Partial Differerential Equations

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1 Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1

$$f : \mathbb{R} \to \mathbb{R}$$
$$\begin{cases} f(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

Solution: $f(t) = a_0 e^{at}$ for all $t \geq 0$. This is a linear ODE (Ordinary differential equation).

Example 1.2 (Non-Linear ODE)

$$f : \mathbb{R} \to \mathbb{R}$$
$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives. Recall $Omega \subseteq \mathbb{R}^d$, $f : \Omega \to \{\mathbb{R}, \mathbb{C}\}$ open,

$$\partial_{x_1} f(x) = \lim \frac{f(x + he_i) - f(x)}{h}, e_i = (0, 0, 1, , 0, 0) \in \mathbb{R}^d$$
$$D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \ddots \partial_{x_d}^{\alpha_d} f(x), |\alpha| = \sum_{i=1}^{d} |\alpha_i|$$
$$Df = \nabla f = \text{ gradient of } f = (\partial x_1 \ldots \partial_{x_d})$$
$$\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$$
$$D^k f = (D^\alpha f)_{|\alpha|=k}$$
$$D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i,j \leq d}$$

Goals: For *solving a PDE* we want to

- Find an explizit solution! In many cases, it is impossible
- Prove a *well-posted theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have to notations of solution:

$Omega \subseteq R^d$ Navier-Stokes equation:

$$\begin{cases} d_t u + u\nabla u - \Delta u = \nabla f, \quad f \text{ is known} \\ \div(u) = \sum_{i=1}^{d} \frac{\partial}{\partial x_1} u_i(x) = 0 \end{cases}$$

open in 3D, exists smooth solution