Chapter 2 : Point Estimation

- Method of Moments (MOM)

is a method which is estimating population moment by sample moment.

Estimation of moment

$$\mu = E[X] \rightarrow \tilde{\mu}_n = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\mu = E[X^k] \rightarrow \tilde{\mu}_n = \overline{X^k}_n = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

Estimation of central moment

$$\sigma^2 = Var[X] = E[(X - E[X])^2] \rightarrow \tilde{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$\sigma^2 = Var[X] = E[X^2] - E[X]^2 \rightarrow \tilde{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

$$\mu_k = E[(X - E[X])^k] \rightarrow \tilde{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X}_n)^k$$

Ex. *Suppose* $X_i's$ *be a indepedent sample random variables with PDF given by*

$$f(x|\beta) = \frac{\beta}{x^{\beta+1}} \; ; x > 1$$

What is the MOM estimator for β

Sol^n

Ex. *Suppose* $X_i's \overset{iid}{\sim} Beta(\alpha = \theta, \beta = 1)$. *What is MOM estimator for* θ ? Midterm 63

Sol^n

Ex. *Suppose* $X_i \sim Bin(n = i, p)$ *for* $i = 1, \ldots, r$ *and* $X_i's$ *are independent R.V.* ; *moreover,* $\sum_{i=1}^{r} X_i \sim Bin\left(n = \frac{r(r+1)}{2}, p\right)$. *Find MOM estimator for p*

Final 66

Sol^n

- Maximum Likelihood Estimators (MLE)

is a method that provide possible parameters which maximize our probability based on a sample

Suppose $X_1, \dots, X_n$ *are iid sample from a population with PDF or PMF*

$$L(\theta|x_i's) = L(\theta_1, \dots, \theta_k|x_i's) = \prod_{i=1}^{n} f(x_i|\theta_1, \dots, \theta_k)$$

$$l(\theta|x_i's) = \log\big(L(\theta|x_i's)\big)$$

Step of finding MLE for parameters

1. $L(\theta|x_i's) = \prod_{i=1}^{n} f(x_i|\theta_1, \dots, \theta_k)$
2. $l(\theta|x_i's) = \log\big(L(\theta|x_i's)\big)$
3. $\frac{\partial}{\partial x} l(\theta|x_i's)$
4. $set\ \frac{\partial}{\partial x} l(\theta|x_i's) = 0$
5. $\hat{\theta}_n\ (MLE\ for\ \theta)$

Note 📌

$\log = \log_e \neq \log_{10}$

Ex. *Suppose* $X_i's \overset{iid}{\sim} Bin(m,p)$. *What is the MLE estimator for* p?

Sol^n

Ex. Suppose $X_i's \overset{iid}{\sim} Geo(p)$. What is the MLE estimator for p ?

Sol^n

Ex. *Suppose* $X_i's \overset{iid}{\sim} Uniform(0,\theta)$. *What is the MLE estimator for* θ ?

Sol^n

Ex. *Suppose* $X_i's \overset{iid}{\sim} Uniform(0, \theta)$. *What is the MLE estimator for* θ ?

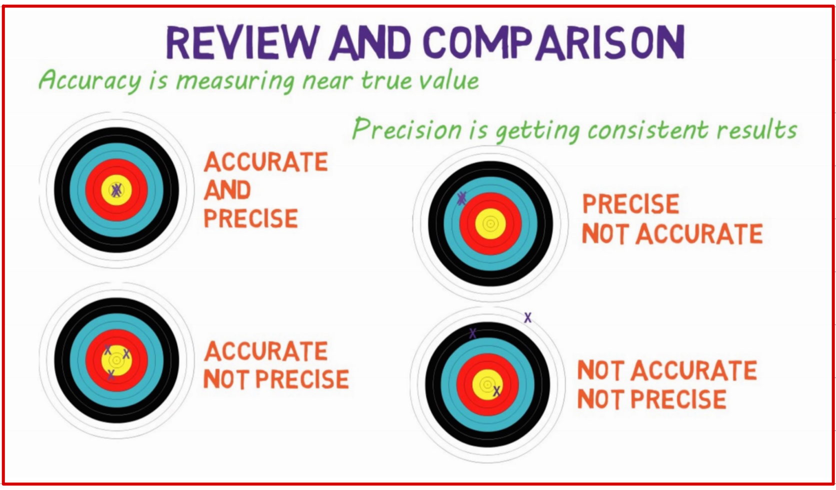
Sol^n

Ex. *Suppose* $X_i's \overset{iid}{\sim} Uniform(2\theta, 0)$ *for* $\theta < 0$. *What is the MLE estimator for* θ ? Midterm 66

Sol^n

- Methods of Evaluating Estimators

Accuracy vs Precision of Estimators



The bias of a point estimator W of parameter theta

$$Bias_\theta(W) = E[W] - \theta$$

- Bias = 0 : unbiased (W is unbiased estimator.)
- Bias > 0 : overestimate
- Bias < 0 : underestimate

The mean square error (MSE) of a point estimator W of parameter theta

$$MSE_\theta(W) = E[(W-\theta)^2] = Var[W] + [Bias_\theta(W)]^2$$

- MSE = Var if and only bias = 0

$$\hat{\theta} \text{ is said to be more precise than } \tilde{\theta} \text{ if } Var[\hat{\theta}] < Var\left[\tilde{\theta}\right]$$

$$\hat{\theta} \text{ is said to be more accurate than } \tilde{\theta} \text{ if } |Bias_\theta(\hat{\theta})| < \left|Bias_\theta\left(\tilde{\theta}\right)\right|$$

Ex. *Suppose* $X \sim Exp(\theta)$ *and* $Y \sim Gamma(\alpha = 2, \beta = \theta)$ *which* $X \perp Y$.

If $\hat{\theta}_1 = X, \hat{\theta}_2 = \dfrac{Y}{2}, \hat{\theta}_3 = \dfrac{X+Y}{3}$ *and* $\hat{\theta}_4 = \dfrac{X+2Y}{6}$ *are an estimator for* θ

1. *Show that* $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ *are unbiased estimators.*
2. *Show that* $\hat{\theta}_4$ *is biased estimators.*
3. *Find* $MSE_\theta(\hat{\theta}_i)$ *for* $i = 1, 2, 3$ *and* 4

Sol^n

True or False ?

Ex. $Suppose\ \hat{\theta}\ and\ \tilde{\theta}\ are\ an\ estimator\ for\ parameter\ \theta.\ Given\ that\ Var[\hat{\theta}] > Var[\tilde{\theta}]$

$and\ \ Bias_{\theta}(\hat{\theta}) < Bias_{\theta}(\tilde{\theta}), we\ can\ show\ that\ \hat{\theta}\ is\ better\ estimator\ than\ \tilde{\theta}.$

Sol^{n}

Ex. *Suppose* $X_i's \overset{iid}{\sim} Uniform(2\theta, 0)$ *for* $\theta < 0$. $\hat{\theta} = X_{(1)}/2$ Midterm 66

Check whether MLE estimator is a unbiased estimator or not.

Sol^n

Relative efficiency

Given two unbiased estimators, $\hat{\theta}_1$ and $\hat{\theta}_2$,

$$RE(\hat{\theta}_1, \hat{\theta}_2) = \frac{Var[\hat{\theta}_2]}{Var[\hat{\theta}_1]}$$

$$RE(\hat{\theta}_1, \hat{\theta}_2)\begin{cases} < 1 \rightarrow \hat{\theta}_1 \text{ is less efficient than } \hat{\theta}_2 \\ = 1 \rightarrow \hat{\theta}_1 \text{ is as efficient as } \hat{\theta}_2 \\ > 1 \rightarrow \hat{\theta}_1 \text{ is more efficient than } \hat{\theta}_2 \end{cases}$$

Ex. Suppose $X_i's \overset{iid}{\sim} Gamma(\alpha = 2, \beta = \theta)$ for $\theta > 0$. $\tilde{\theta}_n$ and $\hat{\theta}_n$ are an estimator from MOM and MLE, respectively. Which one is more efficient? Midterm 62

Sol^n

Asymptotic Evaluation

$\hat{\theta}_n$ *is said to be a consistent estimator of* θ *if*

$$\hat{\theta}_n \xrightarrow{P} \theta \text{ as } n \to \infty \ (\bar{X}_n \text{ use } WLLN)$$

or

$$\lim_{n\to\infty} Bias_\theta(\hat{\theta}_n) = 0 \text{ and } \lim_{n\to\infty} Var[\hat{\theta}_n] = 0$$

Ex. *Suppose* $X_1, \ldots, X_n$ *be independent sample variable with given PDF,*

$$f(x_i|\theta) = \begin{cases} \dfrac{2x_i}{\theta^2} & ; 0 < x_i \le \theta \\ 0 & ; otw. \end{cases}$$

Show that MOM estimator, $\tilde{\theta}_n = \frac{3}{2}\bar{x}_n$, *is a consistent estimator*

Ex. Suppose $X_i \sim Bin(n = i, p)$ for $i = 1, \dots, r$ and $X_i's$ are independent R.V.; moreover, $\sum_{i=1}^{r} X_i \sim Bin\left(n = \frac{r(r+1)}{2}, p\right)$. MOM estimator for p is $\frac{2\sum_{i=1}^{r} X_i}{r(r+1)}$

Final 66

Show that MOM estimator is a consistent estimator.

Asymptotic Relative Efficiency

Given two consistent estimators, $\hat{\theta}_1$ *and* $\hat{\theta}_2$ *of a parameter* θ

$$ARE(\hat{\theta}_1, \hat{\theta}_2) = \lim_{n\to\infty}\left(\frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}\right)$$

Ex. *Suppose* $X_i's \overset{iid}{\sim} Gamma(\alpha = 2, \beta = \theta)$ *for* $\theta > 0$. $\tilde{\theta}_n$ *and* $\hat{\theta}_n$ *are an estimator from MOM and MLE, respectively. Find* $ARE\left(\tilde{\theta}_n, \hat{\theta}_n\right)$

Method of Finding Estimation II

Bayesian Estimation

Bayes Theorem

$$\boxed{P(A|B) = \frac{P(AB)}{P(B)} \rightarrow P(AB) = P(A|B)P(B)}$$

In Bayesian Estimation, we treat parameter θ as a random variable; therefore, $f(x|\theta)$ is a conditional probability.

Joint Distribution Function

$$\boxed{f(x,\theta) = f(x|\theta) \times \underline{f(\theta)}}$$

Prior Distribution Function

Marginal of X

$$\boxed{f(x) = \int_{-\infty}^{\infty} f(x,\theta)\, d\theta = \int_{-\infty}^{\infty} f(x|\theta) f(\theta)\, d\theta}$$

Posterior Distribution Function

$$\boxed{f(\theta|x) = \frac{f(x,\theta)}{f(x)} = \frac{f(x|\theta)f(\theta)}{f(x)}}$$

Ex. Let $X \sim N\left(\mu = 0, \sigma^2 = \frac{1}{\lambda}\right)$ and $Y \sim Exp\left(\theta = \frac{1}{\lambda}\right)$ when $X \perp Y$.

Midterm 66

We assume that the prior distribution of λ is $Gamma(\alpha = 2, \beta = 3)$.

Find $f(x,y), f(\lambda|x,y)$ and $E[\lambda|X,Y]$

Sol^n

Ex. Let $X \sim Geo(p), Y \sim NegBin(r = 5, p)$ and $Z \sim Bin(n = 2, p)$ when $X \perp Y \perp Z$.

We assume that the prior distribution of p is $Beta(\alpha = 2, \beta = 3)$.

Find $f(x, y, z), f(p|x, y, z)$ and $E[p|X, Y, Z]$

Sol^n

- Sufficiency

Sufficient Statistic

$T(\underset{\sim}{X})$ is said to be sufficient for θ if

$$P\left(X_i's \mid T(\underset{\sim}{X}), \theta\right) = P\left(\underset{\sim}{X} \mid T(\underset{\sim}{X})\right)$$

Neyman's Factorization Theorem

$T(\underset{\sim}{X})$ is said to be sufficient for θ if

$$f(x_i's \mid \theta) = g\left(T(\underset{\sim}{x}) \mid \theta\right) h(\underset{\sim}{x})$$

Ex. Suppose $X_i's \overset{iid}{\sim} Ber(p)$. Show that $\sum_{i=1}^{n} X_i$ is a sufficient statistic for p.

Sol^n

Ex. Suppose $X_i's \overset{iid}{\sim} Gamma(\alpha = 2, \beta = \theta)$ for $\theta > 0$.

Show that $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

Midterm 62

Sol^n

Ex. Suppose $X_i's \overset{iid}{\sim} Uniform(\theta, 0)$ for $\theta < 0$

Show that $\min\{X_1, X_2, \dots, X_n\}$ is a sufficient statistic for θ.

Sol^n

Exponential families

PDF or PMF is called an exponential family if

$$f(x|\theta) = h(x)c(\theta)\exp\left\{\sum_{i=1}^{k} w_i(\theta)t_i(x)\right\}$$

where $h(x), c(\theta) > 0$

Note

Uniform Dist.

is not an expo. fam.

Ex. Suppose $X_i's \overset{iid}{\sim} Beta(\alpha = \theta, \beta = 1)$ *for* $\theta > 0$.

Show that $Beta(\alpha = \theta, \beta = 1)$ *is an exponential family.*

Midterm 63

Sol^n

Ex. *Show that* $Bin(n,p)$ *is an exponential family.*

Sol^n