

Partial Differential Equations  
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# Chapter 1

## Introduction

A differential equation is an equation of a function and its derivatives.

**Example 1.1** (Linear ODE) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\begin{cases} f'(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is:  $f(t) = a_0 e^{at}$  for all  $t \geq 0$ .

**Example 1.2** (Non-Linear ODE)  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider  $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$ . Then we have

$$f'(t) = \frac{1}{\cos^2(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in  $(-\pi, \pi)$ . It's a problem to extend this to  $\mathbb{R} \rightarrow \mathbb{R}$ .

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

**Remark 1.3** Recall for  $\Omega \subseteq \mathbb{R}^d$  open and  $f : \Omega \rightarrow \{\mathbb{R}, \mathbb{C}\}$  the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x + he_i) - f(x)}{h}$ , where  $e_i = (0, 0, \dots, 1, \dots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f(x)$ , where  $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial_{x_1}, \dots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \dots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha|=k}$ , where  $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i, j \leq d}$

**Definition 1.4** Given a function  $F$ . Then the equation of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function  $u : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$  is called a *PDE of order  $k$* .

- Equations  $\sum_d a_\alpha(x) D^\alpha u(x) = 0$ , where  $a_\alpha$  and  $u$  are unknown functions are called *Linear PDEs*.
- Equations  $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1} u, D^{k-2} u, \dots, Du, u, x) = 0$  are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explicit solution! This is in many cases impossible.
- Prove a *well-posed theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g.  $\Delta u = f \rightsquigarrow u \in C^2$ )
2. Weak Solutions: The solution is not smooth/continuous

**Definition 1.5** (Spaces of continuous and differentiable functions) Let  $\Omega \subseteq \mathbb{R}^d$  be open

$$\begin{aligned} C(\Omega) &= \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ continuous}\} \\ C^k(\Omega) &= \{f : \Omega \rightarrow \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \leq k\} \end{aligned}$$

Classical solution of a PDE of order  $k \rightsquigarrow C^k$  solutions!

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ Lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{ exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation:  $-\Delta u = f$
- Heat Equation:  $\partial_t u - \Delta u = f$
- Wave Equation:  $\partial_t^2 u - \Delta u = f$
- Schrödinger Equation:  $i\partial_t u - \Delta u = f$

## Chapter 2

# Laplace / Poisson Equation

### 2.1 Laplace Equation

$-\Delta u = 0$  (Laplace) or  $-\Delta u = f(x)$  (Poisson).

**Definition 2.1** (Harmonic Function) Let  $\Omega$  be an open set in  $\mathbb{R}^d$ . If  $u \in C^2(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ , then  $u$  is a harmonic function in  $\Omega$ .

**Theorem 2.2** (Gauss-Green Theorem) Let  $A \subseteq \mathbb{R}^d$  open,  $\vec{F} \in C^1(A, \mathbb{R}^d)$  and  $K \subseteq A$  compact with  $C^1$  boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \, dS(x) = \int_K \operatorname{div}(\vec{F}) \, dx$$

where  $\nu$  is the outward unit normal vector field on  $\partial K$ . Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \, dS(x) = \int_V \operatorname{div}(\nabla u) \, dx = \int_V \Delta u(x) \, dx$$

for any  $V \subseteq \Omega$  open.

**Theorem 2.3** (Green's Identities) Let  $A \subseteq \mathbb{R}^d$  open,  $K \subseteq A$  d-dim. compactum with  $C^1$  boundary and  $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

where  $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f)g \, dx = \int_{\partial K} \left( f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) \, dS$$

**Exercise 2.4** Let  $\Omega \subseteq \mathbb{R}^d$  open, let  $f : \Omega \rightarrow \mathbb{R}$  be continuous. Prove that if  $\int_B f(x) \, dx = 0$ , then  $u \equiv 0$  in  $\Omega$ .

**Theorem 2.5** (Fundamental Lemma of Calculus of Variations) Let  $\Omega \subseteq \mathbb{R}^d$  open, let  $f \in L^1(\Omega)$ . If  $\int_B f(x) \, dx = 0$  for all  $x \in B_r(x) \subseteq \Omega$ , then  $f(x) = 0$  a.e. (almost everywhere)  $x \in \Omega$ .

**Remark 2.6** (Solving Laplace Equation)  $-\Delta u = 0$  in  $\mathbb{R}^d$ . Consider the case when  $u$  is radial, i.e.  $u(x) = v(|x|)$ ,  $v : \mathbb{R} \rightarrow \mathbb{R}$ . Denote  $r = |x|$ , then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i} \left( \sqrt{x_1^2 + \cdots + x_d^2} \right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned} \partial_{x_i} u &= \partial_{x_i} v = (\partial_r v) \frac{\partial r}{\partial x_i} = v'(r) \frac{x_i}{r} \\ \partial_{x_i}^2 u &= \partial_{x_i} \left( v(r) \frac{x_i}{r} \right) = (\partial_{x_i} v(r))' \frac{x_i}{r} + v'(r) \partial_{x_i} \left( \frac{x_i}{r} \right) \\ &= (\partial_r v'(r)) \left( \frac{dr}{\partial x_i} \right) \frac{x_i}{r} + v'(r) \left( \frac{1}{r} - \frac{x_i}{r^2} (\partial_{x_i} r) \right) = v'(r) \frac{x_i^2}{r^2} + v'(r) \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right) \end{aligned}$$

So we have  $\Delta u = \left( \sum_{i=1}^d \partial_{x_i}^2 \right) u = v''(r) + v'(r) \left( \frac{d}{r} - \frac{1}{r} \right)$

Thus  $\Delta u = v'(r) + v(r) \frac{d-1}{r}$ . We consider  $d \geq 2$ . Laplace operator  $\Delta u = 0$  now becomes  $v''(r) + v'(r) \frac{d-1}{r} = 0$

$$\Rightarrow \log(v(r))' = \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)' \quad (\text{recall } \log(f)' = \frac{f'}{f})$$

$$\Rightarrow v'(r) = \frac{1}{v^{d-2} + \text{const.}}$$

$$\begin{cases} \frac{\text{const}}{r^{d-2}} + \text{const} r + \text{const} & , d \geq 3 \\ \text{const} \log(r) + \text{const} r + \text{const} & , d = 2 \end{cases}$$

**Definition 2.7** (Fundamental Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|} \frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where  $|B_1|$  is the Volume of the ball  $B_1(0) = B(0, 1) \subseteq \mathbb{R}^d$ .

**Remark 2.8**  $\Delta \Phi(x) = 0$  for all  $x \in \mathbb{R}^d$  and  $x \neq 0$ .

## 2.2 Poisson-Equation

The Poisson-Equation is  $-\Delta u(x) = f(x)$  in  $\mathbb{R}^d$ . The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy$$

This can be heuristically justified with

$$-\Delta(\Phi \star f) = (-\Delta \Phi) \star f = \delta_0 \star f = f$$

**Theorem 2.9** Assume  $f \in C_c^2(\mathbb{R}^d)$ . Then  $u = \Phi \star f$  satisfies that  $u \in C^2(\mathbb{R}^d)$  and  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$

*Proof.* By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy.$$

First we check that  $u$  is continuous: Take  $x_k \rightarrow x_0$  in  $\mathbb{R}^d$ . We prove that  $u(x_n) \xrightarrow{n} u_0$ , i.e.

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y) dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n \rightarrow \infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x - y)| \leq \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \leq R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where  $R > 0$  depends on  $\{x_n\}$  and  $\text{supp}(f)$  but independent of  $y$ . Now we compute the derivatives:

$$\begin{aligned} \partial_{x_i} u(x) &= \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x - y) dy = \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + h e_i - y) - f(x - y)}{h} dy \\ (\text{dom. conv.}) &= \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x - y) dy \\ \Rightarrow D^\alpha u(x) &= \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x - y) dy \quad \text{for all } |\alpha| \leq 2 \end{aligned}$$

$D^\alpha u(x)$  is continuous, thus  $u \in C^2(\mathbb{R}^d)$ . Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y) (-\Delta_x) f(x - y) dy = \int_{\mathbb{R}^d} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy + \int_{B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x - y) dy - \int_{\partial B(0, \epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x - y) dy \\ &\quad + \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) - \int_{\partial B(0, \epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \end{aligned}$$

We have that  $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$  and  $\vec{n} = \frac{y}{|y|}$  in  $\partial B(0, \epsilon)$ . This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1| \epsilon^{d-1}} \quad \text{for } y \in \partial B(0, \epsilon)$$

Hence:

$$\begin{aligned} \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) &= \frac{1}{d|B_1| \epsilon^{d-1}} \int_{\partial B(0, \epsilon)} f(x - y) dS(y) \\ &= \oint_{\partial B(0, \epsilon)} f(x - y) dS(y) = \oint_{\partial B(x, \epsilon)} f(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} f(x) \end{aligned}$$

We have to regard the following error terms:

$$\begin{aligned}
\bullet \left| \int_{B(0,\epsilon)} \Phi(y) (-\Delta_y) f(x-y) dy \right| &\leq \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leq \|\Delta f\|_{L^\infty} \mathbb{1}(|y| \leq R)} dy \\
&\leq \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)| \mathbb{1}(|y| \leq R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y| \leq \epsilon) dy \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

Where  $R > 0$  depends on  $x$  and the support of  $f$  but is independent of  $y$ .

$$\begin{aligned}
\bullet \left| \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y) dS(y) \right| &\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)| dy \\
&\leq \begin{cases} \text{const} \cdot \epsilon |\log \epsilon| \rightarrow 0, & d = 2 \\ \text{const} \cdot \epsilon \rightarrow 0, & d \geq 3 \end{cases}
\end{aligned}$$

Conclusion:  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$  proved that  $u = \Phi \star f$  and  $f \in C_c^2(\mathbb{R}^d)$ . ■

Thus, if  $f \in C_c^2(\mathbb{R})$ , then  $u = \Phi \star f$  satisfies  $u \in C^2(\mathbb{R}^2)$  and  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$ .

**Remark 2.10** The result holds for a much bigger class of functions  $f$ . For example if  $f \in C_c^1(\mathbb{R})$  we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x-y) dy \in C(\mathbb{R}^d) \Rightarrow u \in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i} \partial_{x_j} u = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) \partial_{x_j} f(x-y) dy = \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_j} f(x-y) dy \in C(\mathbb{R}^d)$$

So we have  $u \in C^2(\mathbb{R}^d)$ . Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_i} f(x-y) dy \stackrel{(IBP)}{=} f(x).$$

**Exercise 2.11** Extend this to more general functions!

## 2.3 Equations in general domains

**Theorem 2.12** (Mean Value Theorem for Harmonic Functions) Let  $\Omega \subseteq \mathbb{R}^d$  be open, let  $u \in C^2(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ . Then

$$u(x) = \oint_{B(x,r)} u = \oint_{\partial B(x,r)} u \quad \text{for all } x \in \Omega, B(x,r) \subseteq \Omega$$

**Exercise 2.13** In 1D:  $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$  (Linear Equation)

*Proof.* (Of theorem) 2.12 Consider all  $r > 0$  s.t.  $B(x,r) \subseteq \Omega$ ,

$$f(r) = \oint_{\partial B(x,r)} u$$



We need to prove that  $f(r)$  is independent of  $r$ . When it is done, then we immediately obtain

$$f(r) = \lim_{t \rightarrow 0} f(t) = u(x)$$

as  $u$  is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr} \left( \oint_{\partial B(0,r)} u(x+y) dS(y) \right) \\
&= \frac{d}{dr} \left( \oint_{\partial B(0,1)} u(x+rz) dS(z) \right) \\
(\text{dom. convergence}) \quad &= \oint_{\partial B(0,1)} \frac{d}{dr} [u(x+rz)] dS(z) \\
&= \oint_{\partial B(0,1)} \nabla u(x+rz) z dS(z) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r} dS(y) \\
&= \frac{1}{|B(x,r)|^{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green 2.2}) \quad &= \frac{1}{|B(x,r)|^{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0} dy = 0 \quad \blacksquare
\end{aligned}$$

**Remark 2.14** Recall the polar decomposition. Let  $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, w \in S^{d-1}$ , then

$$\int_{B(0,r)} g(y) dy = \int_0^r \left( \int_{B(0,s)} g(y) dS(y) \right) ds$$

**Remark 2.15** We already proved that for  $u$  harmonic we have  $u(x) = \oint_{\partial B(x,r)} u dy$ . Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_{B(0,r)} u(x+y) dy \\
(\text{Pol. decomposition}) \quad &= \int_0^r \left( \int_{\partial B(0,s)} u(x+y) dS(y) \right) ds \\
&= \int_0^r \left( \int_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\text{Mean value property}) \quad &= \int_0^r (|\partial B(x,s)| u(x)) ds = |B(x,r)| u(x)
\end{aligned}$$

This implies

$$\oint_{B(x,r)} u(y) dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

**Remark 2.16** The reverse direction is also correct, namely if  $u \in C^2(\Omega)$  and

$$u(x) = \oint_{B(x,r)} u(y) dy = \oint_{\partial B(x,r)} u(y) dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then  $u$  is harmonic, i.e.  $\Delta u = 0$  in  $\Omega$ . (The proof is exactly like before)

**Exercise 2.17** (E 1.1) Proof the Gauss–Green formula: Let  $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ . Prove that for every open ball  $B(y, r) \subseteq \mathbb{R}^d$  we have

$$\int_{\partial B(y, r)} f(y) \cdot \nu_y dS(y) = \int_{B(y, r)} \operatorname{div} f dx.$$

Here  $\nu_y$  is the outward unit normal vector and  $dS$  is the surface measure on the sphere.

*My Solution (wrong).* Let  $\tilde{1} = (1, \dots, 1) \in \mathbb{R}^d$ .

$$\begin{aligned} \int_{B(y, r)} \operatorname{div}(f) dx &= \int_{B(y, r)} \nabla f \cdot \tilde{1} dx \\ &= \int_{B(y, r)} \nabla f \cdot \nabla x dx \\ \text{(Partial Integration 2.3)} \quad &= \int_{\partial B(y, r)} f \cdot \frac{\partial x}{\partial \nu_x} dS(x) - \int_{B(y, r)} f \cdot \underbrace{\Delta x}_{=0} dx \\ &= \int_{\partial B(y, r)} f \cdot \nu_x \cdot \underbrace{\nabla x}_{=\tilde{1}} dS(x) \\ &= \int_{\partial B(y, r)} f \cdot \nu_x dS(x) \quad \blacksquare \end{aligned}$$

**Theorem 2.18** (Maximum Principle) Let  $\Omega \subseteq \mathbb{R}^d$  be open, let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ ,  $\Delta u = 0$  in  $\Omega$ . Then

1.  $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial \Omega} u(x)$
2. Assume that  $\Omega$  is connected. Then if there is a  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$ , then  $u \equiv \text{const.}$  in  $\Omega$ .

*Proof-Idea.* Assume there exists  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$ . We have that  $B(x_0, r) \subseteq \Omega$ , so by the mean value theorem we have

$$u(x_0) = \int_{B(x_0, r)} \underbrace{u(x)}_{\leq u(x_0)} dx \leq \int_{B(x_0, r)} u(x_0) dx = u(x_0)$$

So we get  $u(x) = u(x_0)$  for all  $x \in B(x_0, r)$ . ■

*Proof.* Given  $U \subseteq \mathbb{R}^d$  open, we can write  $U = \bigcup_i U_i$ , where  $U_i$  is open and connected.

- b) Assume that  $\Omega$  is connected and there is a  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{y \in \Omega} u(y)$ . Define  $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$ .  $U$  is closed since  $u$  is continuous. Moreover,  $U$  is open by the mean-value theorem. I.e. for all  $x \in U$  there is a  $r > 0$  s.t.  $B(x, r) \subseteq U \subseteq \Omega$ . Since  $U$  is connected we get  $U = \Omega$ , so  $u$  is constant in  $\Omega$ . On the other hand, if there is no  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$  we have  $\forall x_0 \in \Omega : u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial \Omega} u(x)$

- a) Given  $\Omega \subseteq \mathbb{R}^d$  open, we can write  $\Omega = \bigcup_i \Omega_i$ , where  $\Omega_i$  is open and connected. By b) we have

$$\begin{aligned} \sup_{x \in \bar{\Omega}_i} u(x) &= \sup_{x \in \partial \Omega_i} u(x), \quad \forall i \\ \Rightarrow \sup_{x \in \bar{\Omega}} u(x) &= \sup_{x \in \partial \Omega} u(x) \quad \blacksquare \end{aligned}$$

**Definition 2.19** • If  $\Omega \subseteq \mathbb{R}^d$  is open,  $u \in C^2(\Omega)$ , then  $u$  is *sub-harmonic* if  $\Delta u \geq 0$  in  $\Omega$ .

• If  $\Delta u \leq u$ , then  $u$  is called *super-harmonic*.

**Theorem 2.20** Let  $\Omega$  be open in  $\mathbb{R}^d$ ,  $u \in C^2(\Omega)$ ,  $\Delta u \geq 0$  in  $\Omega$ .

1. We have the mean-value inequality

$$u(x) = \oint_{B(x,r)} u \leq \oint_{\partial B(x,r)} u \quad \text{for all } x \in B(x,r) \subseteq \Omega$$

2. Assume that  $\Omega$  is connected and bounded. Then either

- $u$  is a constant in  $\Omega$
- $u(x) < \sup_{y \in \partial\Omega} u(y)$  for all  $x \in \partial\Omega$

The Poisson Equation for given  $f, g$  on a bounded set is:

$$\begin{cases} -\Delta f = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

**Theorem 2.21** (Uniqueness) Let  $\Omega \subseteq \mathbb{R}^d$  be bounded, open and connected. Let  $f \in C(\Omega), g \in C(\partial\Omega)$ . Then there exists *at most* one solution  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ , s.t.

$$\begin{cases} -\Delta f = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

*Proof.* Assume that we have two solutions  $u_1$  and  $u_2$ . Then  $u := u_1 - u_2$  is a solution to By the maximum principle, we know that  $u = 0$  in  $\Omega$ . More precisely, by the maximum principle we have

$$\sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) = 0 \Rightarrow u(x) \leq 0 \forall x \in \Omega$$

Since  $-u$  satisfies the same property:

$$\begin{aligned} \sup_{x \in \Omega} (-u(x)) &\leq \sup_{x \in \partial\Omega} (-u(x)) = 0 \\ \Rightarrow -u(x) &\leq 0 \text{ for all } x \in \Omega \\ \Rightarrow u(x) &\geq 0 \text{ for all } x \in \Omega \end{aligned}$$

So we get  $u(x) = 0$  in  $\Omega$ . ■

**Exercise 2.22** Let  $\Omega$  be open, connected and bounded in  $\mathbb{R}^d$ . Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof that

1. If  $g \geq 0$  on  $\partial\Omega$ , then  $u \geq 0$  in  $\Omega$ .

2. If  $g \geq 0$  on  $\partial\Omega$  and  $g \neq 0$ , then  $u > 0$  in  $\Omega$ .

**Lemma 2.23** (Estimates for derivatives) If  $u$  is harmonic in  $\Omega \subseteq \mathbb{R}^d$  and  $B(x_0, r) \subseteq \Omega$ , then

$$|D^\alpha u(x_0)| \leq \frac{(c_d N)^N}{r^{d+n}} \int_{B(x_0, r)} |u|$$

**Theorem 2.24** (Regularity) Let  $\Omega$  be open in  $\mathbb{R}^d$ . Let  $u \in C(\Omega)$  satisfy  $u(x) = \int_{\partial B} u$  for any  $x \in B(x, r) \subseteq \Omega$ . Then  $u$  is a harmonic function in  $\Omega$ , namely  $u \in C^2(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ . Moreover,  $u \in C^\infty(\Omega)$  and  $u$  is analytic in  $\Omega$ .

*Proof.*  $\int \eta = 1$ . Define  $\eta_\epsilon(x) = \epsilon^{-d} \eta(\epsilon^{-1}x)$  for all  $\epsilon > 0$ . Then

$$\int_{\mathbb{R}^d} \eta_\epsilon = \int_{\mathbb{R}^d} \eta = 1$$

We prove  $u_\epsilon := \eta_\epsilon \star u = u$  for all  $\epsilon > 0$ . By definition:

$$\begin{aligned} u_\epsilon(x) &= \int_{\mathbb{R}^d} \eta_\epsilon(x-y)u(y) dy = \int_0^\infty \left[ \int_{\partial B(x, r)} \eta_\epsilon(x-y)u(y) dS(y) \right] dr \\ &= \int_0^\infty \left[ \int_{\partial B} n_\epsilon(r) dS(y) \right] dr = \int_0^\infty [\eta_\epsilon(r)|\partial B(x, r)|u(x)] dr \\ &= u(x) \int_0^\infty [\eta_\epsilon(r)|\partial B(0, r)|] dr = u(x) \int_{\mathbb{R}^d} \eta_\epsilon(y) dy = u(x) \end{aligned}$$

On the other hand,  $u_\epsilon = \eta_\epsilon \star u$  is  $C^\infty(\mathbb{R}^d)$ . In fact  $D^\alpha(\eta_\epsilon \star u) = (D^\alpha \eta_\epsilon) \star u$  is continuous for any  $\alpha$  (Exercise). Then  $u \in C^\infty(\mathbb{R}^d)$ , so  $u$  is harmonic in  $\mathbb{R}^d$ , i.e.  $\Delta u = 0$  in  $\mathbb{R}^d$ .

Consider now the general case where  $\Omega \subseteq \mathbb{R}^d$  is open. Take  $\epsilon > 0$  small and define  $\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \epsilon\}$ . Define

$$u_\epsilon(x) = \int_{\mathbb{R}^d} \eta_\epsilon(x-y)u(y) dy \quad \text{for all } x \in \Omega_\epsilon$$

Recall that  $\eta_\epsilon(y) = 0$  if  $|y| \geq \epsilon$ , then:

$$u_\epsilon(x) = \int_{B(x, \epsilon)} \eta_\epsilon(x-y)u(y) dy$$

is well-defined since  $B(x, \epsilon) \subseteq \Omega$  for all  $x \in \Omega_\epsilon$ . Then by the same computation using the polar-decomposition, we find that  $u_\epsilon(x) = u(x)$  for all  $x \in \Omega$ . Note that  $u_\epsilon \in C^\infty(\Omega_\epsilon)$ , so  $\Delta u = 0$ . Taking  $\epsilon \rightarrow 0$ , we get  $u \in C^\infty(\Omega)$ . Then we conclude that  $u$  is harmonic (We need to reverse the proof of the mean-value theorem). Analytically, we need to prove that for all  $x_0 \in \Omega$ , there is a  $r > 0$  s.t.  $B(x_0, r) \subseteq \Omega$  and

$$u(x) = u(x_0) + \sum_{\alpha \neq 0} c_\alpha (x - x_0)^\alpha \quad \text{for all } x \in B(x_0, r)$$

Here  $\alpha = (\alpha_1, \dots, \alpha_d)$ ,  $\alpha_i \in \{0, 1, 2, \dots\}$  and  $y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \dots y_d^{\alpha_d}$ . We want to prove that the series converges uniformly in  $B(x_0, r)$ . Recall the Taylor expansion:

$$u(x) = u(x_0) + \sum_{0 < |\alpha| < N} D^\alpha u(x_0) \frac{(x - x_0)^\alpha}{\alpha!} + R_N(x)$$

where  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$ ,  $\alpha! = \alpha_1! \dots \alpha_d!$  and

$$R_N(x) = \sum_{|\alpha|=N} \int_0^1 D^\alpha u(x_0 + t(x - x_0)) \frac{(x - x_0)^\alpha}{\alpha!} dt$$

We have: Take  $x_0 \in \Omega$ , take  $r > 0$  small and  $L = L_d > 0$  large, s.t.  $B(x_0, (L+1)r) \subseteq \Omega$ . Then for all  $x \in B(x_0, r)$  we have  $B(x, Lr) \subseteq B(x_0, (L+1)r) \subseteq \Omega$ . With lemma 2.23 we get

$$|D^\alpha u(x)| \leq \frac{(c_d N)^N}{(Lr)^{d+N}} \int_{B(x, Lr)} |u|$$

$$\begin{aligned} |R_N(x)| &\leq \sum_{|\alpha|=N} \|D^\alpha u\|_{L^\infty(B(x_0, r))} \frac{r^N}{\alpha!} \quad \text{for all } x \in B(x_0, r) \\ &\leq \sum_{|\alpha|=N} \frac{(c_d N)^N}{(Lr)^{d+N}} \frac{r^N}{\alpha!} \int_{B(x_0, (L+1)r)} |u| \\ &= \sum_{|\alpha|=N} \left( \frac{c_d N}{L} \right)^N \frac{1}{\alpha!} \underbrace{\frac{1}{(Lr)^d} \int_{B(x_0, (L+1)r)} |u|}_M \end{aligned}$$

Multinomial theorem:

$$d^N = (1 + 1 + \dots + 1)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!}$$

Conclusion:

$$|R_N(x)| \leq \left( \frac{dC_d N}{L} \right)^N \frac{1}{N!} M$$

We need  $N^N \ll N!$  (Stirling formula) ■