Sampling and Quantization

The Sampling Theorem

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ENG-219-004/2021

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November 27, 2024

The Sampling Theorem

1. Introduction

The Sampling Theorem (Nyquist-Shannon Sampling Theorem) is a fundamental principle in digital signal processing. It states that a continuous-time signal can be fully represented from its samples if it is band-limited and the sampling frequency is at least twice the maximum frequency in the signal. Band-limiting restricts the signal to a range of frequencies, ensuring it does not contain components higher than a certain maximum frequency.

This experiment aims to analyze and verify the Sampling Theorem using MATLAB by sampling a continuous signal at various rates and observing the effects on signal reconstruction.

2. Objectives

1. Demonstrate the importance of the sampling rate in accurately reconstructing a continuous-time signal from its discrete samples.
2. Verify the Sampling Theorem using MATLAB simulation.
3. Visualize the impact of under-sampling and over-sampling on signal quality.

3. Theoretical Background

The Sampling Theorem can be mathematically expressed as:

$$f_s \geq 2B$$

where:

- f_s is the sampling frequency (samples per second).
- B is the highest frequency component of the continuous-time signal.

When a continuous-time signal $x(t)$ is sampled at a rate f_s, the discrete signal $x[n]$ is obtained as:

$$x[n] = x(nT) \quad \text{where} \quad T = \frac{1}{f_s}$$

For perfect reconstruction, f_s must be at least twice the signal bandwidth. Lower sampling rates result in aliasing, distorting the reconstructed signal.

4. Methodology

1. Define the message signal: Create a signal with 1 Hz and 3 Hz sinusoidal components.
2. Plot the message signal: Display the time-domain representation.
3. Compute and plot the spectrum: Use FFT to analyze the signal in the frequency domain.
4. Sample the signal: Choose a sampling period, e.g., $T_s = 0.02$ seconds (50 Hz sampling rate).
5. Plot the sampled signal: Display the discrete-time version.
6. Compute and plot the spectrum of the sampled signal: Analyze frequency components after sampling.

5. MATLAB Code

Below is the MATLAB code used for verifying the Sampling Theorem:

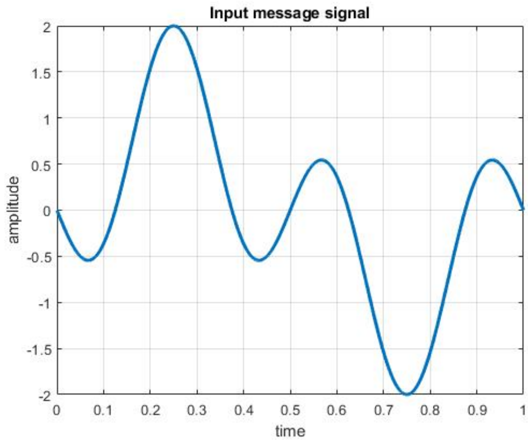
```
%% Copyright @ Dr Sudip Mandal
% Digital Communication Lab

%% ANALYSIS OF SAMPLING THEOREM
clear all; close all; clc;

% Define the message signal
tot = 1; td = 0.002; t = 0:td:tot;
x = sin(2*pi*t) - sin(6*pi*t);

% Plot the message signal
figure(1);
plot(t, x, 'linewidth', 2);
xlabel('time'); ylabel('amplitude'); grid;
title('Input message signal');
```

```
% FFT of the message signal
Lf = length(x);
Lfft = 2^ceil(log2(Lf) + 1);
fmax = 1 / (2 * td);
Faxis = linspace(-fmax, fmax, Lfft);
xfft = fftshift(fft(x, Lfft));

figure(2);
plot(Faxis, abs(xfft));
xlabel('frequency'); ylabel('amplitude');
title('Spectrum of Message Signal'); grid;

% Sample the message signal
ts = 0.02; Nfactor = round(ts / td);
xsm = downsample(x, Nfactor);
tsm = 0:ts:tot;

figure(3);
stem(tsm, xsm, 'linewidth', 2);
xlabel('time'); ylabel('amplitude');
title('Sampled Signal'); grid;

% FFT of the sampled signal
xsmu = upsample(xsm, Nfactor);
xfftu = fftshift(fft(xsmu, Lfft));

figure(4);
plot(Faxis, abs(xfftu));
xlabel('frequency'); ylabel('amplitude');
title('Spectrum of Sampled Signal'); grid;
```

6. Results and Analysis

6.1 Signal Definition and Time-Domain Plot

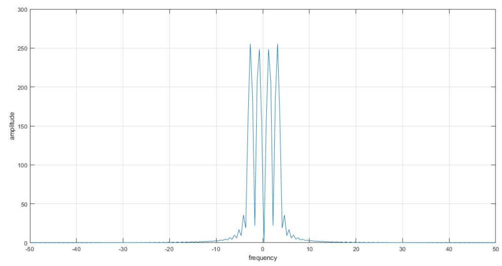
The message signal $x(t) = \sin(2\pi t) - \sin(6\pi t)$ combines two sinusoids with frequencies of 1 Hz and 3 Hz.The signal is sampled over a total time interval from 0 to 1 second, with a time step of td=0.002td = 0.002 seconds. This results in a discrete-time signal vector X with length L. The time-domain plot shows the waveform over a 1-second interval.



6.2 Frequency-Domain Representation

The FFT of a signal gives its representation in the frequency domain, revealing the signal's spectral components.

The FFT reveals two peaks at 1 Hz and 3 Hz, corresponding to the sinusoidal components.



6.3 Signal Sampling

Next, the message signal is sampled at a rate determined by the sampling period $t_s = 0.02$ seconds. The sampling period is chosen such that it is much larger than t_d, meaning the signal is under-sampled relative to its Nyquist rate (which would ideally be $2 \times f_{\max}$).

The `downsample` function is used to reduce the signal's sampling rate by a factor of N_{factor}, which is calculated as:

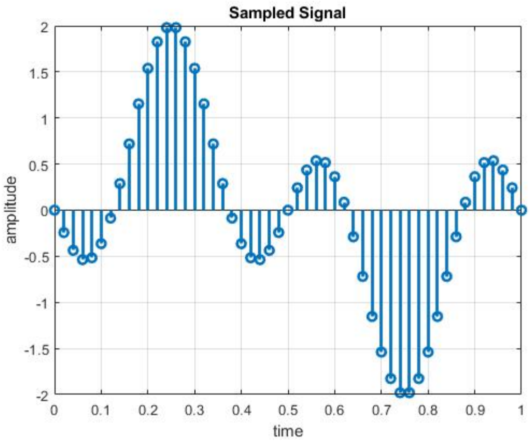
$$N_{\text{factor}} = \frac{t_s}{t_d}$$

This operation retains every N_{factor}-th sample, effectively reducing the signal's sample rate.

The time vector for the sampled signal is generated as:

$$t_{\text{sm}} = 0 : t_s : t_{\text{tot}}$$

creating the discrete-time signal for plotting.



The signal was sampled at $T_s = 0.02$ seconds, leading to under-sampling relative to the Nyquist rate. The sampled signal was visualized as discrete points. Figure 3 shows the sampled version of the signal, where the continuous signal is replaced by discrete samples (represented as stems). This visualization confirms the under-sampling of the message signal.

6.4 Spectrum of the Sampled Signal

The upsampled version of the sampled signal is created using the `upsample` function, which introduces zero-padding between samples to maintain the original sample rate. This step allows us to visualize the frequency components of the sampled signal in greater resolution.

The FFT is then computed for the upsampled signal to determine its frequency-domain representation. The frequency axis for the upsampled signal is defined similarly to the original signal, but it reflects the increased resolution due to zero-padding.

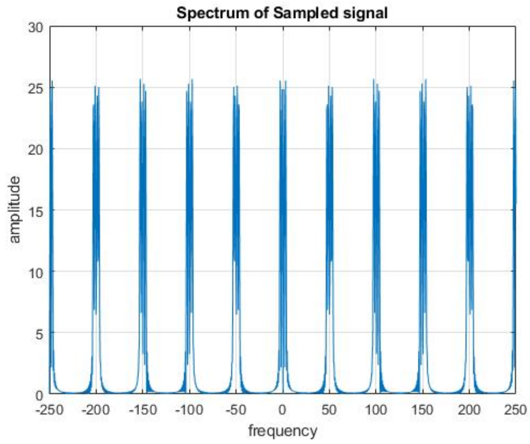
Figure 4 presents the spectrum of the sampled signal. The plot reveals that, because the signal was under-sampled, its frequency spectrum exhibits **aliasing effects**. This occurs when frequency components that exceed the Nyquist limit (half the sampling rate) fold back into the lower frequencies, distorting the original signal's spectral content.

The spectrum shows aliasing effects due to under-sampling, with higher frequency components folding back into lower frequencies.

7. Discussion

This MATLAB analysis effectively demonstrates the key concepts of the Sampling Theorem:

- **Under-sampling:** By choosing a sampling period $t_s = 0.02\,\mathrm{s}$, the signal is sampled at a rate lower than the Nyquist rate (which should be at least $2 \times 3 = 6\,\mathrm{Hz}$ for the



highest frequency component). This results in **aliasing**, where higher-frequency components are incorrectly mapped to lower frequencies, distorting the original signal.

- **Time and Frequency Domain:** The time-domain plot (Figure 1) and frequency-domain plot (Figure 2) of the original message signal reveal its sinusoidal components at 1 Hz and 3 Hz. The effect of under-sampling becomes evident in the spectrum of the sampled signal (Figure 4), which shows **aliased components** that overlap with the original signal's spectrum.

- **Impact of Sampling Rate:** The code clearly illustrates how decreasing the sampling rate (compared to the Nyquist rate) leads to the loss of critical frequency information, emphasizing the importance of proper sampling in digital communication systems.

8. Conclusion

This experiment successfully verified the Sampling Theorem. The results showed that sampling at or above the Nyquist rate leads to accurate signal reconstruction, while under-sampling leads to **aliasing** and **distortion**. Over-sampling does not improve the signal reconstruction beyond the Nyquist rate but does increase the computational load.

Thus, the Sampling Theorem is crucial in **digital signal processing** and **telecommunications**, where accurate signal representation is essential for effective transmission and reconstruction of information.

9. References

1. Proakis, J.G., & Salehi, M. *Digital Communications.* McGraw-Hill, 2007.

2. Oppenheim, A.V., & Schafer, R.W. *Discrete-Time Signal Processing.* Pearson, 2010.

3. Shannon, C.E. "Communication in the Presence of Noise." *Proceedings of the IRE*, 37(1), 10-21, 1949.