Kernel interpolation on sparse grids

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Outline

1. **Problem statement**
2. **Sparse grid approximation**
3. **Samplet kernel matrix compression**
4. **Numerical results**

Setting

- Consider the tensor product Hilbert space

$$\boldsymbol{\mathcal{H}} = \bigotimes_{i=1}^{m} \mathcal{H}^{(i)}.$$

- Each $\mathcal{H}^{(i)}$ is a reproducing kernel Hilbert space defined on a Lipschitz region $\Omega_i \subset \mathbb{R}^{d_i}$ of dimension $d_i \in \mathbb{N}$.
- The corresponding reproducing kernels are denoted by κ_i, $i = 1, \ldots, m$.
- The space $\boldsymbol{\mathcal{H}}$ is a reproducing kernel Hilbert space with product kernel

$$\boldsymbol{\kappa}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{m} \kappa_i(x_i, y_i)$$

defined on the product region $\boldsymbol{\Omega} = \Omega_1 \times \cdots \times \Omega_m$.

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- Given sets of interpolation points $X^{(i)} \subset \Omega_i$, $i = 1, \ldots, m$, we define the product grid $\boldsymbol{X} := \{[x_1, \ldots, x_m] : x_i \in X^{(i)}\}$.
- Associated to $\boldsymbol{X}$, we introduce the subspace $\mathcal{H}_{\boldsymbol{X}} := \operatorname{span}\{\kappa(\boldsymbol{x}, \cdot) : \boldsymbol{x} \in \boldsymbol{X}\}$.
- By the reproducing property, the $\mathcal{H}$-orthogonal projection $f_{\boldsymbol{X}}$ of any $f \in \mathcal{H}$ onto $\mathcal{H}_{\boldsymbol{X}}$ is the interpolant

$$f_{\boldsymbol{X}}(\boldsymbol{x}_i) := \sum_{j=1}^{|\boldsymbol{X}|} \alpha_j \kappa(\boldsymbol{x}_j, \boldsymbol{x}_i) = f(\boldsymbol{x}_i) \quad \text{for all } \boldsymbol{x}_i \in \boldsymbol{X}.$$

- Letting $\boldsymbol{f} := [f(\boldsymbol{x}_i)]_i$, it can be computed by solving the linear system

$$(\boldsymbol{K}^{(1)} \otimes \cdots \otimes \boldsymbol{K}^{(m)})\boldsymbol{\alpha} = \boldsymbol{f}$$

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- **Goal**: Approximately solve the linear system and have control on the approximation error.

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Univariate approximation

- Let $X_0^{(i)} \subset X_1^{(i)} \subset \cdots \subset X^{(i)} \subset \Omega_i$ be a nested sequence of subsets with *fill distance*

$$h_j^{(i)} := \sup_{x\in\Omega_i} \min_{y\in X_j^{(i)}} \|x-y\|_2 \sim 2^{-j}.$$

- Associated to the sequence of points, there is a sequence of subspaces

$$\mathcal{H}_0^{(i)} \subset \mathcal{H}_1^{(i)} \subset \cdots \subset \mathcal{H}^{(i)}, \quad \mathcal{H}_j^{(i)} := \operatorname{span}\left\{\kappa_i(x,\cdot) : x \in X_j^{(i)}\right\}.$$

- Assuming $\mathcal{H}^{(i)} \cong H^{s_i}(\Omega_i)$ for $s_i > d_i/2$, there hold the univariate error estimates

$$\left\|f - P_j^{(i)} f\right\|_{L^2(\Omega_i)} \lesssim \left(h_j^{(i)}\right)^{2s_i} \|f\|_{H^{2s_i}(\Omega_i)}, \quad f \in H^{2s_i}(\Omega_i).$$

Herein $P_j^{(i)} f = f_{X_j^{(i)}}$ denotes the $\mathcal{H}^{(i)}$-orthogonal projection onto $\mathcal{H}_j^{(i)}$.

- Similarly, the *detail projection* $Q_j^{(i)} := P_j^{(i)} - P_{j-1}^{(i)}$, $P_{-1}^{(i)} := 0$, satisfies

$$\left\|Q_j^{(i)} f\right\|_{L^2(\Omega_i)} \lesssim \left(h_j^{(i)}\right)^{2s_i} \|f\|_{H^{2s_i}(\Omega_i)}, \quad f \in H^{2s_i}(\Omega_i).$$

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Sparse grid approximation

- Based on the detail projections, we introduce the *sparse grid projection*

$$\widehat{\boldsymbol{P}}_J^{\boldsymbol{w}}\colon \boldsymbol{\mathcal{H}} \to \widehat{\boldsymbol{\mathcal{H}}}_J^{\boldsymbol{w}}, \quad \widehat{\boldsymbol{P}}_J^{\boldsymbol{w}} f = \sum_{\boldsymbol{j}^T\boldsymbol{w}\leq J} \big(Q_{j_1}^{(1)} \otimes \cdots \otimes Q_{j_m}^{(m)}\big) f.$$

Theorem. Let $f \in \boldsymbol{H}^{2\boldsymbol{s}}(\Omega)$ and let $\boldsymbol{X}$ be quasi-uniform. Denote by $N := \dim \widehat{\boldsymbol{\mathcal{H}}}_J^{\boldsymbol{w}}$ the number of degrees of freedom in the sparse tensor product space $\widehat{\boldsymbol{\mathcal{H}}}_J^{\boldsymbol{w}}$ and set

$$\beta := 2\frac{\min\{s_1/w_1, \ldots, s_m/w_m\}}{\max\{d_1/w_1, \ldots, d_m/w_m\}}.$$

Assume that the minimum in the enumerator is attained $P \in \mathbb{N}$ times and the maximum in the denominator is attained $R \in \mathbb{N}$ times. Then, the sparse grid kernel interpolant in $\widehat{\boldsymbol{\mathcal{H}}}_J^{\boldsymbol{w}}$ satisfies the error estimate

$$\big\|(I - \widehat{\boldsymbol{P}}_J^{\boldsymbol{w}})f\big\|_{L^2(\Omega)} \lesssim N^{-\beta}(\log N)^{(P-1)+\beta(R-1)}\|f\|_{\boldsymbol{H}^{2\boldsymbol{s}}(\Omega)}.$$

Optimal choice of weights

- For all $\boldsymbol{w} > \mathbf{0}$, there holds

$$\beta \leq \beta^\star := 2\min\{s_1/d_1, \dots, s_m/d_m\}.$$

- Let $2s_\ell/d_\ell = \beta^\star$. Then the maximum rate $\beta = \beta^\star$ is attained for all $\boldsymbol{w} > \mathbf{0}$ such that

$$\frac{s_m}{s_i} \leq \frac{w_\ell}{w_i} \leq \frac{d_\ell}{d_i}, \quad i = 1, 2, \dots, m.$$

- **Equilibration of accuracy**: Set $w_i \sim s_i$ such that

$$2^{-2Js_1/w_1} = 2^{-2Js_2/w_2} = \cdots = 2^{-2Js_m/w_m}.$$

- **Equilibration of DOFs**: Set $w_i \sim d_i$ such that

$$2^{Jd_1/w_1} = 2^{Jd_2/w_2} = \cdots = 2^{Jd_m/w_m}.$$

- **Equilibration of cost-benefit rate**: Set $w_i \sim (d_i + 2s_i)$ such that

$$2^{j_1(d_1+2s_1)} \cdot 2^{j_2(d_2+2s_2)} \cdots 2^{j_m(d_m+2s_m)} = 2^{J\cdot const.}, \quad \boldsymbol{j}^T\boldsymbol{w} = J.$$

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Sparse grid combination technique

- Due to the Galerkin orthogonality, the detail projections satisfy

$$(\boldsymbol{Q_j}u, \boldsymbol{Q_{j'}}v)_{\boldsymbol{\mathcal{H}}} = 0 \quad \text{for } \boldsymbol{j} \neq \boldsymbol{j}' \text{ and any } u, v \in \boldsymbol{\mathcal{H}},$$

  as a consequence, the sparse grid combination technique is exact.

- We introduce the tensor product projector $\boldsymbol{P_j} := \sum_{\boldsymbol{\ell} \leq \boldsymbol{j}} \boldsymbol{Q_\ell}$ and define the *combination technique index set* $\mathcal{J}_J^{\boldsymbol{w}} := \{\boldsymbol{j} \in \mathbb{N}_0^m : J - |\boldsymbol{w}| < \boldsymbol{j}^T\boldsymbol{w} \leq J\}$.

- There holds

$$\widehat{\boldsymbol{P}}_J^{\boldsymbol{w}} f = \sum_{\boldsymbol{j} \in \mathcal{J}_J^{\boldsymbol{w}}} c_{\boldsymbol{j}}^{\boldsymbol{w}} \boldsymbol{P_j} f, \quad \text{where } c_{\boldsymbol{j}}^{\boldsymbol{w}} := \sum_{\substack{\boldsymbol{j}' \in \{0,1\}^m \\ (\boldsymbol{j}+\boldsymbol{j}')^T\boldsymbol{w} \leq J}} (-1)^{|\boldsymbol{j}'|}.$$

- As a consequence, we only need to solve the smaller tensor product problems

$$\boldsymbol{K_j}\boldsymbol{\alpha_j} := \big(\boldsymbol{K}_{j_1}^{(1)} \otimes \cdots \otimes \boldsymbol{K}_{j_m}^{(m)}\big)\boldsymbol{\alpha_j} = \boldsymbol{f_j}.$$

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- As a consequence, we only need to solve the smaller tensor product problems

$$\boldsymbol{K}_{\boldsymbol{j}}\boldsymbol{\alpha}_{\boldsymbol{j}} := \big(\boldsymbol{K}_{j_1}^{(1)} \otimes \cdots \otimes \boldsymbol{K}_{j_m}^{(m)}\big)\boldsymbol{\alpha}_{\boldsymbol{j}} = \boldsymbol{f}_{\boldsymbol{j}}.$$

Sparse grid combination technique

- Due to the Galerkin orthogonality, the detail projections satisfy

$$(\boldsymbol{Q}_{\boldsymbol{j}} u, \boldsymbol{Q}_{\boldsymbol{j}'} v)_{\boldsymbol{\mathcal{H}}} = 0 \quad \text{for } \boldsymbol{j} \neq \boldsymbol{j}' \text{ and any } u, v \in \boldsymbol{\mathcal{H}},$$

  as a consequence, the sparse grid combination technique is exact.

- We introduce the tensor product projector $\boldsymbol{P}_{\boldsymbol{j}} := \sum_{\boldsymbol{\ell} \leq \boldsymbol{j}} \boldsymbol{Q}_{\boldsymbol{\ell}}$ and define the *combination technique index set* $\mathcal{J}_J^{\boldsymbol{w}} := \big\{\boldsymbol{j} \in \mathbb{N}_0^m : J - |\boldsymbol{w}| < \boldsymbol{j}^T \boldsymbol{w} \leq J\big\}$.

- There holds

$$\widehat{\boldsymbol{P}}_J^{\boldsymbol{w}} f = \sum_{\boldsymbol{j} \in \mathcal{J}_J^{\boldsymbol{w}}} c_{\boldsymbol{j}}^{\boldsymbol{w}} \boldsymbol{P}_{\boldsymbol{j}} f, \quad \text{where } c_{\boldsymbol{j}}^{\boldsymbol{w}} := \sum_{\substack{\boldsymbol{j}' \in \{0,1\}^m \\ (\boldsymbol{j}+\boldsymbol{j}')^T \boldsymbol{w} \leq J}} (-1)^{|\boldsymbol{j}'|}.$$

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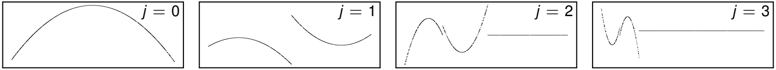
Samplets



- Let $\mathcal{X}_0' \subset \mathcal{X}_1' \subset \cdots \subset \mathcal{X}_J' := \mathcal{X}' := \operatorname{span}\{\delta_x : x \in X\}$.
- Since $\mathcal{X}_j' \subset \mathcal{X}_{j+1}'$, we can (orthogonally) decompose $\mathcal{X}_{j+1}' = \mathcal{X}_j' \oplus \mathcal{S}_j'$.
- For $\mathcal{S}_j'$, we introduce the orthonormal bases $\{\sigma_{j,k}\}_k$.
- Recursively applying the decomposition yields a *samplet basis* for $\mathcal{X}_J'$.
- For data compression, we may construct samplets with vanishing moments

$$(\sigma_{j,k}, p)_\Omega = 0 \quad \text{for all } p \in \mathcal{P}_q(\Omega).$$

- Since $\mathcal{X}_J' \subset \mathcal{H}'$, samplets induce a multiresolution basis for $\mathcal{H}_X$ via the embedding

$$\sigma_{j,k} = \sum_\ell u_{j,k,\ell} \delta_{x_\ell} \mapsto \psi_{j,k} = \sum_\ell u_{j,k,\ell} \kappa(x_\ell, \cdot).$$

- There holds $[\langle \psi_{j,k}, \psi_{j',k'} \rangle_{\mathcal{H}}]_{j,j',k,k'} = \boldsymbol{T}\boldsymbol{K}\boldsymbol{T}^\intercal$.

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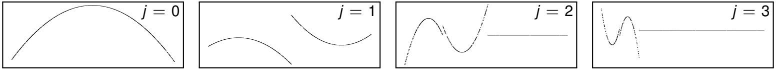

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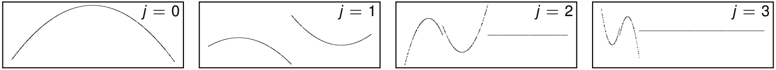

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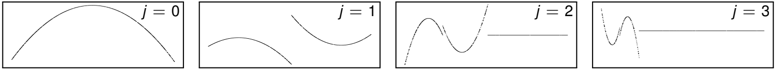

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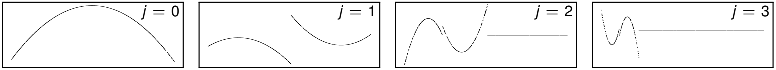

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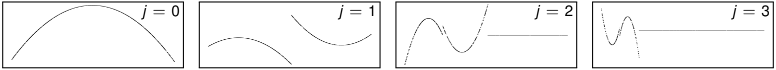

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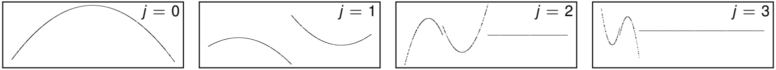

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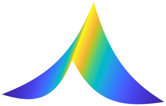
Samplet kernel matrix compression



- For the compression of kernel matrices we make the *asymptotical smoothness* assumption

$$\frac{\partial^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}}{\partial x^{\boldsymbol{\alpha}} \partial y^{\boldsymbol{\beta}}} \kappa(x, y) \leq c_{\mathcal{K}} \frac{(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|)!}{\rho^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \|x-y\|_2^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}}, \quad c_{\mathcal{K}}, \rho > 0.$$

Theorem. Let the data set X be quasi-uniform. Then, setting all coefficients of the transformed kernel matrix $\boldsymbol{K}^{\Sigma} := \boldsymbol{T}\boldsymbol{K}\boldsymbol{T}^{\intercal}$ to zero which satisfy

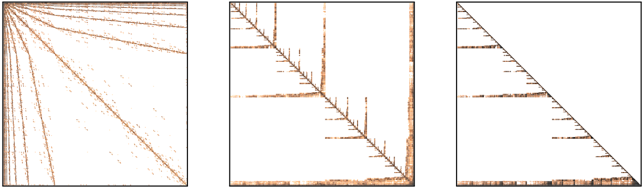
$$\operatorname{dist}(\tau, \tau') \geq \eta \max\{\operatorname{diam}(\tau), \operatorname{diam}(\tau')\}, \quad \eta > 0,$$

results in a consistency error of

$$\frac{\|\boldsymbol{K}^{\Sigma} - \boldsymbol{K}_{\eta}^{\Sigma}\|_F}{\|\boldsymbol{K}^{\Sigma}\|_F} \lesssim (\eta\rho/d)^{-2(q+1)}.$$

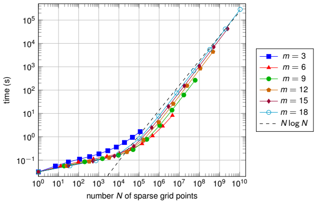
The compressed matrix $\boldsymbol{K}_{\eta}^{\Sigma}$ only contains $\mathcal{O}(|X| \log |X|)$ entries.

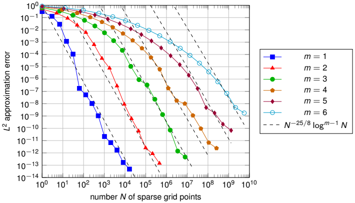
Sparse direct solver



- The samplet compression yields an essentially sparse matrix.
- Corresponding linear systems can be solved by first performing a fill-in reducing reordering (nested dissection) and the Cholesky factorization.
- In the numerical examples, we employ `CHOLMOD`.

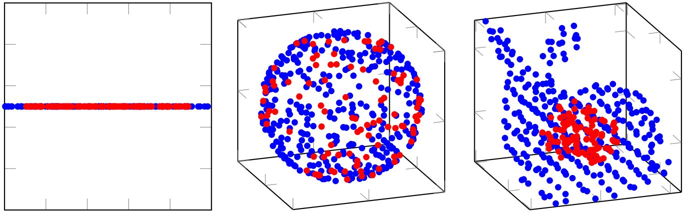
Benchmark in $(0, 1)^m$





- In each direction, we consider the Matérn-17/16 kernel and set $f = 1$.
- The energy space is $H^{25/16}(0, 1)$ and the expected rate of convergence is $25/8$.
- We use equidistant points $X_j = \{2^{-(j+1)}k : k = 1, 2, \dots, 2^{j+1} - 1\}$.
- The error is computed using a 4-point Gaussian quadrature rule in each direction.

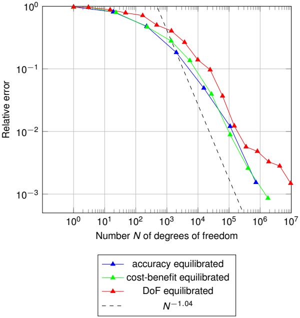
Tensor product of $1 + 2 + 3$ dimensional regions



- On $[0, 1]$ and $\mathbb{S}^2$, we create nested point sets from uniform random points.
- For the Stanford bunny nested point sets are created from a volume mesh.
- We consider the Matérn-$\left(\frac{25}{16} - \frac{d}{2}\right)$ kernel, $d = 1, 2, 3$.
- The energy space is $H^{25/16}(\Omega_i)$ and the expected rate of convergence is $25/8$.
- The interpolation error is evaluated at 100 random points in each region.

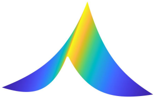
Convergence

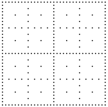
	Interval		Sphere		Rabbit	
	q_{X_j}	h_{X_j,X_J}	q_{X_j}	h_{X_j,X_J}	q_{X_j}	h_{X_j,X_J}
$j=0$	—	$6.80\cdot10^{-1}$	—	2.00	—	1.01
$j=1$	$6.25\cdot10^{-2}$	$4.45\cdot10^{-1}$	$4.03\cdot10^{-1}$	1.24	$1.01\cdot10^{-1}$	$5.82\cdot10^{-1}$
$j=2$	$2.34\cdot10^{-2}$	$2.03\cdot10^{-1}$	$5.49\cdot10^{-2}$	$5.66\cdot10^{-1}$	$3.57\cdot10^{-2}$	$3.01\cdot10^{-1}$
$j=3$	$7.81\cdot10^{-3}$	$1.41\cdot10^{-1}$	$1.62\cdot10^{-2}$	$2.87\cdot10^{-1}$	$1.28\cdot10^{-2}$	$1.76\cdot10^{-1}$
$j=4$	$7.81\cdot10^{-3}$	$1.33\cdot10^{-1}$	$7.01\cdot10^{-4}$	$1.62\cdot10^{-1}$	$8.12\cdot10^{-3}$	$1.06\cdot10^{-1}$
$j=5$	$7.81\cdot10^{-3}$	$4.68\cdot10^{-2}$	$6.48\cdot10^{-4}$	$8.22\cdot10^{-2}$	$3.63\cdot10^{-3}$	$4.69\cdot10^{-2}$
$j=6$	$7.81\cdot10^{-3}$	$7.81\cdot10^{-3}$	$4.22\cdot10^{-5}$	$2.92\cdot10^{-2}$	$1.99\cdot10^{-3}$	$1.83\cdot10^{-2}$

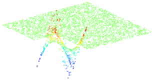


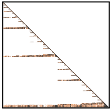
- The fill distance approximately halves from one level to another.
- We (almost) observe the expected rate $\beta = 25/24 \approx 1.04$ for all choices of weights.

Conclusion









- We have extended existing approximation results to kernel interpolation on sparse grids, including super convergence.
- In the considered setting, the sparse grid combination technique is exact and can be used for computations.
- The resulting smaller tensor product problems have been solved in samplet coordinates using a sparse direct solver.
- The numerical results in $1 + 2 + 3$ dimensions confirm the theoratical rates for general data sets.

Thank you

- H.-J. Bungartz and M. Griebel. Sparse grids. Acta Numer., 13:147–269, 2004
- Y. Chen, T. Davis, W. Hager, and S. Rajamanickam, Algorithm 887: CHOLMOD, supernodal sparse Cholesky factorization and update/downdate, ACM Trans. Math. Software, 35(3):22:1–22:14, 2008
- M. Griebel and H. Harbrecht. A note on the construction of L-fold sparse tensor product spaces. Constr. Approx., 38(2):235–251, 2013
- H. Harbrecht and M. Multerer. Samplets: Construction and scattered data compression. J. Comput. Phys., 2022
- H. Harbrecht, M. Multerer, O. Schenk, and C. Schwab. Multiresolution kernel matrix algebra. Numer. Math., 156:1085–1114, 2024
- H. Harbrecht, M. Peters, and M. Siebenmorgen. Combination technique based k-th moment analysis of elliptic problems with random diffusion. J. Comput. Phys., 252:128–141, 2013
- R. Schaback. Superconvergence of kernel-based interpolation. J. Approx. Theory, 235:1–19, 2018
- I. Sloan and V. Kaarnioja. Doubling the rate: Improved error bounds for orthogonal projection with application to numerical analysis. BIT Numer. Math., 65:10, 2025