Partial Differerential Equations

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Contents

Please note that I write this lecture notes for my personal use. I may write things differently than presented in the lecture. This script also contains solutions for exercises (which may be wrong). Of course, I don't push them to GitHub while the exercises can be handed in.

Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} f(t) = af(t) \text{ for all } t \geqslant 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geqslant 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \to \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \to \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \to 0} \frac{f(x+he_i) - f(x)}{h}$, where $e_i = (0, 0, \ldots, 1, \ldots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial x_1, \ldots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha| = k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leqslant i,j \leqslant d}$

Definition 1.4 Given a function F. Then the equation of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u: \ \Omega \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$ is called a *PDE of order k*.

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1}u, D^{k-2}u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explizit solution! This is in many cases impossible.
- Prove a *well-posted theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continous and differentiable functions) Let $\Omega \subseteq R^d$ be open

$$\begin{aligned} C(\Omega) &= \{f: \ \Omega \to \mathbb{R} \mid f \text{ continuous}\} \\ C^k(\Omega) &= \{f: \ \Omega \to \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \ \leqslant k\} \end{aligned}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f: \ \Omega \to \mathbb{R} \text{ lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leqslant p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leqslant k : D^\alpha f \in L^p(\Omega) \text{exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in $\mathbb{R}^d$. If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then u is a harmonic function in Ω.

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \, dS(x) = \int_K \operatorname{div}(\vec{F}) \, dx$$

where ν is the outward unit normal vector field on ∂K. Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \, dS(x) = \int_V \operatorname{div}(\nabla u) \, dx = \int_V \Delta u(x) \, dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

   where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f) g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \to \mathbb{R}$ be continuous. Prove that if $\int_B f(x) \, dx = 0$, then $u \equiv 0$ in Ω.

Theorem 2.5 (Fundamential Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x) \, dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in $\mathbb{R}^d$. Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \to \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2 + \cdots + x_d^2}\right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned}
\partial_{x_i} u &= \partial_{x_i} v = (\partial_r v)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r} \\
\partial_{x_i}^2 u &= \partial_{x_i}\left(v(r)'\frac{x_i}{r}\right) = (\partial_{x_i} v(r)')\frac{x_i}{r} + v'(r)\partial_{x_i}\left(\frac{x_i}{r}\right) \\
&= (\partial_r v'(r))\left(\frac{dr}{\partial_{x_i}}\right)\frac{x_i}{r} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}(\partial_{x_i} r)\right) = v'(r)\frac{x_i^2}{r^2} + v'r(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)
\end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d d_{x_i}^2\right) u = v''(r) + v'(r)(\frac{d}{r} - \frac{1}{r})$
Thus $\Delta u = v'(r) + v(r)\frac{d-1}{r}$. We consider $d \geqslant 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r)\frac{d-1}{r} = 0$
$\Rightarrow \log(v(r))' = \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)'$ (recall $log(f)' = \frac{f'}{f}$)
$\Rightarrow v'(r) = \frac{1}{v^{d-2} + \text{const.}}$
$\begin{cases} \frac{const}{r^{d-2}} + const x x + const & , d \geqslant 3 \\ const \log(r) + const x x r + const & , d = 2 \end{cases}$

Definition 2.7 (Fundamentitial Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|}\frac{1}{|x|^{d-2}}, & d \geqslant 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0,1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta\Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in $\mathbb{R}^d$. The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y)f(y)\ dy = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\ dy$$

This can be heuristically justifyfied with

$$-\Delta(\Phi \star f) = (-\Delta\Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C_c^2(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\,dy.$$

First we check that u is continuous: Take $x_k \to x_0$ in $\mathbb{R}^d$. We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y)\ dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y)\ dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n\to\infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{ for all } y \in \mathbb{R}^d \backslash \{0\}$$

and

$$|\Phi(y) f(x-y)| \ \leqslant \|f||_{L^\infty} \cdot \mathbb{1}(|y| \leqslant R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\mathrm{supp}(f)$ but independent of y. Now we compute the derivatives:

$$\partial_{x_i} u(x) = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x-y)\ dy = \lim_{h\to 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + he_i - y) - f(x-y)}{h}\ dy$$

$$\text{(dom. conv.)} \quad = \int \Phi(y) \partial_{x_i} f(x-y)\ dy$$

$$\Rightarrow \quad D^\alpha u(x) = \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x-y)\ dy \quad \text{for all } |\alpha| \leqslant 2$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned}
-\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y)(-\Delta_x) f(x-y)\, dy = \int_{\mathbb{R}^d} \Phi(y)(-\Delta_y) f(x-y)\, dy \\
&= \int_{\mathbb{R}^d \backslash B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy + \int_{B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy \quad (\epsilon > 0 \text{ small})
\end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned}
&\int_{\mathbb{R}^d \backslash B(0,\epsilon)} \Phi(y)(-\Delta_y) f(x-y)\, dy \\
&\quad = \int_{\mathbb{R}^d \backslash B(0,\epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x-y)\, dy - \int_{\partial B(0,\epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \\
&\quad = \int_{\mathbb{R}^d \backslash B(0,\epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x-y)\, dy \\
&\qquad + \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\, dS(y) - \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y)
\end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0,\epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1|\epsilon^{d-1}} \quad \text{ for } y \in \partial B(0,\epsilon)$$

Hence:

$$\begin{aligned}
\int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\ dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(0,\epsilon)} f(x-y)\ dS(y) \\
= \fint_{\partial B(0,\epsilon)} f(x-y)\ dS(y) &= \fint_{\partial B(x,\epsilon)} f(y)\ dS(y) \xrightarrow{\epsilon \to 0} f(x)
\end{aligned}$$

We have to regard the following error terms:

- $$\left|\int_{B(0,\epsilon)} \Phi(y)(-\Delta_y)f(x-y)\ dy\right| \leqslant \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leqslant \|\Delta f\|_{L^\infty} \mathbb{1}(|y|\leqslant R)}\ dy$$
$$\leqslant \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)|\mathbb{1}(|y|\leqslant R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y|\leqslant \epsilon) \xrightarrow{\epsilon\to 0} 0$$

  Where $R>0$ depends on x and the support of f but is independent of y.

- $$\left|\int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f}{\partial \vec{n}}(x-y)\ dS(y)\right| \leqslant \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)|\ \ dy$$
$$\leqslant \begin{cases} const\cdot \epsilon|\log\epsilon| \ \to 0, & d=2 \\ const\cdot\epsilon \to 0, & d\geqslant 3 \end{cases}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x\in\mathbb{R}^d$ proved that $u=\Phi\star f$ and $f\in C_c^2(\mathbb{R}^d)$. ∎

Thus, if $f\in C_c^2(\mathbb{R})$, then $u=\Phi\star f$ satisfies $u\in C^2(\mathbb{R}^2)$ and $-\Delta u(x)=f(x)$ for all $x\in\mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f. For example if $f\in C_c^1(\mathbb{R})$ we can easily extend the previous proof:

$$\partial_{x_i}u = \int_{\mathbb{R}^d} \Phi(y)\partial_{x_i}f(x-y)\,dy \in C(\mathbb{R}^d) \Rightarrow u\in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i}\partial_{x_j}u = \partial_{x_i}\int_{\mathbb{R}^d}\Phi(y)\partial_{x_j}f(x-y)\,dy = \int_{\mathbb{R}^d}\partial_{x_i}\Phi(y)\partial_{x_j}f(x-y)\,dy\in C(\mathbb{R}^d)$$

So we have $u\in C^2(\mathbb{R}^d)$. Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i}\Phi(y)\partial_{x_i}f(x-y)\,dy \overset{(IBP)}{=} f(x).$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega\subseteq\mathbb{R}$ be open, let $u\in C^2(\Omega)$ and $\Delta u=0$ in Ω. Then

$$u(x) = \fint_{B(x,r)} u = \fint_{\partial B(x,r)} u \quad \text{for all } x\in\Omega, B(x,r)\subseteq\Omega$$

Proof. Consider all $r>0$ s.t. $B(x,r)\subseteq\Omega$,

$$f(r) = \fint_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r. When it is done, then we immediately obtain

$$f(r) = \lim_{t\to 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$
\begin{aligned}
f'(r) &= \frac{d}{dr}\left(\fint_{\partial B(0,r)} u(x+y)\,dS(y)\right) \\
&= \frac{d}{dr}\left(\fint_{\partial B(0,1)} u(x+rz)\,dS(z)\right) \\
\text{(dom. convergence)} \quad &= \fint_{\partial B(0,1)} \frac{d}{dr}[u(x+rz)]\,dS(z) \\
&= \fint_{\partial B(0,1)} \nabla u(x+rz)z\,dS(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y)\frac{y-x}{r}\,dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n_y}\,dS(y) \\
\text{(Gauss-Green 2.2)} \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0}\,dy = 0 \qquad \blacksquare
\end{aligned}
$$

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$ (Linear Equation)

Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, \omega \in S^{d-1}$, then

$$\int_{B(0,r)} g(y)\,dy = \int_0^r \left(\int_{B(0,r)} g(y)\,dS(y)\right) dr$$

Remark 2.15 We already proved that for u harmonic we have $u(x) = \fint_{\partial B(x,r)} u\,dy$. Now we have

$$
\begin{aligned}
\int_{B(x,r)} u(y)\,dy &= \int_{B(0,r)} u(x+y)\,dy \\
\text{(Pol. decomposition)} \quad &= \int_0^r \left(\int_{\partial B(0,s)} u(x+y)\,dS(y)\right) ds \\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y)\,dS(y)\right) ds \\
\text{(Mean value property)} \quad &= \int_0^r (|\partial B(x,s)|\,u(x))\,ds = |B(x,r)|\,u(x)
\end{aligned}
$$

This implies

$$\fint_{B(x,r)} u(y)\,dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.16 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = \fint_{B(x,r)} u(y)\,dy = \fint_{\partial B(x,r)} u(y)\,dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then u is harmonic, i.e. $\Delta u = 0$ in Ω. (The proof is exactly like before)

Theorem 2.17 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω. Then

a) $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$

b) Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$, then $u \equiv$ *const.* in Ω.

Proof. Given $U \subseteq \mathbb{R}^d$ open, we can write $U = \bigcup_i U_i$, where U_i is open and connected.

b) Assume that Ω is connected and there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{y \in \Omega} u(x)$. Define $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$. U is closed since u is continuous. Moreover, U is open by the mean-value theorem. I.e. for all $x \in U$ there is a $r > 0$ s.t. $B(x, r) \subseteq U \subseteq \Omega$. Since U is connected we get $U = \Omega$, so u is constant in Ω. On the other hand, if there is no $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega}$ we have $\forall x_0 \in \Omega : \quad u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$

a) Given $\Omega \subseteq \mathbb{R}^d$ open, we can write $\Omega = \bigcup_i \Omega_i$, where Ω_i is open and connected. By b) we have
$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$
So we can conclude
$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x).$$
■

Definition 2.18
- If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$, then u is called *sub-harmonic* if $\Delta u \geqslant 0$ in Ω.
- If $\Delta u \leqslant 0$, then u is called *super-harmonic*.

Exercise 2.19 (E 1.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$ be subharmonic.

a) Prove that u satisfies the Mean Value Inequality
$$\fint_{\partial B(x,r)} u(y)\, dS(y) \geqslant \fint_{B(x,r)} u(y)\, dy \geqslant u(x)$$
for all $B(x, r) \subseteq \mathbb{R}^d$.

b) Assume further that Ω is connected and $u \in C(\bar{\Omega})$. Prove that u satisfies the strong maximum principle, namely either
   - u is constant in Ω, or
   - $\sup_{y \in \partial\Omega} u(y) > u(x)$ for all $x \in \Omega$.

My Solution. a) Let $f(r) = \fint_{\partial B(x,r)} u(y)\,dS(y)$, then we have

$$
\begin{aligned}
\partial_r f(r) &= \partial_r \fint_{\partial B(x,r)} u(y)\,dS(y) \\
\text{(Dom. Convergence)} \quad &= \fint_{\partial B(x,r)} \partial_r u(y)\,dS(y) \\
&= \fint_{\partial B(0,1)} \partial_r u(x+yr)\,dS(y) \\
&= \fint_{\partial B(0,1)} \nabla u(x+yr)\cdot y\,dS(y) \\
&= \fint_{\partial B(x,r)} \nabla u(y)\cdot \frac{y-x}{r}\,dS(y) \\
&= \fint_{\partial B(x,r)} \nabla u(y)\cdot \vec{n}_y\,dS(y) \\
\text{(Gauss-Green)} \quad &= \fint_{B(x,r)} \operatorname{div}(\nabla u(y))\,dS(y) \\
&= \fint_{B(x,r)} \underbrace{\Delta u(y)}_{\geqslant 0}\,dS(y) \geqslant 0
\end{aligned}
$$

So we can conclude that

$$\fint_{\partial B(x,r)} u(y)\,dS(y) = f(r) \geqslant \lim_{r\to 0} f(r) = u(x).$$

Now regard

$$
\begin{aligned}
\int_{B(x,r)} u(y)\,dy &= \int_0^r \left(\int_{\partial B(x,r)} u(y)\,dS(y)\right) ds \\
&= \int_0^r \left(|\partial B(x,r)| \fint_{\partial B(x,r)} u(y)\,dS(y)\right) ds \\
&\geqslant \int_0^r |\partial B(x,r)|\cdot u(x)\,dS(y) \\
&= u(x)\int_0^r |\partial B(x,r)|\,dS(y) = u(x)|B(x,r)|.
\end{aligned}
$$

Thus we have

$$u(x) \leqslant \fint_{B(x,r)} u(y)dy.$$

Finally, lets regard

$$
\begin{aligned}
\int_{B(x,r)} u(y)\,dy &= \int_0^r \left(|\partial B(x,s)| \fint_{\partial B(x,s)} u(y)\,dS(y)\right) ds \\
(\partial_r f(r) \geqslant 0) \quad &\leqslant \int_0^r \left(|\partial B(x,s)| \fint_{\partial B(x,r)} u(y)\,dS(y)\right) ds \\
&= \fint_{\partial B(x,r)} u(y)\,dS(y) \int_0^r |\partial B(x,s)|\,ds \\
&= \fint_{\partial B(x,r)} u(y)\,dS(y)\cdot |B(x,s)|
\end{aligned}
$$

and we conclude

$$\fint_{B(x,r)} u(y)\,dy \leqslant \fint_{\partial B(x,r)} u(y)\,dS(y).$$

b) Let $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x\in\Omega} u(x)$. Now,

$$\begin{aligned}\sup_{x\in\Omega} u(x) = u(x_0) &\leqslant \fint_{\partial B(x_0,r)} u(y)\,dy \\ &\leqslant \fint_{\partial B(x_0,r)} \sup_{x\in\Omega} u(x)\,dy = \sup_{x\in\Omega} u(x)\end{aligned}$$

Since u is continous we get $u(y) = u(x_0)$ for all $y \in B(x_0, r)$, so u is constant. ∎

Definition 2.20 The *Poisson Equation* for given f, g on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Theorem 2.21 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open and connected. Let $f \in C(\Omega), g \in C(\partial\Omega)$. Then there exists *at most* one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof. Assume that we have two solutions u_1 and u_2. Then $u := u_1 - u_2$ is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that $u = 0$ in Ω. More precisely, by the maximum principle we have $\forall x \in \Omega$

$$\sup_{x\in\Omega} u(x) \leqslant \sup_{x\in\partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leqslant 0$$

Since $-u$ satisfies the same property we have $\forall x \in \Omega$:

$$\sup_{x\in\Omega}(-u(x)) \leqslant \sup_{x\in\partial\Omega}(-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leqslant 0 \quad \Rightarrow \quad u(x) \geqslant 0$$

So we geht $u(x) = 0$ in Ω. ∎

Exercise 2.22 (Bonus 1) Let Ω be open, connected and bounded in $\mathbb{R}^d$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Prove that

a) If $g \geqslant 0$ on $\partial\Omega$, then $u \geqslant 0$ in Ω.

b) If $g \geqslant 0$ on $\partial\Omega$ and $g \neq 0$, then $u > 0$ in Ω.

My Solution. a) We have that $\Delta(-u) = 0$, so $-u$ is harmonic in Ω. Since Ω is open and bounded we can apply the Maximum Principle (2.17) and get that

$$\sup_{x\in\bar{\Omega}} -u(x) \leqslant \sup_{x\in\partial\Omega} -g(x) \leqslant 0.$$

This implies $\inf_{x\in\Omega} u(x) \geqslant 0$, so $u \geqslant 0$ in Ω.

b) We prove this by contraposition. Assume there is a $x_0 \in \Omega$ s.t. $u(x_0) = 0$. Since we have $u \geqslant 0$ on Ω by a), it follows that

$$0 = -u(x_0) = \sup_{x\in\bar{\Omega}} -u(x) \leqslant \sup_{x\in\partial\Omega} -g(x) \leqslant 0,$$

so $-u$ attains its maximum on Ω. Hence $-u = 0 = u$ is constant by the strong maximum principle because Ω is connected, in fact $0 = u|_{\partial\Omega} = g$. ■

Lemma 2.23 (Estimates for derivatives) If u is harmonic in $\Omega \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| = N$ and $B(x_0, r) \subseteq \Omega$, then

$$|D^\alpha u(x)| \leqslant \frac{(c_d N)^N}{r^{d+N}} \int_{B(x,r)} |u| \, dy$$

Proof. Induction: Assume $|\alpha| = N - 1$, Take $|\alpha| = N$

$$|D^\alpha u(x_0)| \leqslant \frac{|S_1|}{|B_1|\frac{r}{N}} \|D^\beta u\|_{L^\infty(B(x_0,\frac{r}{n}))}, \quad D^\alpha u = \partial_{x_i}(D^\beta u)_{|\beta|=N-1}$$

Note: $x \in B(x_0, \frac{r}{N})$, so $B(x, \frac{r(N-1)}{N}) \subseteq B(x_0, r)$. By the induction hypothesis:

$$\|D^\beta u\|_{L^\infty(B(x_0,\frac{r}{N}))} \leqslant \frac{[c_d(N-1)]^{N-1}}{[r\frac{(N-1)}{N}]^{d+N-1}} \int_{B(x_0,r)} |u| \, dy$$

The conclusion is:

$$\begin{aligned}
|D^\alpha u(x_0)| &\leqslant \frac{|S_1|}{|B_1|\frac{r}{N}} \frac{[c_d(N-1)]^{N-1}}{\left(r\frac{N-1}{N^d}\right)^{d+N-1}} \int_{B(x_0,r)} |u| \, dy \\
&= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{\left(\frac{r}{N}\right)^{d+N} (N-1)^d} \int_{B(x_0,r)} |u| \, dy \\
&= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{\left(\frac{r}{N}\right)^{d+N} N^d} \left(\frac{N}{N-1}\right)^d \int_{B(x_0,r)} |u| \, dy \\
&\leqslant \frac{2^d|S_1|}{|B_1|} \frac{c_d^{N-1} N^N}{r^{d+N}} \int_{B(x_0,r)} |u| \; dy \qquad \text{if } c_d \geqslant \frac{2^d|S_1|}{|B_1|}
\end{aligned}$$

■

Theorem 2.24 (Regularity) Let Ω be open in $\mathbb{R}^d$. Let $u \in C(\Omega)$ satisfy $u(x) = \fint_{\partial B} u \, dy$ for any $x \in B(x,r) \subseteq \Omega$. Then u is a harmonic function in Ω. Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω.

Exercise 2.25 (E 1.1: Proof the Gauss–Green formula) Let $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that for every open ball $B(y,r) \subseteq \mathbb{R}^d$ we have

$$\int_{\partial B(y,r)} f(y) \cdot \nu_y \, dS(y) = \int_{B(y,r)} \operatorname{div} f \, dx.$$

Here ν_y is the outward unit normal vector and dS is the surface measure on the sphere.

Solution. We proof this in d=3. Let $f \in C^1(\mathbb{R}^3)$

$$\int_{B(0,1)} \partial_{x_3} f \, dx = \int_{\partial B(0,1)} f x_3 \, dS(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \vec{n} = \frac{x}{|x|} \text{ on } \partial B(0,1)$$

$$\begin{aligned} B(0,1) &= \{x_1^2 + x_2^2 + x_3^2 \leqslant 1\} \\ &= \{x_1^2 + x_2^2 \leqslant 1 - \sqrt{1 - x_1^2 - x_2^2} \leqslant x_3 \leqslant \sqrt{1 - x_1^2 - x_2^2}\} \end{aligned}$$

Then:

$$\begin{aligned} \int_{B(0,1)} \partial_{x_3} f \, dx &= \int_{x_1^2 + x_2^2 \leqslant 1} \left(\int_{-\sqrt{1-x_1^2-x_2^2} \leqslant x_3 \leqslant \sqrt{1-x_1^2-x_2^2}} \partial_{x_3} f \, dx_3 \right) dx_1 \, dx_2 \\ &= \int_{x_1^2 + x_2^2 \leqslant 1} \Big[f(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) \\ &\qquad - f(x_1, x_2, -\sqrt{1 - x_1^2 - x_2^2}) \Big] \, dx_1 \, dx_2 \end{aligned}$$

Lets take polar coordinates in 2D:

$$\begin{aligned} x_1 &= r \cos \phi & \qquad r > 0, \phi \in [0, 2\pi) \\ x_2 &= r \sin \phi & \qquad \det \frac{\partial(x_1, x_2)}{\partial(r, \phi)} = r \end{aligned}$$

$$(\star) = \int_0^1 \int_0^{2\pi} [f(r\cos\phi, r\sin\phi, r) - f(r\cos\theta, r\sin\phi, -r)] r \, dr \, d\phi$$

On the other hand:

$$\int_{\partial B(0,1)} f x_3 \, dS$$

The polar coordinates in 3D are:

$$\begin{aligned} x_1 &= r \cos \phi sin\theta & \qquad r > 0, \phi \in (0, 2\pi), \theta \in (0, \pi) \\ x_2 &= r \sin \phi \sin \theta & \qquad \det \frac{\partial x_1, x_2, x_3}{\partial(r, \phi, t)} = r^2 \sin \theta \\ x_3 &= \cos \theta \end{aligned}$$

Then:

$$\begin{aligned} (\star\star) &= \int_0^{2\pi} \int_0^{\pi} f(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \sin\theta \cos\theta \, d\theta \, d\phi \\ &= \int_0^{2\pi} \left(\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} d\theta \right) d\phi \\ (r = \sin\theta) \quad &= \int_0^{2\pi} \int_0^1 f(r\cos\phi, r\sin\phi, \sqrt{1-r^2}) r \, dr \, d\phi \\ &\quad - f(r\cos\phi, r\sin\phi, -\sqrt{1-r^2}) r \, dr \, d\phi \end{aligned}$$

■

Exercise 2.26 (E 1.2) Let $u \in C(\mathbb{R}^d)$ and $\int_{B(x,r)} u\, dy = 0$ for every open ball $B(x,r) \subseteq \mathbb{R}^d$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

My Solution. Assume there is a $x_0 \in \mathbb{R}^d$ s.t. w.l.o.g. $u(x_0) > 0$. Since u is continous there is a ball $B(x_0, r)$ s.t. $u(y) > \frac{u(x_0)}{2}$ for all $y \in B(x_0, r)$. But then we get

$$\int_{B(x_0,r)} u(y)\, dy \geqslant \int_{B(x_0,r)} \frac{u(x_0)}{2}\, dy = \frac{u(x_0)}{2} |B(x_0, r)| > 0. \qquad \blacksquare$$

Exercise 2.27 (E 1.3) Let $f \in C_c^1(\mathbb{R}^d)$ with $d \geqslant 2$ and $u(x) := (\Phi \star f)(x)$. Prove that $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ (2.9 was the same for $f \in C_1(\mathbb{R})$)

Theorem 2.28 (Liouville's Theorem) If $u \in C^2(\mathbb{R}^d)$ is harmonic and bounded, then $u = const.$

Proof. By the bound of the derivative 2.23 we have

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\leqslant \frac{c_d}{r^{d+1}} \int_{B(x_0,r)} |u|\, dy \quad \forall x_0 \in \mathbb{R}^d\ \forall r > 0 \\ &\leqslant \|u\|_{L^\infty} \frac{c_d}{r^{d+1}} |B(x_0, r)| \\ &\leqslant \|u\|_{L^\infty} \frac{c_d}{r} \xrightarrow{r\to\infty} 0 \end{aligned}$$

Thus $\partial_{x_i} u = 0$ for all $i = 1, 2, \dots d$ and $u = const.$ in $\mathbb{R}^d$ $\blacksquare$

Theorem 2.29 (Uniqueness of solutions to Poisson Equation in $\mathbb{R}^d$) If $u \in C^2(\mathbb{R}^d)$ is a bounded function and satisfies $-\Delta u = f$ in $\mathbb{R}^d$ where $f \in C_c^2(\mathbb{R}^d)$, then we have

$$u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x - y) f(y)\, dy + C \quad \forall x \in \mathbb{R}^d$$

where C is a constant and Φ is the fundamental solution of the Laplace equation in $\mathbb{R}^d$.

Proof. If we can prove that v is bounded, then $v = const.$. We first need to show that $\Phi \star f$ is bounded.

$$\begin{aligned} \Phi = \Phi_1 + \Phi_2 = \Phi \mathbb{1}(|x| \leqslant 1) + \Phi(|x| \geqslant 1) \\ \Phi \star f = \Phi_1 \star f + \Phi_2 \star f \end{aligned}$$

We have $\Phi_1 \star f \in L^1(\mathbb{R}^d)$ and $\Phi_2 \star f$ is bounded since $\Phi \to 0$ as $|x| \to \infty$ in $d \geqslant 3$. $\blacksquare$

Exercise 2.30 (Hanack's inequality) Let $u \in C^2(\mathbb{R}^d)$ be harmonic and non-negative. Prove that for all open, bounded and connected $\Omega \subseteq \mathbb{R}^d$, we have

$$\sup_{x\in\Omega} u(x) \leqslant C_\Omega \inf_{x\in\Omega} u(x),$$

where C_∞ is a finite constant depending only on Ω.

Proof. (Exercise) Hint: $\Omega = B(x, r)$. General case cover Ω by finitely many balls, one ball is inside Ω. $\blacksquare$

Chapter 3

Convolution, Fourier Transform and Distributions

3.1 Convolutions

Definition 3.1 (Convolution) Let $f, g : \mathbb{R}^d \to \mathbb{R}$ or $\mathbb{C}$.

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy = (g \star f)(x)$$

Remark 3.2 (Properties of the Convolution)

- $(f \star g)(x) = f \star (g \star h)$
- $f \hat{\star} g = \hat{f} \star \hat{g}$

Theorem 3.3 (Young Inequality) If $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$, where $1 \leqslant p \leqslant \infty$, then $f \star g \in L^p(\mathbb{R}^d)$ and $\|f \star g\|_{L^p} \leqslant \|f\|_{L^1}\|g\|_{L^p}$. More generally, if $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^1(\mathbb{R}^d)$, $\|f \star g\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^q}$, where $1 \leqslant p, q, r, \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

Proof. Let $f \in L^1, g \in L^p$. With the Hölder Inequality **??**, we have:

$$\begin{aligned}
\|f \star g\|_{L^p}^p &= \int_{\mathbb{R}^d} |f \star g(x)|^p\,dx \\
&\leqslant \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)||g(y)|^p\,dy\,dx \\
&= \|f\|_{L^1}^{\frac{p}{q}+1}\|g\|_{L^p}^p
\end{aligned}$$

So we have $\|f \star g\|_{L^p} \leqslant \|f\|_{L^1}\|g\|_{L^p}$ ∎

Theorem 3.4 (Smoothness of the Convolution) If $f \in C_c^\infty(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$. Then $f \star g \in C^\infty(\mathbb{R})$ and

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for all $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \{0, 1, 2, \dots\}$

Proof. First we note that $x \mapsto (f \star g)$ is continous as $x_n \to x$ in $\mathbb{R}^d$ since

$$(f \star g)(x_n) = \int_{\mathbb{R}^d} f(x_n - y)g(y)\,dy \xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = (f \star g)(x)$$

We can apply Dominated convergence because

$$f(x_n - y)g(y) \to f(x-y)g(y) \quad \forall y \text{ as } f \text{ is continuous and } x_n \to x$$

and

$$|f(x_n - y)\ g(y)| \leqslant \|f\|_{L^\infty}|g(y)|\ \mathbb{1}(|y| \leqslant R) \in L^1(\mathbb{R}^d).$$

Where $R > 0$ satisfies $B(0,R) \supseteq \operatorname{supp} f + \sup_n |x_n|$. Now we can compute the derivatives:

$$\begin{aligned}
\partial_{x_i}(f \star g)(x) &= \lim_{h\to 0} \frac{(f \star g)(x + he_i) - (f \star g)(x)}{h} \\
&= \lim_{h\to 0} \int_{\mathbb{R}^d} \frac{f(x + he_i - y) - f(x-y)}{h} g(y)\, dy \\
\text{(Dominated Convergence)} \quad &= \int_{\mathbb{R}^d} \lim_{h\to 0} \frac{f(x + he_i - y) - f(x-y)}{h} g(y)\, dy \\
&= \int_{\mathbb{R}^d} (\partial_{x_i} f)(x-y) g(y)\ \ dy
\end{aligned}$$

We could apply Dominated Convergence since

$$\begin{aligned}
&\frac{f(x + he_i - y) - f(x-y)}{h} g(y) \xrightarrow{h\to 0} (\partial_{x_i} f)(x-y)g(y) \quad \text{as } f \in C^1 \\
&\left|\frac{f(x + he_i - y) - f(x-y)}{h} g(y)\right| \leqslant \|\partial_{x_i} f\|_{L^\infty} |g(y)|\ \mathbb{1}(|y| \leqslant R) \in L^1(\mathbb{R}^d)
\end{aligned}$$

where $B(0,R) \supseteq \operatorname{supp}(f) + B(0, |x|+1)$ and $\partial_{x_i}(f \star g) = (\partial_{x_i} f) \star g \in C(\mathbb{R}^d)$ since $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$. By induction we get $D^\alpha(f \star g) = (D^\alpha f \star g) \in C(\mathbb{R}^d)$. ■

Remark 3.5 Question: Is there a f s.t. $f \star g = g$ for all g. In fact there is no regular function f that solves this formally:

$$f \star g = g \Rightarrow \widehat{f \star g} = \hat{g} \Rightarrow \hat{f}\hat{g} = \hat{g} \ \Rightarrow \hat{f} = 1 \Rightarrow f \text{ is not a regular function!}$$

However, if f is the Dirac-Delta Distribution, $f = \delta_0$ then $\delta_0 \star g = g$ for all g. Formally:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \\ \int \delta_0 = 1 & \end{cases}$$

In fact, if $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$, then $f_\epsilon \to \delta_0$ in an appropriate sense and $f_\epsilon \star g \to g$ for all g nice enough.

Theorem 3.6 (Approximation by convolution) Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon})$. Then for all $g \in L^p(\mathbb{R}^d)$, where $1 \leqslant p < \infty$, then

$$f_\epsilon \star g \to g \quad \text{in } L^p(\mathbb{R}^\mathrm{d})$$

Proof.

Step 1: Let $f, g \in C_c(\mathbb{R}^d)$. Then

$$
\begin{aligned}
(f_\epsilon \star g)(x) - g(x) &= \int_{\mathbb{R}^d} f_\epsilon(y) g(x-y)\, dy - \int_{\mathbb{R}^d} f_\epsilon(y) g(x)\, dy \\
&= \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x))\, dy \\
|(f_\epsilon \star g)(x) - g(x)| &= \left| \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x))\, dy \right| \\
&\leqslant \int_{\mathbb{R}^d} |f_\epsilon(y)||g(x-y) - g(x)| \ \ dy \\
&\leqslant \int_{|y| \leqslant R_\epsilon} |f_\epsilon(y)||g(x-y) - g(x)|\, dy \\
&\leqslant \underbrace{\int_{|y| \leqslant R_\epsilon} |f_\epsilon(y)| \ \ dy}_{\leqslant \|f_\epsilon\|_{L^1} = \|f\|_{L^1}} \left[\sup_{|z| \leqslant R} |g(x-z) - g(x)| \right] \xrightarrow{\epsilon \to 0} 0
\end{aligned}
$$

We have Dominated Convergence since:

$$
(f_\epsilon \star g)(x) - g(x) \to 0 \quad \text{as } \epsilon \to 0
$$

and

$$
|f_\epsilon \star g(x) - g(x)| \leqslant \|f\|_{L^1} \sup_{|z| \leqslant R_\epsilon} |g(x-z) - g(x)| \leqslant 2\|f\|_1 \|g\|_{L^\infty} \mathbb{1}(|x| \leqslant R_1).
$$

Where $B(0, R_1) \supseteq \mathrm{supp}(g) + B(0, R_\epsilon)$, thus $f_\epsilon \star g \to g$ in $L^p(\mathbb{R}^d)$. To remove the technical assumptions $f, g \in C_c(\mathbb{R}^d)$, then we use a density argument. We use the fact that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$.

Step 2: Let $g \in C_c(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$. Then there is $\{g_m\} \subseteq L^p(\mathbb{R}^d), g_m \to g$ in $L^p(\mathbb{R}^d)$. Then

$$
\begin{aligned}
\|f_\epsilon \star g - g\|_{L^p} &\leqslant \|f_\epsilon \star (g - g_m)\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
\text{(Young)} \quad &\leqslant \|f_\epsilon\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leqslant \|f\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leqslant (\|f\|_{L^1} + 1)\|g - g_m\|_{L^p} + \|f \star g_m - g_m\|_{L^p}
\end{aligned}
$$

So we get:

$$
\limsup_{\epsilon \to 0} \|f_\epsilon \star g - g\|_{L^p} \leqslant (\|f\|_{L^p} + 1)\|g - g_m\|_{L^p} + \underbrace{\limsup_{\epsilon \to 0} \|f_\epsilon \star g_m - g_m\|_{L^p}}_{0 \text{by step 1.}}
$$

$$
\xrightarrow{m \to \infty} 0
$$

Step 3: Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Take $\{f_m\} \subseteq C_c(\mathbb{R}^d)$, s.t.

$$
\begin{cases}
F_m \to g in L^1(\mathbb{R}) \text{ as } m \to \infty \\
\int_{\mathbb{R}^d} F_m = 1 (\text{it is possible since } \int_{\mathbb{R}^d}) f = 1)
\end{cases}
$$

Define $F_{m,\epsilon}(x) = \epsilon^{-d} F_m(\epsilon^{-1} x)$ (recall $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1} x)$). Then:

$$
\begin{aligned}
f_\epsilon \star g - g &= (f_\epsilon - F_{m,\epsilon}) \star g + F_{m,\epsilon} \star g - g \\
\Rightarrow \|f_\epsilon - g\|_{L^p} &\leqslant \underbrace{\|f_\epsilon - F_{m,\epsilon} \star g\|_{L^p}}_{\overset{\text{Young}}{\leqslant} \|f_\epsilon - F_{m,\epsilon}\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p}} + \|F_{m,\epsilon} \star g - g\|_{L^p}
\end{aligned}
$$

$$
\Rightarrow \limsup_{\epsilon \ \to 0} \|f_\epsilon \star g - g\|_{L^p} \leqslant \|f - F_m\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \qquad \blacksquare
$$

Lemma 3.7 $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$

Proof. For all $g \in L^p(\mathbb{R}^d)$ there are g_m step functions and $g_m \to m$ in $L^p(\mathbb{R}^d)$, We can assume that Ω is open and bounded and we want to approximate χ_Ω by $C_c(\mathbb{R}^d)$. ∎

Lemma 3.8 (Urnson) Define

$$\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}$$

Then there is a $\eta_\epsilon \in C_c(\mathbb{R}^d)$ s.t.

$$\begin{cases} 0 \leqslant \eta(x) \leqslant 1 & \forall x \in \mathbb{R}^d \\ \eta_\epsilon(x) = 1 & \text{if } x \in \Omega_\epsilon \\ \eta_\epsilon(x) = 0 & \text{if } x \notin \Omega \end{cases}$$

Lemma 3.9 (Gernal Version of Urnson) If $A, B \subseteq \mathbb{R}^d$, A closed, B closed, $A \cap B = \varnothing$. Then

$$\eta(x) = \frac{\operatorname{dist}(x, A)}{\operatorname{dist}(x, A) + \operatorname{dist}(x, B)}$$

Then $\eta \in C(\mathbb{R}^d)$, $0 \leqslant \eta \leqslant 1$ and $\eta = 0$if $x \in B$, $\eta = 1$ if $x \in A$. App to $A = \overline{\Omega_\epsilon} \subset\subset \Omega$ and $B = \mathbb{R}^d \backslash \Omega$.

Theorem 3.10 (Appendix C4 in Evans) Let Ω be open in $\mathbb{R}^d$ and for $\epsilon > 0$ define

$$\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \mathbb{R}^d \setminus \Omega) > \epsilon\}$$

Let $f \in C_c^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} f = 1, \operatorname{supp} f \subseteq B(0,1), f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x) \operatorname{supp}$ is $B(0,\epsilon)$. Then for all $g \in L^p_{loc}(\Omega)$ (i.e. $\mathbb{1}_K g \in L^p(\Omega) \forall K$ compakt set in Ω), then:

a) $g_\epsilon(x) = (f_\epsilon \star g)(x) = \int_{\mathbb{R}^d} f_\epsilon(x-y) g(y)\, dy - \int_\Omega f_\epsilon(x-y) g(y)\, dy$ is well-defined in Ω_ϵ and $g_\epsilon \in C^\infty(\Omega_\epsilon)$.

b) $g_\epsilon \to g$ in $L^p_{loc}(\Omega)$ if $1 \leqslant p < \infty$ and $g_\epsilon(x) \to g(x)$ almost everywhere $x \in \Omega$.

c) If $g \in C(\Omega)$, then $g_\epsilon(x) \to g(x)$ uniformly in any compact subset of Ω.

Proof. a) $D^\alpha(g_\epsilon) = (D^\alpha f_\epsilon) \star g \in C(\Omega_\epsilon)$

b) Already proved in $\mathbb{R}^d$ space.

∎

Corrolary 3.11 (Lebesgue differentation theorem) If $f \in L^P_{loc}(\mathbb{R}^d)$, then

$$\fint_{B(x,\epsilon)} |f(y) - f(x)|^p\, dy \to 0 \quad \text{as } \epsilon \to 0$$

Exercise 3.12 (E 2.1) Let $u \in C^2(\mathbb{R}^2)$ be convex. I.e.

$$tu(x) + u(y)(1-t) \geqslant u(tx + (1-t)y) \forall x, y \in \mathbb{R}^d \forall t \in [0,1]$$

a) Prove for all $x \in \mathbb{R}^d$ that $H(x) = \ldots$

Solution.

a In 1D: If u is convex $\Leftrightarrow u''(x) \geqslant 0$ for all $x \in \mathbb{R}$. In general: Taylor expansion for all $x, z \in \mathbb{R}^d$:

$$u(x) = u(z) + \nabla u(z)(x-y) + \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z)) \frac{(x-z)^\alpha}{\alpha!}\, ds$$

$x = z + s(x-z), s = 1$ Use $z = tx + (t-1)y \Rightarrow x - z = (1-t)(x-y)$

$$tu(x) = tu(z) + t\nabla u(z)(1-t)(x-y) + t\int_0^1 \sum_{|\alpha|=2} D^\alpha u(z+s(x-z)) \frac{[(1-t)(x-y)]^\alpha}{\alpha!}\, ds$$

$$(1-t)u(y) = (1-t)u(z) + (1-t)\nabla u(z)t(y-x) + (1-t)\int_0^r \sum_{|\alpha|=2} D^\alpha u(z+s(y-z)) \frac{[t(y-x)]^\alpha}{\alpha!}\, ds$$

$$\Rightarrow tu(x) + (1-t)u(y) = u(z) + t\int_0^1 ... + (1-t)\int_0^1 ...$$

$$\Rightarrow t\int_0^1 ... + (1-t)\int_0^1 ... \geqslant 0 \forall x, y, t, z = tx + (1-t)y$$

$$t(1-t)^2 \int_0^1 \sum_{|\alpha|\ =2} D^\alpha u(z+s(x-z)) \frac{(x-y)}{\alpha!}\, ds + (1-t)t^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z+s(y-z)) \frac{(y-z)^\alpha}{\alpha!}\, ds \geqslant 0$$

for all $x, y \in \mathbb{R}^d, t \in [0,1], z = tx + (1-t)y$. Divides for $t(1-t)$

$$(1-t)\int_0^1 \cdots + + \int_0^1 \cdots \geqslant 0$$

Take $t \to 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(y + s(x-y)) \frac{(x-y)^\alpha}{\alpha!}\, ds \geqslant 0 \forall x, y \in \mathbb{R}^d$$

Take $y = x + a, a \in \mathbb{R}^d$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x + a + sa) \frac{a^\alpha}{\alpha!}\, ds \geqslant 0 \forall \epsilon > 0, \forall x, a \in \mathbb{R}^d$$

Take $\epsilon \to 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x) \frac{a^\alpha}{\alpha!} \geqslant 0 \Rightarrow \sum_{i,j=1, i\neq j} \partial_{x_i}\partial_{x_j} u(x) a_i a_j + \sum_{i=j=1}^{d} \partial^2_{x_i} u(x) \frac{a_i^2}{2}$$

We get

$$\frac{1}{2} a^T H a \geqslant 0 \forall a (a_i)_{i=1}^d \in \mathbb{R}^d$$

b $H(x) \geqslant 0 \Rightarrow (\partial_i \partial_j u) \geqslant 0 \Rightarrow TrH(x) \geqslant 0 \Rightarrow \sum_{i=1}^d \partial^2_{x_i} u(x) \geqslant 0 \Rightarrow \Delta u(x) \geqslant 0 \forall x \in \mathbb{R}^d$

■

Exercise 3.13 (E 2.2)

Solution. Regard $d = 3$. De function $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \backslash \{0\}$. We prove

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{\max(|x|, r)}$$

If $|x| > r$, then $0 \notin B(x, r + \epsilon)$. Then

$$y \mapsto \frac{1}{|y|}$$

is harmonic in $B(x, r + \epsilon)$. Then by the Mean Value Property:

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{|x|}$$

If $|x| < r$: Then $\frac{1}{|y|}$ is not harmonic in $B(x, r)$ since $0 \in B(x, r)$. Note

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \fint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$$

This function depends on x only via $|x|$.

$$... = \fint_{\partial B(0,r)} \frac{dS(y)}{|Rx - Ry|}$$

for all R rotation $SO(3)$, $dS(R_y) = dS(y)$

$$\begin{aligned}
&= \fint_{\partial B(0,r)} \frac{dS(y)}{|Rx - y|} \\
&= \fint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \\
\text{(Radial in z)} \quad &= \fint_{\partial B(0,|x|)} \left(\fint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \right) dS(z) \\
\text{(Fubini)} \quad &= \fint_{\partial B(0,r)} \left(\fint_{\partial B(0,|x|)} \frac{dS(z)}{|z-y|} \right) dS(y) \\
\text{(case 1 since } |y| = r > |x|) \quad &= \fint_{\partial B(0,r)} \frac{1}{|y|} \, dS(y) = \frac{1}{r}
\end{aligned}$$

If $|x| = r$: Continuity: $x \mapsto \fint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$ ∎

Remark 3.14 For $f \in C^{|\alpha|}, g \in C^{|\beta|}$:

$$D^{\alpha+\beta}(f \star g) = (D^\alpha f) \star (D^\beta g)$$

Lemma 3.15 If $d \geqslant 3$ and $f : \mathbb{R}^d \to \mathbb{R}$ radial. Then:

$$\begin{aligned}
\left(\frac{1}{|x|^{d-2}} \star f \right)(x) &= \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} \, dy \\
&= \int_{\mathbb{R}^d} \frac{f(y)}{\max(|x|^{d-2}, |y|^{d-2})} \, dy
\end{aligned}$$

Proof. (d=3) Polar coordinates:

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\,dy &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{1}{|x-rw|}\,d\omega\right] f(r)\,dr\\ (a)\quad &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{d\omega}{\max(|x|,r)}\right] f(r)\,dr\\ &= \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|,|y|)}\,dy\end{aligned}$$

(b) (d=3) If f radial and non-negative

$$\int_{\mathbb{R}^3} \frac{f(y))}{|x-y|} = \int_{\mathbb{R}^3} \frac{f(y)}{|x|}\,dy = \frac{(Sf?)}{|x|}$$

Then

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|}\,dx\,dy &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_1(x)f_2(y)}{|x+z_1-y-z_2|}\,dx\,dy\\ &= \int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} f_1(x)\,dx\right) f_2(y)\,dy \leqslant \int_{\mathbb{R}^3} \frac{(\int_{\mathbb{R}^3} f_1)}{|y+z_2-z_1|} f_2(y)\,dy\\ &\leqslant \frac{(\int_{\mathbb{R}^3} f_1)(\int_{\mathbb{R}^3} f_2)}{|z_1-z_2|}\end{aligned}$$

∎

Exercise 3.16 (Bonus 2) a) Prove that $u(x) = \frac{1}{|x|}$ is sub-harmonic in $\mathbb{R}^2\backslash\{0\}$.

b) Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ radial, non-negative, measurable:

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\,dy \geqslant \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|,|y|)}\,dy$$

My Solution. a) Let $x \in \mathbb{R}\backslash\{0\}$.

$$\begin{aligned}\partial_{x_i} u &= \partial_{x_i}|x|^{-1} = -|x|^{-2}\frac{x_i}{|x|} = -x_i|x|^{-3}\\ \Rightarrow\ \partial^2_{x_i} u &= \partial_{x_i}(-x_i|x|^{-3}) = -|x|^{-3} + 3x_i^2|x|^{-5}\\ \Rightarrow\ \Delta u &= -|x|^{-3} + 3x_1^2|x|^{-5} - |x|^{-3} + 3x_2^2|x|^{-5}\\ &= -2|x|^{-3} + 3\underbrace{(x_2^2+x_2^2)}_{=|x|^2}|x|^{-5} = \frac{1}{|x|^3} > 0\end{aligned}$$

So u is sub-harmonic in $\mathbb{R}^2\backslash\{0\}$.

b) Let $r > 0, x \in \mathbb{R}^2$ and $|x| < r$. First we show that

$$\fint_{\partial B(x,r)} \frac{1}{|y|}\,dS(y) \geqslant \frac{1}{r} \qquad (\star)$$

Now,

$$\fint_{\partial B(0,r)} \frac{1}{|y|}\,dS(y) = \fint_{\partial B(0,r)} \frac{1}{|x-y|}dS(y) =: \tilde{u}(x)$$

Take $z \in \mathbb{R}^2\backslash\{0\}$ such that $z = |x|$, then $\tilde{u}(x) = \tilde{u}(z)$. Let $0 < \epsilon < r$ be small. Then we get

$$
\begin{aligned}
\tilde{u}(z) &= \fint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \\
\begin{pmatrix} |y| = r > |x| = |z| \\ \tilde{u} \text{ radial function} \end{pmatrix} \quad &= \fint_{\partial B(0,|x|-\epsilon)} \left(\fint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \right) dS(z) \\
\text{(Fubini)} \quad &= \fint_{\partial B(0,r)} \left(\fint_{\partial B(0,|x|-\epsilon)} \frac{dS(z)}{|z-y|} \right) dS(y) \\
&= \fint_{\partial B(0,r)} \left(\fint_{\partial B(y,|x|-\epsilon)} \frac{dS(z)}{|z|} \right) dS(y) \\
\left(\frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2\backslash\{0\} \right) \quad &\geqslant \fint_{\partial B(0,r)} \frac{1}{|y|}\, dS(y) \\
&= \fint_{\partial B(0,r)} \frac{1}{r}\, dS(y) \\
&= \frac{1}{r}
\end{aligned}
$$

Now,

$$
\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\, dy = \int_{|x| \geqslant |y|} \frac{f(y)}{|x-y|}\, dy + \int_{|x|<|y|} \frac{f(y)}{|x-y|}\, dy,
$$

where

$$
\begin{aligned}
\int_{|x| \leqslant |y|} \frac{f(y)}{|x-y|}\, dy &= \int_0^\infty \int_{\partial B(0,r)} \frac{f(y)}{|x-y|} \mathbb{1}(|x| \leqslant |y|)\, dS(y)\, dr \\
(f \text{ radial}) \quad &= \int_0^\infty f(r) \int_{\partial B(0,r)} \frac{\mathbb{1}(|x| \leqslant r)}{|x-y|}\, dS(y)\, dr \\
&= \int_0^\infty f(r) \int_{\partial B(x,r)} \frac{\mathbb{1}(|x| \leqslant r)}{|y|}\, dS(y)\, dr \\
(\star) \quad &\geqslant \int_0^\infty \frac{f(r)}{r} |\partial B(x,r)| \mathbb{1}(|x| \leqslant r)\, dr \\
&= \int_0^\infty \int_{\partial B(x,r)} \frac{f(r)}{r} \mathbb{1}(|x| \leqslant r)\, dS(y)\, dr \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{|y|} \mathbb{1}(|x| \leqslant |y|)\, dy \\
&= \int_{|x| \leqslant |y|} \frac{f(y)}{|y|}\, dy
\end{aligned}
$$

and

$$\begin{aligned}
\int_{|x|>|y|} \frac{f(y)}{|x-y|}\, dy &= \int_0^\infty \left(\int_{\partial B(0,r)} \frac{f(r)}{|x-y|} \mathbb{1}(|x|>|y|)\, dS(y) \right) dr \\
(f \text{ radial}) \quad &= \int_0^\infty f(r)\mathbb{1}(|x|>r) \left(\int_{\partial B(x,r)} \frac{1}{|y|}\, dS(y) \right) dr \\
\left(\begin{aligned} \frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2, \\ \text{MVP and } |x|>r \end{aligned} \right) \quad &\geqslant \int_0^\infty f(r)\mathbb{1}(|x|>r)|\partial B(x,r)| \frac{1}{|x|}\, dr \\
&= \int_0^\infty \int_{\partial B(x,r)} f(r)\mathbb{1}(|x|>r) \frac{1}{|x|}\, dS(y)\, dr \\
&= \int_{\mathbb{R}^2} f(y)\mathbb{1}(|x|>|y|) \frac{1}{|x|}\, dy \\
&= \int_{|x|>|y|} f(y) \frac{1}{|x|}\, dy.
\end{aligned}$$

So we can conclude,

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\, dy &= \int_{|x|>|y|} \frac{f(y)}{|x-y|}\, dy + \int_{|x|\leqslant|y|} \frac{f(y)}{|x-y|}\, dy \\
&\geqslant \int_{|x|>|y|} \frac{f(y)}{|x|}\, dy + \int_{|x|\leqslant|y|} \frac{f(y)}{|y|}\, dy \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|,|y|)}\, dy
\end{aligned}$$ ■

3.2 Fourier Transformation

Definition 3.17 (Fourier Transform) For $f \in L^1(\mathbb{R}^d)$ define

$$\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k \cdot x}\, dx, \quad k \cdot x = \sum_{i=1}^d k_i x_i$$

Theorem 3.18 (Basic Properties)

1. If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^\infty(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$
2. For all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$. Moreover, $\mathcal{F}$ can be extended to be a unitary transforamtion $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ s.t.
$$\|\mathcal{F}g\|_{L^2} = \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^d)$$
3. The inverse of F can be defined as
4. $$(F^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(x) e^{2\pi i k x}\, dk$$
for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$
5. $\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k)$ as $(2\pi i k)^\alpha f(k) \in L^2(\mathbb{R}^d)$ $(k^\alpha = k_1^{\alpha_1} \cdots k_\alpha^{\alpha_k})$
6. $\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k)$ if f, g are nice enough.

Theorem 3.19 (Hausdorff-Young-Inequality) If $1 \leqslant p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^n)$ then

$$\|\hat{f}\|_{L^{p'}} \leqslant \|f\|_{L^p}$$

and

$$\|\hat{f}\|_{L^p} \leqslant \|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^d)$$

Remark 3.20 We want to apply the Fourier transform to find the solution of a PDE, e.g. the Poisson-Equation:

$$-\Delta u = f \text{ in } \mathbb{R}^d \Rightarrow |2\pi k|^2 \hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{1}{|2\pi k|^2}\hat{f}(k)$$

If we can find G s.t. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$, then

$$\begin{aligned} \hat{u}(k) &= \hat{G}(k)\hat{f}(k) = \widehat{G \star f} \\ \Rightarrow u(x) &= (G \star f)(x) = \int_{\mathbb{R}^d} G(x-y) f(y)\, dy \end{aligned}$$

In fact G is the fundamential solution of laplace quation.

Theorem 3.21 (Fourier Transform of $\frac{1}{|x|^\alpha}$ for $0 < \alpha < d$) We have formally

$$\widehat{\frac{c_\alpha}{|x|^\alpha}} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \quad \forall\ 0 < \alpha < d$$

Here

$$c_\alpha = \pi^{-\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1}\, d\lambda$$

More precisely, for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{c_\alpha}{|x|^\alpha} \star f = \left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)\right)^\vee$$

Moreover if $\alpha > \frac{d}{2}$, then we also have

$$\left(\frac{c_\alpha}{|x|^\alpha} \star f\right)^\wedge = \frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k)$$

Lemma 3.22 (Fourier Transform of Gaussians) In $\mathbb{R}^d$,

$$\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$$

More generally for all $\lambda > 0$:

$$\widehat{e^{-\pi\lambda^2|x|^2}} = \lambda^{-d} e^{-\pi\frac{|k|^2}{\lambda^2}}$$

(exercise)

Proof of Theorem. Formally:

$$
\begin{aligned}
\frac{c_\alpha}{|x|^\alpha} &= \frac{1}{|x|^\alpha}\pi^{-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda}\lambda^{\frac{\alpha}{2}-1}\,d\,\lambda = \int_0^\infty e^{-\pi\lambda|x|^2}\lambda^{\frac{\alpha}{2}-1}\,d\lambda \\
\Rightarrow \frac{\hat{c}_\alpha}{|x|^\alpha}(k) &= \int_0^\infty \widehat{e^{-\pi\lambda|x|^2}}(k)\lambda^{\frac{\alpha}{2}-1}\,d\lambda = \int_0^\infty \lambda^{-\frac{d}{2}}e^{-\pi\frac{|k|^2}{\lambda}}\lambda^{\frac{\alpha}{2}-1}\,d\lambda \\
(\lambda\to\frac{1}{\lambda}) \quad &= \int_0^\infty \lambda^{\frac{d}{2}e^{-\pi|k|^2\lambda}}\lambda^{-\frac{\alpha}{2}+1}\lambda^{-2}\,d\lambda \\
&= \frac{c_{d-\alpha}}{|k|^{d-\alpha}}
\end{aligned}
$$

Let $f\in C_c(\mathbb{R}^d)$. Then $\left(\frac{1}{|x|^\alpha}\star f\right)(x) = \int_{\mathbb{R}^d}\frac{1}{|x-y|^\alpha}f(y)\,dy$ is well defined as $\frac{1}{|x-y|}\in L^1_{loc}(\mathbb{R}^d,dy)$. It is bounded

$$
\frac{1}{|x|^\alpha}\star f = \frac{1}{|x|^\alpha}\underbrace{\mathbb{1}(|x|\leqslant 1)}_{\in L^\infty(\mathbb{R}^d)}\star\underbrace{f}_{L^\infty} + \underbrace{\frac{1}{|x|}\mathbb{1}(|x|>1)}_{\in L^\infty}\star\underbrace{f}_{\in L^1}\in L^\infty(\mathbb{R}^d)
$$

When $|x|\ \to\infty$:

$$
\left(\frac{1}{|x|^\alpha}\star f\right)(x) = \int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^\alpha}\,dy = \int_{|y|\leqslant R}\frac{f(y)}{|x-y|^\alpha}\,dy \sim \frac{\int_{\mathbb{R}^d}f(y)\,dy}{|x|^\alpha}
$$

Note that $\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\underbrace{\hat{f}(k)}_{\text{bounded}}\in L^1(\mathbb{R}^d)$.

$$
(...)\mathbb{1}(|k|\leqslant 1) + (...)\mathbb{1}(|k|>1)\frac{1}{|k|^{d-\alpha}}|\hat{f}(k)|\ \mathbb{1}(|k|\leqslant 1)\leqslant \|f\|_{L^1}\frac{\mathbb{1}(|k|\leqslant 1)}{|k|^{d-\alpha}}\in L^1(\mathbb{R}^d,dk)
$$

$$
\frac{1}{|k|^{d-\alpha}}|\hat{f}(k)|\mathbb{1}(k>1)\leqslant|\hat{f}(k)|\in L^2(\mathbb{R}^d,dK)\text{ as }f\in L^2(\mathbb{R}^d)
$$

Lemma 3.23 If $f\in C_c^\infty(\mathbb{R}^d)$, then $\hat{f}\in L^1(\mathbb{R}^d)$

Proof. (Exercise) Hint: $|\widehat{D^\alpha f}| = |2\pi k|^{|\alpha|}|\hat{f}(k)| \rightsquigarrow |\hat{f}(k)|\leqslant\frac{1}{|k|^{|k|}}$ as $|k|\to\infty$. ■

Compute:

$$
\begin{aligned}
\left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k)\right)^\vee(x) &= \int_{\mathbb{R}^d}\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k)e^{2\pi ikx}\,dk \\
&= \int_{\mathbb{R}^d}\left(\int_0^\infty e^{-\pi|k|^2\lambda}\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda\right)\hat{f}(k)e^{2\pi ikx}\,dk \\
&= \int_0^\infty\left(\int_{\mathbb{R}^d}e^{-\pi|k|^2\lambda}\hat{f}(k)e^{2\pi ikx}\,dk\right)\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty\left(e^{-\pi k^2\lambda}\hat{f}(x)\right)^\vee\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty\left(\widehat{\lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}}}(k)\hat{f}(k)\right)^\vee\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty\left(\lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}}\star f\right)\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \left(\int_0^\infty\lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}}\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda\right)\star f
\end{aligned}
$$

Assume $d > \alpha > \frac{d}{2}$. Then $\frac{c_\alpha}{|x|^\alpha} \star f \in L^\infty$ and behaves $\frac{c_\alpha(\int f)}{|x|^\alpha}$ as $|x| \to \infty$. This implies:

$$\int_{\mathbb{R}^d} \left| \frac{c_\alpha}{|x|^\alpha} \star f \right|^2 \leqslant c + \int_{|x| \geqslant R} \frac{c}{|x|^{2d}} \, dx < \infty$$

Thus the Fourier Transform $\widehat{\frac{c_\alpha}{|x|^\alpha} \star f}$ exists. Combining with

$$\begin{aligned} \frac{c_\alpha}{|x|^\alpha} \star f &= \left(\frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k) \right)^\vee \\ \Rightarrow \widehat{\frac{c_\alpha}{|x|^\alpha} \star f} &= \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \end{aligned}$$

■

Remark 3.24 If $d \geqslant 3$

$$\begin{aligned} \hat{G}(k) &= \frac{1}{|2\pi k|^2} \\ \Rightarrow G(x) &= \left(\frac{1}{|2\pi k|^2} \right)^\vee = \frac{1}{d(d-2(k)|x|^{d-2})} = \Phi(x) \end{aligned}$$

3.3 Theory of Distribution

Let $\Omega \subseteq \mathbb{R}^d$ be open.

- $D(\Omega) = C_c^\infty(\Omega)$ the space of test functions.
- $\phi_n \to \phi$ in $D(\Omega)$ if $\exists K \subseteq \Omega$, $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K$ and $\|D^\alpha(\phi_n - \phi)\|_{L^\infty} \to 0$ for all $\alpha = (\alpha_1, \ldots, \alpha_d)$, $d_i \in \{0, 1, 2, \ldots\}$.

  $$D'(\Omega) = \{T : D(\Omega) \to \mathbb{R} \text{ on } \mathbb{C} \text{ linear and continuous}\}$$

  the space of distributions.

Motivation: $L^2(\Omega)' = L^2(\Omega)$, $(L^p(\Omega))' = (L^q(\Omega))$, $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3.25 ("normal functions" are distributions) If $f \in L^1_{loc}(\Omega)$, then $T = T_f$ defined by:

$$T(\phi) = \int_\Omega f(x)\phi(x) \, dx$$

is a distribution for all $\phi \in D(\Omega)$, i.e. $T \in D'(\Omega)$. Indeed, it is clear that $T(\phi)$ is well-defined for all $\phi \in D(\Omega)$ and $\phi \mapsto T(\phi)$ is linear. Let us check that $\phi \mapsto T(\phi)$ is continuous. Take $\phi_n \to \phi$ in $D(\Omega)$ and prove that $T(\phi_n) \to T(\phi)$. Since $\phi_n \to \phi$ in $D(\Omega)$, there is a compact K s.t. $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K \subseteq \Omega$.

Question: Why is $f \mapsto T_f$ injective?

Lemma 3.26 (Fundamental lemma of calculus of variants) Let $\Omega \subseteq \mathbb{R}^d$ be open. If $f, g \in L^1_{loc}(\Omega)$ and $\int_\Omega f\phi \, dy = \int_\Omega g\phi \, dy$ for all $\phi \in D(\Omega)$, then $f = g$ in $L^1_{loc}(\Omega)$

Example 3.27 (Dirac delta function) Let $\Omega \subseteq \mathbb{R}^d$ open. Define $T : D(\Omega) \to \mathbb{R}$ or $\mathbb{C}$ by $T(\phi) = \phi(x_0)$. Let $x_0 \in \Omega$. Then $T \in D'(\Omega)$ and we denote it by δ_{x_0}. It is clear that $\phi \mapsto T(\phi) = \phi(x_0)$ is well-defined and linear for all $\phi \in D(\Omega)$. Take $\phi_n \to \phi$ in $D(\Omega)$ and prove $T(\phi_n) \to T(\phi)$, i.e. $\phi_n(x_0) \to \phi(x_0)$ (obvious.)

Example 3.28 (Principle Value) The function $f(x) = \frac{1}{x}$ is not in $L^1_{loc}(\mathbb{R})$, but we can still define

$$\int_{\mathbb{R}} f(x)\phi(x)\,dx = \int_{\mathbb{R}} \frac{\phi(x)}{x}\,dx$$

for all $\phi \in D(\mathbb{R})$ s.t. $\phi(0) = 0$. In fact,

$$\phi(x) = |\phi(x) - \phi(0)| \leqslant (\sup|\phi'|)(x),$$

so $\frac{|\phi(x)|}{|x|} \in L^\infty(\mathbb{R})$ and compactly supported. So $\frac{\phi(x)}{x} \in L^1(\mathbb{R})$. Define $T : D(\mathbb{R}) \to \mathbb{R}$ or $\mathbb{C}$ by

$$T(\phi) = \lim_{\epsilon \to 0} \int_{|x| \geqslant \epsilon} \frac{\phi(x)}{x}\,dx \quad \forall \phi \in D(\mathbb{R}) \text{ s.t. } \phi(0) = 0$$

We denote $T = p.v.\left(\frac{1}{x}\right)$. We check that $T \in D'(\mathbb{R})$: For all $\epsilon > 0$ we have

$$\left|\frac{\phi(x)}{x}\right| \leqslant \frac{\|\phi\|_{L^\infty}}{\epsilon}$$

for all $|x| \geqslant \epsilon$ and ϕ is compactly supported. So we get for all $\epsilon > 0$:

$$\mathbb{1}(|x| \geqslant \epsilon)\frac{\phi(x)}{x} \in L^1(\mathbb{R}) \rightsquigarrow \int_{|x| \geqslant \epsilon} \frac{\phi(x)}{x}\,dx < \infty$$

We can write:

$$\int_{|x| \geqslant \epsilon} \frac{\phi(x)}{x}\,dx = \int_{|x| \geqslant 1} \frac{\phi(x)}{x}\,dx + \int_{\epsilon \leqslant |x| \leqslant 1} \frac{\phi(x)}{x}\,dx$$

The second part can be written as:

$$\int_{\epsilon \leqslant |x| \leqslant 1} \frac{\phi(x)}{x}\,dx = \int_{\epsilon}^{1} \frac{\phi(x)}{x}\,dx + \int_{-1}^{-\epsilon} \frac{\phi(x)}{x}\,dx = \int_{\epsilon}^{1} \frac{\phi(x) - \phi(-x)}{x}\,dx$$

Since $\phi \in C_c^\infty(\mathbb{R})$ it holds that $|\phi(x) - \phi(-x)| \leqslant 2\|\phi'\|_{L^\infty}(x)$.

$$\Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^\infty(\mathbb{R}) \Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^1(0,1)$$

$$\Rightarrow \int_0^1 \frac{\phi(x) - \phi(-x)}{x}\,dx = \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{\phi(x) - \phi(-x)}{x}\,dx$$

Remark 3.29 The function $\frac{1}{|x|^d}$ is not in $L^1_{loc}(\mathbb{R}^d)$ but $\exists T \in D'(\mathbb{R}^d)$ s.t. $T(\phi) = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^d}\,dx$ for all $\phi \in C_c^\infty(\mathbb{R}^d)$ s.t. $\phi(0) = 0$

Let in the following $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 3.30 (Derivatives of distributions) Let $\Omega \subseteq \mathbb{R}^d$ and $T \in D'(\Omega)$. Define for $\alpha \in \mathbb{N}^d$:

$$\begin{aligned} D^\alpha T :\ D(\Omega) &\longrightarrow \mathbb{K} \\ \phi &\longmapsto (-1)^{|\alpha|} T(D^\alpha \phi) \end{aligned}$$

Motivation: $f \in C_c^\infty(\Omega)$

$$\int_\Omega (D^\alpha f)\phi = (-1)^{|\alpha|} \int_\Omega f(D^\alpha \phi)$$

„If the classical derivative exists, then it is the same as the distributional derivative.“
We write

$$(D^\alpha T)(\phi) = T_{D^\alpha f}(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi).$$

Remark 3.31 For all $T \in D'(\Omega)$ it holds $D^\alpha T \in D'(\Omega)$ for all $\alpha \in \mathbb{N}^d$. Clearly

$$\phi \longmapsto (D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

is linear. Moreover, if $\phi_n \to \phi$ in $D(\Omega)$, then $D^\alpha \phi_n \to D^\alpha \phi$ in $D(\Omega)$, so

$$(D^\alpha T)(\phi_n) = (-1)^{|\alpha|} T(D^\alpha \phi_n) \xrightarrow{n\to\infty} (-1)^{|\alpha|} T(D^\alpha \phi) = (D^\alpha T)(\phi)$$

Example 3.32 Consider $f : x \mapsto |x|$, then $f \in C(\mathbb{R})$ but $f \notin C^1(\mathbb{R})$. However,

$$f'(x) = g(x) = \begin{cases} 1 & x \geqslant 0 \\ -1 & x < 0 \end{cases} \in L^1_{loc}$$

Lets check $f' = g$, i.e. $-f(\phi') = f'(\phi) = g(\phi)$ for all $\phi \in D(\mathbb{R})$. Thus we need to prove:

$$-\int_{\mathbb{R}} f(x)\phi'(x)\,dx = \int_{\mathbb{R}} g(x)\phi(x)\,dx \quad \forall \phi \in D(\mathbb{R})$$

namely:

$$\underbrace{-\int_{\mathbb{R}} |x|\phi'(x)\,dx}_{:=(\star)} = \int_0^\infty \phi(x)\,dx - \int_{-\infty}^0 \phi(x)\,dx$$

Now we have

$$(\star) = -\int_0^\infty x\phi'(x)\,dx + \int_{-\infty}^0 x\phi'(x)\,dx.$$

By integration by parts:

$$\int_0^\infty x\phi'(x)\,dx = \underbrace{[x\phi(x)]_0^\infty}_{=0} - \int_0^\infty \phi(x)\,dx = -\int_0^\infty \phi(x)\,dx$$

and similary:

$$\int_{-\infty}^0 x\phi'(x)\,dx = -\int_{-\infty}^0 \phi(x)\,dx$$

Thus $f' = g$ in $D'(\Omega)$. We claim that $g' = 2\delta_0$ in $D'(\mathbb{R})$. In fact, for all $\phi \in D(\mathbb{R})$, then:

$$\begin{aligned} g'(\phi) &= -g(\phi') = -\int_{\mathbb{R}} g\phi'\,dx = -\int_{-\infty}^0 (-1)\phi'\,dx - \int_0^\infty (1)\phi'\,dx \\ &= -\int_0^\infty \phi'\,dx + \int_{-\infty}^0 \phi'\,dx = [\phi(0) - \underbrace{\phi(\infty)}_{=0}] + [\phi(0) - \underbrace{\phi(-\infty)}_{=0}] = 2\phi(0) = 2\delta_0(\phi) \end{aligned}$$

So $g' = 2\delta_0$ in $D'(\mathbb{R})$.

Exercise 3.33 Prove that $(D^\alpha \delta_x)(\phi) = (-1)^{|\alpha|}(D^\alpha \phi)(x)$ for all $\phi \in D(\mathbb{R})$ for all $x \in \mathbb{R}$.

Definition 3.34 (Convergence of distributions) Let $\Omega \subseteq \mathbb{R}^d$ be open, then

$$T_n \xrightarrow{n\to\infty} T$$

in $D'(\Omega)$ if $T_n(\phi) \xrightarrow{n\to\infty} T(\phi)$ for all $\phi \in D(\Omega)$.

Exercise 3.35 Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ For $\epsilon > 0$, define $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$. Then: $f_\epsilon \to \delta_0$ in $D'(\Omega)$.

Exercise 3.36 Let $\Omega \subseteq \mathbb{R}^d$ be open and $T_n \to T$ in $D'(\Omega)$. Then: $D^\alpha T_n \to D^\alpha T$ in $D'(\Omega)$ for all $\alpha = (\alpha_1, \dots, \alpha_d)$

Definition 3.37 (Convolution of distributions) Let $T \in D'(\mathbb{R})$ and $f \in L_c^\infty(\mathbb{R}^d)$. Define

$$(T \star f)(y) = T(f_y)$$

We write $f_y(x) = f(x-y)$ and $\tilde{f}(x) = f(-x)$.

Theorem 3.38 Let $T \in D'(\mathbb{R})$. Then for all $f \in D(\mathbb{R})$:

1. $y \mapsto T(f_y)$ is $C^\infty(\mathbb{R}^d)$ and

$$D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D^\alpha f_y)$$

2. For all $g \in L^1(\mathbb{R}^d)$ and compactly supported, then

$$\int_{\mathbb{R}^d} g(y) T(f_y)\, dy = T(\underbrace{f \star g}_{\in C_c^\infty(\mathbb{R})})$$

Proof. 1. We prove that $y \mapsto T(f_y)$ is continuous. Take $y_n \to y$ in $\mathbb{R}^d$, then:

$$T(f_{y_n}) \to T(f_y)$$

since $f_{y_n} \to f_y$ in $D(\mathbb{R}^d)$. We check this: Since $f \subseteq C_c^\infty(\mathbb{R}^d)$, it holds that $\operatorname{supp} f \subseteq B(0,R) \subseteq \mathbb{R}^d$. $\underline{\text{Since } y_n \to y \text{ in } \mathbb{R}^d}$. We have $\sup_n |y_n| < \infty$. Thus f_{y_n}, f_y are supported in $B(0, R + \sup_n |y_n|) = K$ compact. Moreover

$$|f_{y_n}(x) - f_y(x)| = |f(x-y_n) - f(x-y))| \leqslant \|\nabla f\|_{L^\infty} \|y_n - y\| \to 0$$

So we get $\|f_{y_n} - f_y\|_{L^\infty} \to 0$ Similary:

$$\|D^\alpha f_{y_n} - D^\alpha f_n\|_{L^\infty} \to 0$$

■

Exercise 3.39 (E 3.1 Lebesgue Differentiation Theorem) Let $f \in L^1_{loc}(\mathbb{R}^d)$. Prove that that for almost every $x \in \mathbb{R}^d$:

$$\fint_{B(x,r)} |f(y)|\, dy \xrightarrow{r\to 0} 0$$

Proof. Clearly the same result holds with $\mathbb{R}^d \rightsquigarrow \Omega \subseteq \mathbb{R}^d$ open. Also it suffices to consider $f \in L^1(\mathbb{R}^d)$. From the last time discussion, by a density argument there exists $r_n \to 0$ s.t.

$$\fint_{B(x,r_n)} |f(y) - f(x)|\, dy = 0$$

for a.e. $x \in \mathbb{R}^d$. We prove that for all $\epsilon > 0$, te set $A_\epsilon = \{x \in \mathbb{R}^d \mid \limsup_{r\to 0} ⨍_{B(x,r)} |f(y) - f(x)|\, dy > \epsilon\}$ has measure 0. This will imply that

$$\bigcup_{n=1}^{\infty} A_{\frac{1}{n}} = \left\{ x \in \mathbb{R}^d \mid \limsup_{r\to 0} \int_{B(x,r)} |f(y) - f(x)|\, dy > 0 \right\}$$

has measure 0, which is what wie want to show. First, we show that $|A_\epsilon| = 0$: Take $\{f_n\} \subseteq C_c^\infty$, $f_n \to f$ in $L^1(\mathbb{R}^d)$. By the triangle inequality:

$$|f(y) - f(x)| \leqslant |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

So we get

$$\begin{aligned} &⨍_{B(x,r)} |f(y) - f(x)|\, dy \\ &\leqslant ⨍_{B(x,r)} |f(y) - f_n(y)|\, dy + ⨍_{B(x,r)} |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \end{aligned}$$

$$\Rightarrow \quad \limsup_{r\to 0} ... \leqslant \limsup_{r\to 0} (\dots) + 0 + |f_n(x) - f(x)|$$

Thus, for all $x \in A_\epsilon$, then:

$$\limsup_{r\to 0} ⨍_{B(x,r)} |f_n(y) - f(y)|\, dy + |f_n(x) - f(x)| > 2\epsilon$$

Observation: If $a, b \geqslant 0$, $a + b > 2\epsilon$ then either $a > \epsilon$ or $b > \epsilon$. Therefore $A_\epsilon \subseteq \left(S_{n,\epsilon} \bigcup \tilde{S}_{n,\epsilon}\right)$, where

$$\begin{aligned} S_{n,\epsilon} &= \{x \mid |f_n(x) - f(x)| > \epsilon\} \\ \tilde{S}_{n,\epsilon} &= \{x \mid \limsup_{r\to 0} ⨍_{B(x,r)} |f_n(y) - f(y)|\, dy > \epsilon\} \end{aligned}$$

Consequently: $|A_\epsilon| \leqslant |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}|$ for all $n \geqslant 1$. By the Markov / Chebyshev inequality:

$$|S_{n,\epsilon}| \leqslant \int_{S_{n,\epsilon}} \frac{|f_n(x) - f(x)|}{\epsilon}\, dx = \int_{\mathbb{R}^d} \frac{|f_n(x) - f(x)|}{\epsilon}\, dx = \frac{\|f_n - f\|_{L^1}}{\epsilon}$$

We want to prove a simpler bound for $\tilde{S}_{n\epsilon}$. For all $x \in \tilde{S}_{n\epsilon}$:

$$\limsup_{r\to 0} ⨍_{B(x,r)} |f_n(x) - f(y)|\, dy > \epsilon$$

So there is a $r_x \in (0, 1)$ s.t.

$$⨍_{B(x,r_x) = B_x} |f_n(y) - f(y)|\, dy > \epsilon$$

Thus $\tilde{S}_{n\epsilon} \subseteq \left(\bigcup_{x\in\tilde{S}_{n,\epsilon}} B_x\right)$.

Lemma 3.40 (Vitali Covering) If F is a collection of balls in $\mathbb{R}^d$ with bounded radius, then there exists a sub-collection $G \subseteq F$ s.t.

- G has disjoint balls

- $\bigcup_{B\in F} B \subseteq \bigcup_{B\in G} 5B, 5B(x,r) = B(x,5r)$

Remark 3.41 The condition of the boundedness of the radius is necessary. Otherwise, consider $\{B(0,n)\}_{n=1}^{\infty}$

Here consider $F = \{B_x\}_{x\in\tilde{S}_{n\epsilon}}$. With the vitali covering leamm there is a $G \subseteq F$ s.t. G contains disjoint balls and:

$$\tilde{S}_{n,\epsilon} \subseteq \bigcup_{B\in F} B \subseteq \bigcup_{B\in G} 5B$$

So we get

$$|\tilde{S}_{n,\epsilon}| \leqslant |\bigcup_{B\in G} 5B| \leqslant \sum_{B\in G} |5B| = \sum_{B\in G} 5^d |B|$$

On the other hand, for all $B \in G \subseteq F$:

$$⨍_B |f_n(y) - f(y)| \ dy > \epsilon \Rightarrow \int_B |f_n - f| > \epsilon |B|$$

This implies:

$$\sup_{B\in G} \int_B |f_n - f| > \epsilon \sum_{B\in G} |B|$$

Since balls in G are disjoint:

$$\int_{\mathbb{R}^d} \geqslant \int_{\bigcup_{B\in G}} |f_n - f| \, dy > \epsilon \sum_{B\in G} |B| \geqslant \frac{\epsilon}{5^d} |\tilde{S}_{n,\epsilon}|$$

So

$$|\tilde{S}_{n\epsilon}| \leqslant \frac{5^d}{\epsilon} \|f_n - f\|_{L^1}$$

In summary:

$$|A_\epsilon| \leqslant |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}| \leqslant \frac{5^d + 1}{\epsilon} \|f_n - f\|_{L^1} \to 0$$

as $n \to \infty$. So $|A_\epsilon| = 0$ for all $\epsilon > 0$ ■

Remark 3.42
1. The proof can be done by using the Besicovitch covering lemma: For all $E \subseteq \mathbb{R}^d$ s.t. E is bounded. Let F = collection of balls s.t. for all $x \in E$ there is a $B_x \in F$ s.t. x is the center of B_x. There is a sub-collection $G \subseteq F$ s.t.

   - $E \subseteq \bigcup_{B\in G} B$
   - Any point in E belongs to at most C_d balls in C_T (C_d depends only on $\mathbb{R}^d$), i.e.

   $$\mathbb{1}_E(x) \leqslant \sum_{B\in G} \mathbb{1}_B(x) \leqslant C_d \mathbb{1}_E(x) \forall x$$

2. By a simpler argument we can prove the weak L^1-estimate:

   $$\{x \mid f^\star(x) > \epsilon\} \leqslant \frac{c_d}{\epsilon} \|f\|_{L^1(\mathbb{R}^d)}$$

   (Hardy-Littlewood maximal function)

Exercise 3.43 (E 3.2) Let $1 \leqslant p, q, r \leqslant 2$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Recall that if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^r(\mathbb{R}^d)$ by Young's Inequality, and its Fourier transform is well-defined by the Hausdroff-Young inequality. Prove that

$$\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k) \quad \forall k \in \mathbb{R}^d$$

Hint: In the lecture we already discussed the case $f, g \in C_c(\mathbb{R}^d)$.

Solution.

Step 1) $f, g \in C_c^\infty(\mathbb{R}^d)$ (Fubini)

Step 2) $f \in L^p, g \in L^q$, find $f_n, g_n \in C_c^\infty$ s.t. $f_n \to f$ in L^p, $g_n \to g$ in L^q. $\widehat{f_n \star g_n} = \hat{f}_n \hat{g}_n$ pointwise a.e. we have

$$\begin{aligned}
\text{(Hausdorff-Young)} \quad & \|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^{r'}} \\
& \leqslant \|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^r} \\
& = \|(f - f_n) \star g_n + f_n \star (g_n - g)\|_{L^r} \\
& \leqslant \|(f - f_n) \star g_n\|_{L^r} + \|f_n \star (g_n - g)\|_{L^r} \\
\text{(Young)} & \leqslant \|f - f_n\|_{L^p} \|g_n\| + \|f_n\|_{L^p} \|g_n - g\|_{L^p} \xrightarrow{n \to \infty} 0
\end{aligned}$$

Moreover:

$$\begin{aligned}
\|\hat{f}_n \hat{g}_n - \hat{f}\hat{g}\|_{L^{r'}} & = \|(\hat{f}_n \hat{f})\hat{g}_n + \hat{f}(\hat{g}_n - \hat{g})\|_{L^{r'}} \\
\text{(Hölder)} \quad & \leqslant \|\hat{f}_n - \hat{f}\|_{L^{p'}} \|\hat{g}_n\|_{L^{q'}} + \|\hat{f}\|_{L^{q'}} \\
\text{(Hausdorff-Young (3.19))} \quad & \leqslant \|f_n - f\|_{L^p} \|g_n\|_{L^q} + \|f\|_{L^p} \|g_n - g\|_{L^p} \xrightarrow{n \to \infty} 0
\end{aligned}$$

So $\hat{f}_n \hat{g}_n \to \hat{f}\hat{g}$ in $L^{r'}$ $\widehat{f \star g} = \hat{f}\hat{g}$ in $L^{r'}$ $\frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'}$ ■

Exercise 3.44 (E 3.3) $f \in C_c^\infty(\mathbb{R}^d)$. Prove $|\hat{f}(k)| \leqslant \frac{C_N}{(1+|k|)^N}$

Solution. Since $f \in C_c^\infty$ we have that $D^\alpha f \in C_c^\infty$. Recall

$$\widehat{D^\alpha f}(k) = (-2\pi i k)^\alpha \hat{f}(k)$$

For example

$$\begin{aligned}
\widehat{-\Delta f}(k) &= |2\pi i k|^2 \hat{f}(k) \\
\text{(Induction)} \rightsquigarrow \widehat{(-\Delta)^N f}(k) &= |2\pi k|^{2N} \hat{f}(k)
\end{aligned}$$

So we can conclude

$$\hat{f}(k) = \frac{\widehat{(-\Delta)^N f}(k)}{|2\pi k|^{2N}} \forall k \in \mathbb{R}^d$$

1. $f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in L^\infty$
2. $(-\Delta)^N f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \widehat{(-\Delta)^N} f \in L^\infty$

Conclusion: $\hat{f}(k) \leqslant \begin{cases} C & \forall k \\ \frac{C_N}{|k|^{2N}} & \forall k \end{cases}$ So $\hat{f}(k) \leqslant \frac{C_N}{(1+|k|)^N}$ ■

Exercise 3.45 (E 3.4)

Proof. Siehe Goodnotes ■

Exercise 3.46 (Bonus 3) Let $f \in L^1(\mathbb{R}^d)$ such that

$$|\hat{f}(k)| \leqslant \frac{C_N}{(1+|k|)^N}$$

for all $k \in \mathbb{R}^d$, for all $N \geqslant 1$. (C_N is independent of k). Prove that $f \in C^\infty(\mathbb{R}^d)$

($f \in C^\infty$) i.e. $\exists \tilde{f} \in C^\infty$ s.t. $f = \tilde{f}$ a.e.

My Solution. First we regard for $N \in \mathbb{N}$ and $|k| \geqslant 1$:

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{1}{(1+|k|)^N}\, dk &= \int_0^\infty \left(\int_{\partial B(0,r)} \frac{1}{(1+r)^N}\, dS(y) \right) dr \\
&= \int_0^\infty \frac{1}{(1+r)^N} |\partial B(0,r)|\, dr \\
&\leqslant \int_1^\infty \frac{1}{r^N} \underbrace{|\partial B(0,r)|}_{cr^{d-1}}\, dr \quad \text{for a } c \in \mathbb{R} \\
&= c \int_1^\infty \frac{1}{r^{N-d+1}}\, dr
\end{aligned}$$

From Ana I we know that $\int_1^\infty \frac{1}{r^{N-d+1}}\, dr < \infty$ is equivalent to $N - d + 1 > 1 \Leftrightarrow N > d$, so for $N > d$ we have

$$\frac{1}{(1+|k|)^N} \in L^1(\mathbb{R}^d).$$

Now let $\alpha \in \mathbb{N}^d$, then we have

$$k^\alpha = k_1^{\alpha_1} \cdots k_d^{\alpha_d} = |k|^{\alpha_1} \cdots |k|^{\alpha_d} = |k|^{\alpha_1 + \cdots + \alpha_d} = |k|^{|\alpha|} = (1+|k|)^{|\alpha|}.$$

By assumption we have for all $N \geqslant 1$:

$$k^\alpha \hat{f}(k) \leqslant k^\alpha \frac{C_n}{(1+|k|)^N} \leqslant (1+|k|)^{|\alpha|} \frac{C_n}{(1+|k|)^N} = \frac{C_n}{(1+|k|)^{N-|\alpha|}}$$

If we set N such that $N - |\alpha| > d$, for example $N = d + |\alpha| + 1$, then we can conclude that $k^\alpha \hat{f} \in L^1(\mathbb{R}^d)$. This implies $\widehat{k^\alpha \hat{f}} \in L^\infty(\mathbb{R}^d)$, so

$$\widehat{k^\alpha \hat{f}}(k) = \partial^\alpha \hat{\hat{f}}(k) = \partial^\alpha (\hat{f})^\vee(-k) = \partial^\alpha f(-k) \in L^\infty(\mathbb{R}^d).$$

This implies $f \in C^\infty(\mathbb{R}^d)$. ■

Theorem 3.47 Take $T \in D'(\mathbb{R}), f \in C_c^\infty(\mathbb{R}^d) = D(\mathbb{R}^d)$, $f_y(x) = f(x-y)$

a) $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D_x^\alpha f_y)$

b) $\forall g \in L^1(\mathbb{R}^d)$ and compactly supported

$$\int_{\mathbb{R}^d} g(y) T(f_y)\, dy = T(\underbrace{f \star g}_{\in C_c^\infty})$$

Proof. a) $y \mapsto T(f_y)$ is continuous since $y_n \to y$ in $\mathbb{R}^d$, then $f_{y_n} \to f_y$ implies $T(f_{y_n}) \to T(f_y)$. Let's check that $y \mapsto T(f_y) \in C^1$:

$$\lim_{h\to 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} = \lim_{h\to 0} T\left(\frac{f_{y-he_i} - f_y}{h}\right)$$

We have $\frac{f_{y-he_i}-f_y}{h} \xrightarrow{h\to 0} (\partial_i f)_y$ in $D(\mathbb{R}^d)$

- $\exists K$ compact set such that $\operatorname{supp}(f_{y-e_i} - f_y)$, $\operatorname{supp} \partial_i f \subseteq K$ as $|h|$ small.
- $\dfrac{f_{y-he_i}(x) - f_y(x)}{h} - (\partial_i f)_y(x)$
$$= \frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y)$$
$$\left|\int_0^1 \partial_i f(x-y+the_i)\,dt - \partial_i f(x-y)\right| \xrightarrow{h\to 0} 0 \text{ uniformly in } x$$
Similary:
$$\left|D_x^\alpha\left(\frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y)\right)\right|$$
$$= \left|\frac{D^\alpha f(x-y+he_i) - D^\alpha f(x-y)}{h} - \partial_i(D^\alpha f)(x-y)\right| \xrightarrow{h\to 0} 0$$
uniformly in x. Conclude:
$$\lim_{h\to 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} \xrightarrow{h\to 0} T((\partial_i f)_y) \in C(\mathbb{R}^d)$$
So we geht that $y \mapsto T(f_y) \in C^1$ and $-\partial_{y_i} T(f_y) = T((\partial_i f)_y)$

By induction:
$$D_y^\alpha T(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y) = (D^\alpha T)(f_y) \quad \forall \alpha \in \mathbb{N}^d$$

b) Heuristic: $T = T(x)$

$$\begin{aligned}\int_{\mathbb{R}^d} g(y)T(f_y)\,dy &= \int_{\mathbb{R}^d} g(y)\left(\int_{\mathbb{R}^d} T(x)f(x-y)\,dx\right) dy\\ &= \int_{\mathbb{R}^d} T(x)\left(\int_{\mathbb{R}^d} g(y)f(x-y)\,dy\right) dx\\ &= \int_{\mathbb{R}^d} T(x)(f\star g)(x)\,dx = T(f\star g)\end{aligned}$$

Step 1: $g \in C_c^\infty(\mathbb{R}^d)$

$$\text{(Rieman Sum)} \quad \begin{aligned}\int_{\mathbb{R}^d} g(y)T(f_y)\,dy &= \lim_{\Delta_N\to 0} \Delta_N \sum_{j=1}^N g(y_j)T(f_{y_j})\\ &= \lim_{\Delta_N\to 0} T\left(\Delta_N \sum_{j=1}^N g(y_j) f_{y_j}\right)\\ &= T(f\star g)\end{aligned}$$

because

$$\lim_{\Delta_N\to 0} \Delta_N \sum_{j=1}^N g(y_j) f_{y_j}(x) \to (f\star g)(x) \text{ in } D(\mathbb{R}^d)$$

$$\lim_{\Delta_N\to 0} \Delta_N \sum_{j=1}^N g(y_j) f(x-y_j) \xrightarrow{\text{Riemann}} \int_{\mathbb{R}^d} g(y)f(x-y)\ dy = (f\star g)(x)$$

Proof of:

$$\lim_{\Delta_N \to 0} \Delta_N \sum_{j=1}^{N} g(y_j) f(x - y_j) \to (f \star g)(x) \text{ in } D(\mathbb{R}^d)$$

1) Since $f, g \in C_c^\infty$ we have $f \star g \in C_c^\infty$. And we have

$$x \mapsto \Delta_N \sum_{j=1}^{N} g(y_j) f(x - y_j) \in C^\infty$$

since $f \in C^\infty$ supported in $(\operatorname{supp} g + \operatorname{supp} f)$. So all functions are C_c^∞ and supported in $(\operatorname{supp} g + \operatorname{supp} f)$.

2)

$$\left| \lim_{\Delta_N \to 0} \Delta_N \sum_{j=1}^{N} g(y_j) f(x - y_j) - \int_{\mathbb{R}^d} g(y) f(x - y)\, dy \right| \xrightarrow{\Delta_N \to 0} 0$$

uniformly in x. (Result from the Riemann-Sum)

3)

$$\begin{aligned} &\left| D_x^\alpha (\Delta_N \sum_{j=1}^{N} g(y_j) f(x - y) - (f \star g)(x)) \right| \\ &= \left| \Delta_N \sum_{j=1}^{N} g(y_j) D^\alpha f(x - y) - (D^\alpha f) \star g(x) \right| \xrightarrow{\Delta_N \to 0} 0 \end{aligned}$$

uniformly in x for all α.

Step 2: Take $g \in L^1(\mathbb{R}^d)$ and compactly supported. Then $\exists \{g_n\} \subseteq C_c^\infty(\mathbb{R}^d)$, $\operatorname{supp} g_n \subseteq \operatorname{supp} g + B(0,1)$ such that $g_n \to g$ in $L^1(\mathbb{R}^d)$. By Step 1:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y)\, dy = T(g_n \star f)$$

Take $n \to \infty$:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y)\, dy \to \int_{\mathbb{R}^d} g(y) T(f_y)\, dy$$

since $g_n \to g$ in L^1 compactly supported and $y \mapsto T(f_y) \in C^\infty \subseteq L^\infty(K)$. Moreover (exercise):

$$\underbrace{g_n \star f}_{\in C_c^\infty} \to g \star f \quad \text{in } D(\mathbb{R}^d)$$

So $T(g_n \star f) \xrightarrow{n \to \infty} T(g \star f)$. Finally we optain:

$$\int g(y) T(f_n)\, dy = T(g \star f) \qquad \blacksquare$$

Theorem 3.48 Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $T \in D'(\Omega)$ and $f \in C_c^\infty(\Omega)$. Denote

$$\Omega_f = \{y \in \mathbb{R}^d \mid \operatorname{supp} f_y = y + \operatorname{supp} f \subseteq \Omega\}$$

a) $y \mapsto T(f_y) \in C^\infty(\Omega_f)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y)$

b) For all $g \in L^1(\Omega_g)$ compactly supported in Ω_f and it holds:

$$\int_\Omega g(y)T(f_y)\,dy = T(f \star g).$$

Theorem 3.49 Let $T \in D'(\Omega)$ s.t. $\nabla T = 0$ in $D'(\Omega)$. Then: $T = const.$ in Ω.

Proof. ($\Omega = \mathbb{R}^d$) for all $f \in C_c^\infty$, $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $\partial_{y_i} T(f_y) = (\partial_j T)(f_y) = 0$ for all $i = 1, \dots, d$. Then by the result of the theorem for C^∞ functions, $y \mapsto T(f_y) = const$ independent of y. Consequently:

$$T(f_y) = T(f_0) = T(f) \quad \forall y \in \mathbb{R}^d\ \forall f \in C_c^\infty(\mathbb{R}^d)$$

For any $g \in C^\infty(\mathbb{R}^d)$:

$$\left(\int_{\mathbb{R}^d} g\,dy\right) T(f) = \int_{\mathbb{R}^d} g(y)T(f_y)\,dy = T(f \star g) = T(g \star f) = \left(\int_{\mathbb{R}^d} f\,dy\right) T(g)$$

So $\frac{T(f)}{\int_{\mathbb{R}^d} f}$ is independent of f (as soon as $\int f \neq 0$). So we get that $T(f) = const \int_{\mathbb{R}^d} f$, where const is independent of f. ∎

Remark 3.50 If $u \in C^1(\mathbb{R}^d)$, then:

$$u(x+y) - u(x) = \int_0^1 \sum_{j=1}^d y_j (\partial_j u)(x + ty_j)\,dt = \int_0^1 y \nabla u(x+ty)\,dt$$

So we get that if $\nabla u = 0$, then $u(x+y) - u(x) = 0$ for all x, y, so $u = const$.

Theorem 3.51 (Taylor expansion for distributions) Let $T \in D'(\mathbb{R}^d)$ and $f \in C_c^\infty(\mathbb{R}^d)$. Then $y \mapsto T(f_y) \in C^\infty$ and

$$T(f_y) - T(f) = \int_0^1 \sum_{j=1}^d y_j (\partial_j T)(f_{ty})\,dt.$$

In particular, if $g \in L^1_{loc}$ and $\nabla g \in L^1_{loc}$, then $\forall y \in \mathbb{R}^d$:

$$g(x+y) - g(x) = \int_0^1 g(x+ty)y\,dt$$

for a.e. $x \in \mathbb{R}^d$.

Proof. $y \mapsto T(f_y)$ is C^∞ and $\frac{d}{dt}[T(f_{ty})] = (\nabla T)(f_{ty})y$ So we get

$$\begin{aligned} T(f_y) - T(f) &= \int_0^1 \frac{d}{dt}(T(f_{ty}))\,dt \\ &= \int_0^1 (\nabla T)(f_{ty})y\,dt \\ &= \int_0^1 \sum_{j=1}^d (\partial_j T)(f_{ty}) y_j\,dt \end{aligned}$$ ∎

Corrolary 3.52 Let $g \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\partial_j g \in L^1_{loc}(\mathbb{R}^d)$ for all $j = 1, 2, \dots, d$ (i.e. $g \in W^{1,1}_{loc}(\mathbb{R}^d)$). Then for all $y \in \mathbb{R}^d$:

$$\begin{aligned} g(x+y) - g(x) &= \int_0^1 y \cdot \nabla g(x+ty)\,dt \\ &= \int_0^1 \sum_{j=1}^d y_j \partial g(x+ty)\,dt \end{aligned}$$

for a.e. x.

Proof. For all $f \in C_c^\infty$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)[g(x+y) - g(x)]\,dx &= \int_{\mathbb{R}^d} g(x)[f(x-y) - f(x)]\,dx \\ &= g(f_y) - g(f) \\ &= \int_0^1 \sum_{j=1}^d y_j (\partial_j g)(f_{ty})\,dt \\ &= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \sum_{j=1}^d y_j \left[\int_{\mathbb{R}^d} (\partial_j g)(x) f_{ty}(x)\,dx \right] \\ &= \int_0^1 \sum_{j=1}^d y_i \left[\int_{\mathbb{R}^d} (\partial_j g)(x+ty) f(x)\,dx \right] dt \\ &= \int_{\mathbb{R}^d} f(x) \left[\int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty)\,dt \right] dx \end{aligned}$$

For all $\phi \in C_c^\infty$: $= g(x+y) - g(x)$ a.e. $x \in \mathbb{R}^d$. ■

Remark 3.53 If $T \in D'(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open, if $y\nabla T = 0$, then $T = \mathit{const}$.

Theorem 3.54 (Equivalence of the classical and distributional derivatives) Let $\Omega \subseteq \mathbb{R}^d$. Then the following are equivalent:

1. $T \in D'(\Omega)$ s.t. $\partial_{x_i} T = g_i \in C(\Omega)$ for all $i = 1, \dots, d$.
2. $T = f \in C^1(\Omega)$ and $g_i = \partial_{x_i} f$

Proof.

(2) ⇒ (1): If $T = f \in C^1(\Omega)$, then: $\partial_{x_i} f \in C(\Omega)$.

$$\partial_{x_i} T(\phi) = -T(\partial_{x_i}\phi) = -\int_\Omega f(\partial_{x_i}\phi) \int_\Omega (\partial_{x_i} f)\phi$$

for all $\phi \in D(\Omega)$, so $\partial_{x_i} T = \partial_{x_i} f$.

(1) ⇒ (2): Why is $T = f$ with f continous? As $\partial_{x_i} f = g_i$:

$$f(x+y) - f(x) = \int_0^1 \nabla f(x+ty) y\,dt = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i\,dt$$

So we get

$$f(y) = f(0) + \int_0^1 \sum_{i=1}^d g_i(ty) g_i \, dt.$$

We expect that $f \in C^1$ and $\partial_{x_i} f = g_i$. But this is not trivial to prove.

$$\begin{aligned}
\frac{f(y+he_i) - f(y)}{h} &= \int_0^1 \sum_{i=1}^d [g_i(ty + the_i)(y_i + h\delta_{ij})] \, dt \\
&= \int_0^1 g_i(ty + the_i) \, dt + \int_0^1 \sum_{j \neq i} \frac{[g_i(ty + the_i) - g_i(ty)]}{h} y_i \, dt \\
&\xrightarrow{h \to 0} \int_0^1 g_i(ty) \, dt + \text{ is difficult ...}
\end{aligned}$$

Lets take $\phi \in C_c^\infty$, then:

$$\begin{aligned}
T(\phi_y) - T(\phi) &= \int_0^1 \underbrace{\nabla T}_{(g_i)_{i=1}^d} (\phi_{ty}) y \, dt \\
&= \int_0^1 \sum_{i=1}^d \left(\int_\Omega g_i(x) \underbrace{\phi_{ty}}_{=\phi(x-ty)} dx \right) dt \\
&= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_0^1 g_i(x) \phi(x - ty) y_i \; dt \right) dx \\
&= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_0^1 g_i(x + ty) \phi(x) y_i \, dt \right) dx \\
&= \int_{\mathbb{R}^d} \left(\sum_i \int_0^1 g_i(x + ty) y_i \, dt \right) \phi(x) \, dx
\end{aligned}$$

Integrating against $\psi(y)$ with $\psi \in C_c^\infty$:

$$\begin{aligned}
&\int_{\mathbb{R}^d} T(\phi_y)\psi(y) \, dy - T(\phi) \int_{\mathbb{R}^d} \psi(y) \, dy \\
&\quad = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \sum_i \int_0^1 g_i(x + ty) y_i \psi(y) \, dt \, dy \right) \psi(x) \, dx
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow T(\phi \star \psi) - T(\phi) \int \psi = \dots \\
&\Rightarrow \int_{\mathbb{R}^d} T(\psi_y) \phi(y) \, dy - T(\phi) \int \; \psi = \dots
\end{aligned}$$

Take $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\int \psi = 1$. Then:

$$T(\phi) = \int_{\mathbb{R}^d} \underbrace{T(\psi_x) - \left(\int_{\mathbb{R}^d} \sum_{i=1}^d \int_0^1 g_i(x + ty) y_i \psi(y) \, dt \, dy \right)}_{f(x)} \phi(x) \, dx$$

for all $\phi \in C_c^\infty$, so $T = f \in C(\Omega)$. Thus $T = f \in C(\Omega)$ and $\partial_{x_i} T = g_i \in C(\Omega)$. Then we need to prove that $f \in C^1(\Omega)$ and $\partial_{x_i} f = g_i$ (classical derivative). Since

$f \in W^{1,1}_{loc}$:

$$f(x+y) - f(x) = \int_0^1 \sum_{i=1}^d g_i(x+ty)y_i \, dt \quad \forall x, y$$

In particular:

$$\begin{aligned}\frac{f(x+he_i) - f(x)}{h} &= \int_0^1 \frac{1}{h} \sum_{i=1}^d g_i(x+the_i)h\delta_{ij} \, dt \\ &= \int_0^1 g_i(x+the_i) \, dt \xrightarrow{h \to 0} g_i(x)\end{aligned}$$

So we get $\partial_{x_i} f(x) = g_i(x) \in C(\Omega)$ in the classical sense. So $f \in C^1(\Omega)$. ■

Definition 3.55 (Sobolev Spaces) Let $\Omega \subseteq \mathbb{R}^d$ be open. We define for $1 \leqslant p \leqslant \infty$:

$$\begin{aligned}W^{1,p}(\Omega) &= \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega) \ \forall i = 1, \ldots, d\} \\ W^{k,p}(\Omega) &= \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leqslant k\} \\ W^{k,p}_{loc}(\Omega) &= \{f \in L^p_{loc}(\Omega) \mid D^\alpha f \in L^p_{loc}(\Omega) \ \forall |\alpha| \leqslant k\}\end{aligned}$$

Theorem 3.56 (Approximation of $W^{1,p}_{loc}(\Omega)$ by $C^\infty(\Omega)$) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $f \in W^{1,p}_{loc}(\Omega)$. Then there exists $\{f_n\} \subseteq C^\infty(\Omega)$ such that $f_n \to f$ in $W^{1,p}_{loc}(\Omega)$, i.e. for all $K \subseteq \Omega$ compact: $\|f_n - f\|_{L^p(K)} + \sum_{i=1}^d \|\partial_{x_i}(f_n - f)\|_{L^p(K)} \to 0$.

Proof. Case $\Omega = \mathbb{R}^d$: Take $g \in C_c^\infty$, $\int g = 1$, $g_\epsilon(x) = \epsilon^{-d} g(\epsilon^{-1}x)$. Then $g_\epsilon \star f \in C_c^\infty$. Since $f \in L^p_{loc}(\Omega)$ we have $g_\epsilon \star f \to f$ in L^p_{loc} as $\epsilon \to 0$. Moreover $\partial_{x_i}(g_\epsilon \star f) = (g_\epsilon \star \partial_{x_i} f) \xrightarrow{\epsilon \to 0} \partial_{x_i} f$ in L^p_{loc}. Then we can take $f_n = g_{\frac{1}{n}\star f}$. ■

Remark 3.57 In general, if we want to compute the distributional derivative $D^\alpha f$, then we can find $f_n \to f$ in $D'(\Omega)$ and compute $D^\alpha f$. Then $D^\alpha f_n \to D^\alpha f$ in $D^\alpha(\Omega)$. As an example we can compute $\nabla |f|$ with $f \in W^{1,p}_{loc}(\Omega)$.

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Theorem 3.58 (Chain Rule) Let $G \in C^1(\mathbb{R}^d)$ with $|\nabla G|$ is bounded. Let $f = (f_i)_{i=1}^d \subseteq W^{1,p}_{loc}(\Omega)$. Then $x \mapsto G(f(x)) \in W^{1,p}_{loc}(\Omega)$ and

$$\partial_{x_i} G(f) = \sum_{k=1}^d (\partial_k G)(f) \cdot \partial_{x_i} f_k \quad \text{in } D'(\Omega)$$

Moreover, if $G(0) \in L^p(\Omega)$ (i.e. either $|\Omega| < \infty$ or $G(0) = 0$), then if $f = (f_i)_{i=1}^d \subseteq W^{1,p}(\Omega)$, then $G(f) \in W^{1,p}(\Omega)$.

Proof. Since $G \in C^1$ we have that G is bounded in any compact set. Moreover $\|\nabla G\|_{L^\infty} < \infty$ implies:

$$|G(f) - G(0)| \leqslant \|\nabla G\|_{L^\infty} |f| \in L^p_{loc}$$

So $G(f) \in L^p_{loc}$. Let us compute $\partial_{x_i} G(f)$. Let $\{f^{(n)}\}_{n=1}^\infty \subseteq C^\infty$ such that $f^{(n)} \to f$ in $W^{1,p}_{loc}$, then:

$$|G(f(^n)) - G(f)| \leqslant \|\nabla G\|_{L^\infty} |f^{(n)} - f| \to 0 \text{ in } L^p_{loc}$$

So $G(f^{(n)}) \to G(f)$ in L^p_{loc}, thus $\partial_{x_i} G(f^{(n)}) \to \partial_{x_i} G(f)$ in $D'(\Omega)$. On the other hand, by the standard Chain-Rule for C^1-functions:

$$\begin{aligned} \partial_{x_i} G(f^{(k)}) &= \sum_{k=1}^d \underbrace{\partial_k G(f^{(k)})}_{\text{b.d.}\to \partial_k G(f)} \underbrace{\partial_i f_k^{(n)}}_{\to \partial_i f_k \text{ in } L^p(\Omega)} \\ &\to \sum_{k=1}^d \partial_k G(f) \partial_i f_k \text{ in } L^p_{loc}(\Omega) \end{aligned}$$

Thus

$$\partial_{x_i} G(f) = \sum_{k=1}^d \underbrace{\partial_k G(f)}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p_{loc}} \in L^p_{loc} \partial_{x_i} G(f) = \sum_{k=1}^d \underbrace{\partial_k G(f)}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p_{loc}} \in L^p_{loc} \text{ in } D'(\Omega)$$

So $G(f) \in W^{1,p}_{loc}(\Omega)$. Aussume that $G(0) \in L^p(\Omega)$ (i.e. $|\Omega| < \infty$ or $G(0) = 0$). If $f \in W^{1,p}(\Omega)$, then $G(f) \in W^{1,p}(\Omega)$ since

$$|G(f) - G(0)| \leqslant \|\nabla G\|_{L^\infty} |f| \in L^p \Rightarrow G(f) \in L^p$$

and

$$\partial_{x_i} G(f) = \sum_k \underbrace{\partial_k G}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p} \in L^p \Rightarrow G(f) \in W^{1,p}(\Omega)$$

■

Theorem 3.59 (Derivative of absolute value) Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $f \in W^{1,p}(\Omega)$. Then $|f| \in W^{1,p}(\Omega)$ and if f is real-valued:

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Proof. Exercise. Hint: Use the Chain-Rule for $G_\epsilon(x) = \sqrt{\epsilon^2 + x^2} - \epsilon \to |x|$ as $\epsilon \to 0$ ■

3.4 Distribution vs. measures

Let μ be a Borel measure in $\mathbb{R}^d$ s.t. $\mu(K) < \infty$ for all compact $K \subseteq \mathbb{R}^d$. Then define

$$\begin{aligned} T: \ D(\mathbb{R}^d) &\longrightarrow \mathbb{C} \\ \phi &\longmapsto \int_{\mathbb{R}^d} \phi(x)\, d\mu(x) \quad \forall \phi \in C_c^\infty \end{aligned}$$

$\rightsquigarrow$ T is a distribution since if $\phi_n \to \phi$ in $D(\Omega)$, then

$$|T(\phi_n) - T(\phi)| \leqslant \int_{\mathbb{R}^d} |\phi_n - \phi|\, d\mu(x) \leqslant \|\phi_n - \phi\|_{L^\infty} \left(\int_K d\mu \right) \xrightarrow{n\to\infty} 0$$

Example 3.60 ∂_0 in $D'(\mathbb{R}^d)$ is a Borel probability measure.

Theorem 3.61 (Positive distributions are measures) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $T \in D'(\Omega)$. Assume $T \geqslant 0$, i.e. $T(\phi) \geqslant 0$ for all $\phi \in D(\Omega)$ satisfying $\phi(x) \geqslant 0$ for all x. Then there is a Borel positive measure μ on Ω such that $\mu(K) < \infty$ for all $K \subseteq \Omega$ compact and:

$$T(\phi) = \int_\Omega \phi(x)\, d\mu(x) \quad \forall \phi \in D^(\Omega)$$

Proof. See Lieb-Loss Analysis. Sketch: If $O \subseteq \mathbb{R}^d$ is open, then

$$\mu(O) = \sup\{T(\phi) \mid \phi \in D(\Omega), 0 \leqslant \phi \leqslant 1, \operatorname{supp} \phi \subseteq O\}$$

For all $A \subseteq \Omega$ (not necessarily open),

$$\mu(A) = \inf\{\mu(O) \mid O \text{ open}, A \subseteq O\}$$

The mapping $\mu : 2^\Omega \to [0, \infty]$ is an outer measure, i.e.

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leqslant \mu(B)$ if $A \subseteq B$
3. $\mu\left(\bigcup_{i=1}^\infty A_i\right) \leqslant \sum_{i=1}^\infty \mu(A_i)$

From the outer measure we can find a σ-algebra Σ and μ is a measure on Ω s.t. E is measurable iff

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^\complement)$$

. So all open sets are measurable, thus outer regularity (by def $\mu(A) = \inf\{\mu(O) \mid O \text{ open } \supseteq A)$, so inner regularity $\mu(A) = \sup\{\mu(K) \mid K \text{ compact } \subseteq A\}$. ■

Exercise 3.62 (E 4.1) Prove that if $T_n \to T$ in $D'(\mathbb{R}^d)$, then $D^\alpha T_n \to D^\alpha T$ in $D^\alpha(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$.

My Solution. See Goodnotes. ■

Exercise 3.63 (E 4.2)

My Solution. See Goodnotes. ■

Exercise 3.64 (E 4.3) $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$. Then $f_\epsilon \to \delta_0$ in $D'(\mathbb{R}^d)$.

My Solution. See Goodnotes. ■

Exercise 3.65 (E 4.4) Let $\{f_n\} \subseteq L^1$, $\operatorname{supp} f \subseteq B(0,1)$, $f_n \to f$ in L^1. Prove for all $g \in C_c^\infty$ that $f_n \star g \to f \star g$ in $D(\mathbb{R}^d)$.

Solution. Since $f_n \in L^1$, $\operatorname{supp} f \subseteq B(0,1)$ and $g \in C_c^\infty$ we have $f_n \star g \in C_c^\infty$ and

$$\operatorname{supp}(f_n \star g) \subseteq (\operatorname{supp} g) + \overline{B(0,1)} = K.$$

Since $f_n \to f$ in L^1 there is a subsequence $f_{n_k} \to f$ almost everywhere, so f supp in $\overline{B(0,1)}$, so $f \star g \in C_c^\infty$, $\operatorname{supp}(f \star g) \subseteq K$. We have:

$$\begin{aligned}
|f_n \star g(x) - f \star g(x)| &= \left| \int_{\mathbb{R}^d} (f_n(y) - f(y)) g(x-y) \ dy \right| \\
&\leqslant \int_{\mathbb{R}^d} |f_n(y) - f(y)||g(x-y)| \, dy \\
&\leqslant \|g\|_{L^\infty} \|f_n - f\|_{L^1} \xrightarrow{n \to \infty} 0
\end{aligned}$$

thus $\|f_n \star g - f \star g\|_{L^\infty} \to 0$. Similary:

$$\|D^\alpha(f_n \star g) - D^\alpha(f \star g)\|_{L^\infty} = \|f_n \star \underbrace{(D^\alpha g)}_{\in C_c^\infty} - f \star (D^\alpha g)\|_{L^\infty} \xrightarrow{n \to \infty} 0$$

for all $\alpha \in \mathbb{N}^d$, so $f_n \star g \to f \star g$ in $D(\mathbb{R}^d)$. ■

Exercise 3.66 (E 4.5) Compute distributional derivatives f', f'' of $f(x) = x|x-1|$.

Solution. We prove $f'(x) = g(x) := \begin{cases} 2x - 1 & x > 1 \\ 1 - 2x & x < 1 \end{cases}$. Take $\phi \in C_c^\infty(\mathbb{R}^d)$.

$$\begin{aligned}
-f'(\phi) &= -\int_{\mathbb{R}^d} f \phi' \, dy \\
&= -\int_{-\infty}^1 f \phi' \, dy - \int_1^\infty f \phi' \, dy \\
&= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 f' \phi \, dy + [f\phi]_1^\infty - \int_1^\infty f' \phi \, dy \\
&= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 g \phi \, dy + [f\phi]_1^\infty - \int_1^\infty g \phi \, dy \\
&= f(1-)\phi(1) - f(1+)\phi(1) - \int_{\mathbb{R}^d} g\phi \, dy \\
&= 0 - \int_{\mathbb{R}^d} g \phi \, dy
\end{aligned}$$

Now we compute $f'' = g'$. Take $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$
\begin{aligned}
-(g')(\phi) &= \int_{\mathbb{R}^d} g\phi' \, dy \\
&= \int_{-\infty}^{1} g\phi' \, dy + \int_{1}^{\infty} g\phi' \, dy \\
&= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^{1} g'\phi \, dy - \int_{1}^{\infty} g'\phi \, dy \\
&= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^{1} (-2)\phi \, dy - \int_{1}^{\infty} 2\phi \, dy \\
&= -2\phi(1) + \int_{-\infty}^{\infty} [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x) \, dx \\
&= -2\delta_1(\phi) + \int_{-\infty}^{\infty} [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x) \, dx \\
\Rightarrow g' &= \underbrace{2\delta_1}_{\notin L^1_{loc}} - \underbrace{2\mathbb{1}_{(-\infty,1)} + 2\mathbb{1}_{(1,\infty)}}_{\int L^1_{loc}}
\end{aligned}
$$

■

Chapter 4

Weak Solutions and Regularity

Definition 4.1 Consider the linear PDE:

$$\sum_\alpha c_\alpha D^\alpha u(x) = F(x), \quad c_\alpha \text{ constant}, F \text{ given}$$

A function u is called a weak solution (a distributional solution) if

$$\sum_\alpha c_\alpha D^\alpha u = F \quad \text{in } D'(\Omega).$$

Namely,

$$\sum_\alpha (-1)^{|\alpha|} c_\alpha \int_\Omega u(D^\alpha \phi) = \int_\Omega F\phi, \quad \forall \phi \in D(\Omega)$$

Regularity: Given some condition on the data F, what can we say about the smoothness of u? Can we say that the equation holds in the classical sense? We derived G (the solution of the Laplace Equation) before in two ways:

1. $\Delta G(x) = 0$ for all $x \neq 0$, assuming $G(x) = G(|x|)$ and $d \geqslant 2$
2. $\hat{G}(k) = \frac{1}{|2\pi k}|^2$, $d \geqslant 3$

Theorem 4.2 For all $d \geqslant 1$ we have $G \in L^1_{loc}(\mathbb{R}^d)$ and $-\Delta G = \delta_0$ in $D'(\mathbb{R}^d)$.

Proof. Take $\phi \in D(\mathbb{R}^d)$. Then:

$$\begin{aligned}(-\Delta G_y)(\phi) = G_y(-\Delta\phi) &= \int_{\mathbb{R}^d} G_y(x)(-\Delta\phi)(x)\,dx \\ &= \int_{\mathbb{R}^d} G(y-x)(-\Delta\phi)(x)\,dx \\ &= [G \star (-\Delta\phi)](y) = (-\Delta)(G \star \phi)(y)\end{aligned}$$

Recall for all $f \in C^2$, $-\Delta(G \star f) = f$ pointwise. So we can conclude $-\Delta G_y = \delta_y$ in $D'(\mathbb{R}^d)$. ∎

Remark 4.3 In $d = 1$, $G(x) = -\frac{1}{2}|x|$, so $-G'(x) = \mathrm{sgn}(x)/2$, so $-G''(x) = \delta_0$.

Remark 4.4 Formally: $-\Delta(G_y \star \phi) = (-\Delta G_y)\star\phi(x) = (\delta_0 \star \phi)(x) = \int \delta_0(y)\phi(xy)\,dy = \delta_0(\phi(x-\bullet))$

Theorem 4.5 (Poisson's equation with L^1_{loc} data) Let $f \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\omega_d f \in L^1(\mathbb{R}^d)$ where

$$\omega_d(x) = \begin{cases} 1+|x| & d=1 \\ \log(1+|x|) & d=2 \\ \frac{1}{1+|x|^{d-2}} & d \geqslant 3, \end{cases}$$

then $u(x) = (G \star f)(x) \in L^1_{loc}(\mathbb{R}^d)$. Moreover $-\Delta u = f$ in $D'(\mathbb{R}^d)$. In fact, $u \in W^{1,1}_{loc}(\mathbb{R}^d)$ and:

$$\partial_{x_i} u(x) = (\partial_{x_i} G) \star f(x) = \int_{\mathbb{R}^d} (\partial_{x_i} G)(x-y) f(y)\,dy$$

Remark 4.6 We can also replace $\mathbb{R}^d$ by Ω and get $-\Delta u = f$ in $D'(\Omega)$.

Proof of Theorem 4.5. First we check that $u \in L^1_{loc}$. Take any Ball $B(0,R) \subseteq \mathbb{R}^d$, prove $\int_B |u|\,dy < \infty$. We have

$$\begin{aligned} \int_B |u|\,dy &= \int_B \left| \int_{\mathbb{R}^d} G(x-y)f(y)\,dy \right| dx \\ &\leqslant \int_B \int_{\mathbb{R}^d} |G(x-y)||f(y)|\,dy\,dx \\ &= \int_{\mathbb{R}^d} \left(\int_B |G(x-y)\,dx \right) |f(y)|\,dy \end{aligned}$$

If $y \notin B = B(0,R)$, then by Newtons's theorem (Mean-value theorem):

$$\int_{B(0,R)} |G(x-y)|\,dx = |B(0,r)||G(y)| \leqslant C|B|\omega_d(y)$$

If $y \in B$, then $|y| \leqslant R$, so $|x-y| \leqslant 2R$ if $x \in B$.

$$\int_{B(0,R)} |G(x-y)|\,dx \leqslant \int_{|x-y|\ \leqslant 2R} |G(x-y)|\,dx = \int_{|z| \leqslant 2R} |G(z)|\,dz \leqslant c_R \text{ as } G \in L^1_{loc}$$

Thus

$$\int_B |u|\,dy \leqslant c_B \int_{|y|\ \geqslant R} \omega_d(y)|f(y)|\,dy + c_B \int_{|y|\ \leqslant R} |f(y)|\,dy < \infty$$

Let us prove $-\Delta = f$ in $D'(\mathbb{R}^d)$. Take $\phi \in D(\mathbb{R}^d)$. Then:

$$\begin{aligned}
(-\Delta u)(\phi) &= u(-\Delta\phi) \\
&= \int_{\mathbb{R}^d} u(x)(-\Delta\phi)(x)\,dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(y)(-\Delta\phi)(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(y-x)f(y)(-\Delta\phi)(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d} [G\star(-\Delta\phi)](y)f(y)\,dy \\
&= \int_{\mathbb{R}^d} -\Delta(G\star\phi)(y)f(y)\,dy \\
&= \int_{\mathbb{R}^d} \phi(y)f(y)\,dy
\end{aligned}$$

So $-\Delta u = f$ in $D'(\mathbb{R}^d)$. We check that $\partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$. Note that

$$|\partial_i G(x)| \leqslant c\frac{1}{|x|^{d-1}} \in L^1_{loc}(\mathbb{R}^d)$$

and

$$\int_{B(0,R)} |\partial_i G(x-y)|\ dx \leqslant \begin{cases} C_r\omega_d(y) & |y|\ \geqslant R \\ C_r & |y| \leqslant R \end{cases}$$

So $\int_{B(0,R)} |(\partial_i G\star f)|(y) < \infty$ for all $R > 0$. For all $\phi \in D(\mathbb{R}^d)$:

$$\begin{aligned}
-(\partial_i u)(\phi) = u(\partial_i\phi) &= \int_{\mathbb{R}^d} u(x)\partial_i\phi(x)\,dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(y)\partial_i\phi(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(y-x)f(y)\partial_i\phi(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d} (G\star\partial_i^y\phi)(y)f(y)\ \ dy \\
&= \int_{\mathbb{R}^d} (\partial_i^y G\star\phi)(y)f(y)\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_i^y G(y-x)f(y)\phi(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} -(\partial_i G)(x-y)f(y)\phi(x)\,dx\,dy \\
&= -\int_{\mathbb{R}^d} (\partial_i G\star f)(x)\phi(x)\,dx
\end{aligned}$$

So $\partial_i u = \partial_i G\star f \in L^1_{loc}(\mathbb{R}^d)$. Thus $u \in L^1_{loc}$, $\partial_i u \in L^1_{loc}$ for all i. So $u \in W^{1,1}_{loc}(\mathbb{R}^d)$. ■

Regularity: We consider the Laplace Equation $\Delta u = 0$ in $\mathbb{R}^d$.

Lemma 4.7 (Weyl) If $\Omega \subseteq \mathbb{R}^d$ open and $T \in D'(\Omega)$ s.t. $\Delta T = 0$ in $D'(\Omega)$, then: $T = f \in C^\infty(\Omega)$ and f is a harmonic function.

Proof. ($\Omega = \mathbb{R}^d$). Take $\phi \in C_c^\infty$, then $y \mapsto T(\phi_y) = T(\phi(-y))$ is C^∞ and $\Delta_y T(\phi_y) = T((\Delta\phi)_y) = (\Delta T)(\phi_y) = 0$. Take $g \in C_c^\infty$, g is radial. Then:

$$\int_{\mathbb{R}^d} T(\phi_y)g(y)\,dy \overset{\text{(exercise)}}{=} \int_{\mathbb{R}^d} T(\phi)g(y)\,dy = T(\phi)\left(\int_{\mathbb{R}^d} g\,dy\right)$$

Exercise 4.8 Let $f \in C^\infty(\mathbb{R}^d)$ be a harmonic function and $g \in C_c^\infty$, g is radial. Then:

$$\int_{\mathbb{R}^d} f(x)g(x)\,dx = f(0)\left(\int_{\mathbb{R}^d} g(x)\,dx\right)$$

On the other hand:

$$\int_{\mathbb{R}^d} T(\phi_y)g(y)\,dy = T(\phi \star g) = T(g \star \phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y)\,dy$$

Take $\int_{\mathbb{R}^d} g\,dy = 1$, then:

$$T(\phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y)\ \ dy$$

For all $\phi \in C_c^\infty$. Then $T = T(g_y) \in C^\infty$ ■

Now lets regard the Poisson Equation $-\Delta u = f$ in $D'(\mathbb{R}^d)$.

Remark 4.9 Any solution has the form $u = G \star g + h$ where $\Delta h = 0$ in $D'(\mathbb{R}^d)$. By Weyls Lemma (4.7), $h \in C^\infty$, then we only need to consider the regularity of $G \star f$.

Remark 4.10 The regularity is a *local question*, namely if we write

$$f = f_1 + f_2 = f\phi + f(1-\phi),$$

where $\phi = 1$ in a ball B and $\phi \in C_c^\infty$.

Then $G \star f = G \star f_1 + G \star f_2$. Here $f_2 = f(1-\phi) = 0$ in B. With Weyls Lemma (4.7), $G \star f_2 \in C^\infty$.

Theorem 4.11 (Low Regularity of Poisson Equation) Lef $f \in L^p(\mathbb{R}^d)$ and compactly supported. Then

a) If $p \geqslant 1$, then
   - $G \star f \in C^1(\mathbb{R}^d)$ if $d = 1$.
   - $G \star f \in L^q_{loc}(\mathbb{R}^d)$ for any $q < \infty$ if $d = 2$.
   - $G \star f \in L^q_{loc}(\mathbb{R}^d)$ for $q < \frac{d}{d-2}$ if $d \geqslant 3$.

b) If $\frac{d}{2} < p \leqslant d$, then $G \star f \in C^{0,\alpha}_{loc}(\mathbb{R}^d)$ for all $0 < \alpha < 2 - \frac{d}{p}$, i.e.

$$|(G \star f)(x) - (G \star f)(y)| \leqslant C_k|x-y|^\alpha \quad \forall x, y \in K$$

with K compact in $\mathbb{R}^d$.

c) If $p > d$, then $G \star f \in C^{1,\alpha}_{loc}(\mathbb{R}^d)$ for all $0 < \alpha < 1 - \frac{d}{p}$.

where G is den fundamental solution of the laplace equation.

Example 4.12 Let $r = |x|$

$$u(x) = \omega(r) = \log(|\log(r)|)$$

if $0 < r < \frac{1}{2}$, so u is well-defined in $B = B(0, \frac{1}{2})$. We conclude:

$$-\Delta_{\mathbb{R}^3} u(x) = -\omega''(r) - \frac{2\omega'(r)}{r} = f(x) \in L^{\frac{3}{2}(B)}$$

But the Theorem (b) tells us that if $f \in L^{\frac{3}{2}}$ then u is continuous but $u \notin C(B)$.

Proof of theorem 4.11. a) ($p = 1$) Why is $G \star f \in L^q_{loc}$? Recall from the proof of Youngs inequality:

$$\begin{aligned} |(G \star f)(x)| &= \left| \int_{\mathbb{R}^d} G(x-y) f(y)\, dy \right| \\ \text{(Hölder)} \quad &= \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)|\, yd \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(y)|\, dy \right)^{\frac{1}{q'}} \end{aligned}$$

Where $\frac{1}{q} + \frac{1}{q'} = 1$. Then:

$$|(G \star f)(x)|^q \leqslant C \int_{\mathbb{R}^d} |G(x-y)|^q |f(y)|\, dy$$

For any Ball $B = B(0, R) \subseteq \mathbb{R}^d$:

$$\begin{aligned} \int_B |G \star f(x)|^q\, dx &\leqslant C \int_B \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)|\, dy \right) dx \\ &= C \int_{\mathbb{R}^d} \left(\int_B |G(x-y)|^q\, dx \right) |f(y)|\, dy \end{aligned}$$

$G(x) \sim \frac{1}{|x|^{d-2}} \rightsquigarrow |G|^q = \frac{1}{|x|^{(d-2)q}} \in L^1_{loc}(\mathbb{R}^d)$ if $(d-2)q < 2 \Leftrightarrow q < \frac{d}{d-2}$. Here, $y \in \operatorname{supp} f$, so $|y| \leqslant R_1$, then $|x - y| \leqslant R + R$ if $|x| \leqslant R$. With $y \in \operatorname{supp} f$, this implies:

$$\int_{B(0,R)} |G(x-y)|^q\, dx \leqslant \int_{|z| \leqslant R + R_1} |G(z)|^q\, dz < \infty$$

b)

$$(G \star f)(x) - (G \star f)(y) = \int_{\mathbb{R}^d} (G(x-z) - G(y-z)) f(z)\, dz$$

So

$$|G \star f(x) - (G \star f)(y)| \leqslant C \int_{\mathbb{R}^d} \left| \frac{1}{|x-z|^{d-2}} - \frac{1}{|y-z|^{d-2}} \right| |f(z)|\, dz$$

for all $x, y \in \mathbb{R}^d$:

$$\begin{aligned} \left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| &= \left| \left(\frac{1}{|x|} - \frac{1}{|y|} \right) \left(\frac{1}{|x|^{d-3}} + \cdots + \frac{1}{|y|^{d-3}} \right) \right| \\ &\leqslant C \frac{||x| - |y||}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &= C \frac{|x-y|}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &\leqslant C \max(|x|, |y|)^{1-\alpha} \frac{|x-y|^\alpha}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \end{aligned}$$

as

$$||x| - |y|| \leqslant \min\left(|x-y|, \max(|x|,|y|)\right) \leqslant |x-y|^\alpha \max(|x|,|y|)^{1-\alpha}$$

Thus, for all $x, y \in \mathbb{R}^d$:

$$\begin{aligned}\left|\frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}}\right| &\leqslant C|x-y|^\alpha \frac{\max(|x|,|y|)^{1-\alpha}}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}}\right) \\ &\leqslant C|x-y|^\alpha \max\left(\frac{1}{|x|^{d-2+\alpha}}, \frac{1}{|y|^{d-2+\alpha}}\right)\end{aligned}$$

So we get

$$\left|\frac{1}{|x-y|^{d-2}} - \frac{1}{|y-z|^{d-2}}\right| \leqslant C|x-y|^\alpha \max\left(\frac{1}{|x-z|^{d-2+\alpha}}, \frac{1}{|y-z|^{d-2+\alpha}}\right)$$

Therefore:

$$\begin{aligned}&|G \star f(x) - G \star f(y)| \\ &\leqslant C \int_{\mathbb{R}^d} |x-y|^\alpha \max\left(\frac{1}{|x-z|^{d-2+\alpha}}, \frac{1}{|y-z|^{d-2+\alpha}}\right) |f(z)|\, dz \\ &\leqslant C|x-y|^\alpha \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)|\, dz\right)\end{aligned}$$

Claim: If $f \in L^p(\mathbb{R}^d)$ is compactly supported, $d \geqslant p > \frac{d}{2}$, then:

$$\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+d}} |f(z)| \ dz < \infty$$

for all $0 < \alpha < 2 - \frac{d}{p}$. Assume $\operatorname{supp} f \subseteq \overline{B(0, R_1)}$. Consider 2 cases:

- If $|\xi| > 2R_1$, then: $|\xi - z| \ \geqslant R_1$ for all $z \in B(0, R_1)$. Hence:

$$\int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)|\, dz \leqslant \frac{1}{R_1^{d-2+\alpha}} \|f\|_{L^1} < \infty$$

- If $|\xi| \leqslant 2R_1$, then: $|\xi - z| \leqslant 3R_1$ for all $z \in B(0, R_1)$:

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)|\, dz &\leqslant \int_{|\xi - z| \ \leqslant 3R_1} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)|\, dz \\ \text{(Hölder)}, \left(\frac{1}{p} + \frac{1}{q} = 1\right) \quad &\leqslant \left(\int_{\mathbb{R}^d} |f(z)|^p\, dz\right)^{\frac{1}{p}} \\ &\quad \cdot \left(\int_{|\xi - z| \ \leqslant 3R_1} \frac{1}{|\xi - z|^{(d-2+\alpha)q}}\right)^{\frac{1}{q}} \\ &= \|f\|_{L^p} \left(\int_{|z| \leqslant 3R_1} \frac{1}{|z|^{(d-2+\alpha)q}}\, dz\right)^{\frac{1}{q}} < \infty\end{aligned}$$

c) $(d \geqslant 3)$ We already know:

$$\partial_i(G \star f) = (\partial_i G \star f) \in L^1_{loc}(\mathbb{R}^d)$$

as $\omega_d f \in L^1(\mathbb{R}^d)$. We claim that $\partial_i G \star f \in C^{0,\alpha}(\mathbb{R}^d)$. So $G \star f \in C^{1,\alpha}(\mathbb{R}^d)$ by the equivalence between the classical and the distributional derivatives. Exercise. Hint:

$$|\partial_i G \star f(x) - \partial_i G \star f(y)| \leqslant \int_{\mathbb{R}^d} |\partial_i G(x-z) - \partial_i G(y-z)||f(z)| \; dz,$$

$\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$. $\rightsquigarrow$ Need to estimate $|\partial_i G(x) - \partial_i G(y)| \leqslant C|x-y|^\alpha$. ■

Theorem 4.13 (High regularity for Poisson's equation) Let $f \in C^{0,\alpha}(\mathbb{R}^d)$, $0 < \alpha < 1$ be compactly supported. Then $G \star f \in C^{2,\alpha}(\mathbb{R}^d)$.

Remark 4.14 $(-\Delta u = f)$ and $f \in C(\mathbb{R}^d)$ does not imply that $u \in C^2(\mathbb{R}^d)$. (exercise)

Remark 4.15 If $f \in C^{k,\alpha}(\mathbb{R}^d)$, $k \in \{0, 1, \dots\}$, $0 < \alpha < 1$ is compactly supported, then $G \star f \in C^{k+2,\alpha}(\mathbb{R}^d)$. This more general statement is a consequence of the theorem since

$$D^\beta(G \star f) = G \star \underbrace{(D^\beta f)}_{\in C^{0,\alpha}}$$

for all $\beta = (\beta_1, \dots, \beta_d)$, $|\beta| \leqslant k$.

Proof of theorem 4.13. Since $f \in L^p$ for all $p \leqslant \infty$ by the low regularity (4.11) we have $G \star f \in C^{1,\alpha}$ and $\partial_i(G \star f) = \partial_i G \star f$ in the classical sense. We will compute the distributional derivatives $\partial_i \partial_j (G \star f)$ and prove that they are Hölder continuous. Compute $\partial_j \partial_i (G \star f)$: For all $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
-(\partial_j \partial_i G \star f)(\phi) &= (\underbrace{\partial_i (G \ \star f)}_{\in C})(\partial_j \phi) \\
&= \int_{\mathbb{R}^d} ((\partial_i G) \star f)(x) \partial_j \phi(x) \, dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i G(x-y) f(y) \partial_j \phi(x) \, dx \, dy \\
&= \int_{\mathbb{R}^d} f(y) \left[\int_{\mathbb{R}^d} \partial_i G(x-y) \partial \phi(x) \, x \right] dy \\
&\overset{?}{=} \int_{\mathbb{R}^d} \square \phi(y) \, dy
\end{aligned}$$

Recall: $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$, $\partial_i \partial_j G(x) = \frac{1}{|B_1|} \left[\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{d} \right] \frac{1}{|x|^d}$. We have:

$$\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) \, dx = \lim_{\epsilon \to 0^+} \int_{|x-y| \geqslant \epsilon} \partial_i G(x-y) \partial_j \phi(x) \, dx$$

By dominated convergence we have $|\partial_i G(x-y) \partial_j \phi(x)| \in L^1(dx)$. By the Gauss-Green-Theorem (2.2) for all $\epsilon > 0$:

$$\begin{aligned}
&\int_{|x-y| \ \geqslant \epsilon} \partial_i G(x-y) \partial_j \phi(x) \, dx \\
&\quad = \int_{\partial B(y,\epsilon)} \partial_i G(x-y) \phi(x) \omega_j \, dS(x) - \int_{|x-y| \ \geqslant \epsilon} \partial_j \partial_i G(x-y) \phi(x) \, dx
\end{aligned}$$

Where $\omega = \frac{x-y}{|x-y|}$. For the boundary term:

$$\begin{aligned}
-\int_{\partial B(y,\epsilon)} \partial_i G(x-y)\phi(x)\omega_j \, dS(x) &= -\epsilon^{d-1}\int_{\partial B(0,1)} \partial_i G(\epsilon\omega)\phi(y+\epsilon\omega)\omega_j \, d\omega \\
(\star) \quad &= \int_{\partial B(0,1)} \frac{1}{d|B_1|}\omega_i\omega_j\phi(y+\epsilon\omega)\, d\omega \\
&\xrightarrow{\epsilon\to 0} \int_{\partial B(0,1)} \frac{1}{d|B_1|}\; \omega_i\omega_j\phi(y)\, d\omega \\
&= \frac{1}{d}\delta_{i,j}\phi(y)
\end{aligned}$$

$(\star)$ $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$, so $\partial_i G(\epsilon\omega) = -\frac{-\omega_i}{d|B_1|}\frac{1}{\epsilon^{d-1}}$. for all $|\omega| = 1$.

Now we split:

$$\begin{aligned}
&-\int_{|x-y|\;\geqslant\epsilon} \partial_i\partial_j G(x-y)\phi(x)\, dx \\
&\quad = -\int_{|x-y|\;\geqslant 1} \partial_i\partial_j G(x-y)\phi(x)\, dx - \int_{1\geqslant|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)\phi(x)\, dx
\end{aligned}$$

The key observation is: $\int_{\partial B(0,r)} \partial_i\partial_j G(x)\, dx = 0$ since

$$\partial_i\partial_j G(x) = \frac{1}{|B_1|}\left(\omega_i\omega_j - \frac{\partial_{ij}}{d}\right)\frac{1}{|x|^d},$$

$\omega = \frac{x}{|x|}$. For example if $i=1, j=2, r=1$:

$$\int_{\partial B(0,1)} \partial_1\partial_2 G(x)\, dS(x) = \frac{1}{|B_1|}\int_{\partial B(0,1)} \omega_1\omega_2\, d\omega,$$

$\partial B(0,1) = \{\omega \mid |\omega| = 1\}$. Consider: $\omega \mapsto R\omega, (\omega_1,\dots,\omega_d) \mapsto (-\omega_1,\omega_2,\dots,\omega_d)$. Then

$$-\int_{1\geqslant|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)\phi(y)\, dx = 0.$$

So

$$-\int_{1\geqslant|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)\phi(x)\, dx = -\int_{1\geqslant|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)(\phi(x)-\phi(y))\, dx$$

In summary:

$$\begin{aligned}
\partial_i\partial_j(G\star f)(\phi) &= \int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}^d} \partial_i G(x-y)\partial_j\phi(x)\, dx\right) dy \\
&= \int_{\mathbb{R}^d} f(y)\frac{1}{d}\partial_{ij}\phi(y)\, dy \\
&\quad -\int_{\mathbb{R}^d} f(y)\left(\int_{|x-y|>1} \partial_i\partial_j G(x-y)\phi(x)\, dx\right) \\
&\quad -\int_{\mathbb{R}^d}\left[\lim_{\epsilon\to 0}\int_{1\geqslant|x-y|\geqslant\epsilon} \underbrace{\partial_i\partial_j G(x-y)(\phi(x)-\phi(y))\, dx}_{\leqslant\frac{C}{|x-y|^d}|x-y|\|\nabla\phi\|_{L^\infty}\leqslant\frac{C}{|x-y|^{d-1}}\in L^1_{loc}(dx)\forall y}\right] dy \\
&= \int_{\mathbb{R}^d} \frac{\delta_{ij}}{d} f(x)\phi(x)\, dx - \int_{\mathbb{R}^d} \phi(x)\left(\int_{|x-y|>1} \partial_i\partial_j G(x-y)f(y)\, dy\right) dx \\
&\quad -\int_{\mathbb{R}^d} \phi(x)\left[\int_{|x-y|\leqslant 1} \partial_i\partial_j G(x-y)(f(y)-f(x))\, dy\right] dx
\end{aligned}$$

Conclusion:

$$\partial_i\partial_j(G\star f)(x) = -\frac{\delta_{ij}}{d}f(x) + \int_{|x-y|>1}\partial_i\partial_j G(x-y)f(y)\,dy$$
$$+\int_{|x-y|\ \leqslant 1}\partial_i\partial_j G(x-y)\left(f(y)-f(x)\right)\,dy$$

The first term $f\in C^{0,\alpha}$. The second term is also at least $C^{0,\alpha}$ since $\partial_i\partial_j G(x)$ is smooth as $|x|\ >1$. We need to prove that the thirt term

$$W_{ij}(x) = \int_{|x-y|\ \leqslant 1}\partial_i\partial_j G(x-y)(f(y)-f(x))\,dy$$

is Hölder-continuous, $|W_{ij}(x)-W_{ij(y)}|\leqslant C|x-y|^\alpha$. Recall:

$$|\partial_i\partial_j G(x-y)(f(y)-f(x))|\leqslant C\frac{1}{|x-y|^d}|x-y|^\alpha = \frac{C}{|x-y|^{d-\alpha}}\in L^1_{loc}(dy)$$

We write

$$W_{ij}(x) = \int_{|x-y|\leqslant 1}\partial_i\partial_j G(x-y)(f(y)-f(x))\,dy$$
$$= \int_{|z|\leqslant 1}\partial_i\partial_j G(z)(f(x+z)-f(x))\,dz$$

So we get:

$$W_{ij}-W_{ij}(y) = \int_{|z|\leqslant 1}\partial_i\partial_j G(z)(f(x+z)-f(y+z)-f(x)+f(y))\,dz$$

Easy thought: Use $\partial_i\partial_j G(z)|\leqslant\frac{C}{|z|^d}$ and

$$|f(x+z)-f(y+z)-f(x)+f(y)|$$
$$\leqslant\begin{cases}|f(x+z)-f(x)|+|f(y+z)-f(y)|\leqslant C|z|^\alpha\\ |f(x+z)-f(y+z)|+|f(x)-f(y)|\leqslant C|x-y|^\alpha\end{cases}$$

Thus:

$$|W_{ij}(x)-W_{ij}(y)|\leqslant C\int_{|z|\leqslant 1}\frac{1}{|z|^d}\min(|z|^\alpha,|x-y|^\alpha)\,dz$$
$$\leqslant C\int_{|z|\leqslant 1}\frac{1}{|z|^d}(|z|^\alpha)^\epsilon(|x-y|^\alpha)^{1-\epsilon},\quad 0<\epsilon<1$$
$$\leqslant C\left(\int_{|z|\leqslant 1}\frac{1}{|z|^{d-\alpha\epsilon}}\right)|x-y|^{\alpha(1-\epsilon)}$$
$$\leqslant C_\epsilon|x-y|^{\alpha(1-\epsilon)}$$

thus it is easy to prove $|W_{ij}(x)-W_{ij}(y)|\ \leqslant C_\alpha|x-y|^\alpha$ for all $\alpha'\leqslant\alpha$. However, to get $\alpha'=\alpha$ we need a more precise estimate. We split:

$$W_{ij}(x)-W_{ij}(y) = \int_{|z|\leqslant 1}\cdots = \int_{|z|\leqslant min(4|x-y|,1)} + \int_{4|x-y|<|z|\leqslant 1}$$

For the first domain:

$$\int_{|z|\leqslant 4|x-y|} |\partial_{ij}G(z)||f(x+z)-f(y+z)-f(y)+f(x)|\,dz$$
$$\leqslant C\int_{|z|\leqslant 4|x-y|} \frac{1}{|z|^d}|z|^\alpha\,dz = const\cdot|x-y|^\alpha$$

For the second domain:

$$\int_{4|x-y|<|z|\leqslant 1} \partial_{ij}G(z)(f(x+z)-f(y+z)+f(y)f(x))\,dz$$
$$= \int_{4|x-y|<z\leqslant 1} \partial_{ij}G(z)(f(x+z)-f(y+z))\,dz = (\dots)$$

since $\int_{4|x-y|<|z|\leqslant 1}\partial_{ij}G(z)\,dz = 0$. Then

$$(\dots) = \int_{4|x-y|<|z-x|\ \leqslant 1} \partial_{ij}G(z-x)f(z)\,dz - \int_{4|x-y|<|z-y|\ \leqslant 1} \partial_{ij}G(z-y)f(z)\,dz.$$

Denote $A = \{z \mid 4|x-y| < |z-x| \leqslant 1\}$, $B = \{z \mid 4|x-y|\ < |z-y| \leqslant 1\}$. Consider

$$\int_A \partial_{ij}G(z-x)f(z)\,dz - \int_B \partial_{ij}G(z-y)f(z)\,dz$$
$$= \int_{A\setminus B} + \int_{B\setminus A} + \int_{A\cap B}(\partial_{ij}G(z-x)-\partial_{ij}G(z-y))f(z)\,dz$$

Lets regard the intersection. We have

$$\partial_{ij}G(x) = \frac{1}{|B_1|}\frac{1}{|x|^d}(\omega_i\omega_j - \frac{1}{d}\delta_{ij})$$
$$|\partial_{ij}G(x)-\partial_{ij}G(y)| \leqslant C|x-y|\left(\frac{1}{|x|^{d+1}}+\frac{1}{|y|^{d+1}}\right)$$

Now,

$$|\partial_{ij}G(z-x)-\partial_{ij}G(z-y)| \leqslant C|x-y|\left(\frac{1}{|z-x|^{d+1}}+\frac{1}{|z-y|^{d+1}}\right)$$

So we have

$$\left|\int_{A\cap B}(\partial_{ij}G(z-x)-\partial_{ij}G(z-y))f(z)\,dz\right|$$
$$\leqslant C\int_{A\cap B}|x-y|\left(\frac{1}{|z-x|^{d+1}}+\frac{1}{|z-y|^{d+1}}\right)|f(z)|\ dz = (\dots)$$

Now we replace $f(z)$ by $f(z)-f(x)$, then:

$$\left|\int_{A\cap B}(\partial_{ij}G(z-x)-\partial_{ij}G(z-y))(f(z)-f(x))\,dz\right|$$
$$\leqslant C\int_{A\cap B}|x-y|\left(\frac{1}{|z-x|^{d+1}}+\frac{1}{|z-y|^{d+1}}\right)|z-x|^\alpha\ dz$$
$$= \underbrace{C\int_{A\cap B}|x-y|\ \frac{1}{|z-x|^{d+1-\alpha}}\,dz}_{(I)} + \underbrace{C\int_{A\cap B}|x-y|\frac{1}{|z-y|^{d+1}}|z-x|^\alpha\,dz}_{(II)}$$

Now,

$$\begin{aligned}
(I) &\leqslant C|x-y| \int_{4|x-y|<|z-x|\leqslant 1} \frac{1}{|z-x|^{d+1-\alpha}}\, dz \\
&= C|x-y| \int_{4x-y<|z|\leqslant 1} \frac{1}{|z|^{d+1-\alpha}}\, dz \\
&\leqslant C|x-y| \int_{4|x-y|}^{1} \frac{1}{r^{d+1-\alpha}} r^{d-1}\, dr \\
&= C|x-y| \int_{4|x-y|}^{1} \frac{1}{r^{2-\alpha}}\, dr \\
&\leqslant C|x-y| \left[-1 + \frac{1}{(4|x-y|)^{1-\alpha}}\right] \\
&\leqslant C|x-y|^{\alpha}
\end{aligned}$$

$$\begin{aligned}
(II) &\leqslant C|x-y| \int_{A\cap B} \frac{1}{|z-y|^{d+1}} |z-x|^{\alpha}\, dz \\
&\leqslant C|x-y| \int_{A\cap B} \frac{1}{|z-y|^{d+1}} \left(|z-y|^{\alpha} + |x-y|^{\alpha}\right)\, dz \\
&\leqslant \underbrace{C|x-y| \int_{B} \frac{1}{|z-y|^{d+1-\alpha}}\, dz}_{\text{similar to (I)}} + C|x-y|^{1+\alpha} \int_{B} \frac{1}{|z-y|^{d+1}}\, dz
\end{aligned}$$

and

$$C|x-y|^{1+\alpha} \int_{B} \frac{1}{|z-y|^{d+1}}\, dz \leqslant \int_{4|x-y|} \frac{1}{r^{d+1}} r^{d-1}\, dr \leqslant \frac{C}{|x-y|}$$

■

Exercise 4.16 (E 5.1) Prove that if f is a harmonic function in $\mathbb{R}^d$ and $g \in C_c(\mathbb{R}^d)$ is radial, then

$$\int_{\mathbb{R}^d} f(x)g(x)\, dx = f(0) \int_{\mathbb{R}^d} g(x)\, dx$$

Solution. $x = r\omega, r > 0, |\omega| = 1$

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)g(x)\, dx &\overset{\text{(Polar)}}{=} \int_0^\infty \left(\int_{\partial B(0,1)} f(r\omega)g(r\omega)\, d\omega\right) dr \\
&= \int_0^\infty \left(g_0(r) \int_{\partial B(0,1)} f(r\omega)\, d\omega\right) dr \\
\text{(Mean value theorem (2.12))} \quad &= \int_0^\infty \left(g_0(r) f(0) \int_{\partial B(0,1)} d\omega\right) dr \\
&= f(0) \int_0^\infty \left(\int_{\partial B(0,1)} g(r\omega)\, d\omega\right) dr \\
&= f(0) \int_{\mathbb{R}^d} g(x)\, dx
\end{aligned}$$

■

Remark 4.17 Let $g \in C_c(\mathbb{R}^d)$ be radial. Why is $\int_{\mathbb{R}^3} \frac{q(x)}{|x|}\, dx \neq \infty$? Because $f(x) = \frac{1}{|x|}$ is harmonic in $\mathbb{R}^d \backslash \{0\}$ and sub-harmonic in $\mathbb{R}^d$, $-\Delta = c\delta_0$.

Exercise 4.18 (E 5.2) Let $1 \leqslant p < \infty$. Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider the Sobolev Space

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega),\ \forall i = 1, 2, \dots, d\}$$

with the norm

$$\|f\|_{W^{1,p}} = \|f\| + \sum_{i=1}^{d} \|\partial_{x_i} f\|_{L^p(\Omega)}.$$

Prove that $W^{1,p}(\Omega)$ is a Banach space. Here $x = (x_i)_{i=1}^d \in \mathbb{R}^d$. Hint: You can use the fact that $L^p(\Omega$ is a Banach Space.

Solution. $W^{1,p}(\Omega) \subseteq L^p(\Omega) \times L^p(\Omega) \cdots \times L^p(\Omega) = (L^p(\Omega))^{d+1}$. For an element $f \in W^{1,p}(\Omega)$ we can think of it as $f \mapsto (f, \partial_1 f, \partial_2 f, \dots, \partial_d f)$, so $W^{1,p}(\Omega)$ is a subspace of $(L^p(\Omega))^{d+1}$, which is a norm-space. Why is $W^{1,p}(\Omega)$ closed in $(L^p(\Omega))^{d+1}$? Take $\{f_n\}_{n=1}^\infty \subseteq W^{1,p}(\Omega)$ such that $f_n \to f$ in L^p an $\partial_i f_n \to g_i$ in L^p for all $i = 1, \dots, d$. We prove that $(f, g_1, \dots, g_d) \in W^{1,p}(\Omega)$, i.e. $f \in W^{1,p}$ and $g_i = \partial_i f$ for all $i = 1, \dots, d$. We know that $f_n \to f$ in $L^p(\Omega)$, so $f_n \to f$ in $D'(\Omega)$ and $\partial_i f_n \to \partial_i f$ in $D'(\Omega)$. On the other hand we have $partial_i f_n \to g_i$ in $L^p(\Omega)$, so $\partial_i f_n \to g_i$ in $D'(\Omega)$. So we get $\partial_i f = g_i \in L^p(\Omega)$ for all $i = 1, \dots, d$ in $D'(\Omega)$. So we can conclude $f \in W^{1,p}(\Omega)$ and $\partial_i f = g_i$ for all $i = 1, \dots, d$. ∎

Exercise 4.19 (E 5.3) Let f be a real-values function in $W^{1,p}(\mathbb{R}^d)$ for some $1 \leqslant p < \infty$. Prove that $|f| \in W^{1,p}(\mathbb{R}^d)$ and

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}.$$

Solution. Consider $G_\epsilon(t) = \sqrt{\epsilon^2 + t^2} - \epsilon$ for $\epsilon > 0$, $t \in \mathbb{R}$. Clearly we have $G_\epsilon(t) \to |t|$ as $\epsilon \to 0$ and

$$G'_\epsilon(t) = \frac{2t}{2\sqrt{\epsilon^2 + t^2}} = \frac{t}{\sqrt{\epsilon^2 + t^2}},$$

so $|G'_\epsilon(t)| \leqslant 1$, $G_\epsilon(0) = 0$. By the chain rule, $G_\epsilon(f) \in W^{1,p}(\mathbb{R}^d)$ and

$$\partial_i G_\epsilon(f)(x) = G'_\epsilon(f), \partial_i f(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \partial_i f(x) \in L^p(\mathbb{R}^d)$$

for all $i = 1, \dots, d$. Note then when $\epsilon \to 0$ that $G_\epsilon(f)(x) \to |f(x)|$ pointwise, so $G_\epsilon(f) \to |f|$ in $L^p(\mathbb{R}^d)$. $|G_\epsilon(f)(x) - G_\epsilon(0)| \leqslant |f(x)| \in L^p(\mathbb{R}^d)$ by dominated convergence.

$$\partial_i G_\epsilon(f)(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \partial_i f(x) \xrightarrow{\epsilon \to 0} g_i(x) := \begin{cases} \partial f_i(x) & f(x) > 0 \\ -\partial_i f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

$$|\partial_i G_\epsilon(f)(x)| \leqslant \left| \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \right| |\partial_i f(x)| \leqslant |\partial_i f(x)| \ \in L^p(\mathbb{R}^d)$$

So we get $\partial_i G_\epsilon(f) \xrightarrow{\epsilon \to 0} g_i$ in $L^p(\mathbb{R}^d)$ by Dominated Convergence. So we conclude: $\partial_i(|f|) = g_i \in L^p(\mathbb{R}^d)$ for all $i = 1, \dots, d$, so $|f| \in W^{1,p}(\mathbb{R}^d)$, $|f| \in L^p$. ∎

Exercise 4.20 (E 5.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $f \in L^1(\Omega)$,

$$u(x) = \int_\Omega G(x-y)f(y)\,dy$$

Prove that $-\Delta u = f$ in $D'(\Omega)$ and $u \in L^1_{loc}(\Omega)$. Recall $f \in L^1_{loc}(\mathbb{R}^d)$ and $\omega_d f \in L^1(\mathbb{R}^d)$,

$$\omega_d(x) = \begin{cases} 1+|x| & d=1 \\ \log(1+|x|) & d=1 \\ \frac{1}{(1+|x|)^{d-2}} & d \geqslant 3 \end{cases}$$

Then

$$G \star f = \int_{\mathbb{R}^d} G(x-y)f(y)\,dy \in L^1_{loc}(\mathbb{R}^d)$$

and $-\Delta(G \star f) = f$ in $D'(\mathbb{R}^d)$.

Solution. Define $\tilde{f} = \mathbb{1}_\Omega(x)f(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$. Then

$$u(x) = \int_\Omega G(x-y)f(y)\,dy = \int_{\mathbb{R}^d} G(x-y)\tilde{f}(y)\,dy = (G \star \tilde{f})(x)$$

We have $u \in L^1_{loc}(\mathbb{R}^d)$, so $u \in L^1(\Omega)$. Then $-\Delta u = \tilde{f}$ in $D'(\mathbb{R}^d)$, so $-\Delta u = f$ in $D'(\Omega)$. Claim: $-\Delta u = f$ in $D'(\mathbb{R}^d)$, so $-\Delta u = f$ in $D'(\Omega)$ if $\Omega \subseteq \mathbb{R}^d$, $\tilde{f}|_\Omega = f$. Take $\phi \in C_c^\infty(\Omega)$. We need: $(-\Delta u)(\phi) \stackrel{?}{=} \int_\Omega f\phi$. We have $\phi \in C_c^\infty(\Omega)$, so $\phi C_c^\infty(\mathbb{R}^d)$. This implies:

$$(-\Delta u)(\phi) = \int_{\mathbb{R}^d} \tilde{f}\phi = \int_{\substack{\Omega \\ \operatorname{supp}\phi \subseteq \Omega}} \tilde{f}\phi = \int_\Omega f\phi$$

■

Exercise 4.21 (E 5.5) $B = B(0, \frac{1}{2}) \subseteq \mathbb{R}^3$, $r = |x|$, $u(x) = \omega(r) = \log|\log(r)|$ for all $r \in \left(0, \frac{1}{2}\right)$. Compute $-\Delta u$ in $D'(B)$,

Solution.

$$\begin{aligned}
\omega(r) &= \log(-\log(r)), \quad \text{for } r \in \left(0, \frac{1}{2}\right) \\
\omega'(r) &= \frac{1}{-\log(r)}\left(-\frac{1}{r}\right) = \frac{1}{r\log r} \\
\omega''(r) &= -\frac{1}{(r\log(r))^2}(r\log(r))' = -\frac{\log(r)+1}{(r\log r)^2}
\end{aligned}$$

$-\Delta u = -\omega''(r) - \frac{2\omega'(r)}{r} = \frac{\log(r)+1}{(r\log(r))^2} - \frac{2}{r^2\log(r)} = \frac{1}{(r\log r)^2} - \frac{1}{r^2\log(r)} = f(r)\ f \in L^{\frac{3}{2}}:$

$$\int_B |f(x)|^{\frac{3}{2}}\ dx = const \int_0^{\frac{1}{2}} \left| \frac{1}{r^2\log r^2} - \frac{1}{r^2\log r}\right|^{\frac{3}{2}} r^2\, dr$$

$$\begin{pmatrix} r = e^{-x}, \\ x \in (\log(2),\infty), \\ dr = -e^{-x}\ dxs \end{pmatrix} \quad \tilde{<} \int_0^{\frac{1}{2}} \frac{1}{r} \left| \frac{1}{(\log(r))^2} - \frac{1}{(\log(r))}\right|^{\frac{3}{2}} dr$$

$$\tilde{<} \int_{\log(2))}^{\infty} e^x \left(\frac{1}{x^2} + \frac{1}{x}\right)^{\frac{3}{2}} e^{-x}\ dx$$

$$\tilde{<} \int_{\log(2)}^{\infty} \frac{1}{x^{\frac{3}{2}}}\, dx < \infty$$

Where $\tilde{<}$ means *up to a constant*. Now, $u(x) = \omega(r) = \log(-\log(r))$.

$$-\Delta u(x) = f(r) = \frac{1}{r^2(\log(r))^2} - \frac{1}{r^2\log(r))}$$

for all $x \neq 0, |x| = r < \frac{1}{2}$. Why is $-\Delta u(x) = f$ in $D'(B)$? Take $\phi \in C_c^\infty(B)$, check: $\int_B(-\Delta\phi) = \int_B f d\phi$.

$$\int_{|x|<\frac{1}{2}} u(-\Delta\phi)\, dx = \lim_{\epsilon\to 0^+} \int_{\epsilon<|x|<\frac{1}{2}} u(x)(-\Delta\phi)(x)\, dx$$

by Dominated convergence. $u \in L^1(B)$. For all $\epsilon > 0$:

$$\begin{aligned}\int_{\epsilon<|x|<\frac{1}{2}} u(x)(-\Delta\phi)(x)\, dx &= \int_{|x|\ >\epsilon} u(x)(-\Delta\phi)(x)\, dx \\ &= \int_{\partial B(0,\epsilon)} u(x)\nabla\phi(x)\frac{x}{|x|}\, dS(x) + \int_{|x|>\epsilon} \nabla u(x)\nabla\phi(x)\, dx\end{aligned}$$

The boundary term vanishes as $\epsilon \to 0$ since

$$\left|u(x)\nabla\phi(x)\frac{x}{|x|}\right| \leqslant \|\nabla\phi\|_{L^\infty}|u(x)| = c\log|\log(r)|$$

$$\begin{aligned}\left|\int_{\partial B(0,\epsilon)} u(x)\nabla\phi(x)\frac{x}{|x|}\, dS(x)\right| &\leqslant C\int_{\partial B(0,\epsilon)} \log|\log(\epsilon)|\, dS(x) \\ &= C\log|\log\epsilon|\,\underbrace{|\partial B(0,\epsilon)}_{\sim\epsilon^2} \xrightarrow{\epsilon\to 0} 0\end{aligned}$$

$$\begin{aligned}\int_{|x|\ >\epsilon} \nabla u(x)\nabla\phi(x)\, dx &= \sum_{i=1}^d \int_{|x|\ >\epsilon} \partial_i u(x)\partial_i\phi(x)\, dx \\ &= \sum_{i=1}^d \left(-\int_{\partial B(0,\epsilon)} \partial_i u(x)\phi(x)\frac{x_i}{|x|}\, dS(x) - \int_{|x|>\epsilon} \underbrace{\partial_i\partial_i u(x)}_{f(x)}\phi(x)\, dx\right)\end{aligned}$$

The boundary term vanishes as $\epsilon \to 0$ as

$$\begin{aligned}\left|\int_{\partial B(0,\epsilon)} \partial u(x)\phi(x)\frac{x_i}{|x|}\, dS(x)\right| &\leqslant \|\phi\|_{L^\infty}\int_{\partial B(0,\epsilon)} |\partial_i u(x)|\ dS(x) \\ (\star) \quad &\leqslant C\frac{1}{|\epsilon\log(r)|}|\partial B(0,\epsilon)| \to 0\end{aligned}$$

as $\epsilon \to 0$. $(\star) u = u(r), u(x) = \omega(|x|), \partial_i u(x) = \omega(|x|)\frac{x_i}{|x|}$, $|\partial_i u(x)| \leqslant |\omega(|x|)| = \left|\frac{1}{r\log(r)}\right|$. Finally:

$$\int_{|x|\ >\epsilon} f(x)\phi(x)\,dx \xrightarrow{\epsilon\to 0} \int_{\mathbb{R}^d} f(x)\phi(x)\,dx$$

Since $f\phi \in L^1$ and Dominated Convergence. ■

Exercise 4.22 (Bonus 5) Construct $u \in L^1(\mathbb{R}^3)$ compactly supported s.t. $-\Delta u \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and u is not continuous at 0.

Hint: Related to E 5.5. $u_0(x) = \omega(r) = \log(|\log(r)|)$ if $0 < r = |x| < \frac{1}{2}$. Consider χu_0 where $\chi \in C_c^\infty$, $\chi = 0$ if $|x|\ > \frac{1}{2}$, $\chi = 1$ if $|x| < \frac{1}{4}$. You can prove that $\Delta(\chi u_0) = (\Delta\chi)u_0 + 2\nabla\chi\nabla u_0 + \chi(\underbrace{\Delta u_0}_{\in L^{\frac{3}{2}}})$ in $D'(\mathbb{R}^3)$. (almost everywhere, in distributional sense, integration by parts)