and Linear Approximations

Suppose f has continuous partial derivatives. An equation of the **tangent plane** to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then the previous formula gives us the equation of the tangent plane at $(1, 1, 3)$ as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$= 4x + 2y - 3$$

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a,b) and the approximation

$$L(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b) If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz (**increment** of z) can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it approximate $f(1.1, -.1)$

Solution

The partial derivatives are

$$f_x(x,y) = e^{xy} + xye^{xy} \qquad f_y(x,y) - x^2e^{xy}$$

$$f_x(1,0) = 1 \qquad f_y(1,0) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable. The linearization is

$$\begin{aligned} L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 1 + 1(x-1) + 1 \cdot y = x + y \end{aligned}$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

or

$$f(1.1, -.1) \approx 1.1 - .1 = 1$$

Differentials

we define the **differentials** dx and dy to be independent variables. Then the **total differential** dz is defined by

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

15.5 The Chain Rule

Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentaible function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

We can also write the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Chain Rule[**Case 2**] Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Write out the Chain Rule for teh case where $w = f(x,y,z,t)$ and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, and $t = t(u,v)$. We can now write the required expressions:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

Implicit Differentiation

Given the function x and y being both functions of x,

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

We solve for dy/dx and obtain

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

giving us the **Implicit Function Theorem**

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution

Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

15.6 Directional Derivatives and The Gradient Vector

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ can be defined as

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Example

Find the directional derivative $D_u f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and u is the unit vector given by angle $\theta = \pi/6$. What is $D_u f(1, 2)$? Using the equation gives

$$D_u f(x, y) = f_x(x, y)\cos\frac{\pi}{6} + f_y(x, y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$
$$= \frac{1}{2}\left[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3}y)\right]$$

Therefore

$$D_u f(1,2) = \frac{1}{2}\left[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3}y)\right] = \frac{13 - 3\sqrt{3}}{2}$$

The Gradient Vector

If f is a function of three variables x, y, and z then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x,y,z) = \langle f_x(x,y), f_y(x,y), f_z(x,y)\rangle = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}$$

The relationship between the directional derivative and the gradient vector is described below:

$$D_u f(x,y,z) = \nabla f(x,y,z)\cdot u$$

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(x)|$ and it occurs when u has teh same direction as the gradient vector.
Tangent planes to the level surface can be wrriten in the form

$$F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0$$

The **normal line** to S at P is the line passing through P perpendicular to the tangent plane.

15.7 Maximum and Minimum Values

A function of two variables has a **local maximum** at (a,b) if $f(x,y) \le f(a,b)$. If $f(x,y) \ge f(a,b)$, then $f(a,b)$ is a **local minimum value**.
If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

a If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

b If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

c if $D < 0$, the point is known as a **saddle point** and is neither a max or a min.

d If $D = 0$, the test is indeterminate.

To find the absolute maximum and minimum values of a continuous function f on a closed, bonded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from the first two steps is the absolute maximum value; the smallest of these values is the absolute minimum value.

Chapter 16

Multiple Integrals

16.1 Double Integrals over Rectangles

16.2 Iterated Integrals

16.3 Double Integrals Over General Regions

16.4 Double Integrals in Polar Coordinates

16.5 Applications of Double Integrals

16.6 Surface Area

16.7 Triple Integrals

16.8 Triple Integrals in Cylindrical and Spherical Coordinates

16.9 Change of Variables in Multiple Integrals

Chapter 17

Vector Calculus

17.1 Vector Fields

Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field** on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$. Since $\mathbf{F}$ is a two dimensional vector, we can write it in terms of its **component** functions P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}} = \langle P(x, y), Q(x, y)\rangle$$

or, for short,

$$\mathbf{F} = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}}$$

Let E be a set in $\mathbb{R}^3$ (a plane region). A **vector field** on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a two-dimensional vector $\mathbf{F}(x, y, z)$.

Example

Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}x$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z\rangle$ is

$$\mathbf{F}x = \frac{\epsilon q Q}{|\vec{x}|^3}\vec{x}$$

The force is repulsive for like charges ($qQ > 0$) and attractive for unlike charges ($qQ < 0$). This vector field is an example of a **force field**.

Gradient Fields

If f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^2$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}}$$

Example

Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f.

Solution

The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} = 2xy\,\hat{\mathbf{i}} + (x^2 - 3y^2)\,\hat{\mathbf{j}}$$

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation, f is called a potential function for $\mathbf{F}$.

17.2 Line Integrals

If f is defined on a smooth curve C, then the **line integral of** f **along** C is

$$\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if the limit exists. Since f is defined as a continuous function, the following formula can be used to evaluate the line integral:

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

Example

Evaluate $\int_C (2 + x^2y)ds$, where C is the upper half of the unit cirlce $x^2 + y^2 = 1$.

Solution

We first need parametric equations to represent C. Recall that the unit circle can be parametrized by means of the equation

$$x = \cos t \qquad y = \sin t$$

and the upper half of the circle is represented by the parameter interval $0 \le t \le \pi$. Therefore,

$$\int_C (2 + x^2y)ds = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}dt$$

$$= \int_0^{\pi} (2 + \cos^2 t \sin t)dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^{\pi}$$

$$2\pi + \frac{2}{3}$$

The **line integrals of f along C with respect to x and y** are obtained by replacing Δs_i with either Δx_i or Δy_i. The **line integral with respect to arc length** can be written also in respect to x and y as shown below:

$$\int_C f(x, y)dx = \int_a^b f(x(t), y(t))x'tdt$$

$$\int_C f(x, y)dy = \int_a^b f(x(t), y(t))y'tdt$$

The parametrization of a line segment follows the formula below:

$$r(t) = (1 - t)r_0 \qquad 0 \le t \le 1$$

Example

Evaluate $\int_C y^2dx + xdy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. **Solution** (a)A parametric representiation for the line segment is

$$x = 5t - 5 \qquad y = 5t - 3 \qquad 0 \le t \le 1$$

Use the parametrization of a line segment formula with $r_0 = \langle -5, -3\rangle$ and $r_1 = \langle 0, 2\rangle$. Then $dx = 5dt$, $dy = 5dt$, and

$$\int_C y^2dx + xdy = \int_0^1 (5t - 3)^2(5dt) + (5t - 5)(5dt)$$

$$= 5\int_0^1 (25t^2 - 25t + 4)$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}$$

(b) Since the parabola is given as a function of y, let's take y as the parameter and write C_2 as

$$x = 4 - y^2 \qquad y = y \qquad -3 \le y \le 2$$

Then $dx = -2ydy$, and we have

$$\int_C y^2dx + xdy = \int_{-3}^2 y^2(-2y)dy + (4 - y^2)dy$$

$$= \int_{-3}^2 (-2y^3 - y^2 + 4)$$

$$= \left[-\frac{y^4}{4} - \frac{y^3}{3} + 4y \right]_{-3}^{2} = 40\frac{5}{6}$$

If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** as

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$$

Example

Evaluate $\int_C y \sin z ds$, where C is the circular helix given by the equation $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$. **Solution** Using the line integral of f along C with respect to y

$$\int_C y \sin z ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$$

$$= \int_0^{2\pi} (\sin t) \sin t \sqrt{\sin^2 t + \cos^2 t + 1} dt$$

$$= \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)dt = \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin 2t \right]_0^{2\pi} = \sqrt{2}\pi$$

Line Integrals in Vector Fields

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $r(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot dr = \int_a^b \mathbf{F}(r(t)) \cdot r'(t) = \int_C F \cdot T ds$$

Example

Find the work done by the force field $\mathbf{F}(x,y) = x^2\, \hat{\mathbf{i}} - xy\, \hat{\mathbf{j}}$ in moving a particle along the quarter-cirrcle $r(t) = \cos t\, \hat{\mathbf{i}} + \sin t\, \hat{\mathbf{j}}$. $0 \le t \le \pi/2$ **Solution** Since $x = \cos t$ and $y = \sin t$, we have

$$F(r(t)) = \cos^2 t\, \hat{\mathbf{i}} - \cos t \sin t\, \hat{\mathbf{j}}$$

$$r'(t) = -\sin t\, \hat{\mathbf{i}} + \cos t\, \hat{\mathbf{j}}$$

Therefore, the work done is

$$\int_C \mathbf{F} \cdot dr = \int_0^{\pi/2} F(r(t)) \cdot r'(t) = \int_0^{\pi/2} (-2\cos^2 t \sin t)dt$$

$$= 2 \left[\frac{\cos^3 t}{3} \right]_0^{\pi/2} = -\frac{2}{3}$$

The connection between lien integrals of vector fields and line integrals of scalar fields can be noted with the connection below:

$$\int_C \mathbf{F} \cdot dr = \int_C Pdx + Qdy + Rdz \qquad where F = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$$

17.3 The Fundamental Theorem for Line Integrals

Let C be a smooth curve given by the vector function $r(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variable whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f = f(r(b)) - f(r(a))$$

Independence of Path

Suppose C_1 and C_2 are two piecewise-smooth curves(which are called **paths**) that have the same initial point A and terminal point B.

$$\int_{C_1} \nabla f \cdot dr = \int_{C_2} \nabla f \cdot dr$$

F is a continuous vector field with domain D, we say that the line integral $\int_c F \cdot dr$ is **independent of path** if $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$ for any two paths with the The *conversative* vector field depends only on the initial point and terminal point of a curve, and $\int_{C_2} F \cdot dr = 0$f for every closed path C in D.

If $F(x, y) = P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}}$ is a conservative vector field, where P and Q have continuous first-order partial derviatives on a domain D, then throughoout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

A **simple curve** is a curve that doesn't intersect itself anywhere between its endpoints (where $r(a) = r(b)$). A **simply-connected region** is the plane in which every simple closed curve in D encloses only points that are in D. The theorem mentioned before can be used to see if a function is conservative. If the equation is met, then we can say F is conservative

Example

Determine whether or not the vector field

$$F(x, y) = (x - y)\,\hat{\mathbf{i}} + (x - 2)\,\hat{\mathbf{j}}$$

is conservative. **Solution** Let $P(x, y) = (x - y)$ and $Q(x, y) = (x - 2)$. Then

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

Since $\partial P/\partial y \neq \partial Q/\partial x$, F is not conservative.

If F is defined to be conservative, we can write $F = \nabla f$. The **potential energy** of an object at a point is defined as $F = -\nabla P$. An equation relating work and potential energy is shown below

$$W = \int_C F \cdot dr = -\int_C \nabla P \cdot dr$$

17.4 Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derviatives on an open region that contains D, then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Example

Find $\oint_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$ where C is the circle $x^2 + y^2 = 9$.

Solution

The region D bounded by C is the disk $x^2 + y^2 \leq 9$, so let's change to polar coordinates after applying Green's Theorem: $\oint_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$

$$= \iint_D \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4 + 1}) - \frac{\partial}{\partial y}(3y - e^{\sin x}) \right]$$

$$= \int_0^{2\pi} \int_0^3 (7 - 3) r dr d\theta$$

$$= 4 \int_0^{2\pi} \int_0^3 r dr = 36\pi$$

Green's Theorem gives the following forumlas for the area of D:

$$A = \oint_C x dy = -\oint_C y dx = \frac{1}{2} \oint_C x dy - y dx$$

17.5 Curl and Divergence

Curl

If $F = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of $P, Q, andR$ all exist, then the **curl** of **R** is the vector field on $\mathbb{R}^3$ defined by

$$curlF = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

The easiest way to remember this formula is of the symbolic expression

$$curl\mathbf{F} = \nabla \times \mathbf{F}$$

If f is a function of three variables that has continuous second-rorder partial deriviates,

$$curl(\nabla f) = 0$$

Example

If $F(x, y, z) = xz\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} - y^2k\,\hat{\mathbf{k}}$, find curl $\mathbf{F}$.

Solution

Using the equation, we have

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\hat{\mathbf{i}} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\hat{\mathbf{j}}$$
$$+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\hat{\mathbf{k}}$$
$$= (-2y - xy)\,\hat{\mathbf{i}} - (0 - x)\,\hat{\mathbf{j}} + (yz - 0)\,\hat{\mathbf{k}}$$
$$= -y(2 + x)\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + yz\,\hat{\mathbf{k}}$$

Divergence

If $P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

$$div\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

A more condensed form is $div\mathbf{F} = \nabla \cdot \mathbf{F}$

Example

If $F(x, y, z) = xz\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} - y^2\,\hat{\mathbf{k}}$, find div $\mathbf{F}$.

Solution

By the definition of divergence, we have

$$\begin{aligned} div\mathbf{F} = \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) \\ &= z + xz \end{aligned}$$

If $F = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$div(curl\mathbf{F} = 0)$$

17.5.1 Vector Forms of Green's Theorem

1. $\int_C \mathbf{F} \cdot dr = \iint_D (curlF) \cdot k\,dA$
2. $\int_C F \cdot nds = \iint_D div\mathbf{F}(x, y)\,dA$

17.6 Parametric Surfaces and Their Areas

We can describe a surface by a vector function $r(u, v)$ of two parameters u and v.

$$r(u, v) = x(u, v)\,\hat{\mathbf{i}} + y(u, v)\,\hat{\mathbf{j}} + z(u, v)\,\hat{\mathbf{k}}$$

x, y, z are the component functions of $\mathbf{r}$. The set of component functions and (u, v) varies throughout D, is called **parametric surface S** and the component functions are called the **parametric equations**. If we keep v constant by putting $v = v_0$, we get a curve C_2 given by $r(u, v_0)$ that lies on S. This is called a **grid curve**.

Example

Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \leq z \leq 1$$

Solution

The cylinder has a simple representation $r = 2$ in cylindrical coordinates, so we choose as a parameters θ and z in cylindrical coordinates. Then the parametric equations of the cylinder are

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = z$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$

Chapter 18

Second-Order Differential Equations

18.1 Second-Order Linear equations

A **second-order differential equation** has the form

$$P(x)\frac{d^y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

The base auxillary equation can be given in the form

$$ay'' + by' + cy = 0$$

. The equation can also be written as

$$ar^2 + br + c = 0$$

Determinants of the auxillary equation:

1. Case 1: $b^2 - 4ac > 0$ $\qquad y = c_1e^{r^1x} + c_2e^{r^2x}$
2. Case 2: $b^2 - 4ac = 0$ $\qquad y = c_1e^{r^1x} + c_2xe^{r^2x}$
3. Case 3: $b^2 - 4ac > 0$ $\qquad y = e^{\alpha x}(c_1cos\beta x + c_2sin\beta x)$
   $\alpha = \frac{-b}{(2a)}$ and $\beta = \frac{\sqrt{4ac-b^2}}{(2a)}$

Example

Solve the equation $y'' + y' - 6y = 0$ with the initial values $y(0) - 1$, $y'(0) = 0$

Solution

The auxillary equation is

$$r^2 + r + 6 = (r+3)(r-2) = 0$$

So the solution is

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Differentiatng this, we get

$$y' = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

Satisfying initial conditions

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = 2c_1 - 3c_2 = 0$$

$$c_1 = \frac{3}{5} \qquad c_2 = \frac{2}{5}$$

$$y = \frac{3}{5}e^{2x} + \frac{2}{5}e^{-3x}$$

18.2 Nonhomogeneous Linear Equations

Method of Undetermined Coefficients

If the differential equation is given in the form $ay'\prime + by' + cy = G(x)$, then the solution is given by

$$y(x) = y_p(x) + y_c(x)$$

where $y_c(x)$ is the general solution and $y_p(x)$ is the particular solution.

Example

Solve the equation $y'\prime + y' - 2y = x^2$

Solution

The auxillary equation becomes

$$r^2 + r - 2 = (r-1)(r+2) = 0$$

which gives a complementary solution of

$$y_c = c_1 e^x + c_2 e^{-2x}$$

Since $G(x)$ is a polynomial of the 2nd degree, this gives a particular solution

$$y_p(x) = Ax^2 + Bx + C$$

$$y_p'(x) = 2Ax + B$$
$$y_p''(x) = 2A$$

Substituting into the original differential equation gives

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

$$A = \frac{-1}{2} \qquad B = \frac{-1}{2} \qquad C = \frac{-3}{4}$$

which gives the particular solution

$$y_p(x) = \frac{-1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

so the final solution is

$$y = c_1 e^x + c_2 e^{-2x} + \frac{-1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

Method of Variation of Parameters

If we have already solved the homogeneous equation $ay'\prime + by' + cy = 0$ in the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

We can replace the constants and write this with an arbitrary function $u(x)$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$
$$y'(x) = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

Applications of Second-Order Differential Equations

If we ignore any external resisting forces, then, by Newton's Second Law,

$$m\frac{d^2x}{dy^2} = -kx$$

The general solution can be written as

$$x(t) = c_1 cos(wt) + c_2 sin(wt)$$

or

$$x(t) = Acos(wt) + \delta$$

where

$$w = \sqrt{k/m}A = \sqrt{c_1^2 + c_2^2}cos\delta = \frac{c_1}{A} \qquad sin\delta = \frac{-c_2}{A}$$

Dampened vibrations follow the form:

$$m\frac{d^2x}{dy^2} + c\frac{dy}{dx} + kx = 0$$

The roots of the equation follow different cases and show different things about the damping.

1. Case 1: $c^2 - 4mk > 0$ (overdamping)
2. Case 2: $c^2 - 4mk = 0$ (critical damping)
3. Case 3: $c^3 - 4mk < 0$ (underdamping)

In electrical circuits, a second-order differential equation that relates resistors, inductors and conductors and their sum being equal to the supplied voltage is as follows:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + Q/C = E(t)$$

18.3 Series Solution

When differential equations can not be applied explicity, we use the method of power series; in the solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c + 0 + c_1 x + c_2 x^2 + \dots$$

Example

Use the power series to solve the equation $y'' + y = 0$

Solution

Differentiating the power series above gives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + \dots$$

$$y'\prime = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = 2c_2 x + 6c_3 x + \dots$$

Substituting expressions...

$$\sum_{n=0}^{\infty} ((n+2)(n+1)c_{n+1} + c_n) x^n = 0$$

The corresponding coefficients must be equal to

$$c_{n+2} = \frac{-c_0}{((n+1)(n+2))}$$

By plugging in values for n, the pattern disovered shows
For even coefficients: $c_{2n} = \frac{(-1)^n c_)}{(2n)!}$ **For odd coefficients**: $c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}$

Part II

Linear Algebra

Chapter 1

Vectors

1.1 The Geometry and Algebra of Vectors

Vector: a directed line segment that cooresponds to a displacement from one point A to another point B.
Vector Addition: $u + v = [u_1 + v_1, u_2 + v_2]$

Example

If $u = [3, -1]$ and $v = [1.4]$, compute $u + v$.

Solution

$$u + v = [3 + 1, -1 + 4] = [4, 3]$$

Scalar Multiplication: $cv = c[v_1, v_2]$

Example

If $v = [2, -4]$, compute $2v$, $\frac{1}{2}v$ and $-2v$

Solution

$$2v = [2(-2), 2(4)] = [-4.8]$$
$$\frac{1}{2}v = [\frac{1}{2}(2), \frac{1}{2} - 4] = [1, -2]$$
$$-2v = [-2(2), -2(-4)] = [-4, 8]$$

Algebraic Properties of Vectors in $\mathbb{R}^n$

Let u, v, and w be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

1. $u + v = v + u$
2. $(u + v) + w = u + (v + w)$
3. $u + 0 = u$
4. $u + (-u) = 0$
5. $c(u + v) = cu + cv$
6. $(c + d)u = cu + du$
7. $c(du) = (cd)u$
8. $1u = u$

1.2 Length and Angle: The Dot Product

If

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$$

and

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$$

then the dot product of $u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

Example

Compute $u \cdot v$ when

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

and

$$\vec{v} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

$$u \cdot v = 1 \cdot (-3 + 2(5) + (-3)2 = 1)$$

Let u, v, and w be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

1. $u \cdot v = v \cdot u$
2. $(u \cdot v) \cdot w = u \cdot (v \cdot w)$
3. $(cu) \cdot v = c(u \cdot v)$
4. $u \cdot u \geq 0$ and $u \cdot u = 0$ if any only if $u = 0$.

The length of vector v in $\mathbb{R}^n$ is the nonnegative scalar $||v||$ defined by

$$||v|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{(v_1)^2 + (v_2)^2 + \cdots + (v_n)^2}$$

Theorem: Let v be a vector in $\mathbb{R}^n$ and let c be a scalar. Then

1. $||v|| = 0$ if and only if $v = 0$.
2. $||cv|| = |c|||v||$

Cauchy-Schwarz Inequality: For all vectors u and v, $||u \cdot v|| \leq ||u||||v||$ **Triangle Inequality**: $||u + v|| \leq ||u|| + ||v||$ For nonzero vectors u and v in $\mathbb{R}^n$,

$$cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}$$

$$proj_u(\vec{v}) = (\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}})\vec{u}$$

1.3 Lines and Planes

$$d(B, l) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$

$$d(B, P) = \frac{|ax_0 + by_0 - cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Chapter 2

Systems of Linear Equations

2.1 Introduction to Systems of Linear Equations

Recall that the general equation of a line in $\mathbb{R}^2$ is of the form

$$ax + by = c$$

and the general equation of a plane $\mathbb{R}^3$ is of the form

$$ax + by + cz = d$$

Equations of this form called **linear equations**.

A **linear equation** in the n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the **coefficient** and the **constant term** b are constants.

Example

Solve the system

$$\begin{aligned} x - y - z &= 2 \\ y + +3z &= 5 \\ 5z &= 10 \end{aligned}$$

Solution

Using the bottom equation, we get $z = 2$, and then plug it into the middle equation. From $y + 6 = -5 \to y = -1$. From the first equation, $x = 3$.

2.2 Direct Methods for Solving Linear Systems

The **coefficient matrix** contains the coefficients of the variables, and the **augmented matrix** is the coefficient matrix augmented by an extra column containing the constant terms. A matrix is in **row echelon form** if it satisfies the following properties:

1. Any rows consisting entirely of zeroes are at the bottom.
2. In each nonzero row, the first nonzero entry(called the **leading entry**) is in a column to the left of any leading entries below it.

Gaussian Elimination

When row reduction is applied to the augmented matrix of a system of a linear equations, we create an equivalent system that can be solved by back substitution. The entire process is known as **Gaussian elimination**.

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to reduce the augmented matrix to row echelon form.
3. Using back subsitution, solve the equivalent system that corresponds to the row-reduced matrix.

The **rank** of a matrix is the number of nonzero rows in its row echelon form.

Example

Solve the system:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 - 2x_3 &= 1 \end{aligned}$$

Solution

Gauss-Jordan Elimination

Steps for this:

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to reduce the augmented matrix to row echelon form.
3. If the resulting system is consistent, solve for the leading variables in terms of any remaining free variables.

A matrix is in **reduced row echelon form** if it satisfies the following properties:

1. It is in row echelon form
2. The leading entry in each nonzero row is a 1(called a leading 1).
3. Each column containing a leading 1 has zeroes everywhere else.

Example

Determine whether the lines $x = p + su$ and $x = q + tv$ intersect and, ifso find their point of intersection when

$$p = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, andv = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

Solution

$x = p + su = q + tv$, From this, we find out that $s = \frac{5}{4}, t = \frac{3}{4}$. The point of intersection is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{5}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} \\ \frac{5}{4} \\ \frac{1}{4} \end{bmatrix}$$

A system of linear equations is called **homogeneous** if the constant term in each equation is zero.

2.3 Spanning Sets and Linear Independence

Is the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

Solution

We want to find scalars x and y such that
x $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
Expanding, we obtain the system

$$x - y = 1$$

$$y = 2$$

$$3x - 3y = 3$$

Whose augmented matrix is

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ )0 & 0 & 0 \end{bmatrix}$$

. So the solution is $x = 3$, $y = 2$. If $S = \{v_1, v_2, \ldots, v_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations is called the **span** of $v_1, v_2, \ldots, v_k$ and is denoted by $span(v_1, v_2, \ldots, v_k)$. If $span(S) = R^n$, then S is called a **spanning set** for $\mathbb{R}^n$.

2.4 Applications

The combustion of ammonia (NH_3) in oxygen produces nitrogen (N_2) and water. Find a balanced chemical equation for this reaction. $wNH_3 + xO_2 \rightarrow yN_2 + zH_2O$
Nitrogen: $w = 2y$
Hydrogen: $3w = 2z$
Oxygen: $2x = z$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{bmatrix}$$

rref$\rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-2}{3} & 0 \\ 0 & 1 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & \frac{-1}{3} & 0 \end{bmatrix}$$

$$w = \frac{2}{3}z, x = \frac{1}{2}z, y = \frac{1}{3}z$$

$$w = 4, x = 3, y = 2, z = 6$$

$$4NH_3 + 3O_2 \rightarrow 2N_2 + 6H_2O$$

Chapter 3

Matricies

3.1 Matrix Operations

Properties of a Matrix

A **matrix** is a rectangular array of numbers called the entries, or elements of the matrix. The **size** of a matrix is a description of the number of rows and columns it has. A matrix can be described by $m \times n$, with m rows and n columns.

Matrix Addition

The sum of matrix A and B is obtained by adding the corresponding entries.

$$A + B = [a_{ij} + b_{ij}]$$

NOTE: Matricies can be only added together if they have the same dimensions.

Matrix Multiplication

If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then the product $C = AB$ is an $m \times r$ matrix.

Partitioned Marticies: We can partition a matrix A into submatricies, making the matrix a collection of submatricies. Partitions are represented by a —.

Transpose of a Matrix

The **Transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A. Find the transpose of $A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$

The transpose is $A^T = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}$.
A square matrix is **symmetric** if $A^T = A$.

3.2 Matrix Algebra

Algebraic Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be matricies of the same size and let c and d be scalars. Then

a $A + B = B + A$

b $(A + B) + C = A + (B + C)$

c $A + O = A$

d $A + (-A) = O$

e $c(A + B) = cA + cB$

f $c(dA) = (cd)A$

g $1A = A$

A **linear combination** of matricies looks like

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k$$

Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

. Is $B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$ a linear combination of A_1, A_2, and A_3?

We want to find scalars c_1, c_2, c_3 such that the equation aboive is satisfied equal to B.

$$c_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$$

Comparing entries and using the definition of matrix equality, we have four linear equations

$$\begin{aligned} c_2 + c_3 &= 1 \\ c_1 \qquad + c_3 &= 4 \end{aligned}$$

$$-c_1 \qquad + c_3 = 2$$
$$c_2 + c_3 = 1$$

Placing these entries in a matrix, and using Gauss-Jordan elimnation, we get

$$\begin{bmatrix} 1 & 0 & 0| & 1 \\ 0 & 1 & 0| & -2 \\ 0 & 0 & 1| & 3 \\ 0 & 0 & 0| & 0 \end{bmatrix}$$

so this gives $c_1 = 1$, $c_2 = -2$, and $c_3 = 3$.
The **span** of a set of matricies is the set of all linear combinations of the matricies. If asked to describe the span of the matricies A_1, A_2, and A_3 as in the previous example, a good way to do this would be to write out a general linear combination of these matricies equal to a generic matrix.
After solving the matrix, make sure there are solutions for every possible row.

Properties of Matrix Multiplication

Let A, B, and C be matricies and let k be a scalar. Then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $k(AB) = (kA)B = A(kB)$
5. $I_m A = A = AI_n$ if A is $m \times n$

Properties of a Transpose

Let A and b matricies and k be a scalar. Then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = k(A^T)$
4. $(AB)^T = A^T B^T$
5. $(A^r)^T = (A^T)^r$

3.3 Inverse of a Matrix

If A is an $n \times n$ matrix, an **inverse** of A is an $n \times n$ matrix A' with the property that

$$AA' = I$$

and

$$A'A = I$$

where $I = I_n$ is the $n \times n$ identity matrix. If such an A' exists, then A is called **invertible**.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if $detA \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Find the inverses of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 12 & -15 \\ 4 & -5 \end{bmatrix}$, if they exist.

Solution

We have $detA = 1(4) - 2(3) = -2 \neq 0$, so A is invertible, with

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

On the other hand, $detB = 12(-5) - (-15)4 = 0$, so B is not invertible.

Properties of Invertible Matricies

a If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1}) = A$

b If A is an invertible matrix, and c is a nonzero scalar, then cA is an invertible matrix and

$$(cA)^{-1} = \frac{1}{c} A^{-1}$$

c If A and B are invertible matricies of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

d If A is an invertible matrix, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

e If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

3.4 LU Factorization

Let A be a square matrix. A factorization of A as $A = LU$, where L is the unit lower triangular, and U is upper triangular, is called an **LU factorization**.

Row reduction of A proceeds as follows:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} = U \quad (1)$$

The three elementary matrices E_1, E_2, E_3 that accomplish this reduction of A to echelon form U are (in order):

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Hence,

$$E_3E_2E_1A = U$$

Solving for A, we get

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}U = LU$$

Thus, A can be factored as

$$A = LU$$

NOTE: The row multipliers used in the preceding example are the entries of L that are below its diagonal.

P^TLU Factorization

This method is an alternative form of the LU factorization in which it handles row changes during Gaussian elimination. P is called the **permutation matrix**.

Let A be a square matrix. A factorization of A as $A = P^TLU$. This called a factorization of A.

Example

Find a $P^T LU$ factorization of $A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$.

Solution

After reduing A to row echelon form, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 6 \end{bmatrix}$$

We have used two row interchanges ($R_1 \leftrightarrow R_2$ and then $R_2 \leftrightarrow R_3$), so the required permutation matrix is

$$P = P_2 P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

We now find an LU factorization of PA.

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{bmatrix} = U$$

Hence $L_{21} = 2$, and so

$$A = P^T LU = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{bmatrix}$$

3.5 Subspaces, Basis, Dimension, and Rank

Subspaces

A **Subspace** of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that

1. The zero vector 0 in S.
2. If u and v are in S, then $u + v$ is in S. (S is ***closed under addition***.)
3. If u is in S and c is a scalar, then cu is in S. (S is ***closed under scalar multiplication***.)

Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Then $span(v_1, v_2, \ldots, v_k)$ is a subspace of $\mathbb{R}^n$.
Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace row(A) of $\mathbb{R}^n$ spanned by the rows of A.

2. The **column space** of A is the subspace col(A) of $\mathbb{R}^n$ spanned by the columns of A.

Example

Consider the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$.

a Determine whether $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the column space of A.

b Determine whether $w = \begin{bmatrix} 4 & 5 \end{bmatrix}$ is in the row space of A.

c Describe $row(A)$ and $col(A)$.

Solution

(a) We augment the matrix $[A|b]$ to determine if b is in col(A). After row reduction within the augmented matrix,

$$\begin{bmatrix} 1 & 0| & 3 \\ 0 & 1| & 2 \\ 0 & 0|0 & \end{bmatrix}$$

Since the system is consistent, and has a unique solution, **b** is in $col(A)$.
(b) By solving the augmented matrix, $[\frac{A}{\mathbf{w}}]$, and apply elementary row operations to reduce it to the form $\frac{A'}{0}$, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

These calculations show that w is in $row(A)$.
Let A be an $m \times n$ matrix. The **null space** of A is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $Ax = 0$. It is denoted by $null(A)$.

Basis

A **basis** for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that

1. span S and

2. is linearly independent.

Steps for finding the row, column, and the null spaces of a matrix A

1. Find the rref form R of A.
2. Use the nonzero row vectors of R to form a basis for $row(A)$.
3. Use the column vectors of A that correspond to the columns of R containing the leading 1s to form a basis for $col(A)$.
4. Solve for the leading variables of $Rx = 0$ in terms of the free variables, set the free variables equal to parameters, substitute back into x, and write the result as a linear combination of f vectors where f is the number of free variables. These f vectors form a basis for $null(A)$.

Dimension and Rank

The Basis Theorem
Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

If S is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for S is called the **dimension** of S, denoted $dimS$. Since the standard basis for $\mathbb{R}^n$ has n vectors, $dimR^n = n$.

The **rank** of a matrix A is the dimension of its row and column spaces and is denoted by $rank(A)$. The**nullity** of a matrix A is the dimension of its null space and is denoted by $nullity(A)$.

Coordinates

Let S be a subspace of $\mathbb{R}^n$ and let $\beta = \{v_1, v_2, \ldots, v_k\}$ be a basis for S. Let v be a vector in S and write $v = c_1v_1 + c_2v_2 + \cdots + c_kv_k$. Then $c_1, c_2, \ldots, c_k$ are called the **coordinates of v with respect to** β and the column vector

$$[v]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

is called the **coordinate vector of v with respect to** β.

Example

Let $\epsilon = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$. Find the coordinate vector of
$v = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$
with respect to ϵ.

Solution

Since $v = 2e_1 + 7e_2 + 4e_3$,

$$[v]_\epsilon = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

3.6 Intro to Linear Transformations

A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector v in $\mathbb{R}^n$ a unique vector $T(v)$ in $\mathbb{R}^n$.

Linear Transformations

1. $T(u + v) = T(u) + T(v)$
2. $T(cv) = cT(v)$

Theorem: Let $T : R^m \to R^n$ and $S : R^n \to R^p$ be linear transformations. Then $S \circ T : R^m \to R^p$ is a linear transformation. Moreover, their standard matricies related by
$[S \circ T] = [S][T]$ **Inverse Transformations**: Let S and T be linear transformations from $\mathbb{R}^n$ to R_n. Then S and T are inverse transformations if $S \circ T = I_n$ and $T \circ S = I_n$.

Chapter 4

Eigenvalues and Eigenvectors

4.1 Introduction to Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there is a nonzero vector x such that $Ax = \lambda x$. Such a vector x is called an **eigenvector** of A corresponding to λ.

The **eigenspace** of λ is denoted by E_λ and is the collection of all eigenvectors corresponding to each other.

Example

Show that $\lambda = 6$ is an eigenvalue of $A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$ and find a basis for its eigenspace.

Solution

$A - 6I = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 7 \\ 2 & 2 & -4 \end{bmatrix}$

Row reduction produces

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which we can see that the null space of this is nonzero. Hence, 6 is an eigenvalue of A, and the eigenvectors corresponding to this eigenvalue satisfy

$x_1 + x_2 - 2x_3 = 0$. It follows that

$$E_6 = \{\begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix}\} = \{x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\} = span\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\}$$

4.2 Determinants

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the **determinant** of A is the scalar $detA = |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a31 & a_{32} \end{vmatrix}$. This equation can also be wrote as

$$\sum_{j=1}^{3}(-1)^{1+j} a_{1j} detA_{1j}$$

The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = a_{ij}$, where $n \geq 2$ can be computed as

$$detA = a_{i1}C_{i1} + \cdots + a_{in}C_{in} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

(which is the **cofactor expansion along the ith row**) and also as

$$detA = a_{1j}C_{1j} + \cdots + a_{nj}C_{nk} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(the **cofactor expansion along the jth column**)

Example

Compute the determinant of the matrix

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

by (a) cofactor expansion along the third row and (b) cofactor expansion along the second column

Solution

(a) We compute

$$detA = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= 2\begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & 3 \\ 1 & 0 \end{vmatrix} = 5$$

(b) In this case, we have

$$detA = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= -(-3)\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 0\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} = 5$$

Cramer's Rule and the Adjoint

Let A be an invertibel $n \times n$ matrix and let **b** be a vector in R^n. Then the unique solution **x** of the system Ax = b is given by

$$x_i = \frac{det(A_i(b))}{detA} \qquad for i = 1, \ldots, n$$

Example

Use Cramer's Rule to solve the system

$$x_1 + 2x_2 = 2$$

$$-x_1 + 4x_2 = 1$$

Solution

We compute

$$detA = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 6, \qquad det(A_1(b)) = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 6, and \qquad det(A_2(b)) = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$$

By Cramer's Rule,

$$x_1 = \frac{det(A_1(b))}{detA} = \frac{6}{6} = 1 and x_2 = \frac{det(A_2(b))}{detA} = \frac{3}{6} = \frac{1}{2}$$

The matrix

$$[C_{ji}] = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix}$$

is called the **adjoint** of A and is denoted by adj A.

4.3 Eigenvalues and Eigenvectors of $n \times n$ matricies

The eigenvalues of a square matrix A are precisely the solutions λ of the equation

$$det(A - \lambda I) = 0$$

Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $det(A - \lambda I) of A$.
2. Find the eigenvalues of A by solving the characteristic equation $det(A - \lambda I) = 0 for \lambda$.
3. For each eigenvalue λ, find the null space of the matrix $A - \lambda I$. This is the eigenspace E_λ, the nonzero vectors of which are the eigenvectors of A corresponding to λ.
4. Find a basis for each eigenspace.

Refer back to Section 4.1 for an example. **Theorem**: The eigenvalues of a triangular matrix are the entries on its main diagonal.

4.4 Similarity and Diagonalization

Let A and B be $n \times n$ matricies. We say that A **is similar to** B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A \sim B$.

Let A and B be $n \times n$ matricies with $A \sim B$. Then

a $detA = detB$

b A is invertible if and only if B is invertible

c A and B have the same rank.

d A and B have the same characteristic polynomial.

e A and B have the same eigenvalues.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$.

Solution

Then $A \sim B$, since

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Thus, $AP = PB$.

Diagonalization

An $n \times n$ matrix is A is diagonalizable if there is a diagonal matrix D such that A is similar to D, that is, if there is an invertble matrix P such that $P^{-1}AP = D$.

Example

Find if $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is diagonalizable.

Solution

$AP = PD$, From this, since $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, A is diagonalizable. **The Diagonalization Theorem** Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_2, \ldots, \lambda_k$. The following statements are equivalent.

a A is diagonalizable.

b The union β of the bases of the eigenspaces of A contains n vectors.

c The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

Gerschgorin's Disk Theorem

Let $A = [a_{ij}]$ be a real or complex $n \times n$ matrix. Then every eigenvalue of A is contained within a Gerschgorin disk. Let r_i denote the sum of the absolute values of the off-diagonal entries in the *ith* row of A; that is, $r_i = \sum_{j \neq i} |a_{ij}|$. The **ith Gerschgorin disk** is the circular disk D_i in the complex plane with center a_{ii} and radius r_i. That is,

$$D_i = \{zin\mathfrak{C} : |z - a_{ii}| \leq r_i\}$$

Example

Sketch the Gerschgorin disks and the eigenvalues for the following matrix.

$$A = \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}$$

Solution

The Gerschgorin disk is centered at 2 with radii 1. The characteristic polynomial of A is $\lambda^2 + \lambda - 8$

The Perron-Frobenius Theorem

Let A be an irreducible nonnegative $n \times n$ matrix. Then A has a real eigenvalue λ with the following properties:

a $\lambda_1 > 0$

b λ_1 has a corresponding positive eigenvector.

c λ has any other eigenvalue of A, then $|\lambda| \leq \lambda_1$. If A is primitive, then this inequality is strict.

d λ is an eigenvalue of A such that $|\lambda| = \lambda_1$, then λ is a complex root of the equation$\lambda^n = \lambda_1^n = 0$

e λ_1 has algebraic multiplicity 1.

Chapter 5

Orthogonality

5.1 Orthogonal and Orthonormal Sets of Vectors

A set of vectors in $\mathbb{R}^n >$ is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. If $v_1, v_2, \ldots, v_k$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then these vectors are linearly independent.

A set of vectors in $\mathbb{R}^n$ is an northonormal set if it is an **orthogonal set** of unit vectors. An **orthonormal basis** for a subspace W is a basis of W.

Orthogonal Marticies

The columns of $m \times n$ matrix Q form an orthonormal set if any only if

$$Q^T Q = I_n$$

An $n \times n$ matrix Q whose column form an orthonormal set is called an **orthogonal matrix**.
Theorem: A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.

Example

Show that the follow matrix is orthogonal

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution

$$A^{-1} = A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Properties of an orthogonal matrix

1. $Q^{-1} isorthogonal$
2. $detQ \pm 1$
3. If λ has an eigenvalue of Q, then $|\lambda| = 1$.
4. If Q_1 and Q_2 are orthogonal $n \times n$ matricies, then so is $Q_1 Q_2$.

5.2 Orthogonal Complements and Orthogonal Projections

Properties of an Orthogonal Complement

Let W be a subspace of $\mathbb{R}^n$.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. $(W^\perp)^\perp = W$
3. $W \cap W^\perp = 0$
4. If $W = span w_1, \ldots, w_k$, then v is in $W^\perp$.

Orthogonal Projections

Let W be a subspace of $\mathbb{R}^n$ and let $u_1, \ldots, u_k$ be orthogonal basis for W. For any vector $\vec{v}$ in $\mathbb{R}^n$, the **the orthogonal project of v onto W** is given by

$$proj_W(v) = (\frac{u_1 \cdot v}{u_1 \cdot u_1})u_1 + \cdots + (\frac{u_k \cdot v}{u_k \cdot u_k})u_k$$

The **component of v orthogonal to W** is the vector

$$perp_W(v) = v - proj_W(v)$$

Example

Let $v = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ and an orthogonal basis for W be $\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$. Find the orthogonal project of v onto W and the component of v orthogonal to W.

Solution

$$proj_W(v) = (\frac{u_1 \cdot v}{u_1 \cdot u_1})u_1 + \cdots + (\frac{u_k \cdot v}{u_k \cdot u_k})u_k$$

$$u_1 \cdot v = 7 \qquad u_2 \cdot v = 0 \qquad u_1 \cdot u_1 = 2 \qquad u_2 \cdot u_2 = 3$$

$$proj_W(v) = \frac{7}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 1 \\ - \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{7}{2} \\ frac72 \end{bmatrix}$$

$$perp_W(v) = v - proj_W(v) = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{7}{2} \\ frac72 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{-3}{2} \end{bmatrix}$$

The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and let v be a vector in $\mathbb{R}^n$. Then there are unique vectors w in W and $\mathbf{w}^\perp$ in $W^\perp$ such that

$$v = w + w^\perp$$

If W is a subspace of $\mathbb{R}^n$, then $dimW + dimW^\perp = n$

The Rank Theorem: If A is an $m \times n$ matrix, then

$$rank(A) + nullity(A) = n$$

5.3 The Gram-Schmidt Process and the QR Factorization

5.3.1 The Gram-Schmidt Process

Let $x_1, \ldots, x_K$ be a basis for a subspace of W of $\mathbb{R}^n$ and define the following.

$$v_1 = x_1 \qquad W_1 = span(x_1)$$

$$v_2 = x_2 - (\frac{v_1 \cdot x_2}{v_1 \cdot v_1})v_1 \qquad W_2 = span(x_1, x_2)$$

$$v_3 = x_3 - (\frac{v_1 \cdot x_3}{v_1 \cdot v_1})v_1 - (\frac{v_2 \cdot x_3}{v_2 \cdot v_2})v_2 \qquad W_3 = span(x_1, x_2, x_3)$$

$$\vdots$$

$$v_k = x_k - (\frac{v_1 \cdot x_k}{v_1 \cdot v_1}v_1) - (\frac{v_2 \cdot x_k}{v_2 \cdot v_2})v_2 - \cdots - (\frac{v_{k-1} \cdot x_k}{v_{k-1} \cdot v_{k-1}}v_{k-1}) \qquad W_k = span(x_1, \ldots, x_k)$$

Example

Apply the Gram-Schmidt Process to construct an orthonormal basis for the subspace $W = span(x_1, x_2, x_3) of R^4$, where $x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

Solution

We begin by setting $v_1 = x_1$.

$$v_1 = x_1 = \begin{bmatrix} v_1 \cdot x_2 = 11 \\ -1 \\ 1 \end{bmatrix} \qquad v_1 \cdot v_1 = 3$$

Next, we compute the component of x_2 orthogonal to W_1.

$$v_2 = x_2 - (\frac{v_1 \cdot x_2}{v_1 \cdot v_1})v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$v_1 \cdot x_3 = -1 \qquad v_2 \cdot x_3 = \frac{19}{3} \qquad v_2 \cdot v_2 = \frac{14}{3}$$

$$\begin{aligned} v_3 &= x_3 - (\frac{v_1 \cdot x_3}{v_1 \cdot v_1})v_1 - (\frac{v_2 \cdot x_3)}{v_2 \cdot v_2})v_2 \\ &= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \frac{19}{14} \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{14} \\ \frac{-1}{7} \\ frac314 \end{bmatrix} \end{aligned}$$

$\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{14} \\ \frac{-1}{7} \\ \frac{3}{14} \end{bmatrix} \}$ is an orthogonal basis for W.

QR Factorization

Let A be an $m \times n$ matrix with linearly independent columns. Then A can be factored as $A = QR$, where Q is a $m \times n$ matrix with orthonormal columns and R is an invertible upper triangular matrix.

Example

Find a QR factorization of

$$A = \begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{14} \\ -1 & \frac{4}{3} & \frac{-1}{7} \\ -1 & \frac{1}{3} & \frac{3}{14} \end{bmatrix}$$

Solution

Since the columns of A are the vectors from the prevoius example, an orthogonal basis for $col(A)$ is

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} \qquad v_3 = \begin{bmatrix} \frac{1}{14} \\ \frac{-1}{7} \\ frac314 \end{bmatrix}$$

To obtain an orthogonal basis, we normalize each vector.

$$q_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\end{bmatrix} \qquad Q = [q_1, q_2, q_3]$$

$$q_2 = \frac{v_2}{||v_2||} = \frac{\sqrt{3}}{\sqrt{14}}\begin{bmatrix}\frac{5}{3}\\\frac{4}{3}\\\frac{1}{3}\end{bmatrix} \qquad = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{5}{3}\frac{\sqrt{3}}{\sqrt{14}} & \frac{1}{14\sqrt{14}}\\\frac{-1}{\sqrt{3}} & \frac{4}{3}\frac{\sqrt{3}}{\sqrt{14}} & \frac{1}{-7\sqrt{14}}\\\frac{-1}{\sqrt{3}} & \frac{1}{3}\frac{\sqrt{3}}{\sqrt{14}} & \frac{3}{14\sqrt{14}}\end{bmatrix}$$

$$q_3 = \frac{v_3}{||v_3||} = \frac{1}{\sqrt{14}}\begin{bmatrix}\frac{1}{14}\\\frac{-1}{7}\\frac314\end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{5}{3}\frac{\sqrt{3}}{\sqrt{14}} & \frac{1}{14\sqrt{14}}\\\frac{-1}{\sqrt{3}} & \frac{4}{3}\frac{\sqrt{3}}{\sqrt{14}} & \frac{1}{-7\sqrt{14}}\\\frac{-1}{\sqrt{3}} & \frac{1}{3}\frac{\sqrt{3}}{\sqrt{14}} & \frac{3}{14\sqrt{14}}\end{bmatrix}\begin{bmatrix}1 & \frac{5}{3} & \frac{1}{14}\\-1 & \frac{4}{3} & \frac{-1}{7}\\-1 & \frac{1}{3} & \frac{3}{14}\end{bmatrix} = \begin{bmatrix}\frac{3}{\sqrt{3}} & 0 & 0\\0 & \frac{14}{3}\frac{\sqrt{3}}{\sqrt{14}} & 0\\0 & 0 & \frac{3}{14\sqrt{14}}\end{bmatrix}$$

5.4 Orthogonal Diagonalization of Symmetric Matricies

A square matrix A is **orthogonally diagonalizable** if there is an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$. If A is orthogonally diagonalizable, then A is symmetric.

The Spectral Theorem and Spectral Decomposition

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable. The **spectral decomposition** of A or sometimes referred to as the **projection form of the spectral theorem** is

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

Example

Given $A = \begin{bmatrix}4 & 1\\1 & 4\end{bmatrix}$.

1. Orthogonally diagonalizable the matrix A.
2. Find the spectral decomposition of the matrix A.

Solution

$$det(A - \lambda I) = det \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} = (\lambda - 5)(\lambda - 3) = 0$$

$$\lambda = 3, 5 \to D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(A - 5I)v_1 = 0 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - 3I)v_2 = 0 \qquad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Q = [q_1, q_2] = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad D = Q^T A Q = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T = 5 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{2} & \frac{1}{-2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

5.5 Applications

Quadratic Forms

A quadratic form in n variables is a function of the form

$$f(x) = x^T A x$$

where A is a symmetric $n \times n$ matrix and x is in $\mathbb{R}^n$.
We can represent quadratic forms using matricies as follows:

$$ax^2 + by^2 + cxy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & \frac{c}{2} \\ \frac{c}{2} & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example

What is the quadratic form associated with matrix $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Solution

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \qquad a = 2, b = 5, c = -6$$

$$f(x) = x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2(x_1)^2 + 5(x_2)^2 - 6x_1x_2$$

Example

$f(x_1, x_2, x_3) = 2(x_1)^2 - (x_2)^2 + 5(x_3)^2 + 6x_1x_2 - 3x_1x_3$

Solution

$$A = \begin{bmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{bmatrix} = \begin{bmatrix} 2 & 3 & \frac{-3}{2} \\ 3 & -1 & 0 \\ \frac{-3}{2} & 0 & 5 \end{bmatrix}$$

.

Chapter 6

Vector Spaces

6.1 Vector Spaces and Subspaces

Let V be a set on two operations, called addition and scalar multiplication, have been defined. The *sum* is defined by $u + v$. The scalar multiple of **u** is defined by c.
If the following axioms hold for all u, v, and w in V and for all scalars c and d, then V is called a **vector space** and its elements are called vectors.

1. $u + v$ is in V.
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There exists an element **0** in V, called a **zero vector**, such that $u+0 = u$.
5. For each **u** in V, there is an element **-u** in V such that **u + (-u) = 0**.
6. c**u** is in V.
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. c(du) = (cd)u
10. 1u = u

Properties of Vector Spaces

Let V be a vector space, **u** a vector in V, and c a scalar.

1. 0u = 0
2. c0 = 0

3. (-1)u = -u
4. If cu = 0, then c = 0 or u = 0

Example

Show that the set W of all vectors of the form

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$$

is a subspace of $\mathbb{R}^4$.

Solution

Let u and v be in W

$$v = \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix}$$

Then

$$u + v = \begin{bmatrix} a+c \\ b+d \\ -(b+d) \\ a+c \end{bmatrix}$$

so u + v is also in W because it has the right form. Similarly, if k is a scalar, then

$$ku = \begin{bmatrix} ka \\ kb \\ -kb \\ ka \end{bmatrix}$$

so k**u** is in W.
Since W is closed under vector addition and scalar multiplication, W is a subspace of $\mathbb{R}^4$.

6.2 Linear Independece, Basis, and Dimension

A set of vectors $v_1, v_2, \ldots, v_k$ in a vector space V is **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_k$ at least one of which is not zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

A set of vectors that is not linearly dependent is said to be **linearly independent**.

A set of vectors $v_1, v_2, \ldots, v_k$ in a vector space V is linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

Example

In $\mathfrak{P}^2$, determine whether the set $\{1+x, x+x^2, 1+x^2\}$ is linearly independent.

Solution

Let c_1, c_2, and c_3 be scalars such that

$$c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = 0$$

Then

$$(c_1+c_3) + (c_1+c_2)x + (c_2+c_3)x^2 = 0$$

This implies that

$$\begin{aligned} c_1 + \quad\;\; c_3 &= 0 \\ c_1 + c_2 \quad\;\; &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

the solution to which $c_1 = c_2 = c_3 = 0$. It follows that $\{1+x, x+x^2, 1+x^2\}$ is linearly independent.

Basis

A subset β of a vector space V is a **basis** for V if

1. $\beta spans V$ and

2. β is linearly independent.

If e_i is the ith column of the $n \times n$ identity matrix, then $\{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$ called the **standard basis**.

Coordinates

For every vector v in V, there is exactly one way to write v as a linear combination of the basis vectors in β. Let $\beta = v_1, v_2, \ldots, v_n$ be a basis for a vector space V. Let $\mathbf{v}$ be a vector in V. Then $c_1, c_2, \ldots, c_n$ are called the **coordinates of v with respect to** β and the column vector

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector of v with respect to** β.

Dimension

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ ne a nasis for a vector space V.

1. Any set of more than n vectors in V must be linearly dependent.
2. Any set of fewer than n vectors in V cannot span V.

The Basis Theorem: If a vector space V has a basis with n vectors, then every basis for V has exactly n vectors.

A vector space V is called **finite-dimensional** if it has a basis consitting of finitely many vectors. The **dimension** of V, denoted by dim V, si the number of vectors in a basis for V. The deimension of the zaero vector space is defined to be zero. A vector space that has no finite basis is called **infite-dimensional**.

Let V be a vector space with $dimV = n$. Then

a Any linearly independent set in $V4$ contains at most n vectors.

b Any spanning set for V contains at least n vectors.

c Any linearly independent set of exactly n vectors in V is a basis for V.

d Any spanning set for V consiting of exactly n vectors is a basis for V.

e Any linearly independent set in V can be extended for a basis in V.

f Any spanning set for V can be reduced to a basis for V.

Change of Basis

Let $\beta = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ be bases for a vector space V. The $n \times n$ matrix whose columns are the coordinate vectors $[u_1]_C, \ldots, [u_n]_C$ of the vectors in β with respect to C is denoted by
$P_{C \leftarrow \beta}$.
Think of β as the "old basis" and C as the "new basis".

Let $\beta = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ be bases for a vector space V and let $P_{C \leftarrow \beta}$ be the change-of-basis matrix from β to C. Then

a $P_{C \leftarrow \beta}[x]_\beta = [x]_C forallxinV$.

b $P_{C \leftarrow \beta}$ is the unique matrix P with the property that $P[x]_\beta = [x]_C forallxinV$

c P is invertible and $(P_{C \leftarrow \beta})^{-1} = P_{\beta \leftarrow C}$

Example

Find the change of basis matricies $P_{C \leftarrow \beta}$ and $P_{\beta \leftarrow C}$ for the bases $\beta = \{1, x, x^2\}$ and $C = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathfrak{P}_2$. Then find the coordinate vector of $p(x) = 1 + 2x - x^2$ with respect to C.

Solution

$[1{+}x]_\beta = \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad [x{+}x^2]_\beta = \begin{bmatrix}0\\1\\1\end{bmatrix} \qquad [1{+}x^2]_\beta = \begin{bmatrix}1\\0\\1\end{bmatrix} \qquad P_{\beta\leftarrow C} = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 0\\0 & 1 & 1\end{bmatrix}$

$P_{C\leftarrow\beta} = (P_{\beta\leftarrow C})^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2}\end{bmatrix}$

$[p(x)]_C = P_{C\leftarrow\beta}[p(x)]_\beta = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}2\\0\\-1\end{bmatrix}$

6.2.1 The Gauss-Jordan Method for Computing a Change-of-Basis Matrix

Let $\beta = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ be bases for the vector space V. Now let $\beta = \{[u_1]_E, \ldots, [u_n]_E\}$ and $C = \{[v_1]_E, \ldots, [v_n]_E\}$. Then row reduction applied to the $n \times 2n$ augmented matrix [C—B] produces

$$[C|\beta] = [I|P_{C\leftarrow\beta}]$$

Example

In $M_2 2$, let B be the basis$\{E_{11}, E_{21}, E_{12}, E_{22}\}$ and let C be the basis, where $\{A, B, C, D\}$ Find the change-of-basis matrix $P_{C\leftarrow\beta}$ and verify that $[X]_C = P_{C\leftarrow\beta}[X]_\beta$ for $X = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

Solution

$B = P_{\epsilon\leftarrow B} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$ and $C = P_{\epsilon\leftarrow C} = \begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix}$ Row reduction produces

$P_{C\leftarrow\beta} = \begin{bmatrix}1 & 0 & -1 & 0\\0 & -1 & 1 & 0\\0 & 1 & 0 & -1\\0 & 0 & 0 & 1m\end{bmatrix}$

6.3 Linear Transformations

A **linear transformation** from a vector space V to a vector space W is a mapping $T : V \to W$ such that, for all u and v in V and for all scalars c,

1. $T(u + v) = T(u) + T(v)$

2. $T(cu) = cT(u)$

$T : V \to W$ is a linear transformation if any only if $T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = c_1T(v_1) + c_2T(v_2) + \cdots + c_kT(v_k)$.

Example

Define $T : M_{nn} \to M_{nn} by T(A) = A^T$. Show that T is a linear transformation.

Solution

We check that, for A and B in M_{nn} and scalars c.

$$T(A + B) = (A + B)^T = A^T + B^T = T(A) + T(B)$$

and

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T is a linear transformation.
Let $T : V \to W$ be a linear transformation. Then,

1. $T(0) = 0$
2. $T(-v) = -T(v)$
3. $T(u - v) = T(u) - T(v)$

Composition of Linear Transformations

If $T : U \to V$ and $S : V \to W$ are linear transformations, then the **composition of S with T** is the mapping $S \circ T$, defined by

$$(S \circ T)(u) = S(T(u))$$

where u is in U.

Inverses of Linear Transformations

A linear transformation $T : W \to V$ is invertible if there is a linear transformation $T' : V \to W$ such that

$$T' \circ T = I_V$$

and

$$T \circ T' = I_W$$

In this case, $T' is called an inverse for T$

6.4 Kernel and Range of a Linear Transformation

The **kernel** of T, denoted $ker(T)$, is the set of all vectors in V that are mapped by T to 0 in W. That is,

$$ker(T) = \{vinV : T(v) = 0\}$$

The **range** of T, denoted $range(T)$, is the set of all vectors in W that are images of vectors in V under T. That is,
$range(T) = \{T(v) : vinV\}$
$= \{winW : w = T(v)forsomevinV\}$

Example

Find the kernel and range of the differential operator $D : \mathfrak{P}_3 \to \mathfrak{P}_2$ defined by $D(p(x)) = p'(x)$.

Solution

$$\begin{aligned} ker(D) = \{a + bx + cx^2 + dx^3 : D(a + bx + cx^2 + dx^3) = 0\} \\ \{a + bx + cx^2 + dx^3 : b + 2cx + 3dx^2 = 0\} \\ \{a + bx + cx^2 + dx^3 : b = c = d = 0\} \\ = \{a : ain\mathfrak{R}\} \end{aligned}$$

$range(D)$ in all polynomials in P_2

Rank and Nullity

The **rank** of T is the dimension of range of T and is denoted by rank(T). The **nullity** of T is dimension of the kernel of T and is denoted by nullity(T).

Example

Find the rank and the nullity of the linear transformation $D : \mathfrak{P}_3 \to \mathfrak{P}_2$ defined by $D(p(x)) = p'(x)$.

Solution

rank(D) = dim $\mathfrak{P}_2 = 3$
nullity(D) = dim(ker(D)) = 1
Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V into a vector space W. Then

$$rank(T) + nullity(T) = dimV$$

One-to-One and Onto Linear Transformations

A linear transformation $T : V \to W$ is called **one-to-one** if T maps distinct vectors in V to distinct vectors in W. If range(T) $= W$, then T is called **onto**.

a $T : V \to W$ is one-to-one of, for all u and v in V, $u \neq v$ implies that $T(u) \neq T(v)$.

b $T : V \to W$ is one-to-one of, for all u and v in V, $T(u) = T(v)$ implies that $u = v$.

c $T : V \to W$ is onto if, for all u and v in V, there is at least one v in V such that $w = T(v)$

A linear transformation $T : V \to W$ is one-to-one if and only if $ker(T) = \{0\}$.

Let $dimV = dimW = n$. Then a linear transformation $T : V \to W$ is one-to-one if and only if it is onto.

Isomorphisms of Vector Spaces

A linear transformation $T : V \to W$ is called an **isomorphism** if it is one-to-one and onto. If V and W are two vector spaces such that there is an isomorphism from V to W, then we say V is **isomorphic** to W and write $V = W$.

Example

Let W be the vector space of all symmetric 2×2 matricies. Show that W is isomorphic to $\mathbb{R}^3$.

Solution

W is represented by the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, so $dimW = 3$.

The Matrix of a Linear Transformation

Let V and W be two finite-dimensional vector spacfes with bases B and C, respectively, where B $= \{v_1, \ldots, v_n\}$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by
$A = [T(v_1)_C][T(v_2)_C] \ldots [T(v_n)_C$

Example

Let $T : R^3 \to R^n$ be the linear transformation defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$

and let $B = \{e_1, e_2, e_3\}$ and $C = \{e_2, e_1\}$ be bases for R_3 and R_2, respectively. Find the matrix of T with respect to B and C.

Solution

$$T(e_1) = T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}, \qquad T(e_2) = T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}, \qquad T(e_3) = T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix}$$

$$e_2 + e_1 = a\begin{bmatrix}0\\1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \qquad [T(e_1)]_C = \begin{bmatrix}1\\1\end{bmatrix}$$

$$e_2 + e_1 = a\begin{bmatrix}0\\1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix} \qquad [T(e_2)]_C = \begin{bmatrix}1\\-2\end{bmatrix}$$

$$e_3 + e_1 = a\begin{bmatrix}0\\1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-3\end{bmatrix} \qquad [T(e_2)]_C = \begin{bmatrix}-3\\0\end{bmatrix}$$

$$A = [T(e_1)]_C[T(e_2)]_C[T(e_3)]_C = \begin{bmatrix}1 & 1 & -3\\1 & -2 & 0\end{bmatrix}$$

$$A[v]_B = [T(v)]_C \qquad v_B = \begin{bmatrix}1\\3\\-2\end{bmatrix}$$

$$[T(v)]_C = \begin{bmatrix}-5\\10\end{bmatrix}_C = \begin{bmatrix}10\\-5\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -3\\1 & -2 & 0\end{bmatrix}\begin{bmatrix}1\\3\\-2\end{bmatrix} = \begin{bmatrix}10\\-5\end{bmatrix}$$

Matricies of Composite and Inverse Linear Transformations

Let U, V, and W be finite-dimensional vector spaces with bases B, C, and D, respectively. A linear transformation $T : U \to V$ and $S : V \to W$ be linear transformation. Then

$$[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C}[T]_{C \leftarrow B}$$

Example

The linear transformation $T : R^2 \to \mathfrak{P}_1$ defined by $T\begin{bmatrix}a\\b\end{bmatrix} = a + (a+b)x$ were shown to be one to one and onto and hence invertible. Find T^-1.

6.5 Applications

Homogeneous Libear Differential Equations

Let S be the solution space of $y'' + ay' + by = 0$
and let λ_1 and Λ_2 be the roots of the characteristic equation $\lambda^2 + a\lambda + b = 0$

a If $\lambda_1 \neq \lambda_2$, then $\{e^{\lambda_1 t}, e^{\lambda_2} t\}$ is a basis for S.

b If $\lambda_1 \neq \lambda_2$, then $\{e^{\lambda_1 t}, te^{\lambda_1} t\}$ is a basis for S.

Therefore, the solutions are

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} and y = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

Example

Find all solutions of

$$y'' - 5y' + 6y = 0$$

Solution

Make the Equation in terms of lambda.

$$y'' - 5y' + 6y = 0 \rightarrow \lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0 \qquad \lambda = 2, 3$$

Basis is $\{e^{2t}, e^{3t}\}$ $\qquad y = c_1 e^{2t} + c_2 e^{3t}$

Chapter 7

Distance and Approximation

7.1 Inner Product Spaces

We can define the dot product $u \bullet v$ of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$. An **inner product** on a vector space V is an operation that assigns to every pair of vectors $\vec{u}$ and $\vec{v}$ in V a real number $< u, v >$ such that the following properties hold for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and all scalars c.

A vector space with an inner product is called an inner product space. $< u, v >$

7.1.1 Properties of Inner Products

Let u, v, and w be vectors in an inner product space V and let c be a scalar.

1. $< u + v, w > = < u, w > + < v, w >$
2. $< u, cv > = c < u, v >$
3. $< u, 0 > = < 0, v > = 0$

7.1.2 Length, Distance, and Orthogonality

Let u and v be vectors in an inner product space V.

1. The **length** (or **norm**) of v is $||v|| = \sqrt{< v, v >}$
2. The **distance** between u and v is $d(u, v) = ||u - v||$.
3. u and v are **orthogonal** if $< u, v > =$.

Example

Consider the inner product on $\mathfrak{L}[0,1]$. If $f(x)=x$ and $g(x)=3x-2$, find (a)$||f||$ (b)$d(f,g)$ (c) $< f,g >$

Solution

1. $$< f,f >= \int_0^1 f^2(x)dx = \int_0^1 x^2dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$
   so $||f|| = \sqrt{< f,f >} = \frac{1}{\sqrt{3}}$

2. $d(f,g) = ||f-g|| = \sqrt{< f-g, f-g >}$ and
   $$f(x) - g(x) = x - (3x-2) = 2 - 2x$$
   we have
   $$< f-g, f-g >= \int_0^1 (f(x)-g(x))^2dx = 4[x - x^2 + \frac{x^3}{3}]_0^1 = \frac{4}{3}$$
   $$d(f,g) = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

3. $< f,g >= \int_0^1 f(x)g(x)dx = \int_0^1 (3x^2-2x)dx = [x^3-x^2]_0^1 = 0$

Thus, f and g are orthogonal.

7.1.3 Pythagoras' Theorem

Let u and v be vectors in an inner product space V. Then u and v are orthogonal if and only if
$$||u+v||^2 = ||u||^2 + ||v||^2$$

7.1.4 Legendre Polynomials

Construct an orthogonal basis for P with respect to the inner product
$$< f,g >= \int_{-1}^1 f(x)g(x)dx$$

Let $x_1 = 1$, $x_2 = x$, $x_3 = x^2$ We begin by setting $v_1 = x_1 = 1$. Next we compute
$$< v_1, v_1 >= \int_{-1}^1 dx = x\Big|_{-1}^1$$
and
$$< v_1, x_2 >= \int_{-1}^1 xdx = \frac{x^2}{2}\Big|_{-1}^1 = 0$$

Therefore,

$$v_2 = x_2 - \frac{< v_1, x_2 >}{< v_1, v_1 >} v_1 = x$$

To find v_3, we first compute

$$< v_1, x_3 >= \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$< v_2, x_3 >= \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

$$< v_2, v_2 >= \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

Then,

$$v_3 = x_3 - \frac{< v_1, x_3 >}{< v_1, v_1 >} v_1 - \frac{< v_2, x_3 >}{< v_2, v_2 >} v_2 = x^2 - \frac{\frac{2}{3}}{2}(1) - \frac{0}{\frac{2}{3}}(x) = x^2 - \frac{1}{3}$$

The polynomials 1, x, $x^2 - \frac{1}{3}$ are the first three **Legendre Polynomials**.

7.1.5 Cauchy-Schwarz and Triangle inequalities

Let u and v be vectors in an inner product space V. Then
Cauchy-Schwarz: $| < u, v > | \leq ||u|| ||v||$
Triangle: $||u + v|| \leq ||u|| + ||v||$

7.2 Norms and Distance Functions

A norm on a vector space V is a mapping that associates with each vector $\vec{v}$ a real number $||v||$, called the **norm** of v, such that the following properties are satisfied for all vectors $\vec{u}$ and $\vec{v}$ and all scalars c:

1. $||v|| \geq 0$, and $||v|| = 0$ if any only if $v = 0$.
2. $||cv|| = |c| ||v||$
3. $||u + v|| \leq ||u|| + ||v||$

A vector space with a norm is called a **normed linear space**.

7.2.1 Vector Norms

1. The **Sum Norm** is the sum of the absolute values of its components. $||v_s|| = |v_1| + \cdots + |v_n|$
2. The **Max Norm** is the largest number along the absolute values of its components. $||v_m|| = max(|v_1|, \ldots, |v_n|)$
3. The **Euclidean Norm** is the value of the distance between the two vectors.

Example

Let $u =$

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

and

$$v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Compute $d(u, v)$ relative to (a) the Euclidean norm, (b) the sum norm, and (c) the max norm.

Solution

Each calculation requires knowing that $u - v =$

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

1. $d_E(u, v) = ||u - v||_E = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$
2. $d_S(u, v) = ||u - v||_S = |4| + | - 3| = 7$
3. $d_m(u, v) = ||u - v||_m = max[|4|, | - 3|] = 4$

7.2.2 Matrix Norms

A **Matrix Norm** on M_{nn} is a mapping that associates with each matrix A, called the norm of A, and satisfies the following properties.

1. $||A|| \geq 0$ and $||A|| = 0$ if any only if $A = O$.
2. $||cA|| = |c|||A||$
3. $||A + B|| \leq ||A|| + ||B||$
4. $||AB|| \leq ||A||||B||$

7.2.3 Condition Number of a Matrix

A matrix A is ill-conditioned if small changes in its entries can produce large changes in the solutions of $A\mathbf{x} = \mathbf{b}$. The relative error of measured the condition number is defined as $\frac{||\Delta x||}{||x'||} \leq cond(A) \frac{||\Delta A||}{||A||}$

7.3 Least Squares Approximation

The Best Approximation Theorem If W is a subscape of a normed linear space V and if v is a vector in V, then the best approximation to v in W such that

$$||v - \vec{v}|| < ||v - w||$$

for every vector w in W different from $\vec{v}$.
The Least Squares Theorem Let A be an $m \times n$ and let $\mathbf{b}$ be in $\mathbb{R}^n$. Then $A\mathbf{x} = \mathbf{b}$ always has at least one least squares solution. Moreover,

1. $\bar{x}$ is a least square solution of $A\mathbf{x} = \mathbf{b}$ if any only if $\bar{x}$ is a solution of the normal equations $A^T A\bar{x} = A^T\mathbf{b}$.

2. A has linearly independent coloumns if any only if $A^T A$ is invertible. In this case, the least squares solution is unique and is given by $\bar{x} = (A^T A)^{-1} A^T\mathbf{b}$.

Least Squares via the QR Factorization $\bar{x} = R^{-1}Q^T\mathbf{b}$
Orthogonal Projection can be written as a byproduct of the least squares method. $proj_w(v) = A(A^T A)^{-1}A^T\mathbf{v}$

7.3.1 Penrose Conditions

The **pseudoinverse** of A is defined by $A^+ = (A^T A)^{-1}A^T$. Note that if A is $m \times n$, then A^+ is $n \times m$. The **Penrose Conditions** for A are as follow

1. $AA^+A = A$

2. $A^+AA^+ = A^+$

3. AA^+ and A^+A are symmetric.

7.4 The Singular Value Decomposition

7.4.1 Singular Values of a Matrix

If A is an $m \times n$, the **singular values** of A are the square roots of the eigenvalues of $A^T A$ and are denoted by $\sigma_1, \ldots, \sigma_n$.

7.4.2 Singular Value Decomposition

Let A be an $m \times n$ with singular values $\sigma_1, \ldots, \sigma_n$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$. Then there exist an $m \times m$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and an $m \times n$ matrix sigma such that

$$A = U \sum V^T$$

Example

Find a singular value deconomposition for the following matrix. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and find that its eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = 0$, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

. Since these vectors are orthogonal, we normalize them to obtain

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

. The singular values of A are $\sigma_1 = \sqrt{2}$, $\sigma_2 = \sqrt{1} = 1$, and $\sigma_3 = 0$. Thus,

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$\sum = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

To find U, we compute

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These vectors form an orthonormal basiss, so we have

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This yields the SVD where $A = U \sum V^T$. **The Outer Product Form of the SVD**

$$a = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$$

Byproduct of Penrose Conditions $A^+ = V\sum^+ U^T$ where $\sum^+$ is the $n \times m$ matrix $\begin{bmatrix} D^{-1} & O \\ O & O \end{bmatrix}$

The Fundamental Theorem of Invertible Matricies: Final Version

1. A is invertible.
2. $Ax = b$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
3. $Ax = 0$ has only the trivial solution.
4. The reduced row echelon form of A is I_n.
5. A is a product of elementary matricies.
6. $\mathbb{R}ank(A) = n$
7. $nullity(A) = 0$
8. The column vectors of A are linearly independent.
9. The column vectors of A span $\mathbb{R}^n$.
10. The column vectors of a form a basis for $\mathbb{R}^n$.
11. The row vectors of A are linearly independent.
12. The row vectors of A span $\mathbb{R}^n$.
13. The row vectors of a form a basis for $\mathbb{R}^n$.
14. $detA \neq 0$
15. 0 is not an eigenvalue of A.
16. T is invertible.
17. T is one-to-one.
18. T is onto.
19. $ker(T) = 0$
20. $\mathbb{R}ange(T) = W$
21. 0 is not a singular value of A.

7.5 Applications

Example

Find the best linear approximation to $f(x) = e^x$ on the interval $[-1, 1]$.

Solution

$< f, g >= \int_{-1}^{1} f(x)g(x)dx$. A basis for $P_1[-1,1]$ is given by $1, x$. Since $< 1, x >= \int_{-1} 1xdx = 0$, this is an orthogonal basis, so the best approximation to f in W is

$$
\begin{aligned}
g(x) = proj_W(e^x) &= \frac{< 1, e^x >}{< 1, 1 >}1 + \frac{< x, e^x >}{< x, x >}x \\
&= \frac{\int_{-1}^{1} 1 \bullet e^x dx}{\int_{-1}^{1} 1 \bullet 1 dx} + \frac{\int_{-1}^{1} xe^x dx}{\int_{-1}^{1} x^2 dx}x \\
&= \frac{e - e^{-1}}{2} + \frac{2e^{-1}}{\frac{2}{3}}x \approx 1.18 + 1.10x
\end{aligned}
$$

Example

Find the fourth-order Fourier approximation to $f(x) = x$ on $[-\pi, \pi]$

Solution

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = \frac{1}{2\pi}[\frac{x^2}{2}]_{-\pi}^{\pi} = 0 \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} xcoskxdx = \frac{1}{\pi}[\frac{x}{k}sinkx + \frac{1}{k^2}coskx]_{-\pi}^{\pi} = 0 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} xsinkxdx = \frac{1}{\pi}[\frac{-x}{k}coskx + \frac{1}{k^2}sinkx]_{-\pi}^{\pi} \\
&= \frac{2(-1)^{k+1}}{k}
\end{aligned}
$$