Partial Differerential Equations

Thành Nam Phan

Winter Semester 2021/2022

Lecture notes TEXed by Thomas Eingartner

Wednesday 27th October, 2021, 14:51

Contents

1 Introduction **2**

2 Laplace / Poisson Equation **4**

2.1 Laplace Equation 4

2.2 Poisson-Equation 5

2.3 Equations in general domains 7

Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} f(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geq 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \to \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \to \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \to 0} \frac{f(x+he_i) - f(x)}{h}$, where $e_i = (0, 0, \ldots, 1, \ldots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial x_1, \ldots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha|=k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i,j \leq d}$

Definition 1.4 Given a function F. Then the equation of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u:\ \Omega \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$ is called a *PDE of order k*.

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1}u, D^{k-2}u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explizit solution! This is in many cases impossible.
- Prove a *well-posted theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continous and differentiable functions) Let $\Omega \subseteq R^d$ be open

$$\begin{aligned} C(\Omega) &= \{f:\ \Omega \to \mathbb{R} \mid f \text{ continuous}\} \\ C^k(\Omega) &= \{f:\ \Omega \to \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha|\ \leq k\} \end{aligned}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f:\ \Omega \to \mathbb{R} \text{ lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in $\mathbb{R}^d$. If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then u is a harmonic function in Ω.

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \; dS(x) = \int_K \mathrm{div}(\vec{F}) \; dx$$

where ν is the outward unit normal vector field on ∂K. Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \; dS(x) = \int_V \mathrm{div}(\nabla u) \; dx = \int_V \Delta u(x) \; dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

   $$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

   where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

   $$\int_K f \Delta g - (\Delta f) g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \to \mathbb{R}$ be continuous. Prove that if $\int_B f(x) \; dx = 0$, then $u \equiv 0$ in Ω.

Theorem 2.5 (Fundamential Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x) \; dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in $\mathbb{R}^d$. Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \to \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2 + \cdots + x_d^2}\right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned}
&\partial_{x_i} u = \partial_{x_i} v = (\partial_r v)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r} \\
&\partial_{x_i}^2 u = \partial_{x_i}\left(v(r)'\frac{x_i}{r}\right) = (\partial_{x_i} v(r)')\frac{x_i}{r} + v'(r)\partial_{x_i}\left(\frac{x_i}{r}\right) \\
&\quad = (\partial_r v'(r))\left(\frac{dr}{\partial_{x_i}}\right)\frac{x_i}{r} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}(\partial_{x_i} r)\right) = v'(r)\frac{x_i^2}{r^2} + v'r(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)
\end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d d_{x_i}^2\right) u = v''(r) + v'(r)(\frac{d}{r} - \frac{1}{r})$
Thus $\Delta u = v'(r) + v(r)\frac{d-1}{r}$. We consider $d \geq 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r)\frac{d-1}{r} = 0$
$\Rightarrow \log(v(r))' = \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)'$ (recall $log(f)' = \frac{f'}{f}$)
$\Rightarrow v'(r) = \frac{1}{v^{d-2} + \text{const.}}$

$$\begin{cases} \frac{const}{r^{d-2}} + const x x + const & , d \geq 3 \\ const \log(r) + const x x r + const & , d = 2 \end{cases}$$

Definition 2.7 (Fundamential Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|}\frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0,1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta\Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in $\mathbb{R}^d$. The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y)\; dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y)\; dy$$

This can be heuristically justifyfied with

$$-\Delta(\Phi \star f) = (-\Delta\Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C_c^2(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y) f(x-y)\, dy.$$

First we check that u is continuous: Take $x_k \to x_0$ in $\mathbb{R}^d$. We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y)\ dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y)\ dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n\to\infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x-y)| \ \leq \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \leq R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\operatorname{supp}(f)$ but independent of y. Now we compute the derivatives:

$$\partial_{x_i} u(x) = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x-y)\ dy = \lim_{h\to 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + he_i - y) - f(x-y)}{h}\ dy$$

$$\text{(dom. conv.)} \quad = \int \Phi(y) \partial_{x_i} f(x-y)\ dy$$

$$\Rightarrow \quad D^\alpha u(x) = \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x-y)\ dy \quad \text{for all } |\alpha| \leq 2$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned}
-\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y)(-\Delta_x) f(x-y)\, dy = \int_{\mathbb{R}^d} \Phi(y)(-\Delta_y) f(x-y)\, dy \\
&= \int_{\mathbb{R}^d \setminus B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy + \int_{B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy \quad (\epsilon > 0 \text{ small})
\end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus B(0,\epsilon)} \Phi(y)(-\Delta_y) f(x-y)\, dy \\
&\quad = \int_{\mathbb{R}^d \setminus B(0,\epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x-y)\, dy - \int_{\partial B(0,\epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \\
&\quad = \int_{\mathbb{R}^d \setminus B(0,\epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x-y)\, dy \\
&\qquad + \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\, dS(y) - \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y)
\end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0,\epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1|\epsilon^{d-1}} \quad \text{for } y \in \partial B(0,\epsilon)$$

Hence:

$$\begin{aligned}
\int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\ dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(0,\epsilon)} f(x-y)\ dS(y) \\
&= ⨍_{\partial B(0,\epsilon)} f(x-y)\ dS(y) = ⨍_{\partial B(x,\epsilon)} f(y)\ dS(y) \xrightarrow{\epsilon \to 0} f(x)
\end{aligned}$$

We have to regard the following error terms:

- $$\left|\int_{B(0,\epsilon)} \Phi(y)(-\Delta_y) f(x-y)\ dy\right| \leq \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leq \|\Delta f\|_{L^\infty} \mathbb{1}(|y|\leq R)}\ dy$$
$$\leq \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)|\mathbb{1}(|y| \leq R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y| \leq \epsilon) \xrightarrow{\epsilon\to 0} 0$$
Where $R > 0$ depends on x and the support of f but is independent of y.

- $$\left|\int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y)\ dS(y)\right| \leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)|\ \ dy$$
$$\leq \begin{cases} const \cdot \epsilon |\log \epsilon| \ \to 0, & d = 2 \\ const \cdot \epsilon \to 0, & d \geq 3 \end{cases}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ proved that $u = \Phi \star f$ and $f \in C_c^2(\mathbb{R}^d)$.
■

Thus, if $f \in C_c^2(\mathbb{R})$, then $u = \Phi \star f$ satisfies $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f. For example if $f \in C_c^1(\mathbb{R})$ we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x-y)\, dy \in C(\mathbb{R}^d) \Rightarrow u \in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i}\partial_{x_j} u = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) \partial_{x_j} f(x-y)\, dy = \int_{\mathbb{R}^d} \partial_{x_i}\Phi(y)\partial_{x_j} f(x-y)\, dy \in C(\mathbb{R}^d)$$

So we have $u \in C^2(\mathbb{R}^d)$. Now we can compute

$$\Delta u = \sum_{i=1}^{d} \int_{\mathbb{R}^d} \partial_{x_i}\Phi(y)\partial_{x_i} f(x-y)\, dy \overset{(IBP)}{=} f(x).$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega \subseteq \mathbb{R}$ be open, let $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω. Then

$$u(x) = \fint_{B(x,r)} u = \fint_{\partial B(x,r)} u \quad \text{for all } x \in \Omega, B(x,r) \subseteq \Omega$$

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$ (Linear Equation)

Proof. (Of theorem)2.12 Consider all $r > 0$ s.t. $B(x,r) \subseteq \Omega$,

$$f(r) = \fint_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r. When it is done, then we immediately obtain

$$f(r) = \lim_{t \to 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr}\left(⨍_{\partial B(0,r)} u(x+y)\, dS(y) \right) \\
&= \frac{d}{dr}\left(⨍_{\partial B(0,1)} u(x+rz)\, dS(z) \right) \\
\text{(dom. convergence)} \quad &= ⨍_{\partial B(0,1)} \frac{d}{dr}[u(x+rz)]\, dS(z) \\
&= ⨍_{\partial B(0,1)} \nabla u(x+rz) z\, dS(z) \\
&= ⨍_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r}\, dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n_y}\, dS(y) \\
\text{(Gauss-Green 2.2)} \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0}\, dy = 0
\end{aligned}$$

■

Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, \omega \in S^{d-1}$, then

$$\int_{B(0,r)} g(y)\, dy = \int_0^r \left(\int_{B(0,r)} g(y)\, dS(y) \right) dr$$

Remark 2.15 We already proved that for u harmonic we have $u(x) = ⨍_{\partial B(x,r)} u\, dy$. Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y)\, dy &= \int_{B(0,r)} u(x+y)\, dy \\
\text{(Pol. decomposition)} \quad &= \int_0^r \left(\int_{\partial B(0,s)} u(x+y)\, dS(y) \right) ds \\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y)\, dS(y) \right) ds \\
\text{(Mean value property)} \quad &= \int_0^r (|\partial B(x,s)|\, u(x))\, ds = |B(x,r)|\, u(x)
\end{aligned}$$

This implies

$$⨍_{B(x,r)} u(y)\, dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.16 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = ⨍_{B(x,r)} u(y)\, dy = ⨍_{\partial B(x,r)} u(y)\, dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then u is harmonic, i.e. $\Delta u = 0$ in Ω. (The proof is exactly like before)

Theorem 2.17 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω. Then

a) $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$

b) Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$, then $u \equiv const.$ in Ω.

Proof-Idea. Assume there exists $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$. We have that $B(x_0, r) \subseteq \Omega$, so by the mean valum theorem we have

$$u(x_0) = \fint_{B(x_0,r)} \underbrace{u(x)}_{\leq u(x_0)} \leq \fint_{B(x_0,r)} u(x_0)\, dx = u(x_0)$$

So we get $u(x) = u(x_0)$ for all $x \in B(x_0, r)$. ■

Proof. Given $U \subseteq \mathbb{R}^d$ open, we can write $U = \bigcup_i U_i$, where U_i is open and connected.

b) Assume that Ω is connected and there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{y \in \Omega} u(x)$. Define $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$. U is closed since u is continuous. Moreover, U is open by the mean-value theorem. I.e. for all $x \in U$ there is a $r > 0$ s.t. $B(x, r) \subseteq U \subseteq \Omega$. Since U is connected we get $U = \Omega$, so u is constant in Ω. On the other hand, if there is no $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega}$ we have $\forall x_0 \in \Omega : \quad u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$

a) Given $\Omega \subseteq \mathbb{R}^d$ open, we can write $\Omega = \bigcup_i \Omega_i$, where Ω_i is open and connected. By b) we have

$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$

So we can conclude

$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x).$$ ■

Definition 2.18

- If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$, then u is called *sub-harmonic* if $\Delta u \geq 0$ in Ω.
- If $\Delta u \leq 0$, then u is called *super-harmonic*.

Theorem 2.19 Let Ω be open in $\mathbb{R}^d$, $u \in C^2(\Omega)$, $\Delta u \geq 0$ in Ω.

1. We have the mean-value inequality

$$u(x) = \fint_{B(x,r)} u \leq \fint_{\partial B(x,r)} u \quad \text{for all } x \in B(x,r) \subseteq \Omega$$

2. Assume that Ω is connected and bounded. Then either
   - u is a constant in Ω
   - $u(x) < \sup_{y \in \partial\Omega} u(y)$ for all $x \in \Omega$

Definition 2.20 The *Poisson Equation* for given f, g on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Theorem 2.21 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open and connected. Let $f \in C(\Omega), g \in C(\partial\Omega)$. Then there exists *at most* one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof. Assume that we have two solutions u_1 and u_2. Then $u := u_1 - u_2$ is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that $u = 0$ in Ω. More precisely, by the maximum principle we have $\forall x \in \Omega$

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leq 0$$

Since $-u$ satisfies the same property we have $\forall x \in \Omega$:

$$\sup_{x\in\Omega}(-u(x)) \leq \sup_{x\in\partial\Omega}(-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leq 0 \quad \Rightarrow \quad u(x) \geq 0$$

So we geht $u(x) = 0$ in Ω. ■

Exercise 2.22 Let Ω be open, connected and bounded in $\mathbb{R}^d$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof that

1. If $g \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.
2. If $g \geq 0$ on $\partial\Omega$ and $g \neq 0$, then $u > 0$ in Ω.

Lemma 2.23 (Estimates for derivatives) If u is harmonic in $\Omega \subseteq \mathbb{R}^d$ and $B(x_0, r) \subseteq \Omega$, then

$$|D^\alpha u(x_0)| \leq \frac{(c_d N)^N}{r^{d+n}} \int_{B(x_0,r)} |u|$$

Theorem 2.24 (Regularity) Let Ω be open in $\mathbb{R}^d$. Let $u \in C(\Omega)$ satisfy $u(x) = \fint_{\partial B} u$ for any $x \in B(x,r) \subseteq \Omega$. Then u is a harmonic function in Ω, namely $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω. Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω.

Exercise 2.25 (E 1.2) Let $u \in C(\mathbb{R}^d)$ and $\int_{B(x,r)} u = 0$ for every open ball $B(x,r) \subseteq \mathbb{R}^d$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

My Solution. Assume there is a $x_0 \in \mathbb{R}^d$ s.t. w.l.o.g. $u(x_0) > 0$. Since u is continous there is a ball $B(x_0, r)$ s.t. $u(y) > \frac{u(x_0)}{2}$ for all $y \in B(x_0, r)$. But then we get

$$\int_{B(x_0,r)} u(y)\, dy \geq \int_{B(x_0,r)} \frac{u(x_0)}{2}\, dy = \frac{u(x_0)}{2}\, |B(x_0,r)| > 0. \quad ■$$

Exercise 2.26 (E 1.3) Let $f \in C_c^1(\mathbb{R}^d)$ with $d \geq 2$ and $u(x) := (\Phi \star f)(x)$. Prove that $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ (2.9 was the same for $f \in C_1(\mathbb{R})$)

Solution.

$$\partial x_i u(x) = \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x-y)\, dy$$

$$= \int_{\mathbb{R}^d} \Phi(x-y) \partial y_i f(y)\, dy$$

$$\partial x_j \partial x_i u(x) = \int \partial x_j \Phi(x-y) \partial y_i f(y)\, dy,$$

$\nabla \Phi \cos \frac{y}{|y|^d}$ locally integrable. We can apply dominated convergence because

$$\lim_{h \to 0} \int_{\mathbb{R}^d} \frac{\Phi(x + he_j - y) - \Phi(x-y)}{h} \partial_{y_i} f(y)\, dy$$

For every y:

$$\frac{\Phi(x + he_j - y) - \Phi(x-y)}{h} \partial_{y_i} f(y) \to \partial_{x_i} \Phi(x-y)\, \partial_{y_i}(y)$$

$$\begin{aligned} \left| \frac{\Phi(x + he_j - y) - \Phi(x-y)}{h} \right| &= \left| \frac{1}{h} \int_0^h \partial_{x_j} \Phi(x + te_j - y)\, dt \right| \\ &\leq \frac{1}{h} \int_0^h \frac{|(x_j + te - j - y_i)}{|\dots|^d}\, dt \\ &\leq \frac{1}{h} \int_0^h \frac{1}{|x + te_j - y|^{d-1}}\, dt \\ &\leq \sup_{t \in (0,\epsilon]} \frac{1}{|x + te_j - y|^{d-1}} \end{aligned}$$

■

Exercise 2.27 (E 1.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$ be subharmonic.

1. Proof that u satisfies the Mean-Value-Inequality
2. Proof that u satisfies the SMP

Solution. 1.

$$f(r) = ⨍_{\partial B(x,r)} u(y)\, dS(y)$$

$$f'(r) = \frac{r}{d} ⨍_{B(x,r)} \underbrace{\Delta u(y)}_{\geq 0}\, dy \geq 0$$

So f is increasing, so we get

$$f(r) \geq \lim_{t \to 0} f(t) = u(x)$$

$$⨍_{\partial B(x,r)} u \geq u(x)$$

$$
\begin{aligned}
\int_{B(x,r)} u &= \int_0^r \left(\int_{\partial B(x,s) u} \right) ds \\
&= \int_0^r |\partial B(x,s)| \left(\fint_{\partial B(x,s)} u \right) ds \\
&\geq \int_0^r |\partial B(x,r)| u(x)\, ds \\
&= u(x) \int_{B(x,r)} = u(x)|B(x,r)|
\end{aligned}
$$

So we can conclude

$$
\fint_{B(x,r)} u \geq u(x)
$$

Use $\fint_{\partial B(x,s)} u \leq \fint_{\partial B(x,r)} u$:

$$
\int_{B(x,r)} \leq \int_0^r |\partial B(x,r)| \left(\int_{\partial B(x,r)} u \right) ds = \left(\fint_{\partial B(x,r)} u \right) |B(x,r)|
$$

2. $\fint_{B(x_0,r)} u \geq u(x_0) = \sup u \Rightarrow u(x) = u(x_0) \forall x \in B(x,r)$

■

Exercise 2.28 (Bonus 1) Let Ω be open, bounded and connected in $\mathbb{R}^d$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$
\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{in } \partial\Omega \end{cases}
$$

with $g \in C(\partial\Omega)$. Prove

1. Prove if $g \geq 0$ on $\partial\Omega$ then $u \geq 0$ in Ω
2. Prove that if $g \geq 0$ on $\partial\Omega$, $g \neq 0$, then $u > 0$ in Ω

Theorem 2.29 (Liouville's Theorem) If $u \in C^2(\mathbb{R}^d)$ is harmonic and bounded, then $u = const.$

Proof. By the bound of the derivative 2.23 we have

$$
\begin{aligned}
|\partial_{x_i} u(x_0)| &\leq \frac{c_d}{r^{d+1}} \int_{B(x_0,r)} |u| \quad \forall x_0 \in \mathbb{R}^d \forall r > 0 \\
&\leq \|u\|_{L^\infty} \frac{c_d}{r^{d+1}} |B(x_0,r)| \leq \|u\|_{L^\infty} \frac{c_d}{r} \xrightarrow{r\to\infty} 0
\end{aligned}
$$

Thus $\partial_{x_i} u = 0$ for all $i = 1, 2, \ldots d$ and $u = const.$ in $\mathbb{R}^d$ ■

Theorem 2.30 (Uniqueness of solutions to Poisson Equation in $\mathbb{R}^d$) If $u \in C^2(\mathbb{R}^d)$ is a bounded function and satisfies $-\Delta u = f$ in $\mathbb{R}^d$ where $f \in C_c^2(\mathbb{R}^d)$, then we have

$$
u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y)\, dy + C \quad \forall x \in \mathbb{R}^d
$$

where C is a constant and Φ is the fundamental solution of the Laplace equation in $\mathbb{R}^d$.

Proof. If we can prove that v is bounded, then $v = const.$. We first need to show that $\Phi \star f$ is bounded.

$$\Phi = \Phi_1 + \Phi_2 = \Phi\mathbb{1}(|x| \leq 1) + \Phi(|x| \geq 1)$$
$$\Phi \star f = \Phi_1 \star f + \Phi_2 \star f$$

We have $\Phi_1 \star f \in L^1(\mathbb{R}^d)$ and $\Phi_2 \star f$ is bounded since $\Phi \to 0$ as $|x| \to \infty$ in $d \geq 3$. ■

Exercise 2.31 (Hanack's inequality) Let $u \in C^2(\mathbb{R}^d)$ be harmonic and non-negative. Prove that for all open, bounded and connected $\Omega \subseteq \mathbb{R}^d$, we have

$$\sup_{x \in \Omega} u(x) \leq C_\Omega \inf_{x \in \Omega} u(x),$$

where C_∞ is a finite constant depending only on Ω.

Proof. (Exercise) Hint: $\Omega = B(x, r)$. General case cover Ω by finitely many balls, one ball is inside Ω. ■

Definition 2.32 (Convolution) Let $f, g : \mathbb{R}^d \to \mathbb{R}$ or $\mathbb{C}$.

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y)\, dy = \int_{\mathbb{R}^d} f(y) g(x-y)\, dy = (g \star f)(x)$$

Remark 2.33 (Properties of the Convolution)

- $(f \star g)(x) = f \star (g \star h)$
- $\widehat{f \star g} = \hat{f} \star \hat{g}$

Theorem 2.34 (Young Inequality) If $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$, where $1 \leq p \leq \infty$, then $f \star g \in L^p(\mathbb{R}^d)$ and $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$. More generally, if $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^1(\mathbb{R}^d)$, $\|f \star g\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$, where $1 \leq p, q, r, \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

Proof. Let $f \in L^1, g \in L^p$. With the Hölder Inequality **??**, we have:

$$\begin{aligned} \|f \star g\|_{L^p}^p = \int_{\mathbb{R}^d} |f \star g(x)|^p\, dx \qquad &\leq \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)| |g(y)|^p\, dy\, dx \\ = \|f\|_{L^1}^{\frac{p}{q}+1} \|g\|_{L^p}^p \end{aligned}$$

So we have $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$ ■