Partial Differerential Equations

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Winter Semester 2021/2022

Lecture notes TEXed by Thomas Eingartner

Tuesday 26th October, 2021, 19:11

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Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} f(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geq 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \to \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \to \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \to 0} \frac{f(x+he_i) - f(x)}{h}$, where $e_i = (0, 0, \ldots, 1, \ldots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial x_1, \ldots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha| = k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i,j \leq d}$

Definition 1.4 Given a function F. Then the equation of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u: \ \Omega \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$ is called a *PDE of order k*.

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1}u, D^{k-2}u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explizit solution! This is in many cases impossible.
- Prove a *well-posted theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continous and differentiable functions) Let $\Omega \subseteq R^d$ be open

$$C(\Omega) = \{f: \ \Omega \to \mathbb{R} \mid f \text{ continuous}\}$$
$$C^k(\Omega) = \{f: \ \Omega \to \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \ \leq k\}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f: \ \Omega \to \mathbb{R} \text{ lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in $\mathbb{R}^d$. If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then u is a harmonic function in Ω.

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \; dS(x) = \int_K \mathrm{div}(\vec{F}) \; dx$$

where ν is the outward unit normal vector field on ∂K. Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \; dS(x) = \int_V \mathrm{div}(\nabla u) \; dx = \int_V \Delta u(x) \; dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

   where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f) g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \to \mathbb{R}$ be continuous. Prove that if $\int_B f(x) \; dx = 0$, then $u \equiv 0$ in Ω.

Theorem 2.5 (Fundamential Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x) \; dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in $\mathbb{R}^d$. Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \to \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2 + \cdots + x_d^2}\right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned}
\partial_{x_i} u &= \partial_{x_i} v = (\partial_r v)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r} \\
\partial_{x_i}^2 u &= \partial_{x_i}\left(v(r)'\frac{x_i}{r}\right) = (\partial_{x_i} v(r)')\frac{x_i}{r} + v'(r)\partial_{x_i}\left(\frac{x_i}{r}\right) \\
&= (\partial_r v'(r))\left(\frac{dr}{\partial_{x_i}}\right)\frac{x_i}{r} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}(\partial_{x_i} r)\right) = v'(r)\frac{x_i^2}{r^2} + v'r(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)
\end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d d_{x_i}^2\right) u = v''(r) + v'(r)(\frac{d}{r} - \frac{1}{r})$
Thus $\Delta u = v'(r) + v(r)\frac{d-1}{r}$. We consider $d \geq 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r)\frac{d-1}{r} = 0$
$\Rightarrow \log(v(r))' = \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)'$ (recall $log(f)' = \frac{f'}{f}$)
$\Rightarrow v'(r) = \frac{1}{v^{d-2} + \text{const.}}$
$$\begin{cases} \frac{const}{r^{d-2}} + const x x + const & , d \geq 3 \\ const \log(r) + const x x r + const & , d = 2 \end{cases}$$

Definition 2.7 (Fundamential Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|}\frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0,1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta\Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in $\mathbb{R}^d$. The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y)f(y)\ dy = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\ dy$$

This can be heuristically justifyfied with

$$-\Delta(\Phi \star f) = (-\Delta\Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C_c^2(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\,dy.$$

First we check that u is continuous: Take $x_k \to x_0$ in $\mathbb{R}^d$. We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y)\ dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y)\ dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n\to\infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \qquad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x-y)| \ \leq \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \leq R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\mathrm{supp}(f)$ but independent of y. Now we compute the derivatives:

$$\partial_{x_i} u(x) = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x-y)\ dy = \lim_{h\to 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x+he_i-y) - f(x-y)}{h}\ dy$$

$$\text{(dom. conv.)} \quad = \int \Phi(y) \partial_{x_i} f(x-y)\ dy$$

$$\Rightarrow \quad D^\alpha u(x) = \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x-y)\ dy \quad \text{for all } |\alpha| \leq 2$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned}
-\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y)(-\Delta_x) f(x-y)\, dy = \int_{\mathbb{R}^d} \Phi(y)(-\Delta_y) f(x-y)\, dy \\
&= \int_{\mathbb{R}^d \setminus B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy + \int_{B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy \quad (\epsilon > 0 \text{ small})
\end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus B(0,\epsilon)} \Phi(y)(-\Delta_y) f(x-y)\, dy \\
&\quad = \int_{\mathbb{R}^d \setminus B(0,\epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x-y)\, dy - \int_{\partial B(0,\epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \\
&\quad = \int_{\mathbb{R}^d \setminus B(0,\epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x-y)\, dy \\
&\qquad + \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\, dS(y) - \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y)
\end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0,\epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1|\epsilon^{d-1}} \qquad \text{for } y \in \partial B(0,\epsilon)$$

Hence:

$$\begin{aligned}
\int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\ dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(0,\epsilon)} f(x-y)\ dS(y) \\
&= \fint_{\partial B(0,\epsilon)} f(x-y)\ dS(y) = \fint_{\partial B(x,\epsilon)} f(y)\ dS(y) \xrightarrow{\epsilon \to 0} f(x)
\end{aligned}$$

We have to regard the following error terms:

- $$\left|\int_{B(0,\epsilon)} \Phi(y)(-\Delta_y)f(x-y)\ dy\right| \leq \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leq \|\Delta f\|_{L^\infty} \mathbb{1}(|y|\leq R)}\ dy$$
$$\leq \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)|\mathbb{1}(|y|\leq R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y|\leq\epsilon) \xrightarrow{\epsilon\to 0} 0$$
  Where $R > 0$ depends on x and the support of f but is independent of y.

- $$\left|\int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f}{\partial \vec{n}}(x-y)\ dS(y)\right| \leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)|\ \ dy$$
$$\leq \begin{cases} const \cdot \epsilon|\log\epsilon| \ \to 0, & d=2 \\ const\cdot\epsilon \to 0, & d\geq 3 \end{cases}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ proved that $u = \Phi \star f$ and $f \in C_c^2(\mathbb{R}^d)$.

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Thus, if $f \in C_c^2(\mathbb{R})$, then $u = \Phi \star f$ satisfies $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f. For example if $f \in C_c^1(\mathbb{R})$ we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y)\partial_{x_i} f(x-y)\,dy \in C(\mathbb{R}^d) \Rightarrow u \in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i}\partial_{x_j} u = \partial_{x_i}\int_{\mathbb{R}^d} \Phi(y)\partial_{x_j} f(x-y)\,dy = \int_{\mathbb{R}^d} \partial_{x_i}\Phi(y)\partial_{x_j} f(x-y)\,dy \in C(\mathbb{R}^d)$$

So we have $u \in C^2(\mathbb{R}^d)$. Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i}\Phi(y)\partial_{x_i} f(x-y)\,dy \overset{(IBP)}{=} f(x).$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega \subseteq \mathbb{R}$ be open, let $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω. Then

$$u(x) = ⨍_{B(x,r)} u = ⨍_{\partial B(x,r)} u \quad \text{for all } x \in \Omega, B(x,r) \subseteq \Omega$$

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$ (Linear Equation)

Proof. (Of theorem)2.12 Consider all $r > 0$ s.t. $B(x,r) \subseteq \Omega$,

$$f(r) = ⨍_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r. When it is done, then we immediately obtain

$$f(r) = \lim_{t \to 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr}\left(\fint_{\partial B(0,r)} u(x+y)\, dS(y) \right) \\
&= \frac{d}{dr}\left(\fint_{\partial B(0,1)} u(x+rz)\, dS(z) \right) \\
\text{(dom. convergence)} \quad &= \fint_{\partial B(0,1)} \frac{d}{dr}[u(x+rz)]\, dS(z) \\
&= \fint_{\partial B(0,1)} \nabla u(x+rz) z\, dS(z) \\
&= \fint_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r}\, dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n_y}\, dS(y) \\
\text{(Gauss-Green 2.2)} \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0}\, dy = 0
\end{aligned}$$

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Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, \omega \in S^{d-1}$, then

$$\int_{B(0,r)} g(y)\, dy = \int_0^r \left(\int_{B(0,r)} g(y)\, dS(y) \right) dr$$

Remark 2.15 We already proved that for u harmonic we have $u(x) = \fint_{\partial B(x,r)} u\, dy$. Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y)\, dy &= \int_{B(0,r)} u(x+y)\, dy \\
\text{(Pol. decomposition)} \quad &= \int_0^r \left(\int_{\partial B(0,s)} u(x+y)\, dS(y) \right) ds \\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y)\, dS(y) \right) ds \\
\text{(Mean value property)} \quad &= \int_0^r (|\partial B(x,s)|\, u(x))\, ds = |B(x,r)|\, u(x)
\end{aligned}$$

This implies

$$\fint_{B(x,r)} u(y)\, dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.16 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = \fint_{B(x,r)} u(y)\, dy = \fint_{\partial B(x,r)} u(y)\, dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then u is harmonic, i.e. $\Delta u = 0$ in Ω. (The proof is exactly like before)

Theorem 2.17 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω. Then

a) $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$

b) Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$, then $u \equiv \mathit{const.}$ in Ω.

Proof-Idea. Assume there exists $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$. We have that $B(x_0, r) \subseteq \Omega$, so by the mean valum theorem we have

$$u(x_0) = \fint_{B(x_0,r)} \underbrace{u(x)}_{\leq u(x_0)} \leq \fint_{B(x_0,r)} u(x_0)\, dx = u(x_0)$$

So we get $u(x) = u(x_0)$ for all $x \in B(x_0, r)$. ∎

Proof. Given $U \subseteq \mathbb{R}^d$ open, we can write $U = \bigcup_i U_i$, where U_i is open and connected.

b) Assume that Ω is connected and there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{y \in \Omega} u(x)$. Define $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$. U is closed since u is continuous. Moreover, U is open by the mean-value theorem. I.e. for all $x \in U$ there is a $r > 0$ s.t. $B(x, r) \subseteq U \subseteq \Omega$. Since U is connected we get $U = \Omega$, so u is constant in Ω. On the other hand, if there is no $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega}$ we have $\forall x_0 \in \Omega : \quad u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$

a) Given $\Omega \subseteq \mathbb{R}^d$ open, we can write $\Omega = \bigcup_i \Omega_i$, where Ω_i is open and connected. By b) we have

$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$

So we can conclude

$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x).$$ ∎

Definition 2.18
- If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$, then u is called *sub-harmonic* if $\Delta u \geq 0$ in Ω.
- If $\Delta u \leq 0$, then u is called *super-harmonic*.

Theorem 2.19 Let Ω be open in $\mathbb{R}^d$, $u \in C^2(\Omega)$, $\Delta u \geq 0$ in Ω.

1. We have the mean-value inequality

$$u(x) = \fint_{B(x,r)} u \leq \fint_{\partial B(x,r)} u \quad \text{for all } x \in B(x,r) \subseteq \Omega$$

2. Assume that Ω is connected and bounded. Then either
   - u is a constant in Ω
   - $u(x) < \sup_{y \in \partial\Omega} u(y)$ for all $x \in \Omega$

Definition 2.20 The *Poisson Equation* for given f, g on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Theorem 2.21 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open and connected. Let $f \in C(\Omega), g \in C(\partial\Omega)$. Then there exists *at most* one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof. Assume that we have two solutions u_1 and u_2. Then $u := u_1 - u_2$ is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that $u = 0$ in Ω. More precisely, by the maximum principle we have $\forall x \in \Omega$

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leq 0$$

Since $-u$ satisfies the same property we have $\forall x \in \Omega$:

$$\sup_{x\in\Omega}(-u(x)) \leq \sup_{x\in\partial\Omega}(-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leq 0 \quad \Rightarrow \quad u(x) \geq 0$$

So we geht $u(x) = 0$ in Ω. ∎

Exercise 2.22 Let Ω be open, connected and bounded in $\mathbb{R}^d$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof that

1. If $g \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.
2. If $g \geq 0$ on $\partial\Omega$ and $g \neq 0$, then $u > 0$ in Ω.

Lemma 2.23 (Estimates for derivatives) If u is harmonic in $\Omega \subseteq \mathbb{R}^d$ and $B(x_0, r) \subseteq \Omega$, then

$$|D^\alpha u(x_0)| \leq \frac{(c_d N)^N}{r^{d+n}} \int_{B(x_0,r)} |u|$$

Theorem 2.24 (Regularity) Let Ω be open in $\mathbb{R}^d$. Let $u \in C(\Omega)$ satisfy $u(x) = ⨍_{\partial B} u$ for any $x \in B(x,r) \subseteq \Omega$. Then u is a harmonic function in Ω, namely $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω. Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω.

Proof. We use the convolution. For simlicity consider the case $\Omega = \mathbb{R}^d$ first. Take $\eta \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \eta \leq 1$, $\eta(x) = 0$ if $|x| \geq 1$, η radial and $\int \eta = 1$. Define $\eta_\epsilon(x) = \epsilon^{-d}\eta(\epsilon^{-1}x)$ for all $\epsilon > 0$. Then

$$\int_{\mathbb{R}^d} \eta_\epsilon = \int_{\mathbb{R}^d} \eta = 1$$

We prove $u_\epsilon \coloneqq \eta_\epsilon \star u = u$ for all $\epsilon > 0$. By definition:

$$\begin{aligned}
u_\epsilon(x) &= \int_{\mathbb{R}^d} \eta_\epsilon(x-y)u(y)\,dy \\
&= \int_0^\infty \left[\int_{\partial B(x,r)} \eta_\epsilon(x-y)u(y)\,dS(y)\right] dr \\
(\eta \text{ radial}) \qquad &= \int_0^\infty \left[\eta_\epsilon(r)\int_{\partial B(x,r)} u(y)\,dS(y)\right] dr \\
(\text{Assumption}) \qquad &= \int_0^\infty \eta_\epsilon(r)\,|\partial B(x,r)|\,u(x)\,dr \\
&= u(x)\int_0^\infty \eta_\epsilon(r)|\partial B(0,r)|\,dr \\
&= u(x)\int_{\mathbb{R}^d} \eta_\epsilon(y)\,dy = u(x)
\end{aligned}$$

On the other hand, $u_\epsilon = \eta_\epsilon \star u$ is $C^\infty(\mathbb{R}^d)$. In fact $D^\alpha(\eta_\epsilon \star u) = (D^\alpha \eta_\epsilon) \star u$ is continuous for any α (Exercise). Then $u \in C^\infty(\mathbb{R}^d)$, so u is harmonic in $\mathbb{R}^d$, i.e. $\Delta u = 0$ in $\mathbb{R}^d$.

Consider now the general case where $\Omega \subseteq \mathbb{R}^d$ is open. Take $\epsilon > 0$ small and define $\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}$. Define

$$u_\epsilon(x) = \int_{\mathbb{R}^d} \eta_\epsilon(x-y)u(y)\ \ dy \qquad \text{for all } x \in \Omega_\epsilon$$

Recall that $\eta_\epsilon(y) = 0$ if $|y| \geq \epsilon$, then:

$$u_\epsilon(x) = \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)\,dy$$

is well-defined since $B(x,\epsilon) \subseteq \Omega$ for all $x \in \Omega_\epsilon$. Then by the same computation using the polar-decomposition, we find that $u_\epsilon(x) = u(x)$ for all $x \in \Omega$. Note that $u_\epsilon \in C^\infty(\Omega_\epsilon)$. Taking $\epsilon \to 0$, we get $u \in C^\infty(\Omega)$. Then we conclude that u is harmonic (We need to reverse the proof of the mean-value theorem).

To proof that u is analytic, we need to show that for all $x_0 \in \Omega$, there is a $r > 0$ s.t. $B(x_0, r) \subseteq \Omega$ and

$$u(x) = u(x_0) + \sum_{\alpha \neq 0} c_\alpha (x-x_0)^\alpha \quad \text{for all } x \in B(x_0, r)$$

Here $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \{0,1,2,\dots\}$ and $y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \dots y_d^{\alpha_d}$. We want to prove that the series converges uniformly in $B(x_0, r)$. Recall the Taylor expansion:

$$u(x) = u(x_0) + \sum_{0<|\alpha|<N} D^\alpha u(x_0)\frac{(x-x_0)^\alpha}{\alpha!} + R_N(x)$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $\alpha! = \alpha_1! \cdots \alpha_d!$ and

$$R_N(x) = \sum_{|\alpha|=N} \int_0^1 D^\alpha u(x_0 + t(x-x_0))\frac{(x-x_0)^\alpha}{\alpha!}\,dt$$

We have: Take $x_0 \in \Omega$, take $r > 0$ small and $L = L_d > 0$ large, s.t. $B(x_0, (L+1)r) \subseteq \Omega$. Then for all $x \in B(x_0, r)$ we have $B(x, L_r) \subseteq B(x_0, (L+1)r) \subseteq \Omega$. With lemma 2.23 we get

$$|D^\alpha u(x)| \leq \frac{(c_d N)^N}{(Lr)^{d+N}} \int_{B(x,Lr)} |u|$$

$$
\begin{aligned}
|R_N(x)| &\leq \sum_{|\alpha|=N} \|D^\alpha u\|_{L^\infty(B(x_0,r))} \frac{r^N}{\alpha!} \quad \text{for all } x \in B(x_0,r) \\
&\leq \sum_{|\alpha|=N} \frac{(c_d N)^N}{(Lr)^{d+N}} \frac{r^N}{\alpha!} \int_{B(x_0,(L+1)r)} |u| \\
&= \sum_{|\alpha|=N} \left(\frac{c_d N}{L}\right)^N \frac{1}{\alpha!} \underbrace{\frac{1}{(Lr)^d} \int_{B(x_0,(L+1)r)} |u|}_{M}
\end{aligned}
$$

Multinomial theorem:

$$
d^N = (1+1+\cdots+1)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!}
$$

Conclusion:

$$
|R_N(x)| \leq \left(\frac{dC_d N}{L}\right)^N \frac{1}{N!} M
$$

We need $N^N << N!$ (Stirly formula) ■

Exercise 2.25 (E 1.1) Proof the Gauss–Green formula: Let $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that for every open ball $B(y,r) \subseteq \mathbb{R}^d$ we have

$$
\int_{\partial B(y,r)} f(y) \cdot \nu_y \, dS(y) = \int_{B(y,r)} \operatorname{div} f \, dx.
$$

Here ν_y is the outward unit normal vector and dS is the surface measure on the sphere.

Exercise 2.26 (E 1.2) Let $u \in C(\mathbb{R}^d)$ and $\int_{B(x,r) u=0}$ for every open ball $B(x,r) \subseteq \mathbb{R}^d$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

My Solution. Assume there is a $x_0 \in \mathbb{R}^d$ s.t. w.l.o.g. $u(x_0) > 0$. Since u is continous there is a ball $B(x_0,r)$ s.t. $u(y) > \frac{u(x_0)}{2}$ for all $y \in B(x_0,r)$. But then we get

$$
\int_{B(x_0,r)} u(y) \, dy \geq \int_{B(x_0,r)} |u(x_0)| \, dy = |u(x_0)| \, |B(x_0,r)| > 0. \qquad ■
$$