Final

1. (15p) Let A be a set and let R and S be symmetric relations defined on A. Determine whether R ∘ S (the composition of R and S) is symmetric or not.

Solution

$\forall(x,y) \in R \circ S\,; \exists u \in A$, $(x,u) \in S$ *and* $(u,y) \in R$. *Since R and S symmetric,*
$(u,x) \in S$ *and* $(y,u) \in R$. *The last statement does not guarantee* $(y,x) \in R \circ S$
since $(y,x) \in R \circ S$ *iff* $(y,u) \in S$ *and* $(u,x) \in R$, $\exists u \in A$. *Thus,* $R \circ S$ *is not symmetric.*
$(However, \forall(x,y) \in R \circ S, (y,x) \in S \circ R)$

or consider the following counter example

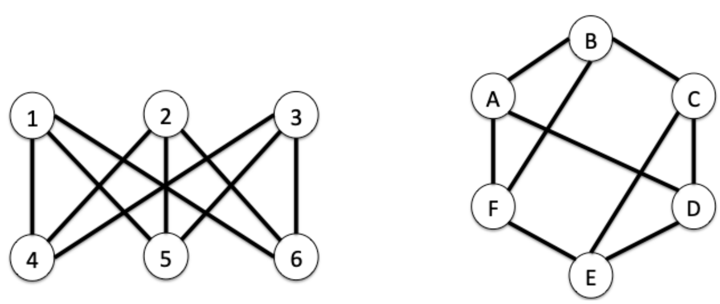
$A = \{a,b,c\}$,
$R = \{(b,b),(c,c),(b,c),(c,b)\}$
$S = \{(a,a),(a,b),(b,a)\}$
$R \circ S = \{(a,b),(a,c)\}$

as you see, R and S symmetric relation on A, but $R \circ S$ *is not.*

2. (15p) Determine whether the graphs G and H given with the following adjacency lists are isomoprhic or not.

graph G		graph H	
1	: 4, 5, 6	A	: B, D, F
2	: 4, 5, 6	B	: A, C, F
3	: 4, 5, 6	C	: B, D, E
4	: 1, 2, 3	D	: A, C, E
5	: 1, 2, 3	E	: C, D, F
6	: 1, 2, 3	F	: A, B, E

Solution



Both of them contains 6 vertices and 9 edges
All the degrees matching, i.e. both of them have 9 edges of degree 3
The graph H contains a simple circuit of length 3, but the graph G does not contain such circuit (the minimum length of the simple circuits is 4 in the graph G)

3. (15p) Consider an ordinary deck of 52 playing cards such that there are 4 suits: diamond, heart, spade, and club, and there are 13 kinds for each suit: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. The cards are to be drawn successively at random and without replacement. What is the probability that the second diamond (not second of diamonds) appears on the sixth draw?

Solution

There are 13 diamonds, and 39 non-diamonds. D stands for diamonds, ND stands for non-dimanods. We are looking for the outcomes in the following forms:

$$D - ND - ND - ND - ND - D$$
$$ND - D - ND - ND - ND - D$$
$$ND - ND - D - ND - ND - D$$
$$ND - ND - ND - D - ND - D$$
$$ND - ND - ND - ND - D - D$$

$$5\frac{13}{52}\cdot\frac{39}{51}\cdot\frac{38}{50}\cdot\frac{37}{49}\cdot\frac{36}{48}\cdot\frac{12}{47}=\frac{C(5,1)P(39,4)P(13,2)}{P(52,6)}$$

4. (15p) The final exam of Algorithm course consist of 50 true/false questions, each worth one point, and 25 multiple choice questions, each worth two points. In how many ways can a student get a grade of 96 on the test?

Solution

Assume the student answers A questions from 50 true/false questions and B questions from 25 multiple choice questions. Then A + 2B = 96. There are three possible cases:

A = 50 and B = 23, A = 48 and B = 24, A = 46 and B = 25

$$\binom{50}{50}\cdot\binom{25}{23} \quad + \quad \binom{50}{48}\cdot\binom{25}{24} \quad + \quad \binom{50}{46}\cdot\binom{25}{25}$$

5. (40p) Employ your id to calculate a specific number that will be used in the question as follows ('14290519' will be used here as an example to show you how the number is calculated):

- take the square of your id

  $14290519^2 = 204218933289361$

- remove all the zeros from the resulting number

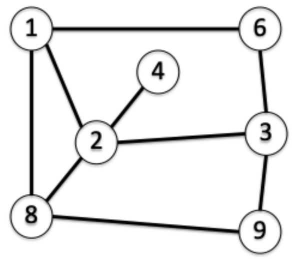
  $204218933289361 \longrightarrow 24218933289361$

- consider each consecutive two numbers as an edge in the graph

  $2 \longrightarrow 4 \longrightarrow 2 \longrightarrow 1 \longrightarrow 8 \longrightarrow 9 \longrightarrow 3 \longrightarrow 3 \longrightarrow 2 \longrightarrow 8 \longrightarrow 9 \longrightarrow 3 \longrightarrow 6 \longrightarrow 1$

- remove the reflexive edges (having same starting and ending nodes) from the graph

  $2 \longrightarrow 4 \longrightarrow 2 \longrightarrow 1 \longrightarrow 8 \longrightarrow 9 \longrightarrow 3 \longrightarrow 2 \longrightarrow 8 \longrightarrow 9 \longrightarrow 3 \longrightarrow 6 \longrightarrow 1$



Note that for the duplicate edges, your consider only one of them for the graph. For the duplicate $2 \longrightarrow 4 \longrightarrow 2 \cdots \longrightarrow 8 \longrightarrow 9 \longrightarrow \cdots 8 \longrightarrow 9 \cdots$, we used only one of them in the graph.

a) Show the adjacency matrix of your graph .

b) Does the graph contain a Euler path? If your answer is YES, provide one such Euler path. If NO, what will be the minimum number edges that should be removed to form a graph that contains a Euler path?

c) What is the chromatic number of your graph?

d) What will be the minimum number of vertices in a vertex cut of your graph?

e) What will be the minimum bumber of edges in an edge cut of your graph?

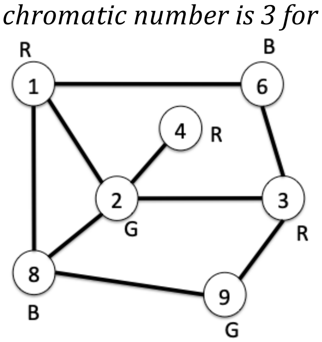
<u>Solution</u>

a)

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 8 \\ 9 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 6 & 8 & 9 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}$$

b) Since there are 4 odd vertices, it does not contain a Euler path. Removing one edge – the edge (1, 8) (connecting two odd vertices) would be enough to form a graph containing a Euler path.

c) the chromatic number is 3 for this graph



d) $\kappa(G) = 1$, *vertex cut {2}*

e) $\lambda(G) = 1$, *edge cut {(2,4)}*