1. (a)

$$w\begin{bmatrix}x'\\y'\\1\end{bmatrix}=\begin{bmatrix}h_{11}&h_{12}&h_{13}\\h_{21}&h_{22}&h_{23}\\h_{31}&h_{32}&1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{bmatrix}w&0&0\\0&w&0\\0&0&2\end{bmatrix}\begin{bmatrix}x'\\y'\\1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x'\\y'\\1\end{bmatrix}=\begin{bmatrix}w&0&0\\0&w&0\\0&0&2\end{bmatrix}^{-1}\begin{bmatrix}h_{11}&h_{12}&h_{13}\\h_{21}&h_{22}&h_{23}\\h_{31}&h_{32}&1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x_i'\\y_i'\\1\end{bmatrix}=\begin{bmatrix}h_{11}/w&h_{12}/w&h_{13}/w\\h_{21}/w&h_{22}/w&h_{23}/w\\h_{31}/w&h_{32}/w&1/w\end{bmatrix}\begin{bmatrix}x_i\\y_i\\1\end{bmatrix}$$

$$\Rightarrow\begin{cases}\frac{h_{11}x_i+h_{12}y_i+h_{13}}{w}=x_i\\\frac{h_{21}x_i+h_{22}y_i+h_{23}}{w}=y_i\\\frac{h_{31}x_i+h_{31}y_i+h_{33}}{w}=1\end{cases}$$

If we have n pairs of homography data $\{(x_1,y_1),(x_1',y_1')\}$, $\{(x_2,y_2),(x_2',y_2')\},\cdots$, $\{(x_n,y_n),(x_n',y_n')\}$, we can first construct a sub-matrix A_i as

$$A_i=\begin{bmatrix}x_i&y_i&1&0&0&0&0&0&0\\0&0&0&x_i&y_i&1&0&0&0\\0&0&0&0&0&0&x_i&y_i&1\end{bmatrix}$$

If we set x as $x=[\frac{h_{11}}{w},\frac{h_{12}}{w},\frac{h_{13}}{w},\frac{h_{21}}{w},\frac{h_{22}}{w},\frac{h_{23}}{w},\frac{h_{31}}{w},\frac{h_{32}}{w},\frac{1}{w}]^T$, we will have

$$A=\begin{bmatrix}A_1\\A_2\\\vdots\\A_n\end{bmatrix}\in\mathbb{R}^{3n\times9},\text{and }\mathbf{y}=\begin{bmatrix}x_1'\\y_1'\\1\\\vdots\\x_n'\\y_n'\\1\end{bmatrix}\in\mathbb{R}^{3n}$$

(b) To solve least-square problem $x^*=\text{argmin}_x\,||Ax-y||_2$, we can say

$$x^*=\underset{x}{\text{argmin}}\,||Ax-y||_2=\underset{x}{\text{argmin}}\,||Ax-y||_2^2$$

$$\Rightarrow x^*=\underset{x}{\text{argmin}}\,E(x),\text{where }E(x)=(Ax-y)^T(Ax-y)=x^TA^TAx-2x^TA^Ty+y^Ty$$

$$\frac{d}{dx}E(x)=2A^TAx-2A^Ty$$

Since $E(x)$ is in the quadratic form, it is convex, and its minimum value exists at

$$\frac{d}{dx}E(x) = 0 \Rightarrow A^T A x^* = A^T y$$

If $A^T A$ is invertible, the we are able to obtain an unique solution for x^* as

$$x^* = (A^T A)^{-1} A^T y$$

To determine if $A^T A$ is invertible, we need to look at the kernel of $A^T A$ as

$$A^T A x = \vec{0} \Rightarrow x^T A^T A x = \vec{0} \Rightarrow (Ax)^T (Ax) = 0$$

$$\Rightarrow Ax = 0 \Rightarrow \ker(A^T A) = \ker(A)$$

$$\because A = \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & 1 \\ & & & & \vdots & & & & \\ x_n & y_n & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_n & y_n & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_n & y_n & 1 \end{bmatrix}$$

Thus, we can have following conclusions:(1) If $n < 3$ the solution for this least square problem is not unique, since $\text{Rank}(A^T A) < 9$. There will be infinity solutions for this questions.
(2) If $n \geq 3$ there may have unique solution for this least square problem, given n distinct pairs of points are not aligned with each other. If n distinct pairs of points are not aligned with each other, it is trivial to see that all column vectors in A are linearly independent, $\text{Rank}(A^T A) = 9$, and $A^T A$ is invertible.

(c) Partial code in function `main(n)` is shown as following

```
yVec = np.ones((3*n)) # vector y
aMat = np.zeros((3*n,9)) # Matrix A

for i in range(n):
    # form matrix a and y for LR
    yVec[3*i] = XY2[i, 0]
    yVec[3*i+1] = XY2[i, 1]
    aMat[3*i, 0:2] = XY1[i, :]
    aMat[3*i, 2]=1
    aMat[3*i+1, 3:5] = XY1[i, :]
    aMat[3*i+1, 5]=1
    aMat[3*i+2, 6:8] = XY1[i, :]
    aMat[3*i+2, 8]=1

```

```
#solve for x*=argmin||Ax-y||_2
ataMatinv = np.linalg.inv((np.matmul(aMat.T,aMat)))
xVec = np.matmul(np.matmul(ataMatinv , aMat.T), yVec)

#Form homogenous transformation matrix
tMat=np.zeros((3,3))
tMat[0,:]=xVec[0:3]
tMat[1,:]=xVec[3:6]
tMat[2,:]=xVec[6:9]
line1MatHomo = np.array([[u[0],u[1]], \
            [v[0],v[1]],[1,1]], dtype=float)

# calculate corresponding points
line2MatHomo = np.matmul(tMat,line1MatHomo)

# check if new line go out of image
u2Vec = line2MatHomo[0,:]
v2Vec = line2MatHomo[1,:]

# plotting the Fooball image 1 with marker 33
fig2 = plt.figure()
ax2 = fig2.add_subplot(111)
ax2.imshow(img2)
ax2.plot(u2Vec, v2Vec,color='yellow')
ax2.set(title='Football image 2')
ax2.set_adjustable('box-forced')
plt.xlim(0,img2.shape[1])
plt.ylim(0,img2.shape[0])
plt.gca().invert_yaxis()
plt.show()
```

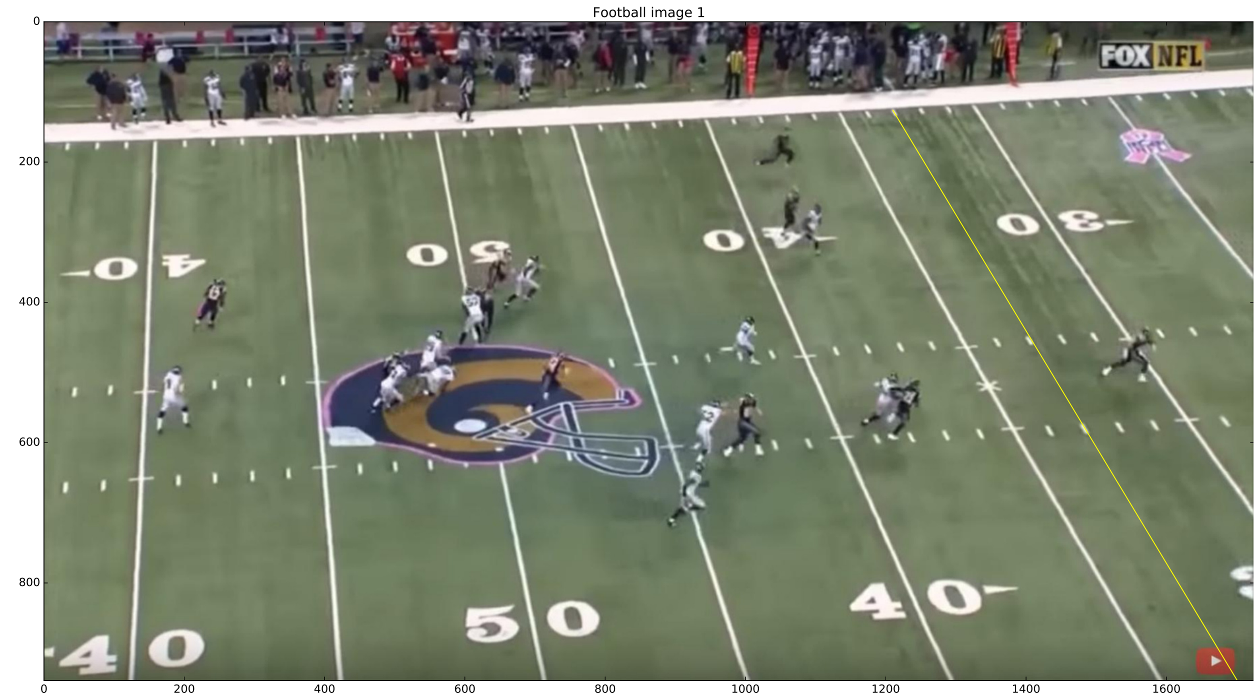


Figure 1: Baseline image with maker 33 highlighted

Figure 2: Output image with maker 33 highlighted

2. (a) Both persons will observe the reflection with the same intensity. Because according to Lambertian model, the reflectance is independent of reflectance direction d.

   (b) It is not a good model, since it does not work on surface such as metal and mirror.

3. (a) One way to deal with the boundary vale is zero-padding the image A, such that we will have

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   Or numerically, we can say that $C = A * B$, where

$$C(i,j) = \sum_{p=1}^{3} \sum_{q=1}^{2} A(p,q) B(i-p+1, j-q+1)$$

$$C = \begin{bmatrix} 4 & 16 & 16 \\ 10 & 23 & 6 \\ 5 & 14 & 12 \\ 2 & 12 & 0 \end{bmatrix}$$

   (b) Assume the size of image is $m \times n$. Set flipped x-derivative kernel as $k_x = [1, -1]$, and flipped y-derivative kernel as $k_y = [1, -1]$. Whereby, the gradient of image in x and y direction can be expressed as $\nabla I = \begin{bmatrix} k_x \otimes I & k_y \otimes I \end{bmatrix}^T$, where convolutions are calculated by

```python
def _convolve(kernel, in_img):
    kernel_size = kernel.shape
    ker_h = kernel_size[0]
    ker_w = kernel_size[1]

    in_img_size = in_img.shape
    img_h = in_img_size[0]
    img_w = in_img_size[1]

    out_h = img_h-ker_h+1
    out_w = img_w-ker_w+1
    out_img = np.zeros((out_h, out_w))

    for i in range(out_h):
        for j in range(out_w):
            temp=0
            for q in range(ker_w):
```

```
                for p in range(ker_h):
                    temp += kernel[p,q]*in_img[i+p,j+q]

            out_img[i,j] = temp
    return out_img
```

We can see that the resultant gradient matrices has dimensions $\nabla_x I \in \mathbb{R}^{m\times(n-1)}$ and $\nabla_y I \in \mathbb{R}^{(m-1)\times n}$. The whole process takes $2m(n-1)$ floating points operations for $\nabla_x I$ and $2(m-1)n$ floating points operations for $\nabla_y I$.

After we obtained the gradient matrices, we are able to calculate Potts energy by iterating through two gradient matrices with following codes:

```
def _calculate_potts_energy(data):
    beta = 1
    Ener = 0
    '''
    calculate potts energy in x
    '''
    xdMat = etai.read(data.x_derivative_path)
    for i in range(xdMat.shape[0]):
        for j in range(xdMat.shape[1]):
            if xdMat[i,j] != 0:
                Ener += beta
    '''
    calculate potts energy in y
    '''
    ydMat = etai.read(data.y_derivative_path)
    for i in range(ydMat.shape[0]):
        for j in range(ydMat.shape[1]):
            if ydMat[i,j] != 0:
                Ener += beta
    return Ener
```

We can see that this operation has $m(n-1)+n(m-1)$ floating points operations. Thus, we can say the the time complexity of this algorithm is $O(n^2)$. Moreover, we get $E(I) = 341202$

(c) After we convolute the original image with a Gaussian kernel, we obtain the new Potts energy $E(I') = 155192$.
There is a large difference in Potts energy before and after Gaussian Kernel, since Gaussian filter is a low pass filter that smooths the image and remove some noise. Therefore, after the image convoluted with the Gaussian Kernel, some of pixel changes are removed and Potts energy decreases.

(d) We found the Potts energy of `img_1.jpg` is $E(I_1) = 1560$ and Potts energy of `img_2.jpg` is $E(I_2) = 1560$. Both images have the same Potts energy, since the total length of edge of both images are same.

4. (a) The image I is a $20 \times 20 \times 3$ images with RGB colorspace. We can extract k-th channel of I with a $3D$ vector whose k-th layer is an 20×20 identity matrix and other two layers are zeros. Then, we have $I_R,\ I_G,\ I_B \in \mathbb{I}^{20\times 20}$.

   - Operation on I_G.
     The green block has its centroid at $C_G = (6, 16)$, it need to rotate by $-90°$ and translate to new centroid $C'_G = (11, 10)$. A good choice for transformation matrix is T_G:
     $$T_G = \begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 16 \\ 0 & 0 & 1 \end{bmatrix}$$
     Then, we can use use T_G to map every pixel of I_G to new image I'_G.
   - Operation on I_B.
     The blue block has its centroid at $C_B = (11, 13)$. It need to scale 2 times along x-axis and 1.5 times along y-axis. Moreover, it needs to translate its centroid to $C'_B = (11, 12.5)$. A good choice for transformation matrix is T_B:
     $$T_B = \begin{bmatrix} 2 & 0 & -11 \\ 0 & 1.5 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$
     Then, we can use use T_B to map every pixel of I_B to new image I'_B. Since we do not need any operation on red channel, we can form the new image by stacking $I_R,\ I'_G,\ I'_B$.

   (b) T_G is an Euclidean transformation.
   T_B is an shear transformation.

5. (a) We first smooth this image with a Gaussian kernel with $\sigma = 0.35$. Then, the gradient intensity image is shown in Figure 3

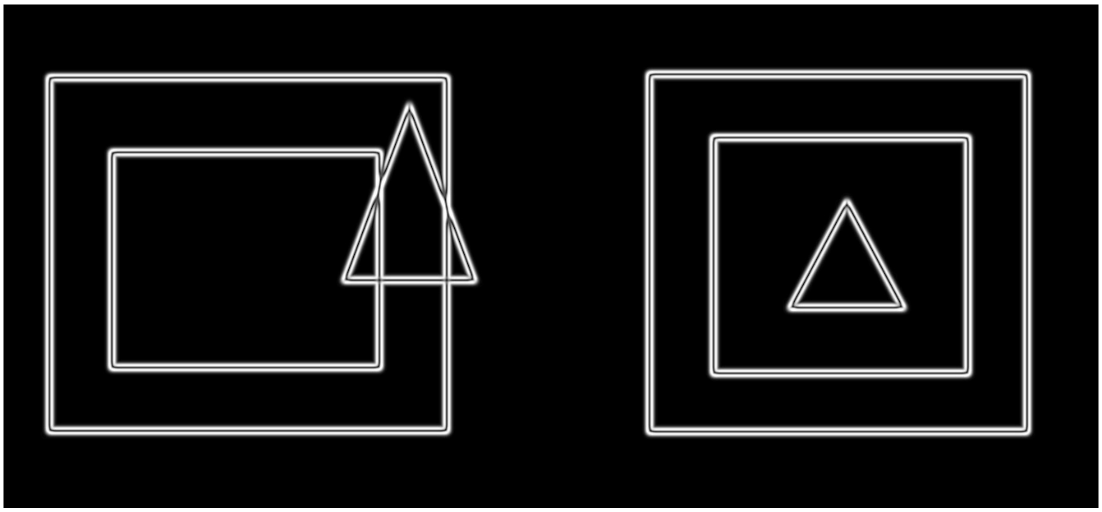


Figure 3: Gradient intensity of Gaussian smoothed image

From this image, we can see that the edge are clear but not sharp. This is caused by Gaussian kernel, which smooths sharp changes in pixels. This leads to thicker edge shown in intensity image and rounded corners.

```
def _create_sobel_horizontal_kernel():
'''Creates the 3x3 horizontal sobel kernel.
Returns:
kernel: the sobel horizontal kernel
'''
    return np.array([[1,0,-1],[2,0,-2],[1,0,-1]])


def _create_sobel_vertical_kernel():
'''Creates the 3x3 vertical sobel kernel.
Returns:
kernel: the sobel vertical kernel
'''
    return np.array([[1,2,1],[0,0,0],[-1,-2,-1]])

def _convolve(kernel, in_img):
'''Convolve the input image "in_img" with the kernel "kernel".
Assume the kernel has already been flipped.

```

```
Args:
kernel: a 2d kernel that is assumed to be flipped appropriately
in_img: the input image

Returns:
out_img: the result of convolving the input image with the specified
kernel
'''
    '''get kernel size'''
    kernel_size = kernel.shape
    ker_h = kernel_size[0]
    ker_w = kernel_size[1]

    '''get input size'''
    in_img_size = in_img.shape
    img_h = in_img_size[0]
    img_w = in_img_size[1]

    '''calculate output size'''
    out_h = img_h-ker_h+1
    out_w = img_w-ker_w+1

    '''init output image'''
    out_img = np.zeros((out_h, out_w))

    for i in range(out_h):
        for j in range(out_w):
        '''loop every centroid'''
            temp=0
            for q in range(ker_w):
                for p in range(ker_h):
                '''element wise multiplication and sum'''
                    temp += kernel[p,q]*in_img[i+p,j+q]
                    out_img[i,j] = temp
    return out_img
```

(b) After non-maximum suppression, detected edges are shown as Figure 4. Code is listed in Appendix.

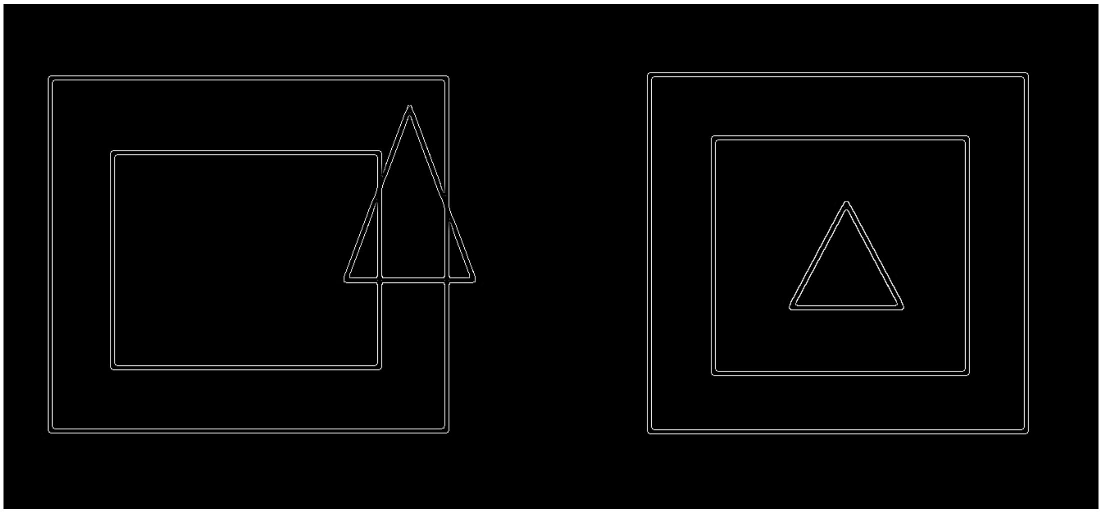


Figure 4: Detected edges after non-maximum suppression

(c) After double threshold and hysteresis, detected edges are shown as Figure 5. Code is listed in Appendix.

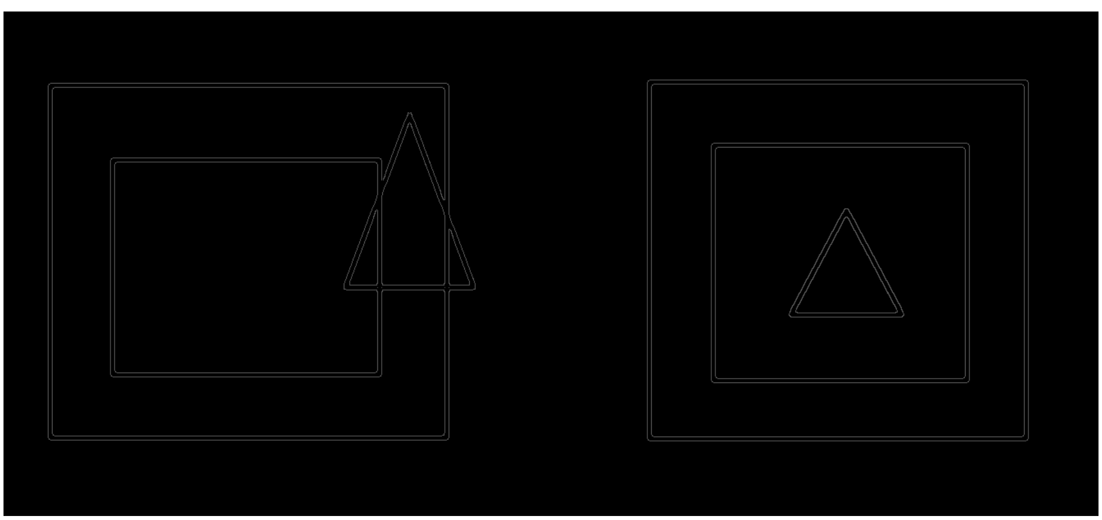
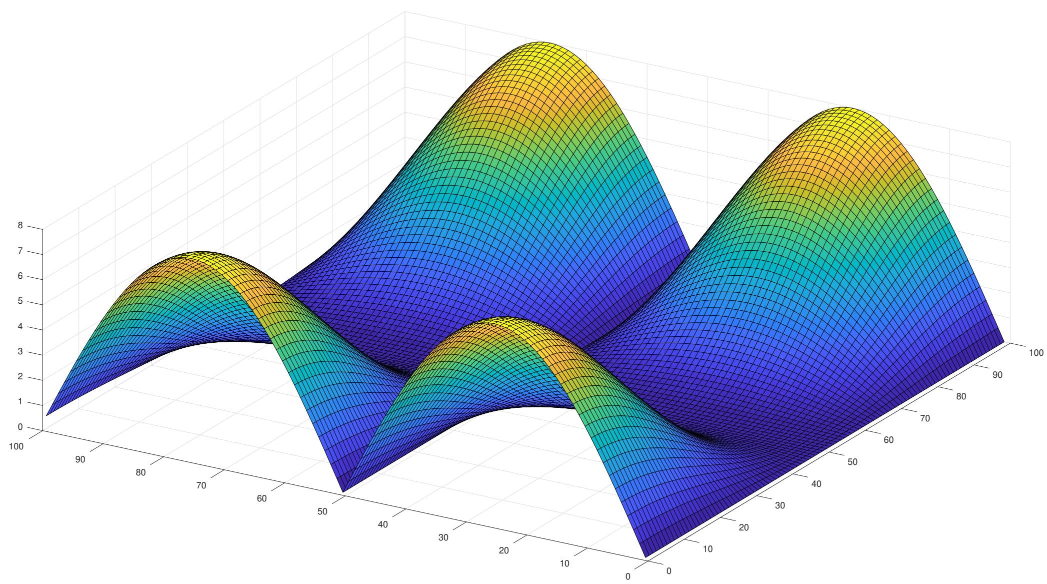


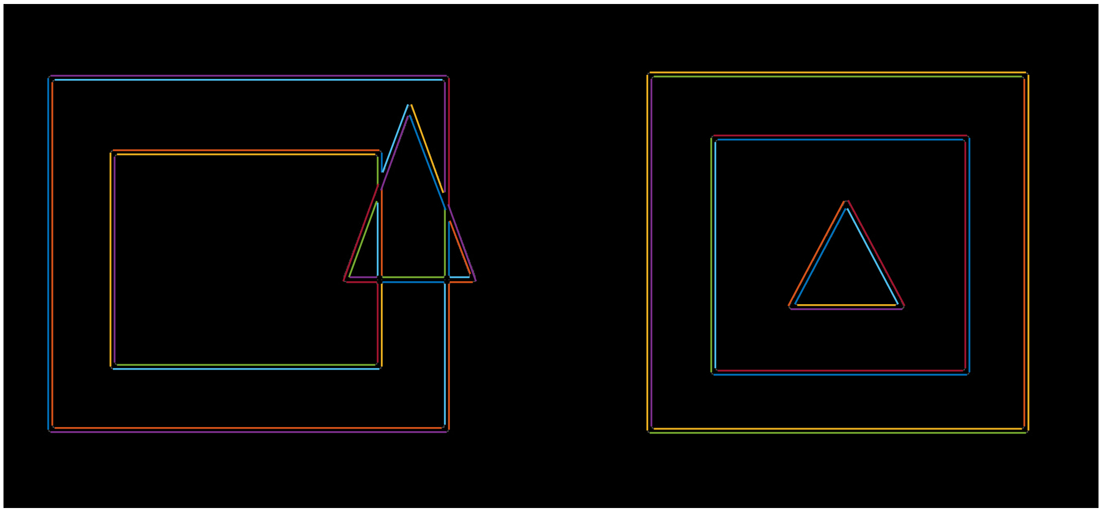
Figure 5: Detected edges after non-maximum suppression

(d) No. The canny edge detector do not have rotation invariance, since the Sobel Operator that obtains image gradient is not an rotational invariant operator. If we look at the Fourier domain of Sobel operator as figure below.



It is clear the Sobel operator is not symmetric in frequency domain. However, during Fourier transform, rotation in spatial domain also preserves in Fourier domain. This means if we rotate the image, then the spectrum in Fourier domain also rotates by the same amount. However, since convolution is the element-wise product in Fourier domain, results of convolution will differ after rotation. Although, we square the gradients in x and y direction to reduce this effect, and rotation will not change edge detection results by a lot. It is still variant under rotation.

(e) By using Hough Line Detector, we are able to detect lines in the edge output. After that a naive search along detected lines will help us to find line segments. The results are shown in figure below, where all line segments are highlighted with different colors.



Here is the list of end points of all 60 line segments

x	y	x'	y'
55	91	55	520
60	96	60	515
131	182	131	443
136	187	136	438
457	187	457	224
457	243	457	332
457	342	457	438
462	182	462	206
462	225	462	332
462	342	462	443
539	96	539	230
539	249	539	332
539	342	539	514
544	91	544	248
544	266	544	332
544	342	544	520
786	87	786	521
791	92	791	516
864	164	864	450
869	169	869	445
1174	169	1174	445
1179	164	1179	450
1246	92	1246	516
1251	87	1251	521

494	137	461	228
456	241	422	334
494	123	463	206
458	220	415	339
1028	250	966	367
1026	241	958	369
789	84	1248	84
58	88	541	88
794	89	1243	89
63	93	536	93
867	161	1176	161
872	166	1171	166
134	179	459	179
139	184	454	184
423	334	456	334
463	334	538	334
545	334	569	334
419	340	456	340
463	340	538	340
545	340	573	340
969	368	1089	368
961	373	1097	373
139	441	454	441
134	446	459	446
872	448	1171	448
867	453	1176	453
63	518	535	518
794	519	1243	519
58	523	541	523
789	524	1248	524
1030	250	1092	367
1031	240	1100	369
496	136	540	252
545	265	570	330
498	123	537	231
543	245	577	339

Non Maximum Suppression

```
def _choose_orientation_mode( theta ):
'''
mode 0: -pi/8 < theta <=  pi/8 U theta > 7/8pi
        U theta<= -7/8pi check horiz
mode 1:  pi/8 < theta <= 3pi/8 U -7pi/8 < theta <= -5pi/8
         check 1,3 quad
mode 2: 3pi/8 < theta <= 5pi/8 U -5pi/8 < theta <= -3pi/8
        check vertical
mode 3: 5pi/8 < theta <= 7pi/8 U -3pi/8 < theta <= -pi/8
        check 2,4 quad
'''
mode = 0
if (-np.pi/8 < theta and theta <= np.pi/8) or theta > 7*np.pi/8
    or theta <= -7*np.pi/8:
    mode = 0
elif (np.pi/8 < theta and theta <= 3*np.pi/8)
    or (-7*np.pi/8 < theta and theta <= -5*np.pi/8):
    mode = 1
elif (3*np.pi/8 < theta and theta <= 5*np.pi/8)
    or (-5*np.pi/8 < theta and theta <= -3*np.pi/8):
    mode = 2
elif (5*np.pi/8 < theta and theta <= 7*np.pi/8)
    or (-3*np.pi/8 < theta and theta <= -np.pi/8):
    mode = 3
return mode


def _non_maximum_suppression(g_intensity, orientation, input_image):
    '''Performs non-maximum suppression. If a pixel is not a local maximum
    (not bigger than it's neighbors with the same orientation), then
    suppress that pixel.

    Args:
    g_intensity: the gradient intensity of each pixel
    orientation: the gradient orientation of each pixel
    input_image: the input image

    Returns:
    g_sup: the gradient intensity of each pixel, with some intensities
    suppressed to 0 if the corresponding pixel was not a local
    maximum
    '''
    input_H = input_image.shape[0]
    input_W = input_image.shape[1]

```

```
    outputImg = np.zeros((input_H,input_W))
    '''y of pixel corresponding to gradient(0,0)'''
    kernel_H_2 = int((input_H-g_intensity.shape[0])/2)
    '''x of pixel corresponding to gradient(0,0)'''
    kernel_W_2 = int((input_W-g_intensity.shape[1])/2)

    for i in range(g_intensity.shape[0]): #vertical(y)
        for j in range(g_intensity.shape[1]): #horizontal(x)
            if(g_intensity[i, j]>0):
                '''determine gradient direction'''
                cur_mode = _choose_orientation_mode(orientation[i,j])
                if cur_mode == 0:
                    prev_x = j-1
                    prev_y = i
                    next_x = j+1
                    next_y = i
                elif cur_mode == 1:
                    prev_x = j-1
                    prev_y = i-1
                    next_x = j+1
                    next_y = i+1
                elif cur_mode == 2:
                    prev_x = j
                    prev_y = i-1
                    next_x = j
                    next_y = i+1
                elif cur_mode == 3:
                    prev_x = j-1
                    prev_y = i+1
                    next_x = j+1
                    next_y = i-1
                cur_bool = True
                '''boolen to check if need to perserve'''
                '''check both side along gradient'''
                if prev_x>=0 and prev_x<g_intensity.shape[1]
                    and prev_y>=0 and prev_y<g_intensity.shape[0]:
                    if(g_intensity[prev_y,prev_x]>g_intensity[i,j]):
                        cur_bool=False
                if next_x>=0 and next_x<g_intensity.shape[1]
                    and next_y>=0 and next_y<g_intensity.shape[0]:
                    if(g_intensity[next_y,next_x]>g_intensity[i,j]):
                        cur_bool=False
                if cur_bool:
                    outputImg[i+kernel_H_2,j+kernel_W_2]
                            = g_intensity[i, j]
    return outputImg
```

Double Thresholding & Hysteresis

```python
def _double_thresholding(g_suppressed, low_threshold, high_threshold):
'''Performs a double threhold. All pixels with gradient intensity larger
than 'high_threshold' are considered strong edges, all pixels with gradient
intensity in between 'high_threshold' and 'low_threshold' are considered
weak edges, and all pixels with gradient intensity smaller than
'low_threshold' are suppressed to 0.

Args:
g_suppressed: the gradient intensities of all pixels, after
non-maxiumum suppression
low_threshold: the lower threshold in double thresholding
high_threshold: the higher threshold in double thresholding

Returns:
g_thresholded: the result of double thresholding
'''
    '''initialize the image'''
    g_thresholded = np.zeros((g_suppressed.shape[0],
        g_suppressed.shape[1]))
    for i in range(g_thresholded.shape[0]):
        for j in range(g_thresholded.shape[1]):
            if g_suppressed[i,j]<low_threshold:
                g_thresholded[i,j]=0
            elif g_suppressed[i,j]>high_threshold:
                g_thresholded[i,j]=high_threshold
            else:
                g_thresholded[i,j]=low_threshold
    return g_thresholded


def _hysteresis(g_thresholded, low_threshold, high_threshold):
'''Performs hysteresis. If a weak pixel is connected to a strong pixel,
then the weak pixel is marked as strong. Otherwise, it is suppressed.
The result will be an image with only strong pixels.

Args:
g_thresholded: the result of double thresholding

Returns:
g_strong: an image with only strong edges
'''
    g_strong = np.zeros((g_thresholded.shape[0],g_thresholded.shape[1]))
    for i in range(1, g_thresholded.shape[0]-1):
        for j in range(1, g_thresholded.shape[1]-1):
            if g_thresholded[i,j]==high_threshold:
```

```
                g_strong[i,j]=high_threshold
                elif g_thresholded[i,j]==low_threshold:
                ''' low threshold'''
                    if (g_thresholded[i,j-1]==high_threshold
                        or g_thresholded[i,j+1]==high_threshold
                        or g_thresholded[i+1,j]==high_threshold
                        or g_thresholded[i-1,j]==high_threshold
                        or g_thresholded[i+1,j+1]==high_threshold
                        or g_thresholded[i+1,j-1]==high_threshold
                        or g_thresholded[i-1,j+1]==high_threshold
                        or g_thresholded[i-1,j-1]==high_threshold):
                        '''check all direction'''
                        g_strong[i,j] = high_threshold
                    else:
                        g_strong[i,j]=0
    return g_strong
```