Boğaziçi University
Department of Industrial Engineering

IE 310 - Operations Research

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Course Project Report

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Contents

1. Modeling

1. Sets and Indices

- S: Set of Supply nodes (Layer 1), indexed by i.
- T^1: Set of First Transshipment nodes (Layer 2), indexed by j.
- T^2: Set of Second Transshipment nodes (Layer 3), indexed by k.
- D: Set of Demand nodes (Layer 4), indexed by l.

2. Parameters

- S_i: Supply capacity available at node $i \in S$.
- D_l: Demand quantity required at node $l \in D$.
- c_{ij}: Unit shipping cost from node $i \in S$ to node $j \in T^1$.
- c_{jk}: Unit shipping cost from node $j \in T^1$ to node $k \in T^2$.
- c_{kl}: Unit shipping cost from node $k \in T^2$ to node $l \in D$.

3. Decision Variables

Let the continuous decision variables representing the flow quantities be:

- $x_{ij} \geq 0$: Amount shipped from $i \in S$ to $j \in T^1$.
- $y_{jk} \geq 0$: Amount shipped from $j \in T^1$ to $k \in T^2$.
- $z_{kl} \geq 0$: Amount shipped from $k \in T^2$ to $l \in D$.

4. Objective Function

Minimize the total transportation cost across all layers:

$$\text{Minimize } Z = \sum_{i \in S} \sum_{j \in T^1} c_{ij} x_{ij} + \sum_{j \in T^1} \sum_{k \in T^2} c_{jk} y_{jk} + \sum_{k \in T^2} \sum_{l \in D} c_{kl} z_{kl}$$

5. Constraints

$$\begin{aligned}
\text{Supply Capacity:} \quad & \sum_{j \in T^1} x_{ij} \leq S_i & \forall i \in S \quad & (1) \\
\text{Flow Conservation (Layer 2):} \quad & \sum_{i \in S} x_{ij} = \sum_{k \in T^2} y_{jk} & \forall j \in T^1 \quad & (2) \\
\text{Flow Conservation (Layer 3):} \quad & \sum_{j \in T^1} y_{jk} = \sum_{l \in D} z_{kl} & \forall k \in T^2 \quad & (3) \\
\text{Demand Satisfaction:} \quad & \sum_{k \in T^2} z_{kl} \geq D_l & \forall l \in D \quad & (4) \\
\text{Non-Negativity:} \quad & x_{ij}, y_{jk}, z_{kl} \geq 0 & \forall i, j, k, l \quad & (5)
\end{aligned}$$

2. Implementation

The optimization model formulated in the modeling part was implemented using the Python `pulp` library. The resulting network flow, showing only the active paths (basic variables) and their respective flow quantities, is visualized below in Figure 1.

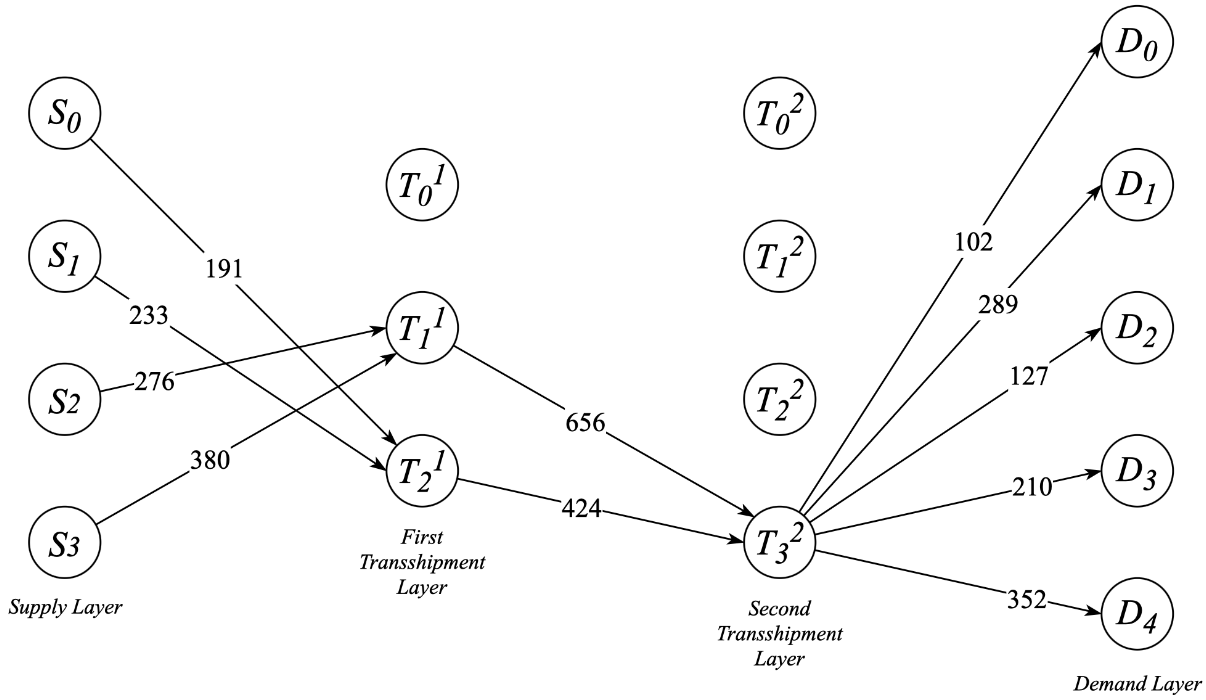

Figure 1: Visual representation of the network.

3. Economic Interpretation and Sensitivity Analysis

a) What if the capacity of S_3 increases by one unit?

In the solution obtained in the implementation part, the flows from the supply nodes are:

$$\sum_{j\in T^1} x_{0j} = 191, \quad \sum_{j\in T^1} x_{1j} = 233, \quad \sum_{j\in T^1} x_{2j} = 276, \quad \sum_{j\in T^1} x_{3j} = 380.$$

The corresponding supply capacities are:

$$S_0 = 191, \quad S_1 = 233, \quad S_2 = 276, \quad S_3 = 385.$$

So, capacity constraints of S_0, S_1 and S_2 are tight, while S_3 has 5 units of unused capacity:

$$\sum_{j\in T^1} x_{3j} = 380 < S_3 = 385.$$

Since the actual flow is strictly less than the available capacity ($380 < 385$), the supply constraint for Node 3 is **non-binding** (it has a slack of 5 units). In linear programming, the shadow price (dual value) of a non-binding constraint is zero. Therefore, increasing the capacity of Supply Node 3 by one unit provides additional surplus to a resource that is already in excess. It will not change the optimal routing or the objective function value.

Expected Change:

$$\Delta Z = 0.$$

b) What if the capacity of S_0 increases by one unit?

The supply constraint for S_0 is **binding** (Flow $x_{0,2} = 191$ equals Capacity $S_0 = 191$):

$$\sum_{j\in T^1} x_{0j} = 191 = S_0 = 191.$$

This suggests that increasing its capacity could reduce the total cost. To estimate the change, the cost of the path used by S_0 against the cost of the current marginal source can be compared.

In part (a) it is observed that S_3 has unused capacity (slack). Therefore, allowing one additional unit of flow from S_0 leads to reducing the flow from S_3 by one unit to maintain the balance of demand.

Path Analysis:

- **Path from S_3 (Current Marginal Source):** The active path is $S_3 \to T^1_1 \to T^2_3$. The cost is $c_{3,1} + c_{1,3} = 17 + 33 = 50$.
- **Path from S_0 (Proposed Source):** The active path is $S_0 \to T^1_2 \to T^2_3$. The cost is $c_{0,2} + c_{2,3} = 11 + 30 = 41$.

Expected Change: By shifting 1 unit of flow from the expensive path (S_3) to the cheaper path (S_0), the total cost is expected to change by:

$$\Delta Z = \text{Cost}_{S0} - \text{Cost}_{S3} = 41 - 50 = -9.$$

The total cost is expected to decrease by **9** units.

c) What if the demand quantity of D_0 increases by one unit?

In the optimal solution, all demand is satisfied exactly by the flow from node T_3^2:

$$z_{3,0} = 102, \quad z_{3,1} = 289, \quad z_{3,2} = 127, \quad z_{3,3} = 210, \quad z_{3,4} = 352.$$

The demand quantities are:

$$D_0 = 102,\ D_1 = 289,\ D_2 = 127,\ D_3 = 210,\ D_4 = 352.$$

Thus, each demand constraint is **binding**, in particular for D_0:

$$\sum_{k \in T^2} z_{k0} = z_{3,0} = 102 = D_0.$$

If we increase the demand of node D_0 by one unit (from 102 to 103), the model must supply one additional unit of flow to D_0. Since there is unused capacity at S_3, it is natural to use this slack to satisfy the extra demand.

Path Analysis: The path is $S_3 \to T_1^1 \to T_3^2 \to D_0$. The total marginal cost is the sum of the unit costs along this path:

- $S_3 \to T_1^1$: $c_{3,1} = 17$
- $T_1^1 \to T_3^2$: $c_{1,3} = 33$
- $T_3^2 \to D_0$: $c_{3,0} = 11$

The per-unit cost of this path is

$$c_{3,1} + c_{1,3} + c_{3,0} = 17 + 33 + 11 = 61.$$

Because no other constraints can be relaxed without incurring extra cost, the cheapest way to provide one more unit to D_0 is precisely to send one extra unit along this path from S_3. Therefore, the marginal (shadow) cost of increasing D_0 by one unit equals this per-unit cost.

Expected Change:

$$\Delta Z = +61.$$

The total cost is expected to increase by **61** units. Consequently, the shadow price for the demand constraint of D_0 is expected to be 61.

d) What if the cost of shipping from S_0 to T_2^1 decreases by one unit?

In the optimal solution from the implementation part, the flow from S_0 to T_2^1 is corresponding to the variable $x_{0,2}$. It is **basic** and strictly positive:

$$x_{0,2} = 191.$$

Its unit cost in the original model is:

$$c_{0,2} = 11.$$

Decreasing the cost of a basic variable makes the current active path strictly more favorable than it was before. Since the supply constraint at S_0 is already binding (all supply is shipped through this link), the flow cannot increased further. Since the link was already optimal at a higher price, it remains optimal at a lower price. Therefore, **no change expected in the basis**.

Expected Change: The total cost will decrease by the reduction amount multiplied by the volume of flow on that link:

$$\Delta Z = (\text{New Cost} - \text{Old Cost}) \times x_{0,2} = (-1) \times 191 = -191.$$

The total cost is expected to decrease by **191** units.

e) What if the cost of shipping from S_0 to T_0^1 decreases by one unit?

In the optimal solution from the implementation part, the flow from S_0 to T_0^1 is corresponding to the variable ($x_{0,0}$). It is **non-basic**:

$$x_{0,0} = 0.$$

To determine if this variable will enter the basis, its **reduced cost** should be checked. The reduced cost represents the amount by which the unit cost $c_{0,0}$ must decrease to make this path competitive with the current optimal path.

Cost Comparison:

- Current active link ($S_0 \to T_2^1$): Unit cost = 11.
- Proposed link ($S_0 \to T_0^1$): Unit cost = 28.

The cost difference in the first layer alone is substantial ($28 - 11 = 17$). A decrease of just 1 unit (reducing $c_{0,0}$ to 27) is insufficient to bridge this gap. The reduced cost of $x_{0,0}$ is undoubtedly greater than 1.

Expected Change: Since the cost reduction is not large enough to make the variable enter the basis, the optimal solution (x^*) remains exactly the same. The variable $x_{0,0}$ continues to have 0 flow.

$$\Delta Z = (\text{New Cost} - \text{Old Cost}) \times x_{0,0} = (-1) \times 0 = 0.$$

There is **no expected change** in the basis or the total cost.

f) What if the same cost of shipping from S_0 to T_0^1 decreases by 27 units instead of one?

In the previous part, the cost difference between the active path and the unused path was on the order of 17 units, so a decrease of 1 unit in $c_{0,0}$ was not enough to change the solution. Now, there is a much larger decrease:

$$c_{0,0}^{\text{new}} = 28 - 27 = 1.$$

In the original optimal solution, all 191 units of supply at S_0 are sent through the arc $S_0 \to T_2^1$, and eventually some of this flow reaches demand node D_3 via T_3^2. A representative path from S_0 to D_3 in the original solution is:

$$S_0 \to T_2^1 \to T_3^2 \to D_3.$$

The per-unit cost of this path is

$$\underbrace{c_{0,2}}_{S_0 \to T_2^1} + \underbrace{c_{2,3}}_{T_2^1 \to T_3^2} + \underbrace{c_{3,3}}_{T_3^2 \to D_3} = 11 + 30 + 12 = 53.$$

After the cost change, the flow can instead be sent from S_0 to D_3 along this cheaper path:

$$S_0 \to T_0^1 \to T_1^2 \to D_3,$$

whose per-unit cost becomes:

$$\underbrace{c_{0,0}^{\text{new}}}_{S_0 \to T_0^1} + \underbrace{c_{0,1}}_{T_0^1 \to T_1^2} + \underbrace{c_{1,3}}_{T_1^2 \to D_3} = 1 + 34 + 13 = 48.$$

For each unit of flow moving from the old path to the new path, the cost decreases by:

$$\text{old path cost} - \text{new path cost} = 53 - 48 = 5.$$

Originally, 191 units of supply from S_0 are shipped along the 53-per-unit route. Rerouting all 191 units to the 48-per-unit route **decreases** the objective value.

Expected Change:

The large cost reduction makes the reduced cost of $x_{0,0}$ negative, so a change in the basis is expected. $x_{0,0}$ enters the basis with positive flow, replacing part of the flow that previously used $x_{0,2}$ and the old route.

$$\Delta Z = (\text{new cost} - \text{old cost}) = -5 \times 191 = -955.$$

g) What if S_0 is no longer allowed to ship to T_2^1?

In the optimal solution, the only active outgoing arc from S_0 is:

$$x_{0,2} = 191.$$

All supply at S_0 is shipped through $S_0 \to T_2^1$. The supply constraint for S_0 is binding:

$$\sum_{j \in T^1} x_{0j} = x_{0,2} = 191 = S_0.$$

Forbidding this shipment effectively imposes a new constraint:

$$x_{0,2} = 0.$$

This removes the currently optimal option from the feasible set. When the optimal choice is removed, the system must resort to a second-best alternative to route the 191 units of supply. By definition, the second-best alternative cannot be cheaper than the optimal one.

Expected Change: It is expected that the total cost **increases**. The model will be forced to use more expensive links (likely diverting flow through T_0^1 or T_1^1 or shifting supply responsibilities to other nodes) to satisfy the demand.

h) What is the maximum amount by which the demand at D_1 can increase?

To determine the maximum possible increase in demand, the total available supply in the system should be compared against the total required demand:

- **Total Supply Capacity:**

$$\sum S_i = 191 + 233 + 276 + 385 = 1085.$$

- **Total Current Demand:**

$$\sum D_l = 102 + 289 + 127 + 210 + 352 = 1080.$$

The system currently has a **surplus capacity** (slack) of:

$$\text{Slack} = \text{Total Supply} - \text{Total Demand} = 1085 - 1080 = 5 \text{ units.}$$

From the optimal solution, it is known that this slack is entirely located at S_3:

$$\sum_{j \in T^1} x_{3j} = 380 < S_3 = 385.$$

If the demand at node D_1 increases by some amount Δ, the total demand becomes:

$$\sum_{l \in D} D_l^{\text{new}} = 1080 + \Delta.$$

For the system to still be able to satisfy *all* demands, this must hold:

$$1080 + \Delta \leq 1085 \quad \Longrightarrow \quad \Delta \leq 5.$$

There are no capacity constraints on the arcs themselves (only on the supply nodes). Therefore, as long as the extra demand does not exceed the remaining 5 units of supply at S_3, the model can reroute additional flow from S_3 through the network to D_1 and still satisfy all demands. Once $\Delta > 5$, the total demand would exceed the total available supply, making the problem infeasible.

Maximum feasible increase:

$$\Delta_{\max} = 5.$$

4. Integrality

In the implemented model, all decision variables are defined as *continuous* and non-negative. However, in the optimal solution, all flow values turn out to be **integers**. This is not a coincidence, but a well-known property of *network flow* models such as transportation and transshipment problems. The main reasons are:

- The constraints of the model are all of the form:
  - supply constraints (sum of flows $\leq$ integer),
  - flow conservation constraints (sum of inflows = sum of outflows),
  - demand constraints (sum of inflows $\geq$ integer),

  and they correspond to a network structure.
- The right-hand sides (supplies and demands) are all **integers**.
- The constraint matrix of a pure network flow problem is **totally unimodular**. This implies that every basic feasible solution of the linear program is integer whenever the right-hand sides are integer.

Because of total unimodularity and integer supplies/demands, the extreme points of the feasible region automatically have integer components. The simplex method returns one of these extreme points as the optimal solution. Therefore, even though we allow the decision variables to take any continuous values, the structure of the problem forces the optimal solution to be integral.