Partial Differererential Equations

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Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} f(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geq 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \to \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \to \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \to 0} \frac{f(x+he_i) - f(x)}{h}$, where $e_i = (0, 0, \ldots, 1, \ldots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial x_1, \ldots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha| = k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i,j \leq d}$

Definition 1.4 Given a function F. Then the equation of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u: \ \Omega \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$ is called a *PDE of order k*.

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1}u, D^{k-2}u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explizit solution! This is in many cases impossible.
- Prove a *well-posted theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continous and differentiable functions) Let $\Omega \subseteq R^d$ be open

$$\begin{aligned} C(\Omega) &= \{f: \ \Omega \to \mathbb{R} \mid f \text{ continuous}\} \\ C^k(\Omega) &= \{f: \ \Omega \to \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \ \leq k\} \end{aligned}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f: \ \Omega \to \mathbb{R} \text{ lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in $\mathbb{R}^d$. If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then u is a harmonic function in Ω.

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \; dS(x) = \int_K \operatorname{div}(\vec{F}) \; dx$$

where ν is the outward unit normal vector field on ∂K. Thus

$$0 = \int_{\partial V} \nabla u \cdot \vec{\nu} \; dS(x) = \int_V \operatorname{div}(\nabla u) \; dx = \int_V \Delta u(x) \; dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

   where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f) g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \to \mathbb{R}$ be continuous. Prove that if $\int_B f(x) \; dx = 0$, then $u \equiv 0$ in Ω.

Theorem 2.5 (Fundamential Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x) \; dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in $\mathbb{R}^d$. Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \to \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2 + \cdots + x_d^2}\right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned}
\partial_{x_i} u &= \partial_{x_i} v = (\partial_r v)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r} \\
\partial_{x_i}^2 u &= \partial_{x_i}\left(v(r)'\frac{x_i}{r}\right) = (\partial_{x_i} v(r)')\frac{x_i}{r} + v'(r)\partial_{x_i}\left(\frac{x_i}{r}\right) \\
&= (\partial_r v'(r))\left(\frac{dr}{\partial_{x_i}}\right)\frac{x_i}{r} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}(\partial_{x_i} r)\right) = v'(r)\frac{x_i^2}{r^2} + v'r(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)
\end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d d_{x_i}^2\right) u = v''(r) + v'(r)(\frac{d}{r} - \frac{1}{r})$
Thus $\Delta u = v'(r) + v(r)\frac{d-1}{r}$. We consider $d \geq 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r)\frac{d-1}{r} = 0$
$\Rightarrow \log(v(r))' = \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)'$ (recall $log(f)' = \frac{f'}{f}$)
$\Rightarrow v'(r) = \frac{1}{v^{d-2} + \text{ const.}}$

$$\begin{cases} \frac{const}{r^{d-2}} + const x x + const & , d \geq 3 \\ const \log(r) + const x x r + const & , d = 2 \end{cases}$$

Definition 2.7 (Fundamential Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|}\frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0,1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta\Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in $\mathbb{R}^d$. The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y)f(y)\ dy = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\ dy$$

This can be heuristically justifyfied with

$$-\Delta(\Phi \star f) = (-\Delta\Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C_c^2(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\,dy.$$

First we check that u is continuous: Take $x_k \to x_0$ in $\mathbb{R}^d$. We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y)\ dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y)\ dy$$

This follows from the Dominated Convegence Theorem. More precisely:

$$\lim_{n\to\infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \qquad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x-y)| \ \le \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \le R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\operatorname{supp}(f)$ but independent of y. Now we compute the derivatives:

$$\partial_{x_i} u(x) = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x-y)\ dy = \lim_{h\to 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + he_i - y) - f(x-y)}{h}\ dy$$

$$\text{(dom. conv.)} \qquad = \int \Phi(y) \partial_{x_i} f(x-y)\ dy$$

$$\Rightarrow \qquad D^\alpha u(x) = \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x-y)\ dy \quad \text{for all } |\alpha| \le 2$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y)(-\Delta_x) f(x-y)\, dy = \int_{\mathbb{R}^d} \Phi(y)(-\Delta_y) f(x-y)\, dy \\ &= \int_{\mathbb{R}^d\setminus B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy + \int_{B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d\setminus B(0,\epsilon)} \Phi(y)(-\Delta_y) f(x-y)\, dy \\ &= \int_{\mathbb{R}^d\setminus B(0,\epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x-y)\, dy - \int_{\partial B(0,\epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \\ &= \int_{\mathbb{R}^d\setminus B(0,\epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x-y)\, dy \\ &\quad + \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}} f(x-y)\, dS(y) - \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0,\epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1| \epsilon^{d-1}} \qquad \text{for } y \in \partial B(0,\epsilon)$$

Hence:

$$\begin{aligned} &\int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}} f(x-y)\ dS(y) = \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(0,\epsilon)} f(x-y)\ dS(y) \\ &= \fint_{\partial B(0,\epsilon)} f(x-y)\ dS(y) = \fint_{\partial B(x,\epsilon)} f(y)\ dS(y) \xrightarrow{\epsilon \to 0} f(x) \end{aligned}$$

Error terms:

1.

$$\left|\int_{B(0,\epsilon)} \Phi(y)(-\Delta_y)f(x-y)\ dy\right| \leq \int_{B(0,\epsilon)} |\Phi(y)|\ \underbrace{|-\Delta_y f(x-y)}_{\leq \|\Delta f\|_{L^\infty}\mathbb{1}(|y|\leq R)}\ dy$$

$$\leq \|\Delta f\|_{L^\infty}\int_{\mathbb{R}^d} \underbrace{|\Phi(y)\mathbb{1}(|y|\leq R)}_{L^1(\mathbb{R}^d)}\mathbb{1}(|y|\leq \epsilon) \xrightarrow{\epsilon\to 0} 0$$

Where $R>0$ depends on x and the support of f but is independent of y.

2.

$$\left|\int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f}{\partial \vec{n}}(x-y)\ dS(y)\right| \leq \|\nabla f\|_{L^\infty}\int_{\partial B(0,\epsilon)} |\Phi(y)|\ \ dy$$

$$\leq \begin{cases} const\cdot\epsilon \to 0, & d\geq 3\\ const\cdot\epsilon|\log\epsilon|\ \to 0, & d=2\end{cases}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x\in\mathbb{R}^d$ proved that $u=\Phi\star f$ and $f\in C_c^2(\mathbb{R}^d)$. ■

Thus, if $f\in C_c^2(\mathbb{R})$, then $u=\Phi\star f$ satisfies $u\in C^2(\mathbb{R}^2)$ and $-\Delta u(x)=f(x)$ for all $x\in\mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f. For example if $f\in C_c^1(\mathbb{R})$ we can easil extend the previous proof:

$$f\in C_c^1 \Rightarrow \partial_{x_i}u = \int_{\mathbb{R}^d}\Phi(y)\partial_{x_i}f(x-y)\,dy \in C(\mathbb{R}^d)\Rightarrow u\in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i}\partial_{x_j}u = \partial_{x_i}\int_{\mathbb{R}^d}\Phi(y)\partial_{x_j}f(x-y)\,dy = \int_{\mathbb{R}^d}\partial_{x_i}\Phi(y)\partial_{x_j}f(x-y)\,dy \in C(\mathbb{R}^d)\Rightarrow u\in C^2(\mathbb{R}^d)$$

The computation

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d}\partial_{x_i}\Phi(y)\partial_{x_i}f(x-y)\,dy \overset{(IBP)}{=} f(x)$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega\subseteq\mathbb{R}$ be open, let $u\in C^2(\Omega)$ and $\Delta u=0$ in Ω. Then

$$u(x) = \fint_{B(x,r)} u = \fint_{\partial B(x,r)} u \quad \text{for all } x\in\Omega, B(x,r)\subseteq\Omega$$

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u''=0\Leftrightarrow u(x)=ax+b$ (Linear Equation)

Proof. Consider all $r > 0$ s.t. $B(x,r) \subseteq \Omega$,

$$f(r) = \fint_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r. When it is done, then we imideatelly obtain

$$f(r) = \lim_{t \to 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr}\left(\fint_{\partial B(0,r)} u(x+y)\,dS(y)\right) = \frac{d}{dr}\left(\fint_{B(0,1)} u(x+rz)\,dS(z)\right) \\
\text{(dom. convergence)} \quad &= \fint_{\partial B(0,1)} \frac{d}{dr}[u(x+rz)]\,dS(z) = \fint_{\partial B(0,1)} \nabla u(x+rz)z\,dS(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y)\frac{y-x}{r}\,dS(y) = \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla u(y)\cdot \vec{n_y}\,dS(y) \\
\text{(Green)} \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0}\,dy = 0
\end{aligned}$$

■

Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r,w), r = |x| > 0, \omega \in S^{d-1}$, then

$$\int_{\mathbb{R}^d} g(y)\,dy = \int_0^r \left(\int_{B(0,r)} g(y)\,dS(y)\right) dr$$

We already proved that $u(x) = \fint_{\partial B(x,r)} u\,dy$ if $B(x,r) \subseteq \Omega$. Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y)\,dy &= \int_{B(0,r)} u(x+y)\,dy \\
\text{(Pol. decomposition)} \quad &= \int_0^r \left(\int_{\partial B(0,r)} u(x+y)\,dS(y)\right) ds \\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y)\,dS(y)\right) ds \\
&= \int_0^r (|\partial B(x,s)|u(x))\,ds = |B(x,r)|u(x)
\end{aligned}$$

This implies

$$\fint_{B(x,r)} u(y)\,dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.15 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = \fint_{B(x,r)} u = \fint_{\partial B(x,r)} u \quad \text{for all } B(x,r) \subseteq \Omega$$

Then u is harmonic, i.e. $\Delta u = 0$ in Ω. (The proof is exactly like before)

Theorem 2.16 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω. Then

1. $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
2. Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$, then $u \equiv \mathit{const}$ in Ω.