Partial Differerential Equations

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Contents

Please note that I write this lecture notes for my personal use. I may write things differently than presented in the lecture. This script also contains some of my personal solutions for exercises (which may be wrong).

Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} f(t) = af(t) \text{ for all } t \geqslant 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geqslant 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \to \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \to \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \to 0} \frac{f(x+he_i) - f(x)}{h}$, where $e_i = (0, 0, \dots, 1, \dots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial x_1, \dots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha| = k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leqslant i,j \leqslant d}$

Definition 1.4 Given a function F. Then the equation of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u: \ \Omega \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$ is called a *PDE of order* k.

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1}u, D^{k-2}u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explizit solution! This is in many cases impossible.
- Prove a *well-posted theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continous and differentiable functions) Let $\Omega \subseteq R^d$ be open

$$\begin{aligned} C(\Omega) &= \{f: \ \Omega \to \mathbb{R} \mid f \text{ continuous}\} \\ C^k(\Omega) &= \{f: \ \Omega \to \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \leqslant k\} \end{aligned}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f: \ \Omega \to \mathbb{R} \text{ lebesgue measurable} \ \middle| \int_\Omega |f|^p d\lambda < \infty, \ 1 \leqslant p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leqslant k : D^\alpha f \in L^p(\Omega) \text{ exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in $\mathbb{R}^d$. If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then u is a harmonic function in Ω.

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu}\; dS(x) = \int_K \operatorname{div}(\vec{F})\; dx$$

where ν is the outward unit normal vector field on ∂K. Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu}\; dS(x) = \int_V \operatorname{div}(\nabla u)\; dx = \int_V \Delta u(x)\; dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Integration by parts):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

   where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f) g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \to \mathbb{R}$ be continuous. Prove that if $\int_B f(x)\; dx = 0$, then $u \equiv 0$ in Ω.

Theorem 2.5 (Fundamential Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x)\; dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in $\mathbb{R}^d$. Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \to \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2 + \cdots + x_d^2}\right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned}
\partial_{x_i} u &= \partial_{x_i} v = (\partial_r v)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r} \\
\partial_{x_i}^2 u &= \partial_{x_i}\left(v(r)'\frac{x_i}{r}\right) = (\partial_{x_i} v(r)')\frac{x_i}{r} + v'(r)\partial_{x_i}\left(\frac{x_i}{r}\right) \\
&= (\partial_r v'(r))\left(\frac{dr}{\partial_{x_i}}\right)\frac{x_i}{r} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}(\partial_{x_i} r)\right) = v'(r)\frac{x_i^2}{r^2} + v'r(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)
\end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d d_{x_i}^2\right) u = v''(r) + v'(r)(\frac{d}{r} - \frac{1}{r})$

Thus $\Delta u = v'(r) + v(r)\frac{d-1}{r}$. We consider $d \geqslant 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r)\frac{d-1}{r} = 0$

$\Rightarrow \log(v(r))' = \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)'$ (recall $log(f)' = \frac{f'}{f}$)

$\Rightarrow v'(r) = \frac{1}{v^{d-2} + \text{const.}}$

$$\begin{cases} \frac{const}{r^{d-2}} + const x x + const & , d \geqslant 3 \\ const \log(r) + const x x r + const & , d = 2 \end{cases}$$

Definition 2.7 (Fundamential Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|}\frac{1}{|x|^{d-2}}, & d \geqslant 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0,1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta\Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in $\mathbb{R}^d$. The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y)f(y)\ dy = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\ dy$$

This can be heuristically justifyfied with

$$-\Delta(\Phi \star f) = (-\Delta\Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C_c^2(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\,dy.$$

First we check that u is continuous: Take $x_k \to x_0$ in $\mathbb{R}^d$. We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y)\ dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y)\ dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n\to\infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{ for all } y \in \mathbb{R}^d \backslash \{0\}$$

and

$$|\Phi(y) f(x-y)| \leqslant \|f||_{L^\infty} \cdot \mathbb{1}(|y| \leqslant R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\operatorname{supp}(f)$ but independent of y. Now we compute the derivatives:

$$\partial_{x_i} u(x) = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x-y)\ dy = \lim_{h\to 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + he_i - y) - f(x-y)}{h}\ dy$$

$$\text{(dom. conv.)} \quad = \int \Phi(y) \partial_{x_i} f(x-y)\ dy$$

$$\Rightarrow \quad D^\alpha u(x) = \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x-y)\ dy \quad \text{for all } |\alpha| \leqslant 2$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y)(-\Delta_x) f(x-y)\, dy = \int_{\mathbb{R}^d} \Phi(y)(-\Delta_y) f(x-y)\, dy \\ &= \int_{\mathbb{R}^d \backslash B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy + \int_{B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d \backslash B(0,\epsilon)} \Phi(y)(-\Delta_y) f(x-y)\, dy \\ &= \int_{\mathbb{R}^d \backslash B(0,\epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x-y)\, dy - \int_{\partial B(0,\epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \\ &= \int_{\mathbb{R}^d \backslash B(0,\epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x-y)\, dy \\ &\quad + \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\, dS(y) - \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0,\epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1|\epsilon^{d-1}} \quad \text{ for } y \in \partial B(0,\epsilon)$$

Hence:

$$\begin{aligned} \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\ dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(0,\epsilon)} f(x-y)\ dS(y) \\ = ⨍_{\partial B(0,\epsilon)} f(x-y)\ dS(y) &= ⨍_{\partial B(x,\epsilon)} f(y)\ dS(y) \xrightarrow{\epsilon\to 0} f(x) \end{aligned}$$

We have to regard the following error terms:

- $$\left|\int_{B(0,\epsilon)} \Phi(y)(-\Delta_y)f(x-y)\ dy\right| \leqslant \int_{B(0,\epsilon)} |\Phi(y)|\underbrace{|-\Delta_y f(x-y)|}_{\leqslant \|\Delta f\|_{L^\infty}\mathbb{1}(|y|\leqslant R)}\ dy$$
$$\leqslant \|\Delta f\|_{L^\infty}\int_{\mathbb{R}^d}\underbrace{|\Phi(y)|\mathbb{1}(|y|\leqslant R)}_{L^1(\mathbb{R}^d)}\mathbb{1}(|y|\leqslant \epsilon)\xrightarrow{\epsilon\to 0} 0$$
Where $R>0$ depends on x and the support of f but is independent of y.
- $$\left|\int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f}{\partial \vec{n}}(x-y)\ dS(y)\right| \leqslant \|\nabla f\|_{L^\infty}\int_{\partial B(0,\epsilon)} |\Phi(y)|\ dy$$
$$\leqslant \begin{cases} const\cdot\epsilon|\log\epsilon|\to 0, & d=2\\ const\cdot\epsilon\to 0, & d\geqslant 3\end{cases}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x\in\mathbb{R}^d$ proved that $u=\Phi\star f$ and $f\in C_c^2(\mathbb{R}^d)$. ∎

Thus, if $f\in C_c^2(\mathbb{R})$, then $u=\Phi\star f$ satisfies $u\in C^2(\mathbb{R}^2)$ and $-\Delta u(x)=f(x)$ for all $x\in\mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f. For example if $f\in C_c^1(\mathbb{R})$ we can easily extend the previous proof:

$$\partial_{x_i}u = \int_{\mathbb{R}^d}\Phi(y)\partial_{x_i}f(x-y)\,dy \in C(\mathbb{R}^d)\Rightarrow u\in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i}\partial_{x_j}u = \partial_{x_i}\int_{\mathbb{R}^d}\Phi(y)\partial_{x_j}f(x-y)\,dy = \int_{\mathbb{R}^d}\partial_{x_i}\Phi(y)\partial_{x_j}f(x-y)\,dy\in C(\mathbb{R}^d)$$

So we have $u\in C^2(\mathbb{R}^d)$. Now we can compute

$$\Delta u = \sum_{i=1}^d\int_{\mathbb{R}^d}\partial_{x_i}\Phi(y)\partial_{x_i}f(x-y)\,dy \overset{(IBP)}{=} f(x).$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega\subseteq\mathbb{R}$ be open, let $u\in C^2(\Omega)$ and $\Delta u=0$ in Ω. Then

$$u(x) = ⨍_{B(x,r)} u = ⨍_{\partial B(x,r)} u \quad \text{for all } x\in\Omega, B(x,r)\subseteq\Omega$$

Proof. Consider all $r>0$ s.t. $B(x,r)\subseteq\Omega$,

$$f(r) = ⨍_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r. When it is done, then we immediately obtain

$$f(r) = \lim_{t\to 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$
\begin{aligned}
f'(r) &= \frac{d}{dr}\left(⨍_{\partial B(0,r)} u(x+y)\,dS(y)\right)\\
&= \frac{d}{dr}\left(⨍_{\partial B(0,1)} u(x+rz)\,dS(z)\right)\\
\text{(dom. convergence)} \quad &= ⨍_{\partial B(0,1)} \frac{d}{dr}[u(x+rz)]\,dS(z)\\
&= ⨍_{\partial B(0,1)} \nabla u(x+rz)z\,dS(z)\\
&= ⨍_{\partial B(x,r)} \nabla u(y)\frac{y-x}{r}\,dS(y)\\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}}\int_{\partial B(x,r)} \nabla \cdot u(y)\cdot \vec{n_y}\,dS(y)\\
\text{(Gauss-Green 2.2)} \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}}\int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0}\,dy = 0 \qquad \blacksquare
\end{aligned}
$$

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$ (Linear Equation)

Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, \omega \in S^{d-1}$, then

$$\int_{B(0,r)} g(y)\,dy = \int_0^r \left(\int_{B(0,r)} g(y)\,dS(y)\right) dr$$

Remark 2.15 We already proved that for u harmonic we have $u(x) = ⨍_{\partial B(x,r)} u\,dy$. Now we have

$$
\begin{aligned}
\int_{B(x,r)} u(y)\,dy &= \int_{B(0,r)} u(x+y)\,dy\\
\text{(Pol. decomposition)} \quad &= \int_0^r \left(\int_{\partial B(0,s)} u(x+y)\,dS(y)\right) ds\\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y)\,dS(y)\right) ds\\
\text{(Mean value property)} \quad &= \int_0^r (|\partial B(x,s)|\,u(x))\,ds = |B(x,r)|\,u(x)
\end{aligned}
$$

This implies

$$⨍_{B(x,r)} u(y)\,dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.16 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = ⨍_{B(x,r)} u(y)\,dy = ⨍_{\partial B(x,r)} u(y)\,dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then u is harmonic, i.e. $\Delta u = 0$ in Ω. (The proof is exactly like before)

Theorem 2.17 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω. Then

a) $\max_{x\in\bar{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$

b) Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \max_{x\in\bar{\Omega}} u(x)$, then $u \equiv \mathit{const.}$ in Ω.

Proof. Given $U \subseteq \mathbb{R}^d$ open, we can write $U = \bigcup_i U_i$, where U_i is open and connected.

b) Assume that Ω is connected and there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{y\in\Omega} u(x)$. Define $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$. U is closed since u is continuous. Moreover, U is open by the mean-value theorem. I.e. for all $x \in U$ there is a $r > 0$ s.t. $B(x,r) \subseteq U \subseteq \Omega$. Since U is connected we get $U = \Omega$, so u is constant in Ω. On the other hand, if there is no $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x\in\Omega}$ we have $\forall x_0 \in \Omega : \quad u(x) < \sup_{x\in\bar{\Omega}} u(x) = \sup_{x\in\partial\Omega} u(x)$

a) Given $\Omega \subseteq \mathbb{R}^d$ open, we can write $\Omega = \bigcup_i \Omega_i$, where Ω_i is open and connected. By b) we have

$$\sup_{x\in\bar{\Omega}_i} u(x) = \sup_{x\in\partial\Omega_i} u(x), \quad \forall i$$

So we can conclude

$$\sup_{x\in\bar{\Omega}} u(x) = \sup_{x\in\partial\Omega} u(x).$$

■

Definition 2.18

- If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$, then u is called *sub-harmonic* if $\Delta u \geqslant 0$ in Ω.
- If $\Delta u \leqslant 0$, then u is called *super-harmonic*.

Exercise 2.19 (E 1.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$ be subharmonic.

a) Prove that u satisfies the Mean Value Inequality

$$⨍_{\partial B(x,r)} u(y)\, dS(y) \geqslant ⨍_{B(x,r)} u(y)\, dy \geqslant u(x)$$

for all $B(x,r) \subseteq \mathbb{R}^d$.

b) Assume further that Ω is connected and $u \in C(\bar{\Omega})$. Prove that u satisfies the strong maximum principle, namely either

- u is constant in Ω, or
- $\sup_{y\in\partial\Omega} u(y) > u(x)$ for all $x \in \Omega$.

My Solution. a) Let $f(r) = \fint_{\partial B(x,r)} u(y)\, dS(y)$, then we have

$$\begin{aligned}
\partial_r f(r) &= \partial_r \fint_{\partial B(x,r)} u(y)\, dS(y) \\
\text{(Dom. Convergence)} \quad &= \fint_{\partial B(x,r)} \partial_r u(y)\, dS(y) \\
&= \fint_{\partial B(0,1)} \partial_r u(x+yr)\, dS(y) \\
&= \fint_{\partial B(0,1)} \nabla u(x+yr) \cdot y\, dS(y) \\
&= \fint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\, dS(y) \\
&= \fint_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}_y\, dS(y) \\
\text{(Gauss-Green)} \quad &= \fint_{B(x,r)} \operatorname{div}(\nabla u(y))\, dS(y) \\
&= \fint_{B(x,r)} \underbrace{\Delta u(y)}_{\geqslant 0}\, dS(y) \geqslant 0
\end{aligned}$$

So we can conclude that

$$\fint_{\partial B(x,r)} u(y)\, dS(y) = f(r) \geqslant \lim_{r \to 0} f(r) = u(x).$$

Now regard

$$\begin{aligned}
\int_{B(x,r)} u(y)\, dy &= \int_0^r \left(\int_{\partial B(x,r)} u(y)\, dS(y) \right) ds \\
&= \int_0^r \left(|\partial B(x,r)| \fint_{\partial B(x,r)} u(y)\, dS(y) \right) ds \\
&\geqslant \int_0^r |\partial B(x,r)| \cdot u(x)\, dS(y) \\
&= u(x) \int_0^r |\partial B(x,r)|\, dS(y) = u(x)|B(x,r)|.
\end{aligned}$$

Thus we have

$$u(x) \leqslant \fint_{B(x,r)} u(y) dy.$$

Finally, lets regard

$$\begin{aligned}
\int_{B(x,r)} u(y)\, dy &= \int_0^r \left(|\partial B(x,s)| \fint_{\partial B(x,s)} u(y)\, dS(y) \right) ds \\
(\partial_r f(r) \geqslant 0) \quad &\leqslant \int_0^r \left(|\partial B(x,s)| \fint_{\partial B(x,r)} u(y)\, dS(y) \right) ds \\
&= \fint_{\partial B(x,r)} u(y)\, dS(y) \int_0^r |\partial B(x,s)|\, ds \\
&= \fint_{\partial B(x,r)} u(y)\, dS(y) \cdot |B(x,s)|
\end{aligned}$$

and we conclude

$$\fint_{B(x,r)} u(y)\,dy \leqslant \fint_{\partial B(x,r)} u(y)\,dS(y).$$

b) Let $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x\in\Omega} u(x)$. Now,

$$\begin{aligned}\sup_{x\in\Omega} u(x) = u(x_0) &\leqslant \fint_{\partial B(x_0,r)} u(y)\,dy \\ &\leqslant \fint_{\partial B(x_0,r)} \sup_{x\in\Omega} u(x)\,dy = \sup_{x\in\Omega} u(x)\end{aligned}$$

Since u is continous we get $u(y) = u(x_0)$ for all $y \in B(x_0, r)$, so u is constant.

■

Definition 2.20 The *Poisson Equation* for given f, g on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Theorem 2.21 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open and connected. Let $f \in C(\Omega), g \in C(\partial\Omega)$. Then there exists *at most* one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof. Assume that we have two solutions u_1 and u_2. Then $u := u_1 - u_2$ is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that $u = 0$ in Ω. More precisely, by the maximum principle we have $\forall x \in \Omega$

$$\sup_{x\in\Omega} u(x) \leqslant \sup_{x\in\partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leqslant 0$$

Since $-u$ satisfies the same property we have $\forall x \in \Omega$:

$$\sup_{x\in\Omega}(-u(x)) \leqslant \sup_{x\in\partial\Omega}(-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leqslant 0 \quad \Rightarrow \quad u(x) \geqslant 0$$

So we geht $u(x) = 0$ in Ω. ■

Exercise 2.22 (Bonus 1) Let Ω be open, connected and bounded in $\mathbb{R}^d$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Prove that

a) If $g \geqslant 0$ on $\partial\Omega$, then $u \geqslant 0$ in Ω.

b) If $g \geqslant 0$ on $\partial\Omega$ and $g \neq 0$, then $u > 0$ in Ω.

Lemma 2.23 (Estimates for derivatives) If u is harmonic in $\Omega \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| = N$ and $B(x_0, r) \subseteq \Omega$, then

$$|D^\alpha u(x)| \leqslant \frac{(c_d N)^N}{r^{d+N}} \int_{B(x,r)} |u|\, dy$$

Proof. Induction: Assume $|\alpha| = N-1$, Take $|\alpha| = N$

$$|D^\alpha u(x_0)| \leqslant \frac{|S_1|}{|B_1|\frac{r}{N}} \|D^\beta u\|_{L^\infty(B(x_0,\frac{r}{n}))}, \quad D^\alpha u = \partial_{x_i}(D^\beta u)_{|\beta|=N-1}$$

Note: $x \in B(x_0, \frac{r}{N})$, so $B(x, \frac{r(N-1)}{N}) \subseteq B(x_0, r)$. By the induction hypothesis:

$$\|D^\beta u\|_{L^\infty(B(x_0,\frac{r}{N}))} \leqslant \frac{[c_d(N-1)]^{N-1}}{[r\frac{(N-1)}{N}]^{d+N-1}} \int_{B(x_0,r)} |u|\, dy$$

The conclusion is:

$$\begin{aligned}
|D^\alpha u(x_0)| &\leqslant \frac{|S_1|}{|B_1|\frac{r}{N}} \frac{[c_d(N-1)]^{N-1}}{\left(r\frac{N-1}{N^d}\right)^{d+N-1}} \int_{B(x_0,r)} |u|\, dy \\
&= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{\left(\frac{r}{N}\right)^{d+N} (N-1)^d} \int_{B(x_0,r)} |u|\, dy \\
&= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{\left(\frac{r}{N}\right)^{d+N} N^d} \left(\frac{N}{N-1}\right)^d \int_{B(x_0,r)} |u|\, dy \\
&\leqslant \frac{2^d|S_1|}{|B_1|} \frac{c_d^{N-1} N^N}{r^{d+N}} \int_{B(x_0,r)} |u|\, dy \qquad \text{if } c_d \geqslant \frac{2^d|S_1|}{|B_1|}
\end{aligned}$$

■

Theorem 2.24 (Regularity) Let Ω be open in $\mathbb{R}^d$. Let $u \in C(\Omega)$ satisfy $u(x) = ⨍_{\partial B} u\, dy$ for any $x \in B(x,r) \subseteq \Omega$. Then u is a harmonic function in Ω. Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω.

Proof. We use the convolution. For simlicity consider the case $\Omega = \mathbb{R}^d$ first. Take $\eta \in C_c^\infty(\mathbb{R}^d)$ with $0 \leqslant \eta \leqslant 1$, $\eta(x) = 0$ if $|x| \geqslant 1$, η radial and $\int \eta = 1$. Define $\eta_\epsilon(x) = \epsilon^{-d}\eta(\epsilon^{-1}x)$ for all $\epsilon > 0$. Then

$$\int_{\mathbb{R}^d} \eta_\epsilon = \int_{\mathbb{R}^d} \eta = 1$$

We prove $u_\epsilon := \eta_\epsilon \star u = u$ for all $\epsilon > 0$. By definition:

$$\begin{aligned}
u_\epsilon(x) &= \int_{\mathbb{R}^d} \eta_\epsilon(x-y)u(y)\,dy \\
&= \int_0^\infty \left[\int_{\partial B(x,r)} \eta_\epsilon(x-y)u(y)\,dS(y)\right] dr \\
(\eta \text{ radial}) \quad &= \int_0^\infty \left[\eta_\epsilon(r)\int_{\partial B(x,r)} u(y)\,dS(y)\right] dr \\
(\text{Assumption}) \quad &= \int_0^\infty \eta_\epsilon(r)\,|\partial B(x,r)|\,u(x)\,dr \\
&= u(x)\int_0^\infty \eta_\epsilon(r)|\partial B(0,r)|\,dr \\
&= u(x)\int_{\mathbb{R}^d} \eta_\epsilon(y)\,dy = u(x)
\end{aligned}$$

On the other hand, $u_\epsilon = \eta_\epsilon \star u$ is $C^\infty(\mathbb{R}^d)$. In fact $D^\alpha(\eta_\epsilon \star u) = (D^\alpha \eta_\epsilon) \star u$ is continuous for any α (Exercise). Then $u \in C^\infty(\mathbb{R}^d)$, so u is harmonic in $\mathbb{R}^d$, i.e. $\Delta u = 0$ in $\mathbb{R}^d$.

Consider now the general case where $\Omega \subseteq \mathbb{R}^d$ is open. Take $\epsilon > 0$ small and define $\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}$. Define

$$u_\epsilon(x) = \int_{\mathbb{R}^d} \eta_\epsilon(x-y)u(y)\,dy \qquad \text{for all } x \in \Omega_\epsilon$$

Recall that $\eta_\epsilon(y) = 0$ if $|y| \geqslant \epsilon$, then:

$$u_\epsilon(x) = \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)\,dy$$

is well-defined since $B(x,\epsilon) \subseteq \Omega$ for all $x \in \Omega_\epsilon$. Then by the same computation using the polar-decomposition, we find that $u_\epsilon(x) = u(x)$ for all $x \in \Omega$. Note that $u_\epsilon \in C^\infty(\Omega_\epsilon)$. Taking $\epsilon \to 0$, we get $u \in C^\infty(\Omega)$. Then we conclude that u is harmonic (We need to reverse the proof of the mean-value theorem).
To proof that u is analytic, we need to show that for all $x_0 \in \Omega$, there is a $r > 0$ s.t. $B(x_0, r) \subseteq \Omega$ and

$$u(x) = u(x_0) + \sum_{\alpha \neq 0} c_\alpha (x - x_0)^\alpha \quad \text{for all } x \in B(x_0, r)$$

Here $\alpha = (\alpha_1, \ldots, \alpha_d), \alpha_i \in \{0, 1, 2, \ldots\}$ and $y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_d^{\alpha_d}$. We want to prove that the series converges uniformly in $B(x_0, r)$. Recall the Taylor expansion:

$$u(x) = u(x_0) + \sum_{0<|\alpha|<N} D^\alpha u(x_0)\frac{(x-x_0)^\alpha}{\alpha!} + R_N(x)$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $\alpha! = \alpha_1! \cdots \alpha_d!$ and

$$R_N(x) = \sum_{|\alpha|=N} \int_0^1 D^\alpha u(x_0 + t(x-x_0))\frac{(x-x_0)^\alpha}{\alpha!}\,dt$$

New: Let $x_0 \in \Omega$, take $r > 0$, $r < \frac{1}{L+1}\operatorname{dist}(x_0, \Omega^c)$ s.t. if $x \in B(x_0, r)$, then

$$B(x, Lr) \subseteq B(x_0, (L+1)r) \subseteq \Omega.$$

With Lemma 2.23 we get:

$$|D^\alpha u(x_0 + t(x - x_0)) \leqslant \frac{(c_d N)^N}{(Lr)^{d+N}} \int_{B(x,Lr)} |u|$$

With $(x_0, r) \rightsquigarrow (x, Lr)$

$$|R_N(x)| \leqslant \sum_{|\alpha|=N} \frac{(c_d N)^N}{(Lr)^{d+N}} \frac{1}{\alpha!} \frac{1}{\alpha!} \tau^N \int_{B(x_0,(L+1)r)|u|}$$

Thus

$$\begin{aligned} \left(\frac{\tilde{c}_d N}{L}\right) \frac{1}{N!} &\leqslant \left(\frac{\tilde{c}_d N}{L}\right) \left(\frac{e}{N}\right)^N \qquad \text{if } N \text{ large} \\ &= \left(\frac{\tilde{c}_d e}{L}\right)^N \xrightarrow{N\to\infty} 0 \quad \text{if } L > \tilde{c}_d e (L = L_d) \end{aligned}$$

We conclude that

$$u(x) = u(x_0) + \sum_{\alpha \neq 0} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha$$

The series converges uniformly $x \in B(x_0, r)$. Now we proof the bound on derivatives. For $\alpha = 0$

$$|u(x_0)| = \left| \fint_{B(x_0,r)} u \right| \leqslant \frac{1}{|B_1| r^d} \int_{B(x_0,r)} |u|$$

For $\alpha = 1 : \Delta u = 0$ in $\Omega \Rightarrow 0 = \partial_{x_i}(\Delta u) = \Delta(\partial_{x_i} u)$, so $\partial_{x_i} u$ is harmonic in Ω. Hence, by the mean-varlue theorem again:

$$\partial_{x_i} u(x_0) = \fint_{B(x_0,\frac{r}{2})} \partial_{x_i} u = \frac{1}{|B_1| \frac{\frac{r}{2}}{2}^d} \int_{B(x_0,\frac{r}{2})} \partial_{x_i} u = \frac{1}{|B_1| \frac{r}{2}^d} \int_{\partial B(x_0,\frac{r}{2})} u n_i \, dS$$

So we get:

$$\begin{aligned} |\partial_{x_i} u(x_0)| &= \frac{1}{|B_1| r^d} \int_{\partial B(x,r)} dS \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))} \\ &= \frac{|S_1|}{|B_1| \frac{r}{2}} \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))} \end{aligned}$$

For any $y \in \partial B(x_0, \frac{r}{2})$ by the mean value theorem, we get:

$$|u(y)| = \left| \fint_{B(y,\frac{r}{2})} u \right| \leqslant \frac{1}{|B_1| \left(\frac{r}{2}\right)^d} \int_{B(y,\frac{r}{2})} |u| \leqslant \frac{1}{|B_1| \left(\frac{r}{2}\right)^d} \int_{B(x_0,r)} |u|$$

Thus,

$$|\partial_{x_i} u(x_0)| \leqslant \frac{|S_1|}{|B_1| \left(\frac{r}{2}\right)} \frac{1}{|B_1| \left(\frac{r}{2}\right)^d} \int_{B(x_0,r)} |u| \leqslant \frac{c_d}{r^{d+1}} \int_{B(x_0,r)} |u|$$

Induction: Assume that we already proved the bound when $|\alpha| = N - 1$. Then:

$$\partial_{x_i} D^\alpha u = D^\alpha (\underbrace{\partial_{x_i} u}_{\text{harmonic}}) = 0 \quad \Rightarrow \quad D^\alpha u \text{ is harmonic}$$

So we get

$$\begin{aligned} \partial_{x_i}(D^\alpha u) &= \fint_{B(x_0,\frac{r}{4})} \partial_{x_i}(D^\alpha u) \\ \Rightarrow \quad |\partial_{x_i}(D^\alpha u)| &\leqslant \frac{C_d}{r^{d+1}} \int_{B(x_0,\frac{r}{2})} |D^\alpha u| \end{aligned}$$

and by the induction hypothesis:

$$\begin{aligned} |D^\alpha u(x_0)| &\leqslant \frac{c_d}{r} \|D^\alpha u\|_{L^\infty B(x_0,\frac{r}{2})} \\ &\leqslant \frac{c_d}{r^{d+N-1}} \int_{B(x_0,r)} |u| \quad \forall x \in B\left(x_0, \frac{r}{2}\right) \end{aligned}$$

Then: $|\partial_{x_i} D^\alpha u(x_0)| \leqslant \frac{c_d}{r^{d+N}} \int_{B(x_0,r)} |u|$ ■

Exercise 2.25 (E 1.1: Proof the Gauss–Green formula) Let $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that for every open ball $B(y,r) \subseteq \mathbb{R}^d$ we have

$$\int_{\partial B(y,r)} f(y) \cdot \nu_y \, dS(y) = \int_{B(y,r)} \operatorname{div} f \, dx.$$

Here ν_y is the outward unit normal vector and dS is the surface measure on the sphere.

Solution. We proof this in d=3. Let $f \in C^1(\mathbb{R}^3)$

$$\int_{B(0,1)} \partial_{x_3} f \, dx = \int_{\partial B(0,1)} f x_3 \, dS(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \vec{n} = \frac{x}{|x|} \text{ on } \partial B(0,1)$$

$$\begin{aligned} B(0,1) &= \{x_1^2 + x_2^2 + x_3^2 \leqslant 1\} \\ &= \{x_1^2 + x_2^2 \leqslant 1 - \sqrt{1 - x_1^2 - x_2^2} \leqslant x_3 \leqslant \sqrt{1 - x_1^2 - x_2^2}\} \end{aligned}$$

Then:

$$\begin{aligned} \int_{B(0,1)} \partial_{x_3} f \, dx &= \int_{x_1^2+x_2^2 \leqslant 1} \left(\int_{-\sqrt{1-x_1^2-x_2^2} \leqslant x_3 \leqslant \sqrt{1-x_1^2-x_2^2}} \partial_{x_3} f \, dx_3 \right) dx_1 \, dx_2 \\ &= \int_{x_1^2+x_2^2 \leqslant 1} \Big[f(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) \\ &\qquad - f(x_1, x_2, -\sqrt{1 - x_1^2 - x_2^2}) \Big] \, dx_1 \, dx_2 \end{aligned}$$

Lets take polar coordinates in 2D:

$$\begin{aligned} x_1 &= r \cos \phi & r > 0, \phi \in [0, 2\pi) \\ x_2 &= r \sin \phi & \det \frac{\partial(x_1, x_2)}{\partial(r, \phi)} = r \end{aligned}$$

$$(\star) = \int_0^1 \int_0^{2\pi} [f(r\cos\phi, r\sin\phi, r) - f(r\cos\theta, r\sin\phi, -r)] r \, dr \, d\phi$$

On the other hand:

$$\int_{\partial B(0,1)} f x_3 \, dS$$

The polar coordinates in 3D are:

$$\begin{aligned} x_1 &= r\cos\phi sin\theta & \qquad & r>0, \phi\in(0,2\pi), \theta\in(0,\pi) \\ x_2 &= r\sin\phi\sin\theta & & \det\frac{\partial x_1, x_2, x_3}{\partial(r,\phi,t)} = r^2\sin\theta \\ x_3 &= \cos\theta \end{aligned}$$

Then:

$$\begin{aligned} (\star\star) &= \int_0^{2\pi}\int_0^{\pi} f(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)\sin\theta\cos\theta\, d\theta\, d\phi \\ &= \int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} d\theta\right) d\phi \\ (r=\sin\theta) \qquad &= \int_0^{2\pi}\int_0^1 f(r\cos\phi, r\sin\phi, \sqrt{1-r^2}) r\, dr\, d\phi \\ &\quad - f(r\cos\phi, r\sin\phi, -\sqrt{1-r^2}) r\, dr\, d\phi \end{aligned}$$

■

Exercise 2.26 (E 1.2) Let $u \in C(\mathbb{R}^d)$ and $\int_{B(x,r)} u\, dy = 0$ for every open ball $B(x,r) \subseteq \mathbb{R}^d$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

My Solution. Assume there is a $x_0 \in \mathbb{R}^d$ s.t. w.l.o.g. $u(x_0) > 0$. Since u is continous there is a ball $B(x_0, r)$ s.t. $u(y) > \frac{u(x_0)}{2}$ for all $y \in B(x_0, r)$. But then we get

$$\int_{B(x_0,r)} u(y)\, dy \geqslant \int_{B(x_0,r)} \frac{u(x_0)}{2}\, dy = \frac{u(x_0)}{2}|B(x_0,r)| > 0.$$

■

Exercise 2.27 (E 1.3) Let $f \in C_c^1(\mathbb{R}^d)$ with $d \geqslant 2$ and $u(x) := (\Phi \star f)(x)$. Prove that $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ (2.9 was the same for $f \in C_1(\mathbb{R})$)

Theorem 2.28 (Liouville's Theorem) If $u \in C^2(\mathbb{R}^d)$ is harmonic and bounded, then $u = const.$

Proof. By the bound of the derivative 2.23 we have

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\leqslant \frac{c_d}{r^{d+1}}\int_{B(x_0,r)} |u|\, dy \quad \forall x_0 \in \mathbb{R}^d\ \forall r > 0 \\ &\leqslant \|u\|_{L^\infty}\frac{c_d}{r^{d+1}}|B(x_0,r)| \\ &\leqslant \|u\|_{L^\infty}\frac{c_d}{r} \xrightarrow{r\to\infty} 0 \end{aligned}$$

Thus $\partial_{x_i} u = 0$ for all $i = 1, 2, \ldots d$ and $u = const.$ in $\mathbb{R}^d$ ■

Theorem 2.29 (Uniqueness of solutions to Poisson Equation in $\mathbb{R}^d$) If $u \in C^2(\mathbb{R}^d)$ is a bounded function and satisfies $-\Delta u = f$ in $\mathbb{R}^d$ where $f \in C_c^2(\mathbb{R}^d)$, then we have

$$u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y)\, dy + C \quad \forall x \in \mathbb{R}^d$$

where C is a constant and Φ is the fundamental solution of the Laplace equation in $\mathbb{R}^d$.

Proof. If we can prove that v is bounded, then $v = const.$. We first need to show that $\Phi \star f$ is bounded.

$$\Phi = \Phi_1 + \Phi_2 = \Phi \mathbb{1}(|x| \leqslant 1) + \Phi(|x| \geqslant 1)$$
$$\Phi \star f = \Phi_1 \star f + \Phi_2 \star f$$

We have $\Phi_1 \star f \in L^1(\mathbb{R}^d)$ and $\Phi_2 \star f$ is bounded since $\Phi \to 0$ as $|x| \to \infty$ in $d \geqslant 3$. ■

Exercise 2.30 (Hanack's inequality) Let $u \in C^2(\mathbb{R}^d)$ be harmonic and non-negative. Prove that for all open, bounded and connected $\Omega \subseteq \mathbb{R}^d$, we have

$$\sup_{x \in \Omega} u(x) \leqslant C_\Omega \inf_{x \in \Omega} u(x),$$

where C_∞ is a finite constant depending only on Ω.

Proof. (Exercise) Hint: $\Omega = B(x, r)$. General case cover Ω by finitely many balls, one ball is inside Ω. ■

Exercise 2.31 (E 3.1 Lebesgue Differentiation Theorem) Let $f \in L^1_{loc}(\mathbb{R}^d)$. Prove that that for almost every $x \in \mathbb{R}^d$:

$$\fint_{B(x,r)} |f(x) - f(y)| \, dy \xrightarrow{r \to 0} 0$$

Proof. Clearly the same result holds with $\mathbb{R}^d \rightsquigarrow \Omega \subseteq \mathbb{R}^d$ open. Also it suffices to consider $f \in L^1(\mathbb{R}^d)$. From the last time discussion, by a density argument there exists $r_n \to 0$ s.t.

$$\fint_{B(x,r_n)} |f(y) - f(x)| \, dy = 0$$

for a.e. $x \in \mathbb{R}^d$. We prove that for all $\epsilon > 0$, te set $A_\epsilon = \{x \in \mathbb{R}^d \mid \limsup_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| \, dy > \epsilon\}$ has measure 0. This will imply that

$$\bigcup_{n=1}^{\infty} A_{\frac{1}{n}} = \left\{ x \in \mathbb{R}^d \mid \limsup_{r \to 0} \int_{B(x,r)} |f(y) - f(x)| \, dy > 0 \right\}$$

has measure 0, which is what wie want to show. First, we show that $|A_\epsilon| = 0$: Take $\{f_n\} \subseteq C_c^\infty$, $f_n \to f$ in $L^1(\mathbb{R}^d)$. By the triangle inequality:

$$|f(y) - f(x)| \leqslant |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

So we get

$$\begin{aligned} &\fint_{B(x,r)} |f(y) - f(x)| \, dy \\ &\leqslant \fint_{B(x,r)} |f(y) - f_n(y)| \, dy + \fint_{B(x,r)} |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \end{aligned}$$
$$\Rightarrow \quad \limsup_{r \to 0} ... \leqslant \limsup_{r \to 0} (\dots) + 0 + |f_n(x) - f(x)|$$

Thus, for all $x \in A_\epsilon$, then:

$$\limsup_{r\to 0} \fint_{B(x,r)} |f_n(y) - f(y)|\, dy + |f_n(x) - f(x)| > 2\epsilon$$

Observation: If $a, b \geqslant 0$, $a + b > 2\epsilon$ then either $a > \epsilon$ or $b > \epsilon$. Therefore $A_\epsilon \subseteq \left(S_{n,\epsilon} \bigcup \tilde{S}_{n,\epsilon}\right)$, where

$$\begin{aligned} S_{n,\epsilon} &= \{x \mid |f_n(x) - f(x)| > \epsilon\} \\ \tilde{S}_{n,\epsilon} &= \{x \mid \limsup_{r\to 0} \fint_{B(x,r)} |f_n(y) - f(y)|\, dy > \epsilon\} \end{aligned}$$

Consequently: $|A_\epsilon| \leqslant |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}|$ for all $n \geqslant 1$. By the Markov / Chebyshev inequality:

$$|S_{n,\epsilon}| \leqslant \int_{S_{n,\epsilon}} \frac{|f_n(x) - f(x)|}{\epsilon}\, dx = \int_{\mathbb{R}^d} \frac{|f_n(x) - f(x)|}{\epsilon}\, dx = \frac{\|f_n - f\|_{L^1}}{\epsilon}$$

We want to prove a simpler bound for $\tilde{S}_{n\epsilon}$. For all $x \in \tilde{S}_{n\epsilon}$:

$$\limsup_{r\to 0} \fint_{B(x,r)} |f_n(x) - f(y)|\, dy > \epsilon$$

So there is a $r_x \in (0, 1)$ s.t.

$$\fint_{B(x,r_x)=B_x} |f_n(y) - f(y)|\, dy > \epsilon$$

Thus $\tilde{S}_{n\epsilon} \subseteq \left(\bigcup_{x\in\tilde{S}_{n,\epsilon}} B_x\right)$.

Lemma 2.32 (Vitali Covering) If F is a collection of balls in $\mathbb{R}^d$ with bounded radius, then there exists a sub-collection $G \subseteq F$ s.t.

- G has disjoint balls
- $\bigcup_{B\in F} B \subseteq \bigcup_{B\in G} 5B, 5B(x,r) = B(x, 5r)$

Remark 2.33 The condition of the boundedness of the radius is necessary. Otherwise, consider $\{B(0,n)\}_{n=1}^{\infty}$

Here consider $F = \{B_x\}_{x\in\tilde{S}_{n_\epsilon}}$. With the vitali covering leamm there is a $G \subseteq F$ s.t. G contains disjoint balls and:

$$\tilde{S}_{n,\epsilon} \subseteq \bigcup_{B\in F} B \subseteq \bigcup_{B\in G} 5B$$

So we get

$$|\tilde{S}_{n,\epsilon}| \leqslant |\bigcup_{B\in G} 5B| \leqslant \sum_{B\in G} |5B| = \sum_{B\in G} 5^d |B|$$

On the other hand, for all $B \in G \subseteq F$:

$$\fint_B |f_n(y) - f(y)|\, dy > \epsilon \Rightarrow \int_B |f_n - f| > \epsilon |B|$$

This implies:

$$\sup_{B\in G}\int_B |f_n - f| > \epsilon \sum_{B\in G}|B|$$

Since balls in G are disjoint:

$$\int_{\mathbb{R}^d} \geqslant \int_{\bigcup_{B\in G}} |f_n - f|\, dy > \epsilon \sum_{B\in G}|B| \geqslant \frac{\epsilon}{5^d}|\tilde{S}_{n,\epsilon}|$$

So

$$|\tilde{S}_{n\epsilon}| \leqslant \frac{5^d}{\epsilon}\|f_n - f\|_{L^1}$$

In summary:

$$|A_\epsilon| \leqslant |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}| \leqslant \frac{5^d+1}{\epsilon}\|f_n - f\|_{L^1} \to 0$$

as $n \to \infty$. So $|A_\epsilon| = 0$ for all $\epsilon > 0$ ∎

Remark 2.34 1. The proof can be done by using the Besicovitch covering lemma: For all $E \subseteq \mathbb{R}^d$ s.t. E is bounded. Let F = collection of balls s.t. for all $x \in E$ there is a $B_x \in F$ s.t. x is the center of B_x. There is a sub-collection $G \subseteq F$ s.t.

- $E \subseteq \bigcup_{B\in G} B$
- Any point in E belongs to at most C_d balls in C_T (C_d depends only on $\mathbb{R}^d$), i.e.

$$\mathbb{1}_E(x) \leqslant \sum_{B\in G}\mathbb{1}_B(x) \leqslant C_d \mathbb{1}_E(x) \forall x$$

2. By a simpler argument we can prove the weak L^1-estimate:

$$\{x \mid f^\star(x) > \epsilon\} \leqslant \frac{c_d}{\epsilon}\|f\|_{L^1(\mathbb{R}^d)}$$

(Hardy-Littlewood maximal function)

Exercise 2.35 (E 3.3) $f \in C_c^\infty(\mathbb{R}^d)$. Prove $|\hat{f}(k)| \leqslant \frac{C_N}{(1+|k|)^N}$

Solution. Since $f \in C_c^\infty$ we have that $D^\alpha f \in C_c^\infty$. Recall

$$\widehat{D^\alpha f}(k) = (-2\pi i k)^\alpha \hat{f}(k)$$

For example

$$\widehat{-\Delta f}(k) = |2\pi i k|^2 \hat{f}(k)$$
$$(\text{Induction}) \rightsquigarrow \widehat{(-\Delta)^N f}(k) = |2\pi k|^{2N}\hat{f}(k)$$

So we can conclude

$$\hat{f}(k) = \frac{\widehat{(-\Delta)^N f}(k)}{|2\pi k|^{2N}} \forall k \in \mathbb{R}^d$$

1. $f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in L^\infty$
2. $(-\Delta)^N f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \widehat{(-\Delta)^N} f \in L^\infty$

Conclusion: $\hat{f}(k) \leqslant \begin{cases} C & \forall k \\ \frac{C_N}{|k|^{2N}} & \forall k \end{cases}$ So $\hat{f}(k) \leqslant \frac{C_N}{(1+|k|)^N}$ ∎

Chapter 3

Convolution, Fourier Transform and Distributions

3.1 Convolutions

Definition 3.1 (Convolution) Let $f, g : \mathbb{R}^d \to \mathbb{R}$ or $\mathbb{C}$.

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy = (g \star f)(x)$$

Remark 3.2 (Properties of the Convolution)

- $f \star g = g \star f$
- $(f \star g) \star h = f \star (g \star h)$ (by Fubini)
- $\widehat{f \star g} = \hat{f}\hat{g}$ (Proof, see exercise 3.21)

Theorem 3.3 (Young Inequality) If $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$, where $1 \leqslant p \leqslant \infty$, then $f \star g \in L^p(\mathbb{R}^d)$ and

$$\|f \star g\|_{L^p} \leqslant \|f\|_{L^1}\|g\|_{L^p}.$$

More generally, if $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^1(\mathbb{R}^d)$,

$$\|f \star g\|_{L^r} \leqslant \|f\|_{L^p}\|g\|_{L^q},$$

where $1 \leqslant p, q, r \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

Proof. Let $f \in L^1, g \in L^p$. With the Hölder Inequality we have:

$$\begin{aligned}
|(f \star g)(x)| &= \left|\int_{\mathbb{R}^d} f(x-y)g(y)\,dy\right| \\
&\leqslant \left(\int_{\mathbb{R}^d} |f(x-y)|\,dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(x-x)||g(y)|^p\,dy\right)^{\frac{1}{p}} \\
&= \|f\|_{L^1}^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(x-x)||g(y)|^p\,dy\right)^{\frac{1}{p}} \\
\|f \star g\|_{L^p}^p &= \int_{\mathbb{R}^d} |f \star g(x)|^p\,dx \\
&\leqslant \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)||g(y)|^p\,dy\,dx \\
&= \|f\|_{L^1}^{\frac{p}{q}+1} \|g\|_{L^p}^p
\end{aligned}$$

So we have $\|f \star g\|_{L^p} \leqslant \|f\|_{L^1}\|g\|_{L^p}$ ■

Theorem 3.4 (Smoothness of the Convolution) If $f \in C_c^\infty(\mathbb{R}^d),\ g \in L^p(\mathbb{R}^d),\ 1 \leqslant p \leqslant \infty$. Then $f \star g \in C^\infty(\mathbb{R})$ and

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for all $\alpha = (\alpha_1, \ldots, \alpha_d), \alpha_i \in \{0, 1, 2, \ldots\}$

Proof. First we note that $x \mapsto (f \star g)$ is continous as $x_n \to x$ in $\mathbb{R}^d$ since

$$(f \star g)(x_n) = \int_{\mathbb{R}^d} f(x_n - y)g(y)\,dy \xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = (f \star g)(x)$$

We can apply Dominated convergence because

$$f(x_n - y)g(y) \to f(x-y)g(y) \quad \forall y \text{ as } f \text{ is continuous and } x_n \to x$$

and

$$|f(x_n - y)\ g(y)| \leqslant \|f\|_{L^\infty}|g(y)|\ \mathbb{1}(|y| \leqslant R) \in L^1(\mathbb{R}^d).$$

Where $R > 0$ satisfies $B(0,R) \supseteq \operatorname{supp} f + \sup_n |x_n|$. Now we can compute the derivatives:

$$\begin{aligned}
\partial_{x_i}(f \star g)(x) &= \lim_{h \to 0} \frac{(f \star g)(x + he_i) - (f \star g)(x)}{h} \\
&= \lim_{h \to 0} \int_{\mathbb{R}^d} \frac{f(x + he_i - y) - f(x-y)}{h} g(y)\,dy \\
\text{(Dominated Convergence)} \qquad &= \int_{\mathbb{R}^d} \lim_{h \to 0} \frac{f(x + he_i - y) - f(x-y)}{h} g(y)\,dy \\
&= \int_{\mathbb{R}^d} (\partial_{x_i} f)(x-y)g(y)\,dy
\end{aligned}$$

We could apply Dominated Convergence since

$$\frac{f(x + he_i - y) - f(x-y)}{h} g(y) \xrightarrow{h \to 0} (\partial_{x_i} f)(x-y)g(y) \quad \text{as } f \in C^1$$

$$\left|\frac{f(x + he_i - y) - f(x-y)}{h} g(y)\right| \leqslant \|\partial_{x_i} f\|_{L^\infty}|g(y)|\ \mathbb{1}(|y| \leqslant R) \in L^1(\mathbb{R}^d)$$

where $B(0,R) \supseteq \operatorname{supp}(f) + B(0, |x| + 1)$ and $\partial_{x_i}(f \star g) = (\partial_{x_i} f) \star g \in C(\mathbb{R}^d)$ since $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$. By induction we get $D^\alpha(f \star g) = (D^\alpha f \star g) \in C(\mathbb{R}^d)$. ■

Remark 3.5 Question: Is there a f s.t. $f \star g = g$ for all g? In fact there is no regular function f that solves this formally:

$$f \star g = g \Rightarrow \widehat{f \star g} = \hat{g} \Rightarrow \hat{f}\hat{g} = \hat{g} \ \Rightarrow \hat{f} = 1 \Rightarrow f \text{ is not a regular function!}$$

However, if f is the Dirac-Delta Distribution, $f = \delta_0$ then $\delta_0 \star g = g$ for all g. Formally:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \\ \int \delta_0 = 1 & \end{cases}$$

In fact, if $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$, then $f_\epsilon \to \delta_0$ in an appropriate sense and $f_\epsilon \star g \to g$ for all g nice enough.

Theorem 3.6 (Approximation by convolution) Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon})$. Then for all $g \in L^p(\mathbb{R}^d)$, where $1 \leqslant p < \infty$, then

$$f_\epsilon \star g \to g \quad \text{in } L^p(\mathbb{R}^{\mathrm{d}})$$

Proof.

Step 1: Let $f, g \in C_c(\mathbb{R}^d)$. Then

$$\begin{aligned}
(f_\epsilon \star g)(x) - g(x) &= \int_{\mathbb{R}^d} f_\epsilon(y) g(x-y)\, dy - g(x) \int_{\mathbb{R}^d} f_\epsilon(y)\, dy \\
&= \int_{\mathbb{R}^d} f_\epsilon(y) g(x-y)\, dy - \int_{\mathbb{R}^d} f_\epsilon(y) g(x)\, dy \\
&= \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x))\, dy \\
|(f_\epsilon \star g)(x) - g(x)| &= \left| \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x))\, dy \right| \\
&\leqslant \int_{\mathbb{R}^d} |f_\epsilon(y)||g(x-y) - g(x)|\, dy \\
&\leqslant \int_{|y| \leqslant R_\epsilon} |f_\epsilon(y)||g(x-y) - g(x)|\, dy \\
&\leqslant \left[\sup_{|z| \leqslant R} |g(x-z) - g(x)| \right] \underbrace{\int_{|y| \leqslant R_\epsilon} |f_\epsilon(y)|\, dy}_{\leqslant \|f_\epsilon\|_{L^1} = \|f\|_{L^1}} \xrightarrow{\epsilon \to 0} 0
\end{aligned}$$

We have Dominated Convergence since:

$$(f_\epsilon \star g)(x) - g(x) \to 0 \quad \text{as } \epsilon \to 0$$

and

$$|f_\epsilon \star g(x) - g(x)| \leqslant \|f\|_{L^1} \sup_{|z| \leqslant R_\epsilon} |g(x-z) - g(x)| \leqslant 2\|f\|_1 \|g\|_{L^\infty} \mathbb{1}(|x| \leqslant R_1).$$

Where $B(0, R_1) \supseteq \mathrm{supp}(g) + B(0, R_\epsilon)$, thus $f_\epsilon \star g \to g$ in $L^p(\mathbb{R}^d)$. To remove the technical assumptions $f, g \in C_c(\mathbb{R}^d)$, then we use a density argument. We use the fact that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$.

Step 2: Let $g \in C_c(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$. Then there is $\{g_m\} \subseteq L^p(\mathbb{R}^d), g_m \to g$ in $L^p(\mathbb{R}^d)$. Then

$$\begin{aligned}
\|f_\epsilon \star g - g\|_{L^p} &\leqslant \|f_\epsilon \star (g - g_m)\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
\text{(Young)} \quad &\leqslant \|f_\epsilon\|_{L^1}\|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leqslant \|f\|_{L^1}\|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leqslant (\|f\|_{L^1} + 1)\|g - g_m\|_{L^p} + \|f \star g_m - g_m\|_{L^p}
\end{aligned}$$

So we get:

$$\begin{aligned}
&\limsup_{\epsilon \to 0} \|f_\epsilon \star g - g\|_{L^p} \\
&\quad \leqslant (\|f\|_{L^p} + 1)\|g - g_m\|_{L^p} + \underbrace{\limsup_{\epsilon \to 0} \|f_\epsilon \star g_m - g_m\|_{L^p}}_{0 \text{ by step 1.}} \xrightarrow{m \to \infty} 0.
\end{aligned}$$

Step 3: Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Take $\{f_m\} \subseteq C_c(\mathbb{R}^d)$, s.t.

$$\begin{cases} F_m \to g \in L^1(\mathbb{R}) \text{ as } m \to \infty \\ \int_{\mathbb{R}^d} F_m = 1 (\text{it is possible since } \int_{\mathbb{R}^d}) f = 1) \end{cases}$$

Define $F_{m,\epsilon}(x) = \epsilon^{-d} F_m(\epsilon^{-1}x)$ (recall $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$). Then:

$$\begin{aligned}
f_\epsilon \star g - g &= (f_\epsilon - F_{m,\epsilon}) \star g + F_{m,\epsilon} \star g - g \\
\Rightarrow \|f_\epsilon - g\|_{L^p} &\leqslant \underbrace{\|f_\epsilon - F_{m,\epsilon} \star g\|_{L^p}}_{\overset{\text{Young}}{\leqslant} \|f_\epsilon - F_{m,\epsilon}\|_{L^1}\|g\|_{L^p} = \|f - F_m\|_{L^1}\|g\|_{L^p}} + \|F_{m,\epsilon} \star g - g\|_{L^p}
\end{aligned}$$

$$\Rightarrow \limsup_{\epsilon \to 0} \|f_\epsilon \star g - g\|_{L^p} \leqslant \|f - F_m\|_{L^1}\|g\|_{L^p} = \|f - F_m\|_{L^1}\|g\|_{L^p} \qquad \blacksquare$$

Lemma 3.7 $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$

Proof. For all $g \in L^p(\mathbb{R}^d)$ there are step functions $(g_m)_m$ and $g_m \to g$ in $L^p(\mathbb{R}^d)$,

$$g_m(x) = \sum_{\substack{\Omega \\ \text{finite sum} \\ \Omega \subseteq \mathbb{R}^d \text{measurable}}} \chi_\Omega(x) a_\Omega.$$

We can assume that Ω is open and bounded and we want to approximate χ_Ω by $C_c(\mathbb{R}^d)$. $\blacksquare$

Lemma 3.8 (Urnson) Define

$$\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}$$

Then there is a $\eta_\epsilon \in C_c(\mathbb{R}^d)$ s.t.

$$\begin{cases} 0 \leqslant \eta(x) \leqslant 1 & \forall x \in \mathbb{R}^d \\ \eta_\epsilon(x) = 1 & \text{if } x \in \Omega_\epsilon \\ \eta_\epsilon(x) = 0 & \text{if } x \notin \Omega \end{cases}$$

Lemma 3.9 (General Version of Urnson) If $A, B \subseteq \mathbb{R}^d$, A closed, B closed, $A \cap B = \varnothing$. Then

$$\eta(x) = \frac{\operatorname{dist}(x, A)}{\operatorname{dist}(x, A) + \operatorname{dist}(x, B)}$$

Then $\eta \in C(\mathbb{R}^d)$, $0 \leqslant \eta \leqslant 1$ and $\eta = 0$ if $x \in B$, $\eta = 1$ if $x \in A$.

For example, this lemma can be applied to $A = \overline{\Omega_\epsilon} \subsetneq \Omega$ and $B = \mathbb{R}^d \backslash \Omega$ for $\Omega \subseteq \mathbb{R}^d$ open.

Theorem 3.10 (Appendix C4 in Evans) Let Ω be open in $\mathbb{R}^d$ and for $\epsilon > 0$ define

$$\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \mathbb{R}^d\backslash\Omega) > \epsilon\}$$

Let $f \in C_c^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} f = 1, \operatorname{supp} f \subseteq B(0,1), f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$. Then $\operatorname{supp} f_\epsilon \subseteq B(0,\epsilon)$ and for all $g \in L^p_{loc}(\Omega)$ (i.e. $\mathbb{1}_K g \in L^p(\Omega)$ for all $K \subseteq \Omega$ compact) we have

a) $g_\epsilon(x) = (f_\epsilon \star g)(x) = \int_{\mathbb{R}^d} f_\epsilon(x-y)g(y)\,dy - \int_\Omega f_\epsilon(x-y)g(y)\,dy$ is well-defined in Ω_ϵ and $g_\epsilon \in C^\infty(\Omega_\epsilon)$,

b) $g_\epsilon \to g$ in $L^p_{loc}(\Omega)$ if $1 \leqslant p < \infty$ and $g_\epsilon(x) \to g(x)$ almost everywhere $x \in \Omega$,

c) If $g \in C(\Omega)$, then $g_\epsilon(x) \to g(x)$ uniformly in any compact subset of Ω.

Proof. a) $D^\alpha(g_\epsilon) = (D^\alpha f_\epsilon) \star g \in C(\Omega_\epsilon)$

b) Replace $g \mapsto \mathbb{1}_K g$ where K is a compact set $\subseteq \Omega$. Then $\mathbb{1}_K g \in L^p$. Then our theorem $f_\epsilon \star (\mathbb{1}_K g) \to \mathbb{1}_K g$ in $L^p(\Omega)$. On the other hand $\mathbb{1}_K[(f_\epsilon \star \mathbb{1}_K g) - (f_\epsilon \star g)] \to 0$ as $\epsilon \to 0$ (exercise)

c) Already proved in $\mathbb{R}^d$ space.

∎

Corrolary 3.11 (Lebesgue differentation theorem) If $f \in L^P_{loc}(\mathbb{R}^d)$, then

$$\fint_{B(x,\epsilon)} |f(y) - f(x)|^p\,dy \to 0 \quad \text{as } \epsilon \to 0$$

Exercise 3.12 (E 2.1) Let $u \in C^2(\mathbb{R}^2)$ be convex, i.e.

$$tu(x) + u(y)(1-t) \geqslant u(tx + (1-t)y)$$

for all $x, y \in \mathbb{R}^d$, $t \in [0,1]$.

a) Prove for all $x \in \mathbb{R}^d$ that the Hessian matrix H is positive semidifinite.

b) Prove that u is sub-harmonic in $\mathbb{R}^d$.

Solution.

a) In 1D: If u is convex $\Leftrightarrow u''(x) \geqslant 0$ for all $x \in \mathbb{R}$. In general: Taylor expansion for all $x, z \in \mathbb{R}^d$:

$$cu(x) = u(z) + \nabla u(z)(x-z) + \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z))\frac{(x-z)^\alpha}{\alpha!}\,ds$$

Note that we have $x = z + s(x-z)$ if $s = 1$. Use $z = tx + (t-1)y \Rightarrow x - z = (1-t)(x-y)$

$$tu(x) = tu(z) + t\nabla u(z)(1-t)(x-y) + t\int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z))\frac{[(1-t)(x-y)]^\alpha}{\alpha!}\,ds$$

$$(1-t)u(y) = (1-t)u(z) + (1-t)\nabla u(z)t(y-x) + (1-t)\int_0^r \sum_{|\alpha|=2} D^\alpha u(z + s(y-z))\frac{[t(y-x)]^\alpha}{\alpha!}\,ds$$

$$\Rightarrow tu(x)+(1-t)u(y)=u(z)+t\int_0^1 ...+(1-t)\int_0^1 ...$$
$$\Rightarrow t\int_0^1 ...+(1-t)\int_0^1 ...\geqslant 0 \forall x,y,t,z=tx+(1-t)y$$

$$t(1-t)^2\int_0^1\sum_{|\alpha|=2}D^\alpha u(z+s(x-z))\frac{(x-y)}{\alpha!}\,ds+(1-t)t^2\int_0^1\sum_{|\alpha|=2}D^\alpha u(z+s(y-z))\frac{(y-z)^\alpha}{\alpha!}\,ds\geqslant 0$$

for all $x,y\in\mathbb{R}^d, t\in[0,1], z=tx+(1-t)y$. Divides for $t(1-t)$

$$(1-t)\int_0^1\cdots++\int_0^1\cdots\geqslant 0$$

Take $t\to 0$

$$\int_0^1\sum_{|\alpha|=2}D^\alpha u(y+s(x-y))\frac{(x-y)^\alpha}{\alpha!}\,ds\geqslant 0\forall x,y\in\mathbb{R}^d$$

Take $y=x+a, a\in\mathbb{R}^d$

$$\int_0^1\sum_{|\alpha|=2}D^\alpha u(x+a+sa)\frac{a^\alpha}{\alpha!}\,ds\geqslant 0\forall\epsilon>0,\forall x,a\in\mathbb{R}^d$$

Take $\epsilon\to 0$

$$\int_0^1\sum_{|\alpha|=2}D^\alpha u(x)\frac{a^\alpha}{\alpha!}\geqslant 0\Rightarrow\sum_{i,j=1,i\neq j}\partial_{x_i}\partial_{x_j}u(x)a_ia_j+\sum_{i=j=1}^{d}\partial^2_{x_i}u(x)\frac{a_i^2}{2}$$

We get

$$\frac{1}{2}a^THa\geqslant 0\forall a(a_i)_{i=1}^d\in\mathbb{R}^d$$

b) $H(x)\geqslant 0\Rightarrow(\partial_i\partial_j u)\geqslant 0\Rightarrow TrH(x)\geqslant 0\Rightarrow\sum_{i=1}^d\partial^2_{x_i}u(x)\geqslant 0\Rightarrow\Delta u(x)\geqslant 0\forall x\in\mathbb{R}^d$

■

Exercise 3.13 (E 2.2, Newton's Theorem) Let $d\geqslant 3$.

a) Prove that for all $r>0$ and $x\in\mathbb{R}^d$, we have

$$⨍_{\partial B(x,r)}\frac{dS(y)}{|y|^{d-2}}=\frac{1}{\max(|x|,r)^{d-2}}$$

where $dS(y)$ is the surface measure on the sphere $\partial B(x,r)\subseteq\mathbb{R}^d$.

b) Let $0\leqslant f_1,f_2\in L^1(\mathbb{R}^d)$ be radial functions with $\int_{\mathbb{R}^d}f_i=M_i$. Prove that for all $z_1,z_2\in\mathbb{R}^d$ we have

$$\int\int_{\mathbb{R}^d}\frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|^{d-2}}\,dx\,dy\leqslant\frac{M_1M_2}{|z_1-z_2|^{d-2}}$$

Moreover, prove that we have the equality if f_1,f_2 are compactly supported and $|z_1-z_2|$ is sufficiently large.

Hint: For a) you may use the mean-value theorem (the function $\frac{1}{|x|^{d-2}}$ is harmonic in Ω if $0 \notin \Omega$). For b) you may use a) and polar coordinates.

Solution. a) Regard $d = 3$. The function $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \backslash \{0\}$. We prove

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{\max(|x|, r)}$$

If $|x| > r$, then $0 \notin B(x, r + \epsilon)$. Then

$$y \mapsto \frac{1}{|y|}$$

is harmonic in $B(x, r + \epsilon)$. Then by the Mean Value Property:

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{|x|}$$

If $|x| < r$: Then $\frac{1}{|y|}$ is not harmonic in $B(x, r)$ since $0 \in B(x, r)$. Note

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \fint_{\partial B(0,r)} \frac{dS(y)}{|x - y|}$$

This function depends on x only via $|x|$.

$$... = \fint_{\partial B(0,r)} \frac{dS(y)}{|Rx - Ry|}$$

for all R rotation $SO(3)$, $dS(R_y) = dS(y)$

$$\begin{aligned}
&= \fint_{\partial B(0,r)} \frac{dS(y)}{|Rx - y|} \\
&= \fint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \\
\text{(Radial in z)} \quad &= \fint_{\partial B(0,|x|)} \left(\fint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \right) dS(z) \\
\text{(Fubini)} \quad &= \fint_{\partial B(0,r)} \left(\fint_{\partial B(0,|x|)} \frac{dS(z)}{|z - y|} \right) dS(y) \\
\text{(case 1 since } |y| = r > |x|) \quad &= \fint_{\partial B(0,r)} \frac{1}{|y|} \, dS(y) = \frac{1}{r}
\end{aligned}$$

If $|x| = r$: Continuity: $x \mapsto \fint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$

b)

Remark 3.14 For $f \in C^{|\alpha|}, g \in C^{|\beta|}$:

$$D^{\alpha+\beta}(f \star g) = (D^\alpha f) \star (D^\beta g)$$

Lemma 3.15 If $d \geqslant 3$ and $f : \mathbb{R}^d \to \mathbb{R}$ radial. Then:

$$\left(\frac{1}{|x|^{d-2}} \star f \right)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-2}} \, dy = \int_{\mathbb{R}^d} \frac{f(y)}{\max(|x|^{d-2}, |y|^{d-2})} \, dy$$

Proof. (d=3) Polar coordinates:

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\,dy &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{d\omega}{|x-rw|}\right] f(r)\,dr \\ (a) \qquad &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{d\omega}{\max(|x|,r)}\right] f(r)\,dr \\ &= \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|,|y|)}\,dy\end{aligned}$$ ■

Now, for $d=3$, if f radial and non-negative with the lemma we get

$$(\star) \qquad \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\,dy = \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|,|y|)}\,dy \leqslant \int_{\mathbb{R}^3} \frac{f(y)}{|x|}\,dy = \frac{\left(\int_{\mathbb{R}^3} f(y)\,dy\right)}{|x|}$$

Then

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|}\,dx\,dy &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_1(x)f_2(y)}{|x+z_1-y-z_2|}\,dx\,dy \\ &= \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{f_1(x)}{|x+z_1-y-z_2|}\,dx\right) f_2(y)\,dy \\ (\star) \qquad &\leqslant \int_{\mathbb{R}^3} \frac{\left(\int_{\mathbb{R}^3} f_1(x)\,dx\right) f_2(y)}{|y+z_2-z_1|}\,dy \\ &\leqslant \frac{(\int_{\mathbb{R}^3} f_1)(\int_{\mathbb{R}^3} f_2)}{|z_1-z_2|}\end{aligned}$$ ■

Exercise 3.16 (Bonus 2) a) Prove that $u(x) = \frac{1}{|x|}$ is sub-harmonic in $\mathbb{R}^2\backslash\{0\}$.

b) Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ radial, non-negative, measurable:

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\,dy \geqslant \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|,|y|)}\,dy$$

3.2 Fourier Transformation

Definition 3.17 (Fourier Transform) For $f \in L^1(\mathbb{R}^d)$ define

$$\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i k\cdot x}\,dx, \quad k\cdot x = \sum_{i=1}^d k_i x_i$$

Theorem 3.18 (Basic Properties) 1. If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^\infty(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$

2. For all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$. Moreover, $\mathcal{F}$ can be extended to be a unitary transformation $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ s.t.

$$\|\mathcal{F}g\|_{L^2} = \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^d)$$

3. The inverse of F can be defined as

$$(F^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(x)e^{2\pi i k x}\,dk$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

4. $\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k)$ as $(2\pi i k)^\alpha f(k) \in L^2(\mathbb{R}^d)$ $(k^\alpha = k_1^{\alpha_1} \cdots k_\alpha^{\alpha_k})$

5. $\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k)$ if f, g are nice enough.

Theorem 3.19 (Hausdorff-Young-Inequality) Let $f \in L^p(\mathbb{R}^d)$. If $1 \leqslant p \leqslant 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\mathcal{F} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is well-defined and

$$\|\mathcal{F}f\|_{L^q(\mathbb{R}^d)} \leqslant \|f\|_{L^p(\mathbb{R}^d)}.$$

Remark 3.20 If $1 \leqslant p \leqslant 2$ and $f \in L^p(\mathbb{R}^d)$ we can write $f = f_1 + f_2$ when $f_1 \in L^1$, $f_2 \in L^2$, e.g.

$$f = \underbrace{f\mathbb{1}(|f| \geqslant 1)}_{f_1} + \underbrace{f\mathbb{1}(|f| < 1)}_{f_2}$$

$$\begin{aligned}
\int_{\mathbb{R}^d} |f_2|^2 \, dy &= \int_{\mathbb{R}^d} |f|^2 \mathbb{1}(|f| < 1) \leqslant \int_{\mathbb{R}^d} |f|^p \, dy < \infty \\
\int_{\mathbb{R}^d} |f_1| \, dy &= \int_{\mathbb{R}^d} |f| \mathbb{1}(|f| \geqslant 1) \leqslant \int_{\mathbb{R}^d} |f|^p < \infty
\end{aligned}$$

thus we can define $\hat{f} = \hat{f}_1 + \hat{f}_2$ well defined in $L^\infty(\mathbb{R}^d) + L^2(\mathbb{R}^d)$.

Proof of the Hausdorff-Young-Inequality 3.19. We need Riesz-Thorin interpolation theorem. If $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$, and $\Omega \subseteq \mathbb{R}^d$ open and

$$T : L^{p_0}(\Omega) + L^{p_1}(\Omega) \longrightarrow L^{q_0}(\Omega) + L^{q_1}(\Omega)$$

is a linear operator and

$$T : \; L^{p_0} \to L^{q_0}$$

and $\|T\|_{L^{p_i} \to L^{q_i}} \leqslant 1$ for $i = 0, 1$. Then,

$$T : \; L^{p_\theta} \to L^{q_\theta} \text{ and } \|T\|_{L^{p_\theta} \to L^{q_\theta}} \leqslant 1$$

for any $0 < \theta < 1$ where

$$\begin{cases} \frac{1}{p_0} = \frac{\theta}{p_0} + \frac{1-\theta}{p_2} \\ \frac{1}{q_0} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1} \end{cases}.$$

Consider the Fourier Transform:

$$F : L^1 + L^1 \to L^2 + L^\infty$$

and

$$\begin{aligned}
& \|F\|_{L^1 \to L^\infty} \leqslant 1 \text{ as } \|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1} && \forall fp \in L^1 \\
& \|F\|_{L^2 \to L^2} = 1 \text{ as } \|\hat{f}\|_{L^2} = \|f\|_{L^2} && \forall f \in L^2 \\
\Rightarrow \quad & \|F\|_{L^{p_\theta} \to L^{p_\theta}} \leqslant 1 && \forall \theta \in (0,1)
\end{aligned}$$

$p_0 = 1, p_1 = 2, q_0 = \infty, q_1 = 2$

$$\begin{aligned}
\frac{1}{p_0} &= \frac{\theta}{p_0} + \frac{1-\theta}{p_2} = \theta + \frac{1-\theta}{2} = \frac{1+\theta}{2} \\
\frac{1}{q_\theta} &= \frac{\theta}{q_0} + \frac{1-\theta}{q_1} = \frac{1-\theta}{2} \\
\Rightarrow \quad 1 &= \frac{1}{p_0} + \frac{1}{q_\theta} = \frac{1+\theta}{2} + \frac{1-\theta}{2}
\end{aligned}$$

■

Exercise 3.21 (E 3.2) Let $1 \leqslant p, q, r \leqslant 2$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Recall that if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^r(\mathbb{R}^d)$ by Young's Inequality, and its Fourier transform is well-defined by the Hausdorff-Young inequality. Prove that

$$\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k) \quad \forall k \in \mathbb{R}^d$$

Solution.

Step 1) $f, g \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
\widehat{f \star g}(k) &= \int_{\mathbb{R}^d} (f \star g)(x) e^{-2\pi i k x}\, dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) g(y) e^{-2\pi i k x}\, dx\, dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) e^{-2\pi i k (x-y)} g(y) e^{-2\pi i k y}\, dx\, dy \\
(z(x) := x - y) \qquad &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(z) e^{-2\pi i k z}\, dz \right) g(y) e^{-2\pi i k y}\, dy \\
&= \left(\int_{\mathbb{R}^d} f(z) e^{-2\pi i k z}\, dz \right) \left(\int_{\mathbb{R}^d} g(y) e^{-2\pi i k y}\, dy \right) = \hat{f}(k)\hat{g}(k)
\end{aligned}$$

Step 2) $f \in L^p, g \in L^q$, find $f_n, g_n \in C_c^\infty$ s.t. $f_n \to f$ in L^p, $g_n \to g$ in L^q. Then we have $\widehat{f_n \star g_n} = \hat{f}_n \hat{g}_n$ almost everywhere. We get with the Hausdorff-Young Inequality (3.19)

$$\begin{aligned}
\|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^{r'}} &\leqslant \|f \star g - f_n \star g_n\|_{L^r} \\
&= \|(f - f_n) \star g_n + f_n \star (g_n - g)\|_{L^r} \\
&\leqslant \|(f - f_n) \star g_n\|_{L^r} + \|f_n \star (g_n - g)\|_{L^r} \\
(\text{Young } 3.3) \qquad &\leqslant \|f - f_n\|_{L^p} \|g_n\|_{L^q} + \|f_n\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \to \infty} 0
\end{aligned}$$

Moreover:

$$\begin{aligned}
\|\hat{f}_n \hat{g}_n - \hat{f}\hat{g}\|_{L^{r'}} &= \|(\hat{f}_n \hat{f})\hat{g}_n + \hat{f}(\hat{g}_n - \hat{g})\|_{L^{r'}} \\
(\text{Hölder}) \qquad &\leqslant \|\hat{f}_n - \hat{f}\|_{L^{p'}} \|\hat{g}_n\|_{L^{q'}} + \|\hat{f}\|_{L^{q'}} \\
(\text{Hausdorff-Young } 3.19) \qquad &\leqslant \|f_n - f\|_{L^p} \|g_n\|_{L^q} + \|f\|_{L^p} \|g_n - g\|_{L^p} \xrightarrow{n \to \infty} 0
\end{aligned}$$

So $\hat{f}_n \hat{g}_n \to \hat{f}\hat{g}$ in $L^{r'}$ $\widehat{f \star g} = \hat{f}\hat{g}$ in $L^{r'}$ $\frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'}$ ■

Remark 3.22 We want to apply the Fourier transform to find the solution of a PDE, e.g. the Poisson-Equation:

$$-\Delta u = f \text{ in } \mathbb{R}^d \Rightarrow |2\pi k|^2 \hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{1}{|2\pi k|^2} \hat{f}(k)$$

If we can find G s.t. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$, then

$$\begin{aligned}
\hat{u}(k) &= \hat{G}(k)\hat{f}(k) = \widehat{G \star f} \\
\Rightarrow u(x) &= (G \star f)(x) = \int_{\mathbb{R}^d} G(x-y) f(y)\, dy
\end{aligned}$$

Thus we need to compute

$$G(x) = \left(\frac{1}{|2\pi k|^2} \right)^{\vee}$$

It turns out for $d \geqslant 3$ that

$$G(x) = \frac{1}{d(d-2)|B_1||x|^{d-2}}.$$

In fact G is the fundamential solution of the Laplace Equation. To make it rigorous, we need to compute the Fourier transform of $\frac{1}{|x|^\alpha}$ for $0 \leqslant \alpha < d$.

Theorem 3.23 (Fourier Transform of $\frac{1}{|x|^\alpha}$ for $0 < \alpha < d$) We have formally

$$\widehat{\frac{c_\alpha}{|x|^\alpha}} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \quad \forall\ 0 < \alpha < d$$

Here

$$c_\alpha = \pi^{-\frac{d}{2}}\Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda}\lambda^{\frac{\alpha}{2}-1}\,d\lambda$$

More precisely, for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{c_\alpha}{|x|^\alpha} \star f = \left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k)\right)^\vee$$

Moreover if $\alpha > \frac{d}{2}$, then we also have

$$\left(\frac{c_\alpha}{|x|^\alpha} \star f\right)^\wedge = \frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k).$$

Proof.

$$\begin{aligned}
\frac{c_\alpha}{|x|^\alpha} &= \frac{1}{|x|^\alpha}\pi^{-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda}\lambda^{\frac{\alpha}{2}-1}\,d\lambda = \int_0^\infty e^{-\pi\lambda|x|^2}\lambda^{\frac{\alpha}{2}-1}\,d\lambda \\
\Rightarrow \frac{\hat{c}_\alpha}{|x|^\alpha}(k) &= \int_0^\infty \widehat{e^{-\pi\lambda|x|^2}}(k)\lambda^{\frac{\alpha}{2}-1}\,d\lambda = \int_0^\infty \lambda^{-\frac{d}{2}}e^{-\pi\frac{|k|^2}{\lambda}}\lambda^{\frac{\alpha}{2}-1}\,d\lambda \\
(\lambda \to \frac{1}{\lambda}) \quad &= \int_0^\infty \lambda^{\frac{d}{2}}e^{-\pi|k|^2\lambda}\lambda^{-\frac{\alpha}{2}+1}\lambda^{-2}\,d\lambda = \frac{c_{d-\alpha}}{|k|^{d-\alpha}}
\end{aligned}$$

Let $f \in C_c(\mathbb{R}^d)$. Then $\left(\frac{1}{|x|^\alpha} \star f\right)(x) = \int_{\mathbb{R}^d}\frac{1}{|x-y|^\alpha}f(y)\,dy$ is well defined as $\frac{1}{|x-y|} \in L^1_{loc}(\mathbb{R}^d, dy)$. It is bounded

$$\frac{1}{|x|^\alpha} \star f = \frac{1}{|x|^\alpha}\underbrace{\mathbb{1}(|x| \leqslant 1)}_{\in L^\infty(\mathbb{R}^d)} \star \underbrace{f}_{L^\infty} + \underbrace{\frac{1}{|x|}\mathbb{1}(|x| > 1)}_{\in L^\infty} \star \underbrace{f}_{\in L^1} \in L^\infty(\mathbb{R}^d)$$

When $|x| \to \infty$:

$$\left(\frac{1}{|x|^\alpha} \star f\right)(x) = \int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^\alpha}\,dy = \int_{|y|\leqslant R}\frac{f(y)}{|x-y|^\alpha}\,dy \sim \frac{\int_{\mathbb{R}^d} f(y)\,dy}{|x|^\alpha}$$

Note that $\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\underbrace{\hat{f}(k)}_{\text{bounded}} \in L^1(\mathbb{R}^d)$.

$$(\ldots)\mathbb{1}(|k| \leqslant 1) + (\ldots)\mathbb{1}(|k| > 1)\frac{1}{|k|^{d-\alpha}}|\hat{f}(k)|\mathbb{1}(|k| \leqslant 1) \leqslant \|f\|_{L^1}\frac{\mathbb{1}(|k| \leqslant 1)}{|k|^{d-\alpha}} \in L^1(\mathbb{R}^d, dk)$$

$$\frac{1}{|k|^{d-\alpha}}|\hat{f}(k)|\mathbb{1}(k > 1) \leqslant |\hat{f}(k)| \in L^2(\mathbb{R}^d, dK) \text{ as } f \in L^2(\mathbb{R}^d)$$

Lemma 3.24 If $f \in C_c^\infty(\mathbb{R}^d)$, then $\hat{f} \in L^1(\mathbb{R}^d)$

Proof. (Exercise) Hint: $|\widehat{D^\alpha f}| = |2\pi k|^{|\alpha|}|\hat{f}(k)| \rightsquigarrow |\hat{f}(k)| \leqslant \frac{1}{|k|^{|k|}}$ as $|k| \to \infty$. ■

Compute:

$$\begin{aligned}
\left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k)\right)^\vee(x) &= \int_{\mathbb{R}^d} \frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k)e^{2\pi ikx}\,dk \\
&= \int_{\mathbb{R}^d}\left(\int_0^\infty e^{-\pi|k|^2\lambda}\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda\right)\hat{f}(k)e^{2\pi ikx}\,dk \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} e^{-\pi|k|^2\lambda}\hat{f}(k)e^{2\pi ikx}\,dk\right)\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty \left(e^{-\pi k^2\lambda}\hat{f}(x)\right)^\vee \lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty \left(\widehat{\lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}}}(k)\hat{f}(k)\right)^\vee \lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}} \star f\right)\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \left(\int_0^\infty \lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}}\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda\right)\star f
\end{aligned}$$

Assume $d > \alpha > \frac{d}{2}$. Then $\frac{c_\alpha}{|x|^\alpha} \star f \in L^\infty$ and behaves $\frac{c_\alpha(\int f)}{|x|^\alpha}$ as $|x| \to \infty$. This implies:

$$\int_{\mathbb{R}^d}\left|\frac{c_\alpha}{|x|^\alpha}\star f\right|^2 \leqslant c + \int_{|x|\geqslant R}\frac{c}{|x|^{2d}}\,dx < \infty$$

Thus the Fourier Transform $\widehat{\frac{c_\alpha}{|x|^\alpha}}\star f$ exists. Combining with

$$\begin{aligned}
\frac{c_\alpha}{|x|^\alpha}\star f &= \left(\frac{c_{d-\alpha}}{|f|^{d-\alpha}}\hat{f}(k)\right)^\vee \\
\Rightarrow \widehat{\frac{c_\alpha}{|x|^\alpha}\star f} &= \frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k)
\end{aligned}$$

■

Remark 3.25 If $f \in L^p$, $1 \leqslant p \leqslant 2$ is such that $\tilde{f}_1 + \tilde{f}_2 = f = f_1 + f_2$ with $f_1, \tilde{f}_1 \in L^1$, $f_2, \tilde{f}_2 \in L^2$. Do we have that $\hat{f}_1 + \hat{f}_2 = \hat{\tilde{f}}_1 + \hat{\tilde{f}}_2 \in L^1 \cap L^2$? In fact, from $f_1 + f_2 = \tilde{f}_1 + \tilde{f}_2$ we get

$$\underbrace{f_1 - \tilde{f}_1}_{\in L^1} = \underbrace{\tilde{f}_2 - f_2}_{\in L^2} \in L^1 \cap L^2$$

$$\Rightarrow \quad \hat{f}_1 - \hat{\tilde{f}}_1 = \hat{\tilde{f}}_2 - \hat{f}_2 \qquad \Rightarrow \quad \hat{f}_1 + \hat{f}_2 = \hat{\tilde{f}}_1 + \hat{\tilde{f}}_1.$$

Lemma 3.26 (Fourier Transform of Gaussians) In $\mathbb{R}^d$,

$$\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$$

More generally for all $\lambda > 0$:

$$\widehat{e^{-\pi\lambda^2|x|^2}} = \lambda^{-d}e^{-\pi\frac{|k|^2}{\lambda^2}}$$

(exercise)

Remark 3.27 If $d \geqslant 3$

$$\hat{G}(k) = \frac{1}{|2\pi k|^2}$$
$$\Rightarrow G(x) = \left(\frac{1}{|2\pi k|^2}\right)^\vee = \frac{1}{d(d-2)(k)|x|^{d-2})} = \Phi(x)$$

3.3 Theory of Distribution

3.3.1 Basic Properties, Convergence, Derivatives

In the following let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\Omega \subseteq \mathbb{R}^d$ be open.

- $D(\Omega) = C_c^\infty(\Omega)$ the space of test functions.
- We say that $\phi_n \to \phi$ in $D(\Omega)$ if for all $K \subseteq \Omega$ compact we have that $\begin{cases} \operatorname{supp}(\phi_n - \phi) \subseteq K \text{ for all } n \\ \|D^\alpha(\phi_n - \phi)\|_{L^\infty(K)} \to 0 \text{ as } n \to \infty \text{ for all } \alpha \end{cases}$
- $D'(\Omega) = \{T : D(\Omega) \to \mathbb{K} \text{ linear and continuous}\}$ the space of distributions.

Motivation: $L^2(\Omega)' = L^2(\Omega)$, $(L^p(\Omega))' = (L^q(\Omega))$, $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3.28 ("normal functions" are distributions) If $f \in L^1_{loc}(\Omega)$, then $T = T_f$ defined by:
$$T(\phi) = \int_\Omega f(x)\phi(x)\,dx$$
is a distribution for all $\phi \in D(\Omega)$, i.e. $T \in D'(\Omega)$. Indeed, it is clear that $T(\phi)$ is well-defined for all $\phi \in D(\Omega)$ and $\phi \mapsto T(\phi)$ is linear. Let us check that $\phi \mapsto T(\phi)$ is continuous. Take $\phi_n \to \phi$ in $D(\Omega)$ and prove that $T(\phi_n) \to T(\phi)$. Since $\phi_n \to \phi$ in $D(\Omega)$, there is a compact K s.t. $\operatorname{supp}(\phi_n), \operatorname{supp}(\phi) \subseteq K \subseteq \Omega$.

Question: Why is $f \mapsto T_f$ injective?

Theorem 3.29 (Fundamental theorem of calculus of variants) Let $\Omega \subseteq \mathbb{R}^d$ be open. If $f, g \in L^1_{loc}(\Omega)$ and $\int_\Omega f\phi\,dy = \int_\Omega g\phi\,dy$ for all $\phi \in D(\Omega)$, then $f = g$ in $L^1_{loc}(\Omega)$

Example 3.30 (Dirac delta function) Let $\Omega \subseteq \mathbb{R}^d$ open and let $x_0 \in \Omega$. Define $T : D(\Omega) \to \mathbb{K}$ by $T(\phi) = \phi(x_0)$. Then $T \in D'(\Omega)$ and we denote it by δ_{x_0}. It is clear that $\phi \mapsto T(\phi) = \phi(x_0)$ is well-defined and linear for all $\phi \in D(\Omega)$. Take $\phi_n \to \phi$ in $D(\Omega)$ and prove $T(\phi_n) \to T(\phi)$, i.e. $\phi_n(x_0) \to \phi(x_0)$ (obvious.)

Example 3.31 (Principle Value) The function $f(x) = \frac{1}{x}$ is not in $L^1_{loc}(\mathbb{R})$, but we can still define
$$\int_\mathbb{R} f(x)\phi(x)\,dx = \int_\mathbb{R} \frac{\phi(x)}{x}\,dx$$
for all $\phi \in D(\mathbb{R})$ s.t. $\phi(0) = 0$. In fact,
$$\phi(x) = |\phi(x) - \phi(0)| \leqslant x \sup |\phi'|,$$

so $\frac{|\phi(x)|}{|x|} \in L^\infty(\mathbb{R})$ and compactly supported. So $\frac{\phi(x)}{x} \in L^1(\mathbb{R})$. Define $T : D(\mathbb{R}) \to \mathbb{K}$ by

$$T(\phi) = \lim_{\epsilon \to 0} \int_{|x| \geqslant \epsilon} \frac{\phi(x)}{x}\, dx \quad \forall \phi \in D(\mathbb{R}) \text{ s.t. } \phi(0) = 0$$

We write $T = \frac{1}{x}$ and check that $T \in D'(\mathbb{R})$: For all $\epsilon > 0$ we have

$$\left| \frac{\phi(x)}{x} \right| \leqslant \frac{\|\phi\|_{L^\infty}}{\epsilon}$$

for all $|x| \geqslant \epsilon$ and ϕ is compactly supported. So we get for all $\epsilon > 0$:

$$\mathbb{1}(|x| \geqslant \epsilon) \frac{\phi(x)}{x} \in L^1(\mathbb{R}) \rightsquigarrow \int_{|x| \geqslant \epsilon} \frac{\phi(x)}{x}\, dx < \infty$$

We can write:

$$\int_{|x| \geqslant \epsilon} \frac{\phi(x)}{x}\, dx = \int_{|x| \geqslant 1} \frac{\phi(x)}{x}\, dx + \int_{\epsilon \leqslant |x| \leqslant 1} \frac{\phi(x)}{x}\, dx$$

The second part can be written as:

$$\int_{\epsilon \leqslant |x| \leqslant 1} \frac{\phi(x)}{x}\, dx = \int_\epsilon^1 \frac{\phi(x)}{x}\, dx + \int_{-1}^{-\epsilon} \frac{\phi(x)}{x}\, dx = \int_\epsilon^1 \frac{\phi(x) - \phi(-x)}{x}\, dx$$

Since $\phi \in C_c^\infty(\mathbb{R})$ it holds that $|\phi(x) - \phi(-x)| \leqslant 2\|\phi'\|_{L^\infty}(x)$.

$$\Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^\infty(\mathbb{R}) \Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^1((0,1))$$

$$\Rightarrow \int_0^1 \frac{\phi(x) - \phi(-x)}{x}\, dx = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{\phi(x) - \phi(-x)}{x}\, dx$$

Remark 3.32 The function $\frac{1}{|x|^d}$ is not in $L^1_{loc}(\mathbb{R}^d)$ but $\exists T \in D'(\mathbb{R}^d)$ s.t. $T(\phi) = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^d}\, dx$ for all $\phi \in C_c^\infty(\mathbb{R}^d)$ s.t. $\phi(0) = 0$

Definition 3.33 (Derivatives of distributions) Let $\Omega \subseteq \mathbb{R}^d$ and $T \in D'(\Omega)$. Define for $\alpha \in \mathbb{N}^d$:

$$\begin{aligned} D^\alpha T : \ D(\Omega) &\longrightarrow \mathbb{K} \\ \phi &\longmapsto (-1)^{|\alpha|} T(D^\alpha \phi) \end{aligned}$$

Motivation: $f \in C_c^\infty(\Omega)$

$$\int_\Omega (D^\alpha f)\phi = (-1)^{|\alpha|} \int_\Omega f(D^\alpha \phi)$$

„If the classical derivative exists, then it is the same as the distributional derivative.“
We write

$$(D^\alpha T)(\phi) = T_{D^\alpha f}(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi).$$

Remark 3.34 For all $T \in D'(\Omega)$ it holds $D^\alpha T \in D'(\Omega)$ for all $\alpha \in \mathbb{N}^d$. Clearly

$$\phi \longmapsto (D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

is linear. Moreover, if $\phi_n \to \phi$ in $D(\Omega)$, then $D^\alpha \phi_n \to D^\alpha \phi$ in $D(\Omega)$, so

$$(D^\alpha T)(\phi_n) = (-1)^{|\alpha|} T(D^\alpha \phi_n) \xrightarrow{n \to \infty} (-1)^{|\alpha|} T(D^\alpha \phi) = (D^\alpha T)(\phi)$$

Example 3.35 Consider $f : x \mapsto |x|$, then $f \in C(\mathbb{R})$ but $f \notin C^1(\mathbb{R})$. However,

$$f'(x) = g(x) := \begin{cases} 1 & x \geqslant 0 \\ -1 & x < 0 \end{cases} \in L^1_{loc} \quad \text{in } D'(\mathbb{R})$$

Lets check $f' = g$, i.e. $-f(\phi') = f'(\phi) \overset{!}{=} g(\phi)$ for all $\phi \in D(\mathbb{R})$. Thus we need to prove:

$$-\int_{\mathbb{R}} f(x)\phi'(x)\,dx = \int_{\mathbb{R}} g(x)\phi(x)\,dx \quad \forall \phi \in D(\mathbb{R})$$

namely we have to show

$$\underbrace{-\int_{\mathbb{R}} |x|\phi'(x)\,dx}_{:=(\star)} = \int_0^\infty \phi(x)\,dx - \int_{-\infty}^0 \phi(x)\,dx.$$

Now we have

$$(\star) = -\int_0^\infty x\phi'(x)\,dx + \int_{-\infty}^0 x\phi'(x)\,dx.$$

By integration by parts we have

$$\int_0^\infty x\phi'(x)\,dx = \underbrace{[x\phi(x)]_0^\infty}_{=0} - \int_0^\infty \phi(x)\,dx = -\int_0^\infty \phi(x)\,dx$$

and similary

$$\int_{-\infty}^0 x\phi'(x)\,dx = -\int_{-\infty}^0 \phi(x)\,dx.$$

Thus $f' = g$ in $D'(\Omega)$. We claim that $g' = 2\delta_0$ in $D'(\mathbb{R})$. In fact, for all $\phi \in D(\mathbb{R})$, then:

$$\begin{aligned} g'(\phi) &= -g(\phi') = -\int_{\mathbb{R}} g\phi'\,dx = -\int_{-\infty}^0 (-1)\phi'\,dx - \int_0^\infty (1)\phi'\,dx \\ &= -\int_0^\infty \phi'\,dx + \int_{-\infty}^0 \phi'\,dx = [\phi(0) - \underbrace{\phi(\infty)}_{=0}] + [\phi(0) - \underbrace{\phi(-\infty)}_{=0}] \\ &= 2\phi(0) = 2\delta_0(\phi) \end{aligned}$$

So $g' = 2\delta_0$ in $D'(\mathbb{R})$.

Exercise 3.36 Prove that $(D^\alpha\delta_x)(\phi) = (-1)^{|\alpha|}(D^\alpha\phi)(x)$ for all $\phi \in D(\mathbb{R})$ for all $x \in \mathbb{R}$.

Definition 3.37 (Convergence of distributions) Let $\Omega \subseteq \mathbb{R}^d$ be open, then

$$T_n \xrightarrow{n\to\infty} T$$

in $D'(\Omega)$ if $T_n(\phi) \xrightarrow{n\to\infty} T(\phi)$ for all $\phi \in D(\Omega)$.

Exercise 3.38 Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ For $\epsilon > 0$, define $f_\epsilon(x) = \epsilon^{-d}f(\epsilon^{-1}x)$. Then: $f_\epsilon \to \delta_0$ in $D'(\Omega)$.

Exercise 3.39 Let $\Omega \subseteq \mathbb{R}^d$ be open and $T_n \to T$ in $D'(\Omega)$. Then: $D^\alpha T_n \to D^\alpha T$ in $D'(\Omega)$ for all $\alpha = (\alpha_1, \ldots, \alpha_d)$

Definition 3.40 (Convolution of distributions) Let $T \in D'(\mathbb{R})$ and $f \in L_c^\infty(\mathbb{R}^d)$. Define

$$(T \star f)(y) = T(f_y)$$

We write $f_y(x) = f(x-y)$ and $\tilde{f}(x) = f(-x)$.

Theorem 3.41 Let $T \in D'(\mathbb{R})$. Then for all $f \in D(\mathbb{R})$:

1. $y \mapsto T(f_y)$ is $C^\infty(\mathbb{R}^d)$ and

$$D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D^\alpha f_y)$$

2. If $g \in L^1(\mathbb{R}^d)$ and g is compactly supported, then

$$\int_{\mathbb{R}^d} g(y) T(f_y)\, dy = T(\underbrace{f \star g}_{\in C_c^\infty(\mathbb{R})})$$

Proof. 1. We prove that $y \mapsto T(f_y)$ is continuous. Take $y_n \to y$ in $\mathbb{R}^d$, then:

$$T(f_{y_n}) \to T(f_y)$$

since $f_{y_n} \to f_y$ in $D(\mathbb{R}^d)$. We check this: Since $f \subseteq C_c^\infty(\mathbb{R}^d)$, it holds that $\operatorname{supp} f \subseteq B(0, R) \subseteq \mathbb{R}^d$. Since $y_n \to y$ in $\mathbb{R}^d$. We have $\sup_n |y_n| < \infty$. Thus f_{y_n}, f_y are supported in $\overline{B(0, R + \sup_n |y_n|)} = K$ compact. Moreover

$$|f_{y_n}(x) - f_y(x)| = |f(x - y_n) - f(x-y))| \leqslant \|\nabla f\|_{L^\infty} \|y_n - y\| \to 0$$

So we get $\|f_{y_n} - f_y\|_{L^\infty} \to 0$ Similary:

$$\|D^\alpha f_{y_n} - D^\alpha f_n\|_{L^\infty} \to 0$$

■

Exercise 3.42 (E 3.4) Compute the Fourier Transform of the Gaussian.

Exercise 3.43 (Bonus 3) Let $f \in L^1(\mathbb{R}^d)$ such that

$$|\hat{f}(k)| \leqslant \frac{C_N}{(1+|k|)^N}$$

for all $k \in \mathbb{R}^d$, for all $N \geqslant 1$. (C_N is independent of k). Prove that $f \in C^\infty(\mathbb{R}^d)$

($f \in C^\infty$) i.e. $\exists \tilde{f} \in C^\infty$ s.t. $f = \tilde{f}$ a.e.

Theorem 3.44 Take $T \in D'(\mathbb{R}), f \in C_c^\infty(\mathbb{R}^d) = D(\mathbb{R}^d)$, $f_y(x) = f(x-y)$

a) $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D_x^\alpha f_y)$

b) $\forall g \in L^1(\mathbb{R}^d)$ and compactly supported

$$\int_{\mathbb{R}^d} g(y) T(f_y)\, dy = T(\underbrace{f \star g}_{\in C_c^\infty})$$

Proof. a) $y \mapsto T(f_y)$ is continuous since $y_n \to y$ in $\mathbb{R}^d$, then $f_{y_n} \to f_y$ implies $T(f_{y_n}) \to T(f_y)$. Let's check that $y \mapsto T(f_y) \in C^1$:

$$\lim_{h\to 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} = \lim_{h\to 0} T\left(\frac{f_{y-he_i} - f_y}{h}\right)$$

We have $\frac{f_{y-he_i}-f_y}{h} \xrightarrow{h\to 0} (\partial_i f)_y$ in $D(\mathbb{R}^d)$

- $\exists K$ compact set such that $\operatorname{supp}(f_{y-e_i} - f_y)$, $\operatorname{supp} \partial_i f \subseteq K$ as $|h|$ small.
- $$\frac{f_{y-he_i}(x) - f_y(x)}{h} - (\partial_i f)_y(x)$$
  $$= \frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y)$$
  $$\left|\int_0^1 \partial_i f(x-y+the_i)\,dt - \partial_i f(x-y)\right| \xrightarrow{h\to 0} 0 \text{ uniformly in } x$$
  Similary:
  $$\left|D_x^\alpha\left(\frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y)\right)\right|$$
  $$= \left|\frac{D^\alpha f(x-y+he_i) - D^\alpha f(x-y)}{h} - \partial_i(D^\alpha f)(x-y)\right| \xrightarrow{h\to 0} 0$$
  uniformly in x. Conclude:
  $$\lim_{h\to 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} \xrightarrow{h\to 0} T((\partial_i f)_y) \in C(\mathbb{R}^d)$$
  So we geht that $y \mapsto T(f_y) \in C^1$ and $-\partial_{y_i} T(f_y) = T((\partial_i f)_y)$

By induction:

$$D_y^\alpha T(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y) = (D^\alpha T)(f_y) \quad \forall \alpha \in \mathbb{N}^d$$

b) Heuristic: $T = T(x)$

$$\begin{aligned}\int_{\mathbb{R}^d} g(y)T(f_y)\,dy &= \int_{\mathbb{R}^d} g(y)\left(\int_{\mathbb{R}^d} T(x)f(x-y)\,dx\right)dy\\ &= \int_{\mathbb{R}^d} T(x)\left(\int_{\mathbb{R}^d} g(y)f(x-y)\,dy\right)dx\\ &= \int_{\mathbb{R}^d} T(x)(f\star g)(x)\,dx = T(f\star g)\end{aligned}$$

Step 1: $g \in C_c^\infty(\mathbb{R}^d)$

$$\text{(Rieman Sum)} \quad \begin{aligned}\int_{\mathbb{R}^d} g(y)T(f_y)\,dy &= \lim_{\Delta_N\to 0} \Delta_N \sum_{j=1}^N g(y_j)T(f_{y_j})\\ &= \lim_{\Delta_N\to 0} T\left(\Delta_N \sum_{j=1}^N g(y_j) f_{y_j}\right)\\ &= T(f\star g)\end{aligned}$$

because

$$\lim_{\Delta_N\to 0} \Delta_N \sum_{j=1}^N g(y_j)f_{y_j}(x) \to (f\star g)(x) \text{ in } D(\mathbb{R}^d)$$

$$\lim_{\Delta_N\to 0} \Delta_N \sum_{j=1}^N g(y_j)f(x-y_j) \xrightarrow{\text{Riemann}} \int_{\mathbb{R}^d} g(y)f(x-y)\,dy = (f\star g)(x)$$

Proof of:

$$\lim_{\Delta_N \to 0} \Delta_N \sum_{j=1}^{N} g(y_j) f(x - y_j) \to (f \star g)(x) \text{ in } D(\mathbb{R}^d)$$

1) Since $f, g \in C_c^\infty$ we have $f \star g \in C_c^\infty$. And we have

$$x \mapsto \Delta_N \sum_{j=1}^{N} g(y_j) f(x - y_j) \in C^\infty$$

since $f \in C^\infty$ supported in $(\operatorname{supp} g + \operatorname{supp} f)$. So all functions are C_c^∞ and supported in $(\operatorname{supp} g + \operatorname{supp} f)$.

2)

$$\left| \lim_{\Delta_N \to 0} \Delta_N \sum_{j=1}^{N} g(y_j) f(x - y_j) - \int_{\mathbb{R}^d} g(y) f(x - y)\, dy \right| \xrightarrow{\Delta_N \to 0} 0$$

uniformly in x. (Result from the Riemann-Sum)

3)

$$\begin{aligned} &\left| D_x^\alpha (\Delta_N \sum_{j=1}^{N} g(y_j) f(x - y) - (f \star g)(x)) \right| \\ &= \left| \Delta_N \sum_{j=1}^{N} g(y_j) D^\alpha f(x - y) - (D^\alpha f) \star g(x) \right| \xrightarrow{\Delta_N \to 0} 0 \end{aligned}$$

uniformly in x for all α.

Step 2: Take $g \in L^1(\mathbb{R}^d)$ and compactly supported. Then $\exists \{g_n\} \subseteq C_c^\infty(\mathbb{R}^d)$, $\operatorname{supp} g_n \subseteq \operatorname{supp} g + B(0,1)$ such that $g_n \to g$ in $L^1(\mathbb{R}^d)$. By Step 1:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y)\, dy = T(g_n \star f)$$

Take $n \to \infty$:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y)\, dy \to \int_{\mathbb{R}^d} g(y) T(f_y)\, dy$$

since $g_n \to g$ in L^1 compactly supported and $y \mapsto T(f_y) \in C^\infty \subseteq L^\infty(K)$. Moreover (exercise):

$$\underbrace{g_n \star f}_{\in C_c^\infty} \to g \star f \quad \text{in } D(\mathbb{R}^d)$$

So $T(g_n \star f) \xrightarrow{n \to \infty} T(g \star f)$. Finally we optain:

$$\int g(y) T(f_n)\, dy = T(g \star f) \qquad \blacksquare$$

Theorem 3.45 Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $T \in D'(\Omega)$ and $f \in C_c^\infty(\Omega)$. Denote

$$\Omega_f = \{y \in \mathbb{R}^d \mid \operatorname{supp} f_y = y + \operatorname{supp} f \subseteq \Omega\}$$

a) $y \mapsto T(f_y) \in C^\infty(\Omega_f)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y)$

b) For all $g \in L^1(\Omega_g)$ compactly supported in Ω_f and it holds:

$$\int_\Omega g(y)T(f_y)\,dy = T(f \star g).$$

Theorem 3.46 Let $T \in D'(\Omega)$ s.t. $\nabla T = 0$ in $D'(\Omega)$. Then: $T = const.$ in Ω.

Proof. ($\Omega = \mathbb{R}^d$) We show for all $f \in C_c^\infty$ that $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $\partial_{y_i} T(f_y) = (\partial_j T)(f_y) = 0$ for all $i = 1, \dots, d$. Then by the result of the theorem for C^∞ functions, $y \mapsto T(f_y) = const$ independent of y. Consequently:

$$T(f_y) = T(f_0) = T(f) \quad \forall y \in \mathbb{R}^d \; \forall f \in C_c^\infty(\mathbb{R}^d)$$

For any $g \in C^\infty(\mathbb{R}^d)$:

$$\left(\int_{\mathbb{R}^d} g\,dy\right) T(f) = \int_{\mathbb{R}^d} g(y)T(f_y)\,dy = T(f \star g) = T(g \star f) = \left(\int_{\mathbb{R}^d} f\,dy\right) T(g)$$

So $\frac{T(f)}{\int_{\mathbb{R}^d} f}$ is independent of f (as soon as $\int f \neq 0$). So we get that $T(f) = const \int_{\mathbb{R}^d} f$, where const is independent of f. ■

Remark 3.47 If $u \in C^1(\mathbb{R}^d)$, then:

$$u(x+y) - u(x) = \int_0^1 \sum_{j=1}^d y_j (\partial_j u)(x + ty_j)\,dt = \int_0^1 y \nabla u(x+ty)\,dt$$

So we get that if $\nabla u = 0$, then $u(x+y) - u(x) = 0$ for all x, y, so $u = const$.

Theorem 3.48 (Taylor expansion for distributions) Let $T \in D'(\mathbb{R}^d)$ and $f \in C_c^\infty(\mathbb{R}^d)$. Then $y \mapsto T(f_y) \in C^\infty$ and

$$T(f_y) - T(f) = \int_0^1 \sum_{j=1}^d y_j (\partial_j T)(f_{ty})\,dt.$$

In particular, if $g \in L^1_{loc}$ and $\nabla g \in L^1_{loc}$, then $\forall y \in \mathbb{R}^d$:

$$g(x+y) - g(x) = \int_0^1 g(x+ty) y\,dt$$

for a.e. $x \in \mathbb{R}^d$.

Proof. $y \mapsto T(f_y)$ is C^∞ and $\frac{d}{dt}[T(f_{ty})] = (\nabla T)(f_{ty})y$ So we get

$$\begin{aligned} T(f_y) - T(f) &= \int_0^1 \frac{d}{dt}(T(f_{ty}))\,dt \\ &= \int_0^1 (\nabla T)(f_{ty}) y\,dt \\ &= \int_0^1 \sum_{j=1}^d (\partial_j T)(f_{ty}) y_j\,dt \end{aligned}$$ ■

Corrolary 3.49 Let $g \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\partial_j g \in L^1_{loc}(\mathbb{R}^d)$ for all $j = 1, 2, \ldots, d$ (i.e. $g \in W^{1,1}_{loc}(\mathbb{R}^d)$). Then for all $y \in \mathbb{R}^d$:

$$\begin{aligned}
g(x+y) - g(x) &= \int_0^1 y \cdot \nabla g(x+ty)\, dt \\
&= \int_0^1 \sum_{j=1}^d y_j \partial g(x+ty)\, dt
\end{aligned}$$

for a.e. x.

Proof. For all $f \in C_c^\infty$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)[g(x+y) - g(x)]\, dx &= \int_{\mathbb{R}^d} g(x)[f(x-y) - f(x)]\, dx \\
&= g(f_y) - g(f) \\
&= \int_0^1 \sum_{j=1}^d y_j (\partial_j g)(f_{ty})\, dt \\
&= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \sum_{j=1}^d y_j \left[\int_{\mathbb{R}^d} (\partial_j g)(x) f_{ty}(x)\, dx\right] \\
&= \int_0^1 \sum_{j=1}^d y_i \left[\int_{\mathbb{R}^d} (\partial_j g)(x+ty) f(x)\, dx\right] dt \\
&= \int_{\mathbb{R}^d} f(x) \left[\int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty)\, dt\right] dx
\end{aligned}$$

For all $\phi \in C_c^\infty$: $= g(x+y) - g(x)$ a.e. $x \in \mathbb{R}^d$. ∎

Remark 3.50 If $T \in D'(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open, if $y\nabla T = 0$, then $T = const$.

Theorem 3.51 (Equivalence of the classical and distributional derivatives) Let $\Omega \subseteq \mathbb{R}^d$. Then the following are equivalent:

1. $T \in D'(\Omega)$ s.t. $\partial_{x_i} T = g_i \in C(\Omega)$ for all $i = 1, \ldots, d$.
2. $T = f \in C^1(\Omega)$ and $g_i = \partial_{x_i} f$

Proof.

(2) ⇒ (1): If $T = f \in C^1(\Omega)$, then: $\partial_{x_i} f \in C(\Omega)$.

$$\partial_{x_i} T(\phi) = -T(\partial_{x_i}\phi) = -\int_\Omega f(\partial_{x_i}\phi) = \int_\Omega (\partial_{x_i} f)\phi$$

for all $\phi \in D(\Omega)$, so $\partial_{x_i} T = \partial_{x_i} f$.

(1) ⇒ (2): Why is $T = f \in C^1(\Omega)$? As $\partial_{x_i} f = g_i$:

$$f(x+y) - f(x) = \int_0^1 \nabla f(x+ty) y\, dt = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i\, dt$$

So we get

$$f(y) = f(0) + \int_0^1 \sum_{i=1}^d g_i(ty) g_i \, dt.$$

We expect that $f \in C^1$ and $\partial_{x_i} f = g_i$. But this is not trivial to prove.

$$\begin{aligned}
\frac{f(y+he_i) - f(y)}{h} &= \int_0^1 \sum_{i=1}^d [g_i(ty + the_i)(y_i + h\delta_{ij})] \, dt \\
&= \int_0^1 g_i(ty + the_i) \, dt + \int_0^1 \sum_{j \neq i} \frac{[g_i(ty + the_i) - g_i(ty)]}{h} y_i \, dt \\
&\xrightarrow{h \to 0} \int_0^1 g_i(ty) \, dt + \text{is difficult ...}
\end{aligned}$$

Lets take $\phi \in C_c^\infty$, then:

$$\begin{aligned}
T(\phi_y) - T(\phi) &= \int_0^1 \underbrace{\nabla T}_{(g_i)_{i=1}^d} (\phi_{ty}) y \, dt \\
&= \int_0^1 \sum_{i=1}^d \left(\int_\Omega g_i(x) \underbrace{\phi_{ty}}_{=\phi(x-ty)} \, dx \right) dt \\
&= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_0^1 g_i(x) \phi(x - ty) y_i \; dt \right) dx \\
&= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_0^1 g_i(x + ty) \phi(x) y_i \, dt \right) dx \\
&= \int_{\mathbb{R}^d} \left(\sum_i \int_0^1 g_i(x + ty) y_i \, dt \right) \phi(x) \, dx
\end{aligned}$$

Integrating against $\psi(y)$ with $\psi \in C_c^\infty$:

$$\begin{aligned}
&\int_{\mathbb{R}^d} T(\phi_y)\psi(y) \, dy - T(\phi) \int_{\mathbb{R}^d} \psi(y) \, dy \\
&\quad = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \sum_i \int_0^1 g_i(x + ty) y_i \psi(y) \, dt \, dy \right) \psi(x) \, dx
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow T(\phi \star \psi) - T(\phi) \int \psi = \dots \\
&\Rightarrow \int_{\mathbb{R}^d} T(\psi_y)\phi(y) \, dy - T(\phi) \int \psi = \dots
\end{aligned}$$

Take $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\int \psi = 1$. Then:

$$T(\phi) = \int_{\mathbb{R}^d} \underbrace{T(\psi_x) - \left(\int_{\mathbb{R}^d} \sum_{i=1}^d \int_0^1 g_i(x + ty) y_i \psi(y) \, dt \, dy \right)}_{f(x)} \phi(x) \, dx$$

for all $\phi \in C_c^\infty$, so $T = f \in C(\Omega)$. Thus $T = f \in C(\Omega)$ and $\partial_{x_i} T = g_i \in C(\Omega)$. Then we need to prove that $f \in C^1(\Omega)$ and $\partial_{x_i} f = g_i$ (classical derivative). Since

$f \in W^{1,1}_{loc}$:

$$f(x+y) - f(x) = \int_0^1 \sum_{i=1}^d g_i(x+ty)y_i \, dt \quad \forall x, y$$

In particular:

$$\begin{aligned}
\frac{f(x+he_i) - f(x)}{h} &= \int_0^1 \frac{1}{h} \sum_{i=1}^d g_i(x+the_i)h\delta_{ij} \, dt \\
&= \int_0^1 g_i(x+the_i) \, dt \xrightarrow{h \to 0} g_i(x)
\end{aligned}$$

So we get $\partial_{x_i} f(x) = g_i(x) \in C(\Omega)$ in the classical sense. So $f \in C^1(\Omega)$. ■

3.3.2 Sobolev Spaces

Definition 3.52 (Sobolev Spaces) Let $\Omega \subseteq \mathbb{R}^d$ be open. We define for $1 \leqslant p \leqslant \infty$:

$$\begin{aligned}
W^{1,p}(\Omega) &= \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega) \ \forall i = 1, \dots, d\} \\
W^{k,p}(\Omega) &= \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leqslant k\} \\
W^{k,p}_{loc}(\Omega) &= \{f \in L^p_{loc}(\Omega) \mid D^\alpha f \in L^p_{loc}(\Omega) \ \forall |\alpha| \leqslant k\}
\end{aligned}$$

Theorem 3.53 (Approximation of $W^{1,p}_{loc}(\Omega)$ by $C^\infty(\Omega)$) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $f \in W^{1,p}_{loc}(\Omega)$. Then there exists $\{f_n\} \subseteq C^\infty(\Omega)$ such that $f_n \to f$ in $W^{1,p}_{loc}(\Omega)$, i.e. for all $K \subseteq \Omega$ compact: $\|f_n - f\|_{L^p(K)} + \sum_{i=1}^d \|\partial_{x_i}(f_n - f)\|_{L^p(K)} \to 0$.

Proof. Case $\Omega = \mathbb{R}^d$: Take $g \in C_c^\infty$, $\int g = 1$, $g_\epsilon(x) = \epsilon^{-d} g(\epsilon^{-1} x)$. Then $g_\epsilon \star f \in C_c^\infty$. Since $f \in L^p_{loc}(\Omega)$ we have $g_\epsilon \star f \to f$ in L^p_{loc} as $\epsilon \to 0$. Moreover $\partial_{x_i}(g_\epsilon \star f) = (g_\epsilon \star \partial_{x_i} f) \xrightarrow{\epsilon \to 0} \partial_{x_i} f$ in L^p_{loc}. Then we can take $f_n = g_{\frac{1}{n}} \star f$. ■

Remark 3.54 In general, if we want to compute the distributional derivative $D^\alpha f$, then we can find $f_n \to f$ in $D'(\Omega)$ and compute $D^\alpha f_n$. Then $D^\alpha f_n \to D^\alpha f$ in $D^\alpha(\Omega)$. As an example we can compute $\nabla |f|$ with $f \in W^{1,p}_{loc}(\Omega)$.

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Theorem 3.55 (Chain Rule) Let $G \in C^1(\mathbb{R}^d)$ with $|\nabla G|$ is bounded. Let $f = (f_i)_{i=1}^d \subseteq W^{1,p}_{loc}(\Omega)$. Then $x \mapsto G(f(x)) \in W^{1,p}_{loc}(\Omega)$ and

$$\partial_{x_i} G(f) = \sum_{k=1}^d (\partial_k G)(f) \cdot \partial_{x_i} f_k \quad \text{in } D'(\Omega).$$

Moreover, if $G(0) \in L^p(\Omega)$ (i.e. either $|\Omega| < \infty$ or $G(0) = 0$), then if $f = (f_i)_{i=1}^d \subseteq W^{1,p}(\Omega)$, then $G(f) \in W^{1,p}(\Omega)$.

Proof. Since $G \in C^1$ we have that G is bounded in any compact set. Moreover $\|\nabla G\|_{L^\infty} < \infty$ implies:

$$|G(f) - G(0)| \leqslant \|\nabla G\|_{L^\infty} |f| \in L^p_{loc}$$

So $G(f) \in L^p_{loc}$. Let us compute $\partial_{x_i} G(f)$. Let $\{f^{(n)}\}_{n=1}^\infty \subseteq C^\infty$ such that $f^{(n)} \to f$ in $W^{1,p}_{loc}$, then:

$$|G(f^{(n)}) - G(f)| \leqslant \|\nabla G\|_{L^\infty} |f^{(n)} - f| \to 0 \text{ in } L^p_{loc}$$

So $G(f^{(n)}) \to G(f)$ in L^p_{loc}, thus $\partial_{x_i} G(f^{(n)}) \to \partial_{x_i} G(f)$ in $D'(\Omega)$. On the other hand, by the standard Chain-Rule for C^1-functions:

$$\partial_{x_i} G(f^{(k)}) = \sum_{k=1}^d \underbrace{\partial_k G(f^{(k)})}_{(\text{b.d.}\to \partial_k G(f))} \underbrace{\partial_i f_k^{(n)}}_{(\to \partial_i f_k \text{ in } L^p(\Omega))} \to \sum_{k=1}^d \partial_k G(f) \partial_i f_k \text{ in } L^p_{loc}(\Omega)$$

Thus

$$\partial_{x_i} G(f) = \sum_{k=1}^d \underbrace{\partial_k G(f)}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p_{loc}} \in L^p_{loc} \text{ in } D'(\Omega)$$

So $G(f) \in W^{1,p}_{loc}(\Omega)$. Aussume that $G(0) \in L^p(\Omega)$ (i.e. $|\Omega| < \infty$ or $G(0) = 0$). If $f \in W^{1,p}(\Omega)$, then $G(f) \in W^{1,p}(\Omega)$ since

$$|G(f) - G(0)| \leqslant \|\nabla G\|_{L^\infty} |f| \in L^p \Rightarrow G(f) \in L^p$$

and

$$\partial_{x_i} G(f) = \sum_k \underbrace{\partial_k G}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p} \in L^p \Rightarrow G(f) \in W^{1,p}(\Omega) \qquad \blacksquare$$

Theorem 3.56 (Derivative of absolute value) Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $f \in W^{1,p}(\Omega)$. Then $|f| \in W^{1,p}(\Omega)$ and if f is real-valued:

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Proof. Exercise. Hint: Use the Chain-Rule for $G_\epsilon(x) = \sqrt{\epsilon^2 + x^2} - \epsilon \to |x|$ as $\epsilon \to 0$ $\blacksquare$

3.4 Distribution vs. measures

Let μ be a Borel measure in $\mathbb{R}^d$ s.t. $\mu(K) < \infty$ for all compact $K \subseteq \mathbb{R}^d$. Then define

$$\begin{aligned} T:\ D(\mathbb{R}^d) &\longrightarrow \mathbb{C} \\ \phi &\longmapsto \int_{\mathbb{R}^d} \phi(x)\, d\mu(x) \quad \forall \phi \in C_c^\infty \end{aligned}$$

$\rightsquigarrow$ T is a distribution since if $\phi_n \to \phi$ in $D(\Omega)$, then

$$|T(\phi_n) - T(\phi)| \leqslant \int_{\mathbb{R}^d} |\phi_n - \phi|\, d\mu(x) \leqslant \|\phi_n - \phi\|_{L^\infty} \left(\int_K d\mu \right) \xrightarrow{n\to\infty} 0$$

Example 3.57 ∂_0 in $D'(\mathbb{R}^d)$ is a Borel probability measure.

Theorem 3.58 (Positive distributions are measures) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $T \in D'(\Omega)$. Assume $T \geqslant 0$, i.e. $T(\phi) \geqslant 0$ for all $\phi \in D(\Omega)$ satisfying $\phi(x) \geqslant 0$ for all x. Then there is a Borel positive measure μ on Ω such that $\mu(K) < \infty$ for all $K \subseteq \Omega$ compact and:

$$T(\phi) = \int_\Omega \phi(x)\, d\mu(x) \quad \forall \phi \in D^(\Omega)$$

Proof. See Lieb-Loss Analysis. Sketch: If $O \subseteq \mathbb{R}^d$ is open, then

$$\mu(O) = \sup\{T(\phi) \mid \phi \in D(\Omega), 0 \leqslant \phi \leqslant 1, \operatorname{supp}\phi \subseteq O\}$$

For all $A \subseteq \Omega$ (not necessarily open),

$$\mu(A) = \inf\{\mu(O) \mid O \text{ open}, A \subseteq O\}$$

The mapping $\mu : 2^\Omega \to [0, \infty]$ is an outer measure, i.e.

1. $\mu(\varnothing) = 0$
2. $\mu(A) \leqslant \mu(B)$ if $A \subseteq B$
3. $\mu\left(\bigcup_{i=1}^\infty A_i\right) \leqslant \sum_{i=1}^\infty \mu(A_i)$

From the outer measure we can find a σ-algebra Σ and μ is a measure on Ω s.t. E is measurable iff

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^\complement).$$

So all open sets are measurable, thus outer regularity (by def $\mu(A) = \inf\{\mu(O) \mid O \text{ open } \supseteq A)$, so inner regularity $\mu(A) = \sup\{\mu(K) \mid K \text{ compact} \subseteq A\}$. ■

Exercise 3.59 (E 4.1) Prove that if $T_n \to T$ in $D'(\mathbb{R}^d)$, then $D^\alpha T_n \to D^\alpha T$ in $D'(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$.

Exercise 3.60 (E 4.2)

Exercise 3.61 (E 4.3) $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$. Then $f_\epsilon \to \delta_0$ in $D'(\mathbb{R}^d)$.

Exercise 3.62 (E 4.4) Let $\{f_n\} \subseteq L^1$, $\operatorname{supp} f \subseteq B(0,1)$, $f_n \to f$ in L^1. Prove for all $g \in C_c^\infty$ that $f_n \star g \to f \star g$ in $D(\mathbb{R}^d)$.

Solution. Since $f_n \in L^1$, $\operatorname{supp} f \subseteq B(0,1)$ and $g \in C_c^\infty$ we have $f_n \star g \in C_c^\infty$ and

$$\operatorname{supp}(f_n \star g) \subseteq (\operatorname{supp} g) + \overline{B(0,1)} = K.$$

Since $f_n \to f$ in L^1 there is a subsequence $f_{n_k} \to f$ almost everywhere, so f supp in $\overline{B(0,1)}$, so $f \star g \in C_c^\infty$, $\operatorname{supp}(f \star g) \subseteq K$. We have:

$$\begin{aligned} |f_n \star g(x) - f \star g(x)| &= \left| \int_{\mathbb{R}^d} (f_n(y) - f(y)) g(x-y)\, dy \right| \\ &\leqslant \int_{\mathbb{R}^d} |f_n(y) - f(y)||g(x-y)|\, dy \\ &\leqslant \|g\|_{L^\infty} \|f_n - f\|_{L^1} \xrightarrow{n\to\infty} 0 \end{aligned}$$

thus $\|f_n \star g - f \star g\|_{L^\infty} \to 0$. Similary:

$$\|D^\alpha(f_n \star g) - D^\alpha(f \star g)\|_{L^\infty} = \|f_n \star \underbrace{(D^\alpha g)}_{\in C_c^\infty} - f \star (D^\alpha g)\|_{L^\infty} \xrightarrow{n\to\infty} 0$$

for all $\alpha \in \mathbb{N}^d$, so $f_n \star g \to f \star g$ in $D(\mathbb{R}^d)$. ■

Exercise 3.63 (E 4.5) Compute distributional derivatives f', f'' of $f(x) = x|x-1|$.

Solution. We prove $f'(x) = g(x) := \begin{cases} 2x-1 & x > 1 \\ 1-2x & x < 1 \end{cases}$. Take $\phi \in C_c^\infty(\mathbb{R}^d)$.

$$\begin{aligned}
-f'(\phi) &= \int_{\mathbb{R}^d} f\phi'\,dy \\
&= \int_{-\infty}^{1} f\phi'\,dy + \int_1^\infty f\phi'\,dy \\
&= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 f'\phi\,dy + [f\phi]_1^\infty - \int_1^\infty f'\phi\,dy \\
&= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 g\phi\,dy + [f\phi]_1^\infty - \int_1^\infty g\phi\,dy \\
&= f(1-)\phi(1) - f(1+)\phi(1) - \int_{\mathbb{R}^d} g\phi\,dy \\
&= 0 - \int_{\mathbb{R}^d} g\phi\,dy
\end{aligned}$$

Now we compute $f'' = g'$. Take $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned}
-(g')(\phi) &= \int_{\mathbb{R}^d} g\phi'\,dy \\
&= \int_{-\infty}^1 g\phi'\,dy + \int_1^\infty g\phi'\,dy \\
&= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 g'\phi\,dy - \int_1^\infty g'\phi\,dy \\
&= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 (-2)\phi\,dy - \int_1^\infty 2\phi\,dy \\
&= -2\phi(1) + \int_{-\infty}^\infty [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x)\,dx \\
&= -2\delta_1(\phi) + \int_{-\infty}^\infty [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x)\,dx \\
\Rightarrow g' &= \underbrace{2\delta_1}_{\notin L^1_{loc}} - \underbrace{2\mathbb{1}_{(-\infty,1)} + 2\mathbb{1}_{(1,\infty)}}_{\int L^1_{loc}}
\end{aligned}$$

■

Chapter 4

Weak Solutions and Regularity

Definition 4.1 Consider the linear PDE:

$$\sum_{\alpha} c_\alpha D^\alpha u(x) = F(x), \quad c_\alpha \text{ constant}, F \text{ given}$$

A function u is called a weak solution (a distributional solution) if

$$\sum_{\alpha} c_\alpha D^\alpha u = F \quad \text{in } D'(\Omega).$$

Namely,

$$\sum_{\alpha} (-1)^{|\alpha|} c_\alpha \int_\Omega u D^\alpha \phi = \int_\Omega F\phi, \quad \forall \phi \in D(\Omega)$$

Regularity: Given some condition on the data F, what can we say about the smoothness of u? Can we say that the equation holds in the classical sense? We derived G (the solution of the Laplace Equation) before in two ways:

1. $\Delta G(x) = 0$ for all $x \neq 0$, assuming $G(x) = G(|x|)$ and $d \geqslant 2$
2. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$ for $d \geqslant 3$

Theorem 4.2 For all $d \geqslant 1$ we have $G \in L^1_{loc}(\mathbb{R}^d)$ and $-\Delta G = \delta_0$ in $D'(\mathbb{R}^d)$.

Proof. Take $\phi \in D(\mathbb{R}^d)$. Then:

$$\begin{aligned}(-\Delta G_y)(\phi) = G_y(-\Delta\phi) &= \int_{\mathbb{R}^d} G_y(x)(-\Delta\phi)(x)\,dx \\ &= \int_{\mathbb{R}^d} G(y-x)(-\Delta\phi)(x)\,dx \\ &= [G \star (-\Delta\phi)](y) = (-\Delta)(G \star \phi)(y)\end{aligned}$$

Recall for all $f \in C^2$, $-\Delta(G \star f) = f$ pointwise. So we can conclude $-\Delta G_y = \delta_y$ in $D'(\mathbb{R}^d)$. ■

Remark 4.3 In $d = 1$, $G(x) = -\frac{1}{2}|x|$, so $-G'(x) = \mathrm{sgn}(x)/2$, so $-G''(x) = \delta_0$.

Remark 4.4 Formally:

$$-\Delta(G_y \star \phi) = (-\Delta G_y) \star \phi(x) = (\delta_0 \star \phi)(x) = \int_{\mathbb{R}^d} \delta_0(y)\phi(x-y)\,dy = \delta_0(\phi(x-\bullet))$$

Theorem 4.5 (Poisson's equation with L^1_{loc} data) Let $f \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\omega_d f \in L^1(\mathbb{R}^d)$ where

$$\omega_d(x) = \begin{cases} 1+|x| & d=1 \\ \log(1+|x|) & d=2 \\ \frac{1}{1+|x|^{d-2}} & d \geqslant 3, \end{cases}$$

then $u(x) = (G \star f)(x) \in L^1_{loc}(\mathbb{R}^d)$. Moreover $-\Delta u = f$ in $D'(\mathbb{R}^d)$. In fact, $u \in W^{1,1}_{loc}(\mathbb{R}^d)$ and:

$$\partial_{x_i} u(x) = (\partial_{x_i} G) \star f(x) = \int_{\mathbb{R}^d} (\partial_{x_i} G)(x-y) f(y)\,dy$$

Remark 4.6 We can also replace $\mathbb{R}^d$ by Ω and get $-\Delta u = f$ in $D'(\Omega)$.

Proof of Theorem 4.5. First we check that $u \in L^1_{loc}$. Take any Ball $B(0,R) \subseteq \mathbb{R}^d$, prove $\int_B |u|\,dy < \infty$. We have

$$\begin{aligned} \int_B |u|\,dy &= \int_B \left| \int_{\mathbb{R}^d} G(x-y) f(y)\,dy \right| dx \\ &\leqslant \int_B \int_{\mathbb{R}^d} |G(x-y)||f(y)|\,dy\,dx \\ &= \int_{\mathbb{R}^d} \left(\int_B |G(x-y)\,dx \right) |f(y)|\,dy \end{aligned}$$

If $y \notin B = B(0,R)$, then by Newtons's theorem (Mean-value theorem):

$$\int_{B(0,R)} |G(x-y)|\,dx = |B(0,R)||G(y)| \leqslant C|B|\omega_d(y)$$

If $y \in B$, then $|y| \leqslant R$, so $|x-y| \leqslant 2R$ if $x \in B$.

$$\int_{B(0,R)} |G(x-y)|\,dx \leqslant \int_{|x-y|\leqslant 2R} |G(x-y)|\,dx = \int_{|z|\leqslant 2R} |G(z)|\,dz \leqslant c_R$$

as $G \in L^1_{loc}$. Thus

$$\int_B |u|\,dy \leqslant c_B \int_{|y|\geqslant R} \omega_d(y)|f(y)|\,dy + c_B \int_{|y|\leqslant R} |f(y)|\,dy < \infty$$

Let us prove $-\Delta u = f$ in $D'(\mathbb{R}^d)$. Take $\phi \in D(\mathbb{R}^d)$. Then:

$$\begin{aligned}
(-\Delta u)(\phi) &= u(-\Delta\phi) \\
&= \int_{\mathbb{R}^d} u(x)(-\Delta\phi)(x)\,dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(y)(-\Delta\phi)(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(y-x)f(y)(-\Delta\phi)(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d} [G \star (-\Delta\phi)](y)f(y)\,dy \\
&= \int_{\mathbb{R}^d} -\Delta(G \star \phi)(y)f(y)\,dy \\
&= \int_{\mathbb{R}^d} \phi(y)f(y)\,dy
\end{aligned}$$

So $-\Delta u = f$ in $D'(\mathbb{R}^d)$. We check that $\partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$. Note that

$$|\partial_i G(x)| \leqslant c\frac{1}{|x|^{d-1}} \in L^1_{loc}(\mathbb{R}^d)$$

and

$$\int_{B(0,R)} |\partial_i G(x-y)|\,dx \leqslant \begin{cases} C_r\omega_d(y) & |y| \geqslant R \\ C_r & |y| \leqslant R \end{cases}$$

So $\int_{B(0,R)} |(\partial_i G \star f)|(y) < \infty$ for all $R > 0$. For all $\phi \in D(\mathbb{R}^d)$:

$$\begin{aligned}
-(\partial_i u)(\phi) = u(\partial_i\phi) &= \int_{\mathbb{R}^d} u(x)\partial_i\phi(x)\,dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(y)\partial_i\phi(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(y-x)f(y)\partial_i\phi(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d} (G \star \partial_i^y\phi)(y)f(y)\,dy \\
&= \int_{\mathbb{R}^d} (\partial_i^y G \star \phi)(y)f(y)\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_i^y G(y-x)f(y)\phi(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} -(\partial_i G)(x-y)f(y)\phi(x)\,dx\,dy \\
&= -\int_{\mathbb{R}^d} (\partial_i G \star f)(x)\phi(x)\,dx
\end{aligned}$$

So $\partial_i u = \partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$. Thus $u \in L^1_{loc}$, $\partial_i u \in L^1_{loc}$ for all i. So $u \in W^{1,1}_{loc}(\mathbb{R}^d)$. ■

Regularity: We consider the Laplace Equation $\Delta u = 0$ in $\mathbb{R}^d$.

Lemma 4.7 (Weyl) If $\Omega \subseteq \mathbb{R}^d$ open and $T \in D'(\Omega)$ s.t. $\Delta T = 0$ in $D'(\Omega)$, then: $T = f \in C^\infty(\Omega)$ and f is a harmonic function.

Proof. ($\Omega = \mathbb{R}^d$). Take $\phi \in C_c^\infty$, then $y \mapsto T(\phi_y) = T(\phi(-y))$ is C^∞ and $\Delta_y T(\phi_y) = T((\Delta\phi)_y) = (\Delta T)(\phi_y) = 0$. Take $g \in C_c^\infty$, g is radial. Then:

$$\int_{\mathbb{R}^d} T(\phi_y) g(y)\, dy \overset{\text{(exercise)}}{=} \int_{\mathbb{R}^d} T(\phi) g(y)\, dy = T(\phi) \left(\int_{\mathbb{R}^d} g\, dy \right)$$

Exercise 4.8 Let $f \in C^\infty(\mathbb{R}^d)$ be a harmonic function and $g \in C_c^\infty$, g is radial. Then:

$$\int_{\mathbb{R}^d} f(x) g(x)\, dx = f(0) \left(\int_{\mathbb{R}^d} g(x)\, dx \right)$$

On the other hand:

$$\int_{\mathbb{R}^d} T(\phi_y) g(y)\, dy = T(\phi \star g) = T(g \star \phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y)\, dy$$

Take $\int_{\mathbb{R}^d} g\, dy = 1$, then:

$$T(\phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y)\, dy$$

For all $\phi \in C_c^\infty$. Then $T = T(g_y) \in C^\infty$ ∎

Now lets regard the Poisson Equation $-\Delta u = f$ in $D'(\mathbb{R}^d)$.

Remark 4.9 Any solution has the form $u = G \star g + h$ where $\Delta h = 0$ in $D'(\mathbb{R}^d)$. By Weyls Lemma (4.7), $h \in C^\infty$, then we only need to consider the regularity of $G \star f$.

Remark 4.10 The regularity is a *local question*, namely if we write

$$f = f_1 + f_2 = f\phi + f(1-\phi),$$

where $\phi = 1$ in a ball B and $\phi \in C_c^\infty$. Then $G \star f = G \star f_1 + G \star f_2$. Here $f_2 = f(1-\phi) = 0$ in B. With Weyls Lemma (4.7), $G \star f_2 \in C^\infty$.

Theorem 4.11 (Low Regularity of Poisson Equation) Lef $f \in L^p(\mathbb{R}^d)$ and compactly supported. Then

a) If $p \geqslant 1$, then

- $G \star f \in C^1(\mathbb{R}^d)$ if $d = 1$.
- $G \star f \in L^q_{loc}(\mathbb{R}^d)$ for any $q < \infty$ if $d = 2$.
- $G \star f \in L^q_{loc}(\mathbb{R}^d)$ for $q < \frac{d}{d-2}$ if $d \geqslant 3$.

b) If $\frac{d}{2} < p \leqslant d$, then $G \star f \in C^{0,\alpha}_{loc}(\mathbb{R}^d)$ for all $0 < \alpha < 2 - \frac{d}{p}$, i.e.

$$|(G \star f)(x) - (G \star f)(y)| \leqslant C_k |x-y|^\alpha \quad \forall x, y \in K$$

with K compact in $\mathbb{R}^d$.

c) If $p > d$, then $G \star f \in C^{1,\alpha}_{loc}(\mathbb{R}^d)$ for all $0 < \alpha < 1 - \frac{d}{p}$.

where G is den fundamental solution of the laplace equation.

Example 4.12 Let $r = |x|$

$$u(x) = \omega(r) = \log(|\log(r)|)$$

if $0 < r < \frac{1}{2}$, so u is well-defined in $B = B(0, \frac{1}{2})$. We conclude:

$$-\Delta_{\mathbb{R}^3} u(x) = -\omega''(r) - \frac{2\omega'(r)}{r} = f(x) \in L^{\frac{3}{2}(B)}$$

But the Theorem (b) tells us that if $f \in L^{\frac{3}{2}}$ then u is continuous but $u \notin C(B)$.

Proof of theorem 4.11. a) $(p = 1)$ Why is $G \star f \in L^q_{loc}$? Recall from the proof of Youngs inequality:

$$\begin{aligned} |(G \star f)(x)| &= \left| \int_{\mathbb{R}^d} G(x-y) f(y)\, dy \right| \\ \text{(Hölder)} \quad &= \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)|\, yd \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(y)|\, dy \right)^{\frac{1}{q'}} \end{aligned}$$

Where $\frac{1}{q} + \frac{1}{q'} = 1$. Then:

$$|(G \star f)(x)|^q \leqslant C \int_{\mathbb{R}^d} |G(x-y)|^q |f(y)|\, dy$$

For any Ball $B = B(0, R) \subseteq \mathbb{R}^d$:

$$\begin{aligned} \int_B |G \star f(x)|^q\, dx &\leqslant C \int_B \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)|\, dy \right) dx \\ &= C \int_{\mathbb{R}^d} \left(\int_B |G(x-y)|^q\, dx \right) |f(y)|\, dy \end{aligned}$$

$G(x) \sim \frac{1}{|x|^{d-2}} \rightsquigarrow |G|^q = \frac{1}{|x|^{(d-2)q}} \in L^1_{loc}(\mathbb{R}^d)$ if $(d-2)q < 2 \Leftrightarrow q < \frac{d}{d-2}$. Here, $y \in \operatorname{supp} f$, so $|y| \leqslant R_1$, then $|x - y| \leqslant R + R$ if $|x| \leqslant R$. With $y \in \operatorname{supp} f$, this implies:

$$\int_{B(0,R)} |G(x-y)|^q\, dx \leqslant \int_{|z| \leqslant R + R_1} |G(z)|^q\, dz < \infty$$

b)

$$(G \star f)(x) - (G \star f)(y) = \int_{\mathbb{R}^d} (G(x-z) - G(y-z)) f(z)\, dz$$

So

$$|G \star f(x) - (G \star f)(y)| \leqslant C \int_{\mathbb{R}^d} \left| \frac{1}{|x-z|^{d-2}} - \frac{1}{|y-z|^{d-2}} \right| |f(z)|\, dz$$

for all $x, y \in \mathbb{R}^d$:

$$\begin{aligned} \left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| &= \left| \left(\frac{1}{|x|} - \frac{1}{|y|} \right) \left(\frac{1}{|x|^{d-3}} + \cdots + \frac{1}{|y|^{d-3}} \right) \right| \\ &\leqslant C \frac{||x| - |y||}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &= C \frac{|x-y|}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &\leqslant C \max(|x|, |y|)^{1-\alpha} \frac{|x-y|^\alpha}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \end{aligned}$$

as

$$||x|-|y|| \leqslant \min\left(|x-y|, \max(|x|,|y|)\right) \leqslant |x-y|^{\alpha} \max(|x|,|y|)^{1-\alpha}$$

Thus, for all $x, y \in \mathbb{R}^d$:

$$\begin{aligned}\left|\frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}}\right| &\leqslant C|x-y|^{\alpha} \frac{\max(|x|,|y|)^{1-\alpha}}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}}\right)\\ &\leqslant C|x-y|^{\alpha} \max\left(\frac{1}{|x|^{d-2+\alpha}}, \frac{1}{|y|^{d-2+\alpha}}\right)\end{aligned}$$

So we get

$$\left|\frac{1}{|x-y|^{d-2}} - \frac{1}{|y-z|^{d-2}}\right| \leqslant C|x-y|^{\alpha} \max\left(\frac{1}{|x-z|^{d-2+\alpha}}, \frac{1}{|y-z|^{d-2+\alpha}}\right)$$

Therefore:

$$\begin{aligned}&|G\star f(x) - G\star f(y)|\\ &\leqslant C\int_{\mathbb{R}^d} |x-y|^{\alpha} \max\left(\frac{1}{|x-z|^{d-2+\alpha}}, \frac{1}{|y-z|^{d-2+\alpha}}\right) |f(z)|\, dz\\ &\leqslant C|x-y|^{\alpha}\left(\sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi-z|^{d-2+\alpha}} |f(z)|\, dz\right)\end{aligned}$$

Claim: If $f \in L^p(\mathbb{R}^d)$ is compactly supported, $d \geqslant p > \frac{d}{2}$, then:

$$\sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi-z|^{d-2+d}} |f(z)|\, dz < \infty$$

for all $0 < \alpha < 2 - \frac{d}{p}$. Assume $\operatorname{supp} f \subseteq \overline{B(0,R_1)}$. Consider 2 cases:

- If $|\xi| > 2R_1$, then: $|\xi - z| \geqslant R_1$ for all $z \in B(0,R_1)$. Hence:

$$\int_{\mathbb{R}^d} \frac{1}{|\xi-z|^{d-2+\alpha}} |f(z)|\, dz \leqslant \frac{1}{R_1^{d-2+\alpha}} \|f\|_{L^1} < \infty$$

- If $|\xi| \leqslant 2R_1$, then: $|\xi - z| \leqslant 3R_1$ for all $z \in B(0,R_1)$:

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1}{|\xi-z|^{d-2+\alpha}} |f(z)|\, dz &\leqslant \int_{|\xi-z|\leqslant 3R_1} \frac{1}{|\xi-z|^{d-2+\alpha}} |f(z)|\, dz\\ \text{(Hölder)}, \left(\frac{1}{p}+\frac{1}{q}=1\right) \quad &\leqslant \left(\int_{\mathbb{R}^d} |f(z)|^p\, dz\right)^{\frac{1}{p}}\\ &\quad \cdot \left(\int_{|\xi-z|\leqslant 3R_1} \frac{1}{|\xi-z|^{(d-2+\alpha)q}}\right)^{\frac{1}{q}}\\ &= \|f\|_{L^p} \left(\int_{|z|\leqslant 3R_1} \frac{1}{|z|^{(d-2+\alpha)q}}\, dz\right)^{\frac{1}{q}} < \infty\end{aligned}$$

c) $(d \geqslant 3)$ We already know:

$$\partial_i(G\star f) = (\partial_i G \star f) \in L^1_{loc}(\mathbb{R}^d)$$

as $\omega_d f \in L^1(\mathbb{R}^d)$. We claim that $\partial_i G \star f \in C^{0,\alpha}(\mathbb{R}^d)$. So $G \star f \in C^{1,\alpha}(\mathbb{R}^d)$ by the equivalence between the classical and the distributional derivatives. Exercise. Hint:

$$|\partial_i G \star f(x) - \partial_i G \star f(y)| \leqslant \int_{\mathbb{R}^d} |\partial_i G(x-z) - \partial_i G(y-z)||f(z)|\,dz,$$

$\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$. $\rightsquigarrow$ Need to estimate $|\partial_i G(x) - \partial_i G(y)| \leqslant C|x-y|^\alpha$. ■

Theorem 4.13 (High regularity for Poisson's equation) Let $f \in C^{0,\alpha}(\mathbb{R}^d)$, $0 < \alpha < 1$ be compactly supported. Then $G \star f \in C^{2,\alpha}(\mathbb{R}^d)$.

Remark 4.14 $(-\Delta u = f)$ and $f \in C(\mathbb{R}^d)$ does not imply that $u \in C^2(\mathbb{R}^d)$. (exercise)

Remark 4.15 If $f \in C^{k,\alpha}(\mathbb{R}^d)$, $k \in \{0, 1, \dots\}$, $0 < \alpha < 1$ is compactly supported, then $G \star f \in C^{k+2,\alpha}(\mathbb{R}^d)$. This more general statement is a consequence of the theorem since

$$D^\beta(G \star f) = G \star \underbrace{(D^\beta f)}_{\in C^{0,\alpha}}$$

for all $\beta = (\beta_1, \dots, \beta_d)$, $|\beta| \leqslant k$.

Proof of theorem 4.13. Since $f \in L^p$ for all $p \leqslant \infty$ by the low regularity (4.11) we have $G \star f \in C^{1,\alpha}$ and $\partial_i(G \star f) = \partial_i G \star f$ in the classical sense. We will compute the distributional derivatives $\partial_i \partial_j (G \star f)$ and prove that they are Hölder continuous. Compute $\partial_j \partial_i (G \star f)$: For all $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
-(\partial_j \partial_i G \star f)(\phi) &= (\underbrace{\partial_i (G \star f)}_{\in C})(\partial_j \phi) \\
&= \int_{\mathbb{R}^d} ((\partial_i G) \star f)(x) \partial_j \phi(x)\,dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i G(x-y) f(y) \partial_j \phi(x)\,dx\,dy \\
&= \int_{\mathbb{R}^d} f(y) \left[\int_{\mathbb{R}^d} \partial_i G(x-y) \partial \phi(x)\, x \right] dy \\
&\overset{?}{=} \int_{\mathbb{R}^d} \square \phi(y)\,dy
\end{aligned}$$

Recall: $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}, \partial_i \partial_j G(x) = \frac{1}{|B_1|}\left[\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{d}\right]\frac{1}{|x|^d}$. We have:

$$\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x)\,dx = \lim_{\epsilon \to 0^+} \int_{|x-y| \geqslant \epsilon} \partial_i G(x-y) \partial_j \phi(x)\,dx$$

By dominated convergence we have $|\partial_i G(x-y) \partial_j \phi(x)| \in L^1(dx)$. By the Gauss-Green-Theorem (2.2) for all $\epsilon > 0$:

$$\begin{aligned}
&\int_{|x-y| \geqslant \epsilon} \partial_i G(x-y) \partial_j \phi(x)\,dx \\
&\quad = \int_{\partial B(y,\epsilon)} \partial_i G(x-y) \phi(x) \omega_j\,dS(x) - \int_{|x-y| \geqslant \epsilon} \partial_j \partial_i G(x-y) \phi(x)\,dx
\end{aligned}$$

Where $\omega = \frac{x-y}{|x-y|}$. For the boundary term:

$$
\begin{aligned}
-\int_{\partial B(y,\epsilon)} \partial_i G(x-y)\phi(x)\omega_j \, dS(x) &= -\epsilon^{d-1}\int_{\partial B(0,1)} \partial_i G(\epsilon\omega)\phi(y+\epsilon\omega)\omega_j \, d\omega \\
(\star) \quad &= \int_{\partial B(0,1)} \frac{1}{d|B_1|}\omega_i\omega_j \phi(y+\epsilon\omega)\, d\omega \\
&\xrightarrow{\epsilon\to 0} \int_{\partial B(0,1)} \frac{1}{d|B_1|}\omega_i\omega_j\phi(y)\, d\omega \\
&= \frac{1}{d}\delta_{i,j}\phi(y)
\end{aligned}
$$

$(\star)$ $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$, so $\partial_i G(\epsilon\omega) = -\frac{-\omega_i}{d|B_1|}\frac{1}{\epsilon^{d-1}}$. for all $|\omega| = 1$.

Now we split:

$$
\begin{aligned}
&-\int_{|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)\phi(x)\, dx \\
&\quad = -\int_{|x-y|\geqslant 1} \partial_i\partial_j G(x-y)\phi(x)\, dx - \int_{1\geqslant|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)\phi(x)\, dx
\end{aligned}
$$

The key observation is: $\int_{\partial B(0,r)} \partial_i\partial_j G(x)\, dx = 0$ since

$$
\partial_i\partial_j G(x) = \frac{1}{|B_1|}\left(\omega_i\omega_j - \frac{\partial_{ij}}{d}\right)\frac{1}{|x|^d},
$$

$\omega = \frac{x}{|x|}$. For example if $i=1, j=2, r=1$:

$$
\int_{\partial B(0,1)} \partial_1\partial_2 G(x)\, dS(x) = \frac{1}{|B_1|}\int_{\partial B(0,1)} \omega_1\omega_2\, d\omega,
$$

$\partial B(0,1) = \{\omega \mid |\omega| = 1\}$. Consider: $\omega \mapsto R\omega, (\omega_1,\ldots,\omega_d) \mapsto (-\omega_1,\omega_2,\ldots,\omega_d)$. Then

$$
-\int_{1\geqslant|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)\phi(y)\, dx = 0.
$$

So

$$
-\int_{1\geqslant|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)\phi(x)\, dx = -\int_{1\geqslant|x-y|\geqslant\epsilon} \partial_i\partial_j G(x-y)(\phi(x)-\phi(y))\, dx
$$

In summary:

$$
\begin{aligned}
\partial_i\partial_j(G\star f)(\phi) &= \int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}^d}\partial_i G(x-y)\partial_j\phi(x)\, dx\right) dy \\
&= \int_{\mathbb{R}^d} f(y)\frac{1}{d}\partial_{ij}\phi(y)\, dy \\
&\quad - \int_{\mathbb{R}^d} f(y)\left(\int_{|x-y|>1}\partial_i\partial_j G(x-y)\phi(x)\, dx\right) \\
&\quad - \int_{\mathbb{R}^d}\left[\lim_{\epsilon\to 0}\int_{1\geqslant|x-y|\geqslant\epsilon} \underbrace{\partial_i\partial_j G(x-y)(\phi(x)-\phi(y))\, dx}_{\leqslant\frac{C}{|x-y|^d}|x-y|\|\nabla\phi\|_{L^\infty}\leqslant\frac{C}{|x-y|^{d-1}}\in L^1_{loc}(dx)\forall y}\right] dy \\
&= \int_{\mathbb{R}^d}\frac{\delta_{ij}}{d}f(x)\phi(x)\, dx - \int_{\mathbb{R}^d}\phi(x)\left(\int_{|x-y|>1}\partial_i\partial_j G(x-y)f(y)\, dy\right) dx \\
&\quad - \int_{\mathbb{R}^d}\phi(x)\left[\int_{|x-y|\leqslant 1}\partial_i\partial_j G(x-y)(f(y)-f(x))\, dy\right] dx
\end{aligned}
$$

Conclusion:

$$\begin{aligned}\partial_i\partial_j(G\star f)(x) &= -\frac{\delta_{ij}}{d}f(x)+\int_{|x-y|>1}\partial_i\partial_j G(x-y)f(y)\,dy\\ &\quad+\int_{|x-y|\leqslant 1}\partial_i\partial_j G(x-y)\left(f(y)-f(x)\right)\,dy\end{aligned}$$

The first term $f\in C^{0,\alpha}$. The second term is also at least $C^{0,\alpha}$ since $\partial_i\partial_j G(x)$ is smooth as $|x|>1$. We need to prove that the thirt term

$$W_{ij}(x)=\int_{|x-y|\leqslant 1}\partial_i\partial_j G(x-y)(f(y)-f(x))\,dy$$

is Hölder-continuous, $|W_{ij}(x)-W_{ij(y)}|\leqslant C|x-y|^{\alpha}$. Recall:

$$|\partial_i\partial_j G(x-y)(f(y)-f(x))|\leqslant C\frac{1}{|x-y|^d}|x-y|^{\alpha}=\frac{C}{|x-y|^{d-\alpha}}\in L^1_{loc}(dy)$$

We write

$$\begin{aligned}W_{ij}(x) &= \int_{|x-y|\leqslant 1}\partial_i\partial_j G(x-y)(f(y)-f(x))\,dy\\ &= \int_{|z|\leqslant 1}\partial_i\partial_j G(z)(f(x+z)-f(x))\,dz\end{aligned}$$

So we get:

$$W_{ij}-W_{ij}(y)=\int_{|z|\leqslant 1}\partial_i\partial_j G(z)(f(x+z)-f(y+z)-f(x)+f(y))\,dz$$

Easy thought: Use $\partial_i\partial_j G(z)|\leqslant\frac{C}{|z|^d}$ and

$$\begin{aligned}&|f(x+z)-f(y+z)-f(x)+f(y)|\\ &\quad\leqslant\begin{cases}|f(x+z)-f(x)|+|f(y+z)-f(y)|\leqslant C|z|^{\alpha}\\ |f(x+z)-f(y+z)|+|f(x)-f(y)|\leqslant C|x-y|^{\alpha}\end{cases}\end{aligned}$$

Thus:

$$\begin{aligned}|W_{ij}(x)-W_{ij}(y)| &\leqslant C\int_{|z|\leqslant 1}\frac{1}{|z|^d}\min(|z|^{\alpha},|x-y|^{\alpha})\,dz\\ &\leqslant C\int_{|z|\leqslant 1}\frac{1}{|z|^d}(|z|^{\alpha})^{\epsilon}(|x-y|^{\alpha})^{1-\epsilon},\quad 0<\epsilon<1\\ &\leqslant C\left(\int_{|z|\leqslant 1}\frac{1}{|z|^{d-\alpha\epsilon}}\right)|x-y|^{\alpha(1-\epsilon)}\\ &\leqslant C_{\epsilon}|x-y|^{\alpha(1-\epsilon)}\end{aligned}$$

thus it is easy to prove $|W_{ij}(x)-W_{ij}(y)|\leqslant C_{\alpha}|x-y|^{\alpha}$ for all $\alpha'\leqslant\alpha$. However, to get $\alpha'=\alpha$ we need a more precise estimate. We split:

$$W_{ij}(x)-W_{ij}(y)=\int_{|z|\leqslant 1}\cdots=\int_{|z|\leqslant min(4|x-y|,1)}+\int_{4|x-y|<|z|\leqslant 1}$$

For the first domain:

$$\begin{aligned}&\int_{|z|\leqslant 4|x-y|} |\partial_{ij}G(z)||f(x+z)-f(y+z)-f(y)+f(x)|\,dz\\&\qquad\leqslant C\int_{|z|\leqslant 4|x-y|}\frac{1}{|z|^d}|z|^\alpha\,dz = const\cdot|x-y|^\alpha\end{aligned}$$

For the second domain:

$$\begin{aligned}&\int_{4|x-y|<|z|\leqslant 1}\partial_{ij}G(z)(f(x+z)-f(y+z)+f(y)f(x))\,dz\\&\qquad=\int_{4|x-y|<z\leqslant 1}\partial_{ij}G(z)(f(x+z)-f(y+z))\,dz=(\ldots)\end{aligned}$$

since $\int_{4|x-y|<|z|\leqslant 1}\partial_{ij}G(z)\,dz=0$. Then

$$(\ldots)=\int_{4|x-y|<|z-x|\leqslant 1}\partial_{ij}G(z-x)f(z)\,dz-\int_{4|x-y|<|z-y|\leqslant 1}\partial_{ij}G(z-y)f(z)\,dz.$$

Denote $A=\{z\mid 4|x-y|<|z-x|\leqslant 1\}$, $B=\{z\mid 4|x-y|<|z-y|\leqslant 1\}$. Consider

$$\begin{aligned}&\int_A\partial_{ij}G(z-x)f(z)\,dz-\int_B\partial_{ij}G(z-y)f(z)\,dz\\&\qquad=\int_{A\setminus B}+\int_{B\setminus A}+\int_{A\cap B}(\partial_{ij}G(z-x)-\partial_{ij}G(z-y))f(z)\,dz\end{aligned}$$

Lets regard the intersection. We have

$$\begin{aligned}\partial_{ij}G(x)&=\frac{1}{|B_1|}\frac{1}{|x|^d}(\omega_i\omega_j-\frac{1}{d}\delta_{ij})\\|\partial_{ij}G(x)-\partial_{ij}G(y)|&\leqslant C|x-y|\left(\frac{1}{|x|^{d+1}}+\frac{1}{|y|^{d+1}}\right)\end{aligned}$$

Now,

$$|\partial_{ij}G(z-x)-\partial_{ij}G(z-y)|\leqslant C|x-y|\left(\frac{1}{|z-x|^{d+1}}+\frac{1}{|z-y|^{d+1}}\right)$$

So we have

$$\begin{aligned}&\left|\int_{A\cap B}(\partial_{ij}G(z-x)-\partial_{ij}G(z-y))f(z)\,dz\right|\\&\qquad\leqslant C\int_{A\cap B}|x-y|\left(\frac{1}{|z-x|^{d+1}}+\frac{1}{|z-y|^{d+1}}\right)|f(z)|\,dz=(\ldots)\end{aligned}$$

Now we replace $f(z)$ by $f(z)-f(x)$, then:

$$\begin{aligned}&\left|\int_{A\cap B}(\partial_{ij}G(z-x)-\partial_{ij}G(z-y))(f(z)-f(x))\,dz\right|\\&\quad\leqslant C\int_{A\cap B}|x-y|\left(\frac{1}{|z-x|^{d+1}}+\frac{1}{|z-y|^{d+1}}\right)|z-x|^\alpha\,dz\\&\quad=\underbrace{C\int_{A\cap B}|x-y|\frac{1}{|z-x|^{d+1-\alpha}}\,dz}_{(I)}+\underbrace{C\int_{A\cap B}|x-y|\frac{1}{|z-y|^{d+1}}|z-x|^\alpha\,dz}_{(II)}\end{aligned}$$

Now,

$$\begin{aligned}
(I) &\leqslant C|x-y| \int_{4|x-y|<|z-x|\leqslant 1} \frac{1}{|z-x|^{d+1-\alpha}}\, dz \\
&= C|x-y| \int_{4x-y<|z|\leqslant 1} \frac{1}{|z|^{d+1-\alpha}}\, dz \\
&\leqslant C|x-y| \int_{4|x-y|}^{1} \frac{1}{r^{d+1-\alpha}} r^{d-1}\, dr \\
&= C|x-y| \int_{4|x-y|}^{1} \frac{1}{r^{2-\alpha}}\, dr \\
&\leqslant C|x-y| \left[-1 + \frac{1}{(4|x-y|)^{1-\alpha}}\right] \\
&\leqslant C|x-y|^{\alpha}
\end{aligned}$$

$$\begin{aligned}
(II) &\leqslant C|x-y| \int_{A\cap B} \frac{1}{|z-y|^{d+1}} |z-x|^{\alpha}\, dz \\
&\leqslant C|x-y| \int_{A\cap B} \frac{1}{|z-y|^{d+1}} \left(|z-y|^{\alpha} + |x-y|^{\alpha}\right)\, dz \\
&\leqslant \underbrace{C|x-y| \int_{B} \frac{1}{|z-y|^{d+1-\alpha}}\, dz}_{\text{similar to (I)}} + C|x-y|^{1+\alpha} \int_{B} \frac{1}{|z-y|^{d+1}}\, dz
\end{aligned}$$

and

$$C|x-y|^{1+\alpha} \int_{B} \frac{1}{|z-y|^{d+1}}\, dz \leqslant \int_{4|x-y|} \frac{1}{r^{d+1}} r^{d-1}\, dr \leqslant \frac{C}{|x-y|}$$

Consider $A \backslash B$:

$$\left| \int_{A\backslash B} \right| \leqslant C\|f\|_{L^\infty} \int_{A\backslash B} \frac{1}{|z-x|^{d}}\, dz$$

where

$$\begin{aligned}
A &= \{z \mid 4|x-y| < |z-x| \leqslant 1\} \\
B &= \{z \mid 4|x-y| < |z-y| \leqslant 1\} \\
A\backslash B &= \{z \in A \mid |z-y| \leqslant 4|x-y|\} \cup \{z \in A \mid |z-y| > 1\} = E_1 \cup E_2
\end{aligned}$$

for

$$\begin{aligned}
E_1 &= \{z \mid |z-y| \leqslant 4|x-y| < |z-x| \leqslant 1\} \\
&\subseteq \{z \mid 4|x-y| \leqslant |x-z| \leqslant 5|x-y|\}.
\end{aligned}$$

$|x-z| \leqslant |x-y| + |y-z| \leqslant 5|x-y|$ in E_1. We have

$$\begin{aligned}
\int_{E_1} \frac{1}{|z-x|^d}\, dz &\leqslant \int_{4|x-y|\leqslant|x-z|\leqslant 5|x-y|} \frac{1}{|z-x|^{d-\alpha}}\, dz \\
&= \int_{4|x-y|\leqslant|z|\leqslant 5|x-y|} \frac{1}{|z|^{d-\alpha}}\, dz \\
&= \int_{4|x-y|} \frac{1}{r^d} r^{d-1}\, dr \\
&= \int_{4|x-y|} \frac{1}{r^{1-\alpha}}\, dr \\
&\leqslant C|x-y|^\alpha
\end{aligned}$$

Now in E_2: $|z-x| \geqslant |z-y| - |y-x| \geqslant 1 - |y-x|$.

$$\begin{aligned}
\int_{E_2} \frac{1}{|z-x|^{d-\alpha}}\, dz \leqslant \int \frac{1}{|z-x|^{d-\alpha}}\, dz &= \int_{1-|x-y|}^{1} \frac{1}{r^{d-\alpha}} r^{d-1}\, dr \\
&\leqslant const. \left| 1 - \frac{1}{(1-|x-y|)^\alpha} \right| \leqslant C|x-y|^\alpha
\end{aligned}$$ ■

Exercise 4.16 (E 5.1) Prove that if f is a harmonic function in $\mathbb{R}^d$ and $g \in C_c(\mathbb{R}^d)$ is radial, then

$$\int_{\mathbb{R}^d} f(x)g(x)\, dx = f(0) \int_{\mathbb{R}^d} g(x)\, dx$$

Solution. $x = r\omega, r > 0, |\omega| = 1$

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)g(x)\, dx &\overset{\text{(Polar)}}{=} \int_0^\infty \left(\int_{\partial B(0,1)} f(r\omega) g(r\omega)\, d\omega \right) dr \\
&= \int_0^\infty \left(g_0(r) \int_{\partial B(0,1)} f(r\omega)\, d\omega \right) dr \\
\text{(Mean value theorem (2.12))} \quad &= \int_0^\infty \left(g_0(r) f(0) \int_{\partial B(0,1)} d\omega \right) dr \\
&= f(0) \int_0^\infty \left(\int_{\partial B(0,1)} g(r\omega)\, d\omega \right) dr \\
&= f(0) \int_{\mathbb{R}^d} g(x)\, dx
\end{aligned}$$ ■

Remark 4.17 Let $g \in C_c(\mathbb{R}^d)$ be radial. Why is $\int_{\mathbb{R}^3} \frac{g(x)}{|x|}\, dx \neq \infty$? Because $f(x) = \frac{1}{|x|}$ is harmonic in $\mathbb{R}^d \backslash \{0\}$ and sub-harmonic in $\mathbb{R}^d$, $-\Delta f = c\delta_0$.

Exercise 4.18 (E 5.2) Let $1 \leqslant p < \infty$. Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider the Sobolev Space

$$W^{1,p}(\Omega) = \{ f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega),\ \forall i = 1, 2, \ldots, d \}$$

with the norm

$$\|f\|_{W^{1,p}} = \|f\| + \sum_{i=1}^d \|\partial_{x_i} f\|_{L^p(\Omega)}.$$

Prove that $W^{1,p}(\Omega)$ is a Banach space. Here $x = (x_i)_{i=1}^d \in \mathbb{R}^d$. Hint: You can use the fact that $L^p(\Omega)$ is a Banach Space.

Solution. $W^{1,p}(\Omega) \subseteq L^p(\Omega) \times L^p(\Omega) \cdots \times L^p(\Omega) = (L^p(\Omega))^{d+1}$. For an element $f \in W^{1,p}(\Omega)$ we can think of it as $f \mapsto (f, \partial_1 f, \partial_2 f, \dots, \partial_d f)$, so $W^{1,p}(\Omega)$ is a subspace of $(L^p(\Omega))^{d+1}$, which is a norm-space. Why is $W^{1,p}(\Omega)$ closed in $(L^p(\Omega))^{d+1}$? Take $\{f_n\}_{n=1}^\infty \subseteq W^{1,p}(\Omega)$ such that $f_n \to f$ in L^p and $\partial_i f_n \to g_i$ in L^p for all $i = 1, \dots, d$. We prove that $(f, g_1, \dots, g_d) \in W^{1,p}(\Omega)$, i.e. $f \in W^{1,p}$ and $g_i = \partial_i f$ for all $i = 1, \dots, d$. We know that $f_n \to f$ in $L^p(\Omega)$, so $f_n \to f$ in $D'(\Omega)$ and $\partial_i f_n \to \partial_i f$ in $D'(\Omega)$. On the other hand we have $\partial_i f_n \to g_i$ in $L^p(\Omega)$, so $\partial_i f_n \to g_i$ in $D'(\Omega)$. So we get $\partial_i f = g_i \in L^p(\Omega)$ for all $i = 1, \dots, d$ in $D'(\Omega)$. So we can conclude $f \in W^{1,p}(\Omega)$ and $\partial_i f = g_i$ for all $i = 1, \dots, d$. ∎

Exercise 4.19 (E 5.3) Let f be a real-valued function in $W^{1,p}(\mathbb{R}^d)$ for some $1 \leqslant p < \infty$. Prove that $|f| \in W^{1,p}(\mathbb{R}^d)$ and

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}.$$

Solution. Consider $G_\epsilon(t) = \sqrt{\epsilon^2 + t^2} - \epsilon$ for $\epsilon > 0$, $t \in \mathbb{R}$. Clearly we have $G_\epsilon(t) \to |t|$ as $\epsilon \to 0$ and

$$G'_\epsilon(t) = \frac{2t}{2\sqrt{\epsilon^2 + t^2}} = \frac{t}{\sqrt{\epsilon^2 + t^2}},$$

so $|G'_\epsilon(t)| \leqslant 1$, $G_\epsilon(0) = 0$. By the chain rule (3.55) we have $G_\epsilon(f) \in W^{1,p}(\mathbb{R}^d)$ and

$$(\partial_i G_\epsilon(f))(x) = G'_\epsilon(f)\partial_i f(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}}\partial_i f(x) \in L^p(\mathbb{R}^d)$$

for all $i = 1, \dots, d$. Note then when $\epsilon \to 0$ that $G_\epsilon(f)(x) \to |f(x)|$ pointwise, so $G_\epsilon(f) \to |f|$ in $L^p(\mathbb{R}^d)$. $|G_\epsilon(f)(x) - G_\epsilon(0)| \leqslant |f(x)| \in L^p(\mathbb{R}^d)$ by dominated convergence.

$$\partial_i G_\epsilon(f)(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}}\partial_i f(x) \xrightarrow{\epsilon \to 0} g_i(x) := \begin{cases} \partial f_i(x) & f(x) > 0 \\ -\partial_i f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

$$|\partial_i G_\epsilon(f)(x)| \leqslant \left| \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \right| |\partial_i f(x)| \leqslant |\partial_i f(x)| \in L^p(\mathbb{R}^d)$$

So we get $\partial_i G_\epsilon(f) \xrightarrow{\epsilon \to 0} g_i$ in $L^p(\mathbb{R}^d)$ by Dominated Convergence. So we conclude: $\partial_i(|f|) = g_i \in L^p(\mathbb{R}^d)$ for all $i = 1, \dots, d$, so $|f| \in W^{1,p}(\mathbb{R}^d)$, $|f| \in L^p$. ∎

Exercise 4.20 (E 5.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $f \in L^1(\Omega)$,

$$u(x) = \int_\Omega G(x - y) f(y)\, dy$$

Let $-\Delta u = f$ in $D'(\Omega)$, $u \in L^1_{loc}(\Omega)$, $f \in L^1_{loc}(\mathbb{R}^d)$ and $\omega_d f \in L^1(\mathbb{R}^d)$, where

$$\omega_d(x) = \begin{cases} 1 + |x| & d = 1 \\ \log(1 + |x|) & d = 1 \\ \frac{1}{(1+|x|)^{d-2}} & d \geqslant 3 \end{cases}.$$

Prove that

$$G \star f = \int_{\mathbb{R}^d} G(x-y) f(y)\, dy \in L^1_{loc}(\mathbb{R}^d)$$

and $-\Delta(G \star f) = f$ in $D'(\mathbb{R}^d)$.

Solution. Define $\tilde{f} = \mathbb{1}_\Omega(x) f(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$. Then

$$u(x) = \int_\Omega G(x-y) f(y)\, dy = \int_{\mathbb{R}^d} G(x-y)\tilde{f}(y)\, dy = (G \star \tilde{f})(x)$$

We have $u \in L^1_{loc}(\mathbb{R}^d)$, so $u \in L^1(\Omega)$. Then $-\Delta u = \tilde{f}$ in $D'(\mathbb{R}^d)$, so $-\Delta u = f$ in $D'(\Omega)$. Claim: $-\Delta u = f$ in $D'(\mathbb{R}^d)$, so $-\Delta u = f$ in $D'(\Omega)$ if $\Omega \subseteq \mathbb{R}^d$, $\tilde{f}|_\Omega = f$. Take $\phi \in C_c^\infty(\Omega)$. We need: $(-\Delta u)(\phi) \overset{?}{=} \int_\Omega f\phi$. We have $\phi \in C_c^\infty(\Omega)$, so $\phi C_c^\infty(\mathbb{R}^d)$. This implies:

$$(-\Delta u)(\phi) = \int_{\mathbb{R}^d} \tilde{f}\phi = \int_{\substack{\Omega, \\ \operatorname{supp}\phi \subseteq \Omega}} \tilde{f}\phi = \int_\Omega f\phi \qquad \blacksquare$$

Exercise 4.21 (E 5.5) Let $B = B\left(0, \frac{1}{2}\right) \subseteq \mathbb{R}^3$. Consider $u : B \to \mathbb{R}$, defined by

$$u(x) = \log|\log|x||.$$

Prove that the distributional derivative $f = -\Delta u$ is a function in $L^{\frac{3}{2}}(B)$.

Solution.

$$\begin{aligned}
\omega(r) &= \log(-\log(r)), \quad \text{for } r \in \left(0, \frac{1}{2}\right) \\
\omega'(r) &= \frac{1}{-\log(r)}\left(-\frac{1}{r}\right) = \frac{1}{r\log r} \\
\omega''(r) &= -\frac{1}{(r\log(r))^2}(r\log(r))' = -\frac{\log(r)+1}{(r\log r)^2}
\end{aligned}$$

So we have

$$-\Delta u = w''(r) = \frac{1}{(r\log r)^2} - \frac{1}{r^2\log(r)} = f(r)$$

We show that $f \in L^{\frac{3}{2}}$:

$$\begin{aligned}
\int_B |f(x)|^{\frac{3}{2}}\, dx &= const \int_0^{\frac{1}{2}} \left| \frac{1}{r^2(\log r)^2} - \frac{1}{r^2\log r} \right|^{\frac{3}{2}} r^2\, dr \\
&\lesssim \int_0^{\frac{1}{2}} \frac{1}{r} \left| \frac{1}{(\log(r))^2} - \frac{1}{(\log(r))} \right|^{\frac{3}{2}} dr \\
\begin{pmatrix} r = e^{-x}, \\ x \in (\log(2), \infty), \\ dr = -e^{-x}\, dx \end{pmatrix} &\lesssim \int_{\log(2)}^\infty e^x \left(\frac{1}{x^2} + \frac{1}{x}\right)^{\frac{3}{2}} e^{-x}\, dx \\
&\lesssim \int_{\log(2)}^\infty \frac{1}{x^{\frac{3}{2}}}\, dx < \infty
\end{aligned}$$

Where $\tilde{<}$ means *up to a constant*. Now, $u(x) = \omega(r) = \log(-\log(r))$.

$$-\Delta u(x) = f(r) = \frac{1}{r^2(\log(r))^2} - \frac{1}{r^2\log(r))}$$

for all $x \neq 0, |x| = r < \frac{1}{2}$. Why is $-\Delta u(x) = f$ in $D'(B)$? Take $\phi \in C_c^\infty(B)$, check: $\int_B u(-\Delta\phi) = \int_B f\phi$.

$$\int_{|x|<\frac{1}{2}} u(-\Delta\phi)\,dx = \lim_{\epsilon\to 0^+}\int_{\epsilon<|x|<\frac{1}{2}} u(x)(-\Delta\phi)(x)\,dx$$

by Dominated convergence. $u \in L^1(B)$. For all $\epsilon > 0$:

$$\begin{aligned}\int_{\epsilon<|x|<\frac{1}{2}} u(x)(-\Delta\phi)(x)\,dx &= \int_{|x|>\epsilon} u(x)(-\Delta\phi)(x)\,dx \\ &= \int_{\partial B(0,\epsilon)} u(x)\nabla\phi(x)\frac{x}{|x|}\,dS(x) + \int_{|x|>\epsilon}\nabla u(x)\nabla\phi(x)\,dx\end{aligned}$$

The boundary term vanishes as $\epsilon \to 0$ since

$$\left|u(x)\nabla\phi(x)\frac{x}{|x|}\right| \leqslant \|\nabla\phi\|_{L^\infty}|u(x)| = C\log|\log(r)|$$

$$\begin{aligned}\left|\int_{\partial B(0,\epsilon)} u(x)\nabla\phi(x)\frac{x}{|x|}\,dS(x)\right| &\leqslant C\int_{\partial B(0,\epsilon)}\log|\log(\epsilon)|\,dS(x) \\ &= C\log|\log\epsilon|\underbrace{|\partial B(0,\epsilon)|}_{\sim\epsilon^2} \xrightarrow{\epsilon\to 0} 0\end{aligned}$$

$$\begin{aligned}\int_{|x|>\epsilon}\nabla u(x)\nabla\phi(x)\,dx &= \sum_{i=1}^d\int_{|x|>\epsilon}\partial_i u(x)\partial_i\phi(x)\,dx \\ &= \sum_{i=1}^d\left(-\int_{\partial B(0,\epsilon)}\partial_i u(x)\phi(x)\frac{x_i}{|x|}\,dS(x) - \int_{|x|>\epsilon}\underbrace{\partial_i\partial_i u(x)}_{f(x)}\phi(x)\,dx\right)\end{aligned}$$

The boundary term vanishes as $\epsilon \to 0$ as

$$\begin{aligned}\left|\int_{\partial B(0,\epsilon)}\partial u(x)\phi(x)\frac{x_i}{|x|}\,dS(x)\right| &\leqslant \|\phi\|_{L^\infty}\int_{\partial B(0,\epsilon)}|\partial_i u(x)|\,dS(x) \\ (\star) \quad &\leqslant C\frac{1}{|\epsilon\log(r)|}|\partial B(0,\epsilon)| \to 0\end{aligned}$$

as $\epsilon \to 0$. $(\star)u = u(r), u(x) = \omega(|x|), \partial_i u(x) = \omega(|x|)\frac{x_i}{|x|}$, $|\partial_i u(x)| \leqslant |\omega(|x|)| = \left|\frac{1}{r\log(r)}\right|$. Finally:

$$\int_{|x|>\epsilon} f(x)\phi(x)\,dx \xrightarrow{\epsilon\to 0}\int_{\mathbb{R}^d} f(x)\phi(x)\,dx$$

Since $f\phi \in L^1$ and Dominated Convergence. ■

Exercise 4.22 (Bonus 5) Construct $u \in L^1(\mathbb{R}^3)$ compactly supported s.t. $-\Delta u \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and u is not continuous at 0.

Hint: Related to E 5.5. $u_0(x) = \omega(r) = \log(|\log(r)|)$ if $0 < r = |x| < \frac{1}{2}$. Consider χu_0 where $\chi \in C_c^\infty$, $\chi = 0$ if $|x| > \frac{1}{2}$, $\chi = 1$ if $|x| < \frac{1}{4}$. You can prove that $\Delta(\chi u_0) = (\Delta\chi)u_0 + 2\nabla\chi\nabla u_0 + \chi(\underbrace{\Delta u_0}_{\in L^{\frac{3}{2}}})$ in $D'(\mathbb{R}^3)$. (almost everywhere, in distributional sense, integration by parts)

Theorem 4.23 (Regularity on Domains) Let $\Omega \subseteq \mathbb{R}^d$ be open. Assume $u, f \in D'(\Omega)$ such that $-\Delta u = f$ in $D'(\Omega)$.

a) If $f \in L^1_{loc}(\Omega)$, then

- $u \in C^1(\Omega)$ if $d = 1$
- $u \in L^q_{loc}(\Omega)$ for all $q < \infty$ if $d = 2$
- $u \in L^q_{loc}(\Omega)$ for all $q < \frac{d}{d-2}$ if $d \geqslant 3$

b) If $f \in L^q_{loc}(\Omega)$, $d \geqslant p < \frac{d}{2}$, then $u \in C^{0,\alpha}_{loc}(\Omega)$, where $0 < \alpha < 2 - \frac{d}{p}$

c) If $f \in L^p_{loc}(\Omega)$, $p > d$, then $u \in C^{1,\alpha}_{loc}(\Omega)$, where $0 \leqslant \alpha < 1 - \frac{d}{p}$

d) If $f \in C^{0,\alpha}_{loc}(\Omega)$ for some $0 < \alpha < 1$, then $u \in C^{2,\alpha}_{loc}(\Omega)$

e) If $f \in C^{m,\alpha}_{loc}(\Omega)$, then $u \in C^{m+2,\alpha}_{loc}(\Omega)$

Proof. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Take a ball $\overline{B} \subseteq \Omega$. Define $f_B : \mathbb{R}^d \to \mathbb{K}$,

$$f_B(x) = (\mathbb{1}_B f)(x) = \begin{cases} f(x) & x \in B \\ 0 & x \notin B \end{cases}$$

Then if $f \in L^1_{loc}(\Omega)$, f_B is compactly supported. From the previous theorems: $G \star f_B \in L^1_{loc}(\mathbb{R}^d)$ and $-\Delta(G \star f_B) = f_B$ in $D'(\mathbb{R}^d)$. On the other hand, $-\Delta u = f$ in $D'(\Omega)$, so $-\Delta(u - G \star f_B) = 0$ in $D'(B)$. Indeed, for all $\phi \in C_c^\infty(B)$, then:

$$(-\Delta u)(\phi) = \int_\Omega f\phi = \int_B f_B\phi = -\int_{\mathbb{R}^d} f_B\phi = (-\Delta)(G \star f_B)(\phi)$$

Then $-\Delta u = -\Delta(G \star f_B)$ in $D'(B)$. Then $u - G \star f_B$ is harmonic in B and by Weyls lemma we have $u - G \star f_B \in C^\infty(B)$. So the smoothness of u in B is the same to that of $G \star f$. ■

Exercise 4.24 (E 6.1) Show that if $\chi \in C^\infty(\mathbb{R}^d)$, $f \in W^{1,p}(\mathbb{R}^d)$, $1 \leqslant p < \infty$, then $\chi f \in W^{1,p}_{loc}(\mathbb{R}^d)$ and

$$\partial_i(\chi f) = (\partial_i\chi)f + \chi(\partial_i f) \quad \text{in } D'(\mathbb{R}^d)$$

Solution. $\chi f \in L^p_{loc}(\mathbb{R}^d)$ obvious. $\partial(\chi f) \in L^p_{loc}(\mathbb{R}^d)$ is nontrivial but follows from $\partial_i(\chi f) = \underbrace{(\partial_i\chi)f + \chi(\partial f)}_{\in L^p_{loc}}$ in $D'(\mathbb{R}^d)$. To compute the distributional derivative $\partial_i(\chi f)$, then: Take $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$-\int_{\mathbb{R}^d} \chi f(\partial\phi)\,dx = \int_{\mathbb{R}^d} (?)\phi\,dx$$

We have

$$
\begin{aligned}
-\int_{\mathbb{R}^d} \chi f(\partial_i \phi)\, dx &= -\int_{\mathbb{R}^d} f(\chi \partial_i \phi)\, dx \\
(\partial_i(\chi\phi) = \chi\partial_i\phi + (\partial_i\chi)\phi) \qquad &= -\int_{\mathbb{R}^d} f\left(\partial_i(\chi\phi) - (\partial_i\chi)\phi\right) dx \\
&= -\int_{\mathbb{R}^d} f\partial_i(\underbrace{\chi\phi}_{\in C_c^\infty})\, dx + \int_{\mathbb{R}^d} f(\partial_i\chi)\phi\, dx \\
&= \int_{\mathbb{R}^d} (\partial_i f)\chi\phi\, dx + \int_{\mathbb{R}^d} f(\partial_i\chi)\phi\, dx \\
&= \int_{\mathbb{R}^d} ((\partial_i f)\chi + f(\partial_i\chi))\phi\, dx
\end{aligned}
$$

So $\partial_i(\chi f) = (\partial_i f)\chi + f(\partial_i \chi)$ in $D'(\mathbb{R}^d)$. ■

Remark 4.25 Question: If $\chi \in C^1(\mathbb{R}^d)$, $f \in W^{1,p}(\mathbb{R}^d)$. Is this it still correct that $\partial_i(\chi f) = (\partial_i\chi)f + \chi(\partial_i f) \quad$ in $D'(\mathbb{R}^d)$?

Proof. It suffices to show that we still can apply integration by parts.

$$(\star) \qquad -\int f\partial_i g \overset{?}{=} \int (\partial_i f)g$$

Approximation: $(\star)$ is correct if $g \in C_c^\infty$

- If $g \in C_c^1$, there is $\{g_n\} \subseteq C_c^\infty$ s.t. $g_n \to g$ in $W^{1,p}_{loc}$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\int (\partial_i g)f \xleftarrow{n\to\infty} -\int \underbrace{f}_{L^p} \underbrace{\partial_i g_n}_{\to \partial_i g \text{ in } L^q} = \int \underbrace{(\partial_i f)}_{\in L^p} \underbrace{g_n}_{\to g \text{ in } L^q} \xrightarrow{n\to\infty} \int (\partial_i f) g$$ ■

Exercise 4.26 (E 6.2) $G(x) = -\frac{1}{2\pi}\log|x|$. Let $f \in L^p(\mathbb{R}^2)$, compactly supported. Define $u(x) = (G \star f)(x) = \int_{\mathbb{R}^2} G(x-y)f(y)\, dy$

1. If $p = 1$, then $u \in L^q_{loc}(\mathbb{R}^2)$ for all $q < \infty$.
2. If $p > 2$, then $u \in C^{1,\alpha}$ with $0 < \alpha < 1 - \frac{2}{p}$.

Solution. 1. Take any ball $B = B(0,R)$ and:

$$
\begin{aligned}
\int_B |u(x)|^q\, dx &= \int_B \left(\int_{\mathbb{R}^d} |G(x-y)||f(y)|\, dy\right)^q dx \\
&\leqslant C\int_B \left(\int_{\mathbb{R}^2} |G(x-y)|^q |f(y)|\, dy\right) dx \\
&= C\int_{\mathbb{R}^2} \left(\int_B |G(x-y)|^q\, dx\right) |f(y)|\, dy
\end{aligned}
$$

Recall from the proof of Youngs inequality:

$$
\begin{aligned}
|u(x)| &= \left|\int_{\mathbb{R}^2} G(x-y)f(y)\, dy\right| \\
&\leqslant \int_{\mathbb{R}^2} |G(x-y)||f(y)|\, dy \\
&\leqslant \left(\int_{\mathbb{R}^2} |G(x-y)|^q |f(y)|\, dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} |f(y)|\, dy\right)^{\frac{1}{q}}, \quad \frac{1}{q} + \frac{1}{q'} = 1
\end{aligned}
$$

Assume $\operatorname{supp} f \subseteq \overline{B(0,R)}$. Then if $y \in \operatorname{supp} f$ and $x \in B(0,R)$, then $|x-y| \leqslant |x| + |y| \leqslant R + R_1$. For all $y \in \operatorname{supp} f$:

$$\begin{aligned}\int_{B(0,R)} |G(x-y)|^q \, dx &\leqslant \int_{|x-y| \leqslant R+R_1} |G(x-y)|^q \, dx \\ &= \int_{|z| \leqslant R+R_1} |G(z)|^q \, dz < \infty\end{aligned}$$

as $G \in L^q_{loc}$ ($|G(z)| = \frac{1}{2\pi} |\log(z)| \leqslant \frac{C_{R+R_1,\epsilon}}{|z|^\epsilon}$ for all $|z| \leqslant R + R_1$), so

$$\int_{|z| \leqslant R+R_1} |G(z)|^q \leqslant C_{R+R_1,\epsilon} \int_{|z| \leqslant R+R_1} \frac{1}{|z|^{\epsilon q}} \, dz < \infty$$

if $\epsilon q < 2$.

2. Recall $\partial_i u \in L^1_{loc}(\mathbb{R}^2)$ and:

$$\partial_i u(x) = (\partial_i G \star f)(x) = c \int_{\mathbb{R}^2} \frac{x_i - y_i}{|x-y|^2} f(y) \, dy$$

First we show $\partial_i u \in C^{0,\alpha}$:

$$\begin{aligned}|\partial_i u(x) - \partial_i u(z)| &= \left| C \int_{\mathbb{R}^2} \left(\frac{x_i - y_i}{|x-y|^2} - \frac{z_i - y_i}{|z-y|^2} \right) f(y) \, dy \right| \\ &\leqslant C \int_{\mathbb{R}^2} \left| \frac{x_i y_i}{|x-y|^2} - \frac{z_i - y_i}{|z-y|^2} \right| |f(y)| \, dy \\ &\overset{?}{\leqslant} C|x-y|^\alpha\end{aligned}$$

Note that

$$\begin{aligned}\left| \frac{x_i - y_i}{|x-y|^2} - \frac{z_i - y_i}{|z-y|^2} \right| &= \left| (x_i - y_i) \left(\frac{1}{|x-y|^2} - \frac{1}{|z-y|^2} \right) + \frac{x_i - z_i}{|z-y|^2} \right| \\ &\leqslant |x_i - y_i| \left| \frac{1}{|x-y|^2} - \frac{1}{|z-y|^2} \right| + \frac{|x_i - z_i|}{|z-y|^2} \\ &\leqslant C|z-x|^\alpha \left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} + \frac{|x-z|}{|z-y|^2} \right)\end{aligned}$$

Here $|x_i - z_i| \leqslant |x-z|$ and $|x_i - y_i| \leqslant |x-y|$ and:

$$\begin{aligned}\underbrace{\left| \frac{1}{|x-y|^2} - \frac{1}{|z-y|^2} \right|}_{\text{sym } x \leftrightarrow z} &= \left| \frac{1}{|x-y|} - \frac{1}{|z-y|} \right| \left| \frac{1}{|x-y|} + \frac{1}{|z-y|} \right| \\ &= \frac{||z-y| - |x-y||}{|x-y||z-y|} \left| \frac{1}{|x-y|} + \frac{1}{|z-y|} \right| \\ &\leqslant |z-x|^\alpha \frac{\max(|z-y|, |x-y|)^{1-\alpha}}{|x-y||z-y|} \left(\frac{1}{|x-y|} + \frac{1}{|z-y|} \right) \\ &\leqslant C|z-x|^\alpha \left(\frac{1}{|x-y|^{2+\alpha}} + \frac{1}{|z-y|^{2+\alpha}} \right)\end{aligned}$$

By the symmetrie $x \leftrightarrow z$:

$$\begin{aligned}LHS &\leqslant C|z-x|^\alpha \left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} \right) + \frac{|x-y|}{|x-y|^2} \\ \Rightarrow LHS &\leqslant C \cdots + |x-z| \min\left(\frac{1}{|z-y|^2}, \frac{1}{|x-y|^2} \right) \\ &\leqslant (|x-y| + |z-y|)^{1-\alpha} \\ &\quad C|z-x|^\alpha \left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} \right)\end{aligned}$$

In summary:

$$\begin{aligned}|\partial_i u(x) - \partial_i u(z)| &\leqslant C\int_{\mathbb{R}^2}\left|\frac{x_i-y_i}{|x-y|^2} - \frac{z_i-y_i}{|z-y|^2}\right| |f(y)|\,dy \\ &= C|x-y|^\alpha \int_{\mathbb{R}^2}\left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}}\right)|f(y)|\,dy\end{aligned}$$

Consider if $|x| > 2R_1$:

$$\int_{\mathbb{R}^2}\frac{1}{|x-y|^{1+\alpha}}|f(y)|\,dy \leqslant \int_{\mathbb{R}^2}\frac{1}{R_1^{1+\alpha}}|f(y)|\,dy \leqslant C$$

$\operatorname{supp} f \subseteq B(0,R_1)$. If $|x| < 2R_1$, then $|x-y| \leqslant 3R$ if $y \in B(0,R_1)$. Hence:

$$\begin{aligned}&\int_{|x-y|\leqslant 3R_1}\frac{1}{|x-y|^{1+\alpha}}|f(y)|\,dy \\ &\leqslant \left(\int_{|x-y|\leqslant 3R_1}\frac{1}{|x-y|^{(1+\alpha)p'}}\right)^{\frac{1}{p'}}\left(\int |f(y)|^p\,dy\right)^{\frac{1}{p}} \\ &= \int_{|z|\leqslant 3R_1}\frac{1}{|z|^{(1+\alpha)p'}}\,dz < \infty\end{aligned}$$

So $\alpha < 1 - \frac{2}{p}$. ■

Exercise 4.27 (E 6.3) Let $f \in C^{0,\alpha}_{loc}$ and $-\Delta u = f$ in $D'(\Omega)$. Prove $u \in C^{2,\alpha}_{loc}(\Omega)$.

Solution. Take an open ball $B \subseteq \bar{B} \subseteq \Omega$. We prove $u \in C^{2,\alpha}(B)$. There is an open Ω_B s.t. $\bar{B} \subseteq \bar{\Omega}_B \subseteq \Omega$. Then there is a $\chi_B \in C_c^\infty(\mathbb{R}^d)$ s.t. $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin \Omega_B$. Define

$$f_B(x) = \chi_B(x) f(x) : \ \mathbb{R}^d \to \mathbb{R}$$

We prove that $f_B \in C^{0,\alpha}(\mathbb{R}^d)$. Since $f \in C^{0,\alpha}_{loc}(\Omega)$ we have $f \in C^{0,\alpha}(\Omega_B)$, so $|f(x) - f(y)| \leqslant C|x-y|^\alpha$ for all $x, y \in \Omega_B$. Then:

$$\begin{aligned}|f_B(x) - f_B(y)| &= |\chi_B(x)f(x) - \chi_B(y)f(y)| \\ &\leqslant |(\chi_B(x) - \chi_B(y))f(x) + \chi_B(y)(f(x) - f(y))| \\ &\leqslant C|x-y|^\alpha \|f\|_{L^\infty(\Omega_B)} + C\|\chi\|_{L^\infty(\Omega_B)}|x-y|^\alpha \leqslant C_{\Omega_B}|x-y|^\alpha\end{aligned}$$

What about other cases? If x, y are bot not in Ω_B, then $|f_B(x) - f_B(y)| = 0$, then if $x \in \Omega_B$ and $y \notin \Omega_B$: $|f_B(x) - f_B(y)| = |f_B(x)| = |\chi_B(x)||f(x)| = |\chi_B(x) - \chi_B(y)||f(x)| \leqslant C|x-y|^\alpha$. Conclusion: $|f_B(x) - f_B(y)| \leqslant C|x-y|^\alpha$ for all $x, y \in \mathbb{R}^d$, i.e. $f_B \in C^{0,\alpha}(\mathbb{R}^d)$. Also f_B is compactly supported. By a theorem in the lecture: $G \star f_B \in C^{2,\alpha}(\mathbb{R}^d)$. Finally: $-\Delta u = f$ in $D'(\Omega)$, $-\Delta(G \star f_B) = f_B$ in $D'(\mathbb{R}^d)$. So we conclude $-\Delta u = f = f_B = -\Delta(G \star f_B)$ in $D'(B)$. $-\Delta(u - G \star f_B) = 0$ in $D'(B)$, so $u - G \star f_B \in C^\infty(B)$, so $u \in C^{2,\alpha}(B)$. ■

Exercise 4.28 (E 6.4) $u, f \in L^2(\mathbb{R}^d)$, $-\Delta u = f$ in $D'(\mathbb{R}^d)$. Prove $u \in W^{2,2}(\mathbb{R}^d)$, $\|u\|_{W^{2,2}(\mathbb{R}^d)} \leqslant C(\|u\|_{L^2} + \|f\|_{L^2})$.

$$\begin{aligned}W^{2,2}(\mathbb{R}^d) &= \{g \in L^2(\mathbb{R}^d) \mid D^\alpha g \in L^2 \text{ for all } |\alpha| \leqslant 2\} \\ &= \{g \in L^2(\mathbb{R}^d) \mid \widehat{D^\alpha g}(k) = (-2\phi i k)^\alpha \hat{g}(k) \in L^2(\mathbb{R}^d) \text{ for all } |\alpha| \leqslant 2\} \\ &= \{g \in L^2(\mathbb{R}^d) \mid (1 + |k|^2)\hat{g}(k) \in L^2(\mathbb{R}^d)\}\end{aligned}$$

$\|u\|_{W^{2,2}(\mathbb{R}^d)}$ is comparable $\int_{\mathbb{R}^d}(1+|k|^2)^2|\hat{g}(k)|^2\,dk$. If $D^\alpha g \in L^2$, then $\widehat{D^\alpha g}(k) = (-2\pi i k)^\alpha \hat{g}(k)$. For any $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned}
\int \widehat{D^\alpha g}(k)\hat{\phi}(k), dk = \int (D^\alpha g)\phi = (-1)^{|\alpha|}\int g(D^\alpha \phi) \\
= (-1)^{|\alpha|}\int \overline{\hat{g}}(k)\widehat{D^\alpha}\phi(k) \\
= (-1)^{|\alpha|}\int \overline{\hat{g}}(k)(-2\pi i k)^\alpha \hat{\phi}(k)\,dk
\end{aligned}$$

so $\hat{D}^\alpha g(k) = (-1)^{|k|}\hat{g}(k)\overline{(-2\pi i k x)^\alpha} = \hat{g}(k)(-2\pi i k)^\alpha$. This implies:

$$\begin{aligned}
\|u\|_{W^{2,2}(\mathbb{R}^d)} &\leqslant C\int_{\mathbb{R}^d}(1+|k|^2)^2|\hat{u}(k)|^2\,dk \\
&= C\left(\|u\|_{L^2}^2 + \int_{\mathbb{R}^d}|k|^4|\hat{u}(k)|^2\,dk\right) \\
&\leqslant C\left(\|u\|_{L^2}^2 + \|f\|_{L^2}^2\right) \\
&\leqslant C(\|u\|_{L^2} + \|f||_{L^2})^2
\end{aligned}$$

Remark 4.29 (Bonus 6) Let $f, g \in W^{1,2}(\mathbb{R}^d)$. Prove that $fg \in W^{1,1}(\mathbb{R}^d)$ and

$$\partial_i(fg) = (\partial_i f)g + f(\partial_i g) \quad \text{ in } D'(\mathbb{R}^d)$$

Chapter 5

Existence for Poisson's Equation on Domains

Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider Poisson's equation.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for given data (f, g) and u the unknown function.

- Classical solutions: $f \in C^2(\bar{\Omega}) \rightsquigarrow$ explicit representation formula.
- Weak solution: $f \in L^p(\Omega)$, $g \in L^p(\partial\Omega) \rightsquigarrow u \in W^{2,p}(\Omega)$. We are going to establish the existence by *Energy Methods.* (Calculus of variations)

Definition 5.1 (C^1-Domains) Let $\Omega \subseteq \mathbb{R}^d$ be open. We say that Ω is of class C^1 (i.e. $\partial\Omega \in C^1$) if for all $x_0 \in \partial\Omega$ there is a bijective function $h : U \to Q$, where

- $x_0 \in U$ open in $\mathbb{R}^d$
- $Q = \{x = (x_1, \ldots, x_d) = (x', x_d)\} \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, |x_d| < 1\}$
- $h \in C^1(\bar{U})$ and $h^{-1} \in C^1(\bar{Q})$ (C^1-diffeomorphism)
- $h(U) = Q$

$$\begin{aligned} h(U \cap \Omega) &= Q_+ = Q \cap \mathbb{R}^d_+ = \{x = (x', x_d) \in Q \mid x_d > 0\} \\ h(U \cap \partial\Omega) &= Q_0 = Q \cap \partial\mathbb{R}^d_+ = \{x = (x', x_d) \in Q \mid x_d = 0\} \\ h(U \backslash \bar{\Omega}) &= Q_- = Q \cap \mathbb{R}^d_- = \{x = (x', x_d) \in Q \mid x_d < 1\} \end{aligned}$$

(From Brezis' book)

Remark 5.2 The set Q can be replaced by a ball, i.e. Ω is of C^1 if for all $x_0 \in \partial\Omega$ there is a function $U \to B(0, 1) \subseteq \mathbb{R}^d$.

- $x_0 \in U$ with $U \subseteq \mathbb{R}^d$ open.
- $h \in C^1(\bar{U}), h^{-1} \in C^1(\overline{B(0, 1)})$
- $h(U \cap \Omega) = B(0, 1) \cap \mathbb{R}^d_+$, $h(U \cap \partial\Omega) = B(0, 1) \cap \mathbb{R}^d$.

Remark 5.3 (An equivalent definition form Evan's book App. C) Let $\Omega \subseteq \mathbb{R}^d$ be open. Then Ω is C^1 if for all $x_0 \in \partial\Omega$ there is a $r > 0$ and a C^1-function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ s.t. (upon relabeling and reorienting the axes if necessary) such that:

$$\Omega \cap B(x_0, r) = \{x = (x', x_d) \in B(x_0, r) \mid x_d < \gamma(x_0)\}$$

Proof of the equivalence of the two definitions.

Def. 2 $\Rightarrow$ Def. 1: In fact, given $x_0 \in \partial\Omega$ and γ we can define

$$h(x', x_d) = (x', x_d - \gamma(x')) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$$
$$h^{-1}(x', x_d) = (x', x_d + \gamma(x')) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$$

Def. 1 $\Rightarrow$ Def. 2: We need the inverse function theorem and the implicit function theorem. Let $x_0 \in \partial\Omega$, let $h : U \to B(0,1)$ as in Def. 1. Denote $h = (h_1, h_2, \dots, h_d)$. Since h is invertible near x_0, by the inverse function theorem we have for the Jacobi matrix $Jh(x_0) = (\partial_j h_i(x_0))_{1 \leqslant i,j \leqslant d}$ is invertible. So we have $\nabla h_d(x_0) = (\partial_j h_d(x_0))_{1 \leqslant j \leqslant d} \neq \vec{0}^{\mathbb{R}^d}$, so there is a $j \in \{1, 2, \dots, d\}$ s.t. $\partial_j h_d(x_0) \neq 0$. By relabeling and reorienting the axes, we can assume that $\partial_d h_d(x_0) > 0$. By continuity there is a $r > 0$ such that $\partial_d h_d(x) > 0$ for all $x \in B(x_0, r)$. Define $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ s.t. in $B(x_0, r)$:

$$x = (x', x_d) \in \partial\Omega \iff h_d(x', x_d) = 0 \iff x_d = \gamma(x'),$$

$h_d : \mathbb{R}^d \to \mathbb{R}$. This gives a solution γ if $\partial_d h_d > 0$ in $B(x_0, r)$. (For implicit function theorem, $\partial_d h_d(x_0) \neq 0$) Question: Why in $B(x_0, r)$?

$$x = (x', x_d) \in \Omega \iff x_d > \gamma(x')$$

Since $\partial_d h_d(x) > 0$ for all $x \in B(x_0, r)$ we have that $x_d \mapsto h_d(x', x_d)$ is strictly increasing, hence

$$\begin{aligned} & x = (x', x_d) \in \Omega \\ \iff & h(x', x_d) \in \mathbb{R}^d_+ \\ \iff & h_d(x', x_d) > 0 = h_d(x', \gamma(x')) \\ \iff & x_d > \gamma(x') \end{aligned}$$

$\blacksquare$

Theorem 5.4 (Gauss-Green formula / Integration by parts) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. Then

1. For all $u, v \in C^1(\bar{\Omega})$:

$$\int_\Omega (\partial_i u) v = -\int_\Omega u(\partial_i v) + \int_{\partial\Omega} u v n_i \, dS,$$

where $\vec{n} = (n_i)_{i=1}^d$ is the outwarded unit normal vector.

2. For all $u, v \in C^2(\bar{\Omega})$:

$$\int_\Omega u(-\Delta v) = \int_\Omega \nabla u \nabla v - \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}} \, dS$$

where $\frac{\partial v}{\partial \vec{n}} = \nabla v \vec{n} = \sum_{i=1}^d \partial_i v n_i$.

Classical solutions via Green's function:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, $\partial\Omega \in C^1$. Assume there exists a $u \in C^2(\bar{\Omega})$, $f \in C(\bar{\Omega})$, $g \in C(\partial\Omega)$. Let G be the fundamential solution of the Laplace Equation in $\mathbb{R}^d$. We use integration by parts in $\Omega \backslash B(x, \epsilon)$:

$$\begin{aligned}
&\int_{\Omega\backslash B(x,\epsilon)} u(y)(-\Delta G)(y-x)\,dy \\
&\quad= \int_{\Omega\backslash B(x,\epsilon)} \nabla u(y)\nabla G(y-x)\,dy - \int_{\partial\Omega\cup\partial B(x,\epsilon)} u(y)\frac{\partial G}{\partial\vec{n}}(y-x)\,dS(y) \\
&\int_{\Omega\backslash B(x,\epsilon)} G(y-x)(-\Delta u)(y)\,dy \\
&\quad= \int_{\Omega\backslash B(x,\epsilon)} \nabla G(y-x)\nabla u(y)\,dy - \int_{\partial\Omega\cup\partial B(x,\epsilon)} G(y-x)\frac{\partial u}{\partial\vec{n}}(y)\,dS(y)
\end{aligned}$$

This implies:

$$\begin{aligned}
&\int_{\Omega\backslash B(x,\epsilon)} \left[u(y)(-\Delta G(y-x)) - G(y-x)(-\Delta u)(y)\right]\,dy \\
&\quad= -\int_{\partial\Omega\cup\partial B(x,\epsilon)} \left[u(y)\frac{\partial G}{\partial\vec{n}}(y-x) - G(y,x)\frac{\partial u}{\partial\vec{n}}(y)\right]\,dS(y)
\end{aligned}$$

for all $x \in \Omega, x \in B(x,\epsilon) \subseteq \Omega$. When $\epsilon \to 0$, then the left hand side converges to $-\int_\Omega G(y-x)f(y)\,dy$ and the right hand side (for $d \geqslant 2$) we have $\partial_j G(y) = \frac{-y_j}{d|B_1||y|^d}$, so

$$\frac{\partial G}{\partial\vec{n}} = \nabla G\vec{n} = \nabla G(y)\left(\frac{-y}{|y|}\right) = \sum_{j=1}^{d} \frac{-y_i}{d|B_1||y|^d}\frac{-y_j}{|y|} = \frac{1}{d|B_1||y|^{d-1}} \text{ on} \partial B(0,\epsilon)$$

so we have

$$\frac{\partial G}{\partial\vec{n}}(y-x) = \frac{1}{d|B_1|\epsilon^{d-1}}$$

on $\partial B(x,\epsilon)$. Hence

$$\begin{aligned}
\int_{\partial B(x,\epsilon)} u(y)\frac{\partial G}{\partial\vec{n}}(y-x)\,dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}}\int_{\partial B(x,\epsilon)} u(y)\,dS(y) \\
&= \fint_{\partial B(x,\epsilon)} u(y)\,dS(y) \xrightarrow{\epsilon\to 0} u(x)
\end{aligned}$$

On the other hand:

$$\left|\int_{\partial B(x,\epsilon)} G(y-x)\frac{\partial u(y)}{\partial\vec{n}}\,dS(y)\right| \leqslant C\epsilon^{d-1}\sup_{|z|=\epsilon}|G(z)| \xrightarrow{\epsilon\to 0} 0$$

since $|G(z)| \leqslant \frac{C}{|z|^{d-2}}$ if $d \geqslant 3$, $|G(z)| \leqslant C|\log(z)|$ if $d = 2$ and $|G(z)| \leqslant C|z|$ if $d = 1$. In summary:

$$\begin{aligned}
-\int_\Omega G(y-x)f(y)\,dy &= -\int_{\partial\Omega}\left[u(y)\frac{\partial G}{\partial\vec{n}}(y-x) - G(y-x)\frac{\partial u}{\partial\vec{n}}(y)\right]\,dS(y) - u(x) \\
\Leftrightarrow u(x) &= \int_\Omega G(y-x)f(y)\,dy + \int_{\partial\Omega}\left[G(y-x)\frac{\partial u}{\partial\vec{n}}(y) - g(y)\frac{\partial G}{\partial\vec{n}}(y-x)\right]\,dS(y)
\end{aligned}$$

Problem: We don't know anything about $\frac{\partial u}{\partial \vec{n}}$ on $\partial\Omega$. Trick: We can resolve that by using the *corrector* function: $\Phi_x = \Phi_x(y)$ which solves:

$$\begin{cases} -\Delta \Phi_x = 0 & \text{in } \Omega \\ \Phi_x(y) = G(y-x) & \text{on } \partial\Omega \end{cases}$$

We assume that Φ_x exists.

Definition 5.5 (Green's function) $\tilde{G}(x-y) = G(y-x) - \Phi_x(y)$ for all $x, y \in \Omega$, $x \neq y$.

Exercise 5.6 (E 7.1) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded with C^1 boundary. For $x \in \Omega$, assume there exist $\Phi_x(y)$, $y \in \bar{\Omega}$, s.t.

$$\begin{cases} \Delta_y \Phi_x(y) = 0 \\ \Phi_x(y) = G(y-x) \end{cases},$$

$G(z) = \frac{1}{d(d-2)|B_1||z|^{d-2}}, d \geqslant 3$. Prove that $\Phi_x(y) = \Phi_y(x)$ for all $x, y \in \Omega$. Then $\tilde{G}(x,y) = G(y-x) - \Phi_x(y)$ is symmetric, i.e. $\tilde{G}(x,y) = \tilde{G}(y,x)$.

Solution. Assume $x \neq y$. Define

$$\begin{aligned} f(z) &= \tilde{G}(x,z) = G(z-x) - \Phi_x(z) \\ g(z) &= \tilde{G}(y,z) = G(z-y) - \Phi_y(z) \end{aligned}$$

Integration by parts:

$$\begin{aligned} \int_{\Omega \backslash (B(x,\epsilon) \cup B(y,\epsilon))} (f\Delta g - g\Delta f) &= \int_{\partial\Omega \cup \partial B(x,\epsilon) \cup \partial B(y,\epsilon)} \left(f\frac{\partial g}{\partial \vec{n_z}} - g\frac{\partial f}{\partial \vec{n_z}} \right) dS(z) \\ &= \int_{\partial B(x,\epsilon) \cup \partial B(y,\epsilon)} \left(f\frac{\partial g}{\partial \vec{n_z}} - g\frac{\partial f}{\partial \vec{n_z}} \right) dS(z) \end{aligned}$$

Consider $f\frac{\partial g}{\partial \vec{n_z}}$ on $\partial B(x,\epsilon)$. Since g is only singular at y, so $\left|\frac{\partial g}{\partial \vec{n}}\right| \leqslant C$ on $\partial B(x,\epsilon)$. This implies:

$$\begin{aligned} \int_{\partial B(x,\epsilon)} \left| f\frac{\partial g}{\partial \vec{n_z}} \right| dS(z) &\leqslant C \int_{\partial B(x,\epsilon)} |f| \, dS(z) \\ &\leqslant C \int_{\partial B(x,\epsilon)} \left(\frac{1}{|x-z|^{d-2}} + \|\Phi_x\|_{L^\infty(\Omega)} \right) dS(z) \\ &\leqslant C\epsilon^{d-1} \left(\frac{1}{\epsilon^{d-2}} + 1 \right) \leqslant C\epsilon \xrightarrow{\epsilon \to 0} 0 \end{aligned}$$

Consider $f\frac{\partial g}{\partial \vec{n_z}}$ on $\partial B(y,\epsilon)$. Decompose $\frac{\partial g}{\partial \vec{n}} = [\nabla_z G(z-y) - \nabla_z \Phi_y(z)] \frac{(z-y)}{|z-y|}$. Since $\Phi_y(z)$ is harmonic in Ω, we have that

$$\int_{\partial B(y,\epsilon)} \left| f \nabla_z \Phi_y(z) \frac{-(z-y)}{|z-y|} \right| \leqslant C \int_{\partial B(y,\epsilon)} |f| \leqslant C\epsilon^{d-1} \xrightarrow{\epsilon \to 0} 0$$

Thus the main contribution from $f\frac{\partial g}{\partial \vec{n}}$ is

$$\begin{aligned}
&\int_{\partial B(y,\epsilon)} f(z)\nabla_z G(z-y)\frac{-(z-y)}{|z-y|}\,dS(z) \\
&\quad= \int_{\partial B(y,\epsilon)} f(z)\frac{-(z-y)}{d|B_1||z-y|^d}\frac{-(z-y)}{|z-y|}\,dS(z) \\
&\quad= \frac{1}{d|B_1|\epsilon^{d-1}}\int_{\partial B(y,\epsilon)} f(z)\,dS(z) \\
&\quad= \fint_{\partial B(y,\epsilon)} f(z)\,dS(z) = f(y)
\end{aligned}$$

In summary:

$$\int_{\partial B(x,\epsilon)\cup\partial B(y,\epsilon)} f\frac{\partial g}{\partial \vec{n_z}}\,dS(z) \xrightarrow{\epsilon\to 0} f(y)$$

Similary:

$$\int_{\partial B(x,\epsilon)\cup\partial B(y,\epsilon)} g\frac{\partial f}{\partial \vec{n_z}}\,dS(z) \xrightarrow{\epsilon\to 0} g(x)$$

So we have that $f(y) = g(x)$, so

$$\begin{aligned}
f(y) &= G(y-x) - \Phi_x(y) \\
g(x) &= G(x-y) - \Phi_y(x).
\end{aligned}$$

So $\Phi_x(y) = \Phi_y(x)$ for all $x \neq y \in \Omega$. This implies $\Phi_x(y) = \Phi_y(x)$ for all $x, y \in \Omega$. ∎

Theorem 5.7 Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and C^1. If $u \in C^2(\Omega)$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases},$$

then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial \tilde{G}}{\partial \vec{n_y}}(x,y)\,dS(y) + \int_\Omega \tilde{G}(x,y)\,dy$$

Proof. We need to prove:

$$\int_\Omega \Phi_x(y)f(y)\,dy + \int_{\partial\Omega}\left(-g(y)\frac{\partial \Phi_x(y)}{\partial \vec{n}_y} + G(y-x)\frac{\partial u}{\partial \vec{n}}(y)\right) = 0$$

By integration by parts:

$$\begin{aligned}
\int_\Omega \Phi_x(y)f(y)\,dy &= \int_\Omega \Phi_x(y)(-\Delta u(y))\,dy \\
&= \int_\Omega \left[\Phi_x(y)(-\Delta u(y)) + (\Delta\Phi_x(y))u(y)\right]\,dy \\
(\Delta\Phi_x(y) = 0) \qquad &= \int_{\partial\Omega}\left(-\Phi_x(y)\frac{\partial u}{\partial \vec{n}} + \frac{\partial \Phi_x(y)}{\partial \vec{n}}\underbrace{u(y)}_{g(y)}\right)\,dS(y)
\end{aligned}$$ ∎

How can we compute $\Phi_x(y)$? It is not easy for general domains. But let us prove on two cases:

- $\Omega = \mathbb{R}^d_+$ (half-space)
- $\Omega = B(0,r)$ (a ball)

5.1 Green's function on the upper half plane

We use the following notation:

$$\begin{aligned}\mathbb{R}^d_+ &= \{x = (x_1, x_2, \dots, x_d) = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid x_d > 0\} \\ \partial\mathbb{R}^d_+ &= \{x = (x', x_d) \mid x_d = 0\} = \mathbb{R}^{d-1} \times \{0\}\end{aligned}$$

For all $x \in \mathbb{R}^d$ we want to find the correction function $\Phi_x(y)$ with $y \in \overline{\mathbb{R}^d_+}$ s.t.

$$\begin{cases} +\Delta_y \Phi_x(y) = 0 & \text{in } \mathbb{R}^d_+ \\ \Phi_x(y) = G(y-x) & \text{in } \partial\mathbb{R}^d_+ \end{cases}$$

Definition 5.8 (Reflection for $\mathbb{R}^d_+$) For all $x = (x', x_d) \in \mathbb{R}^d$, $\tilde{x} = (x', -x_d) \in \mathbb{R}^d$, (if $x \in \mathbb{R}^d_+ \Rightarrow \tilde{x}\mathbb{R}^d_-$)

Claim: $\Delta_y \Phi_x(y) = G(y - \tilde{x})$ is a corrector function.

- $\Delta_y \Phi_x(y) = \Delta_y G(y - \tilde{x}) = 0$ for all $y \in \mathbb{R}^d_+$ for all $x \in \mathbb{R}^d_+$ (as $\tilde{x} \in \mathbb{R}^d_- = \mathbb{R}^d \backslash \overline{\mathbb{R}^d_+}$)
- $\Phi_x(y) = G(y - \tilde{x}) = G(y - x)$ on $y \in \partial\mathbb{R}^d_+$. In fact, $y \in \partial\mathbb{R}^d_+$, so $y_d = 0$, so

$$G(y - \tilde{x}) = G_0(|y - \tilde{x}|) = G_0\left(\sqrt{\sum_{i=1}^{d-1} |x_i - y_i|^2 + |x_d|^2}\right) = G_0(|y - x|)$$

Consider $f = 0$ and

$$\begin{cases} -\Delta = 0 & \text{in } \mathbb{R}^d_+ \\ u = g & \text{on } \partial\mathbb{R}^d_+ \end{cases}$$

Then we expect

$$u(x) = -\int_{\partial\Omega} g(y) \frac{\partial \tilde{G}}{\partial \vec{n_y}}(x, y)\, dS(y)$$

We compute

$$\frac{\partial \tilde{G}}{\partial \vec{n_y}}(x - y) = \sum_{j=1}^{d} \frac{\partial \tilde{G}}{\partial y_j}(x, y)\vec{n_j} = -\frac{\partial \tilde{G}}{\partial y_d}(x, y) = \frac{\partial}{\partial y_d}(G(y - \tilde{x}) - G(y - x)) = \dots$$

because $\tilde{G}(x, y) = G(y - x) - \Phi_x(y) = G(y - x) - G(y - \tilde{x})$.

$$\begin{aligned} \dots &= \frac{1}{d|B_1|}\left[\frac{-(y_d - \tilde{x}_d)}{|y - \tilde{x}|^d} - \frac{-(y_d - x_d)}{|y - x|^d}\right] \\ (y \in \partial\mathbb{R}^d_+) \qquad &= \frac{1}{d|B_1|}\left[\frac{\tilde{x}_d}{|y - x|} - \frac{x_d}{|y - x|^d}\right] = \frac{-2x_d}{d|B_1||y - x|^d} \end{aligned}$$

We expect

$$u(x) = -\int_{\partial\mathbb{R}^d_+} g(y) \frac{\partial \tilde{G}}{\partial \vec{n_y}}(x, y)\, dS(y) = \int_{\partial\mathbb{R}^d_+} g(y) \frac{2x_d}{d|B_1||y - x|^d}\, dS(y)$$

Theorem 5.9 Assume $g \in C(\mathbb{R}^{d-1}) \cap L^\infty(\mathbb{R}^{d-1})$ Then

$$u(x) = \int_{\partial\mathbb{R}^d_+} g(y) K(x, y)\, dS(y)$$

and

$$K(x,y) = \frac{2x_d}{d|B_1||y-x|^d} \quad \text{for all } x \in \mathbb{R}^d_+.$$

satisfies that $u \in C^\infty(\mathbb{R}^d_+) \cap L^\infty(\mathbb{R}^d_+)$ and

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d_+ \\ \lim_{\substack{x \to 0 \\ x \in \mathbb{R}^d_+}} u(x) = g(x_0) & \forall x_0 \in \partial\mathbb{R}^d_+ \end{cases}$$

Proof. Claim: For all $y \in \partial\mathbb{R}^d_+$, $x \mapsto K(x,y)$ is harmonic in $\mathbb{R}^d_+$ (i.e. $\Delta_x K(x,y) = 0$ in $\mathbb{R}^d_+$)

- Argument from Evans:

  $$K(x,y) = -\frac{\partial}{\partial y_d},\ \tilde{G}(y-x) = -\frac{\partial}{\partial y_d}(G(y-x) - G(y-\tilde{x}))$$

  We know that for all $x \in \mathbb{R}^d_+$, $y \mapsto \tilde{G}(y,x)$ is haromnic in $\mathbb{R}^d_+ \backslash \{x\}$. By symmetry we have $\tilde{G}(y,x) = \tilde{G}(x,y)$ for all $x, y \in \mathbb{R}^d_+$. So for all $y \in \mathbb{R}^d_+$, $x \mapsto \tilde{G}(y,x)$ is harmonic in $\mathbb{R}^d_+ \backslash \{y\}$. Then for all $y \in \mathbb{R}^d_+ : -\frac{\partial}{\partial y_d}\tilde{G}(y,x) = K(x,y)$ is harmonic $x \in \mathbb{R}^d_+ \backslash \{y\}$. By a limit argument, for all $y \in \partial\mathbb{R}^d_+$, $x \mapsto K(x,y)$ is harmonic for all $x \in \mathbb{R}^d_+$.

- A direct proof:

  $$K(x,y) = \frac{2x_d}{d|B_1|}\frac{1}{|x-y|^d}$$

  for all $x \in \mathbb{R}^d_+$, $y \in \partial\mathbb{R}^d_+$. For $i \neq d$, $x = (x_1, \ldots, x_d)$,

  $$\begin{aligned} \partial_{x_i} K(x,y) &= \frac{2x_d}{d|B_1|}\frac{(-d)}{|x-y|^{d+1}}\frac{x_i - y_i}{|x-y|} = \frac{-2x_d}{|B_1|}\frac{x_i - y_i}{|x-y|^{d+2}} \\ \partial^2_{x_i} K(x,y) &= -\frac{2x_d}{|B_1|}\left[\frac{1}{|x-y|^{d+1}} - \frac{(d+2)}{|x-y|^{d+3}}(x_i - y_i)\frac{(x_i - y_i)}{|x-y|}\right] \\ &= -\frac{2x_d}{|B_1|}\left[\frac{1}{|x-y|^{d+1}} - \frac{(d+2)}{|x-y|^{d+4}}(x_i - y_i)^2\right] \end{aligned}$$

  Moreover:

  $$\begin{aligned} \partial_{x_d} K(x,y) &= \frac{2}{d|B_1|}\frac{1}{|x-y|^d} + \frac{2x_d}{d|B_1|}(-d)\frac{(x_d - y_d)}{|x-y|^{d+2}} \\ (y_d = 0) \quad &= \frac{2}{d|B_1|}\frac{1}{|x-y|^d} + \frac{2x_d^2}{|B_1||x-y|^{d+2}} \\ \partial^2_{x_d} K(x,y) &= \frac{-2}{|B_1|}\frac{(x_d - y_d)}{|x-y|^{d+2}} + \frac{4x_d}{|B_1||x-y|^{d+2}} - \frac{2(d+2)|B_1|^2}{x}{}_d\frac{(x_d - y_d)}{|x-y|^{d+4}} \end{aligned}$$

Then:

$$
\begin{aligned}
\Delta_x K(x,y) &= \sum_{i=1}^{d-1} \partial_{x_i}^2 K(x,y) + \partial_{x_i}^2 K(x,y) \\
&= -\frac{2x_d}{|B_1|}\left[\frac{d-1}{|x-y|^{d+2}} - (d+2)\sum_{i=1}^{d-1}\frac{(x_i-y_i)^2}{|x-y|^{d+4}}\right. \\
&\quad \left. + \frac{1+2}{|x-y|^{d+2}} - \frac{(d+2)x_d(x_d-y_d)}{|x-y|^{d+4}}\right] \\
&= -\frac{2x_d}{|B_1|}\left[\frac{d+2}{|x-y|^{d+2}} - (d+2)\frac{1}{|x-y|^{d+4}}\left(\underbrace{\sum_{i=1}^{d}|x_i-y_i|^2}_{|x-y|^2}\right)\right] = 0
\end{aligned}
$$

for all $x \in \mathbb{R}_+^d$, $y \in \partial\mathbb{R}_+^d$. Claim (exercise) for all $x \in \mathbb{R}_+^d$,

$$
\int_{\partial\mathbb{R}_+^d} K(x,y)\,dy = 1
$$

Consider

$$
u(x) = \int_{\partial\mathbb{R}_+^d} K(x,y)g(y)\,dy, \quad x \in \mathbb{R}_+^d
$$

Since $g \in L^\infty(\mathbb{R}^{d-1}) = L^\infty(\partial\mathbb{R}_+^d)$ and $K(x,y) \geqslant 0$, hence

$$
|u(x)| \leqslant \left(\int_{\partial\mathbb{R}_+^d} K(x,y)\,dy\right)\|g\|_{L^\infty}
$$

Thus $\|u\|_{L^\infty} \leqslant \|g\|_{L^\infty}$. Moreover

$$
D_x^\alpha u(x) = \int_{\partial\mathbb{R}_+^d} D_x^\alpha K(x,y)g(y)\,dy
$$

bounded, so $u \in C^\infty(\mathbb{R}_+^d)$, $x \mapsto K(x,y)$ is smooth as $x \neq y$.

$$
\Delta_x u(x) = \int_{\partial\mathbb{R}_+^d} \underbrace{\Delta_x K(x,y)}_{=0} g(y)\,dy = 0
$$

So u is harmonic in $\mathbb{R}_+^d$. ($\Rightarrow u \in C^\infty$ by Weyl's lemma). Take $x_0 \in \partial\mathbb{R}_+^d$ and $x \in \mathbb{R}_+^d$. Then:

$$
\begin{aligned}
|u(x) - g(x_0)| &= \left|\int_{\partial\mathbb{R}_+^d} K(x,y)(g(y)-g(x_0))\,dy\right| \\
&\leqslant \int_{\partial\mathbb{R}_+^d} K(x,y)|g(y)-g(x_0)|\,dy \\
&= \underbrace{\int_{|y-x_0|\leqslant L|x-x_0|}}_{(I)} + \underbrace{\int_{|y-x_0|>L|x-x_0|}}_{(II)}
\end{aligned}
$$

$$
\begin{aligned}
(I) &= \int_{|y-x_0|\leqslant L|x-x_0|} K(x,y)|g(y)-g(x_0)|\,dy \\
&= \sup_{|y-x_0|\leqslant L|x-x_0|} |g(y)-g(x_0)| \xrightarrow{x\to x_0} 0 \quad \forall L > 0
\end{aligned}
$$

(II): If $|y-x_0| > L|x-x_0|$, then $|y-x| > \frac{1}{2}|y-x_0| > \frac{L}{2}|x-x_0|$ if $L \geqslant 2$.

$$\int_{|y-x_0|>|L|x-x_0|} K(x,y)|g(y)-g(x_0)|\,dy \leqslant C\int_{y\in\partial\mathbb{R}^d_+} \frac{x_d}{|x_0-y|}\,dy$$
$$Cx_d \int_{\substack{z\in\mathbb{R}^{d-1}\\|z|>L|x-x_0|}} \frac{1}{|z|^d}\,dz = const.\frac{x_d}{L|x-x_0|} \leqslant \frac{const.}{L} \xrightarrow{L\to\infty} 0$$

$x_d = |x_d-(x_0)_d| \leqslant |x-x_0|$

■

5.2 Green's function for a ball

Let $B = B(0,1)$. For all $x \in B$, for all $y \in \bar{B}$ we want to find the corrector function $\Phi_x(y)$ s.t.

$$\begin{cases} \Delta_y \Phi_x(y) = 0 & \text{in } B \\ \Phi_x(y) = G(y-x) & \text{on } \partial B \end{cases}$$

where for $d \geqslant 3$: $G(z) = \frac{1}{d(d-2)|B_1||z|^{d-2}}$.

Definition 5.10 (Reflection / Duality through the sphere ∂B) For all $x \in \mathbb{R}^d\backslash\{0\}$ we define $\tilde{x} = \frac{x}{|x|^2}$. Clearly we have for all $x \in B$ that if $|x| < 1$, then $|\tilde{x}| = \left|\frac{x}{|x|^2}\right| = \frac{1}{|x|} > 1$, so $\tilde{x} \notin \bar{B}$

Lemma 5.11 For $d \geqslant 3$ the function $\Phi_x(y) = G(|x|(y-\tilde{x}))$ is a corrector function.

Proof.

$$\Phi_x(y) = \frac{1}{d(d-2)|B_1||x|^{d-2}|y-\tilde{x}|^{d-2}}$$

for all $x \in B, x \neq 0$, for all $y \in \bar{B}$. Then clearly $y \mapsto \Phi_x(y)$ is harmonic in B (Since $\frac{1}{|z|^{d-2}}$ is harmonic in $\mathbb{R}\backslash 0$). Let's check the boundary: Let $y \in \partial B$, i.e. $|y| = 1$. Then

$$\begin{aligned} ||x|(y-\tilde{x})| &= |x|\left|y - \frac{x}{|x|^2}\right| \\ &= |x|\sqrt{|y|^2 - 2\frac{xy}{|x|^2} + \left|\frac{x}{|x|^2}\right|^2} \\ &= \sqrt{|x|^2|y|^2 - 2xy + 1} \\ (|y|=1) \quad &= \sqrt{|x|^2 - 2xy + |y|^2} = |x-y| \end{aligned}$$

Thus $\Phi_x(y) = G(|x||y-\tilde{x}|) = G(y-x)$ for all $0 \neq x \in B$, for all $y \in \partial B$. Let's compute the Poisson kernel: If want to solve

$$\begin{cases} -\Delta u = 0 & \text{in } B \\ u = g & \text{on } \partial B \end{cases}$$

then

$$u(x) = -\int_{\partial B} \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x,y)g(y)dS(y).$$

$\tilde{G}(x,y) = G(y-x) - \Phi_x(y) = G(y-x) - G(|x|(y-\tilde{x}))$ for all $x \in B\backslash\{0\}$, $y \in \bar{B}$.

$$\frac{\partial \tilde{G}}{\partial \vec{n}_y} = \sum_{i=1}^{d} \partial_{y_i} \tilde{G} y_i$$

Here

$$\begin{aligned}
\partial_{y_i}\tilde{G} &= \partial_{y_i} G(y-x) - \partial_{y_i}[G(|x|(y-\tilde{x}))] \\
&= \frac{-(y_i - x_i)}{d|B_1||y-x|^d} + \frac{y_i - \tilde{x}_i}{d|B_1||x|^{d-2}|y-\tilde{x}|^d} \\
\Rightarrow \frac{\partial \tilde{G}}{\partial \vec{n}_y} &= \sum_{i=1}^{d} [\dots] y_i \\
&= \frac{-y(y-x))}{d|B_1||y-x|^d} + \frac{y(y-\tilde{x})}{d|B_1||x|^{d-2}|y-\tilde{x}|^d} \\
&= \frac{1}{d|B_1||y-x|^d}(-y(y-x) + y(y-\tilde{x})|x|^2) \\
&= \frac{1}{d|B_1||y-x|^d}[-|y|^2 + xy + |y|^2|x|^2 - xy] \\
&= \frac{-1+|x|^2}{d|B_1||y-x|^d}
\end{aligned}$$

as $y \in \partial B$. ■

Theorem 5.12 (Poisson Formula for a Ball) Let $B = B(0,1)$, $g \in C(\partial B)$. Define for all $x \in B$:

$$u(x) = \int_{\partial B} K(x,y) g(y)\, dS(y),$$

$K(x,y) = -\frac{\partial \tilde{G}}{\partial \vec{n}_y}(x,y) = \frac{1-|x|^2}{d|B_1||y-x|^d}$ for all $x \in B$, for all $y \in \partial B$. Then $u \in C^\infty(B)$, $\Delta u = 0$ and for all $x_0 \in \partial B$ we have $\lim_{\substack{x \to x_0 \\ x \in B}} u(x) = g(x_0)$. This holds for all $d \geqslant 2$.

Proof. We need to check:

1. For all $y \in \partial B$, $x \mapsto K(x,y)$ is harmonic in B.
2. $\int_{\partial B} K(x,y)\, dS(y) = 1$ for all $x \in B$ (exercise)

Now for all $x \in B$, for all $y \in \partial B$:

$$\begin{aligned}
K(x,y) &= \frac{1-|x|^2}{d|B_1||y-x|^d} \\
\partial_{x_i} K(x,y) &= \frac{-2x_i}{d|B_1|}\frac{1}{|x-y|^d} - \frac{1-|x|^2}{|B_1|}\frac{x_i-y_i}{|x-y|^{d+2}} \\
\partial_{x_i}^2 K(x,y) &= -\frac{2}{d|B_1|}\frac{1}{|x-y|^d} + \frac{2x_i}{|B_1|}\frac{x_i-y_i}{|x-y|^{d+2}} + \frac{2x_i}{|B_1|}\frac{x_i-y_i}{|x-y|^{d+2}} \\
&\quad - \frac{1-|x|^2}{|B_1|}\frac{1}{|x-y|^{d+2}} + \frac{1+|x|^2}{|B_1|}(d+2)\frac{(x_i-y_i)^2}{|x-y|^{d+4}} \\
\Delta_x K &= \sum_{i=1}^d \partial_{x_i}^2 K = -\frac{2}{|B_1|}\frac{1}{|x-y|^d} + \frac{4x(x-y)}{|B_1||x-y|^{d+2}} \\
&\quad - \frac{d(1-|x|^2)}{|B_1|}\frac{1}{|x-y|^{d+2}} + (d+2)\frac{1-|x|}{|B_1|}\frac{1}{|x-y|^{d+2}} \\
&= \frac{2}{|B_1||x-y|^{d+2}}[-|x|^2+2xy-|y|^2+2|x|^2-2xy+1-|x|^2] \\
&= \frac{2}{|B_1||x-y|^{d+2}}[-|x|^2+2xy-|y|^2+2|x|^2-2xy+1-|x|^2]
\end{aligned}$$

$1-|y|^2 = 0$ as $y \in \partial B$. Thus $\Delta_x K(x,y) = 0$, for all $x \in B$, for all $y \in \partial B$.

$$|u(x)| = \left| \int_{\partial B} K(x,y) g(y)\, dS(y) \right| \leqslant \|g\|_{L^\infty(\partial B)}$$

$\int_{\partial B} K(x,y), dS(y) = \|g\|_{L^\infty}$,

$$\Delta_x u(x) = \int_{\partial B} \underbrace{\Delta_x K(x,y)}_{=0} g(y)\, dS(y) = 0$$

Take $x \in B$, $x \to x_0 \in \partial B$.

$$\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial B} K(x,y)(g(y)-g(x_0))\, dS(y) \right| \\
&\leqslant \int_{A_1} + \int_{A_2} K(x,y)|g(y)-g(x_0)|\, dS(y),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \{y \in \partial B \mid |y-x_0| \leqslant |x-x_0|^\alpha\} \\
A_2 &= \{y \in \partial B \mid |y-x_0| > |x-x_0|^2\}
\end{aligned}$$

On A_1 we have:

$$\int_{A_1} \cdots \leqslant \sup_{\substack{|z-x_0| \leqslant |x-x_0|^\alpha \\ z \in \partial B}} \int_{\partial B} K(x,y)\, dS(y) \xrightarrow{x \to x_0} 0$$

since $G \in C(\partial B)$. On A_2:

$$|y-x_0| > |x-x_0|^\alpha$$

$$\Rightarrow |y-x| \geqslant |y-x_0| - |x-x_0| \geqslant |x-x_0|^\alpha - |x-x_0| \geqslant \frac{1}{2}x-x_0^\alpha$$

if $\alpha < 1$ and $|x-x_0|$ small. So we get

$$K(x,y) = \frac{1-|x|^2}{d|B_1||x-y|^d} \leqslant C\frac{1-|x|^2}{|x-x_0|^{d\alpha}} \leqslant C|x-x_0|^{1-d\alpha}$$

Thus

$$\int_{A_2} K(x,y)|g(y)-g(x_0)|\,dS(y) \leqslant C\|g\|_{L^\infty}|x-x_0|^{1-d\alpha} \xrightarrow{x\to x_0} 0$$

if $1-d\alpha > 0 \Leftrightarrow \alpha < \frac{1}{d}$. ■

Exercise 5.13 (E 7.2) Define $\mathbb{R}^d_+ = \{(x', x_d) \in \mathbb{R}^{d-1}\times\mathbb{R} \mid x_d > 0\}$. Let $K(x,y) = \frac{2x_d}{d|B_1||x-y|^d}$ for all $x \in \mathbb{R}^d_+, y \in \partial\mathbb{R}^d_+ = \{(y',0) \mid y' \in \mathbb{R}^{d-1}\} \simeq \mathbb{R}^{d-1}$. Prove

$$\int_{\partial\mathbb{R}^d_+} K(x,y)\,dS(y) = 1 \quad \forall x \in \mathbb{R}^d_+$$

Solution. Denote $x=(x',x_d)$, $y=(y',0)$, $x',y' \in \mathbb{R}^{d-1}$, $x_d > 0$.

$$\int_{\partial\mathbb{R}^d_+} K(x,y)\,dS(y) = \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|\left(|x'-y'|^2+x_d^2\right)^{\frac{d}{2}}}\,dy' = \dots$$

as $|x-y| = |(x'-y', x_d)| = \sqrt{|x'-y'|^2+x_d^2}$.

$$\begin{aligned}
(y'-x' \mapsto y') \quad \dots &= \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|\left(|y'|^2+x_d^2\right)^{\frac{d}{2}}}\,dy' \\
(y' = x_d z) \quad &= \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|\left(x_d^2(|z|^2+1)\right)^{\frac{d}{2}}}\left(x_d^{d-1}\right)\,dz \\
&= \int_{\mathbb{R}^{d-1}} \frac{2}{d|B_1|\left(|z|^2+1\right)^{\frac{d}{2}}}\,dz \\
&= \int_0^\infty \frac{2\omega_{d-1}}{d|B_1|}\frac{1}{(r^2+1)^{\frac{d}{2}}}r^{d-2}\,dr \\
&= \frac{2\omega_{d-1}}{\omega_d}\int_0^\infty \frac{1}{(r^2+1)^{\frac{d}{2}}}r^{d-2}\,dr
\end{aligned}$$

Set $d=2$: $\omega_1 = 1, |\omega_2| = 2\pi$

$$\frac{2}{\pi}\int_0^\infty \frac{1}{r^2+1}\,dr = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{(\tan t)^2+1}\left[(\tan t)^2+1\right]dt = 1$$

we we set $r = \tan t, t \in \left(0, \frac{\pi}{2}\right), \frac{dr}{dt} = (\tan t)' = 1 + (\tan t)^2$ ■

For $d=3$:

$$\frac{2\cdot 2\pi}{4\pi}\int_0^\infty \frac{1}{(r^2+1)^{\frac{3}{2}}}r\,dr = \int_0^\infty \frac{d}{dr}\left[\frac{-1}{(r^2+1)^{\frac{1}{2}}}\right]dr = \left.\frac{-1}{(r^2+1)^{\frac{1}{2}}}\right]_0^\infty = 1$$

Exercise 5.14 (7.3) Let $g \in C(\partial\mathbb{R}^d_+) \cap L^\infty(\partial\mathbb{R}^d_+)$ $(\partial\mathbb{R}^d_+ \simeq \mathbb{R}^{d-1})$.

$$u(x) = \int_{\partial\mathbb{R}^d_+} K(x,y)g(y)\,dS(y) \qquad K(x,y) = \frac{2x_d}{d|B_1||x-y|^d}, x \in \mathbb{R}^d_+$$

Prove that if $g(y) = |y|$, if $|y| \leqslant 1$, then $|\nabla u|$ is unbounded in $B(0,r)\cap\mathbb{R}^d_+$ for all $r > 0$.

Solution.

$$\begin{aligned}
\partial_{x_d} u(x) &= \int_{\partial\mathbb{R}^d_+} \partial x_d K(x,y) g(y)\, dy \quad \forall x \in \mathbb{R}^d_+ \\
&= \frac{2}{d|B_1|} \int_{\partial\mathbb{R}^d_+} \left[\frac{1}{|x-y|^d} - \frac{dx_d^2}{|x-y|^{d+2}} \right] g(y)\, dy \\
&= \frac{2}{d|B_1|} \int_{\partial\mathbb{R}^d_+} \frac{1}{|x-y|^{d+2}} [|x-y|^2 - dx_d^2] g(y)\, dy \\
&= \frac{2}{d|B_1|} \int_{\partial\mathbb{R}^d_+} \frac{1}{(|x'-y'| + x_d^2)^{\frac{d+2}{2}}} \left[|y'|^2 - (d-1)x_d^2 \right] g(y)\, dy
\end{aligned}$$

Assume that $\partial_d u$ is bounded in $B(0,r) \cap \mathbb{R}^d_+$ Then:

$$|u(0,x_d) - \underbrace{u(0,0)}_{g(0)=0}| \leqslant C|x_d|$$

if x_d small. Consider:

$$\begin{aligned}
\limsup_{x_d \to 0^+} \frac{u(0,x_d)}{x_d} &= \limsup_{x_d \to 0^+} c \int_{\mathbb{R}^{d-1}} \frac{1}{(|y'|^2 + x_d^2)^{\frac{d}{2}}} g(y)\, dy' \\
&\geqslant \int_{\mathbb{R}^{d-1}} \frac{1}{|y'|^d} g(y)\, dy = \int_{|y'| \leqslant 1} + \int_{|y'|>1} \\
&to \int_{\mathbb{R}^{d-1}} \frac{1}{|y'|^{d-1}}\, dy' = \infty
\end{aligned}$$

■

Exercise 5.15 (Bonus 7) Recall the Poisson kernel on a ball $B(0,r) \subseteq \mathbb{R}^d$:

$$K(x,y) = \frac{r^2 - |x|^2}{d|B_1|r} \frac{1}{|x-y|^d}$$

for all $x \in B(0,r)$, $y \in \partial B(0,r)$. Prove:

$$\int_{\partial B(0,r)} K(x,y)\, dS(y) = 1$$

for all $x \in B(0,r)$. (It suffices if you can prove $d=2$ and $d=3$)

5.3 Energy Method

Consider $u \in C^2(\Omega)$ for $\Omega \subseteq \mathbb{R}^d$ open, bounded and with C^1 boundary and

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on} \partial\Omega. \end{cases}$$

Take $\phi \in C_c^\infty(\Omega)$, then by integration by parts:

$$0 = \int_\Omega (-\Delta u - f)\phi = \int_\Omega \nabla u \nabla \phi - \int_\Omega f\phi$$

Key observation: This is the *derivative* of the energy functional

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega fu$$

If u is a minimizer of E, then it solves the equation $-\Delta u = f$ in Ω. The boundary condition $u = g$ does not appear on E, but this is encoded in the set of *admissible functions.* (The set of candidates of solutions). For the classical solutions, we have

Theorem 5.16 (Dirichlet's principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1-boundary. Let $f \in C(\bar{\Omega})$ and $g \in C(\partial B)$. Then the following statements are equivalent:

1. $u \in C^2(\bar{\Omega})$ solves $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

2. u is a minimizer of the variational problem $E = \inf_{v \in A} E(v)$, where
$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv,$$
$A = \{v \in C^2(\bar{\Omega}) \mid v = g \text{ on } \partial\Omega\}$.

Moreover there is at most a solution / minimizer (uniqueness).

Proof. The result holds even for complex-valued functions. Let us write the proof for real-valued functions.

1. $\Rightarrow$ 2.: Let $u \in C^2(\bar{\Omega})$ be a solution of $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$. Then we prove $E(u) \leqslant E(v)$ for all $v \in A$. If $v \in A$, then $u - v = 0$ on $\partial\Omega$. Using this and $-\Delta u = f$ in Ω, we have:
$$\begin{aligned} 0 &= \int_\Omega (-\Delta u - f)\cdot(u - v)\,dy \\ \text{(Part. Int.)} \quad &= \int_\Omega \nabla u(\nabla u - \nabla v)\,dy - \int_\Omega f(u - v)\,dy \\ &= \left[\frac{1}{2}\int_\Omega |\nabla u|^2\,dy - \int_\Omega fu\,dy\right] - \left[\frac{1}{2}\int_\Omega |\nabla v|^2\,dy - \int_\Omega fv\,dy\right] \\ &\quad + \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega \\ &= E(u) - E(v) + \underbrace{\frac{1}{2}\int_\Omega |\nabla u - \nabla v|^2}_{\geqslant 0} \end{aligned}$$
$E(u) \leqslant E(v)$, so u is a minimizer of $\inf_{v \in A} E(v)$. Moreover u is the unique minimizer on A. Since $E(u) = E(v)$ we have $\int_\Omega |\nabla(u - v)^2 = 0$, so $u - v = const.$, so $u - v = 0$ in $\bar{\Omega}$.

2. $\Rightarrow$ 1.: Assume that u is a minimizer of $\inf_{v \in A} E(v)$. Then $E(u) \leqslant E(v)$ for all $v \in A$. Take $\phi \in C_c^\infty(\Omega)$, then $u + t\phi \in A$ for all $t \in \mathbb{R}$.
$$\begin{aligned} &\Rightarrow E(u) \leqslant E(u + t\phi) \text{ for all } t \in \mathbb{R} \\ &\Rightarrow t \mapsto E(u + t\phi) \text{ has a minimizer at } t = 0 \\ &\Rightarrow 0 = \frac{d}{dt}E(u + t\phi)|_{t=0} \\ &\quad = \frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla u + t\nabla\phi|^2 - \int_\Omega f(u + t\phi)\right)\Bigg|_{t=0} \\ &\quad = \frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla u|^2 + t^2|\nabla\phi|^2 + 2t\nabla u\nabla\phi - \int_\Omega f(u + t\phi)\right)\Bigg|_{t=0} \\ &\quad \int_\Omega \nabla u\nabla\phi - \int_\Omega f\phi = \int_\Omega (-\Delta u - f)\phi \end{aligned}$$
for all $\phi \in C_c^\infty(\Omega)$. So $-\Delta u - f = 0$ in Ω and $u = g$ since $u \in A$.

Direct method of calculus of variations. Think $f : \mathbb{R} \to \mathbb{R}$, $f \in C(\mathbb{R})$, $f(x) \to \infty$ as $|x| \to \infty$. There is a $x_0 \in \mathbb{R}$ s.t. $f(x_0) = \inf_{x \in \mathbb{R}} f(x)$.

Step 1: $E = \inf_{x \in \mathbb{R}} f(x) > -\infty$

Step 2: Take a minizing sequence $\{x_n\} \subseteq \mathbb{R}$, $f(x_n) \to E$. Up to a subsequence $x_n \to x_0$ in $\mathbb{R}$ (compactness)

Step 3: Lower semicontinuity $E = \liminf_{n \to \infty} f(x_n) \geqslant f(x_0)$

If we apply the direct method to $\inf_{v \in A} E(v)$,

$$E(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega f v,$$

$A = \{v \in C^2(\bar{\Omega}), v = g \text{ on } \partial\Omega\}$

Step 1: Easy $E = \int_{v \in A} E(v) > -\infty$

Step 2: There is a minimizing sequence $\{v_n\} \subseteq A$ s.t. $E(v_n) \to E$. We don't know if there is a subsequence of $\{v_n\}$ that converges to $u \in A$. The lack of compactness is a serious problem! We need to find the right set A! Consider again

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Consider the simple case $g = 0$. $\Delta u - f$ in $\Omega \Leftrightarrow \nabla u \nabla \phi$... The right set A should be $A = \{v \mid \int_\Omega |\nabla v|^2 < \infty, v = 0 \text{ on } \partial\Omega\}$. Rigorously we take $W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)} W^{1,2}(\Omega)$ (Notation: $H_0^1 = W_0^{1,2}, H^1 = W^{1,2}$) Recall that $W^{1,p}$ is a banach space with norm $\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}$. We know that $C_c^\infty(\Omega)$ is dense in $W_{loc}^{1,p}(\Omega)$, i.e. for all $u \in W_{loc}^{1,p}(\Omega)$ there is $\|u_n\| \subseteq C_c^\infty$ s.t. $u_n \to u$ in $W^{1,p}(K)$ for all $K \subseteq \Omega$ compact. However in general $C_c^\infty(\Omega)$ is not dense in $W^{1,p}(\Omega)$, i.e. $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)} W^{1,p}(\Omega) \subsetneq W^{1,p}(\Omega)$. Clearly $W_0^{1,p}$ is a closed subspace of $W^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ is a Banach space with $\|\cdot\|_{W^{1,p}(\Omega)}$. Why does $W_0^{1,p}(\Omega)$ encode the 0-boundary condition? Note that by definition for all $u \in W_0^{1,p}(\Omega)$ there is a sequence $\{u_n\} \subseteq C_c^\infty(\Omega)$, $u_n \to u$ in $W^{1,p}(\Omega)$ up to a subsequence $u_n(x) \to u(x)$ for almost every $x \in \Omega$. Note $u_n|_{\partial\Omega} = 0 \nrightarrow u|_{\partial\Omega} = 0$ since $\partial\Omega$ must be of 0-measure. ∎

Theorem 5.17 (Characterization for $W_0^{1,p}$) Let Ω be open, bounded with C^1-boundary. Let $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. Then the following statements are equivalent:

a) $u = 0$ on $\partial\Omega$

b) $u \in W_0^{1,p}(\Omega)$

(Later we will remove the condition $C(\bar{\Omega})$ by introducing the *Trace operator*.)

Remark 5.18 If $d = 1$, it holds that $W^{1,p} \subseteq C(\bar{\Omega})$. Then the theorem gives a full characterization for $W_0^{1,p}$, but if $d \geqslant 2$, then in general $W^{1,p} \nsubseteq C(\Omega)$. (later)

Proof of theorem 5.17.

a) $\Rightarrow$ b):

Lemma 5.19 If $u \in W^{1,p}(\Omega)$ and $\operatorname{supp} u \subseteq \Omega$, then $u \in W_0^{1,p}(\Omega)$.

Proof. Since $K := \operatorname{supp} u$ is a compact subset in Ω, we can find a function $\chi \in C_c^\infty(\Omega)$, $\chi = 1$ on K. Moreover since $u \in W^{1,p}(\Omega)$, there is a sequence $\{u_n\} \subseteq C_c^\infty(\Omega)$ s.t. $u_n \to u$ in $W_{loc}^{1,p}(\Omega)$. We claim that $\chi u_n \to \chi u$ in $W_{loc}^{1,p}(\Omega)$. (exercise, $\nabla(\chi u) = \nabla\chi u + \chi \nabla u$). This implies $\chi u_n \to u$ in $W^{1,p}(\operatorname{supp}\chi)$, thus $\chi u_n \to u$ in $W^{1,p}(\Omega)$, so $u \in W_0^{1,p}(\Omega)$. ■

Assume $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$. Take $G \in C^1(\mathbb{R})$ s.t. $|G(t)| \leqslant t$ for all t, $G(t) = t$ if $t \geqslant 2$ and $G(t) = 0$ if $t \leqslant 1$. Then let

$$u_n(x) := \frac{1}{n} G(nu(x)) \in W^{1,p}(\Omega)$$

$$\overset{\text{(Chain-rule)}}{\Rightarrow} \qquad \nabla u_n(x) = \frac{1}{n} G'(nu(x)) n \nabla u(x) = G'(nu(x)) \nabla u(x)$$

Moreover, u_n is compactly supported in Ω, so $u_n \in W_0^{1,p}(\Omega)$ by the lemma and $u_n \to u$ in $W^{1,p}(\Omega)$, so $u \in W_0^{1,p}(\Omega)$ since $W_0^{1,p}$ is a closed space. Recall that $u \in C(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$. Thus for all $\epsilon > 0$ there is a compact $K_\epsilon \subseteq \Omega$ s.t. $\sup_{x \in \Omega \setminus K_\epsilon} |u(x)| \leqslant \epsilon$. For any given $n \in \mathbb{N}$, $u_n(x) \neq 0$, so $G(nu(x)) \neq 0$. This implies $n|u(x)| > 1$, hence $|u(x)| > \frac{1}{n}$. Thus $u_n(x) = 0$ for all x such that $|u(x)| \leqslant \frac{1}{n}$, so $\operatorname{supp} u_n \subseteq K_{\frac{1}{n}}$ compact in Ω. Next, let us check $u_n \to u$ in $W^{1,p}(\Omega)$.

$$\int_\Omega |u_n(x) - u(x)|^p \, dx \to 0$$

since $u_n(x) = \frac{1}{n} G(nu(x)) \xrightarrow{n \to \infty} u(x)$ for all $x \in \Omega$ and $|u_n(x)| \leqslant \frac{1}{n}|G(nu(x))| \leqslant \frac{1}{n}|nu(x)| \leqslant |u(x)| \in L^p(\Omega)$.

$$\int_\Omega |\nabla u_n(x) - \nabla u(x)|^p \, dx = \int_\Omega |G'(nu(x)) - 1|^p |\nabla u(x)|^p \, dx \to 0$$

as $|G'(v(x)) - 1| \to 0$ for all x s.t. $u(x) \neq 0$ and $\nabla u(x) = 0$ on $\{x \mid u(x) = 0\}$. (exercise)

(b) $\Rightarrow$ (a): Let $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $u \in W_0^{1,p}(\Omega)$. Then we prove $u = 0$ on $\partial\Omega$. Lets regard the case $\Omega = Q_+ = \{(x', x_d) \mid \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, 0 < x_d < 1\}$. We prove that if $u \in W_0^{1,p}(Q_+) \cap C(\overline{Q_+})$, then $u = 0$ on $Q_0 = \{(x', 0) \mid x' \in \mathbb{R}^{d-1}, |x'| < 1\}$. Since $u \in W_0^{1,p}(Q_+)$ there is $\{u_n\} \subseteq C_c^\infty(Q_+)$ s.t. $u_n \to u$ in $W^{1,p}(Q_+)$ for all $x = (x', x_d) \in Q_+$, then:

$$u_n(x', x_d) = \underbrace{u_n(x', 0)}_{=0} + \int_0^{x_d} \partial_d u_n(x', t) \, dt$$

Hence

$$|u_n(x', x_d)| \leqslant \int_0^{x_d} |\partial_d u_n(x', t) \, dt$$

This implies:

$$\begin{aligned}
&\int_{0 < x_d < \epsilon} \int_{|x'| \leqslant 1} |u_n(x', x_d)| \, dx' \, dx_d \\
&\leqslant \int_{0 < x_d < \epsilon} \int_{|x'| < 1} \left(\int_0^{x_d} |\partial_d u_n(x', t)| \, dt \right) dx' \, dx_d \\
&\leqslant \epsilon \int_{|x'| < 1} \int_0^\epsilon |\partial_d u_n(x', t)| \, dx' \, dt
\end{aligned}$$

$$\Rightarrow \frac{1}{\epsilon}\int_0^\epsilon \int_{|x'|\leqslant 1} |u_n(x',x_d)|\,dx'\,dx_d \leqslant \int_0^\epsilon \int_{|x'|<1} |\partial_d u_n(x',x_d)|\,dx'\,dx_d$$

for all $n \in \mathbb{N}$, $\epsilon > 0$. Take now $n \to \infty$, use $u_n \to u$ in $W^{1,p}(\Omega)$. Then:

$$\frac{1}{\epsilon}\int_0^\epsilon \int_{|x'|\leqslant 1} |u(x',x_d)|\,dx'\,dx_d \leqslant \int_0^\epsilon \int_{|x'|<1} |\partial_x u_n(x',x_d)|\,dx'\,dy$$

for all $\epsilon > 0$. Take $\epsilon \to 0$:

$$\int_{|x'|\leqslant 1} |u(x',0)|\,dx' \leqslant 0$$

here we use $u \in C(\bar{\Omega})$ for the left side and Dominated Convergence for the right side. Thus $u(x',0) = 0$ for all $|x'| \leqslant 1$, i.e. $u = 0$ on $\partial\Omega$. Let's regard the general case: Let Ω be open, bounded and with C^1-boundary. Lets define *local charts* By definition for all $x \in \partial\Omega$, there is a U_x open, such there is a bijecitve map $h : U_x \to Q$, and h, h^{-1} are C^1. Then clearly $\partial\Omega \subseteq \bigcup_{x\in\partial\Omega} U_x$. Since $\partial\Omega$ is compact, there is a finite subcover $\{U_i\}_{i=1}^N$ s.t. $\partial\Omega \subseteq \bigcup_{i=1}^N U_i$. We can find U_0 open s.t. $\bar{U}_0 \subseteq \Omega$ and $\Omega \subseteq \bigcup_{i=0}^N U_i$.

Lemma 5.20 There is a sequence $\{x_i\}_{i=0}^N \subseteq C^\infty(\mathbb{R}^d)$ s.t.

1. $\chi_i \geqslant 0$, $\sum_{i=0}^N \chi_i = 1$ in $\mathbb{R}^d$ ($\{\chi_i\}$ is a partition of unity)
2. For all $i = 1,\dots,N$, $\operatorname{supp}\chi_i$ is in U_i, i.e. $\chi_i \in C_c^\infty(U_i)$.
3. $i = 0$, $\operatorname{supp}\chi_0 \subseteq \mathbb{R}^d\backslash\partial\Omega$ and $\chi_0\backslash\Omega \in C_c^\infty(\Omega)$. (exercise)

Given $u \in W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$. Then $u = \sum_{i=0}^N \chi_i u$, where $\chi_i \geqslant 0$, $\chi_0 \in C_c^\infty(\Omega)$, $\chi_i \in C_c^\infty(U_i)$. Since $\chi_0 u$ is supported in a copact set inside Ω, $\chi_0 u = 0$ on $\partial\Omega$. It remains to show that for all $i = 1,\dots,N$, $\chi_i u = 0$ on $U_i \cap \partial\Omega$. Then $\chi_i u(h^{-1}x) \in W_0^{1,p}(Q) \cap C(\bar{\Omega})$. This implies $\chi_i u(h^{-1}x) = 0$ on Q_0, so $\chi_i u(x) = 0$ on $U_i \cap \partial\Omega$. Why $W_0^{1,p}(U_i \cap \Omega) \to W_0^{1,p}(Q_+)$. If $v \in W_0^{1,p}(U_i \cap \Omega)$, then $v_n \to v, v_n \in C_c^\infty$. $v_n \circ h^{-1} \to v \circ h^{-1} \Rightarrow v \circ h^{-1} \in W_0^{1,p}(Q_+)$

■

Exercise 5.21 (E 8.1) Let $u \in W_{loc}^{1,1}(\mathbb{R}^d)$. Let $B = u^{-1}(\{0\})$. Prove that $\nabla u(x) = 0$ for a.e. $x \in B$.

Solution. We have already seen that if $f, g \in W_{loc}^{1,1}(\mathbb{R}^d)$, then $\max(f,g) \in W_{loc}^{1,1}$. This implies that if $u = u^+ - u^- \in W_{loc}^{1,1}$, then $u^+, v^+ \in W_{loc}^{1,1}$ since $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$. We have that $\nabla u = \nabla u^+ - \nabla u^-$. Claim:

$$\nabla u^+ = \begin{cases} 0 & u(x) \leqslant 0 \\ \nabla u & u(x) > 0 \end{cases} \quad \nabla u^- = \begin{cases} 0 & u(x) \geqslant 0 \\ \nabla u & u(x) < 0 \end{cases}$$

$$\begin{aligned} \int_{\mathbb{R}^d} (\partial_i u^+)\phi &= -\int_{\mathbb{R}^d} u^+ \partial_i \phi = -\int_{\{u(x)\leqslant 0\}} 0\partial_i\phi - \int_{\{u(x)>0\}} u\partial_i\phi \\ &= \int_{\{u(x)\leqslant 0\}} 0\phi + \int_{\{u(x)>0\}} \partial_i u\phi \end{aligned}$$

Alternative way: We showed for $f \in W^{1,p}(\mathbb{R}^d)$, that

$$\nabla|f|(x) = \begin{cases} (\nabla f)(x) & f(x) > 0 \\ -(\nabla f)(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

$u_+ = \frac{1}{2}(u + |u|)$. Hence $\nabla u_+ = \frac{1}{2}(\nabla u + \nabla|u|)$. Remark: If $A \subseteq \mathbb{R}$ has measure zero, then $\nabla u 1_{\{u(x) \in A\}} = 0$ a.e. (Th. 6.19 Lieb-Loss Analysis) ■

Exercise 5.22 (E 8.2) Let $\Omega, U \subseteq \mathbb{R}^d$ be open, $U \cap \Omega \neq \emptyset$, $u \in W_0^{1,p}(\Omega)$, $1 \leqslant p < \infty$, $\chi \in C_c^\infty(U)$. Prove: $\chi u \in W_0^{1,p}(\Omega \cap U)$ Hint: Recall $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}}}$

Solution. By definition there is a sequence $(u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$ s.t. $u_n \xrightarrow[n\to\infty]{\|\cdot\|_{W^{1,p}}} u$, i.e.

$$\|u_n - u\|_p + \|\nabla u_n - \nabla u\|_p \xrightarrow{n\to\infty} 0.$$

Define $f_n : \mathbb{R}^d \to \mathbb{C}$, $f_n(x) := u_n(x)\chi(x)$. Note $f_n \in C_c^\infty(\Omega \cap U)$ for all $n \in \mathbb{N}$. Claim: $(f_n)_{n\in\mathbb{N}}$ is Cauchy with respect to $\|\cdot\|_{W^{1,p}}$. Proof:

$$\|f_n - f_m\|_p = \|\chi(u_n - u_m)\|_p \leqslant \|\chi\|_\infty \underbrace{\|u_n - u_m\|_p}_{\xrightarrow{n,m\to\infty} 0} \xrightarrow{n,m\to\infty} 0$$

$$\nabla f_n = \nabla(\chi u_n) = (\nabla\chi)u_n + \chi\nabla u_n$$

$$\begin{aligned} \|\nabla f_n - \nabla f_m\|_p &\leqslant \|\nabla\chi(u_n - u_m)\|_p + \|\chi(\nabla u_n - \nabla u_m)\|_p \\ &\leqslant \|\nabla\chi\|_\infty \underbrace{\|u_n - u_m\|_p}_{\xrightarrow{n,m\to\infty} 0} + \underbrace{\|\chi\|}_{<\infty} \underbrace{\|\nabla u_n - \nabla u_m\|_p}_{\xrightarrow{n,m\to\infty} 0} \xrightarrow{n,m\to\infty} 0 \end{aligned}$$

Thus, there is a $f \in W_0^{1,p}(\Omega \cap U)$ s.t. $\|f_n - f\|_{W^{1,p}} \xrightarrow{n\to\infty} 0$. We know:

$$\begin{aligned} \|f_n - \chi u\|_{L^p} &= \|\chi u_n - \chi u\|_p \\ &\leqslant \|\chi\|_\infty \underbrace{\|u_n - u\|_p}_{\to 0} \xrightarrow{n\to\infty} 0 \end{aligned}$$

Since limits in L^p are unique, we get $\chi u = f \in W_0^{1,p}(\Omega \cup U)$. ■

Exercise 5.23 (E 8.3) Let $\Omega, U \subseteq \mathbb{R}^d$ open and bounded, $h : \bar{U} \to \bar{\Omega}$ C^1-diffeomorphisms, $u \in W_0^{1,p}(\Omega)$, $1 \leqslant p < \infty$. Prove $(x \mapsto u(h(x)) \in W_0^{1,p}(U)$.

Solution. Since $u \in W_0^{1,p}(\Omega)$ there is a sequence $(u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$ s.t.

$$\|u - u_n\|_p + \|\nabla u - \nabla u_n\|_p \xrightarrow{n\to\infty} 0$$

Define for all $n \in \mathbb{N}$ $f_n : U \to \mathbb{C}$, $f_n(x) = u_n(h(x))$. Note $f_n \in C_c^1(U)$. Claim 1: $(f_n)_{n\in\mathbb{N}}$ is Cauchy wrt. $\|\cdot\|_{W^{1,p}}$.

$$\begin{aligned} \|f_n - f_m\|_p^p &= \int_U |u_n(h(x)) - u_m(h(x))|^p \, dx \\ &= \int_\Omega |u_n(y) - u_m(y)|^p \, dy \underbrace{|\det(Dh^{-1})(y)|}_{\leqslant C<\infty} \xrightarrow{n,m\to\infty} 0 \end{aligned}$$

$$(\nabla f_n)(x) = \nabla(u_n(h(x))) = (\nabla u_n)(h(x))(Dh)(x)$$

$$
\begin{aligned}
\|\nabla f_n - \nabla f_m\|_p^p &= \int_U |[(\nabla u_n)(h(x)) - (\nabla u_m)(h(x))] \underbrace{(Dh)(x)}_{bdd.}|^p \, dx \\
&\leqslant C \int_U |(\nabla u_n)(h(x)) - (\nabla u_m)(h(x))|^p \, dx \\
&= C \int_\Omega |(\nabla u_n)(y) - (\nabla u_m)(y)|^p \underbrace{|\det Dh^{-1}(y)|}_{\leqslant \tilde{C}} \, dx \xrightarrow{n,m\to 0} 0
\end{aligned}
$$

Claim 2: $\|f_n - u \circ h\|_p \xrightarrow{n\to\infty} 0$.

$$
\begin{aligned}
\|f_n - u \circ h\|_p &= \int_U |u_n(h(x)) - u(h(x))|^p \, dx \\
&= \int_\Omega |u_n(y) - u(y)|^p \underbrace{|\det Dh^{-1}(y)|}_{\leqslant C} \, dy \xrightarrow{n\to\infty} 0
\end{aligned}
$$

Conclusion: Since $(f_n)_{n\in\mathbb{N}} \subseteq C_c^1(U)$ is Cauchy with respect to $\|\cdot\|_{W^{1,p}}$, there is a $f \in W_0^{1,p}(U)$ s.t. $f_n \xrightarrow[\|\cdot\|_{W^{1,p}}]{n\to\infty} f$. Since limits in L^p are unique by claim 2 we get $u \circ h = f \in W_0^{1,p}(U)$. ■

Exercise 5.24 (E 8.4) Let $\Gamma \subseteq \mathbb{R}^d$ be compact, $\{U_i\}_{i=1}^N$ open s.t. $\Gamma \subseteq \bigcup_{i=1}^N U_i$. Prove: There exists $\{\chi_i\}_{i=0}^N \subseteq C^\infty(\mathbb{R}^d)$ s.t.

1. $\chi_i \geqslant 0$ for all i, $\sum_{i=0}^N \chi_i = 1$
2. $\operatorname{supp}(\chi_i) \subseteq U_i$ for all $i \in \{1, \ldots, N\}$
3. $\operatorname{supp}(\chi_0) \subseteq \mathbb{R}^d \backslash \Gamma$

Solution. WLOG assume that $U_i \neq \varnothing$ for all i. If $\Gamma \neq 0$, then $\chi_0 = 1$ does the job. Now suppose $\Gamma \neq \varnothing$. Let $\psi \in C_c^\infty(B_1(0))$, $\psi \geqslant 0$, $\int \psi = 1$, $\psi|_{B_{\frac{1}{2}}(0)} > 0$ and for $\epsilon > 0$ let $\psi_\epsilon(x) = \frac{1}{\epsilon^d}\psi\left(\frac{x}{\epsilon}\right)$, so $\int \psi_\epsilon = 1$. Define

$$
\tilde{d} := \sup\{\tilde{\tilde{d}} > 0 \mid \forall x \in \Gamma \exists i \in \{1, \ldots, N\} s.t. \operatorname{dist}(x, U_i^c) \geqslant \tilde{\tilde{d}}\}
$$

Claim 1: $\tilde{d} > 0$ Suppose this was not true. Then there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \Gamma$ s.t. for all $i \in \{1, \ldots, N\}$,

$$
\operatorname{dist}(x_n, U_i^c) < \frac{1}{n}
$$

Since Γ is compact, there is a subsequence, which we call x_n again, s.t. $x_n \xrightarrow{n\to\infty} \bar{x}$ for asome Γ. By $\Gamma \subseteq \bigcup_{i=1}^N U_i$ there is a $\epsilon_{\bar{x}} > 0$ and $i \in \{1, \ldots, N\}$ s.t. $B_{\epsilon_{\bar{x}}}(\bar{x}) \subseteq U_i$ ↯. Define $d := \min\{\tilde{d}, 1\} > 0$. For all $\epsilon > 0$, for all $A \subseteq \mathbb{R}^d$: $(A)_\epsilon := \{x \in A \mid \operatorname{dist}(x, A^c) \geqslant \epsilon\}$. for every $i \in \{1, \ldots, N\}$ define $\phi_i : U_i \to [0, \infty)$ by

$$
\phi_i(x) := \mathbb{1}_{(U_i \cap B_R(0))_{\frac{d}{4}}} \star \phi_{\frac{d}{4}}
$$

Note $\phi_i \in C_c^\infty(U_i)$ and $(U_i \cap B_R(0))_{\frac{d}{4}} \subseteq (\operatorname{supp}(\phi_i))^0$. Define $\phi_0 : \mathbb{R}^d \backslash \Gamma \to [0, \infty)$ by $\phi_0(x) = \mathbb{1}_{(\mathbb{R}^1 \backslash \Gamma)_{\frac{d}{4}}} \star \psi_{\frac{d}{4}}$. Again, $\phi_0 \in C^\infty(\mathbb{R}^d \backslash \Gamma)$, $\operatorname{supp}(\phi_0))^0 \supseteq (\mathbb{R}^d \backslash \Gamma)_{\frac{d}{4}}$, $\operatorname{supp}(\phi_0) \subseteq \mathbb{R}^d \backslash \Gamma$. Claim 2: For all $x \in \mathbb{R}^d$ there is a $i \in \{0, 1, \ldots, N\} : \phi_i(x) > 0$. Proof: By construction, we know for $i \in \{1, \ldots, N\}$ that ϕ_i is > 0 on $(U_i \cap B_R(0))_{\frac{d}{4}}$. Moreover $\phi_0 > 0$ on $(\mathbb{R}^d \backslash \Gamma)_{\frac{d}{4}}$. thus, we are done if we can show taht $\bigcup_{i=1}^N (U_i \cap B_R(0))_{\frac{d}{4}} \cup$

$(\mathbb{R}^d \backslash \Gamma)_{\frac{d}{4}} = \mathbb{R}^d$. Suppose there is a $x \in \mathbb{R}^d \backslash A$. Then $\operatorname{dist}(x, \Gamma) < \frac{d}{4}$. Since $\Gamma \subseteq B_{\frac{R}{2}}(0)$ and $R > 2$ and $d \leqslant 1$.

$$|x - 0| \leqslant \operatorname{dist}(x, \Gamma) + \frac{R}{2} < \frac{d}{4} + \frac{R}{2} = R - \frac{d}{4} - \frac{R}{2} + \frac{d}{2} < R - \frac{d}{4} - \frac{2}{2} + \frac{1}{2} < R - \frac{d}{4}$$

Thus $x \in (B_R(c))_{\frac{d}{4}}$. Thus, we are done if we can show that $x \in (U_i)_{\frac{d}{4}}$ for some $i \in \{1, \ldots, N\}$. Since $\operatorname{dist}(x, \Gamma) < \frac{d}{4}$, there is a $y \in \Gamma$ s.t. $|x - y| < \frac{d}{4}$. By definition of $\tilde{d}$ there is a $i \in \{1, \ldots, N\}$ s.t. $\operatorname{dist}(y, U_i^c) \geqslant \tilde{d} \geqslant d$, i.e. for all $z \in U_i^c$ we have $|y - z| \geqslant d$. We get

$$|x - z| \geqslant \big|\underbrace{|x - y|}_{< \frac{d}{4}} - \underbrace{|y - z|}_{\geqslant d}\big| \geqslant \frac{3d}{4} < \frac{d}{4}$$

This implies $\operatorname{dist}(x, U_i^c) > \frac{d}{4}$, so $x \in (U_i)_{\frac{d}{4}}$ $\lightning$. Define for all $i \in \{0, \ldots, N\} : \chi_i : \mathbb{R}^d \to [0, \infty)$ by

$$\chi_i(x) = \frac{\phi_i(x)}{\sum_{j=0}^N \phi_j(x)}$$

χ_i is well-defined by Claim 2 and $\chi_i \in C^\infty(\mathbb{R}^d)$. Also note that $\sum \chi_i = 1$, $\chi_i \geqslant 0$, which implies 1. Furthermore, since $\operatorname{supp}(\phi_i) \subseteq U_i$, we have $\operatorname{supp}(\chi_i) \subseteq U_i$ for all $i \in \{1, \ldots, N\}$, which implies 2. Finally, since $\operatorname{supp}(\phi_0) \subseteq \mathbb{R}^d \backslash \Gamma$, we get $\operatorname{supp}(\chi_0) \subseteq \mathbb{R}^d \backslash \Gamma$. This implies 3. ■

5.4 Variational problem for weak solutions

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

("formally") for all $\phi \in C_c^\infty(\Omega)$, then

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi$$

if $\nabla u \in L^2$, $f \in L^2$. By a density argument:

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi$$

for all $\phi \in \overline{C_c^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$.

Theorem 5.25 (Poincare inequality) There is a $C > 0$ s.t.

$$C \int_\Omega |\nabla v|^2 \geqslant \int_\Omega |v|^2$$

for all $v \in H_0^1(\Omega)$.

Remark 5.26 $H^1(\Omega)$ with $\|v\|_{H^1(\Omega)} = \left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right)^{\frac{1}{2}}$ is a Hilbert-Space. This implies that $H_0^1(\Omega) \overset{\text{(closed)}}{\subseteq} H^1(\Omega)$ is also a Hilbert space. By the Poincare inequality (5.25) we have for all $v \in H_0^1(\Omega)$:

$$\|v\|_{H^1(\Omega)} \geqslant \|\nabla v\|_{L^2} \geqslant \frac{1}{2c}\|v\|_{L^2} + \frac{1}{2}\|\nabla v\|_{L^2} \geqslant \frac{1}{C^1}\|v\|_{H^1(\Omega)}$$

We can think of $H_0^1(\Omega)$ as a Hilbert space with $\|v\|_{H_0^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)}$.

Proof. (Of the Poincare inequality (5.25)) We need to prove:

$$\begin{aligned}
&\exists C > 0: \quad C\int_\Omega |\nabla v|^2 \geqslant \int_\Omega |v|^2 \quad \forall v \in H_0^1(\Omega)\\
\Leftrightarrow\quad &\exists C > 0: \quad C\int_\Omega |\nabla v|^2 \geqslant \int_\Omega |v|^2 \quad \forall v \in C_c^\infty(\Omega)
\end{aligned}$$

Assume by contradiction that this does not hold, i.e. there is no $C > 0$ s.t. the statement holds. Thus there is a sequence $\{v_n\} \subseteq C_c^\infty(\Omega)$ s.t.

$$\int_\Omega |v_n|^2 = 1, \quad \int_\Omega |\nabla v_n|^2 \xrightarrow{n\to\infty} 0$$

Since $v_n \in C_c^2(\Omega)$ we can extend v_n by 0 outside Ω, so $v_n \in C_c^\infty(\mathbb{R}^d)$. Then:

$$\int_{\mathbb{R}^d} |v_n|^2 = 1, \quad \int_{\mathbb{R}^d} |\nabla v_n|^2 \to 0, \quad \operatorname{supp} v_n \subseteq \Omega$$

By the Fourier transform:

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2\, dk = 1, \quad \int_{\mathbb{R}^d} |2\pi k|^2 |\hat{v}_n(k)|^2\, dk \to 0, \quad \operatorname{supp} v_n \subseteq \Omega$$

We prove that

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2\, dk \to 0$$

We write

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|\, dk = \int_{|k|\leqslant\epsilon} + \int_{|k|>\epsilon}$$

First, for all $\epsilon > 0$:

$$\int_{|k|>\epsilon} |\hat{v}_n(k)|^2 \leqslant \int_{\mathbb{R}^d} \frac{|k|^2}{\epsilon^2} |\hat{v}_n(k)|^2\, dk \xrightarrow{n\to\infty} 0$$

Second:

$$\begin{aligned}
\int_{|k|\leqslant\epsilon} |\hat{v}_n(k)|^2\, dk &\leqslant \left(\int_{|k|\leqslant\epsilon} 1\, dk\right)^{\frac{1}{q}} \left(\int_{|k|\leqslant\epsilon} |\hat{v}_n(k)|^{2p}\, dk\right)^{\frac{1}{p}}, \quad 1 < p, q < \infty\\
&\leqslant C\epsilon^{\frac{d}{q}} \|\hat{v}_n\|_{L^{2p}}^2, \quad \frac{1}{p} + \frac{1}{q} = 1 \text{ and } 1 \leqslant r \leqslant 2
\end{aligned}$$

Moreover, since Ω is bounded,

$$\|v_n\|_{L^r} \leqslant \left(\int_\Omega |v_n|^r\right)^{\frac{1}{r}} \leqslant \|1_\Omega\|_{L^s} \|v_n\|_{L^2}^{1-\theta} \leqslant C_\Omega \quad \forall 1 \leqslant r \leqslant 2.$$

Thus we can take $r < 1$ but close to 1. Then p is sufficiently large, so q is close to 1. Then

$$\int_{|k|\leqslant\epsilon} |\hat{v}_n(k)|^2 \leqslant C\epsilon^{\frac{d}{q}} \|\hat{v}_n\|_{L^{2p}}^2 \leqslant C\epsilon^{\frac{d}{q}} \|v_n\|_{L^r}^2 \leqslant C\epsilon^{\frac{d}{q}}$$

Conclusion:

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 = \int_{|k|\leqslant\epsilon} + \int_{|k|>\epsilon} \leqslant C\epsilon^{\frac{d}{q}} + \int_{|k|>\epsilon} \xrightarrow{n\to\infty} C\epsilon^{\frac{d}{q}} \xrightarrow{\epsilon\to 0} 0$$

which contradicts to the assumtion $\|\hat{v}\|_{L^2} = \|v\|_{L^2} = 1$. ∎

Exercise 5.27 Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1-boundary. Let $u \in W^{1,p}(\Omega)$, for some $1 \leqslant p < \infty$. Then the following is equivalent:

a) $u \in W_0^{1,p}(\Omega)$

b) $\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^d \backslash \Omega \end{cases} \in W^{1,p}(\mathbb{R}^d)$

Theorem 5.28 (Dirichlet, Riemann, Poincare, Hilbert) Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1-boundary. Let $f \in L^2(\Omega)$. Then there exists a unique solution $u \in H_0^1(\Omega)$ of the variational problem

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f\phi$$

for all $\phi \in H_0^1(\Omega)$. ($\Rightarrow -\Delta u = f$ in $D'(\Omega)$). Moreover, u is the unique minimizer of

$$\inf_{v \in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega fv \right)$$

Proof. Let us prove that there is a solution $u \in H_0^1(\Omega)$ for $\inf_{v \in H_0^1(\Omega)} E(v)$, $E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv$.

Step 1: We prove $E > -\infty$. Take $v \in H_0^1(\Omega)$. By the Poincare and Hölder inequalites:

$$\begin{aligned} E(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv \\ &\geqslant \frac{1}{2C}\|v\|_{L^2(\Omega)} - \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\geqslant \frac{1}{2C}\|v\|^2_{L^2(\Omega)} - \left(\frac{1}{4C}\|v\|^2_{L^2(\Omega)} + C\|f\|^2_{L^2(\Omega)} \right) \\ &\geqslant -C\|f\|^2_{L^2(\Omega)} > -\infty \end{aligned}$$

We can also bound:

$$\begin{aligned} E(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv \\ &\geqslant \frac{1}{4}\int_\Omega |\nabla v|^2 - \frac{1}{4C}\int_\Omega |v|^2 - \|f\|_{L^2}\|v\|_{L^2} \\ &\geqslant \frac{1}{4}\int_\Omega |\nabla v|^2 - C\|f\|^2_{L^2} \end{aligned}$$

Step 2: We can take a minimizing sequence $\{v_n\} \subseteq H_0^1(\Omega)$ s.t. $E(v_n) \xrightarrow{n\to\infty} E$. Then:

$$\frac{1}{4}\int_\Omega |\nabla v_n|^2 \leqslant E(v_n) + C\|f\|^2_{L^2} \longrightarrow const.$$

So $|\nabla v_n|$ is bounded in $L^2(\Omega)$. We know that $H_0^1(\Omega)$ is a Hilbert space with norm $\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$ (and the norm is equivalent to the H^1-norm). Thus $\{v_n\}$ is bounded in $H_0^1(\Omega)$.

Remark 5.29 (Reminder from functional analysis) Let H be a Hilbert space. We say that $v_n \to v$ if $\|v_n - v\| \to 0$ and $v_n \to v$ weakly in H if $\langle v_n, \phi\rangle \to \langle v, \phi\rangle$ for all $\phi \in H$.

Theorem 5.30 (Banach-Alaoglu) If H is a Hilbert space and $\{v_n\}$ is a bounded sequence, then there is a subsequence $\{v_{n_k}\}$ s.t. $v_{n_k} \rightharpoonup v$ weakly in H.

Remark 5.31 – $v_n \rightharpoonup v$ in H iff $f(v_n) \to f(v)$ for all $f \in H^\star = \mathcal{L}(H, \mathbb{R})$.

– If $v_n \rightharpoonup v$ in H, then: $\liminf_{n\to\infty} \|v_n\| \geqslant \|v\|$ (Fatous Lemma)

In fact, for all $\phi \in H$ $\langle v_n, \phi\rangle \to \langle v, \phi\rangle$ and $|\langle v_n, \phi\rangle| \leqslant \|v_n\|\|\phi\|$. This implies

$$\frac{|\langle v, \phi\rangle|}{\|\phi\|} \leqslant \liminf_{n\to\infty} \|v_n\|.$$

So we get

$$\|v\| = \sup_{\phi\neq 0} \frac{\langle v, \phi\rangle|}{\|\phi\|} \leqslant \liminf_{n\to\infty} \|v_n\|$$

By the Banach-Alaoglu theorem (5.30), up to a subsequence, $v_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. We prove that u is a minimizer for $\mathcal{E}$

$$E \longleftarrow \mathcal{E}(v_n) = \frac{1}{2}\int |\nabla v_n|^2 - \int f v_n$$

– Since $v_n \rightharpoonup u$ in $H_0^1(\Omega)$ we have that

$$\liminf_{n\to\infty} \|v_n\|^2_{H_0^1(\Omega)} \geqslant \|u\|^2_{H_0^1(\Omega)}$$

So we have

$$\liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 \geqslant \int_\Omega |\nabla u|^2.$$

– Consider the functional $\mathcal{L} : \phi \in H_0^1(\Omega) \to \int_\Omega f\phi$. We claim that $\mathcal{L}$ is continuous. In fact:

$$|\mathcal{L}| = \left|\int_\Omega f\phi\right| \leqslant \|f\|_{L^2}\|\phi\|_{L^2} \leqslant C\|f\|_{L^2}\|\nabla f\|_{L^2} = C\|f\|_{L^2}\|\phi\|_{H_0^1(\Omega)}$$

Thus from $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ we get $\mathcal{L}(v_n) \to \mathcal{L}(u)$, thus $\int_\Omega f v_n \to \int_\Omega fu$.

Conclusion: $E = \liminf \mathcal{E}(v_n) \geqslant \mathcal{E}(u)$, so u is a minimizer for $\mathcal{E}$.

Step 3: Uniqueness. If E has 2 minimizers u_1, u_2 we can prove that $u_1 = u_2$. This is because of the convexity:

$$\begin{aligned} 0 &\geqslant \frac{\mathcal{E}(u_1) + \mathcal{E}(u_2)}{2} - \mathcal{E}\left(\frac{u_1+u_2}{2}\right) \\ &= \frac{1}{8}\left[2\int_\Omega |\nabla u_1|^2 + 2\int_\Omega |\nabla u_2|^2 - \int_\Omega |\nabla(u_1+u_2)|^2\right] \\ &= \frac{1}{8}\int_\omega |\nabla u_1 - \nabla u_2|^2 \geqslant 0 \end{aligned}$$

This implies that $\nabla(u_1 - u_2) = 0$, so $u_1 - u_2 = \mathit{const} = c_0$. Since $u_1, u_2 \in H_0^1(\Omega)$, we have that $u_1 - u_2 \in H_0^1(\Omega)$ and $c_0 \in C(\bar{\Omega})$. Hence $c_0 = 0$ on $\partial\Omega$, so $c_0 = 0$. ∎

Remark 5.32 We can also prove directly that there is a unique $u \in H_0^1(\Omega)$ s.t.

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f\phi \quad \forall \phi \in H_0^1(\Omega)$$

by Riesz theorem. So we get $\langle u, \phi \rangle_{H_0^1(\Omega)} = \mathcal{L}(\phi)$.

Recall the corrector function for the unit ball:

$$\phi_x(y) = G(|x||y - \tilde{x}|), \quad \tilde{x} = \frac{x}{|x|^2}$$

This is ok if $x \neq 0$. When $x \to 0$:

$$G(|x|(y - \tilde{x})) = G\Bigg(\underbrace{|x|y - \frac{x}{|x|}}_{|\cdot|\to 1}\Bigg) G(z), \quad |z| = 1$$

is well-defined as G is radial. Question: If $u \in H^1(\Omega)$, then how can we define $u|_{\partial\Omega}$?

5.5 Theory of Trace

Theorem 5.33 (Trace Operator) Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1 boundary. Then there is a unique linear bounded operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ such that

- If $u \in H^1(\Omega) \cap C(\bar{\Omega})$, then $Tu = u|_{\partial\Omega}$ in the usual restriction sense.
- There is a $C > 0$ s.t. $\|Tu\|_{L^2(\partial\Omega)} \leqslant C\|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$

Theorem 5.34 If $u \in H^1(\Omega)$, then $u \in H_0^1(\Omega)$ is equivalent to $Tu = 0$ in $L^2(\partial\Omega)$. ($H_0^1(\Omega) = T^{-1}(\{0\})$). The we can discuss about

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega \end{cases}$$

Lemma 5.35 (Trace inequality on $\mathbb{R}_+^d$) if $u \in C_c^\infty(\mathbb{R}^d)$, then:

$$\|u|_{\partial\mathbb{R}_+^d}\|_{L^2(\partial\mathbb{R}_+^d)} \leqslant C\|u\|_{H^1(\mathbb{R}^d)} \quad \text{with } C > 0 \text{ independent of } u.$$

Proof. $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

$$\begin{aligned} |u(x',0)|^2 &= -\int_0^\infty \partial_d(|u(x',x_d)|^2)\, dx_d \\ &= -\int_0^\infty 2\partial_d u(x',x_d)u(x',x_d)\, dx_d \\ &\leqslant \int_0^\infty \left[|\partial_d u(x',x_d)|^2 + |u(x',x_d)|^2\right] dx_d \end{aligned}$$

This implies:

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} |u(x',0)|^2\, dx' &\leqslant \int_{\mathbb{R}^{d-1}} \left(\int_0^\infty [\dots]\, dx_d\right) dx' \\ &= \int_{\mathbb{R}_+^d} \left[|\partial_d u|^2 + |u|^2\right] = \|u\|^2_{H^1(\mathbb{R}_+^d)} \end{aligned}$$

■

Corrolary 5.36 If $u \in H^1(Q)$ and u is compactly supported, then:

$$\|u\|_{L^2(Q_0)} \leqslant \|u\|_{H^1(Q_+)}$$

Here

$$\begin{aligned}
Q &= \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, |x_d| < 1\} \\
Q_+ &= \{x = (x', x_d) \in Q \mid x_d > 0\} \\
Q_0 &= \{x = (x', x_d) \in Q \mid x_d = 0\}.
\end{aligned}$$

Proof. We extend u by 0 outside of Q, so $u \in H^1(\mathbb{R}^d)$. ∎

Theorem 5.37 (Extension) If $\Omega \subseteq \mathbb{R}^d$ is open, bounded with C^1-boundary, then there is a bounded linear operator $B : H^1(\Omega) \to H^1(\mathbb{R}^d)$ s.t.

- $Bu|_\Omega = u$ for all $u \in H^1(\Omega)$
- $\|Bu\|_{H^1(\mathbb{R}^d)} \leqslant C\|u\|_{H^1(\Omega)}$ and $\|Bu\|_{L^2(\mathbb{R}^d)} \leqslant C\|u\|_{L^2(\Omega)}$.

Proof of Theorem 5.33. Since $\partial\Omega$ is C^1 there are open sets $\{U_i\}_{i=1}^N \subseteq \mathbb{R}^d$ such that $\partial\Omega \subseteq \bigcup_{i=1}^N U_i$ and for all i there is a C^1-diffeomorphism $h_i : U_i \to Q$ s.t. $h_i(U_i) = Q$, $h_i(U_i \cap \Omega) = Q_+$, $h_i(U_i \cap \partial\Omega) = Q_0$. Then there exists a partition of unity $\{\theta_i\}_{i=0}^N \subseteq C^\infty(\mathbb{R}^d)$ s.t.

1. $\sum_{i=0}^N \theta_i = 1$ for all $x \in \mathbb{R}^d$
2. For all $i = 1, \dots, N$: $\theta_i \in C_c^\infty(U_i)$
3. $\operatorname{supp} \theta_0 \subseteq \mathbb{R}^d \backslash \partial\Omega$ (in particular $\theta_0|_\Omega \in C_c^\infty(\Omega)$)

Then given $u \in H^1(\Omega)$, we can write $u = \sum_{i=0}^N u_i$, where $u_i = \theta_i u$. By the extension theorem (5.37), $u \longrightarrow$ extended to $Bu \in H^1(\mathbb{R}^d)$, thus

$$Bu = \sum_{i=0}^N \theta_i(Bu) = \sum_{i=0}^N v_i, \quad v_i = \theta_i(Bu)$$

Then $v_i \in H^1(\mathbb{R}^d)$ and v_i is compactly supported in U_i for all $i = 1, 2, \dots, N$ and $\operatorname{supp} v_0 \subseteq \mathbb{R}^d \backslash \partial\Omega$, $v_i \in H^1(\mathbb{R}^d)$ and compactly supported inside U_i. This implies $\tilde{v}_i(y) = v_i(h_i^{-1}(y)) \in H^1(Q)$ and compactly supported inside Q, $y \in Q$. Thus $\|\tilde{v}_i\|_{L^2(Q_0)} \leqslant C\|\tilde{v}_i\|_{H^1(Q_+)}$. So we have $\|v_i\|_{L^2(\partial\Omega)} \leqslant C\|\tilde{v}_i\|_{L^2(Q_0)} \leqslant C'\|\tilde{v}\|_{H^1(Q_+)} \leqslant C''\|v_i\|_{H^1(U_i \cap \Omega)}$. Thus:

$$\begin{aligned}
\|u\|_{L^2(\partial\Omega)} = \left\| \sum_{i=1}^N v_i \right\|_{L^2(\partial\Omega)} &\leqslant \sum_{i=1}^N \|v_i\|_{L^2(\partial\Omega)} \leqslant \sum_{i=1}^N C''\|v_i\|_{H^1(U_i \cap \Omega)} \\
&= C'' \sum_{i=1}^N \|\theta_i u\|_{H^1(\Omega)} \leqslant C'' \sum_{i=1}^N C\|u\|_{H^1(\Omega)}
\end{aligned}$$

This proof works for $u \in C(\bar{\Omega})$. This implies

$$\|u\|_{L^2(\partial\Omega)} \leqslant C\|u\|_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega) \cap C(\bar{\Omega}).$$

This allows us to define

$$\begin{aligned}
T : H^1(\Omega) &\longrightarrow L^2(\partial\Omega) \\
u &\longmapsto u|_{\partial\Omega}
\end{aligned}$$

by continuity. I.e. for all $u \in H^1(\Omega)$ there is $\{u_n\} \subseteq H^1(\Omega) \cap C(\bar{\Omega})$ s.t. $u_n \to u$ in H_0^1. Then $Tu_n \to Tu$ in $L^2(\partial\Omega)$. ∎

Lemma 5.38 (Extension for Q) Let $u \in H^1(Q_+)$. Then we define $Bu : Q \to \mathbb{R}$ by

$$Bu(x) = \begin{cases} u(x) & x \in Q_+ \\ -u(x', -x_d) & x \in Q_- \end{cases},$$

$x = (x, x_d)$. Then $Bu \in H^1(Q)$ and $Bu|_{Q^+} = u$, $\|Bu\|^2_{L^2(Q)} = 2\|u\|^2_{L^2(Q_+)}$, $\|\nabla(Bu)\|^2_{L^2(Q)} = \|\nabla u\|^2_{L^2(Q_+)}$

Proof. It is obvious $Bu|_{Q^+} = u$ and

$$\begin{aligned}\int_Q |Bu|^2 &= \int_{Q_+} |Bu|^2 \int_{Q_-} |Bu|^2 \\ &= \int_Q |u|^2 + \int_{Q_- = \{(x, -x_d) | (x, x_d) \in Q_+\}} |u(x, -x_d)|^2 \\ &= 2\int_{Q_+} |u|^2\end{aligned}$$

We prove:

$$\nabla(Bu)(x) = \begin{cases} \nabla u(x) & u \in Q_+ \\ \nabla u(x', -x_d) & u \in Q_- \end{cases}$$

First, $\partial_d Bu(x) = \partial_d u(x', -x_d)$ if $x \in Q_-$. Take $\phi \in C_c^\infty(Q)$, then:

$$\begin{aligned}\int_Q (Bu(x))(\partial_d\phi)(x)\,dx &= \int_{Q_+} u\partial_d\phi + \int_{Q_-} -u(x', -x_d)\partial_d[\phi(x', x_d)]\,dx \\ (x \to -x_d) \quad &= \int_{Q_+} u\partial_d\phi + \int_{Q_+} [u(x', x_d)(\partial_d\phi)(x', -x_d)]\,dx \\ &\overset{(\phi \notin C_c^\infty(Q_+))}{\approx} -\int_{Q^+} (\partial_d u)\phi(x) + \int_{Q_+} (\partial_d u(x', x_d))\phi(x', -x_d)\,dx \\ &= -\int_{Q_+} (\partial_d u)\phi(x) + \int_{Q_-} \partial_d u(x', -x_d)\phi(x', x_d)\,dx \\ &= -\int_Q f\phi, \quad \text{where } f(x) = \begin{cases} \partial_d u & x \in Q_+ \\ -\partial_d u(x', -x_d) & x \in Q_- \end{cases}\end{aligned}$$

We prove $\int_{Q_+} u\partial_d\tilde{\phi} = -\int_{Q_+} (\partial_d u)\tilde{\phi}$ where $\tilde{\phi}(x, x_d) = \phi(x, x_d) - \phi(x, -x_d)$, $\tilde{\phi} \notin C_c^\infty(Q_+)$. Define $\eta_\epsilon = 0$ when $|x_d| \leqslant \epsilon$, $\eta_\epsilon = 1$ if $|x_d| \geqslant 2\epsilon$, $\eta_\epsilon \in C^\infty$, $\eta_\epsilon(x', x_d) = \eta_0(x', \frac{x_d}{\epsilon}), \eta_0 = \begin{cases} 1 & |x_d| \geqslant 2 \\ 0 & |x_d| \geqslant 1 \end{cases}$. We have

$$\int_{Q_+} u\partial_d(\eta_\epsilon\tilde{\phi}) = -\int_{Q_+} \partial_d u(\eta_\epsilon\tilde{\phi})$$

We take $\epsilon \to 0$,

$$\int_{Q_+} (\partial_d u)(\eta_\epsilon\tilde{\phi}) \to \int_{Q_+} (\partial_d u)\tilde{\phi}$$

by dominated convergence.

$$\int_{Q_+} u\partial_d(\eta_\epsilon\tilde{\phi}) = \int_{Q_+} u(\partial_d\eta_\epsilon)\tilde{\phi} + \int_{Q_+} u\eta_\epsilon\partial_d\tilde{\phi}$$

$$\int_{Q_+} u\eta_\epsilon\partial_d\tilde{\phi} \xrightarrow{\epsilon\to 0} u\partial_d\tilde{\phi}$$

by dominated convergence.

$$\left|\int_{Q_+} u(\partial_d \eta_\epsilon)\tilde{\phi}\right| = \left|\int_Q u\frac{1}{\epsilon}(\partial_d\eta_0)\left(x, \frac{x_d}{\epsilon}\right)\tilde{\phi}\right|$$

$$\begin{pmatrix} |\tilde{\phi}(x', x_d)| \\ = |\phi(x,x_d) - \phi(x,x_d)| \\ \leqslant \|\partial_d\phi\|_{L^\infty}|x_d| \end{pmatrix} \quad \leqslant \frac{1}{\epsilon}\|\partial_d\eta_0\|_{L^\infty}\int_{Q_+\cap\{x_d\leqslant 2\epsilon\}} |u| \underbrace{|\tilde{\phi}|}_{\leqslant C|x_d|\leqslant C\epsilon}$$

$$(\text{Dominated cv } u \in L^1(Q_+)) \quad \leqslant C\int_{Q_+\cap\{0<x_d\leqslant 2\epsilon\}} |u| \xrightarrow{\epsilon\to 0} 0$$

where $u \in L^2(Q_+)$ because $u \in H^1(Q_+)$. ■

Exercise 5.39 (E 9.1) Let Ω be open, bounded with C^1-boundary. Let $u \in H_0^1(\Omega)$, $f \in L^2(\Omega)$. Show that the following statements are equivalent:

1) $-\Delta u = f$ in $D'(\Omega)$
2) $\int_\Omega \nabla u \nabla \phi \, dx = \int_\Omega f\phi \, dx$ for all $\phi \in H_0^1$
3) $E = \inf_v \left(\frac{1}{2}\int_\Omega |\nabla v|^2 \, dx - \int_\Omega fv \, dx\right)$

Solution.

1) ⇒ 2) From $-\Delta u = f$ in $D'(\Omega)$ we get that

$$\int_\Omega u(-\Delta\phi)\, dx = \int_\Omega f\phi\, dx$$

for all $\phi \in C_c^\infty(\Omega)$.

Claim: For $u \in H_0^1, \phi \in C_c^\infty$ we can apply integration by parts, namely

$$\int_\Omega u(-\Delta\phi)\, dx = \int_\Omega \nabla u \nabla\phi\, dx.$$

Density argument: $u \in H_0^1 = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1}}$, so there is a sequence $\{u_n\} \subseteq C_c^\infty(\Omega)$ s.t. $u_n \to u$ in $H^1(\Omega)$. Since $u_n, \phi \in C_c^\infty(\Omega)$, then by the integration by parts:

$$\int_\Omega u_n(-\Delta\phi)\, dx = \int_\Omega (\nabla u_n)\nabla\phi\, dx \quad \forall n$$

Take $n \to \infty$, then,

$$\int_\Omega u(-\Delta\phi)\, dx = \int_\Omega (\nabla u)\nabla\phi\, dx$$

as $u_n \to u$ and $\nabla u_n \to \nabla u$ in L^2.

Claim: If $\int_\Omega \nabla u\nabla\phi\, dx = \int_\Omega f\phi\, dx$ for all $\phi \in C_c^\infty(\Omega)$, then $\int_\Omega \nabla u\nabla\phi\, dx = \int_\Omega f\phi\, dx$ for all $\phi \in H_0^1$. (Given $\nabla u, f \in L^2$). With density argument: For all $\phi \in H_0^1$ there is a sequence $\{\phi_n\} \subseteq C_c^\infty(\Omega)$ s.t. $\phi_n \to \phi$ in H^1. Then:

$$\int_\Omega \nabla u\nabla\phi_n\, dx = \int_\Omega f\phi_n\, dx$$

for all n. Take $n \to \infty$:

$$\int \nabla u \nabla \phi \, dx = \int f \phi \, dx$$

as $\phi_n \to \phi$, $\nabla \phi_n \to \nabla \phi$ in L^2.

2) $\Rightarrow$ 3) We show $E(u) \leqslant E(v)$ for all $v \in H_0^1$, i.e.

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f u \, dx \leqslant \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega f v \, dx$$

for all $v \in H_0^1$. Write $v = u + w$, then:

$$\begin{aligned}
E(v) &= \frac{1}{2}\int |\nabla v|^2 \, dx - \int f v \, dx \\
&= \frac{1}{2}\int_\Omega |\nabla (u+w)|^2 \, dx - \int_\Omega f(u+w) \, dx \\
&= \frac{1}{2}\int_\Omega \left[|\nabla u|^2 + |\nabla w|^2 + 2\nabla u \nabla w \right] \, dx - \int_\Omega f u + f w \, dx \\
&= E(u) + \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx + \underbrace{\left(\int_\Omega \nabla u \nabla w \, dx - \int_\Omega f w \, dx \right)}_{=0}
\end{aligned}$$

as $w = v - u \in H_0^1$ (by (2))

3) $\Rightarrow$ 1)

$$E(u) \leqslant E(u + t\phi)$$

for all $\phi \in H_0^1$ (or C_c^∞) for all $t \in \mathbb{R}$. This implies:

$$\frac{d}{dt} E(u + t\phi)|_{t=0} = 0$$

Here

$$\begin{aligned}
E(u + t\phi) &= \frac{1}{2}\int_\Omega |\nabla (u + t\phi)|^2 \, dx - \int_\Omega f(u + t\phi) \, dx \\
&= \frac{1}{2}\int_\Omega |\nabla u|^2 + t^2 |\nabla \phi|^2 + 2t \nabla u \nabla \phi \, dx - \int_\Omega f(u + t\phi) \, dx \\
&= E(u) + t\left[\int_\Omega \nabla u \nabla \phi \, dx - \int_\Omega f \phi \, dx \right] + t^2 \int_\Omega |\nabla \phi|^2 \, dx
\end{aligned}$$

This implies

$$\frac{d}{dt} E(u + t\phi)|_{t=0} = \int_\Omega \nabla u \nabla \phi \, dx - \int_\Omega f \phi \, dx$$

and we can conclude

$$\int_\Omega \nabla u \nabla \phi \, dx = \int_\Omega f \phi \, dx$$

for all $\phi \in H_0^1$ or C_c^∞. So we get

$$\int_\Omega u(-\Delta \phi) \, dx = \int_\Omega f \phi \, dx$$

for all $\phi \in C_c^\infty$ and hence

$$-\Delta u = f$$

in $D'(\Omega)$. This implies 1). ■

Exercise 5.40 (E 9.2)

$$Q = \{(x', x_d) \mid |x'| < 1, |x_d| < 1\}$$

Given $u \in H^1(Q_+)$, define $Bu : Q \to \mathbb{R}$ as

$$Bu(x) = \begin{cases} u(x) & x \in Q_+ \\ u(\tilde{x}) & x \in Q_- \end{cases},$$

$x = (x', x_d) \Leftrightarrow \tilde{x} = (x', -x_d)$, $x \in Q_- \Leftrightarrow \tilde{x} \in Q_+$. In the lectures:

$$\partial_d(Bu)(x) = \begin{cases} \partial_d u(x) & x \in Q_+ \\ -(\partial_d u)(\tilde{x}) & x \in Q_- \end{cases}$$

This implies $\partial_d(Bu) \in L^2(Q)$.

1. For all $i = 1, \ldots, d-1$, then:

$$\partial_i(Bu)(x) = \begin{cases} \partial_i u(x) & x \in Q_+ \\ \partial_i u(\tilde{x}) & x \in Q_- \end{cases}$$

2. Example $u \in H^2(Q_+)$ but $Bu \notin H^2(Q)$.

Solution. 1. For all $\phi \in C_c^\infty(Q)$:

$$\int_Q Bu(x)\partial_i\phi(x) = \int_{Q_+} u(x)\partial_i\phi(x) + \int_{Q_-} u(\tilde{x})\partial_i\phi(x)$$

Write $\vec{n} = (n_1, \ldots, n_d)$. Here:

$$\int_{Q_+} u(x)\partial_i\phi(x)\,dx = \int_{Q_+} -\partial_i u(x)\phi(x)\,dx + \int_{\partial Q_+} u(x)\phi(x)n_i\,dS$$

$$\begin{aligned}\int_{Q_-} u(x', -x_d)\partial_i\phi(x', x_d)\,dx'dx_d &= -\int_{Q_+} u(x', x_d)\partial\phi(x', -x_d)\,dx'\,dx_d \\ &= \int_{Q_+} \partial_i u(x)\phi(\tilde{x}) - \int_{\partial Q_+} u\phi n_i\,dS \\ &= \int_{Q_-} -\partial_i u(\tilde{x})\phi(x) - \int_{\partial Q_+} u\phi n_i\,dS\end{aligned}$$

with $d(-x_d) = d(x_d)$. Conclude:

$$\begin{aligned}\int_Q (Bu)(x)\partial_i\phi(x)\,dx &= \int_{Q_+} (-\partial_i u)(x)\phi(x) + \int_{Q_-} (-\partial_i u)(\tilde{x})\phi(x) \\ &= \int_Q -h(x)\phi(x)\,dx, \quad h(x) = \begin{cases} \partial_i u(x), & x \in Q_+ \\ \partial_i u(\tilde{x}), & x \in Q_- \end{cases}\end{aligned}$$

for all $\phi \in C_c^\infty(Q)$, so $\partial_i(Bu) \in L^2$ for all $i = 1, 2, \ldots, d-1$. Thus $Bu \in H^1(Q)$.

2. 1D: Take $Q_+(0,1), Q_- = (-1,0), Q_0 = \{0\}, Q = (-1,1)$, $u(x) = x$ in $Q_+ = (0,1)$, $Bu(x) = u(x) = -x$ if $x \in Q_- = (-1,0)$, i.e. $Bu(x) = |x|$ if $x \in Q = (-1,1)$. We know

$$(Bu)'(x) = \begin{cases} 1 & x \in (0,1) \\ -1 & x \in (-1,0) \end{cases} \in L^2(-1,1)$$

i.e. $Bu \in H^1(Q)$.

$$(Bu)''(x) = 2\delta_0(x)$$

in $D'(Q)$ but $\notin L^2(-1,1)$, i.e. $Bu \notin H^2(Q)$. Question: Given $u \in H^2(Q_+)$, can we find an extension $Bu \in H^2(Q)$ Yes! E.g. $u(x) = x$ in $(0,1)$, so $Bu(x) = x$ in $(-1,1)$. In general: $u \in H^2(Q) \rightsquigarrow \tilde{u} \in H^2(Q)$ but $\nabla u = 0$ on ∂Q_+. ■

Exercise 5.41 (Bonus 8) Assume $u \in H^2(Q_+)$ and $\begin{cases} u = 0 \\ \nabla u = 0 \end{cases}$ on ∂Q_+ Prove that $Bu \in H^2(Q)$. (Reflection extension) (Ok in 1D)

Remark 5.42 If $u \in H^2(Q_+)$, then $\nabla u \in H^1(Q_+)$, so $\nabla u|_{\partial Q_+}$ by trace theory. In general: $\Omega \subseteq \mathbb{R}^d$, C^2-boundary condition, then the same result holds.

Remark 5.43 In 1D: $\begin{cases} u \in H^2(0,1) \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$, $u|_{Q_0} \in L^2(Q_0)$, 1D: $Q_0 = \{0\}$. In general: If $u \in H^1(0,1)$, then $u(0)$ is determined by trace theory. If $u \in H^2(0,1)$, $u'(0)$ is determined. Sobolev:

$$H^1(0,1) \subseteq C([0,1])$$
$$H^2(0,1) \subseteq C^1([0,1])$$

Lemma 5.44 (Poincare inequality) Let Ω be open, bounded connected with C^1-boundary. Then for all $g \in L^2(\partial\Omega)$ s.t. $g \neq \textit{constant}$ there is a $C > 0$ s.t.

$$\|u\|_{L^2(\Omega)} \leqslant C\|\nabla u\|_{L^2(\Omega)}$$

for all $u \in M$, where

$$M = \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = g\}.$$

Proof. We assume that the statement does not hold true. Then there is a sequence $\{u_n\} \subseteq H^1(\Omega)$, $u_n|_{\partial\Omega} = g$ s.t.

$$\|\nabla u_n\|_{L^2(\Omega)} \to 0, \quad \|u_n\|_{L^2(\Omega)} = 1.$$

Since $\{u_n\}$ is bounded in $H^1(\Omega)$, by the Banach-Alaoglu theorem (5.30), up to a subsequence

$$u_n \to u_0 \quad \text{weakly in } H^1(\Omega)$$

Since $\nabla u_n \to 0$ strongly in L^2 and $\nabla u_n \to \nabla u_0$ weakly in L^2, we have $\nabla u_0 = 0$, so $u_0|_{\partial\Omega} = \textit{const}$. (here we need Ω to be connected), so $u_0|_{\partial\Omega} = \textit{const}$. On the other hand, note that M is convex and closed in $H^1(\Omega)$ since the trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ is continuous. Therefore, M is also weakly closed in $H^1(\Omega)$ by the Hahn-Banach theorem. Thus from $\{u_n\} \subseteq M$, $u_n \to u_0$ wekly in $H^1(\Omega)$ we get that $u_0 \in M$, so $u_0|_{\partial\Omega} = g$. We get a contradiction since $g \neq \textit{const}$ ■

Theorem 5.45 (Solution for Poisson Equation with inhomogeneous boundary condition) Let Ω be open, bounded with C^1-boundary. Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. Then there is a unique $u \in H^1(\Omega)$ s.t.

$$\begin{cases} -\Delta u = f & \text{in } D'(\Omega) \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega \end{cases}$$

Here $u|_{\partial\Omega} = T(u) \in L^2(\partial\Omega)$ is defined by the trace operator. Moreover if Ω is connected and $g \neq \textit{constant}$, then u is the unique minimizer for the variational problem

$$E = \inf_{v \in M} \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega fv,$$

where $M = \{v \in H^1(\Omega), v|_{\partial\Omega} = g \text{ on } \partial\Omega\}$

Proof. First let us assume that Ω is connected and $g \neq \textit{const}$.

Step 1: We prove that $E = \int_{v \in M} E(v)$ has a minimizer. By Poincares Inequality (5.44), for all $v \in M$:

$$\begin{aligned} E(v) &= \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega fv \\ \text{(Hölder)} \quad &\geqslant \frac{1}{2} \|\nabla v\|^2_{L^2(\Omega)} - \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ \text{(Poincaré 5.44)} \quad &\geqslant \frac{1}{2} \|\nabla v\|^2_{L^2(\Omega)} - C\|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\geqslant \frac{1}{4} \|\nabla v\|_{L^2(\Omega)} - C\|f\|_{L^2(\Omega)} \end{aligned}$$

Thus $E = \inf_{v \in M} E(v) > -\infty$. Moreover, taking a minimizing sequence $\{v_n\} \subseteq M$, $E(v_n) \to E$, we find that $\|\nabla v_n\|_{L^2(\Omega)}$ is bounded, and hence $\|v_n\|_{H^1(\Omega)}$ is bouned (by Poincare inequality) again. By Banach-Alaoglu (5.30), up to a subsequence we have $v_n \to u$ weakly in $H^1(\Omega)$. Hence

$$\begin{cases} \limsup_{n\to\infty} \int_\Omega |\nabla v_n|^2 \geqslant \int_\Omega |\nabla u|^2 & \text{as } \nabla v_n \to \nabla u \text{ in } L^2 \\ \int_\Omega v_n f \to \int_\Omega uf & \text{as } v_n \to u \text{ in } L^2 \end{cases}$$

Note that $\{v_n\} \subseteq M$, $v_n \to u$ in $H^1(\Omega)$ and M is weakly closed in $H^1(\Omega)$ (as argued in the proof of Poincare inequality), therefore $u \in M$. This means that u is a minimizer for $E = \inf_{v \in M} E(v)$.

Step 2: Now we prove that if u is a minimizer for E, then $-\Delta u = f$ in $D'(\Omega)$. In fact, for all $\phi \in C_c^\infty(\Omega)$ we have

$$E(u) \leqslant E(u + t\phi) \quad \forall t \in \mathbb{R}$$

because $u + t\phi \in M$. So we get that

$$0 = \frac{d}{dt} E(u + t\phi)|_{t=0} = \int_\Omega \nabla u \nabla \phi - \int_\Omega f\phi$$

Thus

$$\int_\Omega u(-\Delta\phi) = \int_\Omega \nabla u \nabla \phi, = \int_\Omega f\phi \quad \forall \phi \in C_c^\infty(\Omega).$$

So $-\Delta u = f$ in $D'(\Omega)$.

Step 3: We prove that Poissons equation has at most one solution. Assume that u_1, u_2 are 2 solutions. Then $u = u_1 - u_2$ solves

$$\begin{cases} -\Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \text{ on } \Omega \end{cases}$$

so $u = 0$.

Step 4: If $g = c_0$ is a constant, then Poissons equation can be rewritten with $\tilde{u} = u - c_0$:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = c_0 & \text{on } \Omega \end{cases} \quad \Leftrightarrow \quad \begin{cases} -\Delta \tilde{u} = f & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \Omega \end{cases}$$

If Ω is not connected, then by considering connected components of Ω we can prove that Poisson's equation always has a unique solution (for all $f \in L^2(\Omega), g \in L^2(\partial\Omega)$).

■

5.6 Final Remarks

We can describe $H_0^1(\Omega)$ as the kernel of the trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$

Theorem 5.46 Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1-boundary. Then:

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid T(u) = 0 \text{ on } \partial\Omega\}$$

Recall that if $u \in H^1(\Omega) \cap C(\bar{\Omega})$, then $T(u) = u|_{\partial\Omega}$ is the usual restriction. In this case we recover a result proved before.

Proof. ■

Recall that the varionational characterization of the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f\phi \quad \forall \phi \in M$$

where $M = \{v \in H^1(\Omega) \mid v = g \text{ on } \partial\Omega\}$ In fact, if $u \in H^2(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f\phi \quad \forall \phi \in H^1(\Omega)$$

Then u satisfies the Neumann condition:

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n} = 0 \text{ on } \partial\Omega$$

(justification ...)

For the exercises of sheet 10: Let $\Omega = (a, b) \subseteq \mathbb{R}$ be an open bounded interval. For every $u \in H^1(\Omega)$ the values $u(a)$ and $u(b)$ are determined uniquely by trace theory, or by Sobolev's embedding theorem. Recall: If $u \in H^1((a, b)) \rightsquigarrow \partial\Omega = \{a, b\}$ counting measure iff $g \in L^2(\partial\Omega)$ i.e. $g(a) = g(b)$ are *well-defined.*

Exercise 5.47 (E 10.1) a) Prove $H^1(\mathbb{R}) \subseteq (C(\mathbb{R}) \cap L^\infty(\mathbb{R}))$
Hint: You can use Fourier Transform

b) $H^1(\Omega) \subseteq C(\Omega)$

Solution. a) Let $u \in H^1(\mathbb{R})$. Then $u, u' \in L^2(\mathbb{R}) \Leftrightarrow \hat{u}(k)(1 + |2\pi k|) \in L^2(\mathbb{R})$. Thus:

$$u(x) = \int_{\mathbb{R}} \hat{u}(k) e^{2\pi i k x} \, dk \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

if $\hat{u} \in L^1(\mathbb{R})$. So we have to show $\hat{u} \in L^1(\mathbb{R})$.

$$\begin{aligned} \int_{\mathbb{R}} |\hat{u}(k)| &= \int_{\mathbb{R}} \frac{|g(k)|}{1 + |2\pi k|} \\ &\leqslant \left(\int_{\mathbb{R}} |g(k)|^2 \, dk \right) \left[\int_{\mathbb{R}} \left(\frac{1}{1 + |2\pi k|} \right)^2 dk \right]^{\frac{1}{2}} < \infty \end{aligned}$$

b) Given $u \in H^1(\Omega)$, then there is an extension $\tilde{u} \in H^1(\mathbb{R})$. By a) $\tilde{u} \in C(\mathbb{R})$, so $u = \tilde{u}|_{\bar{\Omega}} \in C(\bar{\Omega})$. Remak: We have $\|u\|_{L^\infty(\Omega)} \leqslant C\|u\|_{H^1(\Omega)}$, where $\Omega = (a, b)$ or $\mathbb{R}$ (but only in 1D) ∎

Recall: If $\Omega \subseteq \mathbb{R}^d (d \geqslant 1)$ open, bounded with C^1-boundary. Then

$$\|u\|_{L^2(\Omega)} \leqslant C\|\nabla u\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega)$$

Actually the same bound holds if $u \in H^1(\Omega)$ and $u|_\Gamma = 0$ for an open subset $\Gamma \subseteq \partial\Omega$. In 1D we have:

Exercise 5.48 (E 10.2 (Poincare inequality)) Let $u \in H^1(\Omega)$, $u(a) = 0$. Prove that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C\|u'\|_{L^2(\Omega)}$$

Solution. Let $u \in C^1(\bar{\Omega})$ and $u(a) = 0$. Then:

$$\begin{aligned} u(x) &= u(a) + \int_0^x u'(t) \, dt \quad \forall x \in (a, b) \\ \Rightarrow \quad |u(x)| &\leqslant \int_a^x |u'(t)| \, dt \leqslant \int_a^b |u'(t)| \, dt = \|u'\|_{L^1(\Omega)} \leqslant C\|u'\|_{L^2(\Omega)} \end{aligned}$$

as Ω is bounded. This implies:

$$\frac{1}{C}\|u\|_{L^2(\Omega)} \leqslant \|u\|_{L^\infty(\Omega)} \leqslant C\|u'\|_{L^2(\Omega)}$$

To extend this for $u \in H^1(\Omega)$, we can use a density argument. More precisely, for all $u \in H^1(\Omega)$ there is a sequence $\{u_n\} \subseteq C^1(\bar{\Omega})$ s.t $u_n \to u$ in $H^1(\Omega)$. Then:

$$\|u\|_{L^2(\Omega)} = \lim_{n\to\infty} \|u_n\|_{L^2(\Omega)} \leqslant C \lim_{n\to\infty} \|u_n'\|_{L^2(\Omega)} = C\|u'\|_{L^2(\Omega)}$$

Recall: For all $f \in W_{loc}^{1,1}(O)$ with O in $\mathbb{R}^d$ we have

$$f(x) - f(y) = \int_0^1 \nabla f(y + t(x - y))(x - y) \, dt$$

if $x, y \in O$, $y + t(x - y) \in O$ for all $t \in [0, 1]$. For 1D: If $u \in H^1(a, b)$:

$$u(x) - u(y) = \int_y^x u'(t) \, dt \quad \forall x, y \in (a, b)$$

∎

Exercise 5.49 (E 10.3 (Poincare inequality)) Let $u \in H^2(\Omega)$ and $f \in L^2(\Omega)$. Prove that the following statements are equivalent:

a) u solves the equation:

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u'(0) = u'(1) = 0 \end{cases}$$

b)

$$\int_\Omega u'\phi' = \int_\Omega f\phi$$

for all $\phi \in H^1(\Omega)$.

Here $u \in H^2(\Omega) \Rightarrow u' \in H^1(\Omega) \Rightarrow u'(0), u'(1)$ determined uniquely by trace theorem / Sobolev inequality $H^1(\Omega) \subseteq C(\bar{\Omega})$

Solution.

b) $\Rightarrow$ a) For all $\phi \in C_c^\infty(\Omega)$:

$$\int_\Omega f\phi = \int_\Omega u'\phi' = -\int_\Omega u\phi''$$

This implies $-u'' = f$ in $D'(\Omega)$ a.e. Thus for all $\phi \in H^1(\Omega)$:

$$\int_\Omega f\phi = \int_\Omega -u''\phi = \int_\Omega u'\phi' - [u'\phi]_a^b$$

By b) we conclude $0 = [u'\phi]_a^b = u'(b)\phi(b) - u'(a)\phi(a)$ for all $\phi \in H^1(\Omega)$. We can choose $\phi \in H^1(\Omega)$ s.t. $\phi(a) = 0$, $\phi(b) = 1$. This implies $\phi'(b) = 0$. Similarly, we can chose $\phi \in H^1(\Omega)$ s.t. $\phi(a) = 1$, $\phi(b) = 0$. This implies $u'(a) = 0$.

a) $\Rightarrow$ b) From a) and Integration by parts:

$$\int_\Omega f\phi = \int_\Omega -u''\phi = \int_\Omega u'\phi' - \underbrace{[u'\phi]_a^b}_{=0 \text{ as } u'(a)=u'(b)=0}$$

This implies:

$$\int_\Omega f\phi = \int_\Omega u'\phi' \quad \forall \phi \in H^1(\Omega)$$

■

Exercise 5.50 (E 10.4 (Robin boundary condition)) Let $f \in L^2(\Omega)$.

a) Prove that there exists a unique $u \in M := \{\phi \in H^1(\Omega), u(a) = 0\}$ s.t.

$$\int_\Omega u'\phi' = \int_\Omega f\phi \quad \forall \phi \in M$$

b) Prove that the above function u is the unique solution to the equation

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = 0 & u'(b) = 0 \end{cases}$$

Solution. a) By 10.2 we have

$$\|\phi\|_{L^2(\Omega)} \leqslant C\|\phi'\|_{L^2(\Omega)} \quad \forall \phi \in M$$

Thus: $(M, \|\phi\|_M := \|\phi'\|_{L^2(\Omega)})$ is a Hilbert space. More precisely, we know $(M, \|\cdot\|_M)$ is a closed subspace of $H^1 \rightsquigarrow$ a Hilbert space. And $\|\cdot\|_M$ is comparable to $\|\cdot\|_{H^1}$. By Riesz representation theorem there is a unique $u \in M$ s.t. $\langle \phi, u \rangle_M = F(\phi)$ for all $\phi \in M$. We use this for

$$F(\phi) = \int_\Omega f\phi \quad \forall \phi \in M$$

Here $|F(\phi)| \leqslant \|f\|_{L^2}\|\phi\|_{L^2}$.

b) Let $u \in M$ be the solution in (a) i.e.

$$\int_\Omega f\phi = \int_\Omega u'\phi' \quad \forall \phi \in M$$

Then we prove that u solves

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = u'(b) = 0 \end{cases}$$

Since $u \in M$ we have $u \in H^1(\Omega)$ and $u(a) = 0$. From

$$\int_\Omega f\phi = \int_\Omega u'\phi' \quad \forall \phi \in M$$

we get for all $\phi \in C_c^\infty(\Omega)$:

$$\int_\Omega f\phi = \int_\Omega u'\phi' = \int_\Omega -u\phi''$$

So we get $-u'' = f$ in $D'(\Omega)$. Since $f \in L^2(\Omega) \Rightarrow u'' \in L^2(\Omega) \Rightarrow u \in H^2(\Omega)$ $\Rightarrow u' \in H^1(\Omega) \Rightarrow u'(b)$ is uniquely determined. For all $\phi \in M$:

$$\int_\Omega f\phi = \int_\Omega -u''\phi = \int_\Omega u'\phi' - \big(u'(b)\phi(b) - u'(a)\phi(a)\big) \quad \text{as } \phi \in M$$

and $\int_\Omega f\phi = \int_\Omega u'\phi'$. This implies:

$$u'(b)\phi(b) = 0 \quad \forall \phi \in M$$

Take $\phi(x) = \frac{x-a}{b-a} \in M$, $\phi(b) = 1$. Uniqueness of the solution: Take u s.t.

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = u'(b) = 0 \end{cases}$$

This implies $u \in H^2(\Omega)$. By integration by parts: For all $\phi \in H^1(\Omega)$, $\phi(a) = 0$.

$$\int_\Omega f\phi = \int_\Omega -u''\phi = \int u'\phi' \quad \forall \phi \in M$$

Thus $u \in M$ and

$$\int_\Omega f\phi = \int_\Omega u'\phi' \quad \forall \phi \in M.$$

■

Exercise 5.51 (Bonus 9) Prove that the solution u in Problem E 10.4 is the unique minimizer for the minimization problem:

$$E = \inf_{v \in M} \left(\int_\Omega |v'|^2 - \int_\Omega fv \right)$$

Chapter 6

Heat Equation

6.1 Fundamental Solution

$$\begin{cases} \partial_t u = \Delta u & (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u = g & (x,t) \in \mathbb{R}^d \times \{0\} \end{cases}$$

The fundamential solution is:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^d, t > 0$$

We have:

$$\begin{cases} \partial_t \Phi = \Delta \Phi & (x,t) \in \mathbb{R}^d \times (0,\infty) \\ \int_{\mathbb{R}^d} \Phi(x,t)\, dx = 1 & \forall t > 0 \\ \lim_{t\to 0} \Phi(x,t) = \delta_0(x) & \text{in } D'(\mathbb{R}^d) \end{cases}$$

Theorem 6.1 If $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then

$$u(x,t) := \int_{\mathbb{R}^d} \Phi(x-y,t) g(y)\, dy$$

satisfies

(i) $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$

(ii) $\partial_t u = \Delta u$ for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$

(iii) $\lim_{t\to 0} u(x,t) = g(x)$ for all $x \in \mathbb{R}^d$

Notation 6.2 For functions of (x,t) we introduce the following notation for different regularity in x and t.

$$f \in C_1^2 \Leftrightarrow f, D_x f, D_x^2 f, \partial_t f \in C$$

Theorem 6.3 (Nonhomogeneous problem) Let $f \in C_1^2(\mathbb{R}^d, [0,\infty))$ be compactly supported. Define

$$u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Phi(x-y,t-s) f(y,s)\, dy\, ds$$

Then

(i) $u \in C_1^2(\mathbb{R}^d \times (0,\infty))$

(ii) $\partial_t u = \Delta u + f$ for all $x \in \mathbb{R}^d, t > 0$

(iii) $\lim_{t\to 0} u(x,t) = 0$ for all $x \in \mathbb{R}^d$.

Proof. We write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y,s) f(x-y,t-s)\,dy\,ds$$

With the Leibniz integral rule we get

$$\partial_t u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y,s)\partial_t f(x-y,t-s)\,dy\,ds + \int_{\mathbb{R}^d} \Phi(y,s) f(x-y,0)\,dy$$

and

$$\partial_{ij} u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y,s)\partial_{ij} f(x-y,t-s)\,dy.$$

This shows that $\partial_t u$, $\partial_{ij} u$ are in $C(\mathbb{R}^d \times (0,\infty))$. Next we calculate:

$$\begin{aligned}
\partial_t u - \Delta u &= \int_0^t \int_{\mathbb{R}^d} \Phi(y,s)(\partial_t - \Delta_x) f(x-y,t-s)\,dy\,ds + \int_{\mathbb{R}^d} \Phi(y,s) f(x-y,0)\,dy \\
&= \underbrace{\int_\epsilon^t \int_{\mathbb{R}^d} \Phi(y,s)(\partial_t - \Delta_x) f(x-y,t-s)\,dy\,ds}_{=:I_\epsilon} \\
&\quad + \underbrace{\int_0^\epsilon \int_{\mathbb{R}^d} \Phi(y,s)(\partial_t - \Delta_x) f(x-y,t-s)\,dy\,ds}_{J_\epsilon} \\
&\quad + \underbrace{\int_{\mathbb{R}^d} \Phi(y,s) f(x-y,0)\,dy}_{K}
\end{aligned}$$

Then

$$\begin{aligned}
|J_\epsilon| &\leqslant \|(\partial_t - \Delta_x) f\|_{L^\infty} \int_0^\epsilon \int_{\mathbb{R}^d} \Phi(y,s)\,dy\,ds \leqslant C\epsilon \xrightarrow{\epsilon\to 0} 0 \\
I_\epsilon &= \int_\epsilon^t \int_{\mathbb{R}^d} \Phi(y,s)(-\partial_s - \Delta_y) f(x-y,t-s)\,dy\,ds \\
\text{(Green (2.3))} \qquad &= \int_\epsilon^t \int_{\mathbb{R}^d} \underbrace{(\partial_s - \Delta_y)\Phi(y,s)}_{=0} f(x-y,t-s)\,dy\,ds \\
&\quad - \left[\int_{\mathbb{R}^d} \Phi(y,s) f(x-y,t-s)\right]_{s=\epsilon}^{s=t}
\end{aligned}$$

This implies:

$$\begin{aligned}
I_\epsilon + K &= \int_{\mathbb{R}^d} \Phi(y,\epsilon) f(x-y,t-\epsilon)\,dy \\
&\xrightarrow{\epsilon\to 0} \int_{\mathbb{R}^d} \delta_0(y) f(x-y,t)\,dy = f(x,t)
\end{aligned}$$

Thus

$$\partial_t u - \Delta u = f(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times (0,\infty)$$

Finally:

$$\|u(\cdot,t)\|_{L^\infty} \leqslant \|f\|_{L^\infty} \int_0^t \int_{\mathbb{R}^d} \Phi(y,s)\,dy\,ds = \|f\|_{L^\infty} t \xrightarrow{t\to 0} 0$$

■

Exercise 6.4 If f, g are given as above, then

$$u(x,t) = \int_{\mathbb{R}^d} \Phi(x-y,t) g(y)\,gy + \int_0^t \int_{\mathbb{R}^d} \Phi(x-y,t-s) f(y,s)\,ds$$

solves

$$\begin{cases} \partial_t - \Delta u = f \\ u(\cdot,t) = g \end{cases}$$

Remark 6.5 (Duhamel formula) Consider the ODE $\partial_t w(t) = Aw(t)$ for all $A \in \mathbb{R}$. Then the solution is

$$w(t) = e^{tA} w(0).$$

More generally: If $\partial_t w(t) = Aw(t) + f(t)$, then

$$\begin{aligned} & \partial_t(e^{-tA} w(t)) = e^{-tA}(\partial_t w(t) - Aw(t)) = e^{-tA} f(t) = e^{-tA} f(t) \\ \Rightarrow \quad & e^{-tA} w(t) = w(0) + \int_0^t e^{-sA} f(s)\,ds \\ \Rightarrow \quad & w(t) = e^{tA} w(0) + \int_0^t e^{(t-s)A} f(s)\,ds \end{aligned}$$

More generally, if A is an operator (independend of time) then:

$$\begin{aligned} & \partial_t w(t) = Aw(t) + f(t) \\ \Rightarrow \quad & w(t) = e^{tA} w(0) + \int_0^t e^{(t-s)A} f(s)\,ds \end{aligned}$$

Application: If $A = \Delta$, then the operator $e^{t\Delta}$ has kernel

$$e^{t\Delta}(x,y) = \Phi(x-y,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}.$$

This is called the *heat kernel*.

Theorem 6.6 (L^2-data) For every $g \in L^2(\mathbb{R}^d)$, define

$$u(t,x) = \int_{\mathbb{R}^d} \Phi(x-y,t) g(y)\,dy$$

Then $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$ and it solves the heat equation

$$\begin{cases} \partial_t u = \Delta_x u & \mathbb{R}^d \times (0,\infty) \\ \lim_{t\to 0} u(\cdot,t) = g & \text{in } L^2(\mathbb{R}^d) \end{cases}$$

Proof. Recall the heuristic computation from the heat equation using the Fourier transform

$$\begin{aligned}
& \partial_t u(x,t) = \Delta_x u(x,t) \\
\Leftrightarrow \quad & \partial_t \hat{u}(k,t) = -|2\pi k|^2 \hat{u}(k,t) \\
\Leftrightarrow \quad & \partial_t (e^{t|2\pi k|^2} \hat{u}(k,t)) = 0 \\
\Leftrightarrow \quad & e^{t|2\pi k|^2} \hat{u}(k,t) = \hat{u}(k,0) = \hat{g}(k) \\
\Leftrightarrow \quad & \hat{u}(k,t) = e^{-t|2\pi k|^2} \hat{g}(k) = \hat{\Phi}(k,t)\hat{g}(k) = \widehat{\Phi \star g} \\
\Leftrightarrow \quad & u(x,t) = \Phi \star g = \int_{\mathbb{R}^d} \Phi(x-y,t) g(y)\, dy
\end{aligned}$$

■

Here we only need the direction $\Leftarrow$ which is rigorous if $g \in L^2(\mathbb{R}^d)$. From the Fourier transform, it is also easy to check that $u(\cdot,t) \to g$ in L^2 as $t \to 0$ (exercise). To see the smoothness, note that for all $t > 0$, and for all $m \in \mathbb{N}$:

$$(1 + |2\pi k|^m)\hat{u}(k,t) = \underbrace{(1 + |2\pi k|^m)e^{-t|2\pi k|^2}}_{\in L^\infty} \underbrace{\hat{g}(k)}_{\in L^2} \in L^2$$

This implies $u(\cdot,t) \in H^m(\mathbb{R}^d)$ for all $m \geqslant 1$, so $u(\cdot,t) \in C^\infty(\mathbb{R}^d)$ by Sobolev embedding (see below). This argument can also be used to show that $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$ (exercise)

Theorem 6.7 (Sobolev embedding) If $m > \frac{d}{2}$, then $H^m(\mathbb{R}^d) \subseteq (C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$.

Proof. We write for all $u \in H^m(\mathbb{R}^d)$:

$$\hat{u}(k) = \underbrace{\hat{u}(k)(1 + |2\pi k|^m)}_{\in L^2 \text{ as } u \in H^m} \underbrace{\frac{1}{1 + |2\pi k|^m}}_{\in L^2 \text{ as } m > \frac{d}{2}}$$

This implies $\hat{u}(k) \in L^1(\mathbb{R}^d)$ and finally $u = (\hat{u})^\vee \in (C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$. ■

Exercise 6.8 (E 11.1) Let $g \in L^2(\mathbb{R}^d)$,

$$u(x,t) = \int_{\mathbb{R}^d} \Phi(x-y,t) g(y)\, dy, \qquad \Phi(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$$

be the fundamential solution of the heat equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \forall (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u(x,t) \to g(x) & \text{as } t \to 0. \end{cases}$$

Prove that

a) $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$.

b) $\|u(\cdot,t) - g\|_{L^2(\mathbb{R}^d)} \xrightarrow{t \to 0^+} 0$

c) If $g \in H^1(\mathbb{R}^d)$, then $\|u(\cdot,t) - g\|_{L^2(\mathbb{R}^d)} \leqslant C\sqrt{t}$ as $t \to 0^+$.

Solution. a) We prove for all $t > 0$:

$$u(x,t) \in \bigcap_{m \geqslant 1} H^m(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$$

We use the Fourier transform:

$$\hat{\Phi}(k,t) = e^{-t|2\pi k|^2}$$

Recall $\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$. From this we get $\widehat{e^{-\pi\lambda|x|^2}} = \lambda^{-\frac{d}{2}} e^{-\frac{\pi|k|^2}{\lambda}}$. Then:

$$\widehat{e^{-\frac{|x|^2}{4t}}} = \widehat{e^{-\pi\frac{1}{4\pi t}|x|^2}} = \left(\frac{1}{4\pi t}\right)^{-\frac{d}{2}} e^{-\pi|k|^2 4\pi t} = (4\pi t)^{\frac{d}{2}} e^{-t|2\pi k|^2}$$

Hence:

$$\hat{u}(k,t) = \hat{\Phi}(k,t)\hat{g}(k) = e^{-t|2\pi k|^2}\hat{g}(k) \in L^1(\mathbb{R}^d, dk) \quad \forall t > 0$$

This implies:

$$u(x,t) = \int_{\mathbb{R}^d} e^{-t|2\pi k|^2}\hat{g}(k)e^{2\pi ikx}\, dk \quad \forall (x,t) \in \mathbb{R}^d \times (0,\infty)$$

Consequently:

$$D_x^\alpha u(x,t) = \int_{\mathbb{R}^d} \underbrace{e^{-t|2\pi k|^2}\hat{g}(k)(2\pi ik)^\alpha}_{L^1(\mathbb{R}^d, dk)} e^{2\pi ikx}\, dk \in C(\mathbb{R}^d, (0,\infty))$$

$$D_t^\alpha u(x,t) = \int_{\mathbb{R}^d} (-|2\pi k|^2)^\alpha e^{-t|2\pi k|^2}\hat{g}(k)e^{2\pi ikx}\, dk \in C(\mathbb{R}^d, (0,\infty))$$

Also:

$$\begin{aligned}
&\partial_t u - \Delta_x u \\
&= \int_{\mathbb{R}^d} -|2\pi k|^2 e^{-t|2\pi k|^2}\hat{g}(k)e^{2\pi ikx}\, dk + \int_{\mathbb{R}^d} e^{-t|2\pi k|^2}\hat{g}(k)|2\pi ik|^2 e^{2\pi ikx}\, dk \\
&= 0
\end{aligned}$$

b) Finally:

$$\begin{aligned}
\int_{\mathbb{R}^d} |u(x,t) - g(x)|^2 &= \int_{\mathbb{R}^d} |\hat{u}(k,t) - \hat{g}(k)|^2\, dk \\
&= \int_{\mathbb{R}^d} \underbrace{|e^{-t|2\pi k|^2} - 1|^2}_{\in[0,1]} \underbrace{|\hat{g}(k)|^2}_{\in L^1(\mathbb{R}^d)}\, dk \xrightarrow{t\to 0} 0
\end{aligned}$$

by dominated convergence. Now,

$$\begin{aligned}
\int_{\mathbb{R}^d} |u(x,t)|^2\, dx &= \int_{\mathbb{R}^d} |\hat{u}(k,t)|^2\, dk \\
&= \int_{\mathbb{R}^d} \underbrace{e^{-2t|2\pi k|^2}}_{\in[0,1] \text{ and } \xrightarrow{t\to 0} 0} |\hat{g}(k)|^2\, dk \xrightarrow{t\to\infty} 0
\end{aligned}$$

c) Assume $g \in H^1(\mathbb{R}^d) \Leftrightarrow \int_{\mathbb{R}^d}(1 + |2\pi k|^2)|\hat{g}(k)|^2\, dk < \infty$. We claim for all $s \geqslant 0$ that $|1 - e^{-s}| \leqslant \min(1, Cs) \leqslant C\sqrt{s}$: We have that $s \mapsto \left|\frac{1-e^{-s}}{s}\right|$ is bounded and continuous in $[0,1]$ as $\left|\frac{1-e^{-s}}{s}\right| \to 1$, so $\frac{1-e^{-s}}{s} \leqslant C$ for all $s \in [0,1]$.

$$\begin{aligned}
\int_{\mathbb{R}^d} |u(x,t) - g(x)|^2\, dx &= \int_{\mathbb{R}^d} \underbrace{\left|1 - e^{-t|2\pi k|^2}\right|^2}_{\leqslant C(t|2\pi k|^2)} |\hat{g}(k)|^2\, dk \\
&\leqslant C \int_{\mathbb{R}^d} t|2\pi k|^2 |\hat{g}(k)|^2\, dk \\
&\leqslant Ct\|g\|_{H^1}^2 \quad \forall t > 0
\end{aligned}$$

■

Step 1: Spectral problem:

$$\begin{cases} -\Delta u_n = \lambda_n u_n & \text{in } \Omega \\ u_n|_{\partial\Omega} = 0 \end{cases}$$

Lemma 6.9 There is a $\lambda_n > 0$, $\lambda_n \xrightarrow{n\to\infty} \infty$ and an orthonormal family $\{u_n\} \subseteq L^2(\Omega)$ s.t. $u_n \in H_0^1(\Omega) \cap C^\infty(\Omega)$ solving this eigenvalue equation.

Step 2:

$$\begin{cases} \partial_t - \Delta_x u = 0 \\ u(x,0) = g(x) \end{cases} \Rightarrow \begin{cases} \partial_t \langle u_n, u\rangle_{L^2(\Omega)} = \langle u_n, \Delta_x u\rangle = \langle \Delta_x u_n, u\rangle = -\lambda_n \langle u_n, u\rangle \\ \langle u_n, u\rangle_{t=0} = \langle u_n, g\rangle \end{cases}$$

$$\Rightarrow \qquad \langle u_n, u\rangle = e^{-t\lambda_n}\langle u_n, g\rangle \qquad \forall t > 0, \forall n = 1, 2, \dots$$

$$\Rightarrow \qquad u = \sum_{n=0}^{\infty} \langle\rangle = -\sum e^{-t\lambda_n}\langle\rangle u$$

Example 6.10 $\Omega = (0,1)$,

$$\begin{cases} -u_n'' = \lambda_n u_n & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

has solution

$$\begin{cases} u_n(x) = \sqrt{2}\sin(\pi n x) & n = 1, 2, \dots \\ \lambda_n = (\pi_n)^2 \end{cases}.$$

has a solution:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-t\lambda_n} \underbrace{\langle u_n, g\rangle}_{g_n} u_n(x) = \sum_{n=1}^{\infty} e^{-t\pi^2 n^2} g_n \sin(\pi n x),$$

$$\int_0^1 \sin(n\pi x)^2\, dx = \frac{1}{2} \quad \forall n > 1$$

$$g_n = \sqrt{2}\langle u_n, g\rangle = 2\int_0^1 \sin(\pi n x) g(x)\, dx$$

Exercise 6.11 (E 11.2) Consider the heat equation in a bounded domain

$$\begin{cases} \partial_t u(x,t) = \Delta_x u(x,t) & \forall x \in \Omega, t > 0 \\ u(x,t) = 0 & \forall x \in \partial\Omega, t > 0 \\ u(x,0) = g(x) & \forall x \in \Omega \end{cases}$$

Let us focus on the simplest case $\Omega = (0,1)$. Prove that for every $g \in C_c^1(0,1)$, the function

$$u(x,t) = \sum_{n=1}^{\infty} g_n e^{-t\pi^2 n^2} \sin(n\pi x), \qquad g_n = 2\int_0^1 g(y)\sin(n\pi y)\, dy$$

is a classical solution to the above heat equation.

Solution. Direct proof of heat equation. $g \in C_c^1(0,1) \subseteq H_0^1(0,1)$, $\Rightarrow \sum_n \pi^2 n^2 |g_n|^2 = c\|g'\|^2_{L^2(0,1)} < \infty$, so $\sum_n |g_n| < \infty$.

$$u(x,0) = \underbrace{\sum_{n=1}^{\infty} g_n \sin(\pi n x)}_{\in C[0,1]} = g(x) \quad \forall x \in [0,1]$$

From $u(x,t) = \sum_{n=1}^{\infty} e^{-tn^2\pi^2} g_n \sin(\pi n x)$ we get

$$\begin{cases} \partial_t u(x,t) = \sum_{n=1}^{\infty} (-n^2\pi^2) e^{-t\pi^2 n^2} g_n \sin(\pi n x) & \forall t > 0, \forall x \in (0,1) \\ \Delta_x u(x,t) = \sum_{n=1}^{\infty} e^{-t\pi^2 n^2} g_n [-(\pi n)^2] \sin(\pi n x) & \forall t > 0, \forall x \in (0,1) \end{cases}$$

So $\partial_t u - \Delta_x u = 0$ for all $t > 0, x \in (0,1)$ ■

Exercise 6.12 (E 11.3) Let $g(t) = e^{-\frac{1}{t^2}}$ and denote $g^{(n)}(t)$ the n-th derivative of g. Define

$$u(x,t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \quad \forall x \in \mathbb{R}, t > 0$$

Prove that u is a classical solution to the heat equation

$$\begin{cases} \partial_t u(x,t) = \Delta_x u(x,t) & \forall x \in \mathbb{R}, t > 0 \\ \lim_{t \to 0} u(x,t) = 0 & \forall x \in \mathbb{R} \end{cases}$$

Solution. Formally:

$$\begin{cases} \partial_t u = \sum_{n=0}^{\infty} \frac{g^{(n+1)}(t)}{(2n)!} x^{2n} \\ -\Delta_x u = \sum_{n=1}^{\infty} \frac{g^{(n)}}{(2n)!} (2n)(2n-1) x^{2n-2} = \sum_{n=1}^{\infty} \frac{g^{(n)}(t)}{(2n-2)!} x^{2n-2} = \sum_{m=0}^{\infty} \frac{g^{(m+1)}(t)}{(2m)!} x^{2m} \end{cases}$$

This implies $\partial_t u = \Delta_x u$ (if the series are convergent) $(x,t) \in B \times \left[\epsilon, \frac{1}{\epsilon}\right]$ for $B \subset \mathbb{R}$ bounded, $\epsilon > 0$. Also

$$\begin{aligned} g(t) &= e^{-\frac{1}{t^2}} \xrightarrow{t \to 0^+} e^{-\infty} = 0 \\ g'(t) &= e^{-\frac{1}{t^2}} \left(\frac{2}{t^3}\right) \xrightarrow{t \to 0^+} 0 \\ g''(t) &= e^{\frac{1}{t^2}} \left(-\frac{3!}{t^4} + \frac{2}{t^3}\right) \xrightarrow{t \to 0^+} 0 \\ g'''(t) &= e^{-\frac{1}{t^2}} \left(\frac{4!}{t^5} - \frac{3!}{t^4} + \frac{2}{t^3}\right) \end{aligned}$$

Let's proof the convergence of the series:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}$$

converges absolutely for $|x| \leqslant C, t \in \left[\epsilon, \frac{1}{\epsilon}\right], \epsilon > 0$. By induction,

$$g^{(n)}(t) = e^{-\frac{1}{t^2}} \underbrace{\left(\frac{(n+1)!}{t^{n+2}} - \frac{n!}{t^{n+1}} + \frac{(n+1)!}{t^n} - \dots\right)}_{\text{pol in } (\frac{1}{t}),\text{ all cos bounded by } (n+1)} (-1)^{n-1}$$

This implies

$$|g^{(n)}(t)| \leqslant e^{-\frac{1}{t^2}}[(n+2)!]\left(\frac{1}{t^{n+2}}+1\right), \quad \frac{1}{t^s} \leqslant \left(\frac{1}{t^{n+2}}+1\right) \forall s = 0, 1, \ldots, n+2$$

Thus

$$\sum_{n \geqslant 0} \left| \frac{g^{(n)}}{(2n)!} x^{2n} \right| \leqslant \sum_{n \geqslant 0} e^{-\frac{1}{t^2}} \frac{(n+2)!}{(2n)!} \left(\frac{1}{t^{n+2}}+1\right) x^{2n}$$

(1)

$$\begin{aligned}
\sum_{n \geqslant 0} \frac{(n+2)!}{(2n)!} x^{2n} &= \sum_{n \geqslant 0} \frac{1}{(n+3)(n+4)\cdots(2n)} \\
&\leqslant \sum_{n \geqslant 0} \frac{1}{n^{n-2}} x^{2n} \\
&\leqslant \sum_{n \geqslant M} + \sum_{n \geqslant M} \frac{1}{M^{n-2}} x^{2n} \\
& M^2 \sum_n \left(\frac{x^2}{M}\right)^n \\
&\leqslant m^2 \frac{1}{1 - \left(\frac{x^2}{M}\right)}
\end{aligned}$$

(2) $t \in [\epsilon, \frac{1}{\epsilon}]$, so $\frac{1}{t} \leqslant \frac{1}{\epsilon}$, so $\frac{1}{t^{n+2}} \leqslant \frac{1}{\epsilon^{n+2}} \longrightarrow \sum_{n \geqslant 0} \frac{(n+2)!}{(2n)!} \frac{1}{t^{n+2}} x^{2n} \leqslant \sum_{n \geqslant 0} \frac{1}{n^{n-2}} \frac{1}{\epsilon^{n-2}} x^{2n}$

■

Remark 6.13 $|u(x,t)| \leqslant \exp\left(\frac{cx^2}{t}\right) \rightsquigarrow$ unphysical solution. Violates $|u(x,t)| \leqslant Ce^{C|x|^2}$ for all $\forall (x,t) \in \mathbb{R} \times [0,T]$

Exercise 6.14 (Bonus 10) Consider

$$u(x,t) = \int_{\mathbb{R}^d} \Phi(x-y,t) g(y)\, dy$$

where $\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{x^2}{4t}}$. Assume $g \in C_c^\infty(\mathbb{R}^d)$. Prove or disprove that

$$\|u(\cdot,t) - g\|_{L^2(\mathbb{R}^d)} \leqslant C_n t^n$$

as $t \to 0^+$ for all $n = 1, 2, \ldots$

6.2 Maximum Principle

Recall the Poisson eqation $-\Delta u \leqslant 0$ in $\Omega \subseteq \mathbb{R}^d$ open, bounded. Then

$$\sup_{\bar{\Omega}} u(x) = \sup_{\partial\Omega} u(x).$$

Theorem 6.15 (Maximum principle for bounded sets) Let $\Omega \subseteq \mathbb{R}^d$ be open and bouneded. Let $T > 0$ and define

$$\begin{aligned}
\Omega_T &= \Omega \times (0,T), \\
\partial^\star \Omega_T &= (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times [0,T])
\end{aligned}$$

If $u \in C^2_1(\Omega_T) \cap C(\bar{\Omega}_T)$ solves $\partial_t u - \Delta_x u \leqslant 0$ in Ω_T, then

$$\max_{\overline{\Omega_T}} u = \max_{\partial^\star \Omega_T} u.$$

Proof. We will use Hopf's argument which is simpler that the mean-value theorem (there exists a mean-value theorem for heat equation, but it is complicated and we will not discuss it). Firstly, to illustrate the principle, we proof the maximum principle for the Poisson Equation: Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$

Step 1) Assume $\Delta u > 0$ in Ω. Since $\bar{\Omega}$ is compact, there is a $x_0 \in \bar{\Omega}$ s.t. $u(x_0) = \max_{x\in\bar{\Omega}} u(x)$. We prove that $x_0 \in \partial\Omega$. In fact, if $x_0 \in \Omega$, then since x_0 is a (local) maximizer of u in Ω, we have $\Delta u(x_0) \leqslant 0$, which contradicts to the assumption that $\Delta u > 0$ in Ω. Thus $x_0 \in \partial\Omega$, and hence

$$\max_{x\in\bar{\Omega}} u(x) = u(x_0) \leqslant \max_{x\in\partial\Omega} u(x).$$

Step 2) Now assume $\Delta u \geqslant 0$ in Ω. Define

$$u_\epsilon(x) = u(x) + \epsilon|x|^2, \quad \epsilon > 0.$$

Then, $\Delta u_\epsilon > 0$ in Ω, hence by Step 1 and

$$u \leqslant u_\epsilon \leqslant u + \epsilon \sup_{x\in\bar{\Omega}} |x|^2$$

we have

$$\begin{aligned}\max_{x\in\bar{\Omega}} u(x) &\leqslant \max_{x\in\bar{\Omega}} u_\epsilon(x) \leqslant \max_{x\in\partial\Omega} u_\epsilon(x)\\ &\leqslant \max_{x\in\partial\Omega} u(x) + \epsilon\left(\sup_{x\in\bar{\Omega}} |x|^2\right) \xrightarrow{\epsilon\to 0} \max_{x\in\partial\Omega} u(x)\end{aligned}$$

Proof for the heat equation:

Step 1) Assume $u \in C^2_1(\Omega \times (0,T]) \cap C(\bar{\Omega} \times [0,T])$ and

$$\partial_t u - \Delta_x u < 0$$

in $\Omega \times (0,T]$. Then, because of compactness, there is $(x_0,t_0) \in \bar{\Omega} \times [0,T]$ s.t.

$$u(x_0,t_0) = \max_{(x,t)\in\bar{\Omega}\times[0,T]} u(x,t).$$

We prove that $(x_0,t_0) \in \partial^\star\Omega_T$. Assume by contradiction that $(x_0,t_0) \notin \partial^\star\Omega_T$, then $x_0 \in \Omega$ and $t_0 \in (0,T]$. Since $x \mapsto u(x,t_0)$ has a (local) maximizer $x_0 \in \Omega$ we have that $\Delta_x u(x_0,t_0) \leqslant 0$. Since $t \mapsto u(x_0,t)$ has a (local) maximizer $t_0 \in (0,T]$ we have that $\partial_t u(x_0,t_0) \geqslant 0$. This implies:

$$(\partial_t u - \Delta_x u)(x_0,t_0) \geqslant 0$$

which is a contradiction to the assumption. Thus $(x_0,t_0) \in \partial^\star\Omega_T$, i.e. $\max_{\bar{\Omega}_T} u = \max_{\partial^\star\Omega_T} u$.

Step 2) Assume $u \in C^2_1(\Omega \times (0,T)) \cap C(\bar{\Omega} \times [0,T])$ and

$$\partial_t u - \Delta_x u \leqslant 0 \quad \text{in } \Omega \times (0,T).$$

Let $\tilde{T} \in (0,T)$ and for $\epsilon > 0$:

$$u_\epsilon(x,t) = u(x,t) + \epsilon|x|^2.$$

Then: $u_\epsilon \in C_1^2(\Omega \times (0,T']) \cap C(\bar{\Omega} \times [0,\tilde{T}])$ and $\partial_t u_\epsilon - \Delta_x u_\epsilon < 0$ in $\Omega \times (0,\tilde{T}]$. By Step 1:

$$\max_{\bar{\Omega}_{\tilde{T}}} u_\epsilon \leqslant \max_{\partial^\star \Omega_{\tilde{T}}} u_\epsilon$$

$$\overset{\epsilon \to 0}{\Rightarrow} \quad \max_{\bar{\Omega}_{\tilde{T}}} u \leqslant \max_{\partial^\star \Omega_{\tilde{T}}} u$$

$$\overset{\tilde{T} \to T}{\Rightarrow} \quad \max_{\bar{\Omega}_T} u \leqslant \max_{\partial^\star \Omega_T} u \qquad \blacksquare$$

Theorem 6.16 (Maximum principle for $\Omega = \mathbb{R}^d$) Let $\Omega_T = \mathbb{R}^d \times (0,T)$, $\bar{\Omega}_T = \mathbb{R}^d \times [0,T]$. Let $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ such that

- $\partial_t u - \Delta_x u \leqslant 0$ in Ω_T
- $u(x,t) \leqslant M e^{M|x|^2}$ for all $(x,t) \in \bar{\Omega}_T$

Then

$$\sup_{(x,t) \in \bar{\Omega}_T} u(x,t) = \sup_{x \in \mathbb{R}^d} u(x,0).$$

Proof.

Step 1: For all $y \in \mathbb{R}^d$ and $\epsilon > 0$ define

$$v(x,t) = u(x,t) - \frac{\epsilon}{(T+\epsilon-t)^{\frac{d}{2}}} \exp\left(\frac{|x-y|^2}{4(T+\epsilon-t)}\right)$$

This implies

$$\partial_t v - \Delta_x v = \partial_t u - \Delta_x u \leqslant 0$$

in Ω_T. For $U = B(y,r), U_T = U \times (0,T), \bar{U}_T = \bar{U} \times [0,T]$, $\partial^\star U_T = (U \times \{0\}) \cup (\partial U \times [0,T])$, by the maximum principle for U bounded we have

$$\max_{\bar{U}_T} v \leqslant \max_{\partial^\star U_T} v.$$

Let us bound $\max_{\partial^\star U_T} v$.

- On $U \times \{0\}$ we use $v \leqslant u$ and hence

$$\max_{x \in \bar{U}} v(x,0) \leqslant \max_{x \in \bar{U}} u(x,0) \leqslant \max_{x \in \mathbb{R}^d} u(x,0).$$

- On $\partial U \times [0,T]$ we use $|x-y| = r \Rightarrow |x| \leqslant |y| + r$.

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\epsilon}{(T+\epsilon-t)^{\frac{d}{2}}} \exp\left(\frac{|x-y|^2}{4(T+\epsilon-t)}\right) \\ &\leqslant M e^{M(|y|+r)^2} - \frac{\epsilon}{(T+\epsilon)^{\frac{d}{2}}} \exp\left(\frac{r^2}{4(T+\epsilon)}\right) \xrightarrow{r \to \infty} -\infty \end{aligned}$$

if $M < \frac{1}{4(T+\epsilon)}$. In particular, we can choose r large s.t.

$$\max_{\substack{x \in \partial U \\ t \in [0,T]}} v(x,t) \leqslant \max_{x \in \mathbb{R}^d} u(x,0).$$

In summary, if $M < \frac{1}{4(T+\epsilon)}$, then:

$$u(y,t) - \frac{\epsilon}{(T+\epsilon-t)^{\frac{d}{2}}} = v(y,t) \leqslant \max_{\bar{U}_T} v \leqslant \max_{x\in\mathbb{R}^d} u(x,0)$$

This holds for all $(y,t) \in \mathbb{R}^d \times [0,T]$. Thus,

$$\max_{\mathbb{R}^d\times[0,T]} u \leqslant \frac{\epsilon}{(T+\epsilon-t)^{\frac{d}{2}}} + \max_{x\in\mathbb{R}^d} u(x,0)$$

Taking $\epsilon \to 0$ we conclude that if $M < \frac{1}{4T}$,

$$\max_{\mathbb{R}^d\times[0,T]} u \leqslant \max_{x\in\mathbb{R}^d} u(x,0)$$

Step 2: For general T, we denote $T_1 = \frac{T}{N}$, $N \in \mathbb{N}$ s.t. $M < \frac{4}{T_1}$. Then by step 1:

$$\begin{aligned}
\max_{\mathbb{R}^d\times[0,T_1]} u &\leqslant \max_{x\in\mathbb{R}^d} u(x,0)\\
\max_{\mathbb{R}^d\times[T_1,2T_1]} u &\leqslant \max_{x\in\mathbb{R}^d} u(x,T_1) \leqslant \max_{x\in\mathbb{R}^d} u(x,0)\\
&\vdots\\
\max_{\mathbb{R}^d\times[(N-1)T_1,NT_1]} &\leqslant \max_{x\in\mathbb{R}^d} u(x,(N-1)T_1) \leqslant \max_{x\in\mathbb{R}^d} u(x,0)\\
\rightsquigarrow \max_{\mathbb{R}^d\times[0,T]} u &\leqslant \max_{x\in\mathbb{R}^d} u(x,0)
\end{aligned}$$

■

Remark 6.17 The condition $u \leqslant Me^{M|x|^2}$ is necessary, otherwise there are solutions $u \neq 0$ s.t. $u(x,0) = 0$

Theorem 6.18 (Uniqueness) If $u \in C_1^2(\mathbb{R}^d \times (0,T)) \cap C(\mathbb{R}^d \times [0,T])$ and

$$\begin{aligned}
u(x,t) &\leqslant Me^{M|x|^2} && \text{in } \mathbb{R}^d \times [0,T],\\
\partial_t u - \Delta_x u &= 0 && \text{in } \mathbb{R}^d \times (0,T),\\
u(x,0) &= 0 && \text{in } \mathbb{R}^d
\end{aligned}$$

Then $u = 0$ in $\mathbb{R}^d \times [0,T]$.

Proof. Use the maximum principle for u and $-u$. ■

Remark 6.19 If $u(\cdot,t) \in L^2(\mathbb{R}^d)$, the proof of uniquness can be done without the maximum principle. Heuristically:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(x,t)|^2\,dx = 2\int_{\mathbb{R}^d} (\partial_t u)u\,dx = 2\int_{\mathbb{R}^d} \Delta_x u u\,dx = -2\int_{\mathbb{R}^d} |\nabla_x u|^2\,dx \leqslant 0$$

This implies

$$e(t) := \int_{\mathbb{R}^d} |u(x,t)|^2\,dx$$

is descreasing. Hence, if $e(0) = 0$, then $e(t) = 0$ for all $t \geqslant 0$. This argument will be helpful below for the heat backward equation.

Remark 6.20 The heat equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(t=0) = g \end{cases}$$

is a well-posed problem:

- Existence
- Uniqueness
- Stability (solution depends continuously on data)

For the latter issue, by the maximum principle we have

$$\|u(\cdot,t)\|_{L^\infty} \leqslant \|u(\cdot,0)\|_{L^\infty} \quad \forall t$$

or in the L^2-situation:

$$\|u(\cdot,t)\|_{L^2} \leqslant \|u(\cdot,0)\|_{L^2} \quad \forall t$$

On the other hand, the heat backward equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(t=T) = g \end{cases}$$

is *not* well-posed.

- Non-Existence: In general, the existence requires some special property on g, e.g. g is very smooth (only $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ or $g \in L^2(\mathbb{R}^d)$ is not enough)
- Uniqueness: On the other hand, the uniqueness still holds.

Lemma 6.21 If $e \in C^2(0,T)$, $e(t) \geqslant 0$, $e'(t) \leqslant 0$, $e''(t) \geqslant 0$ and $|e'(t)|^2 \leqslant e(t)e''(t)$ for $t \in [0,T]$ and $e(T) = 0$, then $e \equiv 0$.

Proof. Since e is monotonly decreasing and $e(T) = 0$ there is a $t_0 \in [0,T]$ s.t. $e(t_0) = 0$ and $e(t) > 0$ if $t \leqslant t_0$. We need to prove that $t_0 = 0$. Assume by contradiction $0 < t_0 \leqslant T$, then for $t \in (0,t_0)$ define $f(t) := \log e(t)$. Then

$$\begin{aligned} f'(t) &= \frac{e'(t)}{e(t)} \\ \Rightarrow \quad f''(t) &= \frac{e''(t)e(t) - |e'(t)|^2}{e(t)^2} \geqslant 0 \end{aligned}$$

This means that f is convex, so for all $t_1, t_2 \in (0,t_0)$ and $\tau \in (0,1)$:

$$\begin{aligned} & f(\tau t_1 + (1-\tau)t_2) \leqslant \tau f(t_1) + (1-\tau) f(t_2) \\ \Rightarrow \quad & e(\tau t_1 + (1-\tau)t_2) \leqslant e(t_1)^\tau e(t_2)^{1-\tau} \end{aligned}$$

Now, $e(\tau t_1 + (1-\tau)t_2) \xrightarrow{t_2 \to t_0} 0$ and $\tau \to 1$ implies $e(t_1) = 0$ for all $t_1 \in (0,t_0)$ which is a contradiction. ■

Theorem 6.22 If $u \in C_1^2(\mathbb{R}^d \times [0,T]) \cap C^1(H^1(\mathbb{R}^d) \times [0,T])$ and

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,T) = 0 \end{cases}$$

Then $u = 0$ in $\mathbb{R}^d \times [0,T]$.

Proof. Recall

$$e(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 \, dx.$$

Then,

$$\begin{aligned} e'(t) &= 2\int_{\mathbb{R}^d} u\partial_t u \, dx = 2\int_{\mathbb{R}^d} u\Delta_x u \, dx = -2\int_{\mathbb{R}^d} |\nabla_x u|^2 \, dx \\ e''(t) &= -4\int_{\mathbb{R}^d} \nabla_x u \nabla_x(\partial_t u) = 4\int_{\mathbb{R}^d} \Delta_x u \partial_t u \, dx = 4\int_{\mathbb{R}^d} |\Delta_x u|^2 \, dx \geqslant 0 \end{aligned}$$

and hence

$$|e'(t)|^2 = 4\left|\int_{\mathbb{R}^d} u\Delta_x u \, dx\right|^2 \leqslant 4\left(\int_{\mathbb{R}^d} |u|^2 \, dx\right)\left(\int_{\mathbb{R}^d} |\Delta_x u|^2 \, dx\right) = e(t)e''(t)$$

Then the statement follows with lemma 6.21. ■

Some remarks about the heat equation in unbounded domains:

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ u(x,0) = 0 & (\text{i.e. } \lim_{t\to 0} u(x,t) = 0 \forall x \in \mathbb{R}^d) \end{cases}$$

There is a classical solution $0 \neq u \in C^1(\mathbb{R}^d \times (0,\infty))$. An example is

$$u(x,t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \quad g(t) = e^{-\frac{1}{t^2}}$$

(s.t. $g \to 0$ as $t \to 0$). Note

$$\begin{aligned} g(t) &= e^{-\frac{1}{t^2}}, \\ g'(t) &= \frac{2}{t^3} g(t) \\ g''(t) &= \left(\frac{2}{t^3}\right)' g(t) + \frac{2}{t^3}\frac{2}{t^3} g(t) \\ g^{(n)}(t) &= P_n\left(\frac{1}{t}\right) g(t) \end{aligned}$$

where

$$\begin{cases} P_0 = 1 \\ P_{n+1}\left(\frac{1}{t}\right) = \left(P_n\left(\frac{1}{t}\right)\right)' + \left(\frac{2}{t^3}\right) P_n\left(\frac{1}{t}\right) = A_1 P_n + A_2 P_n, \begin{cases} A_1 = \partial_t \\ A_2 = \frac{2}{t^3} \end{cases} \\ P_{n+1} = (A_1 + A_2)P_n = (A_1 + A_2)(A_1 + A_2)P_{n-1} = \end{cases}$$

This implies:

$$P_n = (A_1 + A_2)^n P_0 = \sum_{\sigma \in \{1,2\}^n} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)} P_0$$

$$A_1\left(\frac{\alpha}{t^s}\right) = \frac{-s\alpha}{t^{s+1}} \to A_1$$

Multiple coefficients by a factors and + power by 1

$$A_2\left(\frac{\alpha}{t^s}\right) = \frac{2\alpha}{t^{s+3}} \to A_2$$

Mul Cof by a factor 2 and + power by 3

$$|\underbrace{A_{\sigma(1)}\cdots A_{\sigma(n)},1}_{k \text{ times } A_2,\ n-k \text{ times } A_1}| \leqslant \frac{2^k}{t^{3k}} \leqslant \frac{2^k}{t^{3k}}\frac{(3n)^{n-k}}{t^{n-k}} = \frac{2^k(3n)^{n-k}}{t^{n+2k}}$$

This implies

$$|P_n\left(\frac{1}{t}\right)| \leqslant \max_{0\leqslant k\leqslant n} \frac{2^n 2^k(3n)^{n-k}}{t^{n+2k}}$$

Thus:

$$\begin{aligned}
\sum_n \left|\frac{g^{(n)}(t)}{(2n)!}x^{2n}\right| &\leqslant \sum_n \max_{0\leqslant k\leqslant n} \frac{2^n 2^k(3n)^{n-k}}{t^{n+2k}(2n)!}\frac{e^{-\frac{1}{t^2}}}{1}x^{2n} \\
&\leqslant \sum_n \max \frac{2^n 2^k (3n)^{n-k}}{t^{n+2k}(2n)!}(k!)(2t^2)^k e^{-\frac{1}{2t^2}}x^{2n} \\
&= \sum_n \frac{2^n 2^k 2^k (3n)^{n-k}(k!)}{(2n)!t^n} e^{-\frac{1}{2t^2}}x^{2n} \\
&\leqslant \sum_n \frac{(c_n)^n}{(2n)!t^n} e^{-\frac{1}{2t^2}}x^{2n} \\
&\leqslant \sum_n \frac{c^n}{n!t^n} e^{-\frac{1}{2t^2}}x^{2n} \\
&\leqslant \sum_n e^{\frac{cx^2}{t}-\frac{1}{2t^2}}
\end{aligned}$$

Where we used that

$$e^s = \sum_k \frac{s^k}{k!} \geqslant \frac{s^k}{k!}$$

for all $s \geqslant 0$ implies

$$e^{-\frac{1}{2t^2}} = \frac{1}{e^{\frac{1}{2t^2}}} \leqslant \frac{1}{\left(\frac{1}{2t^2}\right)\frac{1}{k!}} = k!(2t^2)^k.$$

We conclude:

- $u(x,t)$ is well-defined, $x \in \mathbb{R}^d$, $t > 0$ real? *to* heat equation.
- $u(x,t) \to 0$ as $t \to 0$ for all $x \in \mathbb{R}^d$.

Exercise 6.23 (E 12.1) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$. Assume that $x_0 \in \Omega$ is a local maximizer of u, namely there exists some $r > 0$ such that $u(x_0) \geqslant u(x)$ for all $x \in B_r(x_0) \subseteq \Omega$.

(a) Prove that the Hessian matrix $H = (D^\alpha u(x_0))_{|\alpha|=2}$ is negative semi-definite, namely

$$yHy \leqslant 0$$

for all $y \in \mathbb{R}^d$.

(b) Prove that $\Delta u(x_0) \leqslant 0$

Hint: Recall that we used (b) for the maximum principle by Hopf's method.

Solution. (a) In 1D this is obvious. If x_0 is a local minimizer of u, then $u'(x_0) = 0, u''(x_0) \leqslant 0$ (Taylor expansion).

In d dimensions:

$$\phi(t) = u(x_0 + t\xi) \quad \xi \in \mathbb{R}^d, t \in \mathbb{R}, |t| \text{ small}$$

So 0 is a local maximizer of ϕ. This implies

$$0 = \phi'(0) = \nabla u(x_0)\xi \quad \forall \xi \in \mathbb{R}^d \Rightarrow H \leqslant 0$$

$$\begin{aligned}
\phi''(0) &= \lim_{t\to 0} \frac{\phi'(t) - \phi'(0)}{t} = \lim_{t\to 0} \frac{(\nabla u(x_0 + t\xi) - \nabla u(x_0))\xi}{t} \\
&= \lim_{t\to 0} \sum_{i=1}^{d} \frac{(\partial_i u(x_0 + t\xi) - \partial_i u(x_0))\xi_i}{t} = \sum_{i=1}^{d}\sum_{j=1}^{d} \partial_j \partial_i u(x_0)\xi_j\xi_i = \langle \xi, H\xi \rangle,
\end{aligned}$$

$H = (\partial_i \partial_j u(x_0))_{i,j=1}^d$.

(b) Consequently

$$\Delta u(x_0) = \sum_{i=1}^{d} \partial_i \partial_i u(x_0) = \mathrm{Tr}(H) \leqslant 0$$ ■

Exercise 6.24 (E 12.2) Let $\Omega \subseteq \mathbb{R}^d$ be open and bouned. We prove the maximum principle for a general elliptic operator

$$Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j u(x) + \sum_{i=1}^{d} b_i(x)\partial_j u(x),$$

$a_{ij}, b_i \in C(\bar{\Omega})$, $A(x) = (a_{ij}(x))_{i,j=1}^d \geqslant \mathbb{1}$ (as matrices). Prove that if $Lu(x) \geqslant 0$ for all $x \in \Omega$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then

$$\max_{x\in\bar{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x).$$

Solution.

Step 1: Assume $Lu(x) > 0$ for all $x \in \Omega$: Since $u \in C(\bar{\Omega})$ there is a $x_0 \in \bar{\Omega}$ s.t.

$$u(x_0) = \max_{x\in\bar{\Omega}} u(x).$$

We prove $x_0 \in \partial\Omega$. Assume by contradiction that $x_0 \notin \partial\Omega$, so $x_0 \in \Omega$ is a local maximizer. We prove $Lu(x_0) \leqslant 0$. Note:

$$\begin{aligned}
Lu(x_0) &= \sum_{i,j=1}^{d} a_{ij}(x_0)\partial_i\partial_j u(x_0) + \sum_{i=1}^{d} b_i(x_0)\partial_i u(x_0) \\
&= \mathrm{Tr}[A(x_0)H(x_0)] + B(x_0)\underbrace{\nabla u(x_0)}_{=0} \leqslant 0 \quad ↯
\end{aligned}$$

$A(x_0) = (a_{ij}(x_0))_{i,j=1}^d$, $B(x_0) = (b_i(x_0))_{i=1}^d$, where $\mathrm{Tr}[AH] = \sum_i (AH)_{ii} = \sum_i \sum_j A_{ij}H_{ij}$

General fact: If $A \geqslant 0, B \geqslant 0$ (matrices), then $Tr(AB) \geqslant 0$.

- $A = (\sqrt{A})^2 \Rightarrow \mathrm{Tr}(AB) = \mathrm{Tr}((\sqrt{A})^2 B) = \mathrm{Tr}(\underbrace{\sqrt{A}B\sqrt{A}}_{\geqslant 0}) \geqslant 0$

- Spectral theorem: $A \geqslant 0$, then there are eigenvectors (α_i) and eigenvalues $\lambda_i \geqslant 0$ s.t.

$$\mathrm{Tr}(AB) = \sum_i \langle \alpha_i, AB\alpha_i \rangle = \sum_i \underbrace{\lambda_i}_{\geqslant 0} \underbrace{\langle \alpha_i, B\alpha_i \rangle}_{\geqslant 0} \geqslant 0$$

- General Case: $Lu(x) \geqslant 0$ for all $x \in \Omega$. Assume that there is a $v \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t. $Lv(x) > 0$ for all $x \in \Omega$. Define for all $\epsilon > 0$ $u_\epsilon = u + \epsilon v$. Then $Lu_\epsilon(x) = Lu(x) + \epsilon Lv(x) > 0$ for all $x \in \Omega$. By Step 1,

$$\begin{aligned} & \max_{x \in \bar{\Omega}} u_\epsilon(x) \leqslant \max_{x \in \partial\Omega} u_\epsilon(x) \\ \xrightarrow{\epsilon \to 0} \quad & \max_{x \in \bar{\Omega}} u(x) \leqslant \max_{x \in \partial\Omega} u(x) \end{aligned}$$

What v? First $v(x) = x^2 = x_1^2 + \cdots + x_d^2$,

$$Lv(x) = \sum_{ij} a_{ij}(x) 2\delta_{ij} + \sum_i b_i(x) 2x_i$$

not clear to be $\geqslant 0$.

$$\begin{aligned} v(x) &= x^{2n} \quad n \text{ large} \\ v(x) &= x_1^{2n} \longrightarrow Lv(x) = a_{11}(x)2n(2n+1)x_1^{2n-2} + b_1(x)2nx_1^{2n-1} \\ &\geqslant 2nx_1^{2n-2}[\underbrace{(2n-1) + \underbrace{b_1(x)x_1}_{\text{b.d. in } \bar{\Omega}}}_{>0}] \geqslant 0 \quad \forall x \in \bar{\Omega} \end{aligned}$$

if n is large enough.

$$v(x) = (x_1 + R)^{2n}$$

where $R > 0$ large s.t. $x_1 + R \geqslant 1$ for all $\forall x \in \bar{\Omega}$. This implies

$$Lv(x) \geqslant 2n \underbrace{(x_1 + R)^{2n-2}}_{>0} [\underbrace{2n - 1 + b_1(x)(x_1 + R)}_{>0}] > 0$$

for all $x \in \bar{\Omega}$ if n is large. ■

Exercise 6.25 (E 12.3) Consider the inhomogeneous heat equation

$$\begin{cases} \partial_t u - \Delta_x u = f(x,t) & \text{in } \mathbb{R}^d \times (0,T) \\ u(t=0) = g & \text{in } \mathbb{R}^d \end{cases},$$

$f \in C_1^2(\mathbb{R}^d \times (0,T))$ and compactly supported and $g \in C(\mathbb{R}^d \times [0,T]) \cap L^\infty(\mathbb{R}^d \times [0,T])$. Assume that there exists a solution $u \in C_1^2(\mathbb{R}^d \times (0,T)) \cap C(\mathbb{R}^d \times [0,T])$ satisfying

$$u(x,t) \leqslant Me^{M|x|^2}, \quad (x,t) \in \mathbb{R}^d \times [0,T].$$

Prove that

$$\max_{(x,t) \in \mathbb{R}^d \times [0,T]} |u(x,t)| \leqslant \|g\|_{L^\infty} + T\|f\|_{L^\infty}.$$

Solution.

Step 1: There is at most one solution u.

Step 2:

$$u(x,t) = \int_{\mathbb{R}^d} \Phi(x-y,t)g(y)\,dy + \int_0^t \int_{\mathbb{R}^d} \Phi(x-y,t-s)f(y,s)\,dy\,ds$$

This implies:

$$\begin{aligned} \|u\|_{L^\infty} &\leqslant \int_{\mathbb{R}^d} \Phi(x-y,t)\|g\|_{L^\infty}\,dy + \int_0^t \int_{\mathbb{R}^d} \Phi(x-y,t-s)\|f\|_{L^\infty}\,dy\,ds \\ \Rightarrow \quad \|u\|_{L^\infty_{x,t}} &\leqslant \int_{\mathbb{R}^d} \Phi(x-y,t)\|g\|_{L^\infty}\,dy + \int_0^T \int_{\mathbb{R}^d} \Phi(x-y,t-s)\|f\|_{L^\infty}\,dy\,ds \\ &= \|g\|_{L^\infty_x} + T\|f\|_{L^\infty_{x,t}} \end{aligned}$$

This is optimal! E.g. $g=0, f=1$, $u(x,t)=u(t)$.

$$\begin{cases} u' = 1 \\ u(0) = 0 \end{cases} \Rightarrow u(t) = t$$

■

Exercise 6.26 (Bonus 11) Denote for all $u \in C^2(\Omega) \cap C(\bar{\Omega})$:

$$Lu(x) = \sum_{i,j=1}^d a_{ij}(x)\partial_i\partial_j u(x)$$

where $a_{ij} \in C(\bar{\Omega})$ s.t. $A(x) = (a_{ij}(x)) \geqslant 1$. Prove that if $\Omega \subseteq \mathbb{R}^d$ is open and bounded, $u \in C_1^2(\bar{\Omega} \times [0,T])$ and

$$\begin{cases} \partial_t u - Lu \leqslant 0 & \text{in } \Omega \times (0,T) \\ u(t=0) = 0 \\ u(x \in \partial\Omega) = 0 \end{cases}$$

Prove that $u(x,t) \leqslant 0$ for all $(x,t) \in \bar{\Omega} \times [0,T]$.

6.3 Backward heat equation

Theorem 6.27 (Instability)
There exist functions $u_\epsilon \in C_1^2(\mathbb{R}^d \times (0,T)) \cap C^1(H^1(\mathbb{R}^d) \times [0,T])$ s.t.

$$\partial_t u_\epsilon - \Delta_x u_\epsilon = 0 \quad \text{in } \mathbb{R}^d \times [0,T]$$

with:

$$\|u_\epsilon(\bullet,T)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon\to 0^+} 0, \quad \|u_\epsilon(\bullet,0)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon\to 0^+} \infty.$$

Proof. Recall by Fourier Transform

$$\begin{aligned} & \partial_t \hat{u}(k,t) + |2\pi k|^2 \hat{u}(k,t) = 0 \\ \Leftrightarrow \quad & \partial_t (e^{|2\pi k|^2 t}\hat{u}(k,t)) = 0 \\ \Rightarrow \quad & e^{|2\pi k|^2 t}\hat{u}(k,t) = u(k,0) \\ \Rightarrow \quad & \hat{u}(k,t) = e^{-t|2\pi k|^2}\hat{u}(k,0) \\ \Rightarrow \quad & \hat{u}(k,0) = e^{T|2\pi k|^2}(k,T). \end{aligned}$$

Now we can take

$$\hat{u}_\epsilon(k,t) = \mathbb{1}\left(|k| \leqslant \frac{1}{\epsilon}\right)\epsilon^{d+1}\,dk$$

Then,

$$\|u(\bullet,T)\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \hat{u}_\epsilon(k,t)\,dk = \lambda^d(\{|k| \leqslant \epsilon^{-1}\})\epsilon^{d+1} \sim \epsilon \xrightarrow{\epsilon\to 0} 0$$

$$\begin{aligned}\|u(\bullet,0)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} e^{2T|2\pi k|^2}\mathbb{1}(|k| \leqslant \epsilon^{-1})\epsilon^{d+1}\,dk \\ &\geqslant \int_{\frac{\epsilon}{2}\leqslant|k|\leqslant\frac{\epsilon}{2}} e^{2T|2\pi k|^2}\mathbb{1}(|k| \leqslant \epsilon^{-1})\epsilon^{d+1}\,dk \tilde{\gtrsim} e^{2T\epsilon^{-2}}\epsilon \xrightarrow{\epsilon\to 0} \infty \qquad \blacksquare\end{aligned}$$

Remark 6.28 This means that a small error of the data at $t = T$ may cause a large error of the output $t = 0$.

Theorem 6.29 (Regularized solution)
Assume that $u \in C_1^2(\mathbb{R}^d \times (0,T)) \cap C^1(H^1(\mathbb{R}^d), [0,T])$

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,T) = g(x) & \text{in } \mathbb{R}^d \end{cases}$$

Then from given data $g_\epsilon \in L^2(\mathbb{R}^d)$ s.t.

$$\|g_\epsilon - g\|_{L^2(\mathbb{R}^d)} \leqslant \epsilon$$

we construct a solution $\tilde{u}_\epsilon$ s.t.

$$\sup_{t\in[0,T]} \|\tilde{u}_\epsilon(\bullet,t) - u(\bullet,t)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon\to 0} 0$$

Proof. Clearly we should not choose $\tilde{u}_\epsilon$ to solve

$$\begin{cases} \partial_t u_\epsilon - \Delta_x u_\epsilon = 0 \\ u_\epsilon(t = T) = g_\epsilon \end{cases},$$

i.e.

$$\hat{u}_\epsilon(k,t) = e^{(T-t)|2\pi k|^2}\hat{g}_\epsilon(k).$$

Rather we take

$$\hat{u}_\epsilon(k,t) = e^{(T-t)|2\pi k|^2}\hat{g}_\epsilon(k)\mathbb{1}(|k| \leqslant \delta_\epsilon^{-1})$$

Where $\delta_\epsilon \to 0$ (chosen later). Then we have for all $t \in [0,T]$:

$$\begin{aligned}\|u_\epsilon(\bullet,t) - u(\bullet,t)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} e^{2(T-t)|2\pi k|^2}|\hat{g}_\epsilon(k)\mathbb{1}(|k| \leqslant \delta_\epsilon^{-1}) - \hat{g}(k)|^2\,dk \\ &\leqslant 2\int_{\mathbb{R}^d} e^{2T|2\pi k|^2}|\hat{g}_\epsilon(k) - \hat{g}(k)|\mathbb{1}(|k| \leqslant \delta_\epsilon^{-1})\,dk \\ &\quad + 2\int_{\mathbb{R}^d} \underbrace{e^{2T|2\pi k|^2}|\hat{g}(k)|^2}_{|\hat{u}(k,0)|^2}\mathbb{1}(|k| > \delta_\epsilon^{-1})\,dk = \text{(I)} + \text{(II)}\end{aligned}$$

We have

$$\text{(I)} \leqslant 2\int_{\mathbb{R}^d} e^{c\delta_\epsilon^{-2}}|\hat{g}_\epsilon(k) - \hat{g}(k)|^2\,dk = 2e^{c\delta_\epsilon^{-2}}\epsilon^{-2} \longrightarrow 0 \qquad \text{if } \delta_\epsilon \gg \frac{1}{\sqrt{|\log\epsilon|}}$$

$$\text{(II)} = 2\int_{\mathbb{R}^d} |\hat{u}(k,0)|^2\mathbb{1}(|k| \geqslant \delta_\epsilon^{-1})\,dk \leqslant 2\int_{\mathbb{R}^d} |k|^2\delta_\epsilon^2|\hat{u}(k,0)|^2\,dk$$

Thus chosing $\frac{1}{\sqrt{|\log\epsilon|}} \ll \delta_\epsilon \ll 1$, e.g. $\delta_\epsilon = (|\log\epsilon|)^{-\frac{1}{4}}$.

$$\sup_{t\in[0,T]} \|u_\epsilon(\bullet,t) - u(\bullet,t)\|_{L^2(\mathbb{R}^d)} \leqslant \text{(I)} + \text{(II)} \xrightarrow{\epsilon\to 0} 0 \qquad \blacksquare$$

Remark 6.30 In application, both u and g are unknown. Only g_ϵ is given. So we have to construct $\tilde{u}_\epsilon$ using only information from g_ϵ.

Chapter 7

Wave Equation

7.1 d'Alembert

Wave equation:

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & x \in \mathbb{R}^d, t > 0 \\ u = g, \partial_t u = h & x \in \mathbb{R}^d, t = 0 \end{cases}$$

In $d = 1$:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & (x,t) \in \mathbb{R} \times (0, \infty) \\ u = g, \partial_t u = h, & x \in \mathbb{R}, t = 0 \end{cases}$$

Key idea: Factorization:

$$\partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x).$$

Then, if we denote $v = (\partial_t - \partial_x)u$, we get the transport equation

$$(\partial_t + \partial_x)v = 0.$$

This implies

$$v(x,t) = a(x-t), \quad a(x) = v(x,0)$$

From this we get the inhomogeneous transport equation

$$(\partial_t - \partial_x)u = a(x-t).$$

Now we decompose $u = u_1 + u_2$, where

$$\begin{cases} (\partial_t - \partial_x)u_1 = 0 \\ (\partial_t - \partial_x)u_2 = a(x-t) \end{cases}.$$

Like above, we get $u_1 = b(x+t)$ and an explicit choice of u_2 is

$$u_2(x,t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy$$

Thus,

$$u(x,t) = b(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy$$

Let's compute a and b:

$$\begin{aligned} b(x) &= u(x,0) = g(x) \\ a(x) &= v(x,0) = (\partial_t u - \partial_x u)_{t=0} = h - g'. \end{aligned}$$

Exercise 7.1 (E 13.1, d'Alembert formula) For $d = 1$ let $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and define u by the *d'Alembert formula*

$$\begin{aligned}u(x,t) &= \int_{x-t}^{x+t} (h(y) - g'(y))\, dy + g(x+t) \\ &= \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy.\end{aligned}$$

Then:

- $u \in C^2(\mathbb{R} \times (0, \infty))$
- $\partial_t^2 u - \partial_x^2 u = 0$
- $u = g, \partial_t u = h$ when $t \to 0$

Solution. We can compute the derivative by regarding $\int_x^{x+t} + \int_{x-t}^{x}$ and $\int_t^{x+t} + \int_{x-t}^{t}$:

$$\begin{aligned}\partial_t u &= \frac{1}{2}\left(g'(x+t) - g'(x-t)\right) + \frac{1}{2}\left(h(x+t) + h(x-t)\right) \\ \partial_t^2 u &= \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \frac{1}{2}\left(h'(x+t) - h'(x-t)\right) \\ \partial_x u &= \frac{1}{2}\left(g'(x+t) + g'(x-t)\right) + \frac{1}{2}\left(h(x+t) - h(x-t)\right) \\ \partial_x^2 u &= \frac{1}{2}\left(g''(x+t) + g''(x-t)\right) + \frac{1}{2}\left(h'(x+t) - h''(x-t)\right) = \partial_t^2 u\end{aligned}$$

Now,

$$\begin{aligned}\lim_{t\to 0} u &= \frac{1}{2}\left(g(x) + g(x)\right) + \frac{1}{2}\int_x^x h = g(x) \\ \lim_{t\to 0} \partial_t u &= \frac{1}{2}\left(g'(x) - g'(x)\right) + \frac{1}{2}\left(h(x) + h(x)\right) = h(x).\end{aligned}$$

■

Remark 7.2 If $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$ (but not better).

Now, let's apply the *Reflection Method*. Replace $\mathbb{R}$ by $\mathbb{R}_+ = (0, \infty)$ and assume

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \mathbb{R}_+ \times (0,\infty) \\ u = g, \partial_t u = h & \text{on } \mathbb{R}_+ \times \{t = 0\}, g(0) = h(0) = 0 \\ u = 0 & \text{on } \{x = 0\} \times \{t > 0\} \end{cases}$$

Define

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & x \geqslant 0, t \geqslant 0 \\ -u(-x,t), & x \leqslant 0, t \geqslant 0 \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & x \geqslant 0 \\ -g(-x) & x \leqslant 0 \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x) & x \geqslant 0 \\ -h(-x) & h \leqslant 0 \end{cases}$$

Then

$$\begin{cases} \partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{u} = \tilde{g}, \partial_t \tilde{u} = \tilde{h} & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}.$$

By d'Alembert formula

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy$$

This imples

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & x \geqslant t \geqslant 0 \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy & t \geqslant x \geqslant 0 \end{cases}.$$

This is the solution of the heat equation in $\mathbb{R}_+ \times (0,\infty)$.

7.2 Euler-Poisson-Darboux

$$(\star) \qquad \begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ u = g, \partial_t u = h & \mathbb{R}^d \times \{t=0\} \end{cases}$$

Idea: Averaging of u over sphere $\rightsquigarrow$ 1D problem. Define for $x \in \mathbb{R}^d$, $t > 0$, $r > 0$,

$$U_r(x,t) := \fint_{\partial B(x,r)} u(y,t)\,dS(y)$$
$$G_r(x) := \fint_{\partial B(x,r)} g(y)\,dS(y)$$
$$H_r(x) := \fint_{\partial B(x,r)} h(y)\,dS(y)$$

Lemma 7.3 (Euler-Poisson-Darboux equation) If $u \in C^2(\mathbb{R}^d \times [0,\infty))$ solves $(\star)$, then for all $x \in \mathbb{R}^d$:

- $(r,t) \mapsto U \in C^2([0,\infty) \times [0,\infty)])$
- $\begin{cases} \partial_t^2 U - \partial_r^2 U - \frac{d-1}{r}\partial_r U = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\ U = G, \partial_t U = H & \text{on } \mathbb{R}_+ \times \{t=0\} \end{cases}$

Note that $\partial_r^2 + \frac{d-1}{r}\partial_r$ is the radial part of Δ.

Proof. We compute for $r > 0$:

$$\begin{aligned}
\partial_r U_r(x,t) &= \partial_r \fint_{\partial B(x,r)} u(y,t)\,dS(y) \\
&= \partial_r \fint_{\partial B(0,1)} u(x+rz,t)\,dS(z) \\
&= \fint_{\partial B(0,1)} \nabla u(x+rz,t) z\,dS(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y,t)\frac{y-x}{r}dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u(y,t)}{\partial \vec{n}}\,dS(y) \\
\text{(Green 2.3)} \qquad &= \frac{1}{|\partial B(0,r)|}\int_{B(x,r)} \Delta_x u(y,t)\,dy \\
\left(|B(0,r)| = \frac{r}{d}|\partial B(0,r)|\right) \qquad &= \frac{r}{d}\fint_{B(x,r)} \Delta_x u(y,t)\,dy
\end{aligned}$$

(The computation is similar to the proof of the mean-value theorem for the Poisson equation.) We compute the second derivative

$$\begin{aligned}
\partial_r^2 U_r(x,t) &= \partial_r \left[\frac{r}{d} \fint_{B(x,r)} \Delta_x u(y,t)\, dy \right] \\
\big(|B(0,r)| = r^d |B(0,1)|\big) \qquad &= \partial_r \left[\frac{1}{d|B_1| r^{d-1}} \int_{B(x,r)} \Delta_x u(y,t)\, dy \right]
\end{aligned}$$

Now, $\partial_r \frac{1}{d|B_1|r^{d-1}} = \frac{-d+1}{d|B_1|r^d} = -\frac{d-1}{d|B(0,r)|}$ and

$$\begin{aligned}
\partial_r \int_{B(x,r)} \Delta_x u(y,t)\, dS(y) &= \partial_r \int_{B(0,r)} \Delta_x u(x+ry,t)\, dy \\
\text{(Green 2.3)} \qquad &= \partial_r \int_{\partial B(0,1)} \nabla_x u(x+ry,t) \frac{y}{|y|}\, dS(y) \\
(|y| = 1) \qquad &= \int_{\partial B(0,1)} \partial_r \nabla_x u(x+ry,t) y\, dS(y) \\
&= \int_{\partial B(0,1)} \Delta_x u(x+ry,t) y \cdot y\, dS(y) \\
(y \cdot y = |y|^2 = 1) \qquad &= \int_{\partial B(0,1)} \Delta_x u(x+ry,t)\, dS(y) \\
&= \int_{\partial B(x,r)} \Delta_x u(y,t)\, dS(y)
\end{aligned}$$

Now with the product rule we get

$$\begin{aligned}
\partial_r^2 U_r(x,t) = &- \left(\frac{d-1}{d}\right) \fint_{B(x,r)} \Delta_x u(y,t)\, dy \\
&+ \frac{1}{d|B_1|r^{d-1}} \int_{\partial B(x,r)} \Delta_x u(y,t)\, dS(y) \\
= &- \left(\frac{d-1}{d}\right) \fint_{B(x,r)} \Delta_x u(y,t)\, dy \\
&+ \fint_{\partial B(x,r)} \Delta_x u\, dS(y)
\end{aligned}$$

And, since u is a solution,

$$\partial_t^2 U = \partial_t^2 \fint_{\partial B(x,r)} u\, dS(y) = \fint_{\partial B(x,r)} (\partial_t^2 u)\, dS(y) = \fint_{\partial B(x,r)} (\Delta_x u)\, dS(y).$$

So we can conclude

$$\partial_t^2 U - \partial_r^2 U - \frac{d-1}{d} U = 0$$

the above computation also shows that $U \in C^2(\mathbb{R}_+ \times [0,\infty))$. Moreover

$$\begin{aligned}
\partial_r U_r(x,t) &\xrightarrow{r \to 0^+} 0 \\
\partial_r^2 U_r(x,t) &\xrightarrow{r \to 0^+} \left(\frac{1}{d} - 1\right) \Delta_x u + \Delta_x u = \frac{1}{d} \Delta_x u
\end{aligned}$$

This imples that $U \in C^2([0,\infty) \times [0,\infty))$. Finally, when $t = 0$,

$$\begin{cases} u = g \\ \partial_t = h \end{cases} \Rightarrow \begin{cases} U = G \\ \partial_t U = H \end{cases} .$$

■

We showed that it is a neccessary condition for u to solve the Euler-Poisson-Darboux equation. Now we try to actually solve the equation. In general, this is easier for odd d than for even d. We will consider the cases $d = 2, 3$.

7.2.1 Solution in three dimensions

Now, for $r > 0$ let $\tilde{U} = rU, \tilde{G} = rG, \tilde{H} = rH$. Then

$$\begin{cases} \partial_t^2 \tilde{U} - \partial_r^2 \tilde{U} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \tilde{U} = \tilde{G}, \partial_t \tilde{U} = \tilde{H} & \text{when } t = 0 \\ \tilde{U} = 0 & \text{when } r = 0 \end{cases}$$

Then, by d'Alembert's formula, for $0 \leqslant r \leqslant t$ we have

$$\tilde{U}_r(x,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\,dy.$$

$$\Rightarrow \quad U_r(x,t) = \frac{1}{2}\left[\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{r}\right] + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy$$

Now, taking $r \to 0$ we get

$$\begin{aligned} u(x,t) &= \tilde{G}'(t) + \tilde{H}(t) \\ &= \partial_t\left(t \fint_{\partial B(x,t)} g(y)\,dS(y)\right) + t\fint_{\partial B(x,t)} h(y)\,dS(y) \end{aligned}$$

Using

$$\fint_{\partial B(x,t)} g(y)\,dS(y) = \fint_{\partial B(0,1)} g(x+tz)\,dS(z)$$

we get

$$\begin{aligned} \partial_t \fint_{\partial B(x,t)} g(y)\,dS(y) &= \fint_{\partial B(0,1)} \nabla g(x+tz) z\,dz \\ &= \fint_{\partial B(x,t)} \nabla g(y)\left(\frac{y-x}{t}\right)\,dS(y) \\ \Rightarrow \quad \partial_t\left(t\fint_{\partial B(x,t)} g(y)\,dS(y)\right) &= \fint_{\partial B(x,t)} \left(g + \nabla g(y-x)\right)\,dS(y) \end{aligned}$$

From that we get:

Remark 7.4 (Kirchhoff's formula in 3D) For all $x \in \mathbb{R}^3$, $t > 0$:

$$u(x,t) = \fint_{\partial B(x,t)} \left(g(y) + \nabla g(y-x) + th(y)\right)\,dS(y)$$

Exercise 7.5 (E 13.3) Let $g \in C_c^2(\mathbb{R}^3)$, $h \in C_c^1(\mathbb{R}^3)$. Assume that $u \in C^2(\mathbb{R}^3 \times [0,\infty])$ satisfies the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \forall x \in \mathbb{R}^3, t > 0 \\ u(x,0) = g(x), \partial_t u(x,0) = h(x) & \forall x \in \mathbb{R}^3 \end{cases}$$

Prove that there exists a constant $C > 0$ such that

$$|u(x,t)| \leqslant \frac{C}{t}, \quad \forall x \in \mathbb{R}^3, t > 0.$$

Solution.

Step 1: Assume

$$(\star) \quad u(x,t) = ⨍_{\partial B_{\mathbb{R}^3}} (g(y) + \nabla g(y)(y-x) + th(y))\, dS(y)$$

Assume $\operatorname{supp} g, h \subseteq B(0,R)$. Then

$$\begin{aligned} |u(x,t)| &\leqslant \frac{1}{|\partial B(x,t)|} \int_{\partial B_{\mathbb{R}^3}(x,t)} (\|g\|_{L^\infty} + \|\nabla g\|_{L^\infty} t + t\|h\|_{L^\infty})\, \mathbb{1}(|y| \leqslant R)\, dS(y) \\ &= \frac{C(1+t)}{t^2} \leqslant \frac{C}{t} \quad \text{as } t \geqslant 1 \end{aligned}$$

and

$$|u(x,t)| \leqslant C ⨍_{\partial B(x,t)} \leqslant C$$

if $t \leqslant 1$.

Step 2: Why is u given by $(\star)$? This follows from the uniqueness of the solution of the wave equation. In fact, if g, h are compactly supported, then for all $t \in [0,T]$, $u(x,t)$ supported in $B(0,R_T)$ a finite ball. ■

7.2.2 Solution in two dimensions

The transformation $\tilde{U} = rU$ does not work! The idea is to think of the 2D problem as 3D problem with x_3 hidden. We write $\bar{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$. Then we get

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_x \bar{u} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{on } \mathbb{R}^3 \times \{t=0\} \end{cases}$$

With Kirchhoff's formula:

$$u(x,t) = \bar{u}(\bar{x},t) = \partial_t \left(t ⨍_{\partial \bar{B}(\bar{x},t)} \bar{g}(y) d\bar{S}(y) \right) + t \int_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\bar{S}(y)$$

Remark 7.6 Let $\gamma(y) = (t^2 - |y-x|^2)^{\frac{1}{2}}, y \in B(x,t)$, then

$$\begin{aligned} ⨍_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} &= \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S}(y) \\ &= \frac{1}{4\pi t^2} \int_{B(x,t)} g(y) 2(1 + |\nabla \gamma|^2)^{\frac{1}{2}}\, dy \\ &= \frac{1}{4\pi t^2} \int_{B(x,r)} g(y) \frac{2t}{\sqrt{t^2 - |y-x|^2}}\, dy \\ &= \frac{t}{2} ⨍_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}}\, dy \end{aligned}$$

Similarly:

$$⨍_{\partial B(\bar{x},t)} \bar{h}\, d\bar{S}(y) = \frac{t}{2} ⨍_{B(x,t)} \frac{h(y)}{\sqrt{t^2 - |y-x|^2}}\, dy$$

This implies:

$$\begin{aligned}
u(x,t) &= \partial_t \left(\frac{t^2}{2} \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy \right) + \frac{t^2}{2} \fint_{B(x,r)} \frac{h(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy \\
&= (I) + (II) \\
(I) &= \partial_t \left(\frac{1}{2} t \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{\frac{1}{2}}}\,dz \right) \\
&= \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{\frac{1}{2}}}\,dz + t \fint_{B(0,1)} \frac{\nabla g(x+tz) z}{(1-|z|^2)^{\frac{1}{2}}}\,dz \\
&= t \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy + t \fint_{B(x,r)} \frac{\nabla g(y)(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy
\end{aligned}$$

Exercise 7.7 (E 13.2, Poisson formula for 2D) For $x \in \mathbb{R}^2$, $t > 0$, $g \in C^2(\mathbb{R}^2)$, $h \in C^2(\mathbb{R}^2)$ let

$$u(x,t) = \frac{t}{2} \fint_{B(x,t)} \frac{g(y) + \nabla g(y)(y-x) + th(y)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy.$$

Prove that $u \in C^2(\mathbb{R}^2 \times (0,\infty))$ and

$$\begin{cases} \partial_t^2 - \Delta_x u = 0 & \forall x \in \mathbb{R}^2, t > 0 \\ \lim_{t\to 0^+} u(x,t) = g(x), \lim_{t\to 0^+} \partial_t u(x,t) = h(x), & \forall x \in \mathbb{R}^2 \end{cases}$$

Solution. Let $\bar{x} = (x, x_3) \in \mathbb{R}^3$, $\bar{u}(\bar{x},t) = u(x,t)$. We claim that

$$(\star) \quad \bar{u}(\bar{x},t) = \partial_t \left(t \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S}(y) \right) + t \fint_{\partial \bar{B}(x,t)} \bar{h}\, d\bar{S}$$

where $\bar{B}(\bar{x},t)$ is a ball in 3D, $\bar{g}(\bar{x}) = g(x)$, $\bar{h}(\bar{x}) = h(x)$. From 7.6 we have

$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{t}{2} \fint_{B_{\mathbb{R}^2}(x,t)} \frac{g(y)}{\sqrt{t^2-|x-y|^2}}\,dy$$

Now,

$$\begin{aligned}
\text{RHS of } (\star) &= \partial_t \left(\frac{t^2}{2} \fint_{B_{\mathbb{R}^2}} \frac{g(y)}{\sqrt{t^2-|x-y|^2}}\,dy \right) + \frac{t^2}{2} \fint_{B_{\mathbb{R}^2}(x,t)} h(y)\,dy \\
?? \quad &= \partial_t \frac{t}{2} \frac{g(y) + \nabla g(y)(y-x)}{\sqrt{t^2-|x-y|^2}} + \frac{t^2}{2} \fint_{B_{\mathbb{R}^2}(x,t)} h(y)\,dy
\end{aligned}$$

3D-Problem: We claim that if $\bar{g} \in C^3(\mathbb{R}^3)$, $\bar{h} \in C^2(\mathbb{R}^3)$ and

$$\bar{u}(\bar{x},t) = \partial_t \left(t \fint_{\partial B_{\mathbb{R}^2}(\bar{x},t)} \bar{g} \right) + t \int_{\partial B_{\mathbb{R}^2}(\bar{x},t)} \bar{h},$$

then $\bar{u} \in C^2(\mathbb{R}^3 \times (0,\infty))$ and

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_{\bar{x}} u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Proof:

Step 1: Assume $\bar{g} = 0$, $\bar{u}(\bar{x},t) = t \fint_{\partial B(\bar{x},t)} \bar{h}$. Lemma (from lecture):

$$\partial_r \int_{\partial B(\bar{x},r)} \bar{h} = \frac{r}{d} \fint_{B(\bar{x},r)} \Delta \bar{h}, \quad d = 3$$

($\bar{U} = \fint_{B(\bar{x},r)} u(x,t)$) Hence:

$$\partial_t \bar{u}(x,t) = \partial_t \left(t \fint_{\partial B(\bar{x},t)} \bar{h} \right) = \fint_{\partial B(\bar{x},t)} \bar{h} + t \frac{t}{3} \fint_{B(x,t)} \Delta \bar{h}$$

$$\partial_t^2 \bar{u}(x,t) = \frac{t}{3} \fint_{B(\bar{,}t)} \Delta \bar{h} + \frac{2}{3} t \fint_{B(x,t)} \Delta \bar{h} + \frac{t^2}{3} \fint_{B(\bar{x},t)} \Delta \bar{h}$$

Where we have

$$\frac{t^2}{3} \fint_{B(\bar{x},t)} \Delta \bar{h} = \frac{1}{|B_1| 3t} \int_{B(\bar{x},t)} \Delta \bar{h}$$

$$\partial_t(\dots) = -\frac{1}{|B_1| 3t^2} \int_{B(\bar{x},t)} \Delta \bar{h} + \frac{1}{|B_1| 3t} \int_{\partial B(\bar{x},t)} \Delta \bar{h}$$

$$= -\frac{t}{3} \fint_{B(\bar{x},t)} \Delta \bar{h} + t \fint_{\partial B(\bar{x},t)} \Delta \bar{h}$$

So we get

$$\partial_t^2 \bar{u}(x,t) = \frac{t}{3} \fint_{B(\bar{,}t)} \Delta \bar{h} + \frac{2}{3} t \fint_{B(x,t)} \Delta \bar{h} - \frac{t}{3} \fint_{B(\bar{x},t)} \Delta \bar{h} + t \fint_{\partial B(\bar{x},t)} \Delta \bar{h}$$

$$= t \fint_{\partial B(\bar{x},t)} \Delta \bar{h}$$

This implies

$$\Delta_{\bar{x}} \bar{u}(x,t) = \Delta_{\bar{x}} \left(t \fint_{\partial B(\bar{x},t)} \bar{h} \right) = \Delta_x \left(t \fint_{\partial B(0,t)} \bar{h}(\bar{x}+y)\, dS(y) \right)$$

$$= t \fint_{\partial B(0,t)} \Delta \bar{h}(\bar{x}+y)\, dS(y) = t \fint_{\partial B(\bar{x},t)} \Delta \bar{h} = \partial_t^2 \bar{u}$$

Thus, $\partial_t \bar{u} - \Delta_{\bar{x}} \bar{u} = 0$ in $\mathbb{R}^3 \times (0,\infty)$. Moreover,

$$\bar{u}(x,t) = t \fint_{\partial B(\bar{x},t)} \bar{h} \xrightarrow{t \to 0} 0$$

and

$$\partial_t \bar{u}(x,t) = \fint_{\partial B(\bar{x},t)} \bar{h} + \frac{t^2}{3} \fint_{B(x,t)} \Delta \bar{h} \xrightarrow{t \to 0} \bar{h}(\bar{x}) \quad \checkmark$$

Step 2: General Case: $\bar{g} \neq 0$. Assume $h = 0$, then

$$\bar{u}(\bar{x},t) = \partial_t \left(t \fint_{\partial B_{\mathbb{R}^3}} \bar{g} \right)$$

By Step 1, $v(\bar{x},t) := t \fint_{\partial B_{\mathbb{R}^3}(0,\infty)} \bar{g}$ satisfies

$$\begin{cases} \partial_t^2 - \Delta_{\bar{x}} v = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ v = 0 \text{ and } \partial_t v = \bar{g} & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Then $\bar{u} = \partial_t \bar{v}$. This implies

$$\partial_t^2 \bar{u} - \Delta_{\bar{x}} \bar{u} = \partial_t^3 \bar{v} - \Delta_{\bar{x}} \partial_t \bar{v} = \partial_t(\underbrace{\partial_t^2 \bar{v} - \Delta_{\bar{x}} \bar{v}}_{=0}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty)$$

and

$$\bar{u}(t=0) = \partial_t \bar{v}(t=0) = \bar{g},$$
$$\partial_t \bar{u}(t=0) = \partial_t^2 \bar{v}(t=0) = t \fint_{\partial B(\bar{x},t)} \Delta \bar{g} \bigg|_{t=0} = 0$$

Step 3: Consider the case $g \neq 0$, $h \neq 0$:

$$\bar{u} = \underbrace{\partial_t \left(t \fint_{\partial B(\bar{x},t)} \bar{g} \right)}_{\bar{u}_1} + \underbrace{t \fint_{\partial B(\bar{x},t)} h}_{\bar{u}_2}$$

Now we have

$$\begin{cases} \partial_t^2 \bar{u}_1 - \Delta_{\bar{x}} \bar{u}_1 = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u}_1 = \bar{g}, \partial_t \bar{u}_1 = 0 & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

and

$$\begin{cases} \partial_t^2 \bar{u}_2 - \Delta_{\bar{x}} \bar{u}_2 = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u}_2 = 0, \partial_t \bar{u}_2 = h & \text{in } \mathbb{R}^3 \times (0, \infty) \end{cases}.$$

This implies

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_{\bar{x}} \bar{u} = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{in } \mathbb{R}^3 \times (0, \infty) \end{cases}.$$

■

7.3 Spectral Method

Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded.

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \Omega \times (0, \infty) \\ u = g, \partial_t u = h & \text{when } t = 0 \\ u = 0 & \text{when } x \in \partial\Omega \end{cases}$$

$-\Delta$ has eigenvectors $(e_i)_{i=1}^{\infty}$ with eigenvalues $(\lambda_i)_{i=1}^{\infty}$, i.e.

$$\begin{cases} -\Delta e_i = \lambda_i e_i \\ e_i|_{\partial\Omega} = 0 \end{cases}$$

s.t. $\lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_i \to \infty$ and $(e_i)_{i \in \mathbb{N}}$ is an orthonormal basis for $L^2(\Omega)$. We write

$$u(x,t) = \sum_{i \in \mathbb{N}} a_i(t) e_i(x)$$

This implies:

$$a_i''(t) + \lambda_i a_i(t) = 0$$
$$\text{(ODE)} \quad \Rightarrow \quad a_i(t) = a_i(0) \cos(\sqrt{\lambda_i} t) + \frac{a_i'(0)}{\sqrt{\lambda_i}} \sin(\sqrt{\lambda_i} t)$$

Here $a_i(0)$, $a_i'(0)$ is determined by

$$\begin{cases} g = u(t=0) = \sum_{i=1}^{\infty} a_i(0) e_i(x) \\ h = \partial_t u(t=0) = \sum_{i=1}^{\infty} a_i'(0) e_i(x) \end{cases} \Rightarrow \begin{cases} a_i(0) = \langle e_i, g \rangle \\ a_i'(0) = \langle e_i, h \rangle \end{cases}$$

7.4 Uniqueness

For $\Omega \subseteq \mathbb{R}^d$ open and bounded with C^1-boundary regard

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \Omega \times (0,T) \\ u = 0, \partial_t u = 0 & \text{in } \Omega \times \{t=0\} \\ u = 0 & \partial\Omega \times [0,T] \end{cases}.$$

If $u \in C^2(\bar{\Omega} \times [0,T])$, then $u = 0$.

Proof. Let

$$e(t) = \int_\Omega (|\partial_t u|^2 + |\nabla_x u|^2)\, dx$$

has $e'(t) = 0$. This implies $e(t) = e(0) = 0$, so $\partial_t u = 0$ and hence $u = 0$. ∎

The same result holds for $\mathbb{R}^d$, i.e.

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \mathbb{R}^d \times (0,T) \\ u = 0, \partial_t = 0 & \mathbb{R}^d \times \{t=0\} \end{cases}$$

and $u \in C^2(H^2(\mathbb{R}^d), [0,T])$ (i.e. $u(t, \bullet) \in H^2(\mathbb{R}^d)$ and $t \mapsto u(t,\bullet)$ continuous).

7.5 Propagation of the wave

Theorem 7.8 Assume that $u \in C^2(\mathbb{R}^d \times [0,\infty)$ and

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \mathbb{R}^d \times (0,\infty) \\ u = 0, \partial_t u = 0 & B(x_0,t_0) \times \{t=0\} \end{cases}$$

Then $u(x,t) = 0$ for $x \in B(x_0, t_0 - t)$.

Proof. Let for $t \in [0,t_0]$

$$e(t) = \int_{B(x_0,t_0-t)} \left(|\partial_t u|^2 + |\nabla_x u|^2\right)\, dx$$

∎

This implies

$$\begin{aligned} e'(t) &= \int_{B(x_0,t_0-t)} 2(\partial_t u \partial_t^2 u + \nabla_x u \partial_t \nabla_x u) - \int_{\partial B(x_0,t_0-t)} (|\partial_t u|^2 + |\nabla_x u|^2)\, dS \\ &= \int_{B(x_0,t_0-t)} 2(\partial_t u \partial_t^2 u - \Delta_x u \partial_t u) + \int_{\partial B(x_0,t_0-t)} [2(\nabla_x u \vec{n})\partial_t u - |\partial_t u|^2 - |\nabla_x u|^2]\, dS \\ &= \int_{B(x_0,t_0-t)} 2(\partial_t u \underbrace{(\partial_t^2 u - \Delta_x u)}_{=0}) + \int_{\partial B(x_0,t_0-t)} \underbrace{[2(\nabla_x u \vec{n})\partial_t u - |\partial_t u|^2 - |\nabla_x u|^2]}_{\leqslant 0}\, dS \end{aligned}$$

where we used

$$|2(\nabla_x u \vec{n})\partial_t u| \leqslant 2|\nabla_x u||\partial_t u| \leqslant |\nabla_x u|^2 + |\partial_t u|^2.$$

Thus $e'(t) \leqslant 0$ for all $t \in (0,t_0)$, so $e(t) \leqslant e(0) = 0$ (as $u = 0$, $\partial_t u = 0$ in $B(x_0,t_0) \times \{t=0\}$). This implies $e(t) = 0$ for all $t \in (0,t_0)$, so

$$\begin{cases} \partial_t u = 0 & x \in B(x_0, t_0 - t) \\ u = 0 & x \in B(x_0, t_0 - t) \times \{t = 0\} \end{cases}.$$

So we get that $u = 0$ for $x \in B(x_0, t_0 - t)$ for all $t \in (0, t_0)$ and $u = 0$ for $x \in B(x_0, t_0 - t)$ for all $t \in [0, t_0]$. (More precisely $u(x_0, t_0) = 0$)

7.6 Wave vs. Heat Equation

Notation 7.9 For $g \in L^2(\mathbb{R}^d)$, $t > 0$ we write $e^{t\Delta} g \in L^2(\mathbb{R}^d)$ for

$$\widehat{e^{t\Delta}} g(k) = e^{-t|2\pi k|^2} \hat{g}(k)$$

$$\Leftrightarrow \quad (e^{t\Delta} g)(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

$$\widehat{\frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}} = e^{-t|2\pi k|^2}$$

Heat Equation:

- Improves smoothness, *i.e.* $g \in L^2(\mathbb{R})$ imples $e^{t\Delta} g \in \bigcap_{m \geqslant 1} H^m(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$ for all $t > 0$.
- Propagation: Speed is ∞.

$t = 0$: $g \in C_c^\infty$

$t > 0$: $e^{t\Delta} g$ does not have compact support

Wave Equation:

- No improvement of smoothnes
- Propagation: Speed is finite

Chapter 8

Schrödinger Equation

$$\begin{cases} -i\partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ u(x,0) = g(x) & x \in \mathbb{R}^d \end{cases}$$

Formally $-i\partial_t = \frac{1}{i}\frac{d}{dt} = \frac{d}{d(it)} \Rightarrow \partial_\xi u - \Delta_x u = 0$, $\xi = it \rightsquigarrow$ Heat equation with *imaginary time*. From the heat equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(x,0) = g(x) \end{cases} \Rightarrow u(x,t) = (e^{t\Delta} g)(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y)$$

$$\rightsquigarrow \begin{cases} -i\partial_t u - \Delta_x u = 0 \\ u(x,0) = g(x) \end{cases} \Rightarrow u(x,t) = (e^{it\Delta} g)(x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y)\, dy$$

if $g \in L^1$.

Theorem 8.1 (Fourier for Schrödinger Solution) For $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, define

$$(e^{it\Delta} g)(x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y)\, dy$$

Then $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)}$. Consequently, for all $g \in L^2(\mathbb{R}^d)$ we can define $e^{it\Delta} g \in L^2(\mathbb{R}^d)$ by a density argument. Moreover,

$$\widehat{e^{it\Delta} g(k)} = e^{-it|2\pi k|^2} \hat{g}(k)$$

for almost every $k \in \mathbb{R}^d$.

Proof. Fourier transform of Gaussian:

$$\frac{1}{(4\pi t)^{\frac{d}{2}}} \widehat{e^{-\frac{|x|^2}{4t}}} = e^{-t|2\pi k|^2}, \quad t > 0$$

Key point: This formula also holds if $t \in \mathbb{C}$ and $\Re(t) > 0$. For all $\epsilon > 0$ consider

$$(e^{(it+\epsilon)\Delta} g)(x) = \left(e^{-(it+\epsilon)|2\pi k|^2} \hat{g}(k) \right)^{\vee} (x) = (\hat{G}_\epsilon g)^{\vee} = (G_\epsilon \star g)(x)$$

where

$$G_\epsilon(x) = \frac{1}{(4\pi(it+\epsilon))^{\frac{d}{2}}} e^{-\frac{|x|^2}{4(it+\epsilon)}}.$$

Since $g \in L^1(\mathbb{R}^d)$:

$$(G_\epsilon \star g)(x) = \frac{1}{(4\pi(it+\epsilon))^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4(it+\epsilon)}} g(y)\, dy$$
$$\longrightarrow \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} g(y)\, dy = (e^{it\Delta} g)(x)$$

for all $x \in \mathbb{R}^d$. Moreover,

$$\|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} = \|\widehat{G_\epsilon \star g}\|_{L^2(\mathbb{R}^d)} = \|\hat{G}_\epsilon \hat{g}\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left|e^{-(it+\epsilon)|2\pi k|^2}\right|^2 |\hat{g}(k)|^2\, dk\right)^{\frac{1}{2}}$$
$$= \left(\int_{\mathbb{R}^d} e^{-\epsilon|2\pi k|^2} |\hat{g}(k)|^2\, dk\right)^{\frac{1}{2}} \xrightarrow{\epsilon \to 0^+} \left(\int_{\mathbb{R}^d} |\hat{g}(k)|^2\right)^{\frac{1}{2}} = \|g\|_{L^2(\mathbb{R}^d)}$$

as $g \in L^2(\mathbb{R}^d)$. Thus, $G_\epsilon \star g \to e^{it\Delta} g$ pointwise. With Fatou:

$$\liminf_{\epsilon \to 0^+} \|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} \geqslant \|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)}$$

Thus, $e^{it\Delta} g \in L^2(\mathbb{R}^d)$ and $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} \leqslant \|g\|_{L^2(\mathbb{R}^d)}$. To get the equality:

$$\widehat{G_\epsilon \star g}(k) = e^{-(it+\epsilon)|2\pi k|^2} \underbrace{\hat{g}(k)}_{L^2} = \underbrace{e^{-\epsilon|2\pi k|^2}}_{\in[0,1]} \left(e^{-it|2\pi k|^2} \hat{g}(k)\right) \xrightarrow{\epsilon \to 0^+} e^{it|2\pi k|^2} \hat{g}(k)$$

in $L^2(\mathbb{R}^d, dk)$. Thus $G_\epsilon \star g$ converges in $L^2(\mathbb{R}^d)$. Then up to a subsequence $\epsilon \to 0^+$, we can assume that

$$(G_\epsilon \star g)(x) \xrightarrow{\epsilon \to 0^+} H(x)$$

almost everywhere. (Dominated convergence) We already proved $G_\epsilon \star g \to e^{it\Delta} g$ pointwise, so $e^{it\Delta} g = H$, i.e. $G_\epsilon \star g \to e^{it\Delta}$ in $L^2(\mathbb{R}^d)$. Conclude

$$\|g\|_{L^2(\mathbb{R}^d)} \xleftarrow{\epsilon \to 0} \|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \to 0^+} \|e^{it\Delta} g\|_{L^2}.$$

This implies $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)}$ for all $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. For $g \in L^2(\mathbb{R}^d)$ there is a sequence $\{g_n\} \subseteq L^1 \cap L^2$ s.t. $g_n \to g$ in $L^2(\mathbb{R}^d)$. Then

$$\|e^{it\Delta} g_n - e^{it\Delta} g_m\|_{L^2} = \|e^{it\Delta} \underbrace{(g_n - g_m)}_{L^1 \cap L^2} = \|g_n - g_m\|_{L^2} \to 0$$

as $m, n \to \infty$. So $e^{it\Delta} g_n$ is a Cauchy sequence in L^2, so it has a limit which we define as $e^{it\Delta} g$. Why the limit $e^{it\Delta} g$ is independent of the choice of g_n: If we have 2 different sequences $\{g_n\}, \{\tilde{g}_n\} \subseteq L^1 \cap L^2$, then $g_n, \tilde{g}_n \to g$ in $L^2(\mathbb{R}^d)$. Then

$$\|e^{it\Delta} g_n - e^{it\Delta} \tilde{g}_n\|_{L^2(\mathbb{R}^d)} = \|e^{it\Delta} \underbrace{(g_n - \tilde{g}_n)}_{L^1 \cap L^2}\|_{L^2} = \|g_n - \tilde{g}_n\|_{L^2} \xrightarrow{n \to \infty} 0$$

$\rightsquigarrow \lim_{n\to\infty} e^{it\Delta} g_n = \lim_{n\to\infty} e^{it\Delta} \tilde{g}_n$. Finally, we have $g \in L^2(\mathbb{R}^d), g_n \in L^1 \cap L^2 \to g$ in L^2.

$$\widehat{e^{it\Delta}} g(k) \xleftarrow{n \to \infty} \widehat{e^{it\Delta} g_n}(k) = e^{-it|2\pi k|^2} \hat{g}_n(k) \xrightarrow{n \to \infty} e^{-it|2\pi k|^2} \hat{g}(k)$$

in $L^2(\mathbb{R}^d)$. ■

Theorem 8.2 (Long term behaviour)

1. If $g \in L^1(\mathbb{R}^d)$, then $(e^{it\Delta}g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y)\, dy$. This implies

$$\|e^{it\Delta}g\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{1}{(4\pi t)^{\frac{d}{2}}} \|g\|_{L^1(\mathbb{R}^d)} \xrightarrow{t\to\infty} 0.$$

   On the other hand, for all $g \in L^2(\mathbb{R}^d)$, $\|e^{it\Delta}g\|_{L^2} = \|g\|_{L^2}$.
   Exercise 8.3: If $g \in L^1 \cap L^2$ then $\|e^{it\Delta}g\|_{L^p} \to 0$ for all $2 < p \leqslant \infty$.

2. For all bounded sets Ω, for all $g \in L^2(\mathbb{R}^d)$ we have $\|\mathbb{1}_\Omega e^{it\Delta}g\|_{L^2(\mathbb{R}^d)} \to 0$ as $t \to \infty$. Equivalently, for all $R > 0$:

$$\int_{|x|\leqslant R} |(e^{it\Delta}g)(x)|^2\, dx \xrightarrow{t\to\infty} 0$$

   (RAGE theorem)

Proof. 1. With the Fourier transform of $e^{it\Delta g}$ (Theorem 8.1) we get

$$\|e^{it\Delta}g - g\|_{L^2}^2 = \|\widehat{e^{it\Delta}g} - \hat{g}\|_{L^2}^2 = \int_{\mathbb{R}^d} \underbrace{(e^{-it|2\pi k|^2} - 1)}_{\substack{\xrightarrow{t\to 0} 0 \\ \text{bd. by } 2\forall t}} | \underbrace{\hat{g}(k)}_{\in L^2} |^2\, dk \xrightarrow{t\to 0} 0$$

by dominated convergence.

2. Decompose $g = g_1 + g_2$, then $e^{it\Delta}g = e^{it\Delta}g_1 + e^{it\Delta}g_2$. Take $g_1 \in L^1$, then by 1. we have

$$\|e^{it\Delta}g_1\|_{L^\infty} \leqslant \frac{\|g_1\|_{L1}}{|4\pi t|^{\frac{d}{2}}} \xrightarrow{t\to\infty} 0$$

$$\Rightarrow \int_{|x|\leqslant R} |(e^{it\Delta}g_1)(x)|^2 \leqslant \frac{CR^d\|g_1\|_{L^1}}{|4\pi t|^{\frac{d}{2}}} \xrightarrow{t\to\infty} 0$$

On the other hand, with a good choice of g_2:

$$\|e^{it\Delta}g_2\|_{L^2(\mathbb{R}^d)} = \|g_2\|_{L^2(\mathbb{R}^d)} \to 0$$

$$\Rightarrow \quad \int_{|x|\leqslant R} |(e^{it\Delta}g_2)(x)|^2\, dx \leqslant \|e^{it\Delta}g_2\|_{L^2(\mathbb{R}^d)}^2 \to 0$$

Conclusion:

$$\int_{|x|\leqslant R} |(e^{it\Delta}g)(x)|^2\, dx \leqslant 2\int_{|x|<R} |(e^{it\Delta}g_1)(x)|^2 + 2\int_{|x|\leqslant R} |e^{it\Delta}g_2(x)|^2 \xrightarrow{t\to\infty} 0$$

■

Exercise 8.3 (E 13.4) Let $g \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Let $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ be the solution of the Schrödinger equation with the initial data g, namely

$$u(x,t) = (e^{it\Delta}g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|}{4t}} g(y)\, dy.$$

Prove that for all $2 \leqslant p \leqslant \infty$ we have

$$\lim_{t\to\infty} \|u(\bullet, t)\|_{L^p(\mathbb{R}^d)} = 0.$$

Solution. We already know:

$$\|e^{it\Delta}g\|_{L^\infty} \leqslant \frac{\|g\|_{L^1}}{(4\pi t)^{\frac{d}{2}}} \xrightarrow{t\to 0} 0$$
$$\|e^{it\Delta}g\|_{L^2} = \|g\|_{L^2}$$

Now, interpolation for all $2 < p < \infty$ gives

$$\|e^{it\Delta}g\|_{L^p} \leqslant \|e^{it\Delta}g\|_{L^2}^{\theta}\|e^{it\Delta}g\|_{L^\infty}^{1-\theta} \xrightarrow{t\to\infty} 0,$$

for $\theta \in (0,1)$. ■

Exercise 8.4 (Bonus 12) Assume $g \in L^p(\mathbb{R}^d)$, for some $1 < p < 2$. Prove that for all $t > 0$, $e^{it\Delta}g$ is well-defined and belongs to $L^{p'}(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover:

$$\|e^{it\Delta}g\|_{L^{p'}} \xrightarrow{t\to\infty} 0$$

Hint: You can use Riesz–Thorin interpolation theorem.

Lemma 8.5 If $g \in L^2(\mathbb{R}^d)$, then for all $\epsilon > 0$ we can write $g = g_1 + g_2$, $g, g_2 \in L^2$, $g_1 \in L^1$. s.t. $\|g_2\|_{L^2} \leqslant \epsilon$ and $\|g_1\|_{L^1} \leqslant C_\epsilon$.

Proof.

$$g = \underbrace{g\mathbb{1}(|g| > L)}_{g_1} + \underbrace{g\mathbb{1}(|g| \leqslant L)}_{g_2}$$

Now $g \in L^2$, then $g_2 \in L^2$ and

$$\int_{\mathbb{R}^d} |g_2|^2 = \int_{\mathbb{R}^d} |g|^2\mathbb{1}(|g| \leqslant L) \xrightarrow{L\to 0} 0$$

and $g_1 \in L^1$ since

$$\int_{\mathbb{R}^d} |g_1| = \int_{\mathbb{R}^d} |g|\mathbb{1}(|g| > L) \leqslant \int_{\mathbb{R}^d} |g|\frac{|g|}{L} = \frac{1}{L}\int_{\mathbb{R}^d} |g|^2 < \infty$$

Consequently:

$$\begin{aligned}\int_{|x|\leqslant R} |(e^{it\Delta}g)(x)|^2\,dx &\leqslant 2\int_{|x|\leqslant R} |e^{it\Delta}g_1(x)|^2 + 2\int_{|x|\leqslant R} |e^{it\Delta}g_2(x)|^2 \\ &\leqslant CR^d\left(\frac{\|g\|_{L^1}}{t^{\frac{d}{2}}}\right) + \|g_2\|_{L^2}^2 \\ &\leqslant \frac{C_\epsilon R^d}{t^d} + \epsilon^2 \xrightarrow{t\to\infty} \epsilon^2 \xrightarrow{\epsilon\to 0} 0\end{aligned}$$

■

Theorem 8.6 If $g \in C_c^\infty(\mathbb{R}^d)$, then

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} g(y)\,dy,$$

$x \in \mathbb{R}^d, t > 0$. satisfies $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$ and

$$\begin{cases}-i\partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ \lim_{t\to 0} u(x,t) = g(x) \text{ uniformly } x \in \mathbb{R}^d\end{cases}.$$

Proof.

$$\hat{u}(k,t) = e^{-it|2\pi k|^2}\hat{g}(k) \quad \forall k \in \mathbb{R}^d, \forall t > 0$$

$$\Rightarrow \quad \widehat{D^\alpha u}(k,t) = e^{-it|2\pi k|^2}(2\pi i k)^\alpha \hat{g}(k), \quad (2\pi i k)^\alpha = \prod_{j=1}^{d}(2\pi i k_j)^{\alpha_j}$$

$$\Rightarrow \quad \|\widehat{D^\alpha u}(k,t)\|_{L^1(\mathbb{R}^d,dk)} = \int_{\mathbb{R}^d} |(2\pi i k)^\alpha||\hat{g}(k)|\,dk \leqslant \int_{\mathbb{R}^d} |k|^{|\alpha|}|\hat{g}(k)|\,dk < \infty$$

as $|\hat{g}(k)| \leqslant \frac{C_n}{(1+|k|)^N}$ for all $N \geqslant 1$. This implies $D^\alpha u(\bullet,t) \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Moreover,

$$\partial_t \hat{u}(k,t) = e^{-it|2\pi k|^2}\hat{g}(k)\underbrace{(-i|2\pi k|^2)}_{\in L^1(\mathbb{R}^d,dk)}$$

$\rightsquigarrow t \mapsto u(x,t) \in C^1$. Similary $t \mapsto u(x,t) \in C^m$ for all $m \geqslant 1$.

$$\begin{aligned}
&-i\widehat{\partial_t u - \Delta_x} u = (-i)e^{-it|2\pi k|^2}(-i|2\pi k|^2)\hat{g}(k) + e^{-it|2\pi k|^2}|2\pi k|^2\hat{g}(k) = 0.\\
\Rightarrow \quad &-i\partial_t u - \Delta_x u = 0 \quad \text{almost everywhere in } \mathbb{R}^d\\
\Rightarrow \quad &-i\partial_t u - \Delta_x u = 0 \quad \text{everywhere in } \mathbb{R}^d
\end{aligned}$$

Before we proved $u(x,t) = (e^{it\Delta}g)(x) \to g(x)$ in $L^2(\mathbb{R}^d)$.. Here we can do better:

$$\int_{\mathbb{R}^d} |\hat{u}(k,t) - \hat{g}(k)|\,dk = \int_{\mathbb{R}^d} |\underbrace{e^{-it|2\pi k|^2} - 1}_{\xrightarrow{t\to+0} 0}|\underbrace{|\hat{g}(k)|}_{\in L^1} \xrightarrow{t\to 0} 0$$

So we get

$$\sup_{x\in\mathbb{R}^d} |u(x,t) - g(x)| \xrightarrow{t\to 0} 0. \qquad \blacksquare$$

Theorem 8.7 1. If $g \in H^2(\mathbb{R}^d)$, then $u = e^{it\Delta}g$ satisfies

- $-i\frac{d}{dt}u(\bullet,t)$ exists as a function in $L^2(\mathbb{R}^d)$ and $= \Delta_x u$ for all $t > 0$.

$$\left(-i\frac{d}{dt}u(\bullet,t) = \frac{u(\bullet,t+h) - u(\bullet,t)}{h} \xrightarrow{h\to 0} \Delta_x u = e^{it\Delta}\underbrace{(\Delta g)}_{\in L^2} \in L^2(\mathbb{R}^d)\right)$$

- $u(\bullet,t) \xrightarrow{t\to 0} g$ in L^2.

2. If $g \in L^2(\mathbb{R}^d)$, then $u = e^{it\Delta}g$ satisfies:

$$-i\frac{d}{dt}\langle \phi, u\rangle_{L^2} = \langle \Delta\phi, u\rangle_{L^2} \quad \forall \phi \in C_c^\infty(\mathbb{R}^d)(\text{or } \phi \in H^2(\mathbb{R}^d))$$

(Here $-i\frac{d}{dt} = \Delta_x u$ is meant in distributional sense)

Proof. 1. $g \in H^2(\mathbb{R}^d)$,

$$\left\| \frac{u(\bullet, t+h) - u(\bullet, t)}{h} e^{it\Delta}(\Delta g) \right\|^2_{L^2(\mathbb{R}^d)} = \left\| \frac{u(\bullet, t+h) - \widehat{u(\bullet}, t)}{h} e^{it\Delta}(\Delta g) \right\|^2_{L^2(k, dk)}$$

$$= \int_{\mathbb{R}^d} \left| \frac{e^{-(t+h)|2\pi k|^2}\hat{g}(k) - e^{-it|2\pi k|^2}\hat{g}(k)}{h} - e^{-it|2\pi k|^2}|2\pi k|^2 \hat{g}(k) \right|^2 dk$$

$$= \int_{\mathbb{R}^d} \underbrace{\left|(-i)\frac{(e^{-ih|2\pi k|^2} - 1)}{h} + |2\pi k|^2 \right|^2 |\hat{g}(k)|^2}_{\xrightarrow{h \to 0} 0} dk \xrightarrow{h\to 0} 0$$

by Dominated Convergence, $|\dots| \leqslant C|k|^2|\hat{(}k)|^2 \in L^1$. We have

$$(-i)\frac{e^{-ih|2\pi k|^2} - 1}{h} \xrightarrow{h \to 0} (-i)^2|2\pi k|^2 = -|2\pi k|^2$$

and $|\dots| \leqslant C|k|^2$. Also

$$\hat{u} = \widehat{e^{it\Delta}g} = e^{-it|2\pi k|^2}\hat{g}(k)$$

$$\Rightarrow \ |2\pi k|^2 \hat{u}(k) = e^{-it|2\pi k|^2} \underbrace{|2\pi k|^2 \hat{g}(k)}_{\in L^2(dk)} \in L^2(dk)$$

$$\Rightarrow \ \widehat{\Delta_x u} = e^{it\widehat{\Delta(\Delta}g)$$

$$\Rightarrow \ \Delta_x u = e^{it\Delta}(\Delta g)$$

2. $g \in L^2$:

$$-i\frac{d}{dt} = \langle \phi, u \rangle = -i\frac{d}{dt}\int_{\mathbb{R}^d} \underbrace{\overline{\hat{\phi}(k)}}_{\leqslant \frac{C_N}{(1+|k|)^N} \forall N \geqslant 1} e^{-it|2\pi k|^2}\hat{g}(k)\, dk$$

$$= \int_{\mathbb{R}^d} \overline{\hat{\phi}(k)}(-|2\pi k|^2)\hat{g}(k)\, dk$$

$$= \int_{\mathbb{R}^d} \overline{\widehat{\Delta\phi}}\hat{g} = \langle \Delta\phi, g \rangle_{L^2}$$

Spectral technique: If $\mathcal{H}$ is a Hilbertspace (separable) and $A : D(A) \to \mathcal{H}$ is a self-adjoint $A = A^\star$, i.e.

$$\begin{cases} \langle u, Av \rangle = \langle Au, v \rangle \\ D(A) = D(A^\star) \end{cases}$$

where

$$D(A^\star) = \{u \in \mathcal{H} \mid \langle u, Av \rangle = \langle A^\star u, v \rangle\}$$

$$= \left\{ u \in \mathcal{H} \,\middle|\, \sup_{v \in D(A)} \frac{|\langle u, Av \rangle}{\|v\|_{\mathcal{H}}} < \infty \right\}$$

Then we can define a unitary operator $e^{-itA} : \mathcal{H} \to \mathcal{H}$ with $t \in \mathbb{R}$. Moreover, for all $g \in \mathcal{H}$, $u = e^{-itA}g$ solves

$$\begin{cases} i\partial_t u = Au \\ u(t \to 0) = g \end{cases}$$

(Stone Theorem) (eg $A = -\Delta$)

∎

Theorem 8.8 (Spectral theorem) If A is a self-adjoint operator on a seperable Hilbert space, then, there is a unitary operator $U : \mathcal{H} \to L^2(\Omega)$ s.t. $UAU^\star = M_f$ on $L^2(\Omega)$, where M_f is a multipilication operator of a real-value function $f : \Omega \to \mathbb{R}$:

$$(M_f)(\phi)(x) = f(x)\phi(x)$$

for all $x \in \Omega, \forall \phi \in L^2(\Omega)$. Consequence: $Ue^{-itA}U^\star = e^{-itM_f}$ (multipilication) on $L^2(\Omega)$ where

$$(e^{-itM_f}\phi)(x) = e^{-itf(x)}\phi(x)$$

for all $x \in \Omega$, $\phi \in L^2(\Omega)$.

Example 8.9 $A = -\Delta$ on $\mathcal{H} = L^2(\mathbb{R}^d)$, $U = \mathcal{F}$ (Fourier Transform), $\mathcal{F}(-\Delta)\mathcal{F}^\star = |2\pi k|^2$ multiplication.

$$\mathcal{F}(e^{-itA}\phi) = e^{-it|2\pi k|^2}\hat{\phi}(k)$$

Example 8.10 Consider $A = -\Delta + V(x)$ in $L^2(\mathbb{R}^d)$, $V(x) \in C_c^\infty(\mathbb{R}^d)$ or $V(x) = |x|^2$. If V is nice, then A can be defined $D(A) \to L^2(\mathbb{R}^d)$ as a self-adjoint operator. $\rightsquigarrow e^{-itA} = e^{-it(-\Delta+v)}$ is well-defined as a unitary operator for all $t \in \mathbb{R}$. In particular, for all $g \in L^2$, $u = e^{-it(-\Delta+v)}g$ solves

$$\begin{cases} i\partial_t u = (-\Delta + v)u \\ u(t \to 0) = g \end{cases}$$

Scattery Theory: $e^{-it(-\Delta+v)}g$ as $t \to \infty$.

1. RAGE: If $-\Delta + V$ has no eigenfunction, then

$$\frac{1}{T}\int_0^T \int_{|x| \leqslant R} \left| \left(e^{-it(-\Delta+v)}g\right)(x) \right|^2 \, dx \, dt \xrightarrow{T \to 0} 0$$

2. (Asymptotic completeness) If $0 \leqslant V \in C_c^\infty$ is small, then there is a unitary operator U on L^2 s.t. for all $g \in L^2$:

$$\left\| e^{-it(-\Delta+v)}g - e^{-it\Delta}(Ug) \right\|_{L^2(\mathbb{R}^d)} \xrightarrow{t \to \infty} 0$$