UNIT 2

DIFFERENTIAL EQUATIONS–2

2.1 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A linear differential equation is that in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus

$$\frac{d^n y}{dx^n} + p_1 \frac{d_y^{n-1}}{dx^{n-1}} + p_2 \frac{dy^{n-2}}{dx^{n-2}} \cdots\cdots\cdots + p_{n-1} \frac{dy}{dx} + p_n y = X$$

where p_1, p_2, p_3 .... p_n and X are functions of x only.

A differential equation of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d_y^{n-1}}{dx^{n-1}} + k_2 \frac{d_y^{n-2}}{dx^{n-2}} + \cdots\cdots\cdots + k_n y = X \qquad ...(1)$$

where k_1, k_2, k_3 .... k_n are constants, is called a linear differential equation with constant coefficients.

Let $\dfrac{d}{dx} = D, \quad \dfrac{d^2}{dx^2} = D^2, \quad \dfrac{d^3}{dx^3} = D^3 \cdots\cdots\cdots \dfrac{d^n}{dx^n} = D^n$

where D is a differential operator.

Then equation (1) becomes

$$\left(D^n + k_1 D^{n-1} + k_2 D^{n-2} + k_3 D^{n-3} + \cdots\cdots\cdots + k_n\right) y = X$$

or $\quad f(D)\, Y = X \qquad ....(2)$

where $\quad f(D) = D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots\cdots\cdots + k_n$

Example 1: Solve the following differential equations.

1) Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 44y = 0$

Solution: The given equation in symbolic form is

$$(D^2 - 7D - 44)\, y = 0$$

$\therefore$ Its AE is $D^2 - 7D - 44 = 0$

$$(D + 4)(D - 11) = 0$$

$\therefore$ $D = -4, \qquad D = 11$

$\therefore$ Its solution is

$$y = c_1 e^{-4x} + c_2\, e^{11x}$$

2) Solve $(D^4 - 5D^2 + 4)\, y = 0$

Solution: The given equation is $(D^4 - 5D^2 + 4)\, y = 0$

Its, AE is $D^4 - 5D^2 + 4 = 0$

$$D^4 - 4D^2 - D^2 + 4 = 0$$

$$D^2 (D^2 - 4) - 1\, (D^2 - 4) = 0$$

$$(D^2 - 4)(D^2 - 1) = 0$$

$\therefore$ $(D - 2)(D + 2)(D - 1)(D + 1) = 0$

$\therefore$ $D = 2, -2, 1, -1$ are the roots.

Hence its solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{x} + c_4 e^{-x}.$$

3) Solve $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$

Solution: The given equation is

$$(D^3 - 2D^2 - 4D + 8)\, y = 0$$

Its $A.E.$ is $D^3 - 2D^2 - 4D + 8 = 0$

$\therefore$ $D^2 (D - 2) - 4(D - 2) = 0$

$$(D - 2)(D - 2)(D + 2) = 0$$

$\therefore$ $D = 2, 2, -2$ are the roots

$\therefore$ Its solution is

$$y = (c_1 x + c_2)\, e^{2x} + c_3\, e^{-2x}$$

4) Solve $\frac{d^4y}{dx^4} - m^4 y = 0$

Solution: The given equation is

$$(D^4 - m^4)\, y = 0$$

$\therefore$ Its AE is $D^4 - m^4 = 0$

$\therefore$ $(D^2 - m^2)\,(D^2 + m^2) = 0$

$\therefore$ $(D - m)\,(D + m)\,(D - mi)\,(D + mi) = 0$

$\therefore$ $D = m,\ -m,\ mi$ and $-mi$

$\therefore$ Its solution is

$$y = c_1 e^{mx} + c_2 e^{-mx} + (c_3 \cos x + c_4 \sin x)$$

5) Solve $(D^4 + m^4)\, y = 0$

Solution: The given equation is

$$(D^4 + m^4)\, y = 0$$

Its AE is $D^4 + m^4 = 0$

$$D^4 + 2m^2 D^2 + m^4 - 2m^2 D^2 = 0$$

$\therefore$ $(D^2 + m^2)^2 - 2m^2 D^2 = 0$

$\therefore$ $(D^2 + m^2 + \sqrt{2}\, mD)\,(D^2 + m^2 - \sqrt{2}\, mD) = 0$

$\therefore$ $D^2 + \sqrt{2}mD + m^2 = 0, \quad D^2 - \sqrt{2}mD + m^2 = 0$

$\therefore$ $D = -\frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2}; \quad D = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2}$

$$= -\frac{m}{\sqrt{2}} \pm \frac{mi}{\sqrt{2}} \quad ; \quad = \frac{m}{\sqrt{2}} \pm \frac{mi}{\sqrt{2}}$$

$\therefore$ Its solution is

$$y = e^{-\frac{mx}{\sqrt{2}}}\left(c_1 \cos\frac{mx}{\sqrt{2}} + c_2 \sin\frac{mx}{\sqrt{2}}\right) + e^{\frac{mx}{\sqrt{2}}}\left(c_3 \cos\frac{mx}{\sqrt{2}} + c_4 \sin\frac{mx}{\sqrt{2}}\right)$$

EXERCISE

Solve the following differential equations.

1. $(D^2 + 1)^2 (D^2 + D + 1)y = 0$

Ans.: $y = (c_1x + c_2) \cos x + (c_3x + c_4) \sin x$

$$+ e^{-\frac{1}{2}x}\left(c_5 \cos\frac{\sqrt{3}}{2}x + c_6 \sin\frac{\sqrt{3}}{2}x\right)$$

2. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} + 6y = 0$

Ans.: $y = c_1e^{-x} + c_2\, e^{-2x} + c_3\, e^{-3x}$

3. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$

Ans.: $y = (c_1x^2 + c_2\, x + c_3)\, e^x$

4. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

Ans.: $y = c_1\, e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$

5. $\dfrac{d^4y}{dx^4} + 4\dfrac{d^3y}{dx^3} - 5\dfrac{d^2y}{dx^2} - 36\dfrac{dy}{dx} - 36y = 0$

Ans.: $y = (c_1x + c_2)\, e^{-2x} + c_3\, e^{3x} + c_4\, e^{-3x}$

2.3 INVERSE OPERATOR $\dfrac{1}{f(D)}$

Definition: $\dfrac{1}{f(D)}X$ is that function of x, free from arbitrary constants which when operated upon by $f(D)$ gives X.

Thus $f(D)\left\{\dfrac{1}{f(D)}X\right\} = X$.

$\therefore \quad f(D)$ and $\dfrac{1}{f(D)}$ are inverse operators.

Note the following important results:

1. $\dfrac{1}{f(D)}X$ is the particular integral of $f(D)\, y = X$.

2. $\dfrac{1}{D}X = \int X\, dx$.

3. $\dfrac{1}{D-a}X = e^{ax}\int X\, e^{-ax} dx$.

2.4 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the differential equation

$$f(D)\, y = X$$

$$\therefore \qquad PI = \frac{1}{f(D)}X$$

Type I(A): When X is of the form e^{ax}, where a is any constant.

$$PI = \frac{1}{f(D)}X = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}. \text{ Provided } f(a) \neq 0.$$

Rule: In $f(D)$, put $D = a$ and PI will be calculated, provided $f(a) \neq 0$.

Type I(B): When X is of the form e^{ax} but $f(D) = 0$ has got 'a' as its root (Failure case of type 1).

$$PI = \frac{1}{f(D)}e^{ax} = \frac{1}{0}e^{ax} \qquad \because \ f(a) = 0$$

$\therefore \quad$ Our method fails.

Now, since 'a' is a root of $f(D)$, $(D - a)$ must be a factor of $f(D)$ which therefore can be written as $(D - a)\ \varphi\ (D)$

$$\therefore \qquad PI = \frac{1}{(D-a)\ \varphi(D)}e^{ax}$$

$$= \frac{e^{ax}}{\varphi(a)} \frac{1}{(D-a+a)}\{1\}$$

$$= \frac{e^{ax}}{\varphi(a)} \frac{1}{D}\{1\}$$

$$= \frac{e^{ax}}{\varphi(a)} . x \qquad \left(\because \frac{1}{D} = \int \right)$$

Similarly, if $f(D) = (D - a)^p \varphi(D)$ then

$$PI = \frac{e^{ax}}{\varphi(a)} \cdot \frac{1}{D^p}\{1\}$$

$$= \frac{e^{ax}}{\varphi(a)} \cdot \frac{x^p}{p!}$$

Rule: Put $D = a$ in those factors of $f(D)$ which do not vanish for $D = a$ and then make the question as PI of a product of e^{ax} and 1 which is calculated.

Note: In case of sinh ax and cosh ax we take

i) $\sinh ax = \dfrac{e^{ax} - e^{-ax}}{2}$ ii) $\cosh ax = \dfrac{e^{ax} + e^{-ax}}{2}$

Example 1: Solve $(D^2 + 3D + 5)\, y = e^{2x}$

Solution: Given equation is

$$(D^2 + 3D + 5)\, y = e^{2x}$$

AE is $D^2 + 3D + 5 = 0$

$$\therefore \qquad D = \frac{-3 \pm \sqrt{9-20}}{2} = \frac{-3}{2} \pm \frac{\sqrt{11}}{2} i$$

$$\therefore \qquad CF = e^{-\frac{3}{2}x}\left(c_1 \cos\frac{\sqrt{11}}{2}x + c_2 \sin\frac{\sqrt{11}}{2}x\right)$$

And, $\qquad PI = \dfrac{1}{D^2 + 3D + 5} e^{2x}$

Put $D = 2$ in $f(D)$

$$= \frac{1}{4+6+5} e^{2x} = \frac{1}{15} e^{2x}$$

$\therefore \quad G.S. = CF + PI$

$$\therefore \quad y = e^{-\frac{3}{2}x}\left(c_1 \cos\frac{\sqrt{11}}{2}x + c_2 \sin\frac{\sqrt{11}}{2}x\right) + \frac{1}{15}e^{2x}$$

Example 2: Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2\sinh 2x.$

Solution: Given equation is

$(D^2 + 4D + 4)\, y = 2 \sinh 2x$

$$= \frac{2(e^{2x} - e^{-2x})}{2} = e^{2x} - e^{-2x}$$

AE is $D^2 + 4D + 4 = 0$

$\therefore \quad (D + 2)^2 = 0 \quad \therefore \quad D = -2, -2$ are the roots.

$\therefore \quad CF = (c_1 x + c_2)\, e^{-2x}$

And PI for $e^{2x} = \dfrac{1}{(D+2)^2} e^{2x} = \dfrac{1}{(2+2)^2} e^{2x} = \dfrac{e^{2x}}{16}$ ...(1)

Also PI for $e^{-2x} = \dfrac{1}{(D+2)^2} e^{-2x} = \dfrac{1}{0} e^{-2x}$

$\therefore$ Our method fails.

Then to find PI put $D - 2$ for D in $f(D)$ and take out e^{-2x}

$$\therefore \quad PI = e^{-2x}\left\{\frac{1}{(D-2+2)^2}\right\}\{1\}$$

$$= e^{-2x}\frac{1}{D^2}\{1\} = e^{-2x}\frac{x^2}{2!} \qquad ....(2)$$

$\therefore \quad GS = CF + PI_1 - PI_2$

$$y = (c_1 x + c_2)\, e^{-2x} + \frac{e^{2x}}{16} - e^{-2x}\frac{x^2}{2!}.$$

Example 3: Solve $(D^3 + 3D^2 + 3D + 1)\, y = e^{-x}$.

Solution: Given equation is

$$(D^3 + 3D^2 + 3D + 1)\, y = e^{-x}$$

$\therefore$ AE is $D^3 + 3D^2 + 3D + 1 = 0$

$\therefore$ $(D + 1)^3 = 0$ $\quad \therefore D = -1, -1, -1$ are the roots.

$\therefore$ $CF = (c_1x^2 + c_2x + c_3)\, e^{-x}$

And
$$PI = \frac{1}{(D+1)^3} e^{-x}$$

$$= e^{-x} \left\{ \frac{1}{(D-1+1)^3} \right\} (1)$$

$$= e^{-x} \frac{1}{D^3}\{1\} = e^{-x} \frac{x^3}{3!}$$

$\therefore$ $G.S. = CF + PI$

$$y = (c_1x^2 + c_2x + c_3)e^{-x} + e^{-x} \frac{x^3}{3!}$$

Example 4: Solve $(D^2 + 4D + 3)y = e^{-3x}$.

Solution: Given equation is

$$(D^2 + 4D + 3)\, y = e^{-3x}$$

AE is $D^2 + 4D + 3 = 0$

$(D + 1)(D + 3) = 0 \qquad D = -1, \qquad D = -3$

$\therefore$ $CF = c_1e^{-x} + c_2e^{-3x}$

$$PI = \frac{1}{(D+1)\,(D+3)} e^{-3x}$$

put $D = -3$ in $f(D)$ then

$$PI = \frac{1}{0} e^{-3x} \qquad \therefore \quad \text{method fails}$$

$\therefore$
$$PI = \frac{e^{-3x}}{(-3+1)\,(D-3+3)}\{1\} = \frac{e^{-3x}}{(-2)} \frac{1}{D}\{1\} = \frac{e^{-3x}}{-2} x$$

$G.S. = CF + PI$

$$y = c_1e^{-x} + c_2e^{-3x} - \frac{e^{-3x}}{2}x$$

Type II(A): When X is of the form sin ax or cos ax.

$$PI = \frac{1}{f(D^2)}\sin ax \quad \text{or} \quad \frac{1}{f(D^2)}\cos ax$$

$$= \frac{1}{f(-a^2)}\sin ax \quad \text{or} \quad \frac{1}{f(-a^2)}\sin ax$$

In this case put the quantity – a^2 in the place of D^2, we can not put any thing for D as is imaginary. D^3 on substitution shall become $D^2 . D = -a^2D$, D^4 will be $D^2 . D^2 = (-a^2)(-a^2)$.

In otherwords $f(D)$ shall be reduced to a linear factor of the form $lD - m$ or $D^1 + m$. Then multiply both numerator and denominator by a conjugate factor $lD + m$ or $lD - m$ respectively, the denominator shall be reduced to $l^2 D^2 - m^2$ in which again put $D^2 = -a^2$ and it will become a constant i.e., $l^2(-a^2) - m^2 = -a^2 l^2 - m^2$.

$$\therefore \qquad PI = \frac{la \cos ax \pm m \sin ax}{-a^2l^2 - m^2}$$

Type II B: (Failure case)

When X is of the form sin ax or cos ax but $f(D)$ becomes zero when we put $D^2 = -a^2$.

$$PI = \frac{1}{f(D^2)}\sin ax \quad \text{or} \quad \frac{1}{f(D^2)}\cos ax$$

Put $D^2 = -a^2$

$$\therefore \qquad PI = \frac{1}{0}\sin ax \quad \text{or} \quad \frac{1}{0}\cos ax$$

$\therefore$ Our method fails

Then $$PI = \frac{1}{D^2 + a^2}\sin ax = \frac{x}{2}\int \sin ax\, dx$$

and $$PI = \frac{1}{D^2 + a^2}\cos ax = \frac{x}{2}\int \cos ax\, dx$$

Example 1: Solve $(D^2 + D + 1)\, y = \sin 2x$.

Solution: Given equation is

$$(D^2 + D + 1)\, y = \sin 2x$$

AE is $D^2 + D + 1 = 0$

$$D = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore \quad CF = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

And $$PI = \frac{1}{D^2 + D + 1}\sin 2x$$

Put $D^2 = -z^2$

$$= \frac{1}{-4 + D + 1}\sin 2x = \frac{1}{D-3}\sin 2x$$

$$= \frac{(D+3)}{D^2 - 9}\sin 2x$$

Put again $D^2 = -2^2$

$$= \frac{D(\sin 2x) + (3\sin 2x)}{-4-9}$$

$$= \frac{2\cos 2x + 3\sin 2x}{-13}$$

$$\therefore \quad G.S. = CF + PI$$

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x)$$

Example 2: Solve $(D^2 - 5D + 6)\, y = \cos 3x$

Solution: Given equation is

$$(D^2 - 5D + 6)\, y = \cos 3x$$

$\therefore$ AE is $D^2 - 5D + 6 = 0$

$(D-2)(D-3) = 0$ $\quad\therefore D = 2, 3$ are the roots.

$$\therefore \quad CF = c_1 e^{2x} + c_2\, e^{3x}$$

And $$PI = \frac{1}{D^2 - 5D + 6}\cos 3x$$

Put $$D^2 = -3^2$$

$$= \frac{1}{-9 - 5D + 6}\cos 3x$$

$$= \frac{-1}{5D + 3}\cos 3x$$

$$= \frac{-(5D - 3)}{25D^2 - 9}\cos 3x$$

Put $D^2 = -3^2$ again

$$= \frac{-[5D(\cos 3x) - 3(\cos 3x)]}{25 \times (-9) - 9}$$

$$= \frac{[-15\sin 3x - 3\cos 3x]}{-234}$$

$$= \frac{-(15\sin 3x + 3\cos 3x)}{234}$$

$\therefore$ $$G.S. = C.F. + P.I.$$

$$y = c_1 e^{2x} + c_2 e^{3x} - \frac{1}{234}(15\sin 3x + 3\cos x)$$

Example 3: Solve $(D^2 + 4)y = \cos 2x$.

Solution: Given equation is

$$(D^2 + 4)\, y = \cos 2x$$

$\therefore$ AE is $D^2 + 4 = 0$ $\qquad \therefore \quad D = \pm\, 2i$

$$C.F. = c_1 \cos 2x + c_2 \sin 2x$$

And

$$PI = \frac{1}{D^2 + 4}\cos 2x$$

Put $$D^2 = -2^2$$

$$= \frac{1}{0}\cos 2x \qquad \therefore \quad \text{Our method fails.}$$

$$\therefore \quad P.I. = \frac{1}{D^2+4}\cos 2x = \frac{x}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2}\frac{\sin 2x}{2} = \frac{x}{4}\sin 2x$$

$$\therefore \quad G.S. = C.F. + P.I.$$

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4}\sin 2x.$$

Example 4: Solve $(D^2 + 4)(D^2 + 1)\, y = \cos 2x + \sin x$.

Solution: Given equation is

$$(D^2 + 4)(D^2 + 1)\, y = \cos 2x + \sin x$$

AE is $(D^2 + 4)(D^2 + 1) = 0$

$$\therefore \quad D = \pm 2i \quad \text{and} \quad D = \pm i$$

$$CF = (c_1 \cos 2x + c_2 \sin 2x) + (c_3 \cos x + c_4 \sin x)$$

Now, PI for $\cos 2x$ $= \dfrac{1}{(D^2+4)(D^2+1)}\cos 2x$

$$= \frac{1}{(-4+4)(-4+1)}\cos 2x = \frac{1}{0}\cos 2x$$

$$\therefore \quad PI = \frac{1}{(-4+1)}\frac{\cos 2x}{D^2+4}$$

$$= \frac{1}{-3}\frac{x}{2}\int \cos 2x\, dx = -\frac{x}{6}\frac{\sin 2x}{2} = -\frac{x}{12}\sin 2x \qquad ...(1)$$

($\because$ put $D^2 = -2^2$ in those factors which do not vanish)

And PI for $\sin x$ $= \dfrac{1}{(D^2+4)(D^2+1)}\sin x$

put $D^2 = -1^2$ in those factors which do not vanish.

$$= \frac{1}{(-1+4)}\frac{x}{2}\int \sin x\, dx = +\frac{1}{3}\frac{x}{2}(-\cos x)$$

$$= -\frac{x}{6}\cos x \qquad ....(2)$$

$$\therefore \quad G.S. = CF + PI_1 + PI_2$$

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos x + c_4 \sin x - \frac{x}{12}\sin 2x - \frac{x}{6}\cos x$$

Type III: When X is of the form x^m.

$$PI = \frac{1}{f(D)} x^m$$

To find the PI of this type one should remember the following expansions.

i) $(1 - D)^{-1} = 1 + D + D^2 + D^3 + \dots\dots\dots$

ii) $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \dots\dots\dots$

iii) $(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots\dots\dots$

iv) $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots\dots\dots$

v) $(1 - D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \dots\dots\dots$

vi) $(1 + D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \dots\dots\dots$

Example 1: Solve $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 5 + 2x$.

Solution: The given equation is

$$(2D^2 + 5D + 2)\, y = 5 + 2x$$

AE is $\quad 2D^2 + 5D + 2 = 0$

$$(2D + 1)(D + 2) = 0 \qquad \therefore \quad D = -\frac{1}{2}, \quad D = -2$$

$$\therefore \quad C.F. = c_1 e^{-\frac{1}{2}x} + c_2 e^{-2x}$$

$$P.I. = \frac{1}{2D^2 + 5D + 2}(5 + 2x)$$

$$= \frac{1}{2}\left[1 + \frac{(2D^2 + 5D)}{2}\right]^{-1} (5 + 2x)$$

$$= \frac{1}{2}\left[1 - \frac{(2D^2 + 5D)}{2} + \dots\dots\right](5 + 2x)$$

$$= \frac{1}{2}\left[5+2x-\frac{5}{2}\,.\,2\right]$$

$$= x$$

$\therefore$ $G.S. = CF + PI$

$$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{-2x} + x$$

Example 2: Solve $(D^2 + 2D + 1)\, y = x^2 + e^x - \sin x + 2^x$.

Solution: Given equation is

$$(D^2 + 2D + 1)\, y = x^2 + e^x - \sin x + 2^x$$

AE is $D^2 + 2D + 1 = 0$

$(D + 1)^2 = 0$ $\therefore$ $D = -1, -1$

$\therefore$ $C.F. = (c_1 x + c_2)\, e^{-x}$

1) PI for $x^2 = \dfrac{1}{(1+D)^2} x^2$

$$= (1 + D)^{-2}\, (x^2)$$

$$= (1 - 2D + 3D^2 - 4D^3 \,......\,)\, x^2$$

$$= x^2 - 4x + 6$$

ii) PI for $e^x = \dfrac{1}{D^2 + 2D + 1}\, e^x$

Put $D = 1$ $= \dfrac{1}{1+2+1} e^x = \dfrac{1}{4} e^x$

iii) PI for $\sin x = \dfrac{1}{D^2 + 2D + 1} \sin x$

Put $D^2 = -1^2$, $= \dfrac{1}{-1+2D+1} = \dfrac{1}{2D} \sin x$

$$= \frac{1}{2}\int \sin x\, dx = -\frac{1}{2}\cos x \qquad \left(\because\ \frac{1}{D} = \int\right)$$

iv) PI for $2^x = \dfrac{1}{D^2 + 2D + 1}\, 2^x$

$$= \frac{1}{D^2 + 2D + 1} e^{x \log 2}$$

$$y = c_1e^{ax} + c_2e^{-ax} + c_3\cos ax + c_4\sin ax - \frac{1}{a^4}\left[x^4 + \frac{24}{a^4}\right] + \frac{1}{b^4 - a^4}\sin bx$$

Type IV: When X is of the form $e^{ax}V$, where V is any function of x.

$$PI = \frac{1}{f(D)}e^{ax}V$$

$$= e^{ax}\left\{\frac{1}{f(D+a)}\right\}V$$

Rule: It means that take out e^{ax} and in $f(D)$ write $D + a$ for every D so that $f(D)$ becomes $f(D + a)$ and then operate $\frac{1}{f(D+a)}$ with V alone by previous method.

Example 1: Solve $(D^2 - 4D + 3)\, y = e^{2x} \sin 3x$.

Solution: Given equation is

$$(D^2 - 4D + 3)\, y = e^{2x} \sin 3x$$

$\therefore$ *A.E.* is $D^2 - 4D + 3 = 0$

$(D - 1)(D - 3) = 0$ $\therefore\ D = 1, 3$

$\therefore$ $C.F. = c_1e^x + c_2\, e^{3x}$

And $PI = \frac{1}{D^2 - 4D + 3}e^{2x}\sin 3x$

$$= e^{2x}\left\{\frac{1}{(D+2)^2 - 4(D+2) + 3}\right\}\sin 3x$$

$$= e^{2x}\left\{\frac{1}{D^2 + 4D + 4 - 4D + 8 + 3}\right\}\sin 3x$$

$$= e^{2x}\left\{\frac{1}{D^2 - 1}\right\}\sin 3x$$

put $D^2 = -3^2$

$$= e^{2x}\left\{\frac{1}{-9 - 1}\right\}\sin 3x$$

$$= -\frac{1}{10}e^{2x}\sin 3x$$

$\therefore$ $$G.S. = CF + PI$$

$$y = c_1e^x + c_2\,e^{3x} - \frac{1}{10}e^{2x}\sin 3x$$

Example 2: Solve $(D^2 - 5D + 6)\,y = x\,(x + e^x)$

Solution:

Given equation is

$$(D^2 - 5D + 6)\,y = x\,(x + e^x)$$

$$= x^2 + xe^x$$

AE is $D^2 - 5D + 6 = 0$

$(D - 2)\,(D - 3) = 0$ $\quad\therefore\quad D = 2, 3$

$\therefore$ $$CF = c_1e^{2x} + c_2e^{3x}$$

And

i) PI for $$x^2 = \frac{1}{D^2 - 5D + 6}x^2$$

$$= \frac{1}{6}\left[1 + \frac{(D^2 - 5D)}{6}\right]^{-1}(x^2)$$

$$= \frac{1}{6}\left[1 - \frac{(D^2 - 5D)}{6} + \frac{(D^2 - 5D)^2}{6} - \ldots\ldots\ldots\ldots\right](x^2)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{25}{36}D^2\ldots\ldots\ldots\right](x^2)$$

$$= \frac{1}{6}\left[x^2 - \frac{2}{6} + \frac{10x}{6} + \frac{50}{36}\right]$$

$$= \frac{1}{6}\left[x^2 + \frac{10x}{6} + \frac{19}{18}\right]$$

$$= e^{-x}\left\{\frac{1}{(D-1+1)^2}\right\} x\sin x$$

$$= e^{-x}\frac{1}{D^2}(x\sin x)$$

$$= e^{-x}\frac{1}{D}\int x\sin x\, dx$$

$$= e^{-x}\frac{1}{D}\left[x(-\cos x)-(-\sin x)\right]$$

$$= e^{-x}\int(-x\cos x+\sin x)\, dx$$

$$= e^{-x}\left[-x\sin x-(-1)(-\cos x)-\cos x\right]$$

$$= e^{-x}\left[-x\sin x-2\cos x\right]$$

$$= -e^{-x}(x\sin x+2\cos x)$$

$$\therefore \qquad G.S = CF + PI$$

$$y = (c_1x + c_2)\, e^{-x} - e^{-x}(x\sin x + 2\cos x)$$

Example 4: Solve $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y = xe^x\cos x$

Solution: Given equation is

$$(D^2 - 2D + 1)\, y = xe^x \cos x$$

AE is $D^2 - 2D + 1 = 0$

$$(D-1)^2 = 0 \qquad \therefore \quad D = 1, 1$$

$$\therefore \qquad CF = (c_1x + c_2)\, e^x$$

And,

$$PI = \frac{1}{(D-1)^2} xe^x\cos x$$

$$= e^x\left\{\frac{1}{(D+1-1)^2}\right\} x\cos x = e^x\left\{\frac{1}{D^2}\right\} x\cos x$$

$$= e^x \cdot \frac{1}{D}\int x\cos x\, dx$$

$$= e^x \frac{1}{D}[x \sin x - 1\,(-\cos x)]$$

$$= e^x \int (x \sin x + \cos x)\, dx$$

$$= e^x [x\,(-\cos x) + \sin x + \sin x]$$

$$= e^x (-x \cos x + 2 \sin x)$$

$\therefore \quad GS = CF + PI$

$$y = (c_1 x + c_2)\, e^x + e^x (-x \cos x + 2 \sin x)$$

Example 5: Solve $e^x \dfrac{d^2 y}{dx^2} + 2e^x \dfrac{dy}{dx} + e^x y = x^2$

Solution: Given equation is

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = x^2\, e^{-x}$$

$\therefore \quad AE$ is $D^2 + 2D + 1 = 0$

$$(D + 1)^2 = 0 \qquad \therefore\ D = -1, 1$$

$\therefore \quad CF = (c_1 x + c_2)\, e^{-x}$

And, $\quad PI = \dfrac{1}{(D+1)^2} x^2\, e^{-x}$

$$= e^{-x} \left\{\frac{1}{D^2}\right\} x^2$$

$$= e^{-x} \frac{1}{D} \int x^2\, dx = e^{-x} \frac{1}{D}\left(\frac{x^3}{3}\right)$$

$$= \frac{e^{-x}}{3} \int x^3 dx = \frac{e^{-x}}{3} \cdot \frac{x^4}{4}$$

$$= \frac{e^{-x} x^4}{12}$$

$\therefore \quad GS = CF + PI$

$$y = (c_1 x + c_2)\, e^{-x} + \frac{1}{12} e^{-x}\, x^4$$

$\therefore$ $G.S. = CF + PI$

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{9}\ (3x \sin x - 2 \cos x)$$

Example 2: Solve $\dfrac{d^2y}{dx^2} + y = x^2 \sin x$

Solution: Given equation is

$(D^2 + 1)\ y = x^2 \sin x$

$\therefore$ AE is $D^2 + 1 = 0$ $\therefore$ $D = \pm i$

$\therefore$ $CF = c_1 \cos x + c_2 \sin x$

$$PI = \frac{1}{D^2+1} x^2 \sin x$$

$$= \frac{1}{D^2+1} \text{ imaginary part of } x^2 e^{ix}$$

$$= \text{Imaginary part of } e^{ix} \left\{\frac{1}{(D+i)^2+1}\right\} x^2$$

$$= \text{Imaginary part of } e^{ix} \left\{\frac{1}{D^2+2Di-1+1}\right\} x^2$$

$$= \text{Imaginary part of } e^{ix} \frac{1}{2Di}\left[1+\frac{D}{2i}\right]^{-1} (x^2)$$

$$= \text{Imaginary part of } e^{ix} \frac{1}{2D_i}\left[1-\frac{D}{2i}+\frac{D^2}{4i^2} \ldots\ldots\ldots\right] (x^2)$$

$$= \text{Imaginary part of } \frac{e^{ix}}{2Di}\left[x^2-\frac{2x}{2i}-\frac{2}{4}\right]$$

$$= \text{Imaginary part of } \frac{e^{ix}}{2Di}\left[x^2+ix-\frac{1}{2}\right]$$

$$= \text{Imaginary part of } \frac{e^{ix}}{2i}\left[\frac{x^3}{3}+\frac{ix^2}{2}-\frac{x}{2}\right]$$

$$= \text{Imaginary part of } \frac{e^{ix}}{12i}(2x^3 - 3x + ix^2)$$

$$= \text{Imaginary part of } \frac{1}{12i} (\cos x + i \sin x)\,(2x^3 - 3x + i\, 3x^2)$$

$$= \text{Imaginary part of } \frac{1}{12i} [(\cos x)\,(2x^3 - 3x) - (\sin x)\, 3x^2$$

$$+ i\,(2x^3 - 3x) \sin x + 3x^2 \cos x]$$

$$= -\frac{1}{12}(2x^3 - 3x) \cos x - 3x^2 \sin x$$

Taking imaginary part only.

$\therefore \quad GS = CF + PI$

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{12}(2x^3 - 3x) \cos x - 3x^2 \sin x.$$

Example 3: Solve $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax$

Solution: Given equation is

$$(D^2 + a^2)\, y = \sec ax$$

$\therefore \quad AE$ is $\quad D^2 + a^2 = 0 \qquad \therefore \; D = \pm\, ai$

$\therefore \quad CF = c_1 \cos ax + c_2 \sin ax$

And, $\quad PI = \dfrac{1}{D^2 + a^2} \sec ax$

$$= \frac{1}{(D + ai)\,(D - ai)} \sec ax$$

Resolve into partial fractions

$$= \frac{1}{2ai}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right] \sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D - ai} \sec ax - \frac{1}{D + ai} \sec ax\right]$$

Now,

$$PI = \frac{1}{D^2+a^2}\tan ax = \frac{1}{(D+ai)\,(D-ai)}\tan ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]\tan ax, \text{ by partial fractions}$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}\tan ax - \frac{1}{D+ai}\tan ax\right]$$

Now $\frac{1}{D-ai}\tan ax = e^{iax}\int \tan ax\, e^{-iax}\,dx$

$$= e^{iax}\int \tan ax(\cos ax - i\sin ax)\,dx$$

$$= e^{iax}\int \left(\sin ax - i\frac{\sin^2 ax}{\cos ax}\right)dx$$

$$= e^{iax}\int [\sin ax - i(\sec ax - \cos ax)]\,dx$$

$$= e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a}\log(\sec ax + \tan ax) + \frac{i\sin ax}{a}\right]$$

$$= -\frac{1}{a}e^{iax}\left[(\cos ax - i\sin ax) + i\log(\sec ax + \tan ax)\right]$$

$$= -\frac{1}{a}e^{iax}\left[e^{-iax} + i\log(\sec ax + \tan ax)\right]$$

$$= -\frac{1}{a}\left[1 + ie^{iax}\log(\sec ax + \tan ax)\right]$$

Changing i to $-i$ we get

$$\frac{1}{D+ai}\tan ax = -\frac{1}{a}\left[1 - ie^{-iax}\log(\sec ax + \tan ax)\right]$$

$$\therefore\ PI = \frac{1}{2ai}\left[-\frac{1}{a}\left(1 + ie^{iax}\log(\sec x + \tan x)\right) + \frac{1}{a}\left(1 - e^{-iax}\log(\sec ax + \tan ax)\right)\right]$$

$$= -\frac{1}{a^2}\log(\sec ax + \tan ax)\,.\,\cos ax$$

$\therefore \quad GS = CF + PI$

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log(\sec ax + \tan ax)$$

2.5 INITIAL VALUE PROBLEMS

Example 1: Solve $y'' + 4y' + 4y = 0$ given $y(0) = 3, y'(0) = 1$

Solution: Given equation is

$(D^2 + 4D + 4)\, y = 0$ with $x = 0, y = 3$ and $x = 0, y' = 1$

AE is $D^2 + 4D + 4 = 0$

$(D + 2)^2 = 0 \qquad \therefore \quad D = -2, -2$

$\therefore$ The solution is

$$y = (c_1 x + c_2)\, e^{-2x} \qquad \text{....(1)}$$

and

$$y' = (c_1 x + c_2)\, e^{-2x}(-2) + c_1 e^{-2x} \qquad \text{....(2)}$$

Given $y = 3$ when $x = 0$ and $y^1 = 1$ when $x = 0$

$\therefore \quad c_2 = 3$ and $c_1 = 7$

$\therefore$ The complete solution is

$$y = (7x + 3)\, e^{-2x}$$

Example 2: Solve the initial value problem $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y + 2\cosh x = 0$

given $y = 0$, $\dfrac{dy}{dx} = 1$ at $x = 0$.

Solution: Given equation is

$$(D^2 + 4D + 5)\, y = -2\cosh x$$

$$= \frac{-2(e^x + e^{-x})}{2} = -(e^x + e^{-x})$$

AE is $D^2 + 4D + 5 = 0$

$$D = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

$$\therefore \quad CF = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$$= \cos a \frac{x}{2} \int \sin x \, dx + \sin a \, . \frac{x}{2} \int \cos x \, dx$$

$$= -\cos a . \frac{x \cos x}{2} + \sin a \frac{x \sin x}{2}$$

$\therefore$ $GS = CF + PI$

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2} \cos a \; x \cos x + \frac{1}{2} \sin a \; x \sin x$$

and $$y' = -c_1 \sin x + c_2 \cos x - \frac{1}{2} \cos a \cos x + \frac{1}{2} \cos a \, . \, x \sin x$$

$$+ \frac{1}{2} \sin a \sin x + \frac{1}{2} \sin a \, . \, x \cos x$$

Given $y = 0$, $y' = 0$ when $x = 0$

$\therefore$ $0 = c_1$ $\quad \therefore \quad c_1 = 0$

and $0 = c_2 - \frac{1}{2} \cos a \quad \therefore \quad c_2 = \frac{1}{2} \cos a$.

$\therefore$ The complete solution is

$$y = \frac{1}{2} \cos a \sin x - \frac{1}{2} \cos a x \cos x + \frac{1}{2} \sin a \; x \sin x$$

$$= \frac{1}{2} \cos a \, (\sin x - x \cos x) + \frac{1}{2} \sin a \, x \sin x$$

Example 4: Solve the initial value problem $\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 0$ given $y\,(0) = 0$, $\frac{dy}{dx}(0) = 15$.

Solution: Given equation is

$$(D^2 + 5D + 6)\, y = 0$$

$\therefore$ AE is $D^2 + 5D + 6 = 0$

$(D + 2)\,(D + 3) = 0$ $\qquad \therefore \; D = -2, -3$

$\therefore$ CF is $y = c_1 e^{-2x} + c_2 \, e^{-3x}$

and $$\frac{dy}{dx} = -\, 2c_1 e^{-2x} - 3c_2 \, e^{-3x}$$

At $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 15$

$\therefore$ $0 = c_1 + c_2$

$15 = -2c_1 - 3c_2$ Solving $c_2 = -15$ and $c_1 = 15$

$\therefore$ The complete solution is

$$y = 15e^{-2x} - 15e^{-3x}$$
$$= 15\,(e^{-2x} - e^{-3x})$$

2.6 SIMULTANEOUS DIFFERENTIAL EQUATIONS OF FIRST ORDER

The differential equation in which there is one independent variable and two or more than two dependent variables are called simultaneous linear differential equations. Here we consider simultaneous linear equation with constant coefficients only.

Example 1: Solve $\dfrac{dx}{dt} + y = \sin t$; $\dfrac{dy}{dt} + x = \cos t$

Solution: The simultaneous equation are

$$Dx + y = \sin t \quad \text{....(1)}$$

$$Dy + x = \cos t \quad \text{....(2)} \qquad \text{where } \frac{d}{dt} = D$$

Multiply (1) by D

$\therefore$ $D^2x + Dy = \cos t$ ....(3)

$\therefore$ $Dy + x = \cos t$ ....(4)

Subtracting (4) from (3) we get

$$D^2x - x = 0$$

$\therefore$ AE is $(D^2 - 1) = 0$ $\therefore$ $D = 1, -1$

$\therefore$ $x = c_1\, e^t + c_2\, e^{-t}$ ....(1)

And $Dx + y = \sin t$

$$\therefore \quad y = \sin t - \frac{d}{dt}\left(c_1e^t - c_2e^{-t}\right)$$

$$y = \sin t - c_1e^t - c_2\, e^{-t}$$

2. Solve $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

Ans.: $y = (c_1 x + c_2) e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3.$

3. Solve $\dfrac{d^4 y}{dx^4} - y = \cos x \cosh x$

Ans.: $y = c_1 e^x + c_2 c^{-x} + c_3 \cos x + c_4 \sin x - \dfrac{1}{5}\cos x \cosh x$.

4. Solve $\dfrac{d^2 y}{dx^2} + y = \operatorname{cosec} x$

Ans.: $y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x \cos x.$

5. Solve $(D^2 - 4D + 3) y = \sin 3x \cos 2x$

$$= \frac{1}{2}[\sin 5x + \sin x]$$

Ans.: $y = c_1 e^x + c_2 e^{3x} + \dfrac{1}{884}(10\cos 5x - 11\sin 5x) + \dfrac{1}{20}(\sin x + 2\cos x)$

6. Solve $(D^2 - 3D + 2) y = 6 e^{-3x} + \sin 2x.$

Ans.: $y = c_1 e^x + c_2 e^{2x} + \dfrac{3}{10}e^{-3x} + \dfrac{1}{20}(3\cos 2x - \sin 2x)$

7. Solve $(D^2 - 4) y = (1 + e^x)^2$

Ans.: $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4} - \dfrac{2}{3}e^x + \dfrac{1}{4}xe^{2x}$

8. Solve $\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

Ans.: $y = c_1 e^x + c_2 e^{2x} + \dfrac{1}{4}e^{3x}(2x - 3) + \dfrac{1}{20}(3\cos 2x - \sin 2x)$

9. Solve $\dfrac{d^2 y}{dx^2} + 4y = 4\tan 2x$

Ans.: $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

Case VII: If $X = e^{ax}(a_0 + a_1x + a_2x^2 + \dots\dots\dots + an\, x^n)$ sin bx or cos bx then the particular integral is of the form.

$y = e^{ax}\{(A_0 + A_1x + \dots\dots + A_nx^n)\sin bx + (B_0 + B_1x + \dots + B_nx^n)\cos bx\}$

Note: However, when X = tan x or sec x, this method fails, since the number of terms obtained by differentiating X = tan x or sec x is infinite.

Example 1: Solve by the method of undetermined coefficients,

$y'' - 3y' + 2y = x^2 + x + 1$

Solution: Given differential equation is

$$y'' - 3y' + 2y = x^2 + x + 1 \qquad \dots(1)$$

AE is $D^2 - 3D + 2 = 0$

$$(D-1)(D-2) = 0 \qquad \therefore\ D = 1, 2$$

$$\therefore \quad CF = c_1e^x + c_2\,e^{2x}$$

PI is of the form

$$y = A_0 + A_1x + A_2x^2$$
$$y' = 0 + A_1 + 2A_2x$$
$$y'' = 0 + 2A_2$$

Substituting in (1)

$$2A_2 - 3(A_1 + 2A_2x) + 2(A_0 + A_1x + A_2x^2) = x^2 + x + 1$$

$$\therefore \quad 2A_2x^2 + (2A_1 - 6A_2)x + 2A_0 - 3A_1 + 2A_2 = x^2 + x + 1$$

Comparing the coefficients

$$2A_2 = 1, \qquad 2A_1 - 6A_2 = 1, \qquad 2A_0 - 3A_1 + 2A_2 = 1$$

Solving, $A_2 = \dfrac{1}{2}$, $\qquad A_1 = 2, \quad A_0 = 3$

$$\therefore \quad PI = 3 + 2x + \frac{1}{2}x^2$$

$$\therefore \quad GS = CF + PI$$

$$y = c_1e^x + c_2e^{2x} + \left(3 + 2x + \frac{1}{2}x^2\right)$$

Example 2: Solving by the method of undetermined coefficients the equation $y'' - 2y' + 5y = e^{2x}$.

Solution: Given equation is

$$y'' - 2y' + 5y = e^{2x} \qquad \text{...(1)}$$

AE is $\quad D^2 - 2D + 5 = 0$

$$\therefore \quad D = 1 \pm 2i$$

$$\therefore \quad CF = e^x (c_1 \cos 2x + c_2 \sin 2x)$$

PI is of the form

$$y = Ae^{2x}$$

$$\therefore \quad y' = 2Ae^{2x} \quad \text{and} \quad y'' = 4Ae^{2x}$$

Substituting in (1)

$$4Ae^{2x} - 2 + 2Ae^{2x} + 5Ae^{2x} = e^{2x}$$

$$\therefore \quad 5A\, e^{2x} = e^{2x}$$

$$\therefore \quad 5A = 1 \qquad \therefore \quad A = \frac{1}{5}$$

$$\therefore \quad PI = \frac{1}{5} e^{2x}$$

$$\therefore \quad G.S. = CF + PI$$

$$y = e^x (c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{5} e^{2x}$$

Example 3: Solve by the method of undetermined coefficients, the equation $y'' - 5y' + 6y = \sin 2x$.

Solution: Given differential equation is

$$y'' - 5y' + 6\,y = \sin 2x \qquad \text{....(1)}$$

AE is $\quad D^2 - 5D + 6 = 0 \qquad \therefore \; D = 2, 3$

$$\therefore \quad CF = c_1 e^{2x} + c_2\, e^{3x}$$

PI is of the form

$$y = A \cos 2x + B \sin 2x$$

$$y' = -2A \sin 2x + 2B \cos 2x$$

$$y'' = -4A \cos 2x - 4B \sin 2x$$

Substituting in (1)

$$\therefore \qquad PI = -\frac{1}{4} - \frac{1}{2}x - \frac{3}{10}\sin x - \frac{1}{10}\cos x$$

$$\therefore \qquad G.S. = CF + PI$$

$$y = c_1 e^x + c_2\, e^{-2x} - \frac{1}{4} - \frac{1}{2}x - \frac{3}{10}\sin x - \frac{1}{10}\cos x$$

Example 5: Solve by the method of undetermined coefficients the equation $y'' + 4y = x^2 + e^{-x}$.

Solution: The given differential equation is

$$y'' + 4y = x^2 + e^{-x} \qquad ....(1)$$

AE is $D^2 + 4 = 0 \qquad \therefore \quad D = \pm\, 2i$

$$\therefore \qquad CF = c_1 \cos 2x + c_2 \sin 2x$$

Particular integral is of the form

$$y = A_0 + A_1 x + A_2 x^2 + Be^{-x}$$

$$\therefore \qquad y' = A_1 + 2A_2 x - Be^{-x}$$

$$y'' = 2A_2 + Be^{-x}$$

Substituting in (1)

$$(2A_2 + Be^{-x}) + 4(A_0 + A_1 x + A_2 x^2 + Be^{-x}) = x^2 + e^{-x}$$

$$\therefore \quad (4A_0 + 2A_2) + 4A_1 x + 4A_2 x^2 + 5Be^{-x} = x^2 + e^{-x}$$

Equating the coefficients, we get

$4A_0 + 2A_2 = 0 \qquad$ Solving

$4A_1 = 0 \qquad A_0 = -\frac{1}{8}, \qquad A_1 = 0$

$4A_2 = 1 \qquad A_2 = \frac{1}{4}, \qquad B = \frac{1}{5}$

$$\therefore \qquad PI = -\frac{1}{8} + \frac{1}{4}x^2 + \frac{1}{5}e^{-x}$$

$$\therefore \qquad G.S. = CF + PI$$

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{8} + \frac{1}{4}x^2 + \frac{1}{5}e^{-x}.$$

Example 6: Solve by the method of undetermined coefficients
$y'' - y' - 4y = x + \cos 2x$.

Solution:

The given differential equation is

$$y'' - y' - 4y = x + \cos 2x \qquad ....(1)$$

$\therefore$ AE is $D^2 - D - 4 = 0$

$$\therefore \quad D = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

$$\therefore \quad CF = c_1\, e^{\frac{(1+\sqrt{17})}{2}x} + c_2\, e^{\frac{(1-\sqrt{17})}{2}x}$$

Particular integral is of the form

$$\begin{aligned} y &= A_0 + A_1 x + B_0 \cos 2x + B_1 \sin 2x \\ y' &= A_1 - 2B_0 \sin 2x + 2B_1 \cos 2x \\ y'' &= -4B_0 \cos 2x - 4B_1 \sin 2x \end{aligned}$$

Substituting in (1)

$$(-4\, B_0 \cos 2x - 4B_1 \sin 2x) - (A_1 - 2B_0 \sin 2x + 2B_1 \cos 2x) \\ - 4\,(A_0 + A_1 x + B_0 \cos 2x + B_1 \sin 2x) = x + \cos 2x.$$

Equating the coefficients

$-4\,A_0 - A_1 = 0$ Solving

$-4\,A_1 = 1$ $\quad A_0 = \dfrac{1}{16},\quad A_1 = -\dfrac{1}{4}$

$-8B_0 - 2B_1 = 1$ $\quad B_0 = \dfrac{-2}{17},\quad B_1 = \dfrac{-1}{34}$

$2B_0 - 8B_1 = 0$

$$\therefore \quad PI = \frac{1}{16} - \frac{1}{4}x - \frac{2}{17}\cos 2x - \frac{1}{34}\sin 2x$$

$$\therefore \quad G.S. = CF + PI$$

$$y = c_1 e^{\frac{(1+\sqrt{17})x}{2}} + c_2 e^{\frac{(1-\sqrt{17})x}{2}} + \frac{1}{16} - \frac{1}{4}x - \frac{2}{17}\cos 2x - \frac{1}{34}\sin 2x$$

Example 7: Solve by the method of undetermined coefficients $\frac{d^2y}{dx^2} + y = \sin x$.

Solution: Given differential equation is

$$y'' + y = \sin x \qquad ...(1)$$

AE is $D^2 + 1 = 0 \qquad \therefore \quad D + \pm i$

$$\therefore \qquad CF = c_1 \cos x + c_2 \sin x$$

Note that '$\sin x$' is common in CF and RHS of the equation and therefore particular integral is of the form

$$y = x(A\cos x + B\sin x)$$

$$y' = x(-A\sin x + B\cos x) + (A\cos x + B\sin x)$$

$$y'' = x(-A\cos x - B\sin x) + (-A\sin x + B\cos x) + (-A\sin x + B\cos x)$$

Substituting in (1)

$$-2A\sin x + 2B\cos x = \sin x$$

Equating the coefficients

$$-2A = 2 \qquad \text{and} \qquad 2B = 0$$

$$\therefore \quad A = -\frac{1}{2} \qquad \text{and} \qquad B = 0$$

$$\therefore \qquad PI = -\frac{1}{2}x\cos x$$

$$\therefore \qquad GS = CF + PI$$

$$y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$$

Example 8: Solve by the method of undetermined coefficients the equation $(D^2 + 1)y = 4x - 2\sin x$.

Solution: The given differential equation is

$$y'' + y = 4x - 2\sin x \qquad ....(1)$$

AE is $D^2 + 1 = 0 \qquad \therefore \; D = \pm i$

$$\therefore \qquad CF = c_1\cos x + c_2\sin x$$

Equating the coefficients

$$2A_2 - 3A_1 + 2A_0 = 0 \qquad A_0 = \frac{7}{4}, \qquad A_1 = \frac{3}{2}$$

$$2A_2 = 1 \qquad A_2 = \frac{1}{2} \qquad B = -1$$

$$\therefore \quad PI = \frac{7}{4} + \frac{3}{2}x + \frac{1}{2}x^2 - xe^x$$

$$\therefore \quad G.S. = CF + PI$$

$$y = c_1e^x + c_2e^{2x} + \frac{7}{4} + \frac{3}{2}x + \frac{1}{2}x^2 - xe^x$$

Example 10: Solve by the method of undetermined coefficients the following equations.

i) $(D^2 + 1)\,y = 4x \cos x - 2 \sin x$ ....(1)

Hint: $CF = c_1 \cos x + c_2 \sin x$

Since 'cos x and sin x' are common in CF and RHS of (1) particular integral is of the form.

$$y_1 = x\,[(A_0 + A_1x) \cos x + (A_2 + A_3x) \sin x]$$

and $y_2 = PI$ for $2 \sin 2x$

$$= x\,\{B_0 \cos x + B_1 \sin x\}$$

$$\therefore \quad y = y_1 + y_2$$

ii) $(D^2 - 1)\,y = 10 \sin^2 x$

Hint: $= \dfrac{10(1-\cos 2x)}{2} = 5 - 5\cos 2x$

PI is of the form

$$y = A_0 + \{B_0 \cos x + B_1 \sin x\}$$

iii) $(D^2 - 1)\,y = e^{-x}\,(2 \sin x + 4 \cos x)$

Hint: PI is of the form

$$y = e^{-x}\,(A \cos x + B \sin x)$$

iv) $(D^3 - D)\,y = 3e^x + \sin x$ ....(1)

Hint: $D(D^2 - 1) = 0 \qquad D = 0, \qquad D = 1 \qquad D = -1$

$$CF = c_1 + c_2e^x + c_3\,e^{-x}$$

e^x is common in CF and RHS of (1)

$\therefore PI$ is of the form

$$y = A_0 xe^x + (B_0 \sin x + B_1 \cos x)$$

v) $(D^2 - 5D + 6)\, y = e^{2x} + \sin x$

Hint:

PI is of the form

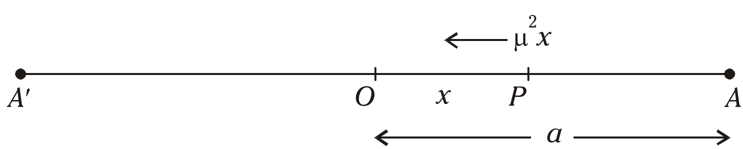
$$y = Axe^{2x} + (B_0 \sin x + B_1 \cos x)$$

2.8 APPLICATIONS TO LINEAR DIFFERENTIAL EQUATIONS

The applications of linear differential equations to various physical problems play a dominant role in uniflying seemingly different theories of mechanical and electrical systems just by renaming the variables. This analogy has an important practical application. Since electrical circuits are easier to assembe, less expensive and accurate measurements can be made of electrical quantities.

A. Simple harmonic mothion (S.H.M.)

A particle is said to execute simple harmonic motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line is proportional to the distance of the particle from the fixed point.



Let O be the fixed point in the line $A'A$. Let P be the position of the particle at any time t where

$$OP = x$$

Since the acceleration is always directed towards O i.e., the acceleration is in the direction opposite to that in which x increases, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu^2 x \quad \text{or} \quad (D^2 + \mu^2)x = 0 \quad \text{where } \frac{d}{dt} = D \qquad ...(1)$$

which is the linear differential equation with constant coefficient.

$\therefore$ The solution of equation (1) is

$$x = c_1 \cos \mu t + c_2 \sin \mu t \qquad \text{....(2)}$$

Velocity of the particle at P is

$$\frac{dx}{dt} = -c_1\mu \sin \mu t + c_2\mu \cos \mu t \text{....(3)}$$

If the particle starts from rest at A, where $OA = a$ then from (2) at $t = 0$, $x = 0$. $\therefore c_1 = a$.

and from (3), at $t = 0$, $\frac{dx}{dt} = 0 \quad \therefore \; c_2 = 0$

$$\therefore \qquad x = -a\,\mu \sin \mu t \qquad \text{...(4)}$$

and
$$\frac{dx}{dt} = -a\mu\sqrt{1-\cos^2 \mu t} = -a\mu\sqrt{1-\frac{x^2}{a^2}} \qquad \text{...(5)}$$

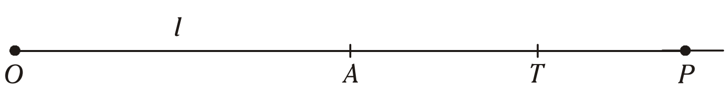
$$\therefore \qquad \frac{dx}{dt} = -\mu\sqrt{a^2 - x^2} \qquad \text{...(6)}$$

Hence equation (4) gives the displacement of the particle from the fixed point O at any time t.

Equation (6) gives the velocity of the particle at any time t. Equation (6) also shows that the velocity is directed towards O and decreases as x increases.

Example 1: In the case of a stretched elastic horizontal string which has one end fixed and a particle of mass m attached to the other, find the equation of motion of the particle given that l is natural length of the string and e is its elongation due to a weight mg. Also find the displacement of the particle when initially $s = s_0$, $v = 0$.

Solution: Let $OA = l$ be the elastic horizontal string with the end O fixed and a particle of mass m attached at A.



Let P be the position of the particle at any time t

Let $OP = s$ so that the elongation $AP = s - l$.

Now, for the elongation e tension $= mg$

$\therefore$ For the elongation $(s - l)$, tension $= \dfrac{mg(s-l)}{e}$

Since tension is the only horizontal force acting on the particle, its equation of motion is

$$m\frac{d^2 s}{dt^2} = -T$$

$$\therefore \quad m\frac{d^2 s}{dt^2} = -\frac{mg(s-l)}{e}$$

$$\text{or} \quad \frac{d^2 s}{dt^2} = -\frac{g}{e}s + \frac{gl}{e}$$

$$\therefore \quad \frac{d^2 s}{dt^2} + \frac{g}{e}s = \frac{gl}{e} \quad \text{or} \quad \left(D^2 + \frac{g}{e}\right) s = \frac{gl}{e} \qquad \text{....(1) when } D = \frac{d}{dt}$$

Which is the linear differential equation

Its AE is $D^2 + \dfrac{g}{e} = 0 \qquad \therefore \; D = \pm i\sqrt{\dfrac{g}{e}}$

$$\therefore \quad CF = c_1 \cos\left(\sqrt{\frac{g}{e}}\right)t + c_2 \sin\left(\sqrt{\frac{g}{e}}\right)t\,.$$

$$\text{And,} \quad PI = \frac{1}{D^2 + \frac{g}{e}}\frac{gl}{e} = \frac{gl}{e}\frac{1}{D^2 + \frac{g}{e}}e^{0t} = \frac{gl}{e}\cdot\frac{1}{g/e} = l$$

$\therefore$ The complete solution of (1) is

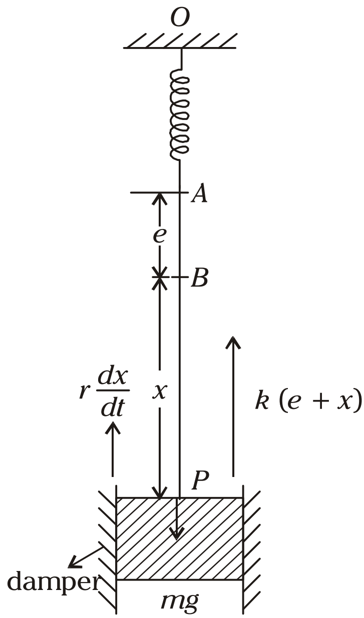
$$s = c_1 \cos\left(\sqrt{\frac{g}{e}}\right)t + c_2 \sin\left(\sqrt{\frac{g}{e}}\right)t + l \qquad ...(2)$$

When $t = 0$, $s = s_0$ from (2) we get

$$s_0 = c_1 + 0 + l \quad \text{or} \quad c_1 = s_0 - l$$

Also,

b) Damped Oscillations



If the motion of the mass m be subject to an additional force of resistance, the oscillations are said to be damped. The damping force may be constant or proporational to velocity. The latter type of damping is important and is usually called viscous damping.

Now, if the damping force be proportional to velocity $\left(\text{say} = r\,\frac{dx}{dt}\right)$ then the equation of motion becomes

$$m\,\frac{d^2x}{dt^2} = mg - k\,(e+x) - r\,\frac{dx}{dt}$$

$$= -kx - r\,\frac{dx}{dt}$$

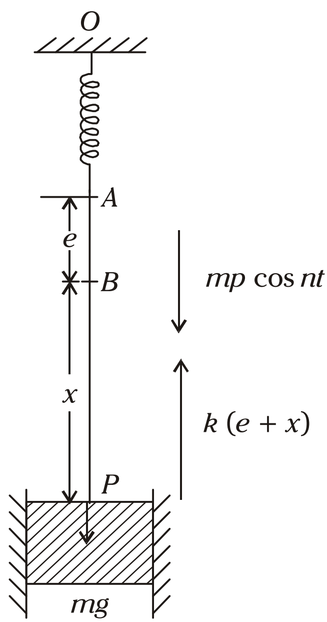
Taking $\frac{r}{m} = 2\lambda$ and $\frac{k}{m} = \mu^2$.

We get

$$\frac{d^2x}{dt^2} + 2\lambda\,\frac{dx}{dt} + \mu^2 x = 0 \quad ....(3)$$

Which is the linear differential equation with constant coefficient.

c) Forced Oscillations (Without dumping)



If the point of the support of the spring is also vibrating with some external periodic force, then the resulting motion is called the forced oscillatory motion.

Taking the external periodic force to be $mp\cos nt$ the equation of the motion is

$$m\,\frac{d^2x}{dt^2} = mg - k\,(e+x) + mp\cos nt$$

$$= -\,kx + mp\cos nt$$

Taking $\frac{k}{m} = \mu^2$ the equation becomes

$$\frac{d^2x}{dt^2} + \mu^2 x = p \cos nt \qquad ....(4)$$

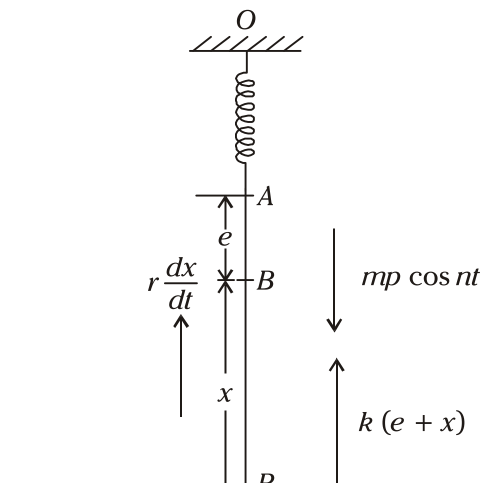
Which is linear differential equation.

The solution of equation (4) is

$$x = c_1 \cos \mu t + c_2 \sin \mu t + \frac{p \cos nt}{n^2 + \mu^2}$$

d) Forced Oscillations (With damping)

If in addition, there is a damping force proportional to velocity $\left(\text{say: } r\frac{dx}{dt}\right)$ then the above equation becomes.



$$m\frac{d^2x}{dt^2} = mg - k(e + x) + mp \cos nt - r\frac{dx}{dt}$$

$$= -kx + mp \cos nt - r\frac{dx}{dt}$$

Taking $\frac{r}{m} = 2\lambda$ and $\frac{k}{m} = \mu^2$ then the equation becomes

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \mu^2 x = p\cos nt$$

Which is a linear differential equation and its solution is

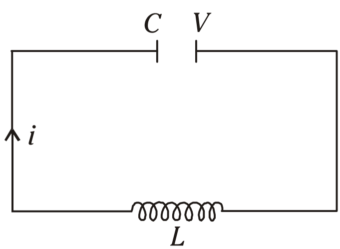
$$x = e^{-st}\left(c_1 e^{+\sqrt{\lambda^2-\mu^2}} + c_2 e^{-t\sqrt{\lambda^2-\mu^2}}\right) + \frac{p(\mu^2 - n^2)\cos nt + 2\lambda n \sin t}{(\mu^2 - n^2) + 4\lambda^2 n^2}$$

With the increase of time, the free oscillations die away while the forced oscillations continue giving the steady state motion.

C. Oscillatory Electrical Circuits

a) *L – C* Circuit

Consider an electrical circuit containing an inductance L and capacitance C.



Let i be the current and q be the charge in the condenser plate at any time t then the voltage drop across

$$L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

and voltage drop across $C = \frac{q}{C}$

As there is no applied e.m.f. in the circuit, therefore by Kirchooff's first law, we have

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

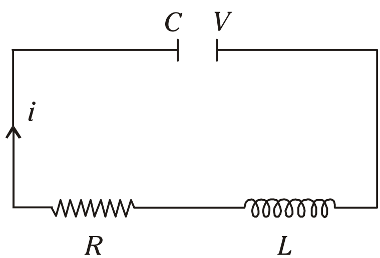
Dividing by L and taking $\frac{1}{LC} = \mu^2$ we get

$$\frac{d^2q}{dt^2} + \mu^2 q = 0 \qquad ...(1)$$

Which is linear differential equation and it reprsents free electrical oscillations of the current having period $\frac{2\pi}{\mu} = 2\pi\sqrt{Lc}$.

b) *L – C – R* Circuit

Consider the discharge of a condenser through an inductance L and the resistance R. Since the voltage drop across L, C and R are respectively.



$$L\frac{d^2q}{dt^2}, \quad \frac{q}{C} \quad \text{and} \quad R\frac{dq}{dt}$$

By Kirchhoff's law, we have

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

Taking $\frac{R}{L} = 2\lambda$ and $\frac{1}{LC} = \mu^2$ we get

$$\frac{d^2q}{dt^2} + 2\lambda\frac{dq}{dt} + \mu^2 q = 0 \qquad ....(2)$$

Which is the linear differential equation.

By Kirchhoff's law the equation is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=p\cos nt$$

Dividing by L

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=\frac{p}{L}\cos nt$$

Taking $\frac{R}{L}=2\lambda$ and $\frac{1}{LC}=\mu^2$, we have

$$\frac{d^2q}{dt^2}+2\lambda\frac{dq}{dt}+\mu^2 q=\frac{p}{L}\cos nt$$

Which is the linear differential equation.

Note: The $L-C-R$ circuit with a source of alternating e.m.f. is an electrical equivalent of the mechanical phenomena of forced oscillations with resistance.

Example 1: In an $L-C-R$ circuit, the charge q on a plate of a condenser is given by

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin pt$$

The circuit is tunned to resource so that $p^2=\frac{1}{LC}$. If initially the current i and the charge q be zero, show that for small values of $\frac{R}{L}$. The current in the circuit at time t is given by $\frac{Et}{2L}\sin pt$.

Solution: Given differential equation is

$$\left(LD^2+RD+\frac{1}{C}\right)q=E\sin pt \qquad ....(1)$$

$\therefore$ AE is $LD^2+RD+\frac{1}{C}=0$

$$\therefore \qquad D = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

As $\frac{R}{L}$ is small, we have

$$D = -\frac{R}{2L} \pm i\frac{1}{\sqrt{LC}} = -\frac{R}{2L} \pm ip$$

$$\therefore \qquad CF = e^{-\frac{Rt}{2L}}(c_1 \cos pt + c_2 \sin pt)$$

$$= \left(1 - \frac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt)$$

$$\left(\because \text{ rejecting the terms in } \left(\frac{R}{L}\right)^2 \text{ etc.}\right)$$

And
$$PI = \frac{1}{LD^2 + RD + \frac{1}{C}} E \sin pt$$

$$= \frac{E \sin pt}{-Lp^2 + RD + \frac{1}{C}} = \frac{E}{R}\int \sin pt\, dt \qquad \because \; p^2 = \frac{1}{LC}$$

$$= -\frac{E}{Rp}\cos pt$$

Thus the complete solution is

$$q = \left(1 - \frac{Rt}{L}\right)\left(c_1 \cos pt + c_2 \sin pt\right) - \frac{E}{Rp}\cos pt \qquad \text{...(ii)}$$

$$\therefore \qquad i = \frac{dq}{dt} = \left(1 - \frac{Rt}{L}\right)(-c_1 \sin pt \times p + c_2 \cos pt \times p)$$

$$+\left(-\frac{R}{L}\right)(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R}\sin pt \qquad \text{....(iii)}$$

When $t = 0$, $q = 0$, $i = 0$

$\therefore$ From (ii) $0 = c_1 - \dfrac{E}{Rp}$ $\qquad\therefore\ c_1 = \dfrac{E}{Rp}$

and from (iii) $0 = c_2 p - c_1 \dfrac{R}{2L}$ $\qquad\therefore\ c_2 = \dfrac{Rc_1}{2Lp} = \dfrac{E}{2Lp^2}$

$$\therefore \quad i = \left(1 - \frac{Rt}{L}\right)\left(-\frac{E}{Rp}\sin pt + c_2 \cos pt\right)p$$

$$- \frac{R}{2L}\left(\frac{E}{Rp}\cos pt + \frac{E}{2Lp^2}\sin pt\right) + \frac{E}{R}\sin pt$$

$$= \frac{Et}{2L}\sin pt \qquad \left(\because \frac{R}{L} \text{ is small}\right)$$

D. Deflection of Beams

Example 1: The deflection of a strut of length l with one end ($x = 0$) built in and the other supported and subjected to end trust p satisfies the equation.

$$\frac{d^2y}{dx^2} + a^2 y = \frac{a^2 R}{p}(l - x)$$

Prove that the deflection curve is $y = \dfrac{R}{p}\left(\dfrac{\sin ax}{a} - l\cos ax + l - x\right)$ and $al = \tan al$.

Solution: Given differential equation is

$$(D^2 + a^2)y = \frac{a^2 R}{p}(l - x) \qquad ....(1)$$

$\therefore$ Its AE is $D^2 + a^2 = 0$ $\qquad\therefore\ D = \pm ai$

$\therefore$ $CF = c_1 \cos ax + c_2 \sin ax$

And $PI = \dfrac{1}{D^2 + a^2}\dfrac{a^2 R}{p}(l - x)$

$$= -\frac{w}{2EIa^2}\left(1-\frac{D^2}{a^2}\right)^{-1}(x^2-lx)$$

$$= -\frac{w}{2p}\left(x^2-lx+\frac{2}{a^2}\right) \text{ where } \quad p = EIa^2$$

Thus the complete solution of (1) is

$$y = c_1 \cosh ax + c_2 \sinh ax - \frac{w}{2p}\left(x^2-lx+\frac{2}{a^2}\right) \qquad \text{...(ii)}$$

At the end O, $t = 0$ when $x = 0$

$\therefore$ (ii) gives $O = c_1 - \dfrac{w}{pa^2}$ $\qquad \therefore \; c_1 = \dfrac{w}{pa^2}$

At the end A, $y = 0$ when $x = l$

$\therefore$ (ii) gives, $O = c_1 \cosh al + c_2 \sinh al - \dfrac{w}{pa^2}$

$$\therefore \quad c_2 \sinh al = \frac{w}{pa^2}(1-\cosh al) \qquad \because \; c_1 = \frac{w}{pa^2}$$

$$\therefore \quad c_1 = \frac{-w}{pa^2}\tanh\frac{al}{2}$$

Substituting in (ii)

$$y = \frac{w}{pa^2}\left(\cosh ax - \tanh\frac{al}{2}\sinh ax\right) - \frac{w}{2p}\left(x^2-lx+\frac{2}{a^2}\right)$$

Which is the deflection of the beam at N.

Thus the central deflection $= y\left(\text{at } x = \dfrac{l}{2}\right)$

$$y = \frac{w}{pa^2}\left(\cosh\frac{al}{2} - \tanh\frac{al}{2}\sinh\frac{al}{2} - 1\right) + \frac{wl^2}{8p}$$

$$= \frac{w}{pa^2}\left(\sec h\,\frac{al}{2} - 1\right) + \frac{wl^2}{8p}$$

Also the bending moment is maximum at the point of maximum deflection $\left(x = \frac{l}{2}\right)$.

$\therefore$ The maximum bending moment

$$EI\frac{d^2y}{dx^2}\left(\text{at } x = \frac{l}{2}\right) = py + \frac{w}{2}(x^2 - lx)\left(\text{at } x = \frac{l}{2}\right)$$

$$= \frac{w}{a}\left(\text{sech } \frac{al}{2} - 1\right)$$

EXERCISES

1. A particle is executing simple harmonic motion with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distance 15 cm and 5 cm from the centre of force and are on the same side of it.

Ans.: 0.38 sec.

2. An elastic string of natural length a is fixed at one end and a particle of mass m hangs freely from the other end. The modulus of elasticity is mg. The particle is pulled down a further distance l below its equilibrium position and released from rest. Show that the motion of the particle is simple harmonic and find the periodicity.

3. The differential equation of a simple pendulum is

$\frac{d^2x}{dt^2} + w_0^2 x = F_0 \sin nt$, where w_0 and F_0 are constants. If initially $x = 0$,

$\frac{dx}{dt} = 0$ determine the motion when $w_0 = n$.

Ans.: $x = \frac{F_0}{2n^2}(\sin nt - n\cos nt)$

4. An e.m.f. $E \sin pt$ is applied at $t = 0$ to a circuit containing a capacitance C and inductance L. The current i satisfies the equation $L\frac{di}{dt} + \frac{1}{C}\int i\, dt$

2.9 QUESTION BANK

A. Choose the correct answer from the given alternatives.

1. A general solution of a linear differential equation of n^{th} order contains.

   (a) one constant (b) n–constants

   (c) zero constants (d) two constants

2. The solution of the differential equation $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} = 0$ is

   (a) $y = c_1x + c_2e^{3x}$ (b) $y = c_1 + c_2e^{3x}$

   (c) $y = (c_1 + c_2x)\, e^{3x}$ (d) $c_1x = c_2$

3. The solution of the differential equation $(D^2 + a^2)\, y = 0$ is

   (a) $y = c_1e^{ax} + c_2e^{-ax}$ (b) $y = c_1 \cos ax + c_2 \sin ax$

   (c) $y = (c_1x + c_2) \cos ax$ (d) $y = e^x (c_1 \cos x + c_2 \sin x)$

4. The solution of the differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$ is

   (a) $y = c_1e^{-x} + c_2e^x$ (b) $y = (c_1x + c_2)\, e^x$

   (c) $y = (c_1x - c_2)\, e^{-x}$ (d) $y = e^x (c_1 \cos x + c_2 \sin x)$

5. The complementary function of $y'' + 2y' + y = xe^{-x} \sin x$ is

   (a) $c_1e^x + c_2e^{-x}$ (b) $(c_1x + c_2)\, e^{-x}$

   (c) $(c_1x + c_2)\, e^x$ (d) $c_1 + e_2\, e^x$

6. Particular integral of the differential equation $(D^2 + 5D + 6)\, y = e^x$ is

   (a) e^x (b) $\dfrac{e^x}{12}$ (c) $\dfrac{e^x}{30}$ (d) $\dfrac{e^x}{6}$

7. Particular integral of the differential equation $(D^2 + 4)\, y = \sin 2x$ is

   (a) $\dfrac{x}{2}\sin 2x$ (b) $-\dfrac{x}{4}\cos 2x$ (c) $\dfrac{x}{2}\cos 2x$ (d) $\dfrac{x}{4}\cos 2x$

8. When $X = e^{ax}V$ where V is any function of x then particular integral is

18. Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

19. Solve $\frac{d^3y}{dx^3} + y = 0$

20. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0,\ y(0) = 4,\ y'(0) = 1$

C. Questions carrying six marks

21. Solve $(D^3 + 1)\, y = \cos(2x - 1)$

22. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

23. Solve $(D - 2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

24. Solve $(D^2 - 4D + 3)\, y = \sin 3x \cos 2x$.

25. Solve $(D^2 - 2D + 1)\, y = xe^x \sin x$.

26. Solve $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

27. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8x^2 e^{2x} \sin 2x$

28. Solve the following simultaneous equations

$$\frac{dx}{dt} = 7x - y, \quad \frac{dy}{dt} = 2x + 5y.$$

29. Solve $\frac{dx}{dt} + y = \sin t$; $\frac{dy}{dt} + x = \cos t$ given $x = 2$, $y = 0$ when $t = 0$.

30. Solve $\frac{dx}{dt} + 5x - 2y = 5,\ \frac{dy}{dx} + 2x + y = 0$ given $x = 0$, $y = 0$, when $t = 0$.

(26) $y = c_1 \cos ax + c_2 \sin ax + \dfrac{1}{a} x \sin ax + \dfrac{1}{a^2} \cos ax \log \cos ax$

(27) $y = (c_1 + c_2 x)\, e^{2x} - e^{2x}\, [4x \cos 2x + (2x^2 - 3) \sin 2x]$

(28) $x = e^{6t}\, [c_1 \cos t + c_2 \sin t]$

$y = e^{6t}\, [(c_1 - c_2) \cos t + (c_1 + c_2) \sin t]$

(29) $x = e^t + e^{-k},\; y = e^{-t} - e^t + \sin t$

(30) $x = -\dfrac{1}{27}(1 + 6t)e^{-3t} + \dfrac{1}{27}(1 + 3t)$

$y = \dfrac{-2}{27}(2 + 3t)e^{-3t} + \dfrac{2}{27}(2 - 3t)$