

Partial Differential Equations  
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Please note that I write this lecture notes for my personal use. I may write things differently than presented in the lecture. This script also contains solutions for exercises (which may be wrong). Of course, I don't push them to GitHub while the exercises still can be handed in.

# Chapter 1

## Introduction

A differential equation is an equation of a function and its derivatives.

**Example 1.1** (Linear ODE) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\begin{cases} f'(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is:  $f(t) = a_0 e^{at}$  for all  $t \geq 0$ .

**Example 1.2** (Non-Linear ODE)  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider  $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$ . Then we have

$$f'(t) = \frac{1}{\cos^2(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in  $(-\pi, \pi)$ . It's a problem to extend this to  $\mathbb{R} \rightarrow \mathbb{R}$ .

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

**Remark 1.3** Recall for  $\Omega \subseteq \mathbb{R}^d$  open and  $f : \Omega \rightarrow \{\mathbb{R}, \mathbb{C}\}$  the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x + he_i) - f(x)}{h}$ , where  $e_i = (0, 0, \dots, 1, \dots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x)$ , where  $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial_{x_1}, \dots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha|=k}$ , where  $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i, j \leq d}$

**Definition 1.4** Given a function  $F$ . Then the equation of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function  $u : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$  is called a *PDE of order  $k$* .

- Equations  $\sum_d a_\alpha(x) D^\alpha u(x) = 0$ , where  $a_\alpha$  and  $u$  are unknown functions are called *Linear PDEs*.
- Equations  $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1} u, D^{k-2} u, \dots, Du, u, x) = 0$  are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explicit solution! This is in many cases impossible.
- Prove a *well-posed theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g.  $\Delta u = f \rightsquigarrow u \in C^2$ )
2. Weak Solutions: The solution is not smooth/continuous

**Definition 1.5** (Spaces of continuous and differentiable functions) Let  $\Omega \subseteq \mathbb{R}^d$  be open

$$C(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ continuous}\}$$

$$C^k(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \leq k\}$$

Classical solution of a PDE of order  $k \rightsquigarrow C^k$  solutions!

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ Lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{ exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation:  $-\Delta u = f$
- Heat Equation:  $\partial_t u - \Delta u = f$
- Wave Equation:  $\partial_t^2 u - \Delta u = f$
- Schrödinger Equation:  $i\partial_t u - \Delta u = f$

## Chapter 2

# Laplace / Poisson Equation

### 2.1 Laplace Equation

$-\Delta u = 0$  (Laplace) or  $-\Delta u = f(x)$  (Poisson).

**Definition 2.1** (Harmonic Function) Let  $\Omega$  be an open set in  $\mathbb{R}^d$ . If  $u \in C^2(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ , then  $u$  is a harmonic function in  $\Omega$ .

**Theorem 2.2** (Gauss-Green Theorem) Let  $A \subseteq \mathbb{R}^d$  open,  $\vec{F} \in C^1(A, \mathbb{R}^d)$  and  $K \subseteq A$  compact with  $C^1$  boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \, dS(x) = \int_K \operatorname{div}(\vec{F}) \, dx$$

where  $\nu$  is the outward unit normal vector field on  $\partial K$ . Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \, dS(x) = \int_V \operatorname{div}(\nabla u) \, dx = \int_V \Delta u(x) \, dx$$

for any  $V \subseteq \Omega$  open.

**Theorem 2.3** (Green's Identities) Let  $A \subseteq \mathbb{R}^d$  open,  $K \subseteq A$  d-dim. compactum with  $C^1$  boundary and  $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

where  $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f)g \, dx = \int_{\partial K} \left( f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) \, dS$$

**Exercise 2.4** Let  $\Omega \subseteq \mathbb{R}^d$  open, let  $f : \Omega \rightarrow \mathbb{R}$  be continuous. Prove that if  $\int_B f(x) \, dx = 0$ , then  $u \equiv 0$  in  $\Omega$ .

**Theorem 2.5** (Fundamental Lemma of Calculus of Variations) Let  $\Omega \subseteq \mathbb{R}^d$  open, let  $f \in L^1(\Omega)$ . If  $\int_B f(x) \, dx = 0$  for all  $x \in B_r(x) \subseteq \Omega$ , then  $f(x) = 0$  a.e. (almost everywhere)  $x \in \Omega$ .

**Remark 2.6** (Solving Laplace Equation)  $-\Delta u = 0$  in  $\mathbb{R}^d$ . Consider the case when  $u$  is radial, i.e.  $u(x) = v(|x|)$ ,  $v : \mathbb{R} \rightarrow \mathbb{R}$ . Denote  $r = |x|$ , then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i} \left( \sqrt{x_1^2 + \cdots + x_d^2} \right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned} \partial_{x_i} u &= \partial_{x_i} v = (\partial_r v) \frac{\partial r}{\partial x_i} = v'(r) \frac{x_i}{r} \\ \partial_{x_i}^2 u &= \partial_{x_i} \left( v(r) \frac{x_i}{r} \right) = (\partial_{x_i} v(r))' \frac{x_i}{r} + v'(r) \partial_{x_i} \left( \frac{x_i}{r} \right) \\ &= (\partial_r v'(r)) \left( \frac{dr}{\partial x_i} \right) \frac{x_i}{r} + v'(r) \left( \frac{1}{r} - \frac{x_i}{r^2} (\partial_{x_i} r) \right) = v'(r) \frac{x_i^2}{r^2} + v'(r) \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right) \end{aligned}$$

So we have  $\Delta u = \left( \sum_{i=1}^d \partial_{x_i}^2 \right) u = v''(r) + v'(r) \left( \frac{d}{r} - \frac{1}{r} \right)$

Thus  $\Delta u = v'(r) + v(r) \frac{d-1}{r}$ . We consider  $d \geq 2$ . Laplace operator  $\Delta u = 0$  now becomes  $v''(r) + v'(r) \frac{d-1}{r} = 0$

$$\begin{aligned} \Rightarrow \log(v(r))' &= \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)' \quad (\text{recall } \log(f)' = \frac{f'}{f}) \\ \Rightarrow v'(r) &= \frac{1}{v^{d-2} + \text{const.}} \end{aligned}$$

$$\begin{cases} \frac{\text{const}}{r^{d-2}} + \text{const}x + \text{const} & , d \geq 3 \\ \text{const} \log(r) + \text{const}x + \text{const} & , d = 2 \end{cases}$$

**Definition 2.7** (Fundamental Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|} \frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where  $|B_1|$  is the Volume of the ball  $B_1(0) = B(0, 1) \subseteq \mathbb{R}^d$ .

**Remark 2.8**  $\Delta \Phi(x) = 0$  for all  $x \in \mathbb{R}^d$  and  $x \neq 0$ .

## 2.2 Poisson-Equation

The Poisson-Equation is  $-\Delta u(x) = f(x)$  in  $\mathbb{R}^d$ . The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy$$

This can be heuristically justified with

$$-\Delta(\Phi \star f) = (-\Delta \Phi) \star f = \delta_0 \star f = f$$

**Theorem 2.9** Assume  $f \in C_c^2(\mathbb{R}^d)$ . Then  $u = \Phi \star f$  satisfies that  $u \in C^2(\mathbb{R}^d)$  and  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$

*Proof.* By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy.$$

First we check that  $u$  is continuous: Take  $x_k \rightarrow x_0$  in  $\mathbb{R}^d$ . We prove that  $u(x_n) \xrightarrow{n} u_0$ , i.e.

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y) dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n \rightarrow \infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x - y)| \leq \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \leq R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where  $R > 0$  depends on  $\{x_n\}$  and  $\text{supp}(f)$  but independent of  $y$ . Now we compute the derivatives:

$$\begin{aligned} \partial_{x_i} u(x) &= \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x - y) dy = \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + h e_i - y) - f(x - y)}{h} dy \\ (\text{dom. conv.}) &= \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x - y) dy \\ \Rightarrow D^\alpha u(x) &= \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x - y) dy \quad \text{for all } |\alpha| \leq 2 \end{aligned}$$

$D^\alpha u(x)$  is continuous, thus  $u \in C^2(\mathbb{R}^d)$ . Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y) (-\Delta_x) f(x - y) dy = \int_{\mathbb{R}^d} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy + \int_{B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x - y) dy - \int_{\partial B(0, \epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x - y) dy \\ &\quad + \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) - \int_{\partial B(0, \epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \end{aligned}$$

We have that  $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$  and  $\vec{n} = \frac{y}{|y|}$  in  $\partial B(0, \epsilon)$ . This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1| \epsilon^{d-1}} \quad \text{for } y \in \partial B(0, \epsilon)$$

Hence:

$$\begin{aligned} &\int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) = \frac{1}{d|B_1| \epsilon^{d-1}} \int_{\partial B(0, \epsilon)} f(x - y) dS(y) \\ &= \oint_{\partial B(0, \epsilon)} f(x - y) dS(y) = \oint_{\partial B(x, \epsilon)} f(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} f(x) \end{aligned}$$



We have to regard the following error terms:

$$\begin{aligned}
\bullet \left| \int_{B(0,\epsilon)} \Phi(y) (-\Delta_y) f(x-y) dy \right| &\leq \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leq \|\Delta f\|_{L^\infty} \mathbb{1}(|y| \leq R)} dy \\
&\leq \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)| \mathbb{1}(|y| \leq R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y| \leq \epsilon) \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

Where  $R > 0$  depends on  $x$  and the support of  $f$  but is independent of  $y$ .

$$\begin{aligned}
\bullet \left| \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y) dS(y) \right| &\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)| dy \\
&\leq \begin{cases} \text{const} \cdot \epsilon |\log \epsilon| \rightarrow 0, & d = 2 \\ \text{const} \cdot \epsilon \rightarrow 0, & d \geq 3 \end{cases}
\end{aligned}$$

Conclusion:  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$  proved that  $u = \Phi \star f$  and  $f \in C_c^2(\mathbb{R}^d)$ . ■

Thus, if  $f \in C_c^2(\mathbb{R})$ , then  $u = \Phi \star f$  satisfies  $u \in C^2(\mathbb{R}^2)$  and  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$ .

**Remark 2.10** The result holds for a much bigger class of functions  $f$ . For example if  $f \in C_c^1(\mathbb{R})$  we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x-y) dy \in C(\mathbb{R}^d) \Rightarrow u \in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i} \partial_{x_j} u = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) \partial_{x_j} f(x-y) dy = \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_j} f(x-y) dy \in C(\mathbb{R}^d)$$

So we have  $u \in C^2(\mathbb{R}^d)$ . Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_i} f(x-y) dy \stackrel{(IBP)}{=} f(x).$$

**Exercise 2.11** Extend this to more general functions!

## 2.3 Equations in general domains

**Theorem 2.12** (Mean Value Theorem for Harmonic Functions) Let  $\Omega \subseteq \mathbb{R}$  be open, let  $u \in C^2(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ . Then

$$u(x) = \oint_{B(x,r)} u = \oint_{\partial B(x,r)} u \quad \text{for all } x \in \Omega, B(x,r) \subseteq \Omega$$

*Proof.* Consider all  $r > 0$  s.t.  $B(x,r) \subseteq \Omega$ ,

$$f(r) = \oint_{\partial B(x,r)} u$$

We need to prove that  $f(r)$  is independent of  $r$ . When it is done, then we immediately obtain

$$f(r) = \lim_{t \rightarrow 0} f(t) = u(x)$$

as  $u$  is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr} \left( \oint_{\partial B(0,r)} u(x+y) dS(y) \right) \\
&= \frac{d}{dr} \left( \oint_{\partial B(0,1)} u(x+rz) dS(z) \right) \\
(\text{dom. convergence}) \quad &= \oint_{\partial B(0,1)} \frac{d}{dr} [u(x+rz)] dS(z) \\
&= \oint_{\partial B(0,1)} \nabla u(x+rz) z dS(z) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r} dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green 2.2}) \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0} dy = 0 \quad \blacksquare
\end{aligned}$$

**Exercise 2.13** In 1D:  $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$  (Linear Equation)

**Remark 2.14** Recall the polar decomposition. Let  $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, w \in S^{d-1}$ , then

$$\int_{B(0,r)} g(y) dy = \int_0^r \left( \int_{B(0,s)} g(y) dS(y) \right) ds$$

**Remark 2.15** We already proved that for  $u$  harmonic we have  $u(x) = \oint_{\partial B(x,r)} u dy$ . Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_{B(0,r)} u(x+y) dy \\
(\text{Pol. decomposition}) \quad &= \int_0^r \left( \int_{\partial B(0,s)} u(x+y) dS(y) \right) ds \\
&= \int_0^r \left( \int_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\text{Mean value property}) \quad &= \int_0^r (|\partial B(x,s)| u(x)) ds = |B(x,r)| u(x)
\end{aligned}$$

This implies

$$\oint_{B(x,r)} u(y) dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

**Remark 2.16** The reverse direction is also correct, namely if  $u \in C^2(\Omega)$  and

$$u(x) = \oint_{B(x,r)} u(y) dy = \oint_{\partial B(x,r)} u(y) dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then  $u$  is harmonic, i.e.  $\Delta u = 0$  in  $\Omega$ . (The proof is exactly like before)

**Theorem 2.17** (Maximum Principle) Let  $\Omega \subseteq \mathbb{R}^d$  be open, let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ ,  $\Delta u = 0$  in  $\Omega$ . Then

- a)  $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
- b) Assume that  $\Omega$  is connected. Then if there is a  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$ , then  $u \equiv \text{const.}$  in  $\Omega$ .

*Proof.* Given  $U \subseteq \mathbb{R}^d$  open, we can write  $U = \bigcup_i U_i$ , where  $U_i$  is open and connected.

- b) Assume that  $\Omega$  is connected and there is a  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{y \in \Omega} u(y)$ . Define  $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$ .  $U$  is closed since  $u$  is continuous. Moreover,  $U$  is open by the mean-value theorem. I.e. for all  $x \in U$  there is a  $r > 0$  s.t.  $B(x, r) \subseteq U \subseteq \Omega$ . Since  $U$  is connected we get  $U = \Omega$ , so  $u$  is constant in  $\Omega$ . On the other hand, if there is no  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$  we have  $\forall x_0 \in \Omega : u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
- a) Given  $\Omega \subseteq \mathbb{R}^d$  open, we can write  $\Omega = \bigcup_i \Omega_i$ , where  $\Omega_i$  is open and connected. By b) we have

$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$

So we can conclude

$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x). \quad \blacksquare$$

**Definition 2.18** • If  $\Omega \subseteq \mathbb{R}^d$  is open,  $u \in C^2(\Omega)$ , then  $u$  is called *sub-harmonic* if  $\Delta u \geq 0$  in  $\Omega$ .

- If  $\Delta u \leq 0$ , then  $u$  is called *super-harmonic*.

**Exercise 2.19** (E 1.4) Let  $\Omega \subseteq \mathbb{R}^d$  be open and  $u \in C^2(\Omega)$  be subharmonic.

- a) Prove that  $u$  satisfies the Mean Value Inequality

$$\oint_{\partial B(x, r)} u(y) dS(y) \geq \int_{B(x, r)} u(y) dy \geq u(x)$$

for all  $B(x, r) \subseteq \mathbb{R}^d$ .

- b) Assume further that  $\Omega$  is connected and  $u \in C(\bar{\Omega})$ . Prove that  $u$  satisfies the strong maximum principle, namely either
  - $u$  is constant in  $\Omega$ , or
  - $\sup_{y \in \partial\Omega} u(y) > u(x)$  for all  $x \in \Omega$ .

*My Solution.* a) Let  $f(r) = \oint_{\partial B(x,r)} u(y) dS(y)$ , then we have

$$\begin{aligned}
\partial_r f(r) &= \partial_r \oint_{\partial B(x,r)} u(y) dS(y) \\
(\text{Dom. Convergence}) \quad &= \oint_{\partial B(x,r)} \partial_r u(y) dS(y) \\
&= \oint_{\partial B(0,1)} \partial_r u(x + yr) dS(y) \\
&= \oint_{\partial B(0,1)} \nabla u(x + yr) \cdot y dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r} dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green}) \quad &= \oint_{B(x,r)} \text{div}(\nabla u(y)) dS(y) \\
&= \oint_{B(x,r)} \underbrace{\Delta u(y)}_{\geq 0} dS(y) \geq 0
\end{aligned}$$

So we can conclude that

$$\oint_{\partial B(x,r)} u(y) dS(y) = f(r) \geq \lim_{r \rightarrow 0} f(r) = u(x).$$

Now regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left( \int_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&= \int_0^r \left( |\partial B(x,r)| \oint_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&\geq \int_0^r |\partial B(x,r)| \cdot u(x) dS(y) \\
&= u(x) \int_0^r |\partial B(x,r)| dS(y) = u(x) |B(x,r)|.
\end{aligned}$$

Thus we have

$$u(x) \leq \oint_{B(x,r)} u(y) dy.$$

Finally, lets regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left( |\partial B(x,s)| \oint_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\partial_r f(r) \geq 0) \quad &\leq \int_0^r \left( |\partial B(x,s)| \oint_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \int_0^r |\partial B(x,s)| ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \cdot |B(x,r)|
\end{aligned}$$

and we conclude

$$\oint_{B(x,r)} u(y) dy \leq \oint_{\partial B(x,r)} u(y) dS(y).$$

b) Let  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{x \in \Omega} u(x)$ . Now,

$$\begin{aligned} \sup_{x \in \Omega} u(x) = u(x_0) &\leq \oint_{\partial B(x_0,r)} u(y) dy \\ &\leq \oint_{\partial B(x_0,r)} \sup_{x \in \Omega} u(x) dy = \sup_{x \in \Omega} u(x) \end{aligned}$$

Since  $u$  is continuous we get  $u(y) = u(x_0)$  for all  $y \in B(x_0, r)$ , so  $u$  is constant. ■

**Definition 2.20** The *Poisson Equation* for given  $f, g$  on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

**Theorem 2.21** (Uniqueness) Let  $\Omega \subseteq \mathbb{R}^d$  be bounded, open and connected. Let  $f \in C(\Omega), g \in C(\partial\Omega)$ . Then there exists *at most* one solution  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ , s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

*Proof.* Assume that we have two solutions  $u_1$  and  $u_2$ . Then  $u := u_1 - u_2$  is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that  $u = 0$  in  $\Omega$ . More precisely, by the maximum principle we have  $\forall x \in \Omega$

$$\sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leq 0$$

Since  $-u$  satisfies the same property we have  $\forall x \in \Omega$ :

$$\sup_{x \in \Omega} (-u(x)) \leq \sup_{x \in \partial\Omega} (-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leq 0 \quad \Rightarrow \quad u(x) \geq 0$$

So we get  $u(x) = 0$  in  $\Omega$ . ■

**Exercise 2.22** (Bonus 1) Let  $\Omega$  be open, connected and bounded in  $\mathbb{R}^d$ . Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Prove that

a) If  $g \geq 0$  on  $\partial\Omega$ , then  $u \geq 0$  in  $\Omega$ .

b) If  $g \geq 0$  on  $\partial\Omega$  and  $g \neq 0$ , then  $u > 0$  in  $\Omega$ .

*My Solution.* a) We have that  $\Delta(-u) = 0$ , so  $-u$  is harmonic in  $\Omega$ . Since  $\Omega$  is open and bounded we can apply the Maximum Principle (2.17) and get that

$$\sup_{x \in \bar{\Omega}} -u(x) \leq \sup_{x \in \partial\Omega} -g(x) \leq 0.$$

This implies  $\inf_{x \in \Omega} u(x) \geq 0$ , so  $u \geq 0$  in  $\Omega$ .

b) We prove this by contraposition. Assume there is a  $x_0 \in \Omega$  s.t.  $u(x_0) = 0$ . Since we have  $u \geq 0$  on  $\Omega$  by a), it follows that

$$0 = -u(x_0) = \sup_{x \in \bar{\Omega}} -u(x) \leq \sup_{x \in \partial\Omega} -g(x) \leq 0,$$

so  $-u$  attains its maximum on  $\Omega$ . Hence  $-u = 0 = u$  is constant by the strong maximum principle because  $\Omega$  is connected, in fact  $0 = u|_{\partial\Omega} = g$ . ■

**Lemma 2.23** (Estimates for derivatives) If  $u$  is harmonic in  $\Omega \subseteq \mathbb{R}^d$ ,  $\alpha \in \mathbb{N}_0^d$ ,  $|\alpha| = N$  and  $B(x_0, r) \subseteq \Omega$ , then

$$|D^\alpha u(x)| \leq \frac{(c_d N)^N}{r^{d+N}} \int_{B(x, r)} |u| dy$$

*Proof.* Induction: Assume  $|\alpha| = N - 1$ , Take  $|\alpha| = N$

$$|D^\alpha u(x_0)| \leq \frac{|S_1|}{|B_1| \frac{r}{N}} \|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))}, \quad D^\alpha u = \partial_{x_i}(D^\beta u)|_{|\beta|=N-1}$$

Note:  $x \in B(x_0, \frac{r}{N})$ , so  $B(x, \frac{r(N-1)}{N}) \subseteq B(x_0, r)$ . By the induction hypothesis:

$$\|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))} \leq \frac{[c_d(N-1)]^{N-1}}{[r \frac{(N-1)}{N}]^{d+N-1}} \int_{B(x_0, r)} |u| dy$$

The conclusion is:

$$\begin{aligned} |D^\alpha u(x_0)| &\leq \frac{|S_1|}{|B_1| \frac{r}{N}} \frac{[c_d(N-1)]^{N-1}}{(r \frac{N-1}{N})^{d+N-1}} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} (N-1)^d} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} N^d} \left(\frac{N}{N-1}\right)^d \int_{B(x_0, r)} |u| dy \\ &\leq \frac{2^d |S_1|}{|B_1|} \frac{c_d^{N-1} N^N}{r^{d+N}} \int_{B(x_0, r)} |u| dy \quad \text{if } c_d \geq \frac{2^d |S_1|}{|B_1|} \end{aligned}$$

■

**Theorem 2.24** (Regularity) Let  $\Omega$  be open in  $\mathbb{R}^d$ . Let  $u \in C(\Omega)$  satisfy  $u(x) = \int_{\partial B} u dy$  for any  $x \in B(x, r) \subseteq \Omega$ . Then  $u$  is a harmonic function in  $\Omega$ . Moreover,  $u \in C^\infty(\Omega)$  and  $u$  is analytic in  $\Omega$ .

**Exercise 2.25** (E 1.1: Proof the Gauss–Green formula) Let  $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ . Prove that for every open ball  $B(y, r) \subseteq \mathbb{R}^d$  we have

$$\int_{\partial B(y, r)} f(y) \cdot \nu_y dS(y) = \int_{B(y, r)} \operatorname{div} f dx.$$

Here  $\nu_y$  is the outward unit normal vector and  $dS$  is the surface measure on the sphere.

*Solution.* We proof this in  $d=3$ . Let  $f \in C^1(\mathbb{R}^3)$

$$\int_{B(0,1)} \partial_{x_3} f dx = \int_{\partial B(0,1)} f x_3 dS(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \vec{n} = \frac{x}{|x|} \text{ on } \partial B(0,1)$$

$$\begin{aligned} B(0,1) &= \{x_1^2 + x_2^2 + x_3^2 \leq 1\} \\ &= \{x_1^2 + x_2^2 \leq 1 - \sqrt{1 - x_1^2 - x_2^2} \leq x_3 \leq \sqrt{1 - x_1^2 - x_2^2}\} \end{aligned}$$

Then:

$$\begin{aligned} \int_{B(0,1)} \partial_{x_3} f dx &= \int_{x_1^2 + x_2^2 \leq 1} \left( \int_{-\sqrt{1-x_1^2-x_2^2} \leq x_3 \leq \sqrt{1-x_1^2-x_2^2}} \partial_{x_3} f dx_3 \right) dx_1 dx_2 \\ &= \int_{x_1^2 + x_2^2 \leq 1} \left[ f(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \right. \\ &\quad \left. - f(x_1, x_2, -\sqrt{1-x_1^2-x_2^2}) \right] dx_1 dx_2 \end{aligned}$$

Lets take polar coordinates in 2D:

$$\begin{aligned} x_1 &= r \cos \phi & r > 0, \phi \in [0, 2\pi) \\ x_2 &= r \sin \phi & \det \frac{\partial(x_1, x_2)}{\partial(r, \phi)} = r \end{aligned}$$

$$(\star) = \int_0^1 \int_0^{2\pi} [f(r \cos \phi, r \sin \phi, r) - f(r \cos \phi, r \sin \phi, -r)] r dr d\phi$$

On the other hand:

$$\int_{\partial B(0,1)} f x_3 dS$$

The polar coordinates in 3D are:

$$\begin{aligned} x_1 &= r \cos \phi \sin \theta & r > 0, \phi \in (0, 2\pi), \theta \in (0, \pi) \\ x_2 &= r \sin \phi \sin \theta & \det \frac{\partial(x_1, x_2, x_3)}{\partial(r, \phi, \theta)} = r^2 \sin \theta \\ x_3 &= r \cos \theta \end{aligned}$$

Then:

$$\begin{aligned} (\star\star) &= \int_0^{2\pi} \int_0^\pi f(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \sin \theta \cos \theta d\theta d\phi \\ &= \int_0^{2\pi} \left( \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^\pi d\theta \right) d\phi \\ (r = \sin \theta) &= \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, \sqrt{1-r^2}) r dr d\phi \\ &\quad - \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, -\sqrt{1-r^2}) r dr d\phi \end{aligned} \quad \blacksquare$$

**Exercise 2.26** (E 1.2) Let  $u \in C(\mathbb{R}^d)$  and  $\int_{B(x,r)} u \, dy = 0$  for every open ball  $B(x,r) \subseteq \mathbb{R}^d$ . Show that  $u(x) = 0$  for all  $x \in \mathbb{R}^d$ .

*My Solution.* Assume there is a  $x_0 \in \mathbb{R}^d$  s.t. w.l.o.g.  $u(x_0) > 0$ . Since  $u$  is continuous there is a ball  $B(x_0, r)$  s.t.  $u(y) > \frac{u(x_0)}{2}$  for all  $y \in B(x_0, r)$ . But then we get

$$\int_{B(x_0, r)} u(y) \, dy \geq \int_{B(x_0, r)} \frac{u(x_0)}{2} \, dy = \frac{u(x_0)}{2} |B(x_0, r)| > 0. \quad \blacksquare$$

**Exercise 2.27** (E 1.3) Let  $f \in C_c^1(\mathbb{R}^d)$  with  $d \geq 2$  and  $u(x) := (\Phi \star f)(x)$ . Prove that  $u \in C^2(\mathbb{R}^d)$  and  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$  (2.9 was the same for  $f \in C_1(\mathbb{R})$ )

**Theorem 2.28** (Liouville's Theorem) If  $u \in C^2(\mathbb{R}^d)$  is harmonic and bounded, then  $u = \text{const.}$

*Proof.* By the bound of the derivative 2.23 we have

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\leq \frac{c_d}{r^{d+1}} \int_{B(x_0, r)} |u| \, dy \quad \forall x_0 \in \mathbb{R}^d \quad \forall r > 0 \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r^{d+1}} |B(x_0, r)| \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r} \xrightarrow{r \rightarrow \infty} 0 \end{aligned}$$

Thus  $\partial_{x_i} u = 0$  for all  $i = 1, 2, \dots, d$  and  $u = \text{const.}$  in  $\mathbb{R}^d$   $\blacksquare$

**Theorem 2.29** (Uniqueness of solutions to Poisson Equation in  $\mathbb{R}^d$ ) If  $u \in C^2(\mathbb{R}^d)$  is a bounded function and satisfies  $-\Delta u = f$  in  $\mathbb{R}^d$  where  $f \in C_c^2(\mathbb{R}^d)$ , then we have

$$u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y) \, dy + C \quad \forall x \in \mathbb{R}^d$$

where  $C$  is a constant and  $\Phi$  is the fundamental solution of the Laplace equation in  $\mathbb{R}^d$ .

*Proof.* If we can prove that  $v$  is bounded, then  $v = \text{const.}$  We first need to show that  $\Phi \star f$  is bounded.

$$\begin{aligned} \Phi &= \Phi_1 + \Phi_2 = \Phi \mathbb{1}(|x| \leq 1) + \Phi(|x| \geq 1) \\ \Phi \star f &= \Phi_1 \star f + \Phi_2 \star f \end{aligned}$$

We have  $\Phi_1 \star f \in L^1(\mathbb{R}^d)$  and  $\Phi_2 \star f$  is bounded since  $\Phi \rightarrow 0$  as  $|x| \rightarrow \infty$  in  $d \geq 3$ .  $\blacksquare$

**Exercise 2.30** (Hanack's inequality) Let  $u \in C^2(\mathbb{R}^d)$  be harmonic and non-negative. Prove that for all open, bounded and connected  $\Omega \subseteq \mathbb{R}^d$ , we have

$$\sup_{x \in \Omega} u(x) \leq C_\Omega \inf_{x \in \Omega} u(x),$$

where  $C_\Omega$  is a finite constant depending only on  $\Omega$ .

*Proof.* (Exercise) Hint:  $\Omega = B(x, r)$ . General case cover  $\Omega$  by finitely many balls, one ball is inside  $\Omega$ .  $\blacksquare$



## Chapter 3

# Convolution, Fourier Transform and Distributions

### 3.1 Convolutions

**Definition 3.1** (Convolution) Let  $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$  or  $\mathbb{C}$ .

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) dy = \int_{\mathbb{R}^d} f(y)g(x-y) dy = (g \star f)(x)$$

**Remark 3.2** (Properties of the Convolution) •  $(f \star g)(x) = f \star (g \star h)$

$$\bullet \hat{f \star g} = \hat{f} \star \hat{g}$$

**Theorem 3.3** (Young Inequality) If  $f \in L^1(\mathbb{R}^d)$  and  $g \in L^p(\mathbb{R}^d)$ , where  $1 \leq p \leq \infty$ , then  $f \star g \in L^p(\mathbb{R}^d)$  and  $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$ . More generally, if  $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$ , then  $f \star g \in L^r(\mathbb{R}^d)$ ,  $\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$ , where  $1 \leq p, q, r \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

*Proof.* Let  $f \in L^1, g \in L^p$ . With the Hölder Inequality ??, we have:

$$\begin{aligned} \|f \star g\|_{L^p}^p &= \int_{\mathbb{R}^d} |f \star g(x)|^p dx \\ &\leq \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)| |g(y)|^p dy dx \\ &= \|f\|_{L^1}^{\frac{p}{q}+1} \|g\|_{L^p}^p \end{aligned}$$

So we have  $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$  ■

**Theorem 3.4** (Smoothness of the Convolution) If  $f \in C_c^\infty(\mathbb{R}^d)$ ,  $g \in L^p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ . Then  $f \star g \in C^\infty(\mathbb{R})$  and

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for all  $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \{0, 1, 2, \dots\}$

*Proof.* First we note that  $x \mapsto (f \star g)$  is continuous as  $x_n \rightarrow x$  in  $\mathbb{R}^d$  since

$$(f \star g)(x_n) = \int_{\mathbb{R}^d} f(x_n - y)g(y) dy \xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^d} f(x - y)g(y) dy = (f \star g)(x)$$

We can apply Dominated convergence because

$$f(x_n - y)g(y) \rightarrow f(x - y)g(y) \quad \forall y \text{ as } f \text{ is continuous and } x_n \rightarrow x$$

and

$$|f(x_n - y)g(y)| \leq \|f\|_{L^\infty} |g(y)| \mathbb{1}(|y| \leq R) \in L^1(\mathbb{R}^d).$$

Where  $R > 0$  satisfies  $B(0, R) \supseteq \text{supp } f + \sup_n |x_n|$ . Now we can compute the derivatives:

$$\begin{aligned} \partial_{x_i}(f \star g)(x) &= \lim_{h \rightarrow 0} \frac{(f \star g)(x + he_i) - (f \star g)(x)}{h} \\ &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\ (\text{Dominated Convergence}) \quad &= \int_{\mathbb{R}^d} \lim_{h \rightarrow 0} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\ &= \int_{\mathbb{R}^d} (\partial_{x_i} f)(x - y)g(y) dy \end{aligned}$$

We could apply Dominated Convergence since

$$\begin{aligned} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) &\xrightarrow{h \rightarrow 0} (\partial_{x_i} f)(x - y)g(y) \quad \text{as } f \in C^1 \\ \left| \frac{f(x + he_i - y) - f(x - y)}{h} g(y) \right| &\leq \|\partial_{x_i} f\|_{L^\infty} |g(y)| \mathbb{1}(|y| \leq R) \in L^1(\mathbb{R}^d) \end{aligned}$$

where  $B(0, R) \supseteq \text{supp}(f) + B(0, |x| + 1)$  and  $\partial_{x_i}(f \star g) = (\partial_{x_i} f) \star g \in C(\mathbb{R}^d)$  since  $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$ . By induction we get  $D^\alpha(f \star g) = (D^\alpha f \star g) \in C(\mathbb{R}^d)$ . ■

**Remark 3.5** Question: Is there a  $f$  s.t.  $f \star g = g$  for all  $g$ . In fact there is no regular function  $f$  that solves this formally:

$$f \star g = g \Rightarrow \widehat{f \star g} = \hat{g} \Rightarrow \hat{f} \hat{g} = \hat{g} \Rightarrow \hat{f} = 1 \Rightarrow f \text{ is not a regular function!}$$

However, if  $f$  is the Dirac-Delta Distribution,  $f = \delta_0$  then  $\delta_0 \star g = g$  for all  $g$ . Formally:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \int \delta_0 = 1$$

In fact, if  $f \in L^1(\mathbb{R}^d)$ ,  $\int f = 1$ ,  $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ , then  $f_\epsilon \rightarrow \delta_0$  in an appropriate sense and  $f_\epsilon \star g \rightarrow g$  for all  $g$  nice enough.

**Theorem 3.6** (Approximation by convolution) Let  $f \in L^1(\mathbb{R}^d)$ ,  $\int f = 1$ ,  $f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon})$ . Then for all  $g \in L^p(\mathbb{R}^d)$ , where  $1 \leq p < \infty$ , then

$$f_\epsilon \star g \rightarrow g \quad \text{in } L^p(\mathbb{R}^d)$$

*Proof.*

Step 1: Let  $f, g \in C_c(\mathbb{R}^d)$ . Then

$$\begin{aligned}
(f_\epsilon \star g)(x) - g(x) &= \int_{\mathbb{R}^d} f_\epsilon(y)g(x-y) dy - \int_{\mathbb{R}^d} f_\epsilon(y)g(x) dy \\
&= \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x)) dy \\
|(f_\epsilon \star g)(x) - g(x)| &= \left| \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x)) dy \right| \\
&\leq \int_{\mathbb{R}^d} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\
&\leq \int_{|y| \leq R_\epsilon} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\
&\leq \underbrace{\int_{|y| \leq R_\epsilon} |f_\epsilon(y)| dy}_{\leq \|f_\epsilon\|_{L^1} = \|f\|_{L^1}} \left[ \sup_{|z| \leq R} |g(x-z) - g(x)| \right] \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

We have Dominated Convergence since:

$$(f_\epsilon \star g)(x) - g(x) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

and

$$|f_\epsilon \star g(x) - g(x)| \leq \|f\|_{L^1} \sup_{|z| \leq R_\epsilon} |g(x-z) - g(x)| \leq 2\|f\|_1 \|g\|_{L^\infty} \mathbf{1}(|x| \leq R_1).$$

Where  $B(0, R_1) \supseteq \text{supp}(g) + B(0, R_\epsilon)$ , thus  $f_\epsilon \star g \rightarrow g$  in  $L^p(\mathbb{R}^d)$ . To remove the technical assumptions  $f, g \in C_c(\mathbb{R}^d)$ , then we use a density argument. We use the fact that  $C_c(\mathbb{R}^d)$  is dense in  $L^p(\mathbb{R}^d)$ ,  $1 \leq p < \infty$ .

Step 2: Let  $g \in C_c(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$ . Then there is  $\{g_m\} \subseteq L^p(\mathbb{R}^d), g_m \rightarrow g$  in  $L^p(\mathbb{R}^d)$ . Then

$$\begin{aligned}
\|f_\epsilon \star g - g\|_{L^p} &\leq \|f_\epsilon \star (g - g_m)\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
(\text{Young}) &\leq \|f_\epsilon\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leq \|f\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leq (\|f\|_{L^1} + 1) \|g - g_m\|_{L^p} + \|f \star g_m - g_m\|_{L^p}
\end{aligned}$$

So we get:

$$\begin{aligned}
\limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq (\|f\|_{L^p} + 1) \|g - g_m\|_{L^p} + \underbrace{\limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g_m - g_m\|_{L^p}}_{\text{by step 1.}} \\
&\xrightarrow{m \rightarrow \infty} 0
\end{aligned}$$

Step 3: Let  $f \in L^1(\mathbb{R}^d)$  and  $g \in L^p(\mathbb{R}^d)$ . Take  $\{f_m\} \subseteq C_c(\mathbb{R}^d)$ , s.t.

$$\begin{cases} F_m \rightarrow g \text{ in } L^1(\mathbb{R}) \text{ as } m \rightarrow \infty \\ \int_{\mathbb{R}^d} F_m = 1 \text{ (it is possible since } \int_{\mathbb{R}^d} f = 1) \end{cases}$$

Define  $F_{m,\epsilon}(x) = \epsilon^{-d} F_m(\epsilon^{-1}x)$  (recall  $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ ). Then:

$$\begin{aligned}
f_\epsilon \star g - g &= (f_\epsilon - F_{m,\epsilon}) \star g + F_{m,\epsilon} \star g - g \\
\Rightarrow \|f_\epsilon - g\|_{L^p} &\leq \underbrace{\|f_\epsilon - F_{m,\epsilon} \star g\|_{L^p}}_{\text{Young}} + \|F_{m,\epsilon} \star g - g\|_{L^p} \\
&\leq \|f_\epsilon - F_{m,\epsilon}\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \\
\Rightarrow \limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq \|f - F_m\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \quad \blacksquare
\end{aligned}$$

**Lemma 3.7**  $C_c(\mathbb{R}^d)$  is dense in  $L^p(\mathbb{R}^d)$ ,  $1 \leq p < \infty$

*Proof.* For all  $g \in L^p(\mathbb{R}^d)$  there are  $g_m$  step functions and  $g_m \rightarrow g$  in  $L^p(\mathbb{R}^d)$ . We can assume that  $\Omega$  is open and bounded and we want to approximate  $\chi_\Omega$  by  $C_c(\mathbb{R}^d)$ . ■

**Lemma 3.8** (Urnson) Define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \epsilon\}$$

Then there is a  $\eta_\epsilon \in C_c(\mathbb{R}^d)$  s.t.

$$\begin{cases} 0 \leq \eta(x) \leq 1 & \forall x \in \mathbb{R}^d \\ \eta_\epsilon(x) = 1 & \text{if } x \in \Omega_\epsilon \\ \eta_\epsilon(x) = 0 & \text{if } x \notin \Omega \end{cases}$$

**Lemma 3.9** (General Version of Urnson) If  $A, B \subseteq \mathbb{R}^d$ ,  $A$  closed,  $B$  closed,  $A \cap B = \emptyset$ . Then

$$\eta(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}$$

Then  $\eta \in C(\mathbb{R}^d)$ ,  $0 \leq \eta \leq 1$  and  $\eta = 0$  if  $x \in B$ ,  $\eta = 1$  if  $x \in A$ . Apply to  $A = \overline{\Omega_\epsilon} \subset \subset \Omega$  and  $B = \mathbb{R}^d \setminus \Omega$ .

**Theorem 3.10** (Appendix C4 in Evans) Let  $\Omega$  be open in  $\mathbb{R}^d$  and for  $\epsilon > 0$  define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \mathbb{R}^d \setminus \Omega) > \epsilon\}$$

Let  $f \in C_c^\infty(\mathbb{R}^d)$ ,  $\int_{\mathbb{R}^d} f = 1$ ,  $\text{supp } f \subseteq B(0, 1)$ ,  $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$   $\text{supp } f_\epsilon \subseteq B(0, \epsilon)$ . Then for all  $g \in L_{loc}^p(\Omega)$  (i.e.  $\mathbb{1}_K g \in L^p(\Omega) \forall K$  compact set in  $\Omega$ ), then:

- a)  $g_\epsilon(x) = (f_\epsilon \star g)(x) = \int_{\mathbb{R}^d} f_\epsilon(x-y)g(y) dy = \int_\Omega f_\epsilon(x-y)g(y) dy$  is well-defined in  $\Omega_\epsilon$  and  $g_\epsilon \in C^\infty(\Omega_\epsilon)$ .
- b)  $g_\epsilon \rightarrow g$  in  $L_{loc}^p(\Omega)$  if  $1 \leq p < \infty$  and  $g_\epsilon(x) \rightarrow g(x)$  almost everywhere  $x \in \Omega$ .
- c) If  $g \in C(\Omega)$ , then  $g_\epsilon(x) \rightarrow g(x)$  uniformly in any compact subset of  $\Omega$ .

*Proof.* a)  $D^\alpha(g_\epsilon) = (D^\alpha f_\epsilon) \star g \in C(\Omega_\epsilon)$

b) Already proved in  $\mathbb{R}^d$  space. ■

**Corollary 3.11** (Lebesgue differentiation theorem) If  $f \in L_{loc}^p(\mathbb{R}^d)$ , then

$$\int_{B(x, \epsilon)} |f(y) - f(x)|^p dy \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

**Exercise 3.12** (E 2.1) Let  $u \in C^2(\mathbb{R}^2)$  be convex. I.e.

$$tu(x) + u(y)(1-t) \geq u(tx + (1-t)y) \forall x, y \in \mathbb{R}^d \forall t \in [0, 1]$$

- a) Prove for all  $x \in \mathbb{R}^d$  that  $H(x) = \dots$

*Solution.*

- a In 1D: If  $u$  is convex  $\Leftrightarrow u''(x) \geq 0$  for all  $x \in \mathbb{R}$ . In general: Taylor expansion for all  $x, z \in \mathbb{R}^d$ :

$$u(x) = u(z) + \nabla u(z)(x - z) + \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{(x - z)^\alpha}{\alpha!} ds$$

$$x = z + s(x - z), s = 1 \text{ Use } z = tx + (1 - t)y \Rightarrow x - z = (1 - t)(x - y)$$

$$tu(x) = tu(z) + t\nabla u(z)(1 - t)(x - y) + t \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{[(1 - t)(x - y)]^\alpha}{\alpha!} ds$$

$$(1 - t)u(y) = (1 - t)u(z) + (1 - t)\nabla u(z)t(y - x) + (1 - t) \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y - z)) \frac{[t(y - x)]^\alpha}{\alpha!} ds$$

$$\begin{aligned} \Rightarrow tu(x) + (1 - t)u(y) &= u(z) + t \int_0^1 \dots + (1 - t) \int_0^1 \dots \\ \Rightarrow t \int_0^1 \dots + (1 - t) \int_0^1 \dots &\geq 0 \forall x, y, t, z = tx + (1 - t)y \end{aligned}$$

$$t(1 - t)^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{(x - y)^\alpha}{\alpha!} ds + (1 - t)t^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y - z)) \frac{(y - x)^\alpha}{\alpha!} ds \geq 0$$

for all  $x, y \in \mathbb{R}^d, t \in [0, 1], z = tx + (1 - t)y$ . Divides for  $t(1 - t)$

$$(1 - t) \int_0^1 \dots + \int_0^1 \dots \geq 0$$

Take  $t \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(y + s(x - y)) \frac{(x - y)^\alpha}{\alpha!} ds \geq 0 \forall x, y \in \mathbb{R}^d$$

Take  $y = x + a, a \in \mathbb{R}^d$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x + a + sa) \frac{a^\alpha}{\alpha!} ds \geq 0 \forall \epsilon > 0, \forall x, a \in \mathbb{R}^d$$

Take  $\epsilon \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x) \frac{a^\alpha}{\alpha!} \geq 0 \Rightarrow \sum_{i,j=1, i \neq j} \partial_{x_i} \partial_{x_j} u(x) a_i a_j + \sum_{i=1}^d \partial_{x_i}^2 u(x) \frac{a_i^2}{2}$$

We get

$$\frac{1}{2} a^T H a \geq 0 \forall a(a_i)_{i=1}^d \in \mathbb{R}^d$$

- b  $H(x) \geq 0 \Rightarrow (\partial_i \partial_j u) \geq 0 \Rightarrow \text{Tr} H(x) \geq 0 \Rightarrow \sum_{i=1}^d \partial_{x_i}^2 u(x) \geq 0 \Rightarrow \Delta u(x) \geq 0 \forall x \in \mathbb{R}^d$

■

**Exercise 3.13** (E 2.2)

*Solution.* Regard  $d = 3$ . De function  $\frac{1}{|x|}$  is harmonic in  $\mathbb{R}^3 \setminus \{0\}$ . We prove

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{\max(|x|, r)}$$

If  $|x| > r$ , then  $0 \notin B(x, r + \epsilon)$ . Then

$$y \mapsto \frac{1}{|y|}$$

is harmonic in  $B(x, r + \epsilon)$ . Then by the Mean Value Property:

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{|x|}$$

If  $|x| < r$ : Then  $\frac{1}{|y|}$  is not harmonic in  $B(x, r)$  since  $0 \in B(x, r)$ . Note

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \oint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$$

This function depends on  $x$  only via  $|x|$ .

$$\dots = \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - Ry|}$$

for all  $R$  rotation  $SO(3)$ ,  $dS(Ry) = dS(y)$

$$\begin{aligned} &= \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - y|} \\ &= \oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \\ \text{(Radial in } z) &= \oint_{\partial B(0,|x|)} \left( \oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \right) dS(z) \\ \text{(Fubini)} &= \oint_{\partial B(0,r)} \left( \oint_{\partial B(0,|x|)} \frac{dS(z)}{|z - y|} \right) dS(y) \\ \text{(case 1 since } |y| = r > |x|) &= \oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) = \frac{1}{r} \end{aligned}$$

If  $|x| = r$ : Continuity:  $x \mapsto \oint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$  ■

**Remark 3.14** For  $f \in C^{|\alpha|}, g \in C^{|\beta|}$ :

$$D^{\alpha+\beta}(f \star g) = (D^\alpha f) \star (D^\beta g)$$

**Lemma 3.15** If  $d \geq 3$  and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  radial. Then:

$$\begin{aligned} \left( \frac{1}{|x|^{d-2}} \star f \right) (x) &= \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} dy \\ &= \int_{\mathbb{R}^d} \frac{f(y)}{\max(|x|^{d-2}, |y|^{d-2})} dy \end{aligned}$$

*Proof.* (d=3) Polar coordinates:

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \left[ \int_{\partial B(0,1)} \frac{1}{|x-rw|} d\omega \right] f(r) dr \\
(a) \quad &= \int_0^\infty \left[ \int_{\partial B(0,1)} \frac{d\omega}{\max(|x|, r)} \right] f(r) dr \\
&= \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|, |y|)} dy
\end{aligned}$$

(b) (d=3) If  $f$  radial and non-negative

$$\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \frac{f(y)}{|x|} dy = \frac{(Sf?)}{|x|}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|} dx dy &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x)f_2(y)}{|x+z_1-y-z_2|} dx dy \\
&= \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} f_1(x) dx \right) f_2(y) dy \leq \int_{\mathbb{R}^3} \frac{(\int_{\mathbb{R}^3} f_1)}{|y+z_2-z_1|} f_2(y) dy \\
&\leq \frac{(\int_{\mathbb{R}^3} f_1)(\int_{\mathbb{R}^3} f_2)}{|z_1-z_2|}
\end{aligned}$$

■

**Exercise 3.16** (Bonus 2) a) Prove that  $u(x) = \frac{1}{|x|}$  is sub-harmonic in  $\mathbb{R}^2 \setminus \{0\}$ .

b) Prove that if  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  radial, non-negative, measurable:

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy \geq \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)} dy$$

*My Solution.* a) Let  $x \in \mathbb{R} \setminus \{0\}$ .

$$\begin{aligned}
\partial_{x_i} u &= \partial_{x_i} |x|^{-1} = -|x|^{-2} \frac{x_i}{|x|} = -x_i |x|^{-3} \\
\Rightarrow \partial_{x_i}^2 u &= \partial_{x_i} (-x_i |x|^{-3}) = -|x|^{-3} + 3x_i^2 |x|^{-5} \\
\Rightarrow \Delta u &= -|x|^{-3} + 3x_1^2 |x|^{-5} - |x|^{-3} + 3x_2^2 |x|^{-5} \\
&= -2|x|^{-3} + 3 \underbrace{(x_2^2 + x_2^2)}_{=|x|^2} |x|^{-5} = \frac{1}{|x|^3} > 0
\end{aligned}$$

So  $u$  is sub-harmonic in  $\mathbb{R}^2 \setminus \{0\}$ .

b) Let  $r > 0, x \in \mathbb{R}^2$  and  $|x| < r$ . First we show that

$$\oint_{\partial B(x,r)} \frac{1}{|y|} dS(y) \geq \frac{1}{r} \quad (\star)$$

Now,

$$\oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) = \oint_{\partial B(0,r)} \frac{1}{|x-y|} dS(y) =: \tilde{u}(x)$$

Take  $z \in \mathbb{R}^2 \setminus \{0\}$  such that  $z = |x|$ , then  $\tilde{u}(x) = \tilde{u}(z)$ . Let  $0 < \epsilon < r$  be small. Then we get

$$\begin{aligned}
\tilde{u}(z) &= \oint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \\
\left( \begin{array}{l} |y| = r > |x| = |z| \\ \tilde{u} \text{ radial function} \end{array} \right) &= \oint_{\partial B(0,|x|-\epsilon)} \left( \oint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \right) dS(z) \\
(\text{Fubini}) &= \oint_{\partial B(0,r)} \left( \oint_{\partial B(0,|x|-\epsilon)} \frac{dS(z)}{|z-y|} \right) dS(y) \\
&= \oint_{\partial B(0,r)} \left( \oint_{\partial B(y,|x|-\epsilon)} \frac{dS(z)}{|z|} \right) dS(y) \\
\left( \frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2 \setminus \{0\} \right) &\geq \oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) \\
&= \oint_{\partial B(0,r)} \frac{1}{r} dS(y) \\
&= \frac{1}{r}
\end{aligned}$$

Now,

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy = \int_{|x| \geq |y|} \frac{f(y)}{|x-y|} dy + \int_{|x| < |y|} \frac{f(y)}{|x-y|} dy,$$

where

$$\begin{aligned}
\int_{|x| \leq |y|} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \int_{\partial B(0,r)} \frac{f(y)}{|x-y|} \mathbb{1}(|x| \leq |y|) dS(y) dr \\
(f \text{ radial}) &= \int_0^\infty f(r) \int_{\partial B(0,r)} \frac{\mathbb{1}(|x| \leq r)}{|x-y|} dS(y) dr \\
&= \int_0^\infty f(r) \int_{\partial B(x,r)} \frac{\mathbb{1}(|x| \leq r)}{|y|} dS(y) dr \\
(\star) &\geq \int_0^\infty \frac{f(r)}{r} |\partial B(x,r)| \mathbb{1}(|x| \leq r) dr \\
&= \int_0^\infty \int_{\partial B(x,r)} \frac{f(r)}{r} \mathbb{1}(|x| \leq r) dS(y) dr \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{|y|} \mathbb{1}(|x| \leq |y|) dy \\
&= \int_{|x| \leq |y|} \frac{f(y)}{|y|} dy
\end{aligned}$$



and

$$\begin{aligned}
\int_{|x|>|y|} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \left( \int_{\partial B(0,r)} \frac{f(r)}{|x-y|} \mathbb{1}(|x|>|y|) dS(y) \right) dr \\
(f \text{ radial}) &= \int_0^\infty f(r) \mathbb{1}(|x|>r) \left( \int_{\partial B(x,r)} \frac{1}{|y|} dS(y) \right) dr \\
\left( \begin{array}{l} \frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2, \\ \text{MVP and } |x|>r \end{array} \right) &\geq \int_0^\infty f(r) \mathbb{1}(|x|>r) |\partial B(x,r)| \frac{1}{|x|} dr \\
&= \int_0^\infty \int_{\partial B(x,r)} f(r) \mathbb{1}(|x|>r) \frac{1}{|x|} dS(y) dr \\
&= \int_{\mathbb{R}^2} f(y) \mathbb{1}(|x|>|y|) \frac{1}{|x|} dy \\
&= \int_{|x|>|y|} f(y) \frac{1}{|x|} dy.
\end{aligned}$$

So we can conclude,

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy &= \int_{|x|>|y|} \frac{f(y)}{|x-y|} dy + \int_{|x|\leq|y|} \frac{f(y)}{|x-y|} dy \\
&\geq \int_{|x|>|y|} \frac{f(y)}{|x|} dy + \int_{|x|\leq|y|} \frac{f(y)}{|y|} dy \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)} dy
\end{aligned}$$

■

## 3.2 Fourier Transformation

**Definition 3.17** (Fourier Transform) For  $f \in L^1(\mathbb{R}^d)$  define

$$\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k \cdot x} dx, \quad k \cdot x = \sum_{i=1}^d k_i x_i$$

**Theorem 3.18** (Basic Properties) 1. If  $f \in L^1(\mathbb{R}^d)$ , then  $\hat{f} \in L^\infty(\mathbb{R}^d)$  and  $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

2. For all  $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ ,  $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ . Moreover,  $\mathcal{F}$  can be extended to be a unitary transformation  $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  s.t.

$$\|\mathcal{F}g\|_{L^2} = \|g\|_{L^2} \quad \forall g \in L^2(\mathbb{R}^d)$$

3. The inverse of  $F$  can be defined as

4.

$$(F^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(x) e^{2\pi i k x} dk$$

for all  $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

5.  $\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k)$  as  $(2\pi i k)^\alpha f(k) \in L^2(\mathbb{R}^d)$  ( $k^\alpha = k_1^{\alpha_1} \dots k_\alpha^{\alpha_k}$ )

6.  $\widehat{f \star g}(k) = \hat{f}(k) \hat{g}(k)$  if  $f, g$  are nice enough.

**Theorem 3.19** (Hausdorff-Young-Inequality) If  $1 \leq p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $f \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^n)$  then

$$\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}$$

and

$$\|\hat{f}\|_{L^p} \leq \|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^d)$$

**Remark 3.20** We want to apply the Fourier transform to find the solution of a PDE, e.g. the Poisson-Equation:

$$-\Delta u = f \text{ in } \mathbb{R}^d \Rightarrow |2\pi k|^2 \hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{1}{|2\pi k|^2} \hat{f}(k)$$

If we can find  $G$  s.t.  $\hat{G}(k) = \frac{1}{|2\pi k|^2}$ , then

$$\begin{aligned} \hat{u}(k) &= \hat{G}(k) \hat{f}(k) = \widehat{G \star f} \\ \Rightarrow u(x) &= (G \star f)(x) = \int_{\mathbb{R}^d} G(x-y) f(y) dy \end{aligned}$$

In fact  $G$  is the fundamental solution of laplace quation.

**Theorem 3.21** (Fourier Transform of  $\frac{1}{|x|^\alpha}$  for  $0 < \alpha < d$ ) We have formally

$$\widehat{\frac{c_\alpha}{|x|^\alpha}} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \quad \forall 0 < \alpha < d$$

Here

$$c_\alpha = \pi^{-\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda$$

More precisely, for all  $f \in C_c^\infty(\mathbb{R}^d)$ ,

$$\frac{c_\alpha}{|x|^\alpha} \star f = \left( \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee$$

Moreover if  $\alpha > \frac{d}{2}$ , then we also have

$$\left( \frac{c_\alpha}{|x|^\alpha} \star f \right)^\wedge = \frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k)$$

**Lemma 3.22** (Fourier Transform of Gaussians) In  $\mathbb{R}^d$ ,

$$\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$$

More generally for all  $\lambda > 0$ :

$$\widehat{e^{-\pi\lambda^2|x|^2}} = \lambda^{-d} e^{-\pi \frac{|k|^2}{\lambda^2}}$$

(exercise)

*Proof of Theorem.* Formally:

$$\begin{aligned}
\frac{c_\alpha}{|x|^\alpha} &= \frac{1}{|x|^\alpha} \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty e^{-\pi\lambda|x|^2} \lambda^{\frac{\alpha}{2}-1} d\lambda \\
\Rightarrow \frac{\hat{c}_\alpha}{|x|^\alpha}(k) &= \int_0^\infty \widehat{e^{-\pi\lambda|x|^2}}(k) \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi\frac{|k|^2}{\lambda}} \lambda^{\frac{\alpha}{2}-1} d\lambda \\
(\lambda \rightarrow \frac{1}{\lambda}) &= \int_0^\infty \lambda^{\frac{d}{2}} e^{-\pi|k|^2\lambda} \lambda^{-\frac{\alpha}{2}+1} \lambda^{-2} d\lambda \\
&= \frac{c_{d-\alpha}}{|k|^{d-\alpha}}
\end{aligned}$$

Let  $f \in C_c(\mathbb{R}^d)$ . Then  $\left(\frac{1}{|x|^\alpha} \star f\right)(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} f(y) dy$  is well defined as  $\frac{1}{|x-y|} \in L^1_{loc}(\mathbb{R}^d, dy)$ . It is bounded

$$\frac{1}{|x|^\alpha} \star f = \frac{1}{|x|^\alpha} \underbrace{\mathbb{1}(|x| \leq 1)}_{\in L^\infty(\mathbb{R}^d)} \star \underbrace{f}_{L^\infty} + \frac{1}{|x|^\alpha} \underbrace{\mathbb{1}(|x| > 1)}_{\in L^\infty} \star \underbrace{f}_{\in L^1} \in L^\infty(\mathbb{R}^d)$$

When  $|x| \rightarrow \infty$ :

$$\left(\frac{1}{|x|^\alpha} \star f\right)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^\alpha} dy = \int_{|y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy \sim \frac{\int_{\mathbb{R}^d} f(y) dy}{|x|^\alpha}$$

Note that  $\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \underbrace{\hat{f}(k)}_{\text{bounded}} \in L^1(\mathbb{R}^d)$ .

$$\begin{aligned}
(\dots)\mathbb{1}(|k| \leq 1) + (\dots)\mathbb{1}(|k| > 1) \frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| \leq 1) &\leq \|f\|_{L^1} \frac{\mathbb{1}(|k| \leq 1)}{|k|^{d-\alpha}} \in L^1(\mathbb{R}^d, dk) \\
\frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| > 1) &\leq |\hat{f}(k)| \in L^2(\mathbb{R}^d, dK) \text{ as } f \in L^2(\mathbb{R}^d)
\end{aligned}$$

**Lemma 3.23** If  $f \in C_c^\infty(\mathbb{R}^d)$ , then  $\hat{f} \in L^1(\mathbb{R}^d)$

*Proof.* (Exercise) Hint:  $|\widehat{D^\alpha f}| = |2\pi k|^\alpha |\hat{f}(k)| \rightsquigarrow |\hat{f}(k)| \leq \frac{1}{|k|^\alpha}$  as  $|k| \rightarrow \infty$ . ■

Compute:

$$\begin{aligned}
\left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)\right)^\vee(x) &= \int_{\mathbb{R}^d} \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_{\mathbb{R}^d} \left(\int_0^\infty e^{-\pi|k|^2\lambda} \lambda^{\frac{d-\alpha}{2}-1} d\lambda\right) \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} e^{-\pi|k|^2\lambda} \hat{f}(k) e^{2\pi i k x} dk\right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(e^{-\pi k^2 \lambda} \hat{f}(x)\right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}}(k) \hat{f}(k)\right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}} \star f\right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \left(\int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}} \lambda^{\frac{d-\alpha}{2}-1} d\lambda\right) \star f
\end{aligned}$$

Assume  $d > \alpha > \frac{d}{2}$ . Then  $\frac{c_\alpha}{|x|^\alpha} \star f \in L^\infty$  and behaves  $\frac{c_\alpha(\int f)}{|x|^\alpha}$  as  $|x| \rightarrow \infty$ . This implies:

$$\int_{\mathbb{R}^d} \left| \frac{c_\alpha}{|x|^\alpha} \star f \right|^2 \leq c + \int_{|x| \geq R} \frac{c}{|x|^{2d}} dx < \infty$$

Thus the Fourier Transform  $\widehat{\frac{c_\alpha}{|x|^\alpha} \star f}$  exists. Combining with

$$\begin{aligned} \frac{c_\alpha}{|x|^\alpha} \star f &= \left( \frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k) \right)^\vee \\ \Rightarrow \widehat{\frac{c_\alpha}{|x|^\alpha} \star f} &= \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \end{aligned}$$

■

**Remark 3.24** If  $d \geq 3$

$$\begin{aligned} \hat{G}(k) &= \frac{1}{|2\pi k|^2} \\ \Rightarrow G(x) &= \left( \frac{1}{|2\pi k|^2} \right)^\vee = \frac{1}{d(d-2)|x|^{d-2}} = \Phi(x) \end{aligned}$$

### 3.3 Theory of Distribution

Let  $\Omega \subseteq \mathbb{R}^d$  be open.

- $D(\Omega) = C_c^\infty(\Omega)$  the space of test functions.
- $\phi_n \rightarrow \phi$  in  $D(\Omega)$  if  $\exists K \subseteq \Omega$ ,  $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K$  and  $\|D^\alpha(\phi_n - \phi)\|_{L^\infty} \rightarrow 0$  for all  $\alpha = (\alpha_1, \dots, \alpha_d)$ ,  $d_i \in \{0, 1, 2, \dots\}$ .

$$D'(\Omega) = \{T : D(\Omega) \rightarrow \mathbb{R} \text{ on } \mathbb{C} \text{ linear and continuous}\}$$

the space of distributions.

Motivation:  $L^2(\Omega)' = L^2(\Omega)$ ,  $(L^p(\Omega))' = (L^q(\Omega))$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Example 3.25** ("normal functions" are distributions) If  $f \in L^1_{loc}(\Omega)$ , then  $T = T_f$  defined by:

$$T(\phi) = \int_{\Omega} f(x)\phi(x) dx$$

is a distribution for all  $\phi \in D(\Omega)$ , i.e.  $T \in D'(\Omega)$ . Indeed, it is clear that  $T(\phi)$  is well-defined for all  $\phi \in D(\Omega)$  and  $\phi \mapsto T(\phi)$  is linear. Let us check that  $\phi \mapsto T(\phi)$  is continuous. Take  $\phi_n \rightarrow \phi$  in  $D(\Omega)$  and prove that  $T(\phi_n) \rightarrow T(\phi)$ . Since  $\phi_n \rightarrow \phi$  in  $D(\Omega)$ , there is a compact  $K$  s.t.  $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K \subseteq \Omega$ .

Question: Why is  $f \mapsto T_f$  injective?

**Lemma 3.26** (Fundamental lemma of calculus of variants) Let  $\Omega \subseteq \mathbb{R}^d$  be open. If  $f, g \in L^1_{loc}(\Omega)$  and  $\int_{\Omega} f\phi dy = \int_{\Omega} g\phi dy$  for all  $\phi \in D(\Omega)$ , then  $f = g$  in  $L^1_{loc}(\Omega)$

**Example 3.27** (Dirac delta function) Let  $\Omega \subseteq \mathbb{R}^d$  open. Define  $T : D(\Omega) \rightarrow \mathbb{R}$  or  $\mathbb{C}$  by  $T(\phi) = \phi(x_0)$ . Let  $x_0 \in \Omega$ . Then  $T \in D'(\Omega)$  and we denote it by  $\delta_{x_0}$ . It is clear that  $\phi \mapsto T(\phi) = \phi(x_0)$  is well-defined and linear for all  $\phi \in D(\Omega)$ . Take  $\phi_n \rightarrow \phi$  in  $D(\Omega)$  and prove  $T(\phi_n) \rightarrow T(\phi)$ , i.e.  $\phi_n(x_0) \rightarrow \phi(x_0)$  (obvious.)

**Example 3.28** (Principle Value) The function  $f(x) = \frac{1}{x}$  is not in  $L^1_{loc}(\mathbb{R})$ , but we can still define

$$\int_{\mathbb{R}} f(x)\phi(x) dx = \int_{\mathbb{R}} \frac{\phi(x)}{x} dx$$

for all  $\phi \in D(\mathbb{R})$  s.t.  $\phi(0) = 0$ . In fact,

$$\phi(x) = |\phi(x) - \phi(0)| \leq (\sup |\phi'|)(x),$$

so  $\frac{|\phi(x)|}{|x|} \in L^\infty(\mathbb{R})$  and compactly supported. So  $\frac{\phi(x)}{x} \in L^1(\mathbb{R})$ . Define  $T : D(\mathbb{R}) \rightarrow \mathbb{R}$  or  $\mathbb{C}$  by

$$T(\phi) = \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx \quad \forall \phi \in D(\mathbb{R}) \text{ s.t. } \phi(0) = 0$$

We denote  $T = p.v. \left(\frac{1}{x}\right)$ . We check that  $T \in D'(\mathbb{R})$ : For all  $\epsilon > 0$  we have

$$\left| \frac{\phi(x)}{x} \right| \leq \frac{\|\phi\|_{L^\infty}}{\epsilon}$$

for all  $|x| \geq \epsilon$  and  $\phi$  is compactly supported. So we get for all  $\epsilon > 0$ :

$$\mathbb{1}(|x| \geq \epsilon) \frac{\phi(x)}{x} \in L^1(\mathbb{R}) \rightsquigarrow \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx < \infty$$

We can write:

$$\int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx = \int_{|x| \geq 1} \frac{\phi(x)}{x} dx + \int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx$$

The second part can be written as:

$$\int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx = \int_{\epsilon}^1 \frac{\phi(x)}{x} dx + \int_{-1}^{-\epsilon} \frac{\phi(x)}{x} dx = \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx$$

Since  $\phi \in C_c^\infty(\mathbb{R})$  it holds that  $|\phi(x) - \phi(-x)| \leq 2\|\phi'\|_{L^\infty}(x)$ .

$$\begin{aligned} \Rightarrow \frac{\phi(x) - \phi(-x)}{x} &\in L^\infty(\mathbb{R}) \Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^1(0, 1) \\ \Rightarrow \int_0^1 \frac{\phi(x) - \phi(-x)}{x} dx &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx \end{aligned}$$

**Remark 3.29** The function  $\frac{1}{|x|^d}$  is not in  $L^1_{loc}(\mathbb{R}^d)$  but  $\exists T \in D'(\mathbb{R}^d)$  s.t.  $T(\phi) = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^d} dx$  for all  $\phi \in C_c^\infty(\mathbb{R}^d)$  s.t.  $\phi(0) = 0$

Let in the following  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ .

**Definition 3.30** (Derivatives of distributions) Let  $\Omega \subseteq \mathbb{R}^d$  and  $T \in D'(\Omega)$ . Define for  $\alpha \in \mathbb{N}^d$ :

$$\begin{aligned} D^\alpha T : D(\Omega) &\longrightarrow \mathbb{K} \\ \phi &\longmapsto (-1)^{|\alpha|} T(D^\alpha \phi) \end{aligned}$$

Motivation:  $f \in C_c^\infty(\Omega)$

$$\int_{\Omega} (D^\alpha f) \phi = (-1)^{|\alpha|} \int_{\Omega} f (D^\alpha \phi)$$

„If the classical derivative exists, then it is the same as the distributional derivative.“  
We write

$$(D^\alpha T)(\phi) = T_{D^\alpha f}(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi).$$

**Remark 3.31** For all  $T \in D'(\Omega)$  it holds  $D^\alpha T \in D'(\Omega)$  for all  $\alpha \in \mathbb{N}^d$ . Clearly

$$\phi \longmapsto (D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

is linear. Moreover, if  $\phi_n \rightarrow \phi$  in  $D(\Omega)$ , then  $D^\alpha \phi_n \rightarrow D^\alpha \phi$  in  $D(\Omega)$ , so

$$(D^\alpha T)(\phi_n) = (-1)^{|\alpha|} T(D^\alpha \phi_n) \xrightarrow{n \rightarrow \infty} (-1)^{|\alpha|} T(D^\alpha \phi) = (D^\alpha T)(\phi)$$

**Example 3.32** Consider  $f : x \mapsto |x|$ , then  $f \in C(\mathbb{R})$  but  $f \notin C^1(\mathbb{R})$ . However,

$$f'(x) = g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \in L^1_{loc}$$

Lets check  $f' = g$ , i.e.  $-f(\phi') = f'(\phi) = g(\phi)$  for all  $\phi \in D(\mathbb{R})$ . Thus we need to prove:

$$-\int_{\mathbb{R}} f(x) \phi'(x) dx = \int_{\mathbb{R}} g(x) \phi(x) dx \quad \forall \phi \in D(\mathbb{R})$$

namely:

$$\underbrace{-\int_{\mathbb{R}} |x| \phi'(x) dx}_{:= (\star)} = \int_0^\infty \phi(x) dx - \int_{-\infty}^0 \phi(x) dx$$

Now we have

$$(\star) = -\int_0^\infty x \phi'(x) dx + \int_{-\infty}^0 x \phi'(x) dx.$$

By integration by parts:

$$\int_0^\infty x \phi'(x) dx = \underbrace{[x \phi(x)]_0^\infty}_{=0} - \int_0^\infty \phi(x) dx = -\int_0^\infty \phi(x) dx$$

and similary:

$$\int_{-\infty}^0 x \phi'(x) dx = -\int_{-\infty}^0 \phi(x) dx$$

Thus  $f' = g$  in  $D'(\Omega)$ . We claim that  $g' = 2\delta_0$  in  $D'(\mathbb{R})$ . In fact, for all  $\phi \in D(\mathbb{R})$ , then:

$$\begin{aligned} g'(\phi) &= -g(\phi') = -\int_{\mathbb{R}} g \phi' dx = -\int_{-\infty}^0 (-1) \phi' dx - \int_0^\infty (1) \phi' dx \\ &= -\int_0^\infty \phi' dx + \int_{-\infty}^0 \phi' dx = [\phi(0) - \underbrace{\phi(\infty)}_{=0}] + [\phi(0) - \underbrace{\phi(-\infty)}_{=0}] = 2\phi(0) = 2\delta_0(\phi) \end{aligned}$$

So  $g' = 2\delta_0$  in  $D'(\mathbb{R})$ .

**Exercise 3.33** Prove that  $(D^\alpha \delta_x)(\phi) = (-1)^{|\alpha|} (D^\alpha \phi)(x)$  for all  $\phi \in D(\mathbb{R})$  for all  $x \in \mathbb{R}$ .

**Definition 3.34** (Convergence of distributions) Let  $\Omega \subseteq \mathbb{R}^d$  be open, then

$$T_n \xrightarrow{n \rightarrow \infty} T$$

in  $D'(\Omega)$  if  $T_n(\phi) \xrightarrow{n \rightarrow \infty} T(\phi)$  for all  $\phi \in D(\Omega)$ .

**Exercise 3.35** Let  $f \in L^1(\mathbb{R}^d)$ ,  $\int f = 1$  For  $\epsilon > 0$ , define  $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ . Then:  $f_\epsilon \rightarrow \delta_0$  in  $D'(\Omega)$ .

**Exercise 3.36** Let  $\Omega \subseteq \mathbb{R}^d$  be open and  $T_n \rightarrow T$  in  $D'(\Omega)$ . Then:  $D^\alpha T_n \rightarrow D^\alpha T$  in  $D'(\Omega)$  for all  $\alpha = (\alpha_1, \dots, \alpha_d)$

**Definition 3.37** (Convolution of distributions) Let  $T \in D'(\mathbb{R})$  and  $f \in L_c^\infty(\mathbb{R}^d)$ . Define

$$(T \star f)(y) = T(f_y)$$

We write  $f_y(x) = f(x - y)$  and  $\tilde{f}(x) = f(-x)$ .

**Theorem 3.38** Let  $T \in D'(\mathbb{R})$ . Then for all  $f \in D(\mathbb{R})$ :

1.  $y \mapsto T(f_y)$  is  $C^\infty(\mathbb{R}^d)$  and

$$D_y^\alpha (T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D^\alpha f_y)$$

2. For all  $g \in L^1(\mathbb{R}^d)$  and compactly supported, then

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(\underbrace{f \star g}_{\in C_c^\infty(\mathbb{R})})$$

*Proof.* 1. We prove that  $y \mapsto T(f_y)$  is continuous. Take  $y_n \rightarrow y$  in  $\mathbb{R}^d$ , then:

$$T(f_{y_n}) \rightarrow T(f_y)$$

since  $f_{y_n} \rightarrow f_y$  in  $D(\mathbb{R}^d)$ . We check this: Since  $f \in C_c^\infty(\mathbb{R}^d)$ , it holds that  $\text{supp } f \subseteq B(0, R) \subseteq \mathbb{R}^d$ . Since  $y_n \rightarrow y$  in  $\mathbb{R}^d$ . We have  $\sup_n |y_n| < \infty$ . Thus  $f_{y_n}, f_y$  are supported in  $\overline{B(0, R + \sup_n |y_n|)} = K$  compact. Moreover

$$|f_{y_n}(x) - f_y(x)| = |f(x - y_n) - f(x - y)| \leq \|\nabla f\|_{L^\infty} \|y_n - y\| \rightarrow 0$$

So we get  $\|f_{y_n} - f_y\|_{L^\infty} \rightarrow 0$  Similarly:

$$\|D^\alpha f_{y_n} - D^\alpha f_y\|_{L^\infty} \rightarrow 0$$

■

**Exercise 3.39** (E 3.1 Lebesgue Differentiation Theorem) Let  $f \in L_{loc}^1(\mathbb{R}^d)$ . Prove that for almost every  $x \in \mathbb{R}^d$ :

$$\oint_{B(x,r)} |f(y)| dy \xrightarrow{r \rightarrow 0} 0$$

*Proof.* Clearly the same result holds with  $\mathbb{R}^d \rightsquigarrow \Omega \subseteq \mathbb{R}^d$  open. Also it suffices to consider  $f \in L^1(\mathbb{R}^d)$ . From the last time discussion, by a density argument there exists  $r_n \rightarrow 0$  s.t.

$$\oint_{B(x,r_n)} |f(y) - f(x)| dy = 0$$

for a.e.  $x \in \mathbb{R}^d$ . We prove that for all  $\epsilon > 0$ , the set  $A_\epsilon = \{x \in \mathbb{R}^d \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy > \epsilon\}$  has measure 0. This will imply that

$$\bigcup_{n=1}^{\infty} A_{\frac{1}{n}} = \left\{ x \in \mathbb{R}^d \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy > 0 \right\}$$

has measure 0, which is what we want to show. First, we show that  $|A_\epsilon| = 0$ : Take  $\{f_n\} \subseteq C_c^\infty$ ,  $f_n \rightarrow f$  in  $L^1(\mathbb{R}^d)$ . By the triangle inequality:

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

So we get

$$\begin{aligned} & \int_{B(x,r)} |f(y) - f(x)| dy \\ & \leq \int_{B(x,r)} |f(y) - f_n(y)| dy + \int_{B(x,r)} |f_n(y) - f_n(x)| + |f_n(x) - f(x)| dy \\ \Rightarrow \quad \limsup_{r \rightarrow 0} \dots & \leq \limsup_{r \rightarrow 0} (\dots) + 0 + |f_n(x) - f(x)| \end{aligned}$$

Thus, for all  $x \in A_\epsilon$ , then:

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy + |f_n(x) - f(x)| > 2\epsilon$$

Observation: If  $a, b \geq 0$ ,  $a + b > 2\epsilon$  then either  $a > \epsilon$  or  $b > \epsilon$ . Therefore  $A_\epsilon \subseteq (S_{n,\epsilon} \cup \tilde{S}_{n,\epsilon})$ , where

$$\begin{aligned} S_{n,\epsilon} &= \{x \mid |f_n(x) - f(x)| > \epsilon\} \\ \tilde{S}_{n,\epsilon} &= \{x \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy > \epsilon\} \end{aligned}$$

Consequently:  $|A_\epsilon| \leq |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}|$  for all  $n \geq 1$ . By the Markov / Chebyshev inequality:

$$|S_{n,\epsilon}| \leq \int_{S_{n,\epsilon}} \frac{|f_n(x) - f(x)|}{\epsilon} dx = \int_{\mathbb{R}^d} \frac{|f_n(x) - f(x)|}{\epsilon} dx = \frac{\|f_n - f\|_{L^1}}{\epsilon}$$

We want to prove a simpler bound for  $\tilde{S}_{n,\epsilon}$ . For all  $x \in \tilde{S}_{n,\epsilon}$ :

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy > \epsilon$$

So there is a  $r_x \in (0, 1)$  s.t.

$$\int_{B(x,r_x)=B_x} |f_n(y) - f(y)| dy > \epsilon$$

Thus  $\tilde{S}_{n,\epsilon} \subseteq \left( \bigcup_{x \in \tilde{S}_{n,\epsilon}} B_x \right)$ .

**Lemma 3.40** (Vitali Covering) If  $F$  is a collection of balls in  $\mathbb{R}^d$  with bounded radius, then there exists a sub-collection  $G \subseteq F$  s.t.

- $G$  has disjoint balls



- $\bigcup_{B \in F} B \subseteq \bigcup_{B \in G} 5B, 5B(x, r) = B(x, 5r)$

**Remark 3.41** The condition of the boundedness of the radius is necessary. Otherwise, consider  $\{B(0, n)\}_{n=1}^{\infty}$

Here consider  $F = \{B_x\}_{x \in \tilde{S}_{n,\epsilon}}$ . With the Vitali covering lemma there is a  $G \subseteq F$  s.t.  $G$  contains disjoint balls and:

$$\tilde{S}_{n,\epsilon} \subseteq \bigcup_{B \in F} B \subseteq \bigcup_{B \in G} 5B$$

So we get

$$|\tilde{S}_{n,\epsilon}| \leq \left| \bigcup_{B \in G} 5B \right| \leq \sum_{B \in G} |5B| = \sum_{B \in G} 5^d |B|$$

On the other hand, for all  $B \in G \subseteq F$ :

$$\int_B |f_n(y) - f(y)| dy > \epsilon \Rightarrow \int_B |f_n - f| > \epsilon |B|$$

This implies:

$$\sup_{B \in G} \int_B |f_n - f| > \epsilon \sum_{B \in G} |B|$$

Since balls in  $G$  are disjoint:

$$\int_{\mathbb{R}^d} \geq \int_{\bigcup_{B \in G}} |f_n - f| dy > \epsilon \sum_{B \in G} |B| \geq \frac{\epsilon}{5^d} |\tilde{S}_{n,\epsilon}|$$

So

$$|\tilde{S}_{n,\epsilon}| \leq \frac{5^d}{\epsilon} \|f_n - f\|_{L^1}$$

In summary:

$$|A_\epsilon| \leq |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}| \leq \frac{5^d + 1}{\epsilon} \|f_n - f\|_{L^1} \rightarrow 0$$

as  $n \rightarrow \infty$ . So  $|A_\epsilon| = 0$  for all  $\epsilon > 0$  ■

**Remark 3.42** 1. The proof can be done by using the Besicovitch covering lemma: For all  $E \subseteq \mathbb{R}^d$  s.t.  $E$  is bounded. Let  $F =$  collection of balls s.t. for all  $x \in E$  there is a  $B_x \in F$  s.t.  $x$  is the center of  $B_x$ . There is a sub-collection  $G \subseteq F$  s.t.

- $E \subseteq \bigcup_{B \in G} B$
- Any point in  $E$  belongs to at most  $C_d$  balls in  $G$  ( $C_d$  depends only on  $\mathbb{R}^d$ ), i.e.

$$\mathbb{1}_E(x) \leq \sum_{B \in G} \mathbb{1}_B(x) \leq C_d \mathbb{1}_E(x) \forall x$$

2. By a simpler argument we can prove the weak  $L^1$ -estimate:

$$\{x \mid f^\star(x) > \epsilon\} \leq \frac{C_d}{\epsilon} \|f\|_{L^1(\mathbb{R}^d)}$$

(Hardy-Littlewood maximal function)

**Exercise 3.43** (E 3.2) Let  $1 \leq p, q, r \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ . Take  $f \in L^p, g \in L^q$ , prove:

$$\widehat{f \star g} = \hat{f} \hat{g}$$

Aufgabenstellung ..

*Solution.*

Step 1)  $f, g \in C_c^\infty(\mathbb{R}^d)$  (Fubini)

Step 2)  $f \in L^p, g \in L^q$ , find  $f_n, g_n \in C_c^\infty$  s.t.  $f_n \rightarrow f$  in  $L^p$ ,  $g_n \rightarrow g$  in  $L^q$ .  $\widehat{f_n \star g_n} = \hat{f}_n \hat{g}_n$  pointwise a.e. We have

Step 3)

$$\begin{aligned} & \|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^{r'}} \\ \text{(Hausdorff-Young)} & \leq \|f \star g\|_{L^r} = \|(f - f_n) \star g_n + f_n \star (g_n - g)\|_{L^r} \\ & \leq \|(f - f_n) \star g_n\|_{L^r} + \|f_n \star (g_n - g)\|_{L^r} \\ \text{(Young)} & \leq \|f - f_n\|_{L^p} \|g_n\|_{L^q} + \|f_n\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Moreover:

$$\begin{aligned} & \|\hat{f}_n \hat{g}_n - \hat{f} \hat{g}\|_{L^{r'}} = \|(\hat{f}_n \hat{f}) \hat{g}_n + \hat{f}(\hat{g}_n - \hat{g})\|_{L^{r'}} \\ \text{(Hölder)} & \leq \|\hat{f}_n - \hat{f}\|_{L^{p'}} \|\hat{g}_n\|_{L^{q'}} + \|\hat{f}\|_{L^{p'}} \|\hat{g}_n - \hat{g}\|_{L^{q'}} \\ \text{(Hausdorff-Young)} & \leq \|f_n - f\|_{L^p} \|g_n\|_{L^q} + \|f\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \rightarrow \infty} 0 \\ \text{So } \hat{f}_n \hat{g}_n & \rightarrow \hat{f} \hat{g} \text{ in } L^{r'} \quad \widehat{f \star g} = \hat{f} \hat{g} \text{ in } L^{r'} \quad \frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'} \end{aligned}$$

■

**Exercise 3.44** (E 3.3)  $f \in C_c^\infty(\mathbb{R}^d)$ . Prove  $|\hat{f}(k)| \leq \frac{C_N}{(1+|k|)^N}$

*Solution.* Since  $f \in C_c^\infty$  we have that  $D^\alpha f \in C_c^\infty$ . Recall

$$\widehat{D^\alpha f}(k) = (-2\pi i k)^\alpha \hat{f}(k)$$

For example

$$\begin{aligned} & \widehat{-\Delta f}(k) = |2\pi i k|^2 \hat{f}(k) \\ \text{(Induction)} & \rightsquigarrow \widehat{(-\Delta)^N f}(k) = |2\pi k|^{2N} \hat{f}(k) \end{aligned}$$

So we can conclude

$$\hat{f}(k) = \frac{\widehat{(-\Delta)^N f}(k)}{|2\pi k|^{2N}} \quad \forall k \in \mathbb{R}^d$$

1.  $f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in L^\infty$
2.  $(-\Delta)^N f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \widehat{(-\Delta)^N f} \in L^\infty$

Conclusion:  $\hat{f}(k) \leq \begin{cases} C & \forall k \\ \frac{C_N}{|k|^{2N}} & \forall k \end{cases}$  So  $\hat{f}(k) \leq \frac{C_N}{(1+|k|)^N}$

■

**Exercise 3.45** (E 3.4)

*Proof.* Siehe Goodnotes ■

**Exercise 3.46** (Bonus 3) Let  $f \in L^1(\mathbb{R}^d)$  such that

$$|\hat{f}(k)| \leq \frac{C_N}{(1 + |k|)^N}$$

for all  $k \in \mathbb{R}^d$ , for all  $N \geq 1$ . ( $C_N$  is independent of  $k$ ). Prove that  $f \in C^\infty(\mathbb{R}^d)$

( $f \in C^\infty$ ) i.e.  $\exists \tilde{f} \in C^\infty$  s.t.  $f = \tilde{f}$  a.e.

*My Solution.* First we regard for  $N \in \mathbb{N}$  and  $|k| \geq 1$ :

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{(1 + |k|)^N} dk &= \int_0^\infty \left( \int_{\partial B(0,r)} \frac{1}{(1 + r)^N} dS(y) \right) dr \\ &= \int_0^\infty \frac{1}{(1 + r)^N} |\partial B(0, r)| dr \\ &\leq \int_1^\infty \frac{1}{r^N} \underbrace{|\partial B(0, r)|}_{cr^{d-1}} dr \quad \text{for a } c \in \mathbb{R} \\ &= c \int_1^\infty \frac{1}{r^{N-d+1}} dr \end{aligned}$$

From Ana I we know that  $\int_1^\infty \frac{1}{r^{N-d+1}} dr < \infty$  is equivalent to  $N - d + 1 > 1 \Leftrightarrow N > d$ , so for  $N > d$  we have

$$\frac{1}{(1 + |k|)^N} \in L^1(\mathbb{R}^d).$$

Now let  $\alpha \in \mathbb{N}^d$ , then we have

$$k^\alpha = k_1^{\alpha_1} \dots k_d^{\alpha_d} = |k|^{\alpha_1} \dots |k|^{\alpha_d} = |k|^{\alpha_1 + \dots + \alpha_d} = |k|^{|\alpha|} = (1 + |k|)^{|\alpha|}.$$

By assumption we have for all  $N \geq 1$ :

$$k^\alpha \hat{f}(k) \leq k^\alpha \frac{C_n}{(1 + |k|)^N} \leq (1 + |k|)^{|\alpha|} \frac{C_n}{(1 + |k|)^N} = \frac{C_n}{(1 + |k|)^{N-|\alpha|}}$$

If we set  $N$  such that  $N - |\alpha| > d$ , for example  $N = d + |\alpha| + 1$ , then we can conclude that  $k^\alpha \hat{f} \in L^1(\mathbb{R}^d)$ . This implies  $\widehat{k^\alpha \hat{f}} \in L^\infty(\mathbb{R}^d)$ , so

$$\widehat{k^\alpha \hat{f}}(k) = \partial^\alpha \hat{\hat{f}}(k) = \partial^\alpha (\hat{f})^\vee(-k) = \partial^\alpha f(-k) \in L^\infty(\mathbb{R}^d).$$

This implies  $f \in C^\infty(\mathbb{R}^d)$ . ■

**Theorem 3.47** Take  $T \in D'(\mathbb{R})$ ,  $f \in C_c^\infty(\mathbb{R}^d) = D(\mathbb{R}^d)$ ,  $f_y(x) = f(x - y)$

a)  $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$  and  $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D_x^\alpha f_y)$

b)  $\forall g \in L^1(\mathbb{R}^d)$  and compactly supported

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(\underbrace{f \star g}_{\in C_c^\infty})$$

*Proof.* a)  $y \mapsto T(f_y)$  is continuous since  $y_n \rightarrow y$  in  $\mathbb{R}^d$ , then  $f_{y_n} \rightarrow f_y$  implies  $T(f_{y_n}) \rightarrow T(f_y)$ . Let's check that  $y \mapsto T(f_y) \in C^1$ :

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} = \lim_{h \rightarrow 0} T\left(\frac{f_{y-he_i} - f_y}{h}\right)$$

We have  $\frac{f_{y-he_i} - f_y}{h} \xrightarrow{h \rightarrow 0} (\partial_i f)_y$  in  $D(\mathbb{R}^d)$

- $\exists K$  compact set such that  $\text{supp}(f_{y-e_i} - f_y), \text{supp } \partial_i f \subseteq K$  as  $|h|$  small.

$$\begin{aligned} & \bullet \frac{f_{y-he_i}(x) - f_y(x)}{h} - (\partial_i f)_y(x) \\ &= \frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y) \end{aligned}$$

$$\left| \int_0^1 \partial_i f(x-y+the_i) dt - \partial_i f(x-y) \right| \xrightarrow{h \rightarrow 0} 0 \text{ uniformly in } x$$

Similary:

$$\begin{aligned} & \left| D_x^\alpha \left( \frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y) \right) \right| \\ &= \left| \frac{D^\alpha f(x-y+he_i) - D^\alpha f(x-y)}{h} - \partial_i(D^\alpha f)(x-y) \right| \xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

uniformly in  $x$ . Conclude:

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} \xrightarrow{h \rightarrow 0} T((\partial_i f)_y) \in C(\mathbb{R}^d)$$

So we get that  $y \mapsto T(f_y) \in C^1$  and  $-\partial_{y_i} T(f_y) = T((\partial_i f)_y)$

By induction:

$$D_y^\alpha T(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y) = (D^\alpha T)(f_y) \quad \forall \alpha \in \mathbb{N}^d$$

b) Heuristic:  $T = T(x)$

$$\begin{aligned} \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \int_{\mathbb{R}^d} g(y) \left( \int_{\mathbb{R}^d} T(x) f(x-y) dx \right) dy \\ &= \int_{\mathbb{R}^d} T(x) \left( \int_{\mathbb{R}^d} g(y) f(x-y) dy \right) dx \\ &= \int_{\mathbb{R}^d} T(x) (f \star g)(x) dx = T(f \star g) \end{aligned}$$

Step 1:  $g \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \text{(Riemann Sum)} \quad \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) T(f_{y_j}) \\ &= \lim_{\Delta_N \rightarrow 0} T \left( \Delta_N \sum_{j=1}^N g(y_j) f_{y_j} \right) \\ &= T(f \star g) \end{aligned}$$

because

$$\begin{aligned} \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f_{y_j}(x) &\rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d) \\ \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x-y_j) &\xrightarrow{\text{Riemann}} \int_{\mathbb{R}^d} g(y) f(x-y) dy = (f \star g)(x) \end{aligned}$$

Proof of:

$$\lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d)$$

1) Since  $f, g \in C_c^\infty$  we have  $f \star g \in C_c^\infty$ . And we have

$$x \mapsto \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \in C^\infty$$

since  $f \in C^\infty$  supported in  $(\text{supp } g + \text{supp } f)$ . So all functions are  $C_c^\infty$  and supported in  $(\text{supp } g + \text{supp } f)$ .

2)

$$\left| \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) - \int_{\mathbb{R}^d} g(y) f(x - y) dy \right| \xrightarrow{\Delta_N \rightarrow 0} 0$$

uniformly in  $x$ . (Result from the Riemann-Sum)

3)

$$\begin{aligned} & \left| D_x^\alpha (\Delta_N \sum_{j=1}^N g(y_j) f(x - y) - (f \star g)(x)) \right| \\ &= \left| \Delta_N \sum_{j=1}^N g(y_j) D^\alpha f(x - y) - (D^\alpha f) \star g(x) \right| \xrightarrow{\Delta_N \rightarrow 0} 0 \end{aligned}$$

uniformly in  $x$  for all  $\alpha$ .

Step 2: Take  $g \in L^1(\mathbb{R}^d)$  and compactly supported. Then  $\exists \{g_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ ,  $\text{supp } g_n \subseteq \text{supp } g + B(0, 1)$  such that  $g_n \rightarrow g$  in  $L^1(\mathbb{R}^d)$ . By Step 1:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy = T(g_n \star f)$$

Take  $n \rightarrow \infty$ :

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy \rightarrow \int_{\mathbb{R}^d} g(y) T(f_y) dy$$

since  $g_n \rightarrow g$  in  $L^1$  compactly supported and  $y \mapsto T(f_y) \in C^\infty \subseteq L^\infty(K)$ . Moreover (exercise):

$$\underbrace{g_n \star f}_{\in C_c^\infty} \rightarrow g \star f \quad \text{in } D(\mathbb{R}^d)$$

So  $T(g_n \star f) \xrightarrow{n \rightarrow \infty} T(g \star f)$ . Finally we obtain:

$$\int g(y) T(f_y) dy = T(g \star f) \quad \blacksquare$$

**Theorem 3.48** Let  $\Omega \subseteq \mathbb{R}^d$  be open. Let  $T_1 D'(\Omega)$  and  $f \in C_c^\infty(\Omega)$ . Denote

$$\Omega_f = \{y \in \mathbb{R}^d \mid \text{supp } f_y = y + \text{supp } f \subseteq \Omega\}$$

a)  $y \mapsto T(f_y) \in C^\infty(\Omega_f)$  and  $D_y^\alpha (T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y)$

b) For all  $g \in L^1(\Omega_g)$  compactly supported in  $\Omega_f$  it holds:

$$\int_{\Omega} g(y)T(f_y) dy = T(f \star g).$$

**Theorem 3.49** Let  $T \in D'(\Omega)$  s.t.  $\nabla T = 0$  in  $D'(\Omega)$ . Then:  $T = \text{const.}$  in  $\Omega$ .

*Proof.* ( $\Omega = \mathbb{R}^d$ ) for all  $f \in C_c^\infty$ ,  $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$  and  $\partial_{y_i} T(f_y) = (\partial_j T)(f_y) = 0$  for all  $i = 1, \dots, d$ . Then by the result of the theorem for  $C^\infty$  functions,  $y \mapsto T(f_y) = \text{const}$  independent of  $y$ . Consequently:

$$T(f_y) = T(f_0) = T(f) \quad \forall y \in \mathbb{R}^d \quad \forall f \in C_c^\infty(\mathbb{R}^d)$$

For any  $g \in C^\infty(\mathbb{R}^d)$ :

$$\left( \int_{\mathbb{R}^d} g dy \right) T(f) = \int_{\mathbb{R}^d} g(y)T(f_y) dy = T(f \star g) = T(g \star f) = \left( \int_{\mathbb{R}^d} f dy \right) T(g)$$

So  $\frac{T(f)}{\int_{\mathbb{R}^d} f}$  is independent of  $f$  (as soon as  $\int f \neq 0$ ). So we get that  $T(f) = \text{const} \int_{\mathbb{R}^d} f$ , where const is independent of  $f$ .  $\blacksquare$

**Remark 3.50** If  $u \in C^1(\mathbb{R}^d)$ , then:

$$u(x+y) - u(x) = \int_0^1 \sum_{j=1}^d y_j u(x + ty_j) dt = \int_0^1 y \nabla u(x + ty) dt$$

So we get that if  $\nabla u = 0$ , then  $u(x+y) - u(x) = 0$  for all  $x, y$ , so  $u = \text{const.}$

**Theorem 3.51** (Taylor expansion for distributions) Let  $T \in D'(\mathbb{R}^d)$  and  $f \in C_c^\infty(\mathbb{R}^d)$ . Then  $y \mapsto T(f_y) \in C^\infty$  and

$$T(f_y) - T(f) = \int_0^1 \sum_{j=1}^d y_j (\partial_j T)(f_{ty}) dt.$$

In particular, if  $g \in L_{loc}^1$  and  $\nabla g \in L_{loc}^1$ , then  $\forall y \in \mathbb{R}^d$ :

$$g(x+y) - g(x) = \int_0^1 g(x+ty)y dt$$

for a.e.  $x \in \mathbb{R}^d$ .

*Proof.*  $y \mapsto T(f_y)$  is  $C^\infty$  and  $\frac{d}{dt}[T(f_{ty})] = (\nabla T)(f_{ty})y$  So we get

$$\begin{aligned} T(f_y) - T(f) &= \int_0^1 \frac{d}{dt}(T(f_{ty})) dt \\ &= \int_0^1 (\nabla T)(f_{ty})y dt \\ &= \int_0^1 \sum_{j=1}^d (\partial_j T)(f_{ty})y_j dt \end{aligned} \quad \blacksquare$$

**Corrolary 3.52** Let  $g \in L^1_{loc}(\mathbb{R}^d)$  s.t.  $\partial_j g \in L^1_{loc}(\mathbb{R}^d)$  for all  $j = 1, 2, \dots, d$  (i.e.  $g \in W^{1,1}_{loc}(\mathbb{R}^d)$ ). Then for all  $y \in \mathbb{R}^d$ :

$$\begin{aligned} g(x+y) - g(x) &= \int_0^1 y \nabla g(x+ty) dt \\ &= \int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \end{aligned}$$

for a.e.  $x$ .

*Proof.* For all  $f \in C_c^\infty$  we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)[g(x+y) - g(x)] dx &= \int_{\mathbb{R}^d} g(x)[f(x-y) - f(x)] dx \\ &= g(f_y) - g(f) \\ &= \int_0^1 \sum_{j=1}^d y_j (\partial_j g)(f_{ty}) dt \\ &= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \sum_{j=1}^d y_j \left[ \int_{\mathbb{R}^d} (\partial_j g)(x) f_{ty}(x) dx \right] \\ &= \int_0^1 \sum_{j=1}^d y_j \left[ \int_{\mathbb{R}^d} (\partial_j g)(x+ty) f(x) dx \right] dt \\ &= \int_{\mathbb{R}^d} f(x) \left[ \int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \right] dx \end{aligned}$$

For all  $\phi \in C_c^\infty$ :  $= g(x+y) - g(x)$  a.e.  $x \in \mathbb{R}^d$ . ■

**Remark 3.53** If  $T \in D'(\Omega)$ ,  $\Omega \subseteq \mathbb{R}^d$  open, if  $y \nabla T = 0$ , then  $T = \text{const.}$

**Theorem 3.54** (Equivalence of the classical and distributional derivatives) Let  $\Omega \subseteq \mathbb{R}^d$ . Then the following are equivalent:

1.  $T \in D'(\Omega)$  s.t.  $\partial_{x_i} T = g_i \in C(\Omega)$  for all  $i = 1, \dots, d$ .
2.  $T = f \in C^1(\Omega)$  and  $g_i = \partial_{x_i} f$

*Proof.*

(2)  $\Rightarrow$  (1): If  $T = f \in C^1(\Omega)$ , then:  $\partial_{x_i} f \in C(\Omega)$ .

$$\partial_{x_i} T(\phi) = -T(\partial_{x_i} \phi) = - \int_{\Omega} f(\partial_{x_i} \phi) = - \int_{\Omega} (\partial_{x_i} f) \phi$$

for all  $\phi \in D(\Omega)$ , so  $\partial_{x_i} T = \partial_{x_i} f$ .

(1)  $\Rightarrow$  (2): Why is  $T = f$  with  $f$  continuous? As  $\partial_{x_i} f = g_i$ :

$$f(x+y) - f(x) = \int_0^1 \nabla f(x+ty) y dt = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i dt$$

So we get

$$f(y) = f(0) + \int_0^1 \sum_{i=1}^d g_i(ty) g_i dt.$$

We expect that  $f \in C^1$  and  $\partial_{x_i} f = g_i$ . But this is not trivial to prove.

$$\begin{aligned} \frac{f(y + he_i) - f(y)}{h} &= \int_0^1 \sum_{i=1}^d [g_i(ty + the_i)(y_i + h\delta_{ij})] dt \\ &= \int_0^1 g_i(ty + the_i) dt + \int_0^1 \sum_{j \neq i} \frac{[g_i(ty + the_i) - g_i(ty)]}{h} y_j dt \\ &\xrightarrow{h \rightarrow 0} \int_0^1 g_i(ty) dt + \text{is difficult ...} \end{aligned}$$

Lets take  $\phi \in C_c^\infty$ , then:

$$\begin{aligned} T(\phi_y) - T(\phi) &= \int_0^1 \underbrace{\nabla T}_{(g_i)_{i=1}^d}(\phi_{ty}) y dt \\ &= \int_0^1 \sum_{i=1}^d \left( \int_{\Omega} g_i(x) \underbrace{\phi_{ty}}_{=\phi(x-ty)} dx \right) dt \\ &= \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_0^1 g_i(x) \phi(x-ty) y_i dt \right) dx \\ &= \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_0^1 g_i(x+ty) \phi(x) y_i dt \right) dx \\ &= \int_{\mathbb{R}^d} \left( \sum_i \int_0^1 g_i(x+ty) y_i dt \right) \phi(x) dx \end{aligned}$$

Integrating against  $\psi(y)$  with  $\psi \in C_c^\infty$ :

$$\begin{aligned} &\int_{\mathbb{R}^d} T(\phi_y) \psi(y) dy - T(\phi) \int_{\mathbb{R}^d} \psi(y) dy \\ &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \sum_i \int_0^1 g_i(x+ty) y_i \psi(y) dt dy \right) \psi(x) dx \\ &\Rightarrow T(\phi \star \psi) - T(\phi) \int \psi = \dots \\ &\Rightarrow \int_{\mathbb{R}^d} T(\psi_y) \phi(y) dy - T(\phi) \int \psi = \dots \end{aligned}$$

Take  $\psi \in C_c^\infty(\mathbb{R}^d)$  such that  $\int \psi = 1$ . Then:

$$T(\phi) = \underbrace{\int_{\mathbb{R}^d} T(\psi_x) - \left( \int_{\mathbb{R}^d} \sum_{i=1}^d \int_0^1 g_i(x+ty) y_i \psi(y) dt dy \right)}_{f(x)} \phi(x) dx$$

for all  $\phi \in C_c^\infty$ , so  $T = f \in C(\Omega)$ . Thus  $T = f \in C(\Omega)$  and  $\partial_{x_i} T = g_i \in C(\Omega)$ . Then we need to prove that  $f \in C^1(\Omega)$  and  $\partial_{x_i} f = g_i$  (classical derivative). Since



$f \in W_{loc}^{1,1}$ :

$$f(x+y) - f(x) = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i dt \quad \forall x, y$$

In particular:

$$\begin{aligned} \frac{f(x+he_i) - f(x)}{h} &= \int_0^1 \frac{1}{h} \sum_{i=1}^d g_i(x+the_i) h \delta_{ij} dt \\ &= \int_0^1 g_i(x+the_i) dt \xrightarrow{h \rightarrow 0} g_i(x) \end{aligned}$$

So we get  $\partial_{x_i} f(x) = g_i(x) \in C(\Omega)$  in the classical sense. So  $f \in C^1(\Omega)$ .  $\blacksquare$

**Definition 3.55** (Sobolev Spaces) Let  $\Omega \subseteq \mathbb{R}^d$  be open. We define for  $1 \leq p \leq \infty$ :

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega) \quad \forall i = 1, \dots, d\}$$

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \quad \forall |\alpha| \leq k\}$$

$$W_{loc}^{k,p}(\Omega) = \{f \in L_{loc}^p(\Omega) \mid D^\alpha f \in L_{loc}^p(\Omega) \quad \forall |\alpha| \leq k\}$$

**Theorem 3.56** (Approximation of  $W_{loc}^{\frac{1}{p}}(\Omega)$  by  $C^\infty(\Omega)$ ) Let  $\Omega \subseteq \mathbb{R}^d$  be open, let  $f \in W_{loc}^{\frac{1}{p}}(\Omega)$ . Then there exists  $\{f_n\} \subseteq C^\infty(\Omega)$  such that  $f_n \rightarrow f$  in  $W_{loc}^{\frac{1}{p}}(\Omega)$ , i.e. for all  $K$  compact in  $\Omega$ :  $\|f_n - f\|_{L^p(K)} + \sum_{i=1}^d \|\partial_{x_i}(f_n - f)\|_{L^p(K)} \rightarrow 0$ .

*Proof.* Case  $\Omega = \mathbb{R}^d$ : Take  $g \in C_c^\infty$ ,  $\int g = 1$ ,  $g_\epsilon(x) = \epsilon^{-d} g(\epsilon^{-1}x)$ . Then  $g_\epsilon \star f$ . Then  $f \in L_{loc}^p(\Omega)$ , so  $g_\epsilon \star f \rightarrow f$  in  $L_{loc}^p$  as  $\epsilon \rightarrow 0$ . Moreover  $\partial_{x_i}(g_\epsilon \star f) \rightarrow \partial_{x_i} f$  in  $L_{loc}^p$ . Then we can take  $f_n = g_{\frac{1}{n}} \star f$ .  $\blacksquare$

**Remark 3.57** In general, if we want to compute the distributional derivative  $D^\alpha f$ , then we can find  $f_n \rightarrow f$  in  $D'(\Omega)$  and compute  $D^\alpha f_n$ . Then  $D^\alpha f_n \rightarrow D^\alpha f$  in  $D'(\Omega)$ . As an example we can compute  $\nabla|f|$  with  $f \in W_{loc}^{\frac{1}{p}}(\Omega)$ .

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

**Theorem 3.58** (Chain Rule) let  $G \in C^1(\mathbb{R}^1)$  with  $|\nabla G|$  is bounded. Let  $f = (f_i)_{i=1}^d \in W_{loc}^{\frac{1}{p}}(\Omega)$ . Then  $x \mapsto G(f(x)) \in W_{loc}^{\frac{1}{p}}(\Omega)$  and

$$\partial_{x_i} G(f) = \sum_{k=1}^d \partial_k G(f) \partial_{x_i} f_k \quad \text{in } D'(\Omega)$$

Moreover, if  $G(0) \in L^p(\Omega)$  (i.e. either  $|\Omega| < \infty$  or  $G(0) = 0$ ), then if  $f = (f_i)_{i=1}^d \in W_{loc}^{\frac{1}{p}}(\Omega)$ , then  $G(f) \in W_{loc}^{\frac{1}{p}}(\Omega)$ .

*Proof.* Since  $G \in C^1$  we have that  $G$  is bounded in any compact set. Moreover  $\|\nabla G\|_{L^\infty} < \infty$  implies:

$$|G(f) - G(0)| \leq \|\nabla G\|_{L^\infty} |f| \in L_{loc}^p$$

So  $G(f) \in L^p_{loc}$ . Let us compute  $\partial_{x_i} G(f)$ . Let  $\{f^{(n)}\}_{n=1}^\infty \subseteq C^\infty$  such that  $f^{(n)} \rightarrow f$  in  $W^{\frac{1}{p}}_{loc}$ , then:

$$|G(f^{(n)}) - G(f)| \leq \|\nabla G\|_{L^\infty} |f^{(n)} - f| \rightarrow 0 \text{ in } L^p_{loc}$$

So  $G(f^{(n)}) \rightarrow G(f)$  in  $L^p_{loc}$ , thus  $\partial_{x_i} G(f^{(n)}) \rightarrow \partial_{x_i} G(f)$  in  $D'(\Omega)$ . On the other hand, by the standard Chain-Rule for  $C^1$ -functions:

$$\begin{aligned} \partial_{x_i} G(f^{(k)}) &= \sum_{k=1}^d \underbrace{\partial_k G(f^{(k)})}_{\text{b.d.} \rightarrow \partial_k G(f)} \underbrace{\partial_i f_k^{(n)}}_{\rightarrow \partial_i f_k \text{ in } L^p(\Omega)} \\ &\rightarrow \sum_{k=1}^d \partial_k G(f) \partial_i f_k \text{ in } L^p_{loc}(\Omega) \end{aligned}$$

Thus

$$\partial_{x_i} G(f) = \sum_{k=1}^d \underbrace{\partial_k G(f)}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p_{loc}} \in L^p_{loc} \partial_{x_i} G(f) = \sum_{k=1}^d \underbrace{\partial_k G(f)}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p_{loc}} \in L^p_{loc} \text{ in } D'(\Omega)$$

So  $G(f) \in W^{\frac{1}{p}}_{loc}(\Omega)$ . Assume that  $G(0) \in L^p(\Omega)$  (i.e.  $|\Omega| < \infty$  or  $G(0) = 0$ ). If  $f \in W^{\frac{1}{p}}(\Omega)$ , then  $G(f) \in W^{\frac{1}{p}}(\Omega)$  since

$$|G(f) - G(0)| \leq \|\nabla G\|_{L^\infty} |f| \in L^p \Rightarrow G(f) \in L^p$$

and

$$\partial_{x_i} G(f) = \sum_k \underbrace{\partial_k G}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p} \in L^p \Rightarrow G(f) \in W^{\frac{1}{p}}(\Omega)$$

■

**Theorem 3.59** (Derivative of absolute value) Let  $\Omega \subseteq \mathbb{R}^d$  be open. Let  $f \in W^{\frac{1}{p}}(\Omega)$ . Then  $|f| \in W^{\frac{1}{p}}(\Omega)$  and if  $f$  is real-valued:

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

*Proof.* Exercise. Hint: Use the Chain-Rule for  $G_\epsilon(x) = \sqrt{\epsilon^2 + x^2} - \epsilon \rightarrow |x|$  as  $\epsilon \rightarrow 0$  ■

### 3.4 Distribution vs. measures

Let  $\mu$  be a Borel measure in  $\mathbb{R}^d$  s.t.  $\mu(K) < \infty$  for all compact  $K \subseteq \mathbb{R}^d$ . Then define

$$\begin{aligned} T : D(\mathbb{R}^d) &\longrightarrow \mathbb{C} \\ \phi &\longmapsto \int_{\mathbb{R}^d} \phi(x) d\mu(x) \quad \forall \phi \in C_c^\infty \end{aligned}$$

$\rightsquigarrow$  T is a distribution since if  $\phi_n \rightarrow \phi$  in  $D(\Omega)$ , then

$$|T(\phi_n) - T(\phi)| \leq \int_{\mathbb{R}^d} |\phi_n - \phi| d\mu(x) \leq \|\phi_n - \phi\|_{L^\infty} \left( \int_K d\mu \right) \rightarrow 0$$

**Example 3.60**  $\delta_0$  in  $D'(\mathbb{R}^d)$  is a Borel probability measure.

**Theorem 3.61** (Positive distributions are measures) Let  $\Omega \subseteq \mathbb{R}^d$  be open, let  $T \in D'(\Omega)$ . Assume  $T \geq 0$ , i.e.  $T(\phi) \geq 0$  for all  $\phi \in D(\Omega)$  satisfying  $\phi(x) \geq 0$  for all  $x$ . Then there is a Borel positive measure  $\mu$  on  $\Omega$  such that  $\mu(K) < \infty$  for all  $K \subseteq \Omega$  compact and:

$$T(\phi) = \int_{\Omega} \phi(x) d\mu(x) \quad \forall \phi \in D(\Omega)$$

*Proof.* See Lieb-Loss Analysis. Sketch: If  $O \subseteq \mathbb{R}^d$  is open, then

$$\mu(O) = \sup\{T(\phi) \mid \phi \in D(\Omega), 0 \leq \phi \leq 1, \text{supp } \phi \subseteq O\}$$

For all  $A \subseteq \Omega$  (not necessarily open),

$$\mu(A) = \inf\{\mu(O) \mid O \text{ open}, A \subseteq O\}$$

The mapping  $\mu : 2^{\Omega} \rightarrow [0, \infty]$  is an outer measure, i.e.

1.  $\mu(\emptyset) = 0$
2.  $\mu(A) \leq \mu(B)$  if  $A \subseteq B$
3.  $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$

From the outer measure we can find a  $\sigma$ -algebra  $\Sigma$  and  $\mu$  is a measure on  $\Sigma$  s.t.  $E$  is measurable iff

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c)$$

. So all open sets are measurable, thus outer regularity (by def  $\mu(A) = \inf\{\mu(O) \mid O \text{ open } \supseteq A\}$ ), so inner regularity  $\mu(A) = \sup\{\mu(K) \mid K \text{ compact } \subseteq A\}$ . ■

**Exercise 3.62** (E 4.1) Prove that if  $T_n \rightarrow T$  in  $D'(\mathbb{R}^d)$ , then  $D^{\alpha}T_n \rightarrow D^{\alpha}T$  in  $D^{\alpha}(\mathbb{R}^d)$  for all  $\alpha \in \mathbb{N}^d$ .

*My Solution.* See Goodnotes. ■

**Exercise 3.63** (E 4.2)

*My Solution.* See Goodnotes. ■

**Exercise 3.64** (E 4.3)  $f \in L^1(\mathbb{R}^d)$ ,  $\int f = 1$   $f_{\epsilon}(x) = \epsilon^{-d}f(\epsilon^{-1}x)$ . Then  $f_{\epsilon} \rightarrow \delta_0$  in  $D'(\mathbb{R}^d)$ .

*My Solution.* See Goodnotes. ■

**Exercise 3.65** (E 4.4)  $\{f_n\} \subseteq L^1$ ,  $\text{supp } f_n \subseteq B(0, 1)$ ,  $f_n \rightarrow f$  in  $L^1$ . Prove for all  $g \in C_c^{\infty}$  that  $f_n \star g \rightarrow f \star g$  in  $D(\mathbb{R}^d)$ .

*Solution.* Since  $f_n \in L^1$ ,  $\text{supp } f_n \subseteq B(0, 1)$ ,  $g \in C_c^{\infty}$  We have  $f_n \star g \in C_c^{\infty}$ ,  $\text{supp}(f_n \star g) \subseteq (\text{supp } g) + \overline{B(0, 1)} = K$ . Since  $f_n \rightarrow f$  in  $L^1$  there is a subsequence  $f_{n_k} \rightarrow f$  almost

everywhere, so  $f$  supp in  $\overline{B(0,1)}$ , so  $f \star g \in C_c^\infty$ ,  $\text{supp}(f \star g) \subseteq K$ . We have:

$$\begin{aligned} |f_n \star g(x) - f \star g(x)| &= \left| \int_{\mathbb{R}^d} (f_n(y) - f(y))g(x-y) dy \right| \\ &\leq \int_{\mathbb{R}^d} |f_n(y) - f(y)| |g(x-y)| dy \\ &\leq \|g\|_{L^\infty} \|f_n - f\|_{L^1} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

thus  $\|f_n \star g - f \star g\|_{L^\infty} \rightarrow 0$ . Similary:

$$\|D^\alpha(f_n \star g) - D^\alpha(f \star g)\|_{L^\infty} = \|f_n \star \underbrace{(D^\alpha g)_{\in C_c^\infty}}_{(D^\alpha g)} - f \star (D^\alpha g)\|_{L^\infty} \xrightarrow{n \rightarrow \infty} 0 \forall \alpha$$

So  $f_n \star g \rightarrow f \star g$  in  $D(\mathbb{R}^d)$ . ■

**Exercise 3.66** (E 4.5) Compute distributional derivatives  $f', f''$  of  $f(x) = x|x-1|$ .

*Solution.* We prove  $f'(x) = g(x) := \begin{cases} 2x-1 & x > 1 \\ 1-2x & x < 1 \end{cases}$ . Take  $\phi \in C_c^\infty(\mathbb{R}^d)$ .

$$\begin{aligned} -f'(\phi) &= - \int_{\mathbb{R}^d} f \phi' dy \\ &= - \int_{-\infty}^1 f \phi' dy - \int_1^\infty f \phi' dy \\ &= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 f' \phi dy + [f\phi]_1^\infty - \int_1^\infty f' \phi \\ &= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 f' \phi dy + [f\phi]_1^\infty - \int_1^\infty g \phi \\ &= f(1-)\phi(1) - f(1+)\phi(1) - \int_{\mathbb{R}^d} g \phi dy \\ &= 0 - \int_{\mathbb{R}^d} g \phi dy \end{aligned}$$

Now we compute  $f'' = g'$ . Take  $\phi \in C_c^\infty(\mathbb{R}^d)$ :

$$\begin{aligned} -(g')(\phi) &= \int_{\mathbb{R}^d} g \phi' \\ &= \int_{-\infty}^1 + \int_1^\infty \\ &= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 (-2)\phi - \int_1^\infty 2\phi \\ &= -2\phi(1) + \int_{-\infty}^\infty [2\mathbb{1}_{(1,\infty)}(x) - 2\mathbb{1}_{(-\infty,1)}(x)]\phi(x) dx \\ &= -2\delta_1(\phi) + \int_{-\infty}^\infty [2\mathbb{1}_{(1,\infty)}(x) - 2\mathbb{1}_{(-\infty,1)}(x)]\phi(x) dx \\ &\Rightarrow g' = \underbrace{2\delta_1}_{\notin L_{loc}^1} - \underbrace{2\mathbb{1}_{(-\infty,1)} + 2\mathbb{1}_{(1,\infty)}}_{\in L_{loc}^1} \end{aligned}$$

■

**Exercise 3.67** (Bonus 4) Let  $T \in D'(\mathbb{R}^d)$  and  $f \in D(\mathbb{R}^d)$ . Recall

$$(f \star T)(y) = T(\tilde{f}_y) = T(f(y - \cdot)) \in C^\infty(\mathbb{R}^d)$$

Prove that if  $\int_{\mathbb{R}^d} f = 1$  and  $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ , then:  $f_\epsilon \star T \rightarrow T$  in  $D'(\mathbb{R}^d)$  as  $\epsilon \rightarrow 0$

## Chapter 4

# Weak Solutions and Regularity

**Definition 4.1** Consider the linear PDE:

$$\sum_{\alpha} c_{\alpha} D^{\alpha} u(x) = F(x), \quad c_{\alpha} \text{ constant, } F \text{ given}$$

A function is called a weak solution (a distributional solution) if

$$\sum_{\alpha} c_{\alpha} D^{\alpha} u = F \quad \text{in } D'(\Omega).$$

Namely,

$$\sum_{\alpha} (-1)^{|\alpha|} c_{\alpha} \int_{\Omega} u(D^{\alpha} \phi) = \int_{\Omega} F \phi, \quad \forall \phi \in D(\Omega)$$

Regularity: Given some condition on the data  $F$ , what can we say about the smoothness of  $u$ ? Can we say that the equation holds in the classical sense? We derived  $G$  (the solution of the Laplace Equation) before in two ways:

1.  $\Delta G(x) = 0$  for all  $x \neq 0$ , assuming  $G(x) = G(|x|)$  and  $d \geq 2$
2.  $\hat{G}(k) = \frac{1}{|2\pi k|^2}$ ,  $d \geq 3$

**Theorem 4.2** For all  $d \geq 1$  we have  $G \in L^1_{loc}(\mathbb{R}^d)$  and  $-\Delta G = \delta_0$  in  $D'(\mathbb{R}^d)$ .

*Proof.* Take  $\phi \in D(\mathbb{R}^d)$ . Then:

$$\begin{aligned} (-\Delta G_y)(\phi) &= G_y(-\Delta \phi) = \int_{\mathbb{R}^d} G_y(x)(-\Delta \phi)(x) dx \\ &= \int_{\mathbb{R}^d} G(y-x)(-\Delta \phi)(x) dx \\ &= [G \star (-\Delta \phi)](y) = (-\Delta)(G \star \phi)(y) \end{aligned}$$

Recall for all  $f \in C^2$ ,  $-\Delta(G \star f) = f$  pointwise. So we can conclude  $-\Delta G_y = \delta_y$  in  $D'(\mathbb{R}^d)$ . ■

**Remark 4.3** In  $d = 1$ ,  $G(x) = -\frac{1}{2}|x|$ , so  $-G'(x) = \text{sgn}(x)/2$ , so  $-G''(x) = \delta_0$ .

**Remark 4.4** Formally:  $-\Delta(G_y \star \phi) = (-\Delta G_y) \star \phi(x) = (\delta_0 \star \phi)(x) = \int \delta_0(y) \phi(xy) dy = \delta_0(\phi(x - \bullet))$

**Theorem 4.5** (Poisson's equation with  $L^1_{loc}$  data) Let  $f \in L^1_{loc}(\mathbb{R}^d)$  s.t.  $\omega_d f \in L^1(\mathbb{R}^d)$  where

$$\omega_d(x) = \begin{cases} 1 + |x| & d = 1 \\ \log(1 + |x|) & d = 2 \\ \frac{1}{1 + |x|^{d-2}} & d \geq 3, \end{cases}$$

then  $u(x) = (G \star f)(x) \in L^1_{loc}(\mathbb{R}^d)$ . Moreover  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ . In fact,  $u \in W^{1,1}_{loc}(\mathbb{R}^d)$  and:

$$\partial_{x_i} u(x) = (\partial_{x_i} G) \star f(x) = \int_{\mathbb{R}^d} (\partial_{x_i} G)(x - y) f(y) dy$$

**Remark 4.6** We can also replace  $\mathbb{R}^d$  by  $\Omega$  and get  $-\Delta u = f$  in  $D'(\Omega)$ .

*Proof of Theorem 4.5.* First we check that  $u \in L^1_{loc}$ . Take any Ball  $B(0, R) \subseteq \mathbb{R}^d$ , prove  $\int_B |u| dy < \infty$ . We have

$$\begin{aligned} \int_B |u| dy &= \int_B \left| \int_{\mathbb{R}^d} G(x - y) f(y) dy \right| dx \\ &\leq \int_B \int_{\mathbb{R}^d} |G(x - y)| |f(y)| dy dx \\ &= \int_{\mathbb{R}^d} \left( \int_B |G(x - y)| dx \right) |f(y)| dy \end{aligned}$$

If  $y \notin B = B(0, R)$ , then by Newtons's theorem (Mean-value theorem):

$$\int_{B(0, R)} |G(x - y)| dx = |B(0, r)| |G(y)| \leq C |B| \omega_d(y)$$

If  $y \in B$ , then  $|y| \leq R$ , so  $|x - y| \leq 2R$  if  $x \in B$ .

$$\int_{B(0, R)} |G(x - y)| dx \leq \int_{|x-y| \leq 2R} |G(x - y)| dx = \int_{|z| \leq 2R} |G(z)| dz \leq c_R \text{ as } G \in L^1_{loc}$$

Thus

$$\int_B |u| dy \leq c_B \int_{|y| \geq R} \omega_d(y) |f(y)| dy + c_B \int_{|y| \leq R} |f(y)| dy < \infty$$

Let us prove  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ . Take  $\phi \in D(\mathbb{R}^d)$ . Then:

$$\begin{aligned}
(-\Delta u)(\phi) &= u(-\Delta \phi) \\
&= \int_{\mathbb{R}^d} u(x)(-\Delta \phi)(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)f(y)(-\Delta \phi)(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y-x)f(y)(-\Delta \phi)(x) dx dy \\
&= \int_{\mathbb{R}^d} [G \star (-\Delta \phi)](y)f(y) dy \\
&= \int_{\mathbb{R}^d} -\Delta(G \star \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} \phi(y)f(y) dy
\end{aligned}$$

So  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ . We check that  $\partial_i G \star f \in L_{loc}^1(\mathbb{R}^d)$ . Note that

$$|\partial_i G(x)| \leq c \frac{1}{|x|^{d-1}} \in L_{loc}^1(\mathbb{R}^d)$$

and

$$\int_{B(0,R)} |\partial_i G(x-y)| dx \leq \begin{cases} C_r \omega_d(y) & |y| \geq R \\ C_r & |y| \leq R \end{cases}$$

So  $\int_{B(0,R)} |(\partial_i G \star f)(y)| dy < \infty$  for all  $R > 0$ . For all  $\phi \in D(\mathbb{R}^d)$ :

$$\begin{aligned}
-(\partial_i u)(\phi) &= u(\partial_i \phi) = \int_{\mathbb{R}^d} u(x) \partial_i \phi(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)f(y) \partial_i \phi(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y-x)f(y) \partial_i \phi(x) dx dy \\
&= \int_{\mathbb{R}^d} (G \star \partial_i^y \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} (\partial_i^y G \star \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i^y G(y-x)f(y)\phi(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} -(\partial_i G)(x-y)f(y)\phi(x) dx dy \\
&= - \int_{\mathbb{R}^d} (\partial_i G \star f)(x)\phi(x) dx
\end{aligned}$$

So  $\partial_i u = \partial_i G \star f \in L_{loc}^1(\mathbb{R}^d)$ . Thus  $u \in L_{loc}^1$ ,  $\partial_i u \in L_{loc}^1$  for all  $i$ . So  $u \in W_{loc}^{1,1}(\mathbb{R}^d)$ . ■