Chapter 2

Matrix Algebra

Determinant of Matrix

In linear algebra, the determinant is a value associated with a square matrix. It can be computed from the entries of the matrix by a specific arithmetic expression. The determinant provides important information when the matrix is that of the coefficients of a system of linear equations. The system has a unique solution if and only if the determinant is nonzero.

Matrix $A = (\quad) = [\quad]$

square $m \times m$ square $n \times n$

Determinant

$\det(A) = |A| = |\quad|$

square

2X2 Matrix: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 2x2

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Ex: Find the value of the determinant for the following Matrices

(i) $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

$$\det(A) = |A| = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= (1)(3) - (-1)(2)$$

$$= 3 - (-2) = \boxed{5}$$

(ii) $B = \begin{pmatrix} 3 & -5 \\ 1 & 6 \end{pmatrix}$

$$\det(B) = |B| = \begin{vmatrix} 3 & -5 \\ 1 & 6 \end{vmatrix}$$

$$= (3)(6) - (1)(-5)$$

$$= 18 - (-5) = 18 + 5 = 23$$

The Determinant for 3x3 Matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

Second Column

$$= -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

Ex: Find the determinant of the following matrices

(i) $A = \begin{bmatrix} -2 & 1 & 6 \\ 3 & -5 & 4 \\ 9 & 1 & -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -2 & 1 & 6 \\ 3 & -5 & 4 \\ 9 & 1 & -3 \end{vmatrix}$$

$$= -2\begin{vmatrix} -5 & 4 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ 9 & -3 \end{vmatrix} + 6\begin{vmatrix} 3 & -5 \\ 9 & 1 \end{vmatrix}$$

$$= -2(15-4) - (-9-36) + 6(3-(-45))$$

$$= -2(11) + 45 + 6(48) = +311$$

(ii) $B = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 5 \\ 6 & 2 & 1 \end{bmatrix}$

$$|B| = \begin{vmatrix} 1 & -1 & 3 \\ 0 & 1 & 5 \\ 6 & 2 & 1 \end{vmatrix}$$

$$= -0\begin{vmatrix} \; & \; \end{vmatrix} + (1)\begin{vmatrix} 1 & 3 \\ 6 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & -1 \\ 6 & 2 \end{vmatrix}$$

$$= 0 + 1 - 18 - 5(2-(-6))$$

$$= 0 + 1 - 18 - 5(8)$$

$$= 1 - 18 - 40 = -57$$

$3x = 9$

solve for x?

$3x = 9$

$\frac{1}{3} \cdot 3x = \frac{1}{3} \cdot 9$

$(1)x = 3$

$3 \cdot \frac{1}{3} = \frac{1}{3} \cdot 3 = 1$

Inverse Matrix A^{-1}

$$AB = BA = I$$

$$B = A^{-1}$$

$$AA^{-1} = A^{-1}A = I$$

$AB \neq BA$

2x3 3x1

$AI = IA = A$

Identity, unit

$$\begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$$

A^{-1} exists if $|A| \neq 0$

If $|A| = 0$
A has no inverse
Singular matrix
non invertible matrix

$$A \xrightarrow{A^{-1}} X = B \xrightarrow{A^{-1}}$$

$$A^{-1} A X = A^{-1} B$$

$$I X = A^{-1} B$$

$$X = A^{-1} B$$

If $|A| \neq 0$
A has inverse
A is non singular
invertible

How to find A^{-1}

2x2 Matrix $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

① Cofactors Method

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} (\text{Cofactors})^T$$

$$= \frac{1}{|A|} \text{Adjoint}(A)$$

$$\text{Adj}(A) = ((\text{Cofactors}))^T$$

$$\text{Adj}(A) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T$$

$C_{11} = +d$

$C_{12} = -c$

$C_{21} = -b$

$C_{22} = +a$

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T$$

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

② Gauss - Jordan Method

① $(A \mid I)$

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

② Elementry row operations
① Multiplying any row by value
② Adding any two row
③ interchanging any two rows

③ $\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & A^{-1} \end{array}\right)$

Find the inverse Matrix of the following Matrices:

(i) $A = \begin{pmatrix} 1 & 3 \\ 5 & 4 \end{pmatrix}$ $\quad A^{-1} = \frac{1}{|A|} \text{Adj}(A)$

$|A| = \begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix} = (1)(4) - (3)(5) = -11$

$\text{Adj}(A) = \begin{pmatrix} +4 & -5 \\ -3 & +1 \end{pmatrix}^T = \begin{pmatrix} 4 & -3 \\ -5 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{-11} \begin{pmatrix} 4 & -3 \\ -5 & 1 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right) \xrightarrow{-5}$$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -11 & -5 & 1 \end{array}\right) \xrightarrow{\frac{1}{-11}}$$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{5}{11} & \frac{-1}{11} \end{array}\right) \xrightarrow{-3}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -\frac{4}{11} & \frac{3}{11} \\ 0 & 1 & \frac{5}{11} & \frac{-1}{11} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{4}{11} & \frac{3}{11} \\ \frac{5}{11} & \frac{-1}{11} \end{pmatrix}$$

(ii) $A = \begin{pmatrix} 5 & -1 \\ 4 & 3 \end{pmatrix}$ $\quad |A| = 15 - (-4) = 19$

$$A^{-1} = \frac{1}{19} \begin{pmatrix} 3 & 1 \\ -4 & 5 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

① Cofactors:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{Adj}(A) = (\text{Cofactors})^{T} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

$C_{11} = +\, ei - hf$

$C_{12} = -\,(di - gf)$

$C_{13} = +\, dh - ge$

$C_{21} = -\,(bi - hc)$

$C_{22} = +\, ai - gc$

$C_{23} = -\,(ah - gb)$

$C_{31} = +\, bf - ec$

$C_{32} = -\,(af - dc)$

$C_{33} = +\, ae - bd$

Gauss Jordan

① $\left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right)$

② Elementary row operations

③ $\left(I \mid A^{-1} \right)$

Ex: $A = \begin{pmatrix} 1 & 5 & 3 \\ 4 & 2 & 1 \\ -1 & 3 & 0 \end{pmatrix}$

$\quad + \quad - \quad +$

$$|A| = (-1)(-1) - 3(-11) + 0(\quad) = 34$$

$$\operatorname{Adj}(A) = \begin{pmatrix} -3 & -1 & 14 \\ 9 & +3 & -8 \\ -1 & +11 & -18 \end{pmatrix}^{T}$$

$$A^{-1} = \frac{1}{34} \begin{pmatrix} -3 & 9 & -1 \\ -1 & 3 & 11 \\ 14 & -8 & -18 \end{pmatrix}$$

Solving the linear system using the inverse matrix

Method:

unique solution

$$\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}$$

① no. of eq = no. of unknowns

② $|A| \neq 0$

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$\boxed{X = A^{-1}B}$$

Solve the following systems using the inverse matrix

$$\boxed{\begin{aligned} x-y+5z &= 5 \\ 3x+y+z &= 5 \\ 2x-y+2z &= 3 \end{aligned}}$$

$$A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}, B = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{Adj}(A) = \begin{pmatrix} +3 & -4 & -5 \\ -3 & -8 & -1 \\ -6 & +14 & +4 \end{pmatrix}^T \qquad |A| = -18$$

$$AX = B$$

$$A^{-1} = \frac{1}{-18}\begin{pmatrix} 3 & -3 & -6 \\ -4 & -8 & 14 \\ -5 & -1 & 4 \end{pmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{-18}\begin{pmatrix} 3 & -3 & -6 \\ -4 & -8 & 14 \\ -5 & -1 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$