Imperial College London

Black holes and branes in supergravity: literature review

Muchen Li
Blackett Laboratory, Imperial College London

05th October 2020

Project Code: THEO-Stelle-1
Supervisor: Professor Kellogg Stelle
Assessor: Professor Daniel Waldram
Word Count: 2469

We give an overview of essential concepts in general relativity, which is a theory that manifests gravity as spacetime curvature. Beginning with manifolds, we see how familiar concepts from relativistic electromagnetism reappear and are generalised. We introduce differential forms, associated operations, and characterise curvature using the Riemann tensor, through the so-called Christoffel connection. Using these mathematical frameworks, we derive an action for gravitation, and solve for the equations of motion. Furthermore, we present the extension of electromagnetism to curved spacetime, and discuss a particular solution: the extreme Reissner–Nordström black hole.

I. Introduction

General relativity (GR) is a *geometrical* theory of gravity which is characterised by spacetime curvature, and succeeds Newtonian gravity. It is the Equivalence Principle, which postulates that locally, gravity is indistinguishable from ordinary acceleration, that gave grounds for the geometrical theory, rather than describing gravity as an independent dynamical field. However, GR is a classical description of nature, which is fundamentally incompatible with quantum field theory. Supergravity aims to resolve this issue by combining elements of supersymmetry and GR, and although presently viewed as an effective theory [1], it is nonetheless a vital ingredient in the formulation of a theory of quantum gravity.

In this literature review, we build the theory of GR from the ground up, assuming knowledge of relativistic electromagnetism, to a level where we'll be able to tackle supergravity in extended dimensions. In particular, we will introduce concepts, tools, and techniques which will be reused and generalised in the context of supergravity.

II. Pseudo-Riemannian manifolds

We begin our mathematical preliminaries with an introduction to manifolds, which have the correct mathematical structure to describe spacetime curvatures. Roughly speaking, an n-manifold M is a set which can be mapped to coordinates in $\mathbf{R}^n$ using one or more mappings [2]. For the purposes of GR, our manifolds must specifically be **pseudo-Riemannian**, satisfying two criterions: they are locally similar to $\mathbf{R}^n$, and come equipped with an inner product, characterizable by an everywhere nondegenerate metric [3]. We shall examine these statements in turn.

The first requirement is such that we may build local linear spaces and use techniques from vector calculus. Consider a parameterised path $x^\mu(\lambda)$ on M, and a function $f : M \to \mathbf{R}$. We define the directional derivative of f by $df/d\lambda$[1] at every point $p \in M$ along the path [2]. For every p, the space of all possible directional derivatives form a linear (vector) space, because the derivative is a linear operator, and satisfies the product rule, such that the space closes. In addition, application of the chain rule yields

$$\frac{df}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} = \frac{dx^\mu}{d\lambda}\partial_\mu f \ . \tag{2.1}$$

Since f is arbitrary, we may write

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu \ , \tag{2.2}$$

[1] Strictly speaking, this doesn't make sense as the domain of f is not $\mathbf{R}$. So, we also require the coordinate map(s) from M to $\mathbf{R}^n$, $\phi : M \to \mathbf{R}^n$, and its inverse. Then the parameterised path $x^\mu(\lambda)$ is simply a curve $\gamma(\lambda)$ on M (note the absence of a μ label, showing it is not labelled by coordinates), which is subsequently set onto the coordinates: $x^\mu = (\phi \circ \gamma)^\mu$. Thus, the directional derivative of f is formally, $d(f \circ \gamma)/d\lambda$. The notation $df/d\lambda$ is a shorthand. Likewise, $\partial_\mu f$ is shorthand for $\partial_\mu(f \circ \phi^{-1})$.

which shows the vectors $d/d\lambda$ have a basis $\{\partial_\mu\}$, and components $V^\mu = dx^\mu/d\lambda$.

Now, we recognise that coordinates are meaningless in the context of GR, since they are arbitrary labels we assign to some physical observable. In other words, observables, and thus the laws of physics, must remain invariant under coordinate transformations [2]. For vectors, a change of coordinates to $x^{\mu'}$ would imply the basis vectors transform as

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \ , \tag{2.3}$$

by the chain rule. For $d/d\lambda$ to remain invariant, we must therefore have

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}V^\mu \ . \tag{2.4}$$

This is a generalisation of the (3+1)D flat spacetime vector (component) transformation law, $V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$. Dual-vectors can then be defined analogously. It turns out their bases are the coordinate gradients $\{\mathrm{d}x^\mu\}$, and their components satisfy the (2.4) dual-analogue [2]

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\omega_\mu \ . \tag{2.5}$$

The familiar notion of tensor (components) $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots}$, their transformation law

$$T^{\mu_1{}'\mu_2{}'\cdots}{}_{\nu_1{}'\nu_2{}'\cdots} = \frac{\partial x^{\mu_1{}'}}{\partial x^{\mu_1}}\frac{\partial x^{\mu_2{}'}}{\partial x^{\mu_2}}\cdots\frac{\partial x^{\nu_1}}{\partial x^{\nu_1{}'}}\frac{\partial x^{\nu_2}}{\partial x^{\nu_2{}'}}\cdots T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\cdots} \ , \tag{2.6}$$

and typical operations on them, such as direct sums and tensor products, immediately follows [2, 4]. We emphasise here that whilst objects may have contra-/co-variant indices, they do not automatically qualify as tensors. General tensors, by definition, must satisfy (2.6), otherwise they do not represent quantities which are coordinate invariant, and thus cannot be named such. An example is the partial derivative of any non-scalar tensor, which transforms like

$$\begin{aligned}\partial_{\mu'}T^{\nu'}{}_{\rho'} &= \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\frac{\partial x^\rho}{\partial x^{\rho'}}T^\nu{}_\rho\right)\\ &\neq \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\frac{\partial x^\rho}{\partial x^{\rho'}}\partial_\mu T^\nu{}_\rho \ ,\end{aligned} \tag{2.7}$$

due to the product rule, showing that $\partial_\mu T^\nu{}_\rho$ is *not* a *general* tensor. The extension to arbitrary (n,p) tensors is simple. We will later deal with this by defining the covariant derivative, which *does* produce proper tensors.

The second requirement imposes the existence of a line element [2]

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \ , \tag{2.8}$$

characterised by the metric tensor $g_{\mu\nu}$, specific to M. The metric must also have an inverse defined everywhere by

$$g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu \ . \tag{2.9}$$

The metric is necessarily symmetric [4]; it raises and lowers indices *only on tensorial objects*. Furthermore, the line element characterises a particle's path:

$$\text{if } ds^2 \begin{cases} < 0, & \text{path is timelike} \\ = 0, & \text{path is null} \\ > 0, & \text{path is spacelike} \end{cases} , \tag{2.10}$$

which is the same characterisation used in special relativity.

Finally, in GR, we also impose the metric to be **Lorentzian**. The consequence is that, in a special coordinate system $x^{\hat{\mu}}$, which can always be found for any given point $p \in M$, we have [2]

$$g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}} , \qquad \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0 , \tag{2.11}$$

where $\eta_{\hat{\mu}\hat{\nu}}$ is the generalisation of the Minkowski metric in $n = \dim(M)$ dimensions:

$$\eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(-1, \underbrace{1, 1, \cdots, 1}_{(n-1) \text{ times}}) . \tag{2.12}$$

This is very useful, because we find all (non-singular) local geometry to be identical to that of special relativity, plus curvature described by second-order metric derivative terms.

III. Differential forms

We define a **differential *p*-form** A as any $(0, p)$ totally antisymmetric tensor. An example is the Levi-Civita *tensor* in n dimensions, $\epsilon_{\mu_1\mu_2\cdots\mu_n}$, defined in terms of the naïve Levi-Civita *symbol*, $\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$ [5]:

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{-g}\ \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} , \tag{3.1}$$

where $g \equiv \det(g_{\mu\nu}) < 0$, because we specified $g_{\mu\nu}$ to be Lorentzian. A p-form A and a q-form B can be combined together into a $(p+q)$-form through the **wedge product**, defined by the components

$$(A \wedge B)_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1\cdots\mu_p} B_{\nu_1\cdots\nu_q]} , \tag{3.2}$$

where $[\cdots]$ indicates the alternating sum of all index permutations, normalised by the number of terms. For example,

$$T_{[\mu\nu\rho]\sigma} = \frac{1}{3!}\left(T_{\mu\nu\rho\sigma} - T_{\mu\rho\nu\sigma} + T_{\nu\rho\mu\sigma} - T_{\nu\mu\rho\sigma} + T_{\rho\mu\nu\sigma} - T_{\rho\nu\mu\sigma}\right) . \tag{3.3}$$

Forms come with an **exterior derivative** operator, defined by the components

$$(\mathrm{d}A)_{\mu_1\mu_2\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]} . \tag{3.4}$$

It can be shown that exterior derivatives *are* tensorial objects with $(p+1)$ indices, unlike partial derivatives [2]. If a form A satisfies

$$\mathrm{d}A = 0 , \tag{3.5}$$

then it is said to be closed. Likewise, if A satisfies

$$A = \mathrm{d}B \ , \tag{3.6}$$

for some $(p-1)$-form B, then it is said to be exact. It can be easily shown that all exact forms are closed, using the identity [2]

$$\mathrm{d}(\mathrm{d}A) = 0 \ . \tag{3.7}$$

Moreover, a result in multivariate calculus, the *Poincaré Lemma*, states that all closed forms are globally exact if the manifold is *contractible*, and locally exact otherwise [6]. Importantly, it can be shown that Minkowski space and its generalisation to D dimensions, is contractible [6].

Forms also come with a **Hodge dual**, defined by the components

$$({}^{*}A)_{\mu_1\mu_2\cdots\mu_{n-p}} = \frac{1}{p!}\epsilon^{\nu_1\nu_2\cdots\nu_p}{}_{\mu_1\mu_2\cdots\mu_{n-p}}A_{\nu_1\nu_2\cdots\nu_p} \ , \tag{3.8}$$

which is itself a $(n-p)$-form [2].

These objects and operations will allow us to later rewrite Maxwell's equations in a form which can be easily generalised to higher dimensions.

IV. CONNECTION COEFFICIENTS AND CURVATURE

As previously discussed, only tensorial equations are adequate descriptions of physical laws. Equations such as $\partial_\mu T^{\mu\nu} = 0$: conservation of the electromagnetic stress-energy tensor in Minkowski space, must be somehow generalised. Let us now do this, and generalise the partial derivative to curved manifolds. We want an operator that satisfies the following properties when acting on vectors V and W:

1. tensorial: $\nabla_{\mu'} V^{\nu'} = \dfrac{\partial x^\mu}{\partial x^{\mu'}}\dfrac{\partial x^{\nu'}}{\partial x^\nu}\nabla_\mu V^\nu$;
2. linearity: $\nabla_\mu (V^\nu + W^\nu) = \nabla_\mu V^\nu + \nabla_\mu W^\nu$;
3. product rule: $\nabla_\mu (V^\nu W^\lambda) = W^\lambda \nabla_\mu V^\nu + V^\nu \nabla_\mu W^\lambda$. (4.1)

The latter two requirements imply we can write [2]

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \ , \tag{4.2}$$

for some matrices $\Gamma^\nu_{\mu\lambda}$. These matrices are known as **connection coefficients**, and the operator we have defined is named the **covariant derivative**. The first requirement implies that $\Gamma^\nu_{\mu\lambda}$ satisfies a transformation law [2]

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} \ , \tag{4.3}$$

showing that $\Gamma^\nu_{\mu\lambda}$ are *not* tensors. This can be understood easily because their role is to cancel the non-tensorial components of the partial derivative under coordinate transformations. In order to further specify the behaviour of ∇_μ and $\Gamma^\nu_{\mu\lambda}$, we impose four additional properties:

$$
\begin{array}{lll}
\text{4. contraction commutation:} & \nabla_\mu (T^\lambda{}_{\lambda\nu}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\nu} \ ; & \\
\text{5. reduction to partials on scalars:} & \nabla_\mu \phi = \partial_\mu \phi \ ; & \\
\text{6. torsion free:} & \Gamma^\nu_{\mu\lambda} = \Gamma^\nu_{\lambda\mu} \ ; & \\
\text{7. metric compatibility:} & \nabla_\lambda g_{\mu\nu} = 0 \ . & (4.4)
\end{array}
$$

These are all certainly reasonable sounding properties to impose, but they do not represent any physical law. For GR, we simply assume they hold, and see that they greatly simplify the resulting algebra.

Here, the former two requirements sets the action of ∇_μ on dual-vectors [2]:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \ , \tag{4.5}$$

and indeed, all tensors behave under the covariant derivative according to (4.2) and (4.5) for every contra-/co-variant index. The latter two requirements *uniquely* specify all of $\Gamma^\lambda_{\mu\nu}$ [2]. The resulting connection is named the **Christoffel symbol**[2], given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \ . \tag{4.6}$$

In fact, the metric compatibility requirement is redundant in GR. We will later derive the Einstein–Hilbert action for gravitation, which has $g_{\mu\nu}$ as its dynamical field. If we were to treat $\Gamma^\lambda_{\mu\nu}$ as an independent field, known as the **Palatini formalism**, then the requirement that $\Gamma^\lambda_{\mu\nu}$ be torsion free automatically guarantees metric compatibility, and thus the Christoffel symbols [2, 5].

Now, we define the **Riemann tensor**

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \ . \tag{4.7}$$

This object characterises the curvature of a given manifold. For example, given (4.6), the following statement regarding flatness is true [2]:

$$R^\rho{}_{\sigma\mu\nu} = 0 \ \Leftrightarrow \ \exists \ x^{\bar{\mu}} \text{ where } g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} \text{ everywhere.} \tag{4.8}$$

Whilst curvature generally depend on a connection, we see through (4.6) that $R^\rho{}_{\sigma\mu\nu}$ is characterised uniquely by the metric. Furthermore, it can be shown, given a Christoffel connection, that $R^\rho{}_{\sigma\mu\nu}$ satisfies a uniqueness theorem that allows itself to *uniquely* characterise curvature [7]. In other words, in GR, curvature is uniquely specified by the metric.

The Riemann tensor is antisymmetric with respect to its first two, and last two indices. This means its unique contraction is $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$, which is named the Ricci tensor. A further contraction forms the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$. These antisymmetry properties also imply [2]

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \ . \tag{4.9}$$

---

[2] Endearingly named the "Christ-awful" symbol by students everywhere.

Contracting four of the five indices yields

$$\nabla^{\mu}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=0\ . \tag{4.10}$$

These are named **Bianchi identities**, and we will see their relevance later on.

Having covered sufficient differential geometry, we are now in a position to do some physics. Namely: invent a theory of gravity, and see how it couples to electromagnetism.

V. Gravitation

Let us derive and vary an appropriate action for GR, which must be the integral of a *scalar* Lagrangian $\mathcal{L}$, and with only the metric as its dynamical field. This follows from the discussions in sections I and IV. We note that, as a consequence of (2.11), $\mathcal{L}$ must be nontrivial, such that it contains at least second derivatives in $g_{\mu\nu}$. However, $\mathcal{L}$ must not contain third and higher derivatives, or nonlinear second derivatives of $g_{\mu\nu}$, otherwise we induce Ostrogradsky instabilities into the system [8]. These instabilities are observed by constructing the corresponding Hamiltonian from $\mathcal{L}$, and noting that it is unbounded from below.

Thus, the only good candidate for the Lagrangian is a linear function in R:

$$\mathcal{L}=aR+b\ , \tag{5.1}$$

because $R^{\rho}{}_{\sigma\mu\nu}$ uniquely contain *linear* second derivatives of $g_{\mu\nu}$. The only unique contraction of $R^{\rho}{}_{\sigma\mu\nu}$ which remains linear in metric second derivatives is given by R.

The term b is related to the cosmological constant [2], which we discard for our purposes. We also set $a=1$ for now, and recover numerical constants later on. Thus, the action for GR, the **Einstein–Hilbert action**, is

$$S[g_{\mu\nu}]=\int d^n x\sqrt{-g}\,R_{\mu\nu}g^{\mu\nu}\ . \tag{5.2}$$

The factor of $\sqrt{-g}$ is due to the generalisation of integrals on curved manifolds. In short, it ensures the naïve volume element $d^n x$ transforms as a proper tensor, similar to our definition of the Levi-Civita tensor [2]. Variation of (5.2) yields three terms

$$\delta S=\int d^n x\left[\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}+R_{\mu\nu}g^{\mu\nu}\delta\sqrt{-g}+g^{\mu\nu}\sqrt{-g}\delta R_{\mu\nu}\right]\ . \tag{5.3}$$

The first term is already in the correct form, and the second term gives

$$\delta\sqrt{-g}=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\ , \tag{5.4}$$

where we have used a linear algebra result $\ln(\sin M)=\mathrm{tr}(\ln M)$ [2]. The third term can be evaluated by firstly varying $R^{\rho}{}_{\sigma\mu\nu}$ with respect to the connection, giving the **Palatini identity** [2]

$$\delta R_{\mu\nu}=\nabla_{\lambda}(\delta\Gamma^{\lambda}_{\nu\mu})-\nabla_{\nu}(\delta\Gamma^{\lambda}_{\lambda\mu})\ , \tag{5.5}$$

implying

$$\int d^n x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} = \int d^n x \sqrt{-g}\, \nabla_\lambda \left[g^{\mu\nu} \delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda} \delta\Gamma^\nu_{\nu\mu} \right] \ . \tag{5.6}$$

But we know $\Gamma^\lambda_{\mu\nu}$ is given by an algebraic equation in metric first derivatives. Suppose we impose that variations of the metric, and their first derivatives, vanish at infinity. Then we may use the generalised Stokes' theorem [2, 6] to convert this hypervolume integral into a hypersurface integral, which must vanish.

Therefore, (5.3) gives

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \ . \tag{5.7}$$

If we now set $a \neq 1$, and add a matter action term S_M associated with a stress-energy tensor $T_{\mu\nu}$, we find the general **Einstein's equations**

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{5.8}$$

where we impose $a = (16\pi G)^{-1}$, in order for Newtonian gravity to be recovered in the Newtonian limit [9]. This set of differential equations describe the curvature of spacetime in response to energy and momentum. Free particles in this spacetime react to the curvature by following trajectories described by the geodesic equation [2]

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \ , \tag{5.9}$$

where $x^\mu(\lambda)$ is the particle's parameterised path. We may choose $\lambda = \tau$, the proper time, defined by $d\tau^2 = -ds^2$, for timelike paths followed by massive particles. Hence, we have successively derived a theory of gravity, describing the interaction between spacetime curvature, energy-momentum, and matter.

Note that the Bianchi identity, through (5.8), becomes a statement about the conservation of energy-momentum:

$$\nabla^\mu T_{\mu\nu} = 0 \ . \tag{5.10}$$

VI. COUPLING TO ELECTROMAGNETISM

In units where $\mu_0 = 4\pi$, Maxwell's equations in flat spacetime are

$$\partial_\mu F^{\nu\mu} = 4\pi J^\nu \ , \qquad \partial_{[\mu} F_{\nu\sigma]} = 0 \ , \tag{6.1}$$

where J^ν is the four-current, and $F^{\mu\nu}$ the antisymmetric Faraday tensor, hence a 2-form. In curved spacetime, we generalise this through the **minimal coupling** procedure, where we simply replace $\partial_\mu \to \nabla_\mu$, and $\eta_{\mu\nu} \to g_{\mu\nu}$ [2]:

$$\nabla_\mu F^{\nu\mu} = 4\pi J^\nu \ , \qquad \partial_{[\mu} F_{\nu\sigma]} = 0 \ , \tag{6.2}$$

where we noted that the second equation still holds because all terms involving the connection cancel by antisymmetry. The resulting equations are, of course, not unique, but this is the simplest way to guarantee coordinate covariance (2.6), and reduction to (6.1) when spacetime is globally/locally flat. Given the notation we have introduced, we can rewrite (6.2) as [2]

$$\mathrm{d}({}^*F) = 4\pi\, {}^*J \ , \qquad \mathrm{d}F = 0 \ . \tag{6.3}$$

In Minkowski space, the second equation implies F is closed, and thus exact globally. For a general curved manifold, we find that *locally*,

$$F = \mathrm{d}A \ , \tag{6.4}$$

for some 1-form (dual-vector) A. This is the familiar equation of $F_{\mu\nu}$ in terms of the four-potential, when written in component form. Gauge invariance is expressed through

$$\begin{aligned} & A \to A + \mathrm{d}\chi(x) \\ \Rightarrow\ & F \to F \ , \end{aligned} \tag{6.5}$$

for some 0-form (scalar) $\chi(x)$. This is a local $U(1)$ gauge symmetry. In the vacuum, $J = 0$, there is additionally a dual invariance in (6.3), expressed through

$$\begin{aligned} & \mathbf{F} \to \mathbf{M} \cdot \mathbf{F} \\ \Rightarrow\ & \mathrm{d}\mathbf{F} = 0 \to \mathrm{d}\mathbf{F} = 0 \ , \end{aligned} \tag{6.6}$$

where $\mathbf{F} = (aF, b\,{}^*F)$, a and b are arbitrary constants, and $\mathbf{M} \in GL(2, \mathbf{R})$.

Defining the charge in a region Σ as

$$q(\Sigma) \equiv \int_\Sigma {}^*J \ , \tag{6.7}$$

we see that it can be rewritten, through (6.3) and Stokes' theorem, as

$$q(\Sigma) = \frac{1}{4\pi} \int_{\partial\Sigma} {}^*F \ , \tag{6.8}$$

and it can be shown to be conserved [10, 11].

All of the above results can be compactly described by the action for electromagnetism in curved spacetime [10]:

$$S[g_{\mu\nu}, A_\mu] = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 4\pi A_\mu J^\mu \right] \ . \tag{6.9}$$

Note that the dynamical field now include also the four-potential, in addition to the metric. In vacuum, variation with respect to $g_{\mu\nu}$ yields the Einstein equations [10]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \ , \tag{6.10}$$

where indeed, the electromagnetic stress-energy tensor appears on the right hand side. Variation with respect to A_μ, of course, give (6.3) with $J = 0$.

Solutions to Einstein's equations like (6.10), are elusive, because they describe highly nonlinear PDEs. We must impose high levels of symmetry to force a solution. Let us enforce spherical and time-reversal symmetry, in addition to time-independence on the metric:

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + e^{2\gamma(r)}r^2 d\Omega^2 \ . \tag{6.11}$$

We now quote a result. A generalised version of Birkhoff's theorem, applicable for $A_\mu \neq 0$, fixes the content of the three functions in (6.11):

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2 d\Omega^2 \ , \tag{6.12}$$

and states that this is the *unique* metric that describes our imposed symmetries, which solve (6.10) and (6.3) [2]. In fact, this spherical solution describes an electromagnetically charged black hole with mass M and charge Q. Whilst such objects would be quickly neutralised by nearby plasma in reality [12], they nonetheless encapsulate interesting physics, which will prove useful to us when we analyse branes in $D = 11$ supergravity.

For us, the case of interest is for $GM^2 = Q^2$, named the **extreme Reissner–Nordström solution**. The metric takes the form

$$ds^2 = -\left(1 - \frac{GM}{r}\right)^2\mathrm{d}t^2 + \left(1 - \frac{GM}{r}\right)^{-2}\mathrm{d}r^2 + r^2 d\Omega^2 \ , \tag{6.13}$$

and we see there is a singularity at $r = 0$, which is a curvature singularity, representing the divergence of physical observables, and another at $r = GM$, which is a coordinate singularity, representing a fault in the chosen coordinate system [2, 10]. Suitable coordinate transformations will allow us to overcome this, and analyse the spacetime causal structure everywhere. Such structures are typically represented by conformal diagrams, which is shown for the $GM^2 = Q^2$ case in Figure 1.

Finally, we mention that electromagnetism is fundamentally a gauge theory. There is a formulation of GR known as the **tetrad formalism**, where rather than using partial derivatives and gradients, we work with sets of local orthonormal vectors as bases. This formalism makes apparent how we should treat general gauge theories on curved manifolds [2]. Sadly[3], this topic is outside the scope of this review.

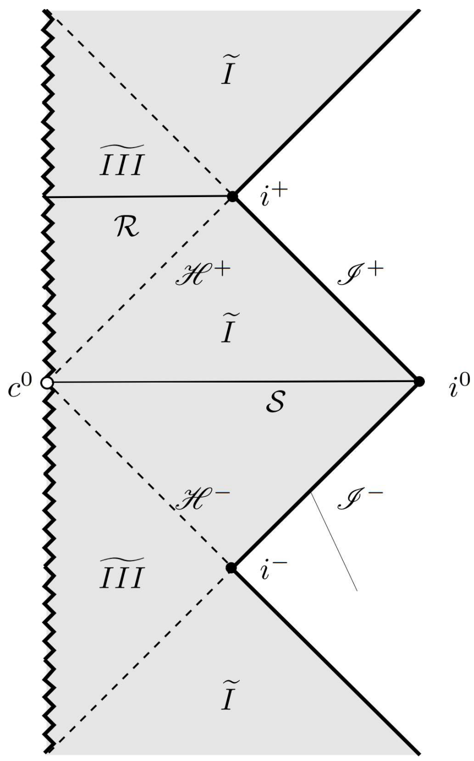

FIG. 1. A conformal diagram showing the casual structure of the extremal Reissner–Nordström solution [13: p.5]. A description of their conveyed information can be found in [10].

[3] Due to an overly cruel word limit, and my inability to overcome this issue through diplomacy.

VII. Conclusions

We are now equipped with the necessary tools to study supergravity. As an example, the four-potential is promoted to a three-form $A_{[3]}$ in $D = 11$ supergravity [14]. In the bosonic sector, this promotion leads to an identical action, plus a term described by wedge products of forms. The analysis of this action will proceed in the same way as what we have thus presented.

Acknowledgements

I would like to give thanks to the brilliant individual who thought it was a good idea setting the deadline of this literature review on the first day of term.

References

[1] Passias A. *Aspects of Supergravity in Eleven Dimensions.* Master's thesis. Imperial College London; 2010.

[2] Carroll SM. *Spacetime and Geometry: An Introduction to General Relativity.* [Google Books Edition]. Cambridge University Press; 2019.

[3] Chen B-Y. Pseudo-Riemannian Manifolds. In: *Pseudo-Riemannian Geometry, δ -Invariants and Applications.* World Scientific; 2011. p.1–24. Available from: doi:10.1142/8003 .

[4] Schutz B. *A First Course in General Relativity.* [Google Books Edition] 2nd ed. Cambridge University Press; 2009.

[5] Weinberg S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* USA: John Wiley & Sons; 1972.

[6] Warner FW. *Foundations of Differentiable Manifolds and Lie Groups.* Graduate Texts in Mathematics. USA: Springer; 1983.

[7] Segura W. *Uniqueness theorem of the curvature tensor.* [Preprint] 2019. Available from: doi:10.13140/RG.2.2.33183.71848 .

[8] Woodard RP. *The Theorem of Ostrogradsky.* 2015. Available from: arXiv:1506.02210 .

[9] Gauntlett J. *General Relativity Lecture Notes.* London: Department of Physics, Imperial College London; 2019.

[10] Dowker F. *Black Holes Lecture Notes.* London: Department of Physics, Imperial College London; 2015. Available from:
http://sns.ias.edu/~bkocsis/Assets/Teaching/BH2015/dowker.pdf
[Accessed 05th Oct 2020].

[11] Mohaupt T. *Black Holes in Supergravity and String Theory.* Halle-Wittenberg: Department of Physics, Martin Luther University Halle-Wittenberg; 2000. Available from: arXiv:hep-th/0004098v2 .

[12] Zajaček M, Tursunov A. Electric charge of black holes: Is it really always negligible? *Observatory.* 2019. Available from: arXiv:1904.04654v2 .

[13] Lübbe C, Kroon JAV. *On the conformal structure of the extremal Reissner-Nordström spacetime.* 2013. Available from: arXiv:1308.1325 .

[14] Stelle KS. *Brane Solutions in Supergravity.* London: Department of Physics, Imperial College London.