Inverse

- If a function both one-to-one and onto, it is called bijection.

  If f is a bijection, then f^{-1} can be defined, i.e. f is invertible

- $f: \mathbb{Z} \to \mathbb{Z}$, defined as $f(x) = x + 1$, f is invertible ?

$$\forall x_1, x_2 \in \mathbb{Z},\ f(x_1) = f(x_2) \to x_1 + 1 = x_2 + 1$$
$$\to x_1 = x_2 \text{ (one-to-one)}$$

$$\forall y \in \mathbb{Z}, f(x) = y \leftrightarrow x + 1 = y$$
$$\leftrightarrow x = y - 1 \in \mathbb{Z} \text{ (onto)}$$

$$f^{-1}(x) = x - 1$$

Inverse

- If a function both one-to-one and onto, it is called bijection.

  If f is a bijection, then f^{-1} can be defined, i.e. f is invertible

- $f: \mathbb{Z} \to \mathbb{Z}$, defined as $f(x) = 2x + 1$, f is invertible ?

$$\forall x_1, x_2 \in \mathbb{Z},\ f(x_1) = f(x_2) \to 2x_1 + 1 = 2x_2 + 1$$
$$\to x_1 = x_2 \text{ (one-to-one)}$$

$$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}\ f(x) = y \leftrightarrow 2x + 1 = y$$
$$\leftrightarrow x = \frac{y-1}{2}$$

but for some $y \in \mathbb{Z}$, $x = \frac{y-1}{2} \notin \mathbb{Z}$ (not onto)

Inverse

- If a function both one-to-one and onto, it is called bijection.

  If f is a bijection, then f^{-1} can be defined, i.e. f is invertible

- $f: \mathbb{Z} \to \mathbb{N}$, defined as $f(x) = \begin{cases} 2x - 1 & if \;\; x > 0 \\ -2x & if \;\; x \leq 0 \end{cases}$, f is invertible ?

$$\forall x_1, x_2 \in \mathbb{Z},\;\; f(x_1) = f(x_2) \to 2x_1 - 1 = 2x_2 - 1$$
$$\to x_1 = x_2$$
$$\forall x_1, x_2 \in \mathbb{Z},\;\; f(x_1) = f(x_2) \to -2x_1 = -2x_2$$
$$\to x_1 = x_2$$ (one-to-one)

$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$, if $y = 2k, \exists k \in \mathbb{Z}$, then $f(x) = y \leftrightarrow -2x = y$

$$\leftrightarrow x = -\frac{y}{2} = -k \in \mathbb{Z}$$

$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$, if $y = 2k + 1, \exists k \in \mathbb{Z}$,

then $f(x) = y \leftrightarrow 2x - 1 = y$

$$\leftrightarrow x = \frac{y+1}{2} = k + 1 \in \mathbb{Z}$$

(onto)

Composition

- $f, g: \mathbb{Z} \to \mathbb{Z}$,
  $f(x) = 3x + 1$ and $g(x) = 2x - 1$
  $g \circ f(x) = g(f(x)) = g(3x + 1) = 2(3x + 1) - 1 = 6x + 1$
  $f \circ g(x) = f(g(x)) = f(2x - 1) = 3(2x - 1) + 1 = 6x - 2$

- $f: A \to B$
  $f \circ f^{-1}(y) = f(f^{-1}(y)) = f(x) = y, \quad f \circ f^{-1} = I_B$
  $f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(y) = x, \quad f^{-1} \circ f = I_A$

- If f and g are one-to-one, then $f \circ g$ is also one-to-one.

$\forall x_1, x_2 \in A, f \circ g(x_1) = f \circ g(x_2) \;\to\; f(g(x_1)) = f(g(x_2))$
$\to\; g(x_1) = g(x_2)$ (f is one-to-one)
$\to\; x_1 = x_2$ (g is one-to-one)

Floor and Ceiling Functions

- floor function of a real number x : is the largest integer that is less than or equal to x, denoted by $\lfloor x \rfloor$.

  $\lfloor 1/5 \rfloor = 0$, $\lfloor -1/5 \rfloor = -1$, $\lfloor 3{,}56 \rfloor = 3$, $\lfloor -3{,}56 \rfloor = -4$

  $\lfloor x \rfloor = n$ if $n \le x < n+1$ or $\lfloor x \rfloor = n$ if $x-1 \le n < x$

- ceiling function of a real number x : is the smallest integer that is greater than or equal to x, denoted by $\lceil x \rceil$.

  $\lceil 1/5 \rceil = 1$, $\lceil -1/5 \rceil = 0$, $\lceil 3{,}56 \rceil = 4$, $\lceil -3{,}56 \rceil = -3$

  $\lceil x \rceil = n$ if $n-1 < x \le n$ or $\lceil x \rceil = n$ if $x \le n < x+1$

Floor and Ceiling Functions

- show that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

  assume $x = n + \varepsilon$ where n is integer and $0 \leq \varepsilon < 1$

$$0 \leq \varepsilon < \frac{1}{2}$$
$$\lfloor 2n + 2\varepsilon \rfloor = \lfloor n + \varepsilon \rfloor + \lfloor n + \varepsilon + 1/2 \rfloor$$
$$2n = n + n$$

$$\frac{1}{2} \leq \varepsilon < 1$$
$$\lfloor 2n + 2\varepsilon \rfloor = \lfloor n + \varepsilon \rfloor + \lfloor n + \varepsilon + 1/2 \rfloor$$
$$2n + 1 = n + n + 1$$

- determine whether $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all $x, y \in \mathbb{R}$.

  assume $0 < x, y < \frac{1}{2}$, then $x + y < 1$.

$$\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$$
$$1 \neq 1 + 1$$

Sequences

Definition : A sequence is a function from $\mathbb{N}$ (or $\mathbb{Z}^+$) to a set S, denoted by $\{a_n\}$ where a_n is the general term of the sequence.

$1, 4, 7, 10, 13, \ldots \qquad \{3n + 1\}$

$0, 1, 3, 7, 15, \ldots \qquad \{2^n - 1\}$

- $a_n = \frac{1}{n} \qquad a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3}, \ldots$
- $a_n = \frac{1}{3^n+2} \qquad a_0 = \frac{1}{2},\ a_1 = \frac{1}{5},\ a_2 = \frac{1}{11}, \ldots$

Sequences

Geometric Sequence :

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

initial term — common ratio — general term

$a_n = (-1)^n$	$a_n = 2.3^n$	$a_n = 3.(1/2)^n$
$1, -1, 1, -1, \ldots$	$2, 2.3, 2.9, 2.27, \ldots$	$3, 3/2, 3/4, 3/8, \ldots$

Sequences

Arithmetic Sequence :

$$\underbrace{a}_{\text{initial term}}, a + \underbrace{d}_{\text{common difference}}, a + 2d, \ldots, \underbrace{a + n.d}_{\text{general term}}, \ldots$$

$a_n = 1 + n$	$a_n = 2 - 4n$	$a_n = -1 + 8n$
$1, 2, 3, 4, \ldots$	$2, -2, -6, -10, \ldots$	$-1, 7, 15, 23, \ldots$

Summations

- $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$

  $\sum_{i=0}^{\infty} a_i = a_0 + a_1 + \ldots + a_n + \ldots$

  $\sum_{i=2}^{5} (i^2 - 1) = 4 - 1 + 9 - 1 + 16 - 1 + 25 - 1 = 50$

- $S = \{2, 3, 4\}, \quad \sum_{x \in S} x^3 = 2^3 + 3^3 + 4^3 = 99$

- $\sum cf(x) = c \sum f(x)$

  $\sum (f(x) + g(x)) = \sum f(x) + \sum g(x)$

  $\sum_{i=m}^{n} f(i) = \sum_{i=m}^{k} f(i) + \sum_{i=k+1}^{n} f(i)$

- $$\begin{aligned}\sum_{i=1}^{n} i &= 1 + 2 + \ldots + \frac{n}{2} + \left(\frac{n}{2} + 1\right) + \ldots + (n - 1) + n \\ &= (\mathrm{n} + 1) + (\mathrm{n} + 1) + \ldots + (n + 1) \\ &= \frac{n}{2}(\mathrm{n} + 1)\end{aligned}$$

Summations

- $a, a+d, a+2d, \ldots, a+n.d$

$$\sum_{i=0}^{n}(a+id) = \sum_{i=0}^{n} a + \sum_{i=0}^{n} id$$
$$= \sum_{i=0}^{n} a + d\sum_{i=0}^{n} i$$
$$= (n+1)a + d\,\frac{n(n+1)}{2}$$

- $a, ar, ar^2, \ldots, ar^n$

$$S_n = \sum_{i=0}^{n} ar^i \rightarrow rS_n = r\sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} ar^{i+1}$$
$$rS_n = \sum_{i=1}^{n+1} ar^i = \sum_{i=1}^{n} ar^i + ar^{n+1}$$
$$rS_n = \sum_{i=0}^{n} ar^i + ar^{n+1} - a$$
$$rS_n = S_n + ar^{n+1} - a \;\rightarrow\; S_n = \frac{ar^{n+1}-a}{r-1}$$

Recurrence Relations

- sometimes the elements of the sequence are defined recursively in terms of previous and the initial elements of the sequence

$$a_0 = 1,\ a_1 = 5,\ a_2 = 13,\ a_3 = 29,\ a_4 = ?$$

$$\begin{aligned} a_1 &= 2a_0 + 3 = 5 \\ a_2 &= 2a_1 + 3 = 13 \\ a_3 &= 2a_2 + 3 = 29 \\ a_4 &= 2a_3 + 3 = 61 \end{aligned}$$

Definition : an equation that express the general term of the sequence in terms of previous terms. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relations

- $a_{n+1} = 3a_n, \;\; a_0 = 5$

$a_1 = 15 = 3.5$
$a_2 = 75 = 3.(3.5)$
$a_3 = 225 = 3.(3.(3.5))$
$\vdots$
$a_n = 3^n 5$; the unique solution of the given recurrence relation

- $a_{n+1} = d.a_n, \;\; a_0 = A$ where d is constant

  the solution of the recurrence relation will be $a_n = A.d^n$

- solve the recurrence relation $a_{n+1} = 7.a_n$ where $n \geq 1$ and $a_2 = 98$

  $a_2 = A.7^2 \rightarrow 98 = A.49 \rightarrow A = 2$

  the solution is $a_n = 2.7^n$

Recurrence Relations

- 3 can be written as a sum of positive integers in 4 different ways:

3
1 + 2
2 + 1
1 + 1 + 1

2
1 + 1

- In how many different ways can n be written as a sum of positive integers ?

4
1 + 3
2 + 2
1 + 1 + 2

3 + 1
1 + 2 + 1
2 + 1 + 1
1 + 1 + 1 + 1

- $a_4 = 2.a_3$, $a_3 = 2.a_2$, and $a_2 = 2$

  $a_{n+1} = 2.a_n$, $a_1 = 1$

  create a new sequence $b_n = a_{n+1}$

  $b_n = 2b_{n-1}$, $b_0 = 1$; the solution will be $b_n = 2^n$; thus $a_n = 2^{n-1}$

first order linear homogeneous recurrence relation

Recurrence Relations

- $a_{n+1} - d.a_n = 0,\ a_0 = A$ where d is constant.
  - first order since a_{n+1} only depends on a_n (the previous term)
  - linear since each variable appears in the first power and there is no product such as $a_{n+1}.a_n$
  - homogeneous since the right hand side is 0
- The second order linear homogeneous recurrence relation :

$$C_0 a_{n+1} + C_1 a_n + C_2 a_{n-1} = 0,\ a_0 = A,\ a_1 = B,\ \ n \geq 2$$

- The Fibonacci sequence:

$$F_{n+1} = F_n + F_{n-1},\ \ F_0 = 1,\ \ F_2 = 1,\ \ n \geq 2$$

Recurrence Relations

- The second order linear homogeneous recurrence relation :

$$C_0 a_{n+1} + C_1 a_n + C_2 a_{n-1} = 0,\ a_0 = A,\ a_1 = B,\ \ n \geq 2$$

$a_{n+1} - d.a_n = 0,\ \ a_0 = A.$ the solution was in the form of $a_n = A.d^n$

- Similarly, we look for a solution in the form of $a_n = c.r^n$

If we place it in the equation:

$$C_0 c.r^{n+1} + C_1 c.r^n + C_2 c.r^{n-1} = 0$$

$$C_0 r^2 + C_1 r + C_2 = 0 \quad \text{(characteristic equation)}$$

The solutions for the characteristic equation are called characteristic roots; r_1 and r_2

Recurrence Relations

- $a_{n+1} + a_n - 6a_{n-1} = 0,\ a_0 = -1,\ a_1 = 8,\ \ n \geq 2$

$r^2 + r - 6 = 0$ (characteristic equation)

$r_1 = 2,\ r_2 = -3$ (characteristic roots)

the solution will be in the form of $a_n = c_1 2^n + c_2(-3)^n$.

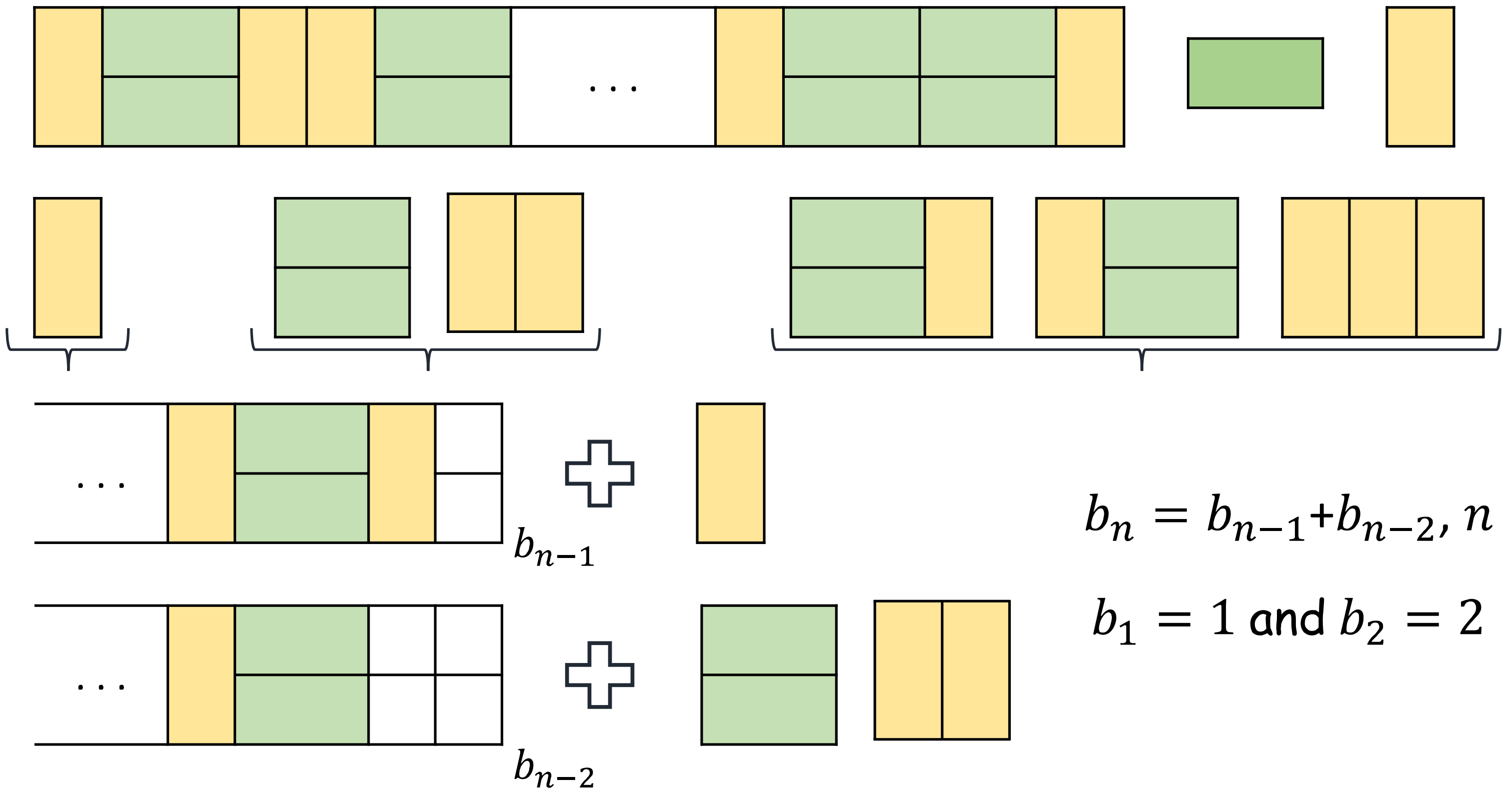
$a_0 = c_1 2^0 + c_2(-3)^0 \ \rightarrow\ \ -1 = c_1 + c_2$

$a_1 = c_1 2^1 + c_2(-3)^1 \ \rightarrow\ \ 8 = 2c_1 - 3c_2$

$$\begin{array}{r} c_1 + c_2 = -1 \\ 2c_1 - 3c_2 = 8 \\ \hline c_1 = 1,\ c_2 = -2 \end{array} \longrightarrow \quad a_n = 2^n - 2.(-3)^n$$

Recurrence Relations

- Suppose we have a 2xn chessboard and we wish to cover it using 2x1 and 1x2 dominoes. In how many different ways can we cover it ?



$$b_n = b_{n-1} + b_{n-2},\ n \geq 3$$

$b_1 = 1$ and $b_2 = 2$

Recurrence Relations

- Suppose we have a 2xn chessboard and we wish to cover it using 2x1 and 1x2 dominoes. In how many different ways can we cover it ?

- $b_n = b_{n-1} + b_{n-2}, n \geq 3, b_1 = 1$ and $b_2 = 2$

$$r^2 - r - 1 = 0 \quad \text{(characteristic equation)}$$

$$r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2} \quad \text{(characteristic roots)}$$

the solution will be in the form of $b_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$

$$b_0 = c_1(\frac{1+\sqrt{5}}{2})^0 + c_2(\frac{1-\sqrt{5}}{2})^0 \rightarrow \quad 1 = c_1 + c_2$$

$$b_1 = c_1(\frac{1+\sqrt{5}}{2})^1 + c_2(\frac{1-\sqrt{5}}{2})^1 \rightarrow \quad 2 = \left(\frac{1+\sqrt{5}}{2}\right) c_1 + (\frac{1-\sqrt{5}}{2}) c_2$$

$$c_1 = 1/\sqrt{5}, c_2 = -1/\sqrt{5} \longrightarrow b_n = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n \right)$$

Recurrence Relations

- 3 can be written as a sum of positive integers in 4 different ways:

3
$1 + 2$
$2 + 1$
$1 + 1 + 1$

palindrome

they are read the same from left to right, right to left

- How many different palindromes can be found for a given $n \in \mathbb{Z}^+$?

3
$1 + 1 + 1$

4
$1 + 2 + 1$
$2 + 2$
$1 + 1 + 1 + 1$

5
$2 + 1 + 2$
$1 + 3 + 1$
$1 + 1 + 1 + 1 + 1$

6
$2 + 2 + 2$
$3 + 3$
$2 + 1 + 1 + 2$
$1 + 4 + 1$
$1 + 1 + 2 + 1 + 1$
$1 + 2 + 2 + 1$
$1 + 1 + 1 + 1 + 1 + 1$

Recurrence Relations

- 3 can be written as a sum of positive integers in 4 different ways:

3
1 + 2
2 + 1
1 + 1 + 1

palindrome

they are read the same from left to right, right to left

- How many different palindromes can be found for a given $n \in \mathbb{Z}^+$?

$$b_n = 2b_{n-2},\ n \geq 3,\ b_1 = 1 \text{ and } b_2 = 2$$

$$r^2 - 2 = 0 \quad \text{(characteristic equation)}$$

$$r_1 = \sqrt{2},\ r_2 = -\sqrt{2} \quad \text{(characteristic roots)}$$

the solution will be in the form of $b_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n$

$$b_0 = c_1(\sqrt{2})^0 + c_2(-\sqrt{2})^0 \rightarrow \quad 1 = c_1 + c_2$$

$$b_1 = c_1(\sqrt{2})^1 + c_2(-\sqrt{2})^1 \rightarrow \quad 2 = (\sqrt{2})c_1 + (-\sqrt{2})c_2$$

$$b_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)(\sqrt{2})^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)(-\sqrt{2})^n$$