

Partial Differential Equations
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Contents

1	Introduction	3
2	Laplace / Poisson Equation	5
2.1	Laplace Equation	5
2.2	Poisson-Equation	6
2.3	Equations in general domains	8
3	Convolution, Fourier Transform and Distributions	19
3.1	Convolutions	19
3.2	Fourier Transformation	26
3.3	Theory of Distribution	31
3.4	Distribution vs. measures	41
4	Weak Solutions and Regularity	44
5	Existence for Poisson's Equation on Domains	64
5.1	Green's function on the upper half plane	69
5.2	Green's function for a ball	72
5.3	Energy Method	76
5.4	Variational problem for weak solutions	83
5.5	Theory of Trace	87
5.6	Final Remarks	95
6	Heat Equation	99
6.1	Fundamental Solution	99
6.2	Maximum Principle	106
6.3	Backward heat equation	115
7	Wave Equation	118
7.1	d'Alembert	118
7.2	Euler-Poisson-Darboux	120
7.2.1	Solution in three dimensions	122
7.2.2	Solution in two dimensions	123
7.3	Spectral Method	127
7.4	Uniqueness	127
7.5	Propagation of the wave	128
7.6	Wave vs. Heat Equation	128
8	Schrödinger Equation	130

Please note that I write this lecture notes for my personal use. I may write things differently than presented in the lecture. This script also contains some of my personal solutions for exercises (which may be wrong).

Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{cases} f'(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geq 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos^2(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \rightarrow \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \rightarrow \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x + he_i) - f(x)}{h}$, where $e_i = (0, 0, \dots, 1, \dots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial_{x_1}, \dots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \dots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha|=k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i, j \leq d}$

Definition 1.4 Given a function F . Then the equation of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$ is called a *PDE of order k* .

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1} u, D^{k-2} u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explicit solution! This is in many cases impossible.
- Prove a *well-posed theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continuous and differentiable functions) Let $\Omega \subseteq \mathbb{R}^d$ be open

$$\begin{aligned} C(\Omega) &= \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ continuous}\} \\ C^k(\Omega) &= \{f : \Omega \rightarrow \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \leq k\} \end{aligned}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ Lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{ exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 u - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in \mathbb{R}^d . If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω , then u is a harmonic function in Ω .

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \, dS(x) = \int_K \operatorname{div}(\vec{F}) \, dx$$

where ν is the outward unit normal vector field on ∂K . Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \, dS(x) = \int_V \operatorname{div}(\nabla u) \, dx = \int_V \Delta u(x) \, dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Integration by parts):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f)g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) \, dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \rightarrow \mathbb{R}$ be continuous. Prove that if $\int_B f(x) \, dx = 0$, then $u \equiv 0$ in Ω .

Theorem 2.5 (Fundamental Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x) \, dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in \mathbb{R}^d . Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \rightarrow \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i} \left(\sqrt{x_1^2 + \cdots + x_d^2} \right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned} \partial_{x_i} u &= \partial_{x_i} v = (\partial_r v) \frac{\partial r}{\partial x_i} = v'(r) \frac{x_i}{r} \\ \partial_{x_i}^2 u &= \partial_{x_i} \left(v(r) \frac{x_i}{r} \right) = (\partial_{x_i} v(r))' \frac{x_i}{r} + v'(r) \partial_{x_i} \left(\frac{x_i}{r} \right) \\ &= (\partial_r v'(r)) \left(\frac{dr}{\partial x_i} \right) \frac{x_i}{r} + v'(r) \left(\frac{1}{r} - \frac{x_i}{r^2} (\partial_{x_i} r) \right) = v'(r) \frac{x_i^2}{r^2} + v'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right) \end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d \partial_{x_i}^2 \right) u = v''(r) + v'(r) \left(\frac{d}{r} - \frac{1}{r} \right)$

Thus $\Delta u = v'(r) + v(r) \frac{d-1}{r}$. We consider $d \geq 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r) \frac{d-1}{r} = 0$

$$\begin{aligned} \Rightarrow \log(v(r))' &= \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)' \quad (\text{recall } \log(f)' = \frac{f'}{f}) \\ \Rightarrow v'(r) &= \frac{1}{v^{d-2} + \text{const.}} \end{aligned}$$

$$\begin{cases} \frac{\text{const}}{r^{d-2}} + \text{const}x + \text{const} & , d \geq 3 \\ \text{const} \log(r) + \text{const}x + \text{const} & , d = 2 \end{cases}$$

Definition 2.7 (Fundamental Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|} \frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0, 1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta \Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in \mathbb{R}^d . The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy$$

This can be heuristically justified with

$$-\Delta(\Phi \star f) = (-\Delta \Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C_c^2(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy.$$

First we check that u is continuous: Take $x_k \rightarrow x_0$ in \mathbb{R}^d . We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y) dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n \rightarrow \infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x - y)| \leq \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \leq R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\text{supp}(f)$ but independent of y . Now we compute the derivatives:

$$\begin{aligned} \partial_{x_i} u(x) &= \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x - y) dy = \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + h e_i - y) - f(x - y)}{h} dy \\ (\text{dom. conv.}) &= \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x - y) dy \\ \Rightarrow D^\alpha u(x) &= \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x - y) dy \quad \text{for all } |\alpha| \leq 2 \end{aligned}$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y) (-\Delta_x) f(x - y) dy = \int_{\mathbb{R}^d} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy + \int_{B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x - y) dy - \int_{\partial B(0, \epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x - y) dy \\ &\quad + \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) - \int_{\partial B(0, \epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0, \epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1| \epsilon^{d-1}} \quad \text{for } y \in \partial B(0, \epsilon)$$

Hence:

$$\begin{aligned} \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) &= \frac{1}{d|B_1| \epsilon^{d-1}} \int_{\partial B(0, \epsilon)} f(x - y) dS(y) \\ &= \oint_{\partial B(0, \epsilon)} f(x - y) dS(y) = \oint_{\partial B(x, \epsilon)} f(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} f(x) \end{aligned}$$

We have to regard the following error terms:

$$\begin{aligned}
\bullet \left| \int_{B(0,\epsilon)} \Phi(y) (-\Delta_y) f(x-y) dy \right| &\leq \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leq \|\Delta f\|_{L^\infty} \mathbb{1}(|y| \leq R)} dy \\
&\leq \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)| \mathbb{1}(|y| \leq R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y| \leq \epsilon) \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

Where $R > 0$ depends on x and the support of f but is independent of y .

$$\begin{aligned}
\bullet \left| \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y) dS(y) \right| &\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)| dy \\
&\leq \begin{cases} \text{const} \cdot \epsilon |\log \epsilon| \rightarrow 0, & d = 2 \\ \text{const} \cdot \epsilon \rightarrow 0, & d \geq 3 \end{cases}
\end{aligned}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ proved that $u = \Phi \star f$ and $f \in C_c^2(\mathbb{R}^d)$. ■

Thus, if $f \in C_c^2(\mathbb{R})$, then $u = \Phi \star f$ satisfies $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f . For example if $f \in C_c^1(\mathbb{R})$ we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x-y) dy \in C(\mathbb{R}^d) \Rightarrow u \in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i} \partial_{x_j} u = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) \partial_{x_j} f(x-y) dy = \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_j} f(x-y) dy \in C(\mathbb{R}^d)$$

So we have $u \in C^2(\mathbb{R}^d)$. Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_i} f(x-y) dy \stackrel{(IBP)}{=} f(x).$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega \subseteq \mathbb{R}$ be open, let $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω . Then

$$u(x) = \oint_{B(x,r)} u = \oint_{\partial B(x,r)} u \quad \text{for all } x \in \Omega, B(x,r) \subseteq \Omega$$

Proof. Consider all $r > 0$ s.t. $B(x,r) \subseteq \Omega$,

$$f(r) = \oint_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r . When it is done, then we immediately obtain

$$f(r) = \lim_{t \rightarrow 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr} \left(\oint_{\partial B(0,r)} u(x+y) dS(y) \right) \\
&= \frac{d}{dr} \left(\oint_{\partial B(0,1)} u(x+rz) dS(z) \right) \\
(\text{dom. convergence}) \quad &= \oint_{\partial B(0,1)} \frac{d}{dr} [u(x+rz)] dS(z) \\
&= \oint_{\partial B(0,1)} \nabla u(x+rz) z dS(z) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r} dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green 2.2}) \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0} dy = 0 \quad \blacksquare
\end{aligned}$$

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$ (Linear Equation)

Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, w \in S^{d-1}$, then

$$\int_{B(0,r)} g(y) dy = \int_0^r \left(\int_{B(0,s)} g(y) dS(y) \right) ds$$

Remark 2.15 We already proved that for u harmonic we have $u(x) = \oint_{\partial B(x,r)} u dy$. Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_{B(0,r)} u(x+y) dy \\
(\text{Pol. decomposition}) \quad &= \int_0^r \left(\int_{\partial B(0,s)} u(x+y) dS(y) \right) ds \\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\text{Mean value property}) \quad &= \int_0^r (|\partial B(x,s)| u(x)) ds = |B(x,r)| u(x)
\end{aligned}$$

This implies

$$\oint_{B(x,r)} u(y) dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.16 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = \oint_{B(x,r)} u(y) dy = \oint_{\partial B(x,r)} u(y) dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then u is harmonic, i.e. $\Delta u = 0$ in Ω . (The proof is exactly like before)

Theorem 2.17 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω . Then

- a) $\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$
- b) Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$, then $u \equiv \text{const.}$ in Ω .

Proof. Given $U \subseteq \mathbb{R}^d$ open, we can write $U = \bigcup_i U_i$, where U_i is open and connected.

- b) Assume that Ω is connected and there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{y \in \Omega} u(y)$. Define $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$. U is closed since u is continuous. Moreover, U is open by the mean-value theorem. I.e. for all $x \in U$ there is a $r > 0$ s.t. $B(x, r) \subseteq U \subseteq \Omega$. Since U is connected we get $U = \Omega$, so u is constant in Ω . On the other hand, if there is no $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega} u(x)$ we have $\forall x_0 \in \Omega : u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
- a) Given $\Omega \subseteq \mathbb{R}^d$ open, we can write $\Omega = \bigcup_i \Omega_i$, where Ω_i is open and connected. By b) we have

$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$

So we can conclude

$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x). \quad \blacksquare$$

Definition 2.18 • If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$, then u is called *sub-harmonic* if $\Delta u \geq 0$ in Ω .

- If $\Delta u \leq 0$, then u is called *super-harmonic*.

Exercise 2.19 (E 1.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$ be subharmonic.

- a) Prove that u satisfies the Mean Value Inequality

$$\oint_{\partial B(x, r)} u(y) dS(y) \geq \int_{B(x, r)} u(y) dy \geq u(x)$$

for all $B(x, r) \subseteq \mathbb{R}^d$.

- b) Assume further that Ω is connected and $u \in C(\bar{\Omega})$. Prove that u satisfies the strong maximum principle, namely either
 - u is constant in Ω , or
 - $\sup_{y \in \partial\Omega} u(y) > u(x)$ for all $x \in \Omega$.

My Solution. a) Let $f(r) = \oint_{\partial B(x,r)} u(y) dS(y)$, then we have

$$\begin{aligned}
\partial_r f(r) &= \partial_r \oint_{\partial B(x,r)} u(y) dS(y) \\
(\text{Dom. Convergence}) &= \oint_{\partial B(x,r)} \partial_r u(y) dS(y) \\
&= \oint_{\partial B(0,1)} \partial_r u(x + yr) dS(y) \\
&= \oint_{\partial B(0,1)} \nabla u(x + yr) \cdot y dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r} dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green}) &= \oint_{B(x,r)} \text{div}(\nabla u(y)) dS(y) \\
&= \oint_{B(x,r)} \underbrace{\Delta u(y)}_{\geq 0} dS(y) \geq 0
\end{aligned}$$

So we can conclude that

$$\oint_{\partial B(x,r)} u(y) dS(y) = f(r) \geq \lim_{r \rightarrow 0} f(r) = u(x).$$

Now regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left(\int_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&= \int_0^r \left(|\partial B(x,r)| \oint_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&\geq \int_0^r |\partial B(x,r)| \cdot u(x) dS(y) \\
&= u(x) \int_0^r |\partial B(x,r)| dS(y) = u(x) |B(x,r)|.
\end{aligned}$$

Thus we have

$$u(x) \leq \oint_{B(x,r)} u(y) dy.$$

Finally, lets regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left(|\partial B(x,s)| \oint_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\partial_r f(r) \geq 0) &\leq \int_0^r \left(|\partial B(x,s)| \oint_{\partial B(x,s)} u(y) dS(y) \right) ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \int_0^r |\partial B(x,s)| ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \cdot |B(x,r)|
\end{aligned}$$

and we conclude

$$\oint_{B(x,r)} u(y) dy \leq \oint_{\partial B(x,r)} u(y) dS(y).$$

b) Let $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega} u(x)$. Now,

$$\begin{aligned} \sup_{x \in \Omega} u(x) = u(x_0) &\leq \oint_{\partial B(x_0,r)} u(y) dy \\ &\leq \oint_{\partial B(x_0,r)} \sup_{x \in \Omega} u(x) dy = \sup_{x \in \Omega} u(x) \end{aligned}$$

Since u is continuous we get $u(y) = u(x_0)$ for all $y \in B(x_0, r)$, so u is constant. \blacksquare

Definition 2.20 The *Poisson Equation* for given f, g on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Theorem 2.21 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open and connected. Let $f \in C(\Omega), g \in C(\partial\Omega)$. Then there exists *at most* one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof. Assume that we have two solutions u_1 and u_2 . Then $u := u_1 - u_2$ is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that $u = 0$ in Ω . More precisely, by the maximum principle we have $\forall x \in \Omega$

$$\sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leq 0$$

Since $-u$ satisfies the same property we have $\forall x \in \Omega$:

$$\sup_{x \in \Omega} (-u(x)) \leq \sup_{x \in \partial\Omega} (-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leq 0 \quad \Rightarrow \quad u(x) \geq 0$$

So we get $u(x) = 0$ in Ω . \blacksquare

Exercise 2.22 (Bonus 1) Let Ω be open, connected and bounded in \mathbb{R}^d . Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Prove that

a) If $g \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω .

b) If $g \geq 0$ on $\partial\Omega$ and $g \neq 0$, then $u > 0$ in Ω .

Lemma 2.23 (Estimates for derivatives) If u is harmonic in $\Omega \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| = N$ and $B(x_0, r) \subseteq \Omega$, then

$$|D^\alpha u(x)| \leq \frac{(c_d N)^N}{r^{d+N}} \int_{B(x, r)} |u| dy$$

Proof. Induction: Assume $|\alpha| = N - 1$, Take $|\alpha| = N$

$$|D^\alpha u(x_0)| \leq \frac{|S_1|}{|B_1| \frac{r}{N}} \|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))}, \quad D^\alpha u = \partial_{x_i}(D^\beta u)_{|\beta|=N-1}$$

Note: $x \in B(x_0, \frac{r}{N})$, so $B(x, \frac{r(N-1)}{N}) \subseteq B(x_0, r)$. By the induction hypothesis:

$$\|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))} \leq \frac{[c_d(N-1)]^{N-1}}{[r \frac{(N-1)}{N}]^{d+N-1}} \int_{B(x_0, r)} |u| dy$$

The conclusion is:

$$\begin{aligned} |D^\alpha u(x_0)| &\leq \frac{|S_1|}{|B_1| \frac{r}{N}} \frac{[c_d(N-1)]^{N-1}}{(r \frac{N-1}{N})^{d+N-1}} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} (N-1)^d} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} N^d} \left(\frac{N}{N-1}\right)^d \int_{B(x_0, r)} |u| dy \\ &\leq \frac{2^d |S_1|}{|B_1|} \frac{c_d^{N-1} N^N}{r^{d+N}} \int_{B(x_0, r)} |u| dy \quad \text{if } c_d \geq \frac{2^d |S_1|}{|B_1|} \end{aligned}$$

■

Theorem 2.24 (Regularity) Let Ω be open in \mathbb{R}^d . Let $u \in C(\Omega)$ satisfy $u(x) = \int_{\partial B} u dy$ for any $x \in B(x, r) \subseteq \Omega$. Then u is a harmonic function in Ω . Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω .

Exercise 2.25 (E 1.1: Proof the Gauss–Green formula) Let $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that for every open ball $B(y, r) \subseteq \mathbb{R}^d$ we have

$$\int_{\partial B(y, r)} f(y) \cdot \nu_y dS(y) = \int_{B(y, r)} \operatorname{div} f dx.$$

Here ν_y is the outward unit normal vector and dS is the surface measure on the sphere.

Solution. We proof this in $d=3$. Let $f \in C^1(\mathbb{R}^3)$

$$\int_{B(0,1)} \partial_{x_3} f dx = \int_{\partial B(0,1)} f x_3 dS(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \vec{n} = \frac{x}{|x|} \text{ on } \partial B(0,1)$$

$$\begin{aligned} B(0,1) &= \{x_1^2 + x_2^2 + x_3^2 \leq 1\} \\ &= \{x_1^2 + x_2^2 \leq 1 - \sqrt{1 - x_1^2 - x_2^2} \leq x_3 \leq \sqrt{1 - x_1^2 - x_2^2}\} \end{aligned}$$

Then:

$$\begin{aligned}\int_{B(0,1)} \partial_{x_3} f \, dx &= \int_{x_1^2 + x_2^2 \leq 1} \left(\int_{-\sqrt{1-x_1^2-x_2^2} \leq x_3 \leq \sqrt{1-x_1^2-x_2^2}} \partial_{x_3} f \, dx_3 \right) dx_1 \, dx_2 \\ &= \int_{x_1^2 + x_2^2 \leq 1} \left[f(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \right. \\ &\quad \left. - f(x_1, x_2, -\sqrt{1-x_1^2-x_2^2}) \right] dx_1 \, dx_2\end{aligned}$$

Lets take polar coordinates in 2D:

$$\begin{aligned}x_1 &= r \cos \phi & r > 0, \phi \in [0, 2\pi) \\ x_2 &= r \sin \phi & \det \frac{\partial(x_1, x_2)}{\partial(r, \phi)} = r\end{aligned}$$

$$(\star) = \int_0^1 \int_0^{2\pi} [f(r \cos \phi, r \sin \phi, r) - f(r \cos \phi, r \sin \phi, -r)] r \, dr \, d\phi$$

On the other hand:

$$\int_{\partial B(0,1)} f x_3 \, dS$$

The polar coordinates in 3D are:

$$\begin{aligned}x_1 &= r \cos \phi \sin \theta & r > 0, \phi \in (0, 2\pi), \theta \in (0, \pi) \\ x_2 &= r \sin \phi \sin \theta & \det \frac{\partial(x_1, x_2, x_3)}{\partial(r, \phi, \theta)} = r^2 \sin \theta \\ x_3 &= r \cos \theta\end{aligned}$$

Then:

$$\begin{aligned}(\star\star) &= \int_0^{2\pi} \int_0^\pi f(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \sin \theta \cos \theta \, d\theta \, d\phi \\ &= \int_0^{2\pi} \left(\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^\pi d\theta \right) d\phi \\ (r = \sin \theta) &= \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, \sqrt{1-r^2}) r \, dr \, d\phi \\ &\quad - \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, -\sqrt{1-r^2}) r \, dr \, d\phi\end{aligned} \quad \blacksquare$$

Exercise 2.26 (E 1.2) Let $u \in C(\mathbb{R}^d)$ and $\int_{B(x,r)} u \, dy = 0$ for every open ball $B(x,r) \subseteq \mathbb{R}^d$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

My Solution. Assume there is a $x_0 \in \mathbb{R}^d$ s.t. w.l.o.g. $u(x_0) > 0$. Since u is continous there is a ball $B(x_0, r)$ s.t. $u(y) > \frac{u(x_0)}{2}$ for all $y \in B(x_0, r)$. But then we get

$$\int_{B(x_0, r)} u(y) \, dy \geq \int_{B(x_0, r)} \frac{u(x_0)}{2} \, dy = \frac{u(x_0)}{2} |B(x_0, r)| > 0. \quad \blacksquare$$

Exercise 2.27 (E 1.3) Let $f \in C_c^1(\mathbb{R}^d)$ with $d \geq 2$ and $u(x) := (\Phi \star f)(x)$. Prove that $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ (2.9 was the same for $f \in C_1(\mathbb{R})$)

Theorem 2.28 (Liouville's Theorem) If $u \in C^2(\mathbb{R}^d)$ is harmonic and bounded, then $u = \text{const.}$

Proof. By the bound of the derivative 2.23 we have

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\leq \frac{c_d}{r^{d+1}} \int_{B(x_0, r)} |u| dy \quad \forall x_0 \in \mathbb{R}^d \quad \forall r > 0 \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r^{d+1}} |B(x_0, r)| \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r} \xrightarrow{r \rightarrow \infty} 0 \end{aligned}$$

Thus $\partial_{x_i} u = 0$ for all $i = 1, 2, \dots, d$ and $u = \text{const.}$ in \mathbb{R}^d ■

Theorem 2.29 (Uniqueness of solutions to Poisson Equation in \mathbb{R}^d) If $u \in C^2(\mathbb{R}^d)$ is a bounded function and satisfies $-\Delta u = f$ in \mathbb{R}^d where $f \in C_c^2(\mathbb{R}^d)$, then we have

$$u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y) dy + C \quad \forall x \in \mathbb{R}^d$$

where C is a constant and Φ is the fundamental solution of the Laplace equation in \mathbb{R}^d .

Proof. If we can prove that v is bounded, then $v = \text{const.}$ We first need to show that $\Phi \star f$ is bounded.

$$\begin{aligned} \Phi &= \Phi_1 + \Phi_2 = \Phi \mathbb{1}(|x| \leq 1) + \Phi(|x| \geq 1) \\ \Phi \star f &= \Phi_1 \star f + \Phi_2 \star f \end{aligned}$$

We have $\Phi_1 \star f \in L^1(\mathbb{R}^d)$ and $\Phi_2 \star f$ is bounded since $\Phi \rightarrow 0$ as $|x| \rightarrow \infty$ in $d \geq 3$. ■

Exercise 2.30 (Hanack's inequality) Let $u \in C^2(\mathbb{R}^d)$ be harmonic and non-negative. Prove that for all open, bounded and connected $\Omega \subseteq \mathbb{R}^d$, we have

$$\sup_{x \in \Omega} u(x) \leq C_\Omega \inf_{x \in \Omega} u(x),$$

where C_Ω is a finite constant depending only on Ω .

Proof. (Exercise) Hint: $\Omega = B(x, r)$. General case cover Ω by finitely many balls, one ball is inside Ω . ■

Exercise 2.31 (E 3.1 Lebesgue Differentiation Theorem) Let $f \in L^1_{loc}(\mathbb{R}^d)$. Prove that that for almost every $x \in \mathbb{R}^d$:

$$\oint_{B(x, r)} |f(x) - f(y)| dy \xrightarrow{r \rightarrow 0} 0$$

Proof. Clearly the same result holds with $\mathbb{R}^d \rightsquigarrow \Omega \subseteq \mathbb{R}^d$ open. Also it suffices to consider $f \in L^1(\mathbb{R}^d)$. From the last time discussion, by a density argument there exists $r_n \rightarrow 0$ s.t.

$$\oint_{B(x, r_n)} |f(y) - f(x)| dy = 0$$

for a.e. $x \in \mathbb{R}^d$. We prove that for all $\epsilon > 0$, the set $A_\epsilon = \{x \in \mathbb{R}^d \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy > \epsilon\}$ has measure 0. This will imply that

$$\bigcup_{n=1}^{\infty} A_{\frac{1}{n}} = \left\{ x \in \mathbb{R}^d \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy > 0 \right\}$$

has measure 0, which is what we want to show. First, we show that $|A_\epsilon| = 0$: Take $\{f_n\} \subseteq C_c^\infty$, $f_n \rightarrow f$ in $L^1(\mathbb{R}^d)$. By the triangle inequality:

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

So we get

$$\begin{aligned} & \int_{B(x,r)} |f(y) - f(x)| dy \\ & \leq \int_{B(x,r)} |f(y) - f_n(y)| dy + \int_{B(x,r)} |f_n(y) - f_n(x)| + |f_n(x) - f(x)| dy \\ \Rightarrow \quad \limsup_{r \rightarrow 0} \dots & \leq \limsup_{r \rightarrow 0} (\dots) + 0 + |f_n(x) - f(x)| \end{aligned}$$

Thus, for all $x \in A_\epsilon$, then:

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy + |f_n(x) - f(x)| > 2\epsilon$$

Observation: If $a, b \geq 0$, $a + b > 2\epsilon$ then either $a > \epsilon$ or $b > \epsilon$. Therefore $A_\epsilon \subseteq (S_{n,\epsilon} \cup \tilde{S}_{n,\epsilon})$, where

$$\begin{aligned} S_{n,\epsilon} &= \{x \mid |f_n(x) - f(x)| > \epsilon\} \\ \tilde{S}_{n,\epsilon} &= \{x \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy > \epsilon\} \end{aligned}$$

Consequently: $|A_\epsilon| \leq |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}|$ for all $n \geq 1$. By the Markov / Chebyshev inequality:

$$|S_{n,\epsilon}| \leq \int_{S_{n,\epsilon}} \frac{|f_n(x) - f(x)|}{\epsilon} dx = \int_{\mathbb{R}^d} \frac{|f_n(x) - f(x)|}{\epsilon} dx = \frac{\|f_n - f\|_{L^1}}{\epsilon}$$

We want to prove a simpler bound for $\tilde{S}_{n,\epsilon}$. For all $x \in \tilde{S}_{n,\epsilon}$:

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy > \epsilon$$

So there is a $r_x \in (0, 1)$ s.t.

$$\int_{B(x,r_x)=B_x} |f_n(y) - f(y)| dy > \epsilon$$

Thus $\tilde{S}_{n,\epsilon} \subseteq \left(\bigcup_{x \in \tilde{S}_{n,\epsilon}} B_x \right)$.

Lemma 2.32 (Vitali Covering) If F is a collection of balls in \mathbb{R}^d with bounded radius, then there exists a sub-collection $G \subseteq F$ s.t.

- G has disjoint balls

- $\bigcup_{B \in F} B \subseteq \bigcup_{B \in G} 5B, 5B(x, r) = B(x, 5r)$

Remark 2.33 The condition of the boundedness of the radius is necessary. Otherwise, consider $\{B(0, n)\}_{n=1}^{\infty}$

Here consider $F = \{B_x\}_{x \in \tilde{S}_{n,\epsilon}}$. With the Vitali covering lemma there is a $G \subseteq F$ s.t. G contains disjoint balls and:

$$\tilde{S}_{n,\epsilon} \subseteq \bigcup_{B \in F} B \subseteq \bigcup_{B \in G} 5B$$

So we get

$$|\tilde{S}_{n,\epsilon}| \leq \left| \bigcup_{B \in G} 5B \right| \leq \sum_{B \in G} |5B| = \sum_{B \in G} 5^d |B|$$

On the other hand, for all $B \in G \subseteq F$:

$$\int_B |f_n(y) - f(y)| dy > \epsilon \Rightarrow \int_B |f_n - f| > \epsilon |B|$$

This implies:

$$\sup_{B \in G} \int_B |f_n - f| > \epsilon \sum_{B \in G} |B|$$

Since balls in G are disjoint:

$$\int_{\mathbb{R}^d} \geq \int_{\bigcup_{B \in G}} |f_n - f| dy > \epsilon \sum_{B \in G} |B| \geq \frac{\epsilon}{5^d} |\tilde{S}_{n,\epsilon}|$$

So

$$|\tilde{S}_{n,\epsilon}| \leq \frac{5^d}{\epsilon} \|f_n - f\|_{L^1}$$

In summary:

$$|A_\epsilon| \leq |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}| \leq \frac{5^d + 1}{\epsilon} \|f_n - f\|_{L^1} \rightarrow 0$$

as $n \rightarrow \infty$. So $|A_\epsilon| = 0$ for all $\epsilon > 0$ ■

Remark 2.34 1. The proof can be done by using the Besicovitch covering lemma: For all $E \subseteq \mathbb{R}^d$ s.t. E is bounded. Let $F =$ collection of balls s.t. for all $x \in E$ there is a $B_x \in F$ s.t. x is the center of B_x . There is a sub-collection $G \subseteq F$ s.t.

- $E \subseteq \bigcup_{B \in G} B$
- Any point in E belongs to at most C_d balls in G (C_d depends only on \mathbb{R}^d), i.e.

$$\mathbb{1}_E(x) \leq \sum_{B \in G} \mathbb{1}_B(x) \leq C_d \mathbb{1}_E(x) \forall x$$

2. By a simpler argument we can prove the weak L^1 -estimate:

$$\{x \mid f^\star(x) > \epsilon\} \leq \frac{C_d}{\epsilon} \|f\|_{L^1(\mathbb{R}^d)}$$

(Hardy-Littlewood maximal function)

Exercise 2.35 (E 3.3) $f \in C_c^\infty(\mathbb{R}^d)$. Prove $|\hat{f}(k)| \leq \frac{C_N}{(1+|k|)^N}$

Solution. Since $f \in C_c^\infty$ we have that $D^\alpha f \in C_c^\infty$. Recall

$$\widehat{D^\alpha f}(k) = (-2\pi i k)^\alpha \hat{f}(k)$$

For example

$$\begin{aligned} \widehat{-\Delta f}(k) &= |2\pi i k|^2 \hat{f}(k) \\ (\text{Induction}) \rightsquigarrow \widehat{(-\Delta)^N f}(k) &= |2\pi k|^{2N} \hat{f}(k) \end{aligned}$$

So we can conclude

$$\hat{f}(k) = \frac{\widehat{(-\Delta)^N f}(k)}{|2\pi k|^{2N}} \quad \forall k \in \mathbb{R}^d$$

1. $f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in L^\infty$
2. $(-\Delta)^N f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \widehat{(-\Delta)^N f} \in L^\infty$

Conclusion: $\hat{f}(k) \leq \begin{cases} C & \forall k \\ \frac{C_N}{|k|^{2N}} & \forall k \end{cases}$ So $\hat{f}(k) \leq \frac{C_N}{(1+|k|)^N}$ ■

Chapter 3

Convolution, Fourier Transform and Distributions

3.1 Convolutions

Definition 3.1 (Convolution) Let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ or \mathbb{C} .

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) dy = \int_{\mathbb{R}^d} f(y)g(x-y) dy = (g \star f)(x)$$

Remark 3.2 (Properties of the Convolution)

- $f \star g = g \star f$
- $(f \star g) \star h = f \star (g \star h)$ (by Fubini)
- $\widehat{f \star g} = \hat{f}\hat{g}$ (Proof, see exercise 3.21)

Theorem 3.3 (Young Inequality) If $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$, where $1 \leq p \leq \infty$, then $f \star g \in L^p(\mathbb{R}^d)$ and

$$\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

More generally, if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^r(\mathbb{R}^d)$,

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q},$$

where $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

Proof. Let $f \in L^1, g \in L^p$. With the Hölder Inequality we have:

$$\begin{aligned}
|(f \star g)(x)| &= \left| \int_{\mathbb{R}^d} f(x-y)g(y) dy \right| \\
&\leq \left(\int_{\mathbb{R}^d} |f(x-y)| dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(x-y)||g(y)|^p dy \right)^{\frac{1}{p}} \\
&= \|f\|_{L^1}^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(x-y)||g(y)|^p dy \right)^{\frac{1}{p}} \\
\|f \star g\|_{L^p}^p &= \int_{\mathbb{R}^d} |f \star g(x)|^p dx \\
&\leq \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)||g(y)|^p dy dx \\
&= \|f\|_{L^1}^{\frac{p}{q}+1} \|g\|_{L^p}^p
\end{aligned}$$

So we have $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$ ■

Theorem 3.4 (Smoothness of the Convolution) If $f \in C_c^\infty(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. Then $f \star g \in C^\infty(\mathbb{R})$ and

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for all $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \{0, 1, 2, \dots\}$

Proof. First we note that $x \mapsto (f \star g)$ is continuous as $x_n \rightarrow x$ in \mathbb{R}^d since

$$(f \star g)(x_n) = \int_{\mathbb{R}^d} f(x_n - y)g(y) dy \xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^d} f(x - y)g(y) dy = (f \star g)(x)$$

We can apply Dominated convergence because

$$f(x_n - y)g(y) \rightarrow f(x - y)g(y) \quad \forall y \text{ as } f \text{ is continuous and } x_n \rightarrow x$$

and

$$|f(x_n - y)g(y)| \leq \|f\|_{L^\infty} |g(y)| \mathbf{1}(|y| \leq R) \in L^1(\mathbb{R}^d).$$

Where $R > 0$ satisfies $B(0, R) \supseteq \text{supp } f + \sup_n |x_n|$. Now we can compute the derivatives:

$$\begin{aligned}
\partial_{x_i}(f \star g)(x) &= \lim_{h \rightarrow 0} \frac{(f \star g)(x + he_i) - (f \star g)(x)}{h} \\
&= \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\
(\text{Dominated Convergence}) \quad &= \int_{\mathbb{R}^d} \lim_{h \rightarrow 0} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\
&= \int_{\mathbb{R}^d} (\partial_{x_i} f)(x - y)g(y) dy
\end{aligned}$$

We could apply Dominated Convergence since

$$\begin{aligned}
&\frac{f(x + he_i - y) - f(x - y)}{h} g(y) \xrightarrow{h \rightarrow 0} (\partial_{x_i} f)(x - y)g(y) \quad \text{as } f \in C^1 \\
&\left| \frac{f(x + he_i - y) - f(x - y)}{h} g(y) \right| \leq \|\partial_{x_i} f\|_{L^\infty} |g(y)| \mathbf{1}(|y| \leq R) \in L^1(\mathbb{R}^d)
\end{aligned}$$

where $B(0, R) \supseteq \text{supp}(f) + B(0, |x| + 1)$ and $\partial_{x_i}(f \star g) = (\partial_{x_i} f) \star g \in C(\mathbb{R}^d)$ since $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$. By induction we get $D^\alpha(f \star g) = (D^\alpha f \star g) \in C(\mathbb{R}^d)$. ■

Remark 3.5 Question: Is there a f s.t. $f \star g = g$ for all g ? In fact there is no regular function f that solves this formally:

$$f \star g = g \Rightarrow \widehat{f \star g} = \widehat{g} \Rightarrow \hat{f} \hat{g} = \hat{g} \Rightarrow \hat{f} = 1 \Rightarrow f \text{ is not a regular function!}$$

However, if f is the Dirac-Delta Distribution, $f = \delta_0$ then $\delta_0 \star g = g$ for all g . Formally:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \\ \int \delta_0 = 1 \end{cases}$$

In fact, if $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$, then $f_\epsilon \rightarrow \delta_0$ in an appropriate sense and $f_\epsilon \star g \rightarrow g$ for all g nice enough.

Theorem 3.6 (Approximation by convolution) Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon})$. Then for all $g \in L^p(\mathbb{R}^d)$, where $1 \leq p < \infty$, then

$$f_\epsilon \star g \rightarrow g \quad \text{in } L^p(\mathbb{R}^d)$$

Proof.

Step 1: Let $f, g \in C_c(\mathbb{R}^d)$. Then

$$\begin{aligned} (f_\epsilon \star g)(x) - g(x) &= \int_{\mathbb{R}^d} f_\epsilon(y) g(x-y) dy - g(x) \int_{\mathbb{R}^d} f_\epsilon(y) dy \\ &= \int_{\mathbb{R}^d} f_\epsilon(y) g(x-y) dy - \int_{\mathbb{R}^d} f_\epsilon(y) g(x) dy \\ &= \int_{\mathbb{R}^d} f_\epsilon(y) (g(x-y) - g(x)) dy \\ |(f_\epsilon \star g)(x) - g(x)| &= \left| \int_{\mathbb{R}^d} f_\epsilon(y) (g(x-y) - g(x)) dy \right| \\ &\leq \int_{\mathbb{R}^d} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\ &\leq \int_{|y| \leq R_\epsilon} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\ &\leq \left[\sup_{|z| \leq R} |g(x-z) - g(x)| \right] \underbrace{\int_{|y| \leq R_\epsilon} |f_\epsilon(y)| dy}_{\leq \|f_\epsilon\|_{L^1} = \|f\|_{L^1}} \xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned}$$

We have Dominated Convergence since:

$$(f_\epsilon \star g)(x) - g(x) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

and

$$|f_\epsilon \star g(x) - g(x)| \leq \|f\|_{L^1} \sup_{|z| \leq R_\epsilon} |g(x-z) - g(x)| \leq 2\|f\|_1 \|g\|_{L^\infty} \mathbf{1}(|x| \leq R_1).$$

Where $B(0, R_1) \supseteq \text{supp}(g) + B(0, R_\epsilon)$, thus $f_\epsilon \star g \rightarrow g$ in $L^p(\mathbb{R}^d)$. To remove the technical assumptions $f, g \in C_c(\mathbb{R}^d)$, then we use a density argument. We use the fact that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

Step 2: Let $g \in C_c(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$. Then there is $\{g_m\} \subseteq L^p(\mathbb{R}^d), g_m \rightarrow g$ in $L^p(\mathbb{R}^d)$. Then

$$\begin{aligned} \|f_\epsilon \star g - g\|_{L^p} &\leq \|f_\epsilon \star (g - g_m)\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\ (\text{Young}) \quad &\leq \|f_\epsilon\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\ &\leq \|f\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\ &\leq (\|f\|_{L^1} + 1) \|g - g_m\|_{L^p} + \|f \star g_m - g_m\|_{L^p} \end{aligned}$$

So we get:

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq (\|f\|_{L^1} + 1) \|g - g_m\|_{L^p} + \underbrace{\limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g_m - g_m\|_{L^p}}_{0 \text{ by step 1.}} \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Step 3: Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Take $\{f_m\} \subseteq C_c(\mathbb{R}^d)$, s.t.

$$\begin{cases} F_m \rightarrow g \in L^1(\mathbb{R}) \text{ as } m \rightarrow \infty \\ \int_{\mathbb{R}^d} F_m = 1 \text{ (it is possible since } \int_{\mathbb{R}^d} f = 1) \end{cases}$$

Define $F_{m,\epsilon}(x) = \epsilon^{-d} F_m(\epsilon^{-1}x)$ (recall $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$). Then:

$$\begin{aligned} f_\epsilon \star g - g &= (f_\epsilon - F_{m,\epsilon}) \star g + F_{m,\epsilon} \star g - g \\ \Rightarrow \|f_\epsilon - g\|_{L^p} &\leq \underbrace{\|f_\epsilon - F_{m,\epsilon} \star g\|_{L^p}}_{\stackrel{\text{Young}}{\leq} \|f_\epsilon - F_{m,\epsilon}\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p}} + \|F_{m,\epsilon} \star g - g\|_{L^p} \\ \Rightarrow \limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq \|f - F_m\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \quad \blacksquare \end{aligned}$$

Lemma 3.7 $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$

Proof. For all $g \in L^p(\mathbb{R}^d)$ there are step functions $(g_m)_m$ and $g_m \rightarrow g$ in $L^p(\mathbb{R}^d)$,

$$g_m(x) = \sum_{\substack{\Omega \\ \text{finite sum} \\ \Omega \subseteq \mathbb{R}^d \text{ measurable}}} \chi_\Omega(x) a_\Omega.$$

We can assume that Ω is open and bounded and we want to approximate χ_Ω by $C_c(\mathbb{R}^d)$. ■

Lemma 3.8 (Urnson) Define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \epsilon\}$$

Then there is a $\eta_\epsilon \in C_c(\mathbb{R}^d)$ s.t.

$$\begin{cases} 0 \leq \eta(x) \leq 1 & \forall x \in \mathbb{R}^d \\ \eta_\epsilon(x) = 1 & \text{if } x \in \Omega_\epsilon \\ \eta_\epsilon(x) = 0 & \text{if } x \notin \Omega \end{cases}$$

Lemma 3.9 (General Version of Urnson) If $A, B \subseteq \mathbb{R}^d$, A closed, B closed, $A \cap B = \emptyset$. Then

$$\eta(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}$$

Then $\eta \in C(\mathbb{R}^d)$, $0 \leq \eta \leq 1$ and $\eta = 0$ if $x \in B$, $\eta = 1$ if $x \in A$.

For example, this lemma can be applied to $A = \overline{\Omega_\epsilon} \subsetneq \Omega$ and $B = \mathbb{R}^d \setminus \Omega$ for $\Omega \subseteq \mathbb{R}^d$ open.

Theorem 3.10 (Appendix C4 in Evans) Let Ω be open in \mathbb{R}^d and for $\epsilon > 0$ define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \mathbb{R}^d \setminus \Omega) > \epsilon\}$$

Let $f \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} f = 1$, $\text{supp } f \subseteq B(0, 1)$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$. Then $\text{supp } f_\epsilon \subseteq B(0, \epsilon)$ and for all $g \in L_{loc}^p(\Omega)$ (i.e. $\mathbb{1}_K g \in L^p(\Omega)$ for all $K \subseteq \Omega$ compact) we have

- a) $g_\epsilon(x) = (f_\epsilon \star g)(x) = \int_{\mathbb{R}^d} f_\epsilon(x-y)g(y) dy = \int_\Omega f_\epsilon(x-y)g(y) dy$ is well-defined in Ω_ϵ and $g_\epsilon \in C^\infty(\Omega_\epsilon)$,
- b) $g_\epsilon \rightarrow g$ in $L_{loc}^p(\Omega)$ if $1 \leq p < \infty$ and $g_\epsilon(x) \rightarrow g(x)$ almost everywhere $x \in \Omega$,
- c) If $g \in C(\Omega)$, then $g_\epsilon(x) \rightarrow g(x)$ uniformly in any compact subset of Ω .

Proof. a) $D^\alpha(g_\epsilon) = (D^\alpha f_\epsilon) \star g \in C(\Omega_\epsilon)$

b) Already proved in \mathbb{R}^d space. ■

Corollary 3.11 (Lebesgue differentiation theorem) If $f \in L_{loc}^p(\mathbb{R}^d)$, then

$$\int_{B(x, \epsilon)} |f(y) - f(x)|^p dy \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

Exercise 3.12 (E 2.1) Let $u \in C^2(\mathbb{R}^2)$ be convex, i.e.

$$tu(x) + u(y)(1-t) \geq u(tx + (1-t)y)$$

for all $x, y \in \mathbb{R}^d$, $t \in [0, 1]$.

- a) Prove for all $x \in \mathbb{R}^d$ that the Hessian matrix H is positive semidefinite.
- b) Prove that u is sub-harmonic in \mathbb{R}^d .

Solution.

- a) In 1D: If u is convex $\Leftrightarrow u''(x) \geq 0$ for all $x \in \mathbb{R}$. In general: Taylor expansion for all $x, z \in \mathbb{R}^d$:

$$u(x) = u(z) + \nabla u(z)(x-z) + \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z)) \frac{(x-z)^\alpha}{\alpha!} ds$$

Note that we have $x = z + s(x-z)$ if $s = 1$. Use $z = tx + (1-t)y \Rightarrow x - z = (1-t)(x-y)$

$$tu(x) = tu(z) + t\nabla u(z)(1-t)(x-y) + t \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z)) \frac{[(1-t)(x-y)]^\alpha}{\alpha!} ds$$

$$(1-t)u(y) = (1-t)u(z) + (1-t)\nabla u(z)t(y-x) + (1-t) \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y-z)) \frac{[t(y-x)]^\alpha}{\alpha!} ds$$

$$\begin{aligned} &\Rightarrow tu(x) + (1-t)u(y) = u(z) + t \int_0^1 \dots + (1-t) \int_0^1 \dots \\ &\Rightarrow t \int_0^1 \dots + (1-t) \int_0^1 \dots \geq 0 \forall x, y, t, z = tx + (1-t)y \end{aligned}$$

$$t(1-t)^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z)) \frac{(x-y)^\alpha}{\alpha!} ds + (1-t)t^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y-z)) \frac{(y-z)^\alpha}{\alpha!} ds \geq 0$$

for all $x, y \in \mathbb{R}^d$, $t \in [0, 1]$, $z = tx + (1-t)y$. Divides for $t(1-t)$

$$(1-t) \int_0^1 \dots + \int_0^1 \dots \geq 0$$

Take $t \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(y + s(x-y)) \frac{(x-y)^\alpha}{\alpha!} ds \geq 0 \forall x, y \in \mathbb{R}^d$$

Take $y = x + a$, $a \in \mathbb{R}^d$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x + a + sa) \frac{a^\alpha}{\alpha!} ds \geq 0 \forall \epsilon > 0, \forall x, a \in \mathbb{R}^d$$

Take $\epsilon \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x) \frac{a^\alpha}{\alpha!} \geq 0 \Rightarrow \sum_{i,j=1, i \neq j} \partial_{x_i} \partial_{x_j} u(x) a_i a_j + \sum_{i=j=1}^d \partial_{x_i}^2 u(x) \frac{a_i^2}{2}$$

We get

$$\frac{1}{2} a^T H a \geq 0 \forall a(a_i)_{i=1}^d \in \mathbb{R}^d$$

$$\text{b) } H(x) \geq 0 \Rightarrow (\partial_i \partial_j u) \geq 0 \Rightarrow \text{Tr} H(x) \geq 0 \Rightarrow \sum_{i=1}^d \partial_{x_i}^2 u(x) \geq 0 \Rightarrow \Delta u(x) \geq 0 \forall x \in \mathbb{R}^d$$

■

Exercise 3.13 (E 2.2, Newton's Theorem) Let $d \geq 3$.

a) Prove that for all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|^{d-2}} = \frac{1}{\max(|x|, r)^{d-2}}$$

where $dS(y)$ is the surface measure on the sphere $\partial B(x, r) \subseteq \mathbb{R}^d$.

b) Let $0 \leq f_1, f_2 \in L^1(\mathbb{R}^d)$ be radial functions with $\int_{\mathbb{R}^d} f_i = M_i$. Prove that for all $z_1, z_2 \in \mathbb{R}^d$ we have

$$\iint_{\mathbb{R}^d} \frac{f_1(x - z_1) f_2(y - z_2)}{|x - y|^{d-2}} dx dy \leq \frac{M_1 M_2}{|z_1 - z_2|^{d-2}}$$

Moreover, prove that we have the equality if f_1, f_2 are compactly supported and $|z_1 - z_2|$ is sufficiently large.

Hint: For a) you may use the mean-value theorem (the function $\frac{1}{|x|^{d-2}}$ is harmonic in Ω if $0 \notin \Omega$). For b) you may use a) and polar coordinates.

Solution. a) Regard $d = 3$. The function $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$. We prove

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{\max(|x|, r)}$$

If $|x| > r$, then $0 \notin B(x, r + \epsilon)$. Then

$$y \mapsto \frac{1}{|y|}$$

is harmonic in $B(x, r + \epsilon)$. Then by the Mean Value Property:

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{|x|}$$

If $|x| < r$: Then $\frac{1}{|y|}$ is not harmonic in $B(x, r)$ since $0 \in B(x, r)$. Note

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \oint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$$

This function depends on x only via $|x|$.

$$\dots = \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - Ry|}$$

for all R rotation $SO(3)$, $dS(Ry) = dS(y)$

$$\begin{aligned} &= \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - y|} \\ &= \oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \\ \text{(Radial in } z) \quad &= \oint_{\partial B(0,|x|)} \left(\oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \right) dS(z) \\ \text{(Fubini)} \quad &= \oint_{\partial B(0,r)} \left(\oint_{\partial B(0,|x|)} \frac{dS(z)}{|z - y|} \right) dS(y) \\ \text{(case 1 since } |y| = r > |x|) \quad &= \oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) = \frac{1}{r} \end{aligned}$$

If $|x| = r$: Continuity: $x \mapsto \oint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$

b)

Remark 3.14 For $f \in C^{|\alpha|}, g \in C^{|\beta|}$:

$$D^{\alpha+\beta}(f \star g) = (D^\alpha f) \star (D^\beta g)$$

Lemma 3.15 If $d \geq 3$ and $f : \mathbb{R}^d \rightarrow \mathbb{R}$ radial. Then:

$$\left(\frac{1}{|x|^{d-2}} \star f \right) (x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} dy = \int_{\mathbb{R}^d} \frac{f(y)}{\max(|x|^{d-2}, |y|^{d-2})} dy$$

Proof. (d=3) Polar coordinates:

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{d\omega}{|x-rw|} \right] f(r) dr \\
(a) \quad &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{d\omega}{\max(|x|, r)} \right] f(r) dr \\
&= \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|, |y|)} dy \quad \blacksquare
\end{aligned}$$

Now, for $d = 3$, if f radial and non-negative with the lemma we get

$$(\star) \quad \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|, |y|)} dy \leq \int_{\mathbb{R}^3} \frac{f(y)}{|x|} dy = \frac{(\int_{\mathbb{R}^3} f(y) dy)}{|x|}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|} dx dy &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x)f_2(y)}{|x+z_1-y-z_2|} dx dy \\
&= \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{f_1(x)}{|x+z_1-y-z_2|} dx \right) f_2(y) dy \\
(\star) \quad &\leq \int_{\mathbb{R}^3} \frac{(\int_{\mathbb{R}^3} f_1(x) dx) f_2(y)}{|y+z_2-z_1|} dy \\
&\leq \frac{(\int_{\mathbb{R}^3} f_1)(\int_{\mathbb{R}^3} f_2)}{|z_1-z_2|} \quad \blacksquare
\end{aligned}$$

Exercise 3.16 (Bonus 2) a) Prove that $u(x) = \frac{1}{|x|}$ is sub-harmonic in $\mathbb{R}^2 \setminus \{0\}$.

b) Prove that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ radial, non-negative, measurable:

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy \geq \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)} dy$$

3.2 Fourier Transformation

Definition 3.17 (Fourier Transform) For $f \in L^1(\mathbb{R}^d)$ define

$$\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k \cdot x} dx, \quad k \cdot x = \sum_{i=1}^d k_i x_i$$

Theorem 3.18 (Basic Properties) 1. If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^\infty(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

2. For all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$. Moreover, \mathcal{F} can be extended to be a unitary transformation $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ s.t.

$$\|\mathcal{F}g\|_{L^2} = \|g\|_{L^2} \quad \forall g \in L^2(\mathbb{R}^d)$$

3. The inverse of \mathcal{F} can be defined as

$$(F^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(x) e^{2\pi i k x} dk$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

4. $\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k)$ as $(2\pi i k)^\alpha f(k) \in L^2(\mathbb{R}^d)$ ($k^\alpha = k_1^{\alpha_1} \dots k_d^{\alpha_d}$)
5. $\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k)$ if f, g are nice enough.

Theorem 3.19 (Hausdorff-Young-Inequality) Let $f \in L^p(\mathbb{R}^d)$. If $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\mathcal{F} : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$ is well-defined and

$$\|\mathcal{F}f\|_{L^q(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}.$$

Remark 3.20 If $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R}^d)$ we can write $f = f_1 + f_2$ when $f_1 \in L^1$, $f_2 \in L^2$, e.g.

$$f = \underbrace{f\mathbb{1}(|f| \geq 1)}_{f_1} + \underbrace{f\mathbb{1}(|f| < 1)}_{f_2}$$

$$\begin{aligned} \int_{\mathbb{R}^d} |f_2|^2 dy &= \int_{\mathbb{R}^d} |f|^2 \mathbb{1}(|f| < 1) \leq \int_{\mathbb{R}^d} |f|^p dy < \infty \\ \int_{\mathbb{R}^d} |f_1| dy &= \int_{\mathbb{R}^d} |f| \mathbb{1}(|f| \geq 1) \leq \int_{\mathbb{R}^d} |f|^p < \infty \end{aligned}$$

thus we can define $\hat{f} = \hat{f}_1 + \hat{f}_2$ well defined in $L^\infty(\mathbb{R}^d) + L^2(\mathbb{R}^d)$.

Proof of the Hausdorff-Young-Inequality 3.19. We need Riesz-Thorin interpolation theorem. If $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and $\Omega \subseteq \mathbb{R}^d$ open and

$$T : L^{p_0}(\Omega) + L^{p_1}(\Omega) \longrightarrow L^{q_0}(\Omega) + L^{q_1}(\Omega)$$

is a linear operator and

$$T : L^{p_0} \rightarrow L^{q_0}$$

and $\|T\|_{L^{p_i} \rightarrow L^{q_i}} \leq 1$ for $i = 0, 1$. Then,

$$T : L^{p_\theta} \rightarrow L^{q_\theta} \text{ and } \|T\|_{L^{p_\theta} \rightarrow L^{q_\theta}} \leq 1$$

for any $0 < \theta < 1$ where

$$\begin{cases} \frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1} \\ \frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1} \end{cases}.$$

Consider the Fourier Transform:

$$F : L^1 + L^1 \rightarrow L^2 + L^\infty$$

and

$$\begin{aligned} \|F\|_{L^1 \rightarrow L^\infty} &\leq 1 \text{ as } \|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1} & \forall f \in L^1 \\ \|F\|_{L^2 \rightarrow L^2} &= 1 \text{ as } \|\hat{f}\|_{L^2} = \|f\|_{L^2} & \forall f \in L^2 \\ \Rightarrow \|F\|_{L^{p_\theta} \rightarrow L^{q_\theta}} &\leq 1 & \forall \theta \in (0, 1) \end{aligned}$$

$$p_0 = 1, p_1 = 2, q_0 = \infty, q_1 = 2$$

$$\begin{aligned} \frac{1}{p_\theta} &= \frac{\theta}{p_0} + \frac{1-\theta}{p_1} = \theta + \frac{1-\theta}{2} = \frac{1+\theta}{2} \\ \frac{1}{q_\theta} &= \frac{\theta}{q_0} + \frac{1-\theta}{q_1} = \frac{1-\theta}{2} \\ \Rightarrow 1 &= \frac{1}{p_\theta} + \frac{1}{q_\theta} = \frac{1+\theta}{2} + \frac{1-\theta}{2} \end{aligned}$$

■

Exercise 3.21 (E 3.2) Let $1 \leq p, q, r \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Recall that if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^r(\mathbb{R}^d)$ by Young's Inequality, and its Fourier transform is well-defined by the Hausdorff-Young inequality. Prove that

$$\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k) \quad \forall k \in \mathbb{R}^d$$

Solution.

Step 1) $f, g \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \widehat{f \star g}(k) &= \int_{\mathbb{R}^d} (f \star g)(x) e^{-2\pi i k x} dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) g(y) e^{-2\pi i k x} dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) e^{-2\pi i k(x-y)} g(y) e^{-2\pi i k y} dx dy \\ (z(x) := x-y) \quad &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(z) e^{-2\pi i k z} dz \right) g(y) e^{-2\pi i k y} dy \\ &= \left(\int_{\mathbb{R}^d} f(z) e^{-2\pi i k z} dz \right) \left(\int_{\mathbb{R}^d} g(y) e^{-2\pi i k y} dy \right) = \hat{f}(k)\hat{g}(k) \end{aligned}$$

Step 2) $f \in L^p, g \in L^q$, find $f_n, g_n \in C_c^\infty$ s.t. $f_n \rightarrow f$ in L^p , $g_n \rightarrow g$ in L^q . Then we have $\widehat{f_n \star g_n} = \hat{f}_n \hat{g}_n$ almost everywhere. We get with the Hausdorff-Young Inequality (3.19)

$$\begin{aligned} \|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^{r'}} &\leq \|f \star g - f_n \star g_n\|_{L^r} \\ &= \|(f - f_n) \star g_n + f_n \star (g_n - g)\|_{L^r} \\ &\leq \|(f - f_n) \star g_n\|_{L^r} + \|f_n \star (g_n - g)\|_{L^r} \\ (\text{Young 3.3}) \quad &\leq \|f - f_n\|_{L^p} \|g_n\|_{L^q} + \|f_n\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Moreover:

$$\begin{aligned} \|\hat{f}_n \hat{g}_n - \hat{f} \hat{g}\|_{L^{r'}} &= \|(\hat{f}_n \hat{f}) \hat{g}_n + \hat{f}(\hat{g}_n - \hat{g})\|_{L^{r'}} \\ (\text{H\"older}) \quad &\leq \|\hat{f}_n - \hat{f}\|_{L^{p'}} \|\hat{g}_n\|_{L^{q'}} + \|\hat{f}\|_{L^{q'}} \\ (\text{Hausdorff-Young 3.19}) \quad &\leq \|f_n - f\|_{L^p} \|g_n\|_{L^q} + \|f\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \rightarrow \infty} 0 \\ \text{So } \hat{f}_n \hat{g}_n &\rightarrow \hat{f} \hat{g} \text{ in } L^{r'} \quad \widehat{f \star g} = \hat{f} \hat{g} \text{ in } L^{r'} \quad \frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'} \quad \blacksquare \end{aligned}$$

Remark 3.22 We want to apply the Fourier transform to find the solution of a PDE, e.g. the Poisson-Equation:

$$-\Delta u = f \text{ in } \mathbb{R}^d \Rightarrow |2\pi k|^2 \hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{1}{|2\pi k|^2} \hat{f}(k)$$

If we can find G s.t. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$, then

$$\begin{aligned} \hat{u}(k) &= \hat{G}(k) \hat{f}(k) = \widehat{G \star f} \\ \Rightarrow u(x) &= (G \star f)(x) = \int_{\mathbb{R}^d} G(x-y) f(y) dy \end{aligned}$$

Thus we need to compute

$$G(x) = \left(\frac{1}{|2\pi k|^2} \right)^\vee$$

It turns out for $d \geq 3$ that

$$G(x) = \frac{1}{d(d-2)|B_1||x|^{d-2}}.$$

In fact G is the fundamental solution of the Laplace Equation. To make it rigorous, we need to compute the Fourier transform of $\frac{1}{|x|^\alpha}$ for $0 \leq \alpha < d$.

Theorem 3.23 (Fourier Transform of $\frac{1}{|x|^\alpha}$ for $0 < \alpha < d$) We have formally

$$\widehat{\frac{c_\alpha}{|x|^\alpha}} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \quad \forall 0 < \alpha < d$$

Here

$$c_\alpha = \pi^{-\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda$$

More precisely, for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{c_\alpha}{|x|^\alpha} \star f = \left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee$$

Moreover if $\alpha > \frac{d}{2}$, then we also have

$$\left(\frac{c_\alpha}{|x|^\alpha} \star f \right)^\wedge = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k).$$

Proof.

$$\begin{aligned} \frac{c_\alpha}{|x|^\alpha} &= \frac{1}{|x|^\alpha} \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty e^{-\pi\lambda|x|^2} \lambda^{\frac{\alpha}{2}-1} d\lambda \\ \Rightarrow \frac{\hat{c}_\alpha}{|x|^\alpha}(k) &= \int_0^\infty \widehat{e^{-\pi\lambda|x|^2}}(k) \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi\frac{|k|^2}{\lambda}} \lambda^{\frac{\alpha}{2}-1} d\lambda \\ (\lambda \rightarrow \frac{1}{\lambda}) &= \int_0^\infty \lambda^{\frac{d}{2}} e^{-\pi|k|^2\lambda} \lambda^{-\frac{\alpha}{2}+1} \lambda^{-2} d\lambda = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \end{aligned}$$

Let $f \in C_c(\mathbb{R}^d)$. Then $\left(\frac{1}{|x|^\alpha} \star f \right)(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} f(y) dy$ is well defined as $\frac{1}{|x-y|} \in L^1_{loc}(\mathbb{R}^d, dy)$. It is bounded

$$\frac{1}{|x|^\alpha} \star f = \frac{1}{|x|^\alpha} \underbrace{\mathbb{1}(|x| \leq 1)}_{\in L^\infty(\mathbb{R}^d)} \star \underbrace{f}_{L^\infty} + \underbrace{\frac{1}{|x|^\alpha} \mathbb{1}(|x| > 1)}_{\in L^\infty} \star \underbrace{f}_{L^1} \in L^\infty(\mathbb{R}^d)$$

When $|x| \rightarrow \infty$:

$$\left(\frac{1}{|x|^\alpha} \star f \right)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^\alpha} dy = \int_{|y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy \sim \frac{\int_{\mathbb{R}^d} f(y) dy}{|x|^\alpha}$$

Note that $\underbrace{\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)}_{\text{bounded}} \in L^1(\mathbb{R}^d)$.

$$\begin{aligned} (...)\mathbb{1}(|k| \leq 1) + (...)\mathbb{1}(|k| > 1) \frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| \leq 1) &\leq \|f\|_{L^1} \frac{\mathbb{1}(|k| \leq 1)}{|k|^{d-\alpha}} \in L^1(\mathbb{R}^d, dk) \\ \frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| > 1) &\leq |\hat{f}(k)| \in L^2(\mathbb{R}^d, dK) \text{ as } f \in L^2(\mathbb{R}^d) \end{aligned}$$

Lemma 3.24 If $f \in C_c^\infty(\mathbb{R}^d)$, then $\hat{f} \in L^1(\mathbb{R}^d)$

Proof. (Exercise) Hint: $|\widehat{D^\alpha f}| = |2\pi k|^{|\alpha|} |\hat{f}(k)| \rightsquigarrow |\hat{f}(k)| \leq \frac{1}{|k|^{|\alpha|}} \text{ as } |k| \rightarrow \infty.$ ■

Compute:

$$\begin{aligned}
\left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee(x) &= \int_{\mathbb{R}^d} \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_{\mathbb{R}^d} \left(\int_0^\infty e^{-\pi |k|^2 \lambda} \lambda^{\frac{d-\alpha}{2}-1} d\lambda \right) \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} e^{-\pi |k|^2 \lambda} \hat{f}(k) e^{2\pi i k x} dk \right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(e^{-\pi k^2 \lambda} \hat{f}(x) \right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi \frac{x^2}{\lambda}}(k) \hat{f}(k) \right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi \frac{x^2}{\lambda}} \star f \right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \left(\int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi \frac{x^2}{\lambda}} \lambda^{\frac{d-\alpha}{2}-1} d\lambda \right) \star f
\end{aligned}$$

Assume $d > \alpha > \frac{d}{2}$. Then $\frac{c_\alpha}{|x|^\alpha} \star f \in L^\infty$ and behaves $\frac{c_\alpha(f)}{|x|^\alpha}$ as $|x| \rightarrow \infty$. This implies:

$$\int_{\mathbb{R}^d} \left| \frac{c_\alpha}{|x|^\alpha} \star f \right|^2 \leq c + \int_{|x| \geq R} \frac{c}{|x|^{2d}} dx < \infty$$

Thus the Fourier Transform $\widehat{\frac{c_\alpha}{|x|^\alpha} \star f}$ exists. Combining with

$$\begin{aligned}
\frac{c_\alpha}{|x|^\alpha} \star f &= \left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee \\
\Rightarrow \widehat{\frac{c_\alpha}{|x|^\alpha} \star f} &= \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)
\end{aligned}$$

Remark 3.25 If $f \in L^p$, $1 \leq p \leq 2$ is such that $\tilde{f}_1 + \tilde{f}_2 = f = f_1 + f_2$ with $f_1, \tilde{f}_1 \in L^1, f_2, \tilde{f}_2 \in L^2$. Do we have that $\hat{f}_1 + \hat{f}_2 = \hat{\tilde{f}}_1 + \hat{\tilde{f}}_2 \in L^1 \cap L^2$? In fact, from $f_1 + f_2 = \tilde{f}_1 + \tilde{f}_2$ we get

$$\begin{aligned}
\underbrace{f_1 - \tilde{f}_1}_{\in L^1} &= \underbrace{\tilde{f}_2 - f_2}_{\in L^2} \in L^1 \cap L^2 \\
\Rightarrow \hat{f}_1 - \hat{\tilde{f}}_1 &= \hat{\tilde{f}}_2 - \hat{f}_2 \quad \Rightarrow \quad \hat{f}_1 + \hat{f}_2 = \hat{\tilde{f}}_1 + \hat{\tilde{f}}_2.
\end{aligned}$$

Lemma 3.26 (Fourier Transform of Gaussians) In \mathbb{R}^d ,

$$\widehat{e^{-\pi |x|^2}} = e^{-\pi |k|^2}$$

More generally for all $\lambda > 0$:

$$\widehat{e^{-\pi \lambda^2 |x|^2}} = \lambda^{-d} e^{-\pi \frac{|k|^2}{\lambda^2}}$$

(exercise)

Remark 3.27 If $d \geq 3$

$$\begin{aligned}\hat{G}(k) &= \frac{1}{|2\pi k|^2} \\ \Rightarrow G(x) &= \left(\frac{1}{|2\pi k|^2} \right)^\vee = \frac{1}{d(d-2)(k)|x|^{d-2}} = \Phi(x)\end{aligned}$$

3.3 Theory of Distribution

In the following let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\Omega \subseteq \mathbb{R}^d$ be open.

- $D(\Omega) = C_c^\infty(\Omega)$ the space of test functions.
- We say that $\phi_n \rightarrow \phi$ in $D(\Omega)$ if for all $K \subseteq \Omega$ compact we have that

$$\begin{cases} \text{supp}(\phi_n - \phi) \subseteq K \text{ for all } n \\ \|D^\alpha(\phi_n - \phi)\|_{L^\infty(K)} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } \alpha \end{cases}$$
- $D'(\Omega) = \{T : D(\Omega) \rightarrow \mathbb{K} \text{ linear and continuous}\}$ the space of distributions.

Motivation: $L^2(\Omega)' = L^2(\Omega)$, $(L^p(\Omega))' = (L^q(\Omega))$, $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3.28 ("normal functions" are distributions) If $f \in L^1_{loc}(\Omega)$, then $T = T_f$ defined by:

$$T(\phi) = \int_{\Omega} f(x)\phi(x) dx$$

is a distribution for all $\phi \in D(\Omega)$, i.e. $T \in D'(\Omega)$. Indeed, it is clear that $T(\phi)$ is well-defined for all $\phi \in D(\Omega)$ and $\phi \mapsto T(\phi)$ is linear. Let us check that $\phi \mapsto T(\phi)$ is continuous. Take $\phi_n \rightarrow \phi$ in $D(\Omega)$ and prove that $T(\phi_n) \rightarrow T(\phi)$. Since $\phi_n \rightarrow \phi$ in $D(\Omega)$, there is a compact K s.t. $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K \subseteq \Omega$.

Question: Why is $f \mapsto T_f$ injective?

Theorem 3.29 (Fundamental theorem of calculus of variants) Let $\Omega \subseteq \mathbb{R}^d$ be open. If $f, g \in L^1_{loc}(\Omega)$ and $\int_{\Omega} f\phi dy = \int_{\Omega} g\phi dy$ for all $\phi \in D(\Omega)$, then $f = g$ in $L^1_{loc}(\Omega)$

Example 3.30 (Dirac delta function) Let $\Omega \subseteq \mathbb{R}^d$ open and let $x_0 \in \Omega$. Define $T : D(\Omega) \rightarrow \mathbb{K}$ by $T(\phi) = \phi(x_0)$. Then $T \in D'(\Omega)$ and we denote it by δ_{x_0} . It is clear that $\phi \mapsto T(\phi) = \phi(x_0)$ is well-defined and linear for all $\phi \in D(\Omega)$. Take $\phi_n \rightarrow \phi$ in $D(\Omega)$ and prove $T(\phi_n) \rightarrow T(\phi)$, i.e. $\phi_n(x_0) \rightarrow \phi(x_0)$ (obvious.)

Example 3.31 (Principle Value) The function $f(x) = \frac{1}{x}$ is not in $L^1_{loc}(\mathbb{R})$, but we can still define

$$\int_{\mathbb{R}} f(x)\phi(x) dx = \int_{\mathbb{R}} \frac{\phi(x)}{x} dx$$

for all $\phi \in D(\mathbb{R})$ s.t. $\phi(0) = 0$. In fact,

$$\phi(x) = |\phi(x) - \phi(0)| \leq x \sup |\phi'|,$$

so $\frac{|\phi(x)|}{|x|} \in L^\infty(\mathbb{R})$ and compactly supported. So $\frac{\phi(x)}{x} \in L^1(\mathbb{R})$. Define $T : D(\mathbb{R}) \rightarrow \mathbb{K}$ by

$$T(\phi) = \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx \quad \forall \phi \in D(\mathbb{R}) \text{ s.t. } \phi(0) = 0$$

We write $T = \frac{1}{x}$ and check that $T \in D'(\mathbb{R})$: For all $\epsilon > 0$ we have

$$\left| \frac{\phi(x)}{x} \right| \leq \frac{\|\phi\|_{L^\infty}}{\epsilon}$$

for all $|x| \geq \epsilon$ and ϕ is compactly supported. So we get for all $\epsilon > 0$:

$$\mathbb{1}(|x| \geq \epsilon) \frac{\phi(x)}{x} \in L^1(\mathbb{R}) \rightsquigarrow \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx < \infty$$

We can write:

$$\int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx = \int_{|x| \geq 1} \frac{\phi(x)}{x} dx + \int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx$$

The second part can be written as:

$$\int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx = \int_{\epsilon}^1 \frac{\phi(x)}{x} dx + \int_{-1}^{-\epsilon} \frac{\phi(x)}{x} dx = \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx$$

Since $\phi \in C_c^\infty(\mathbb{R})$ it holds that $|\phi(x) - \phi(-x)| \leq 2\|\phi'\|_{L^\infty}(x)$.

$$\begin{aligned} \Rightarrow \frac{\phi(x) - \phi(-x)}{x} &\in L^\infty(\mathbb{R}) \Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^1((0, 1)) \\ &\Rightarrow \int_0^1 \frac{\phi(x) - \phi(-x)}{x} dx = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx \end{aligned}$$

Remark 3.32 The function $\frac{1}{|x|^d}$ is not in $L_{loc}^1(\mathbb{R}^d)$ but $\exists T \in D'(\mathbb{R}^d)$ s.t. $T(\phi) = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^d} dx$ for all $\phi \in C_c^\infty(\mathbb{R}^d)$ s.t. $\phi(0) = 0$

Definition 3.33 (Derivatives of distributions) Let $\Omega \subseteq \mathbb{R}^d$ and $T \in D'(\Omega)$. Define for $\alpha \in \mathbb{N}^d$:

$$\begin{aligned} D^\alpha T : D(\Omega) &\longrightarrow \mathbb{K} \\ \phi &\longmapsto (-1)^{|\alpha|} T(D^\alpha \phi) \end{aligned}$$

Motivation: $f \in C_c^\infty(\Omega)$

$$\int_{\Omega} (D^\alpha f) \phi = (-1)^{|\alpha|} \int_{\Omega} f (D^\alpha \phi)$$

„If the classical derivative exists, then it is the same as the distributional derivative.“
We write

$$(D^\alpha T)(\phi) = T_{D^\alpha f}(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi).$$

Remark 3.34 For all $T \in D'(\Omega)$ it holds $D^\alpha T \in D'(\Omega)$ for all $\alpha \in \mathbb{N}^d$. Clearly

$$\phi \longmapsto (D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

is linear. Moreover, if $\phi_n \rightarrow \phi$ in $D(\Omega)$, then $D^\alpha \phi_n \rightarrow D^\alpha \phi$ in $D(\Omega)$, so

$$(D^\alpha T)(\phi_n) = (-1)^{|\alpha|} T(D^\alpha \phi_n) \xrightarrow{n \rightarrow \infty} (-1)^{|\alpha|} T(D^\alpha \phi) = (D^\alpha T)(\phi)$$

Example 3.35 Consider $f : x \mapsto |x|$, then $f \in C(\mathbb{R})$ but $f \notin C^1(\mathbb{R})$. However,

$$f'(x) = g(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \in L^1_{loc} \quad \text{in } D'(\mathbb{R})$$

Lets check $f' = g$, i.e. $-f(\phi') = f'(\phi) \stackrel{!}{=} g(\phi)$ for all $\phi \in D(\mathbb{R})$. Thus we need to prove:

$$-\int_{\mathbb{R}} f(x)\phi'(x) dx = \int_{\mathbb{R}} g(x)\phi(x) dx \quad \forall \phi \in D(\mathbb{R})$$

namely we have to show

$$\underbrace{-\int_{\mathbb{R}} |x|\phi'(x) dx}_{:= (\star)} = \int_0^{\infty} \phi(x) dx - \int_{-\infty}^0 \phi(x) dx.$$

Now we have

$$(\star) = -\int_0^{\infty} x\phi'(x) dx + \int_{-\infty}^0 x\phi'(x) dx.$$

By integration by parts we have

$$\int_0^{\infty} x\phi'(x) dx = \underbrace{[x\phi(x)]_0^{\infty}}_{=0} - \int_0^{\infty} \phi(x) dx = -\int_0^{\infty} \phi(x) dx$$

and similary

$$\int_{-\infty}^0 x\phi'(x) dx = -\int_{-\infty}^0 \phi(x) dx.$$

Thus $f' = g$ in $D'(\Omega)$. We claim that $g' = 2\delta_0$ in $D'(\mathbb{R})$. In fact, for all $\phi \in D(\mathbb{R})$, then:

$$\begin{aligned} g'(\phi) &= -g(\phi') = -\int_{\mathbb{R}} g\phi' dx = -\int_{-\infty}^0 (-1)\phi' dx - \int_0^{\infty} (1)\phi' dx \\ &= -\int_0^{\infty} \phi' dx + \int_{-\infty}^0 \phi' dx = [\phi(0) - \underbrace{\phi(\infty)}_{=0}] + [\phi(0) - \underbrace{\phi(-\infty)}_{=0}] \\ &= 2\phi(0) = 2\delta_0(\phi) \end{aligned}$$

So $g' = 2\delta_0$ in $D'(\mathbb{R})$.

Exercise 3.36 Prove that $(D^\alpha \delta_x)(\phi) = (-1)^{|\alpha|}(D^\alpha \phi)(x)$ for all $\phi \in D(\mathbb{R})$ for all $x \in \mathbb{R}$.

Definition 3.37 (Convergence of distributions) Let $\Omega \subseteq \mathbb{R}^d$ be open, then

$$T_n \xrightarrow{n \rightarrow \infty} T$$

in $D'(\Omega)$ if $T_n(\phi) \xrightarrow{n \rightarrow \infty} T(\phi)$ for all $\phi \in D(\Omega)$.

Exercise 3.38 Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ For $\epsilon > 0$, define $f_\epsilon(x) = \epsilon^{-d}f(\epsilon^{-1}x)$. Then: $f_\epsilon \rightarrow \delta_0$ in $D'(\Omega)$.

Exercise 3.39 Let $\Omega \subseteq \mathbb{R}^d$ be open and $T_n \rightarrow T$ in $D'(\Omega)$. Then: $D^\alpha T_n \rightarrow D^\alpha T$ in $D'(\Omega)$ for all $\alpha = (\alpha_1, \dots, \alpha_d)$

Definition 3.40 (Convolution of distributions) Let $T \in D'(\mathbb{R})$ and $f \in L_c^\infty(\mathbb{R}^d)$. Define

$$(T \star f)(y) = T(f_y)$$

We write $f_y(x) = f(x - y)$ and $\tilde{f}(x) = f(-x)$.

Theorem 3.41 Let $T \in D'(\mathbb{R})$. Then for all $f \in D(\mathbb{R})$:

1. $y \mapsto T(f_y)$ is $C^\infty(\mathbb{R}^d)$ and

$$D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D^\alpha f_y)$$

2. If $g \in L^1(\mathbb{R}^d)$ and g is compactly supported, then

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(\underbrace{f \star g}_{\in C_c^\infty(\mathbb{R})})$$

Proof. 1. We prove that $y \mapsto T(f_y)$ is continuous. Take $y_n \rightarrow y$ in \mathbb{R}^d , then:

$$T(f_{y_n}) \rightarrow T(f_y)$$

since $f_{y_n} \rightarrow f_y$ in $D(\mathbb{R}^d)$. We check this: Since $f \in C_c^\infty(\mathbb{R}^d)$, it holds that $\text{supp } f \subseteq B(0, R) \subseteq \mathbb{R}^d$. Since $y_n \rightarrow y$ in \mathbb{R}^d . We have $\sup_n |y_n| < \infty$. Thus f_{y_n}, f_y are supported in $\overline{B(0, R + \sup_n |y_n|)} = K$ compact. Moreover

$$|f_{y_n}(x) - f_y(x)| = |f(x - y_n) - f(x - y)| \leq \|\nabla f\|_{L^\infty} \|y_n - y\| \rightarrow 0$$

So we get $\|f_{y_n} - f_y\|_{L^\infty} \rightarrow 0$ Similarly:

$$\|D^\alpha f_{y_n} - D^\alpha f_y\|_{L^\infty} \rightarrow 0 \quad \blacksquare$$

Exercise 3.42 (E 3.4) Compute the Fourier Transform of the Gaussian.

Exercise 3.43 (Bonus 3) Let $f \in L^1(\mathbb{R}^d)$ such that

$$|\hat{f}(k)| \leq \frac{C_N}{(1 + |k|)^N}$$

for all $k \in \mathbb{R}^d$, for all $N \geq 1$. (C_N is independent of k). Prove that $f \in C^\infty(\mathbb{R}^d)$

($f \in C^\infty$) i.e. $\exists \tilde{f} \in C^\infty$ s.t. $f = \tilde{f}$ a.e.

Theorem 3.44 Take $T \in D'(\mathbb{R})$, $f \in C_c^\infty(\mathbb{R}^d) = D(\mathbb{R}^d)$, $f_y(x) = f(x - y)$

- a) $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D_x^\alpha f_y)$
- b) $\forall g \in L^1(\mathbb{R}^d)$ and compactly supported

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(\underbrace{f \star g}_{\in C_c^\infty})$$

Proof. a) $y \mapsto T(f_y)$ is continuous since $y_n \rightarrow y$ in \mathbb{R}^d , then $f_{y_n} \rightarrow f_y$ implies $T(f_{y_n}) \rightarrow T(f_y)$. Let's check that $y \mapsto T(f_y) \in C^1$:

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} = \lim_{h \rightarrow 0} T\left(\frac{f_{y-he_i} - f_y}{h}\right)$$

We have $\frac{f_{y-he_i} - f_y}{h} \xrightarrow{h \rightarrow 0} (\partial_i f)_y$ in $D(\mathbb{R}^d)$

- $\exists K$ compact set such that $\text{supp}(f_{y-he_i} - f_y), \text{supp } \partial_i f \subseteq K$ as $|h|$ small.

$$\begin{aligned} & \bullet \frac{f_{y-he_i}(x) - f_y(x)}{h} - (\partial_i f)_y(x) \\ &= \frac{f(x - y + he_i) - f(x - y)}{h} - (\partial_i f)(x - y) \end{aligned}$$

$$\left| \int_0^1 \partial_i f(x - y + the_i) dt - \partial_i f(x - y) \right| \xrightarrow{h \rightarrow 0} 0 \text{ uniformly in } x$$

Similary:

$$\begin{aligned} & \left| D_x^\alpha \left(\frac{f(x - y + he_i) - f(x - y)}{h} - (\partial_i f)(x - y) \right) \right| \\ &= \left| \frac{D^\alpha f(x - y + he_i) - D^\alpha f(x - y)}{h} - \partial_i(D^\alpha f)(x - y) \right| \xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

uniformly in x . Conclude:

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} \xrightarrow{h \rightarrow 0} T((\partial_i f)_y) \in C(\mathbb{R}^d)$$

So we get that $y \mapsto T(f_y) \in C^1$ and $-\partial_{y_i} T(f_y) = T((\partial_i f)_y)$

By induction:

$$D_y^\alpha T(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y) = (D^\alpha T)(f_y) \quad \forall \alpha \in \mathbb{N}^d$$

b) Heuristic: $T = T(x)$

$$\begin{aligned} \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \int_{\mathbb{R}^d} g(y) \left(\int_{\mathbb{R}^d} T(x) f(x - y) dx \right) dy \\ &= \int_{\mathbb{R}^d} T(x) \left(\int_{\mathbb{R}^d} g(y) f(x - y) dy \right) dx \\ &= \int_{\mathbb{R}^d} T(x) (f \star g)(x) dx = T(f \star g) \end{aligned}$$

Step 1: $g \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \text{(Riemann Sum)} \quad \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) T(f_{y_j}) \\ &= \lim_{\Delta_N \rightarrow 0} T \left(\Delta_N \sum_{j=1}^N g(y_j) f_{y_j} \right) \\ &= T(f \star g) \end{aligned}$$

because

$$\begin{aligned} \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f_{y_j}(x) &\rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d) \\ \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) &\xrightarrow{\text{Riemann}} \int_{\mathbb{R}^d} g(y) f(x - y) dy = (f \star g)(x) \end{aligned}$$

Proof of:

$$\lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d)$$

1) Since $f, g \in C_c^\infty$ we have $f \star g \in C_c^\infty$. And we have

$$x \mapsto \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \in C^\infty$$

since $f \in C^\infty$ supported in $(\text{supp } g + \text{supp } f)$. So all functions are C_c^∞ and supported in $(\text{supp } g + \text{supp } f)$.

2)

$$\left| \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) - \int_{\mathbb{R}^d} g(y) f(x - y) dy \right| \xrightarrow{\Delta_N \rightarrow 0} 0$$

uniformly in x . (Result from the Riemann-Sum)

3)

$$\begin{aligned} & \left| D_x^\alpha (\Delta_N \sum_{j=1}^N g(y_j) f(x - y) - (f \star g)(x)) \right| \\ &= \left| \Delta_N \sum_{j=1}^N g(y_j) D^\alpha f(x - y) - (D^\alpha f) \star g(x) \right| \xrightarrow{\Delta_N \rightarrow 0} 0 \end{aligned}$$

uniformly in x for all α .

Step 2: Take $g \in L^1(\mathbb{R}^d)$ and compactly supported. Then $\exists \{g_n\} \subseteq C_c^\infty(\mathbb{R}^d)$, $\text{supp } g_n \subseteq \text{supp } g + B(0, 1)$ such that $g_n \rightarrow g$ in $L^1(\mathbb{R}^d)$. By Step 1:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy = T(g_n \star f)$$

Take $n \rightarrow \infty$:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy \rightarrow \int_{\mathbb{R}^d} g(y) T(f_y) dy$$

since $g_n \rightarrow g$ in L^1 compactly supported and $y \mapsto T(f_y) \in C^\infty \subseteq L^\infty(K)$. Moreover (exercise):

$$\underbrace{g_n \star f}_{\in C_c^\infty} \rightarrow g \star f \quad \text{in } D(\mathbb{R}^d)$$

So $T(g_n \star f) \xrightarrow{n \rightarrow \infty} T(g \star f)$. Finally we obtain:

$$\int g(y) T(f_y) dy = T(g \star f) \quad \blacksquare$$

Theorem 3.45 Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $T \in D'(\Omega)$ and $f \in C_c^\infty(\Omega)$. Denote

$$\Omega_f = \{y \in \mathbb{R}^d \mid \text{supp } f_y = y + \text{supp } f \subseteq \Omega\}$$

a) $y \mapsto T(f_y) \in C^\infty(\Omega_f)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y)$

b) For all $g \in L^1(\Omega_g)$ compactly supported in Ω_f and it holds:

$$\int_{\Omega} g(y)T(f_y) dy = T(f \star g).$$

Theorem 3.46 Let $T \in D'(\Omega)$ s.t. $\nabla T = 0$ in $D'(\Omega)$. Then: $T = \text{const.}$ in Ω .

Proof. ($\Omega = \mathbb{R}^d$) We show for all $f \in C_c^\infty$ that $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $\partial_{y_i} T(f_y) = (\partial_j T)(f_y) = 0$ for all $i = 1, \dots, d$. Then by the result of the theorem for C^∞ functions, $y \mapsto T(f_y) = \text{const}$ independent of y . Consequently:

$$T(f_y) = T(f_0) = T(f) \quad \forall y \in \mathbb{R}^d \quad \forall f \in C_c^\infty(\mathbb{R}^d)$$

For any $g \in C^\infty(\mathbb{R}^d)$:

$$\left(\int_{\mathbb{R}^d} g dy \right) T(f) = \int_{\mathbb{R}^d} g(y)T(f_y) dy = T(f \star g) = T(g \star f) = \left(\int_{\mathbb{R}^d} f dy \right) T(g)$$

So $\frac{T(f)}{\int_{\mathbb{R}^d} f}$ is independent of f (as soon as $\int f \neq 0$). So we get that $T(f) = \text{const} \int_{\mathbb{R}^d} f$, where const is independent of f . \blacksquare

Remark 3.47 If $u \in C^1(\mathbb{R}^d)$, then:

$$u(x+y) - u(x) = \int_0^1 \sum_{j=1}^d y_j (\partial_j u)(x + ty_j) dt = \int_0^1 y \nabla u(x + ty) dt$$

So we get that if $\nabla u = 0$, then $u(x+y) - u(x) = 0$ for all x, y , so $u = \text{const.}$

Theorem 3.48 (Taylor expansion for distributions) Let $T \in D'(\mathbb{R}^d)$ and $f \in C_c^\infty(\mathbb{R}^d)$. Then $y \mapsto T(f_y) \in C^\infty$ and

$$T(f_y) - T(f) = \int_0^1 \sum_{j=1}^d y_j (\partial_j T)(f_{ty}) dt.$$

In particular, if $g \in L_{loc}^1$ and $\nabla g \in L_{loc}^1$, then $\forall y \in \mathbb{R}^d$:

$$g(x+y) - g(x) = \int_0^1 g(x+ty)y dt$$

for a.e. $x \in \mathbb{R}^d$.

Proof. $y \mapsto T(f_y)$ is C^∞ and $\frac{d}{dt}[T(f_{ty})] = (\nabla T)(f_{ty})y$ So we get

$$\begin{aligned} T(f_y) - T(f) &= \int_0^1 \frac{d}{dt}(T(f_{ty})) dt \\ &= \int_0^1 (\nabla T)(f_{ty})y dt \\ &= \int_0^1 \sum_{j=1}^d (\partial_j T)(f_{ty})y_j dt \end{aligned} \quad \blacksquare$$

Corrolary 3.49 Let $g \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\partial_j g \in L^1_{loc}(\mathbb{R}^d)$ for all $j = 1, 2, \dots, d$ (i.e. $g \in W^{1,1}_{loc}(\mathbb{R}^d)$). Then for all $y \in \mathbb{R}^d$:

$$\begin{aligned} g(x+y) - g(x) &= \int_0^1 y \cdot \nabla g(x+ty) dt \\ &= \int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \end{aligned}$$

for a.e. x .

Proof. For all $f \in C_c^\infty$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)[g(x+y) - g(x)] dx &= \int_{\mathbb{R}^d} g(x)[f(x-y) - f(x)] dx \\ &= g(f_y) - g(f) \\ &= \int_0^1 \sum_{j=1}^d y_j (\partial_j g)(f_{ty}) dt \\ &= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \sum_{j=1}^d y_j \left[\int_{\mathbb{R}^d} (\partial_j g)(x) f_{ty}(x) dx \right] \\ &= \int_0^1 \sum_{j=1}^d y_j \left[\int_{\mathbb{R}^d} (\partial_j g)(x+ty) f(x) dx \right] dt \\ &= \int_{\mathbb{R}^d} f(x) \left[\int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \right] dx \end{aligned}$$

For all $\phi \in C_c^\infty$: $= g(x+y) - g(x)$ a.e. $x \in \mathbb{R}^d$. ■

Remark 3.50 If $T \in D'(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open, if $y \nabla T = 0$, then $T = \text{const}$.

Theorem 3.51 (Equivalence of the classical and distributional derivatives) Let $\Omega \subseteq \mathbb{R}^d$. Then the following are equivalent:

1. $T \in D'(\Omega)$ s.t. $\partial_{x_i} T = g_i \in C(\Omega)$ for all $i = 1, \dots, d$.
2. $T = f \in C^1(\Omega)$ and $g_i = \partial_{x_i} f$

Proof.

(2) \Rightarrow (1): If $T = f \in C^1(\Omega)$, then: $\partial_{x_i} f \in C(\Omega)$.

$$\partial_{x_i} T(\phi) = -T(\partial_{x_i} \phi) = - \int_{\Omega} f(\partial_{x_i} \phi) = \int_{\Omega} (\partial_{x_i} f) \phi$$

for all $\phi \in D(\Omega)$, so $\partial_{x_i} T = \partial_{x_i} f$.

(1) \Rightarrow (2): Why is $T = f \in C^1(\Omega)$? As $\partial_{x_i} f = g_i$:

$$f(x+y) - f(x) = \int_0^1 \nabla f(x+ty) y dt = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i dt$$

So we get

$$f(y) = f(0) + \int_0^1 \sum_{i=1}^d g_i(ty) g_i dt.$$

We expect that $f \in C^1$ and $\partial_{x_i} f = g_i$. But this is not trivial to prove.

$$\begin{aligned} \frac{f(y + he_i) - f(y)}{h} &= \int_0^1 \sum_{i=1}^d [g_i(ty + the_i)(y_i + h\delta_{ij})] dt \\ &= \int_0^1 g_i(ty + the_i) dt + \int_0^1 \sum_{j \neq i} \frac{[g_i(ty + the_i) - g_i(ty)]}{h} y_j dt \\ &\xrightarrow{h \rightarrow 0} \int_0^1 g_i(ty) dt + \text{is difficult ...} \end{aligned}$$

Lets take $\phi \in C_c^\infty$, then:

$$\begin{aligned} T(\phi_y) - T(\phi) &= \int_0^1 \underbrace{\nabla T}_{(g_i)_{i=1}^d}(\phi_{ty}) y dt \\ &= \int_0^1 \sum_{i=1}^d \left(\int_{\Omega} g_i(x) \underbrace{\phi_{ty}}_{=\phi(x-ty)} dx \right) dt \\ &= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_0^1 g_i(x) \phi(x-ty) y_i dt \right) dx \\ &= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_0^1 g_i(x+ty) \phi(x) y_i dt \right) dx \\ &= \int_{\mathbb{R}^d} \left(\sum_i \int_0^1 g_i(x+ty) y_i dt \right) \phi(x) dx \end{aligned}$$

Integrating against $\psi(y)$ with $\psi \in C_c^\infty$:

$$\begin{aligned} &\int_{\mathbb{R}^d} T(\phi_y) \psi(y) dy - T(\phi) \int_{\mathbb{R}^d} \psi(y) dy \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \sum_i \int_0^1 g_i(x+ty) y_i \psi(y) dt dy \right) \phi(x) dx \\ &\Rightarrow T(\phi \star \psi) - T(\phi) \int \psi = \dots \\ &\Rightarrow \int_{\mathbb{R}^d} T(\psi_y) \phi(y) dy - T(\phi) \int \psi = \dots \end{aligned}$$

Take $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\int \psi = 1$. Then:

$$T(\phi) = \underbrace{\int_{\mathbb{R}^d} T(\psi_x) - \left(\int_{\mathbb{R}^d} \sum_{i=1}^d \int_0^1 g_i(x+ty) y_i \psi(y) dt dy \right)}_{f(x)} \phi(x) dx$$

for all $\phi \in C_c^\infty$, so $T = f \in C(\Omega)$. Thus $T = f \in C(\Omega)$ and $\partial_{x_i} T = g_i \in C(\Omega)$. Then we need to prove that $f \in C^1(\Omega)$ and $\partial_{x_i} f = g_i$ (classical derivative). Since

$f \in W_{loc}^{1,1}$:

$$f(x+y) - f(x) = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i dt \quad \forall x, y$$

In particular:

$$\begin{aligned} \frac{f(x+he_i) - f(x)}{h} &= \int_0^1 \frac{1}{h} \sum_{i=1}^d g_i(x+the_i) h \delta_{ij} dt \\ &= \int_0^1 g_i(x+the_i) dt \xrightarrow{h \rightarrow 0} g_i(x) \end{aligned}$$

So we get $\partial_{x_i} f(x) = g_i(x) \in C(\Omega)$ in the classical sense. So $f \in C^1(\Omega)$. \blacksquare

Definition 3.52 (Sobolev Spaces) Let $\Omega \subseteq \mathbb{R}^d$ be open. We define for $1 \leq p \leq \infty$:

$$\begin{aligned} W^{1,p}(\Omega) &= \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega) \ \forall i = 1, \dots, d\} \\ W^{k,p}(\Omega) &= \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leq k\} \\ W_{loc}^{k,p}(\Omega) &= \{f \in L_{loc}^p(\Omega) \mid D^\alpha f \in L_{loc}^p(\Omega) \ \forall |\alpha| \leq k\} \end{aligned}$$

Theorem 3.53 (Approximation of $W_{loc}^{1,p}(\Omega)$ by $C^\infty(\Omega)$) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $f \in W_{loc}^{1,p}(\Omega)$. Then there exists $\{f_n\} \subseteq C^\infty(\Omega)$ such that $f_n \rightarrow f$ in $W_{loc}^{1,p}(\Omega)$, i.e. for all $K \subseteq \Omega$ compact: $\|f_n - f\|_{L^p(K)} + \sum_{i=1}^d \|\partial_{x_i}(f_n - f)\|_{L^p(K)} \rightarrow 0$.

Proof. Case $\Omega = \mathbb{R}^d$: Take $g \in C_c^\infty$, $\int g = 1$, $g_\epsilon(x) = \epsilon^{-d} g(\epsilon^{-1}x)$. Then $g_\epsilon \star f \in C_c^\infty$. Since $f \in L_{loc}^p(\Omega)$ we have $g_\epsilon \star f \rightarrow f$ in L_{loc}^p as $\epsilon \rightarrow 0$. Moreover $\partial_{x_i}(g_\epsilon \star f) = (g_\epsilon \star \partial_{x_i} f) \xrightarrow{\epsilon \rightarrow 0} \partial_{x_i} f$ in L_{loc}^p . Then we can take $f_n = g_{\frac{1}{n}} \star f$. \blacksquare

Remark 3.54 In general, if we want to compute the distributional derivative $D^\alpha f$, then we can find $f_n \rightarrow f$ in $D'(\Omega)$ and compute $D^\alpha f_n$. Then $D^\alpha f_n \rightarrow D^\alpha f$ in $D'(\Omega)$. As an example we can compute $\nabla|f|$ with $f \in W_{loc}^{1,p}(\Omega)$.

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Theorem 3.55 (Chain Rule) Let $G \in C^1(\mathbb{R}^d)$ with $|\nabla G|$ is bounded. Let $f = (f_i)_{i=1}^d \in W_{loc}^{1,p}(\Omega)$. Then $x \mapsto G(f(x)) \in W_{loc}^{1,p}(\Omega)$ and

$$\partial_{x_i} G(f) = \sum_{k=1}^d (\partial_k G)(f) \cdot \partial_{x_i} f_k \quad \text{in } D'(\Omega).$$

Moreover, if $G(0) \in L^p(\Omega)$ (i.e. either $|\Omega| < \infty$ or $G(0) = 0$), then if $f = (f_i)_{i=1}^d \in W^{1,p}(\Omega)$, then $G(f) \in W^{1,p}(\Omega)$.

Proof. Since $G \in C^1$ we have that G is bounded in any compact set. Moreover $\|\nabla G\|_{L^\infty} < \infty$ implies:

$$|G(f) - G(0)| \leq \|\nabla G\|_{L^\infty} |f| \in L_{loc}^p$$

So $G(f) \in L^p_{loc}$. Let us compute $\partial_{x_i} G(f)$. Let $\{f^{(n)}\}_{n=1}^\infty \subseteq C^\infty$ such that $f^{(n)} \rightarrow f$ in $W^{1,p}_{loc}$, then:

$$|G(f^{(n)}) - G(f)| \leq \|\nabla G\|_{L^\infty} |f^{(n)} - f| \rightarrow 0 \text{ in } L^p_{loc}$$

So $G(f^{(n)}) \rightarrow G(f)$ in L^p_{loc} , thus $\partial_{x_i} G(f^{(n)}) \rightarrow \partial_{x_i} G(f)$ in $D'(\Omega)$. On the other hand, by the standard Chain-Rule for C^1 -functions:

$$\partial_{x_i} G(f^{(k)}) = \sum_{k=1}^d \underbrace{\partial_k G(f^{(k)})}_{(\text{b.d.} \rightarrow \partial_k G(f))} \underbrace{\partial_i f_k^{(n)}}_{(\rightarrow \partial_i f_k \text{ in } L^p(\Omega))} \rightarrow \sum_{k=1}^d \partial_k G(f) \partial_i f_k \text{ in } L^p_{loc}(\Omega)$$

Thus

$$\partial_{x_i} G(f) = \sum_{k=1}^d \underbrace{\partial_k G(f)}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p_{loc}} \in L^p_{loc} \text{ in } D'(\Omega)$$

So $G(f) \in W^{1,p}_{loc}(\Omega)$. Assume that $G(0) \in L^p(\Omega)$ (i.e. $|\Omega| < \infty$ or $G(0) = 0$). If $f \in W^{1,p}(\Omega)$, then $G(f) \in W^{1,p}(\Omega)$ since

$$|G(f) - G(0)| \leq \|\nabla G\|_{L^\infty} |f| \in L^p \Rightarrow G(f) \in L^p$$

and

$$\partial_{x_i} G(f) = \sum_k \underbrace{\partial_k G}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p} \in L^p \Rightarrow G(f) \in W^{1,p}(\Omega) \quad \blacksquare$$

Theorem 3.56 (Derivative of absolute value) Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $f \in W^{1,p}(\Omega)$. Then $|f| \in W^{1,p}(\Omega)$ and if f is real-valued:

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Proof. Exercise. Hint: Use the Chain-Rule for $G_\epsilon(x) = \sqrt{\epsilon^2 + x^2} - \epsilon \rightarrow |x|$ as $\epsilon \rightarrow 0$ \blacksquare

3.4 Distribution vs. measures

Let μ be a Borel measure in \mathbb{R}^d s.t. $\mu(K) < \infty$ for all compact $K \subseteq \mathbb{R}^d$. Then define

$$T : D(\mathbb{R}^d) \longrightarrow \mathbb{C} \\ \phi \longmapsto \int_{\mathbb{R}^d} \phi(x) d\mu(x) \quad \forall \phi \in C_c^\infty$$

\leadsto T is a distribution since if $\phi_n \rightarrow \phi$ in $D(\Omega)$, then

$$|T(\phi_n) - T(\phi)| \leq \int_{\mathbb{R}^d} |\phi_n - \phi| d\mu(x) \leq \|\phi_n - \phi\|_{L^\infty} \left(\int_K d\mu \right) \xrightarrow{n \rightarrow \infty} 0$$

Example 3.57 ∂_0 in $D'(\mathbb{R}^d)$ is a Borel probability measure.

Theorem 3.58 (Positive distributions are measures) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $T \in D'(\Omega)$. Assume $T \geq 0$, i.e. $T(\phi) \geq 0$ for all $\phi \in D(\Omega)$ satisfying $\phi(x) \geq 0$ for all x . Then there is a Borel positive measure μ on Ω such that $\mu(K) < \infty$ for all $K \subseteq \Omega$ compact and:

$$T(\phi) = \int_{\Omega} \phi(x) d\mu(x) \quad \forall \phi \in D(\Omega)$$

Proof. See Lieb-Loss Analysis. Sketch: If $O \subseteq \mathbb{R}^d$ is open, then

$$\mu(O) = \sup\{T(\phi) \mid \phi \in D(\Omega), 0 \leq \phi \leq 1, \text{supp } \phi \subseteq O\}$$

For all $A \subseteq \Omega$ (not necessarily open),

$$\mu(A) = \inf\{\mu(O) \mid O \text{ open}, A \subseteq O\}$$

The mapping $\mu : 2^{\Omega} \rightarrow [0, \infty]$ is an outer measure, i.e.

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leq \mu(B)$ if $A \subseteq B$
3. $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

From the outer measure we can find a σ -algebra Σ and μ is a measure on Σ s.t. E is measurable iff

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c).$$

So all open sets are measurable, thus outer regularity (by def $\mu(A) = \inf\{\mu(O) \mid O \text{ open } \supseteq A\}$), so inner regularity $\mu(A) = \sup\{\mu(K) \mid K \text{ compact } \subseteq A\}$. ■

Exercise 3.59 (E 4.1) Prove that if $T_n \rightarrow T$ in $D'(\mathbb{R}^d)$, then $D^{\alpha}T_n \rightarrow D^{\alpha}T$ in $D'(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$.

Exercise 3.60 (E 4.2)

Exercise 3.61 (E 4.3) $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ $f_{\epsilon}(x) = \epsilon^{-d}f(\epsilon^{-1}x)$. Then $f_{\epsilon} \rightarrow \delta_0$ in $D'(\mathbb{R}^d)$.

Exercise 3.62 (E 4.4) Let $\{f_n\} \subseteq L^1$, $\text{supp } f \subseteq B(0, 1)$, $f_n \rightarrow f$ in L^1 . Prove for all $g \in C_c^{\infty}$ that $f_n \star g \rightarrow f \star g$ in $D(\mathbb{R}^d)$.

Solution. Since $f_n \in L^1$, $\text{supp } f \subseteq B(0, 1)$ and $g \in C_c^{\infty}$ we have $f_n \star g \in C_c^{\infty}$ and

$$\text{supp}(f_n \star g) \subseteq (\text{supp } g) + \overline{B(0, 1)} = K.$$

Since $f_n \rightarrow f$ in L^1 there is a subsequence $f_{n_k} \rightarrow f$ almost everywhere, so f supp in $\overline{B(0, 1)}$, so $f \star g \in C_c^{\infty}$, $\text{supp}(f \star g) \subseteq K$. We have:

$$\begin{aligned} |f_n \star g(x) - f \star g(x)| &= \left| \int_{\mathbb{R}^d} (f_n(y) - f(y))g(x-y) dy \right| \\ &\leq \int_{\mathbb{R}^d} |f_n(y) - f(y)| |g(x-y)| dy \\ &\leq \|g\|_{L^{\infty}} \|f_n - f\|_{L^1} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

thus $\|f_n \star g - f \star g\|_{L^\infty} \rightarrow 0$. Similary:

$$\|D^\alpha(f_n \star g) - D^\alpha(f \star g)\|_{L^\infty} = \|f_n \star \underbrace{(D^\alpha g)}_{\in C_c^\infty} - f \star (D^\alpha g)\|_{L^\infty} \xrightarrow{n \rightarrow \infty} 0$$

for all $\alpha \in \mathbb{N}^d$, so $f_n \star g \rightarrow f \star g$ in $D(\mathbb{R}^d)$. ■

Exercise 3.63 (E 4.5) Compute distributional derivatives f', f'' of $f(x) = x|x-1|$.

Solution. We prove $f'(x) = g(x) := \begin{cases} 2x-1 & x > 1 \\ 1-2x & x < 1 \end{cases}$. Take $\phi \in C_c^\infty(\mathbb{R}^d)$.

$$\begin{aligned} -f'(\phi) &= \int_{\mathbb{R}^d} f \phi' dy \\ &= \int_{-\infty}^1 f \phi' dy + \int_1^\infty f \phi' dy \\ &= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 f' \phi dy + [f\phi]_1^\infty - \int_1^\infty f' \phi dy \\ &= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 g \phi dy + [f\phi]_1^\infty - \int_1^\infty g \phi dy \\ &= f(1-)\phi(1) - f(1+)\phi(1) - \int_{\mathbb{R}^d} g \phi dy \\ &= 0 - \int_{\mathbb{R}^d} g \phi dy \end{aligned}$$

Now we compute $f'' = g'$. Take $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned} -(g')(\phi) &= \int_{\mathbb{R}^d} g \phi' dy \\ &= \int_{-\infty}^1 g \phi' dy + \int_1^\infty g \phi' dy \\ &= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 g' \phi dy - \int_1^\infty g' \phi dy \\ &= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 (-2)\phi dy - \int_1^\infty 2\phi dy \\ &= -2\phi(1) + \int_{-\infty}^\infty [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x) dx \\ &= -2\delta_1(\phi) + \int_{-\infty}^\infty [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x) dx \\ &\Rightarrow g' = \underbrace{2\delta_1}_{\notin L_{loc}^1} - \underbrace{2\mathbb{1}_{(-\infty,1)} + 2\mathbb{1}_{(1,\infty)}}_{\in L_{loc}^1} \end{aligned} \quad \blacksquare$$

Chapter 4

Weak Solutions and Regularity

Definition 4.1 Consider the linear PDE:

$$\sum_{\alpha} c_{\alpha} D^{\alpha} u(x) = F(x), \quad c_{\alpha} \text{ constant, } F \text{ given}$$

A function u is called a weak solution (a distributional solution) if

$$\sum_{\alpha} c_{\alpha} D^{\alpha} u = F \quad \text{in } D'(\Omega).$$

Namely,

$$\sum_{\alpha} (-1)^{|\alpha|} c_{\alpha} \int_{\Omega} u D^{\alpha} \phi = \int_{\Omega} F \phi, \quad \forall \phi \in D(\Omega)$$

Regularity: Given some condition on the data F , what can we say about the smoothness of u ? Can we say that the equation holds in the classical sense? We derived G (the solution of the Laplace Equation) before in two ways:

1. $\Delta G(x) = 0$ for all $x \neq 0$, assuming $G(x) = G(|x|)$ and $d \geq 2$
2. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$ for $d \geq 3$

Theorem 4.2 For all $d \geq 1$ we have $G \in L^1_{loc}(\mathbb{R}^d)$ and $-\Delta G = \delta_0$ in $D'(\mathbb{R}^d)$.

Proof. Take $\phi \in D(\mathbb{R}^d)$. Then:

$$\begin{aligned} (-\Delta G_y)(\phi) &= G_y(-\Delta \phi) = \int_{\mathbb{R}^d} G_y(x) (-\Delta \phi)(x) dx \\ &= \int_{\mathbb{R}^d} G(y-x) (-\Delta \phi)(x) dx \\ &= [G \star (-\Delta \phi)](y) = (-\Delta)(G \star \phi)(y) \end{aligned}$$

Recall for all $f \in C^2$, $-\Delta(G \star f) = f$ pointwise. So we can conclude $-\Delta G_y = \delta_y$ in $D'(\mathbb{R}^d)$. ■

Remark 4.3 In $d = 1$, $G(x) = -\frac{1}{2}|x|$, so $-G'(x) = \text{sgn}(x)/2$, so $-G''(x) = \delta_0$.

Remark 4.4 Formally:

$$-\Delta(G_y \star \phi) = (-\Delta G_y) \star \phi(x) = (\delta_0 \star \phi)(x) = \int_{\mathbb{R}^d} \delta_0(y) \phi(x-y) dy = \delta_0(\phi(x - \bullet))$$

Theorem 4.5 (Poisson's equation with L^1_{loc} data) Let $f \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\omega_d f \in L^1(\mathbb{R}^d)$ where

$$\omega_d(x) = \begin{cases} 1 + |x| & d = 1 \\ \log(1 + |x|) & d = 2 \\ \frac{1}{1+|x|^{d-2}} & d \geq 3, \end{cases}$$

then $u(x) = (G \star f)(x) \in L^1_{loc}(\mathbb{R}^d)$. Moreover $-\Delta u = f$ in $D'(\mathbb{R}^d)$. In fact, $u \in W^{1,1}_{loc}(\mathbb{R}^d)$ and:

$$\partial_{x_i} u(x) = (\partial_{x_i} G) \star f(x) = \int_{\mathbb{R}^d} (\partial_{x_i} G)(x-y) f(y) dy$$

Remark 4.6 We can also replace \mathbb{R}^d by Ω and get $-\Delta u = f$ in $D'(\Omega)$.

Proof of Theorem 4.5. First we check that $u \in L^1_{loc}$. Take any Ball $B(0, R) \subseteq \mathbb{R}^d$, prove $\int_B |u| dy < \infty$. We have

$$\begin{aligned} \int_B |u| dy &= \int_B \left| \int_{\mathbb{R}^d} G(x-y) f(y) dy \right| dx \\ &\leq \int_B \int_{\mathbb{R}^d} |G(x-y)| |f(y)| dy dx \\ &= \int_{\mathbb{R}^d} \left(\int_B |G(x-y)| dx \right) |f(y)| dy \end{aligned}$$

If $y \notin B = B(0, R)$, then by Newtons's theorem (Mean-value theorem):

$$\int_{B(0,R)} |G(x-y)| dx = |B(0, R)| |G(y)| \leq C |B| \omega_d(y)$$

If $y \in B$, then $|y| \leq R$, so $|x-y| \leq 2R$ if $x \in B$.

$$\int_{B(0,R)} |G(x-y)| dx \leq \int_{|x-y| \leq 2R} |G(x-y)| dx = \int_{|z| \leq 2R} |G(z)| dz \leq c_R$$

as $G \in L^1_{loc}$. Thus

$$\int_B |u| dy \leq c_B \int_{|y| \geq R} \omega_d(y) |f(y)| dy + c_B \int_{|y| \leq R} |f(y)| dy < \infty$$

Let us prove $-\Delta u = f$ in $D'(\mathbb{R}^d)$. Take $\phi \in D(\mathbb{R}^d)$. Then:

$$\begin{aligned}
(-\Delta u)(\phi) &= u(-\Delta \phi) \\
&= \int_{\mathbb{R}^d} u(x)(-\Delta \phi)(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)f(y)(-\Delta \phi)(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y-x)f(y)(-\Delta \phi)(x) dx dy \\
&= \int_{\mathbb{R}^d} [G \star (-\Delta \phi)](y)f(y) dy \\
&= \int_{\mathbb{R}^d} -\Delta(G \star \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} \phi(y)f(y) dy
\end{aligned}$$

So $-\Delta u = f$ in $D'(\mathbb{R}^d)$. We check that $\partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$. Note that

$$|\partial_i G(x)| \leq c \frac{1}{|x|^{d-1}} \in L^1_{loc}(\mathbb{R}^d)$$

and

$$\int_{B(0,R)} |\partial_i G(x-y)| dx \leq \begin{cases} C_r \omega_d(y) & |y| \geq R \\ C_r & |y| \leq R \end{cases}$$

So $\int_{B(0,R)} |(\partial_i G \star f)(y)| dy < \infty$ for all $R > 0$. For all $\phi \in D(\mathbb{R}^d)$:

$$\begin{aligned}
-(\partial_i u)(\phi) &= u(\partial_i \phi) = \int_{\mathbb{R}^d} u(x) \partial_i \phi(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)f(y) \partial_i \phi(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y-x)f(y) \partial_i \phi(x) dx dy \\
&= \int_{\mathbb{R}^d} (G \star \partial_i^y \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} (\partial_i^y G \star \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i^y G(y-x)f(y)\phi(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} -(\partial_i G)(x-y)f(y)\phi(x) dx dy \\
&= - \int_{\mathbb{R}^d} (\partial_i G \star f)(x)\phi(x) dx
\end{aligned}$$

So $\partial_i u = \partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$. Thus $u \in L^1_{loc}$, $\partial_i u \in L^1_{loc}$ for all i . So $u \in W^{1,1}_{loc}(\mathbb{R}^d)$. \blacksquare

Regularity: We consider the Laplace Equation $\Delta u = 0$ in \mathbb{R}^d .

Lemma 4.7 (Weyl) If $\Omega \subseteq \mathbb{R}^d$ open and $T \in D'(\Omega)$ s.t. $\Delta T = 0$ in $D'(\Omega)$, then: $T = f \in C^\infty(\Omega)$ and f is a harmonic function.

Proof. ($\Omega = \mathbb{R}^d$). Take $\phi \in C_c^\infty$, then $y \mapsto T(\phi_y) = T(\phi(-y))$ is C^∞ and $\Delta_y T(\phi_y) = T((\Delta\phi)_y) = (\Delta T)(\phi_y) = 0$. Take $g \in C_c^\infty$, g is radial. Then:

$$\int_{\mathbb{R}^d} T(\phi_y)g(y) dy \stackrel{(\text{exercise})}{=} \int_{\mathbb{R}^d} T(\phi)g(y) dy = T(\phi) \left(\int_{\mathbb{R}^d} g dy \right)$$

Exercise 4.8 Let $f \in C^\infty(\mathbb{R}^d)$ be a harmonic function and $g \in C_c^\infty$, g is radial. Then:

$$\int_{\mathbb{R}^d} f(x)g(x) dx = f(0) \left(\int_{\mathbb{R}^d} g(x) dx \right)$$

On the other hand:

$$\int_{\mathbb{R}^d} T(\phi_y)g(y) dy = T(\phi \star g) = T(g \star \phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y) dy$$

Take $\int_{\mathbb{R}^d} g dy = 1$, then:

$$T(\phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y) dy$$

For all $\phi \in C_c^\infty$. Then $T = T(g_y) \in C^\infty$ ■

Now lets regard the Poisson Equation $-\Delta u = f$ in $D'(\mathbb{R}^d)$.

Remark 4.9 Any solution has the form $u = G \star g + h$ where $\Delta h = 0$ in $D'(\mathbb{R}^d)$. By Weyls Lemma (4.7), $h \in C^\infty$, then we only need to consider the regularity of $G \star f$.

Remark 4.10 The regularity is a *local question*, namely if we write

$$f = f_1 + f_2 = f\phi + f(1 - \phi),$$

where $\phi = 1$ in a ball B and $\phi \in C_c^\infty$. Then $G \star f = G \star f_1 + G \star f_2$. Here $f_2 = f(1 - \phi) = 0$ in B . With Weyls Lemma (4.7), $G \star f_2 \in C^\infty$.

Theorem 4.11 (Low Regularity of Poisson Equation) Let $f \in L^p(\mathbb{R}^d)$ and compactly supported. Then

a) If $p \geq 1$, then

- $G \star f \in C^1(\mathbb{R}^d)$ if $d = 1$.
- $G \star f \in L_{loc}^q(\mathbb{R}^d)$ for any $q < \infty$ if $d = 2$.
- $G \star f \in L_{loc}^q(\mathbb{R}^d)$ for $q < \frac{d}{d-2}$ if $d \geq 3$.

b) If $\frac{d}{2} < p \leq d$, then $G \star f \in C_{loc}^{0,\alpha}(\mathbb{R}^d)$ for all $0 < \alpha < 2 - \frac{d}{p}$, i.e.

$$|(G \star f)(x) - (G \star f)(y)| \leq C_k |x - y|^\alpha \quad \forall x, y \in K$$

with K compact in \mathbb{R}^d .

c) If $p > d$, then $G \star f \in C_{loc}^{1,\alpha}(\mathbb{R}^d)$ for all $0 < \alpha < 1 - \frac{d}{p}$.

where G is den fundamental solution of the laplace equation.

Example 4.12 Let $r = |x|$

$$u(x) = \omega(r) = \log(|\log(r)|)$$

if $0 < r < \frac{1}{2}$, so u is well-defined in $B = B(0, \frac{1}{2})$. We conclude:

$$-\Delta_{\mathbb{R}^3} u(x) = -\omega''(r) - \frac{2\omega'(r)}{r} = f(x) \in L^{\frac{3}{2}}(B)$$

But the Theorem (b) tells us that if $f \in L^{\frac{3}{2}}$ then u is continuous but $u \notin C(B)$.

Proof of theorem 4.11. a) ($p = 1$) Why is $G \star f \in L_{loc}^q$? Recall from the proof of Youngs inequality:

$$\begin{aligned} |(G \star f)(x)| &= \left| \int_{\mathbb{R}^d} G(x-y)f(y) dy \right| \\ (\text{H\"older}) &= \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(y)| dy \right)^{\frac{1}{q'}} \end{aligned}$$

Where $\frac{1}{q} + \frac{1}{q'} = 1$. Then:

$$|(G \star f)(x)|^q \leq C \int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy$$

For any Ball $B = B(0, R) \subseteq \mathbb{R}^d$:

$$\begin{aligned} \int_B |G \star f(x)|^q dx &\leq C \int_B \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy \right) dx \\ &= C \int_{\mathbb{R}^d} \left(\int_B |G(x-y)|^q dx \right) |f(y)| dy \end{aligned}$$

$G(x) \sim \frac{1}{|x|^{d-2}} \rightsquigarrow |G|^q = \frac{1}{|x|^{(d-2)q}} \in L_{loc}^1(\mathbb{R}^d)$ if $(d-2)q < 2 \Leftrightarrow q < \frac{d}{d-2}$. Here, $y \in \text{supp } f$, so $|y| \leq R_1$, then $|x-y| \leq R+R$ if $|x| \leq R$. With $y \in \text{supp } f$, this implies:

$$\int_{B(0,R)} |G(x-y)|^q dx \leq \int_{|z| \leq R+R_1} |G(z)|^q dz < \infty$$

b)

$$(G \star f)(x) - (G \star f)(y) = \int_{\mathbb{R}^d} (G(x-z) - G(y-z))f(z) dz$$

So

$$|G \star f(x) - (G \star f)(y)| \leq C \int_{\mathbb{R}^d} \left| \frac{1}{|x-z|^{d-2}} - \frac{1}{|y-z|^{d-2}} \right| |f(z)| dz$$

for all $x, y \in \mathbb{R}^d$:

$$\begin{aligned} \left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| &= \left| \left(\frac{1}{|x|} - \frac{1}{|y|} \right) \left(\frac{1}{|x|^{d-3}} + \dots + \frac{1}{|y|^{d-3}} \right) \right| \\ &\leq C \frac{||x| - |y||}{|x||y|} \max \left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &= C \frac{|x-y|}{|x||y|} \max \left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &\leq C \max(|x|, |y|)^{1-\alpha} \frac{|x-y|^\alpha}{|x||y|} \max \left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \end{aligned}$$

as

$$||x| - |y|| \leq \min(|x - y|, \max(|x|, |y|)) \leq |x - y|^\alpha \max(|x|, |y|)^{1-\alpha}$$

Thus, for all $x, y \in \mathbb{R}^d$:

$$\begin{aligned} \left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| &\leq C|x - y|^\alpha \frac{\max(|x|, |y|)^{1-\alpha}}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}}\right) \\ &\leq C|x - y|^\alpha \max\left(\frac{1}{|x|^{d-2+\alpha}}, \frac{1}{|y|^{d-2+\alpha}}\right) \end{aligned}$$

So we get

$$\left| \frac{1}{|x - y|^{d-2}} - \frac{1}{|y - z|^{d-2}} \right| \leq C|x - y|^\alpha \max\left(\frac{1}{|x - z|^{d-2+\alpha}}, \frac{1}{|y - z|^{d-2+\alpha}}\right)$$

Therefore:

$$\begin{aligned} &|G \star f(x) - G \star f(y)| \\ &\leq C \int_{\mathbb{R}^d} |x - y|^\alpha \max\left(\frac{1}{|x - z|^{d-2+\alpha}}, \frac{1}{|y - z|^{d-2+\alpha}}\right) |f(z)| dz \\ &\leq C|x - y|^\alpha \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \right) \end{aligned}$$

Claim: If $f \in L^p(\mathbb{R}^d)$ is compactly supported, $d \geq p > \frac{d}{2}$, then:

$$\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz < \infty$$

for all $0 < \alpha < 2 - \frac{d}{p}$. Assume $\text{supp } f \subseteq \overline{B(0, R_1)}$. Consider 2 cases:

- If $|\xi| > 2R_1$, then: $|\xi - z| \geq R_1$ for all $z \in B(0, R_1)$. Hence:

$$\int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \leq \frac{1}{R_1^{d-2+\alpha}} \|f\|_{L^1} < \infty$$

- If $|\xi| \leq 2R_1$, then: $|\xi - z| \leq 3R_1$ for all $z \in B(0, R_1)$:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz &\leq \int_{|\xi - z| \leq 3R_1} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \\ \text{(H\"older), } \left(\frac{1}{p} + \frac{1}{q} = 1\right) &\leq \left(\int_{\mathbb{R}^d} |f(z)|^p dz \right)^{\frac{1}{p}} \\ &\quad \cdot \left(\int_{|\xi - z| \leq 3R_1} \frac{1}{|\xi - z|^{(d-2+\alpha)q}} dz \right)^{\frac{1}{q}} \\ &= \|f\|_{L^p} \left(\int_{|z| \leq 3R_1} \frac{1}{|z|^{(d-2+\alpha)q}} dz \right)^{\frac{1}{q}} < \infty \end{aligned}$$

c) ($d \geq 3$) We already know:

$$\partial_i(G \star f) = (\partial_i G \star f) \in L_{loc}^1(\mathbb{R}^d)$$

as $\omega_d f \in L^1(\mathbb{R}^d)$. We claim that $\partial_i G \star f \in C^{0,\alpha}(\mathbb{R}^d)$. So $G \star f \in C^{1,\alpha}(\mathbb{R}^d)$ by the equivalence between the classical and the distributional derivatives. Exercise. Hint:

$$|\partial_i G \star f(x) - \partial_i G \star f(y)| \leq \int_{\mathbb{R}^d} |\partial_i G(x-z) - \partial_i G(y-z)| |f(z)| dz,$$

$$\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d} \rightsquigarrow \text{Need to estimate } |\partial_i G(x) - \partial_i G(y)| \leq C|x-y|^\alpha. \quad \blacksquare$$

Theorem 4.13 (High regularity for Poisson's equation) Let $f \in C^{0,\alpha}(\mathbb{R}^d)$, $0 < \alpha < 1$ be compactly supported. Then $G \star f \in C^{2,\alpha}(\mathbb{R}^d)$.

Remark 4.14 $(-\Delta u = f)$ and $f \in C(\mathbb{R}^d)$ does not imply that $u \in C^2(\mathbb{R}^d)$. (exercise)

Remark 4.15 If $f \in C^{k,\alpha}(\mathbb{R}^d)$, $k \in \{0, 1, \dots\}$, $0 < \alpha < 1$ is compactly supported, then $G \star f \in C^{k+2,\alpha}(\mathbb{R}^d)$. This more general statement is a consequence of the theorem since

$$D^\beta(G \star f) = G \star \underbrace{(D^\beta f)}_{\in C^{0,\alpha}}$$

for all $\beta = (\beta_1, \dots, \beta_d)$, $|\beta| \leq k$.

Proof of theorem 4.13. Since $f \in L^p$ for all $p \leq \infty$ by the low regularity (4.11) we have $G \star f \in C^{1,\alpha}$ and $\partial_i(G \star f) = \partial_i G \star f$ in the classical sense. We will compute the distributional derivatives $\partial_i \partial_j(G \star f)$ and prove that they are Hölder continuous. Compute $\partial_j \partial_i(G \star f)$: For all $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} -(\partial_j \partial_i G \star f)(\phi) &= \underbrace{(\partial_i(G \star f))}_{\in C}(\partial_j \phi) \\ &= \int_{\mathbb{R}^d} ((\partial_i G) \star f)(x) \partial_j \phi(x) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i G(x-y) f(y) \partial_j \phi(x) dx dy \\ &= \int_{\mathbb{R}^d} f(y) \left[\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx \right] dy \\ &\stackrel{?}{=} \int_{\mathbb{R}^d} \square \phi(y) dy \end{aligned}$$

Recall: $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$, $\partial_i \partial_j G(x) = \frac{1}{|B_1|} \left[\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{d} \right] \frac{1}{|x|^d}$. We have:

$$\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_{|x-y| \geq \epsilon} \partial_i G(x-y) \partial_j \phi(x) dx$$

By dominated convergence we have $|\partial_i G(x-y) \partial_j \phi(x)| \in L^1(dx)$. By the Gauss-Green-Theorem (2.2) for all $\epsilon > 0$:

$$\begin{aligned} &\int_{|x-y| \geq \epsilon} \partial_i G(x-y) \partial_j \phi(x) dx \\ &= \int_{\partial B(y,\epsilon)} \partial_i G(x-y) \phi(x) \omega_j dS(x) - \int_{|x-y| \geq \epsilon} \partial_j \partial_i G(x-y) \phi(x) dx \end{aligned}$$

Where $\omega = \frac{x-y}{|x-y|}$. For the boundary term:

$$\begin{aligned}
- \int_{\partial B(y, \epsilon)} \partial_i G(x-y) \phi(x) \omega_j dS(x) &= -\epsilon^{d-1} \int_{\partial B(0,1)} \partial_i G(\epsilon \omega) \phi(y + \epsilon \omega) \omega_j d\omega \\
(\star) \quad &= \int_{\partial B(0,1)} \frac{1}{d|B_1|} \omega_i \omega_j \phi(y + \epsilon \omega) d\omega \\
&\xrightarrow{\epsilon \rightarrow 0} \int_{\partial B(0,1)} \frac{1}{d|B_1|} \omega_i \omega_j \phi(y) d\omega \\
&= \frac{1}{d} \delta_{i,j} \phi(y)
\end{aligned}$$

(\star) $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$, so $\partial_i G(\epsilon \omega) = -\frac{-\omega_i}{d|B_1|} \frac{1}{\epsilon^{d-1}}$. for all $|\omega| = 1$.

Now we split:

$$\begin{aligned}
&- \int_{|x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx \\
&= - \int_{|x-y| \geq 1} \partial_i \partial_j G(x-y) \phi(x) dx - \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx
\end{aligned}$$

The key observation is: $\int_{\partial B(0,r)} \partial_i \partial_j G(x) dx = 0$ since

$$\partial_i \partial_j G(x) = \frac{1}{|B_1|} \left(\omega_i \omega_j - \frac{\partial_{ij}}{d} \right) \frac{1}{|x|^d},$$

$\omega = \frac{x}{|x|}$. For example if $i = 1, j = 2, r = 1$:

$$\int_{\partial B(0,1)} \partial_1 \partial_2 G(x) dS(x) = \frac{1}{|B_1|} \int_{\partial B(0,1)} \omega_1 \omega_2 d\omega,$$

$\partial B(0,1) = \{\omega \mid |\omega| = 1\}$. Consider: $\omega \mapsto R\omega, (\omega_1, \dots, \omega_d) \mapsto (-\omega_1, \omega_2, \dots, \omega_d)$. Then

$$- \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(y) dx = 0.$$

So

$$- \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx = - \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) (\phi(x) - \phi(y)) dx$$

In summary:

$$\begin{aligned}
\partial_i \partial_j (G \star f)(\phi) &= \int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx \right) dy \\
&= \int_{\mathbb{R}^d} f(y) \frac{1}{d} \partial_{ij} \phi(y) dy \\
&\quad - \int_{\mathbb{R}^d} f(y) \left(\int_{|x-y| > 1} \partial_i \partial_j G(x-y) \phi(x) dx \right) \\
&\quad - \int_{\mathbb{R}^d} \left[\lim_{\epsilon \rightarrow 0} \int_{1 \geq |x-y| \geq \epsilon} \underbrace{\partial_i \partial_j G(x-y) (\phi(x) - \phi(y)) dx}_{\leq \frac{C}{|x-y|^d} |x-y| \|\nabla \phi\|_{L^\infty} \leq \frac{C}{|x-y|^{d-1}} \in L^1_{loc}(dx) \forall y} \right] dy \\
&= \int_{\mathbb{R}^d} \frac{\delta_{ij}}{d} f(x) \phi(x) dx - \int_{\mathbb{R}^d} \phi(x) \left(\int_{|x-y| > 1} \partial_i \partial_j G(x-y) f(y) dy \right) dx \\
&\quad - \int_{\mathbb{R}^d} \phi(x) \left[\int_{|x-y| \leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy \right] dx
\end{aligned}$$

Conclusion:

$$\begin{aligned}\partial_i \partial_j (G \star f)(x) &= -\frac{\delta_{ij}}{d} f(x) + \int_{|x-y|>1} \partial_i \partial_j G(x-y) f(y) dy \\ &\quad + \int_{|x-y|\leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy\end{aligned}$$

The first term $f \in C^{0,\alpha}$. The second term is also at least $C^{0,\alpha}$ since $\partial_i \partial_j G(x)$ is smooth as $|x| > 1$. We need to prove that the third term

$$W_{ij}(x) = \int_{|x-y|\leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy$$

is Hölder-continuous, $|W_{ij}(x) - W_{ij}(y)| \leq C|x-y|^\alpha$. Recall:

$$|\partial_i \partial_j G(x-y) (f(y) - f(x))| \leq C \frac{1}{|x-y|^d} |x-y|^\alpha = \frac{C}{|x-y|^{d-\alpha}} \in L^1_{loc}(dy)$$

We write

$$\begin{aligned}W_{ij}(x) &= \int_{|x-y|\leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy \\ &= \int_{|z|\leq 1} \partial_i \partial_j G(z) (f(x+z) - f(x)) dz\end{aligned}$$

So we get:

$$W_{ij} - W_{ij}(y) = \int_{|z|\leq 1} \partial_i \partial_j G(z) (f(x+z) - f(y+z) - f(x) + f(y)) dz$$

Easy thought: Use $|\partial_i \partial_j G(z)| \leq \frac{C}{|z|^d}$ and

$$\begin{aligned}&|f(x+z) - f(y+z) - f(x) + f(y)| \\ &\leq \begin{cases} |f(x+z) - f(x)| + |f(y+z) - f(y)| \leq C|z|^\alpha \\ |f(x+z) - f(y+z)| + |f(x) - f(y)| \leq C|x-y|^\alpha \end{cases}\end{aligned}$$

Thus:

$$\begin{aligned}|W_{ij}(x) - W_{ij}(y)| &\leq C \int_{|z|\leq 1} \frac{1}{|z|^d} \min(|z|^\alpha, |x-y|^\alpha) dz \\ &\leq C \int_{|z|\leq 1} \frac{1}{|z|^d} (|z|^\alpha)^\epsilon (|x-y|^\alpha)^{1-\epsilon}, \quad 0 < \epsilon < 1 \\ &\leq C \left(\int_{|z|\leq 1} \frac{1}{|z|^{d-\alpha\epsilon}} \right) |x-y|^{\alpha(1-\epsilon)} \\ &\leq C_\epsilon |x-y|^{\alpha(1-\epsilon)}\end{aligned}$$

thus it is easy to prove $|W_{ij}(x) - W_{ij}(y)| \leq C_\alpha |x-y|^\alpha$ for all $\alpha' \leq \alpha$. However, to get $\alpha' = \alpha$ we need a more precise estimate. We split:

$$W_{ij}(x) - W_{ij}(y) = \int_{|z|\leq 1} \dots = \int_{|z|\leq \min(4|x-y|, 1)} + \int_{4|x-y| < |z|\leq 1}$$

For the first domain:

$$\begin{aligned} & \int_{|z| \leq 4|x-y|} |\partial_{ij}G(z)| |f(x+z) - f(y+z) - f(y) + f(x)| dz \\ & \leq C \int_{|z| \leq 4|x-y|} \frac{1}{|z|^d} |z|^\alpha dz = \text{const} \cdot |x-y|^\alpha \end{aligned}$$

For the second domain:

$$\begin{aligned} & \int_{4|x-y| < |z| \leq 1} \partial_{ij}G(z)(f(x+z) - f(y+z) + f(y)f(x)) dz \\ & = \int_{4|x-y| < |z| \leq 1} \partial_{ij}G(z)(f(x+z) - f(y+z)) dz = (\dots) \end{aligned}$$

since $\int_{4|x-y| < |z| \leq 1} \partial_{ij}G(z) dz = 0$. Then

$$(\dots) = \int_{4|x-y| < |z-x| \leq 1} \partial_{ij}G(z-x)f(z) dz - \int_{4|x-y| < |z-y| \leq 1} \partial_{ij}G(z-y)f(z) dz.$$

Denote $A = \{z \mid 4|x-y| < |z-x| \leq 1\}$, $B = \{z \mid 4|x-y| < |z-y| \leq 1\}$. Consider

$$\begin{aligned} & \int_A \partial_{ij}G(z-x)f(z) dz - \int_B \partial_{ij}G(z-y)f(z) dz \\ & = \int_{A \setminus B} + \int_{B \setminus A} + \int_{A \cap B} (\partial_{ij}G(z-x) - \partial_{ij}G(z-y))f(z) dz \end{aligned}$$

Lets regard the intersection. We have

$$\begin{aligned} \partial_{ij}G(x) &= \frac{1}{|B_1|} \frac{1}{|x|^d} (\omega_i \omega_j - \frac{1}{d} \delta_{ij}) \\ |\partial_{ij}G(x) - \partial_{ij}G(y)| &\leq C|x-y| \left(\frac{1}{|x|^{d+1}} + \frac{1}{|y|^{d+1}} \right) \end{aligned}$$

Now,

$$|\partial_{ij}G(z-x) - \partial_{ij}G(z-y)| \leq C|x-y| \left(\frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right)$$

So we have

$$\begin{aligned} & \left| \int_{A \cap B} (\partial_{ij}G(z-x) - \partial_{ij}G(z-y))f(z) dz \right| \\ & \leq C \int_{A \cap B} |x-y| \left(\frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right) |f(z)| dz = (\dots) \end{aligned}$$

Now we replace $f(z)$ by $f(z) - f(x)$, then:

$$\begin{aligned} & \left| \int_{A \cap B} (\partial_{ij}G(z-x) - \partial_{ij}G(z-y))(f(z) - f(x)) dz \right| \\ & \leq C \int_{A \cap B} |x-y| \left(\frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right) |z-x|^\alpha dz \\ & = C \underbrace{\int_{A \cap B} |x-y| \frac{1}{|z-x|^{d+1-\alpha}} dz}_{(I)} + C \underbrace{\int_{A \cap B} |x-y| \frac{1}{|z-y|^{d+1}} |z-x|^\alpha dz}_{(II)} \end{aligned}$$

Now,

$$\begin{aligned}
(I) &\leq C|x-y| \int_{4|x-y| < |z-x| \leq 1} \frac{1}{|z-x|^{d+1-\alpha}} dz \\
&= C|x-y| \int_{4|x-y| < |z| \leq 1} \frac{1}{|z|^{d+1-\alpha}} dz \\
&\leq C|x-y| \int_{4|x-y|}^1 \frac{1}{r^{d+1-\alpha}} r^{d-1} dr \\
&= C|x-y| \int_{4|x-y|}^1 \frac{1}{r^{2-\alpha}} dr \\
&\leq C|x-y| \left[-1 + \frac{1}{(4|x-y|)^{1-\alpha}} \right] \\
&\leq C|x-y|^\alpha
\end{aligned}$$

$$\begin{aligned}
(II) &\leq C|x-y| \int_{A \cap B} \frac{1}{|z-y|^{d+1}} |z-x|^\alpha dz \\
&\leq C|x-y| \int_{A \cap B} \frac{1}{|z-y|^{d+1}} (|z-y|^\alpha + |x-y|^\alpha) dz \\
&\leq C|x-y| \underbrace{\int_B \frac{1}{|z-y|^{d+1-\alpha}} dz}_{\text{similar to (I)}} + C|x-y|^{1+\alpha} \int_B \frac{1}{|z-y|^{d+1}} dz
\end{aligned}$$

and

$$C|x-y|^{1+\alpha} \int_B \frac{1}{|z-y|^{d+1}} dz \leq \int_{4|x-y|}^1 \frac{1}{r^{d+1}} r^{d-1} dr \leq \frac{C}{|x-y|}$$

Consider $A \setminus B$:

$$\left| \int_{A \setminus B} \right| \leq C \|f\|_{L^\infty} \int_{A \setminus B} \frac{1}{|z-x|^d} dz$$

where

$$\begin{aligned}
A &= \{z \mid 4|x-y| < |z-x| \leq 1\} \\
B &= \{z \mid 4|x-y| < |z-y| \leq 1\} \\
A \setminus B &= \{z \in A \mid |z-y| \leq 4|x-y|\} \cup \{z \in A \mid |z-y| > 1\} = E_1 \cup E_2
\end{aligned}$$

for

$$\begin{aligned}
E_1 &= \{z \mid |z-y| \leq 4|x-y| < |z-x| \leq 1\} \\
&\subseteq \{z \mid 4|x-y| \leq |x-z| \leq 5|x-y|\}.
\end{aligned}$$

$|x - z| \leq |x - y| + |y - z| \leq 5|x - y|$ in E_1 . We have

$$\begin{aligned}
\int_{E_1} \frac{1}{|z - x|^d} dz &\leq \int_{4|x-y| \leq |x-z| \leq 5|x-y|} \frac{1}{|z - x|^{d-\alpha}} dz \\
&= \int_{4|x-y| \leq |z| \leq 5|x-y|} \frac{1}{|z|^{d-\alpha}} dz \\
&= \int_{4|x-y|} \frac{1}{r^d} r^{d-1} dr \\
&= \int_{4|x-y|} \frac{1}{r^{1-\alpha}} dr \\
&\leq C|x - y|^\alpha
\end{aligned}$$

Now in E_2 : $|z - x| \geq |z - y| - |y - x| \geq 1 - |y - x|$.

$$\begin{aligned}
\int_{E_2} \frac{1}{|z - x|^{d-\alpha}} dz &\leq \int \frac{1}{|z - x|^{d-\alpha}} dz = \int_{1-|x-y|}^1 \frac{1}{r^{d-\alpha}} r^{d-1} dr \\
&\leq \text{const.} \left| 1 - \frac{1}{(1 - |x - y|)^\alpha} \right| \leq C|x - y|^\alpha \quad \blacksquare
\end{aligned}$$

Exercise 4.16 (E 5.1) Prove that if f is a harmonic function in \mathbb{R}^d and $g \in C_c(\mathbb{R}^d)$ is radial, then

$$\int_{\mathbb{R}^d} f(x)g(x) dx = f(0) \int_{\mathbb{R}^d} g(x) dx$$

Solution. $x = r\omega, r > 0, |\omega| = 1$

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)g(x) dx &\stackrel{(\text{Polar})}{=} \int_0^\infty \left(\int_{\partial B(0,1)} f(r\omega)g(r\omega) d\omega \right) dr \\
&= \int_0^\infty \left(g_0(r) \int_{\partial B(0,1)} f(r\omega) d\omega \right) dr \\
(\text{Mean value theorem (2.12)}) \quad &= \int_0^\infty \left(g_0(r) f(0) \int_{\partial B(0,1)} d\omega \right) dr \\
&= f(0) \int_0^\infty \left(\int_{\partial B(0,1)} g(r\omega) d\omega \right) dr \\
&= f(0) \int_{\mathbb{R}^d} g(x) dx \quad \blacksquare
\end{aligned}$$

Remark 4.17 Let $g \in C_c(\mathbb{R}^d)$ be radial. Why is $\int_{\mathbb{R}^3} \frac{g(x)}{|x|} dx \neq \infty$? Because $f(x) = \frac{1}{|x|}$ is harmonic in $\mathbb{R}^d \setminus \{0\}$ and sub-harmonic in \mathbb{R}^d , $-\Delta f = c\delta_0$.

Exercise 4.18 (E 5.2) Let $1 \leq p < \infty$. Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider the Sobolev Space

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega), \forall i = 1, 2, \dots, d\}$$

with the norm

$$\|f\|_{W^{1,p}} = \|f\| + \sum_{i=1}^d \|\partial_{x_i} f\|_{L^p(\Omega)}.$$

Prove that $W^{1,p}(\Omega)$ is a Banach space. Here $x = (x_i)_{i=1}^d \in \mathbb{R}^d$. Hint: You can use the fact that $L^p(\Omega)$ is a Banach Space.

Solution. $W^{1,p}(\Omega) \subseteq L^p(\Omega) \times L^p(\Omega) \cdots \times L^p(\Omega) = (L^p(\Omega))^{d+1}$. For an element $f \in W^{1,p}(\Omega)$ we can think of it as $f \mapsto (f, \partial_1 f, \partial_2 f, \dots, \partial_d f)$, so $W^{1,p}(\Omega)$ is a subspace of $(L^p(\Omega))^{d+1}$, which is a norm-space. Why is $W^{1,p}(\Omega)$ closed in $(L^p(\Omega))^{d+1}$? Take $\{f_n\}_{n=1}^\infty \subseteq W^{1,p}(\Omega)$ such that $f_n \rightarrow f$ in L^p and $\partial_i f_n \rightarrow g_i$ in L^p for all $i = 1, \dots, d$. We prove that $(f, g_1, \dots, g_d) \in W^{1,p}(\Omega)$, i.e. $f \in W^{1,p}$ and $g_i = \partial_i f$ for all $i = 1, \dots, d$. We know that $f_n \rightarrow f$ in $L^p(\Omega)$, so $f_n \rightarrow f$ in $D'(\Omega)$ and $\partial_i f_n \rightarrow \partial_i f$ in $D'(\Omega)$. On the other hand we have $\partial_i f_n \rightarrow g_i$ in $L^p(\Omega)$, so $\partial_i f_n \rightarrow g_i$ in $D'(\Omega)$. So we get $\partial_i f = g_i \in L^p(\Omega)$ for all $i = 1, \dots, d$ in $D'(\Omega)$. So we can conclude $f \in W^{1,p}(\Omega)$ and $\partial_i f = g_i$ for all $i = 1, \dots, d$. ■

Exercise 4.19 (E 5.3) Let f be a real-valued function in $W^{1,p}(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Prove that $|f| \in W^{1,p}(\mathbb{R}^d)$ and

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}.$$

Solution. Consider $G_\epsilon(t) = \sqrt{\epsilon^2 + t^2} - \epsilon$ for $\epsilon > 0, t \in \mathbb{R}$. Clearly we have $G_\epsilon(t) \rightarrow |t|$ as $\epsilon \rightarrow 0$ and

$$G'_\epsilon(t) = \frac{2t}{2\sqrt{\epsilon^2 + t^2}} = \frac{t}{\sqrt{\epsilon^2 + t^2}},$$

so $|G'_\epsilon(t)| \leq 1, G_\epsilon(0) = 0$. By the chain rule (3.55) we have $G_\epsilon(f) \in W^{1,p}(\mathbb{R}^d)$ and

$$(\partial_i G_\epsilon(f))(x) = G'_\epsilon(f) \partial_i f(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \partial_i f(x) \in L^p(\mathbb{R}^d)$$

for all $i = 1, \dots, d$. Note then when $\epsilon \rightarrow 0$ that $G_\epsilon(f)(x) \rightarrow |f(x)|$ pointwise, so $G_\epsilon(f) \rightarrow |f|$ in $L^p(\mathbb{R}^d)$. $|G_\epsilon(f)(x) - G_\epsilon(0)| \leq |f(x)| \in L^p(\mathbb{R}^d)$ by dominated convergence.

$$\partial_i G_\epsilon(f)(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \partial_i f(x) \xrightarrow{\epsilon \rightarrow 0} g_i(x) := \begin{cases} \partial_i f(x) & f(x) > 0 \\ -\partial_i f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

$$|\partial_i G_\epsilon(f)(x)| \leq \left| \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \right| |\partial_i f(x)| \leq |\partial_i f(x)| \in L^p(\mathbb{R}^d)$$

So we get $\partial_i G_\epsilon(f) \xrightarrow{\epsilon \rightarrow 0} g_i$ in $L^p(\mathbb{R}^d)$ by Dominated Convergence. So we conclude: $\partial_i(|f|) = g_i \in L^p(\mathbb{R}^d)$ for all $i = 1, \dots, d$, so $|f| \in W^{1,p}(\mathbb{R}^d), |f| \in L^p$. ■

Exercise 4.20 (E 5.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $f \in L^1(\Omega)$,

$$u(x) = \int_{\Omega} G(x-y) f(y) dy$$

Let $-\Delta u = f$ in $D'(\Omega)$, $u \in L^1_{loc}(\Omega)$, $f \in L^1_{loc}(\mathbb{R}^d)$ and $\omega_d f \in L^1(\mathbb{R}^d)$, where

$$\omega_d(x) = \begin{cases} 1 + |x| & d = 1 \\ \log(1 + |x|) & d = 2 \\ \frac{1}{(1+|x|)^{d-2}} & d \geq 3 \end{cases}.$$

Prove that

$$G \star f = \int_{\mathbb{R}^d} G(x-y)f(y) dy \in L_{loc}^1(\mathbb{R}^d)$$

and $-\Delta(G \star f) = f$ in $D'(\mathbb{R}^d)$.

Solution. Define $\tilde{f} = \mathbb{1}_\Omega(x)f(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$. Then

$$u(x) = \int_{\Omega} G(x-y)f(y) dy = \int_{\mathbb{R}^d} G(x-y)\tilde{f}(y) dy = (G \star \tilde{f})(x)$$

We have $u \in L_{loc}^1(\mathbb{R}^d)$, so $u \in L^1(\Omega)$. Then $-\Delta u = \tilde{f}$ in $D'(\mathbb{R}^d)$, so $-\Delta u = f$ in $D'(\Omega)$. Claim: $-\Delta u = f$ in $D'(\mathbb{R}^d)$, so $-\Delta u = f$ in $D'(\Omega)$ if $\Omega \subseteq \mathbb{R}^d$, $\tilde{f}|_\Omega = f$. Take $\phi \in C_c^\infty(\Omega)$. We need: $(-\Delta u)(\phi) \stackrel{?}{=} \int_{\Omega} f\phi$. We have $\phi \in C_c^\infty(\Omega)$, so $\phi C_c^\infty(\mathbb{R}^d)$. This implies:

$$(-\Delta u)(\phi) = \int_{\mathbb{R}^d} \tilde{f}\phi = \int_{\substack{\Omega, \\ \text{supp } \phi \subseteq \Omega}} \tilde{f}\phi = \int_{\Omega} f\phi \quad \blacksquare$$

Exercise 4.21 (E 5.5) Let $B = B(0, \frac{1}{2}) \subseteq \mathbb{R}^3$. Consider $u : B \rightarrow \mathbb{R}$, defined by

$$u(x) = \log |\log |x||.$$

Prove that the distributional derivative $f = -\Delta u$ is a function in $L^{\frac{3}{2}}(B)$.

Solution.

$$\begin{aligned} \omega(r) &= \log(-\log(r)), \quad \text{for } r \in \left(0, \frac{1}{2}\right) \\ \omega'(r) &= \frac{1}{-\log(r)} \left(-\frac{1}{r}\right) = \frac{1}{r \log r} \\ \omega''(r) &= -\frac{1}{(r \log(r))^2} (r \log(r))' = -\frac{\log(r) + 1}{(r \log r)^2} \end{aligned}$$

So we have

$$-\Delta u = \omega''(r) = \frac{1}{(r \log r)^2} - \frac{1}{r^2 \log(r)} = f(r)$$

We show that $f \in L^{\frac{3}{2}}$:

$$\begin{aligned} \int_B |f(x)|^{\frac{3}{2}} dx &= \text{const} \int_0^{\frac{1}{2}} \left| \frac{1}{r^2(\log r)^2} - \frac{1}{r^2 \log r} \right|^{\frac{3}{2}} r^2 dr \\ &\lesssim \int_0^{\frac{1}{2}} \frac{1}{r} \left| \frac{1}{(\log(r))^2} - \frac{1}{(\log(r))} \right|^{\frac{3}{2}} dr \\ \left(\begin{array}{l} r = e^{-x}, \\ x \in (\log(2), \infty), \\ dr = -e^{-x} dx \end{array} \right) &\lesssim \int_{\log(2)}^{\infty} e^x \left(\frac{1}{x^2} + \frac{1}{x} \right)^{\frac{3}{2}} e^{-x} dx \\ &\lesssim \int_{\log(2)}^{\infty} \frac{1}{x^{\frac{3}{2}}} dx < \infty \end{aligned}$$

Where \lesssim means *up to a constant*. Now, $u(x) = \omega(r) = \log(-\log(r))$.

$$-\Delta u(x) = f(r) = \frac{1}{r^2(\log(r))^2} - \frac{1}{r^2 \log(r)}$$

for all $x \neq 0, |x| = r < \frac{1}{2}$. Why is $-\Delta u(x) = f$ in $D'(B)$? Take $\phi \in C_c^\infty(B)$, check: $\int_B u(-\Delta \phi) = \int_B f \phi$.

$$\int_{|x| < \frac{1}{2}} u(-\Delta \phi) dx = \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |x| < \frac{1}{2}} u(x)(-\Delta \phi)(x) dx$$

by Dominated convergence. $u \in L^1(B)$. For all $\epsilon > 0$:

$$\begin{aligned} \int_{\epsilon < |x| < \frac{1}{2}} u(x)(-\Delta \phi)(x) dx &= \int_{|x| > \epsilon} u(x)(-\Delta \phi)(x) dx \\ &= \int_{\partial B(0, \epsilon)} u(x) \nabla \phi(x) \frac{x}{|x|} dS(x) + \int_{|x| > \epsilon} \nabla u(x) \nabla \phi(x) dx \end{aligned}$$

The boundary term vanishes as $\epsilon \rightarrow 0$ since

$$\left| u(x) \nabla \phi(x) \frac{x}{|x|} \right| \leq \|\nabla \phi\|_{L^\infty} |u(x)| = C \log |\log(r)|$$

$$\begin{aligned} \left| \int_{\partial B(0, \epsilon)} u(x) \nabla \phi(x) \frac{x}{|x|} dS(x) \right| &\leq C \int_{\partial B(0, \epsilon)} \log |\log(\epsilon)| dS(x) \\ &= C \log |\log \epsilon| \underbrace{|\partial B(0, \epsilon)|}_{\sim \epsilon^2} \xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned}$$

$$\begin{aligned} \int_{|x| > \epsilon} \nabla u(x) \nabla \phi(x) dx &= \sum_{i=1}^d \int_{|x| > \epsilon} \partial_i u(x) \partial_i \phi(x) dx \\ &= \sum_{i=1}^d \left(- \int_{\partial B(0, \epsilon)} \partial_i u(x) \phi(x) \frac{x_i}{|x|} dS(x) - \int_{|x| > \epsilon} \underbrace{\partial_i \partial_i u(x)}_{f(x)} \phi(x) dx \right) \end{aligned}$$

The boundary term vanishes as $\epsilon \rightarrow 0$ as

$$\begin{aligned} \left| \int_{\partial B(0, \epsilon)} \partial_i u(x) \phi(x) \frac{x_i}{|x|} dS(x) \right| &\leq \|\phi\|_{L^\infty} \int_{\partial B(0, \epsilon)} |\partial_i u(x)| dS(x) \\ (\star) \quad &\leq C \frac{1}{|\epsilon \log(r)|} |\partial B(0, \epsilon)| \rightarrow 0 \end{aligned}$$

as $\epsilon \rightarrow 0$. $(\star)u = u(r), u(x) = \omega(|x|), \partial_i u(x) = \omega'(|x|) \frac{x_i}{|x|}, |\partial_i u(x)| \leq |\omega'(|x|)| = \left| \frac{1}{r \log(r)} \right|$. Finally:

$$\int_{|x| > \epsilon} f(x) \phi(x) dx \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} f(x) \phi(x) dx$$

Since $f \phi \in L^1$ and Dominated Convergence. ■

Exercise 4.22 (Bonus 5) Construct $u \in L^1(\mathbb{R}^3)$ compactly supported s.t. $-\Delta u \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and u is not continuous at 0.

Hint: Related to E 5.5. $u_0(x) = \omega(r) = \log(|\log(r)|)$ if $0 < r = |x| < \frac{1}{2}$. Consider χu_0 where $\chi \in C_c^\infty$, $\chi = 0$ if $|x| > \frac{1}{2}$, $\chi = 1$ if $|x| < \frac{1}{4}$. You can prove that $\Delta(\chi u_0) = (\Delta\chi)u_0 + 2\nabla\chi\nabla u_0 + \chi(\underbrace{\Delta u_0}_{\in L^{\frac{3}{2}}})$ in $D'(\mathbb{R}^3)$. (almost everywhere, in distributional sense, integration by parts)

Theorem 4.23 (Regularity on Domains) Let $\Omega \subseteq \mathbb{R}^d$ be open. Assume $u, f \in D'(\Omega)$ such that $-\Delta u = f$ in $D'(\Omega)$.

- a) If $f \in L_{loc}^1(\Omega)$, then
 - $u \in C^1(\Omega)$ if $d = 1$
 - $u \in L_{loc}^q(\Omega)$ for all $q < \infty$ if $d = 2$
 - $u \in L_{loc}^q(\Omega)$ for all $q < \frac{d}{d-2}$ if $d \geq 3$
- b) If $f \in L_{loc}^q(\Omega)$, $d \geq p < \frac{d}{2}$, then $u \in C_{loc}^{0,\alpha}(\Omega)$, where $0 < \alpha < 2 - \frac{d}{p}$
- c) If $f \in L_{loc}^p(\Omega)$, $p > d$, then $u \in C_{loc}^{1,\alpha}(\Omega)$, where $0 \leq \alpha < 1 - \frac{d}{p}$
- d) If $f \in C_{loc}^{0,\alpha}(\Omega)$ for some $0 < \alpha < 1$, then $u \in C_{loc}^{2,\alpha}(\Omega)$
- e) If $f \in C_{loc}^{m,\alpha}(\Omega)$, then $u \in C_{loc}^{m+2,\alpha}(\Omega)$

Proof. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Take a ball $\bar{B} \subseteq \Omega$. Define $f_B : \mathbb{R}^d \rightarrow \mathbb{K}$,

$$f_B(x) = (\mathbb{1}_B f)(x) = \begin{cases} f(x) & x \in B \\ 0 & x \notin B \end{cases}$$

Then if $f \in L_{loc}^1(\Omega)$, f_B is compactly supported. From the previous theorems: $G \star f_B \in L_{loc}^1(\mathbb{R}^d)$ and $-\Delta(G \star f_B) = f_B$ in $D'(\mathbb{R}^d)$. On the other hand, $-\Delta u = f$ in $D'(\Omega)$, so $-\Delta(u - G \star f_B) = 0$ in $D'(B)$. Indeed, for all $\phi \in C_c^\infty(B)$, then:

$$(-\Delta u)(\phi) = \int_{\Omega} f \phi = \int_B f_B \phi = - \int_{\mathbb{R}^d} f_B \phi = (-\Delta)(G \star f_B)(\phi)$$

Then $-\Delta u = -\Delta(G \star f_B)$ in $D'(B)$. Then $u - G \star f_B$ is harmonic in B and by Weyl's lemma we have $u - G \star f_B \in C^\infty(B)$. So the smoothness of u in B is the same to that of $G \star f$. ■

Exercise 4.24 (E 6.1) Show that if $\chi \in C^\infty(\mathbb{R}^d)$, $f \in W^{1,p}(\mathbb{R}^d)$, $1 \leq p < \infty$, then $\chi f \in W_{loc}^{1,p}(\mathbb{R}^d)$ and

$$\partial_i(\chi f) = (\partial_i \chi)f + \chi(\partial_i f) \quad \text{in } D'(\mathbb{R}^d)$$

Solution. $\chi f \in L_{loc}^p(\mathbb{R}^d)$ obvious. $\partial(\chi f) \in L_{loc}^p(\mathbb{R}^d)$ is nontrivial but follows from $\partial_i(\chi f) = \underbrace{(\partial_i \chi)f + \chi(\partial_i f)}_{\in L_{loc}^p}$ in $D'(\mathbb{R}^d)$. To compute the distributional derivative

$\partial_i(\chi f)$, then: Take $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$- \int_{\mathbb{R}^d} \chi f(\partial_i \phi) dx = \int_{\mathbb{R}^d} (?) \phi dx$$

We have

$$\begin{aligned}
-\int_{\mathbb{R}^d} \chi f(\partial_i \phi) dx &= -\int_{\mathbb{R}^d} f(\chi \partial_i \phi) dx \\
(\partial_i(\chi \phi) = \chi \partial_i \phi + (\partial_i \chi) \phi) &= -\int_{\mathbb{R}^d} f(\partial_i(\chi \phi) - (\partial_i \chi) \phi) dx \\
&= -\int_{\mathbb{R}^d} f \underbrace{\partial_i(\chi \phi)}_{\in C_c^\infty} dx + \int_{\mathbb{R}^d} f(\partial_i \chi) \phi dx \\
&= \int_{\mathbb{R}^d} (\partial_i f) \chi \phi dx + \int_{\mathbb{R}^d} f(\partial_i \chi) \phi dx \\
&= \int_{\mathbb{R}^d} ((\partial_i f) \chi + f(\partial_i \chi)) \phi dx
\end{aligned}$$

So $\partial_i(\chi f) = (\partial_i f) \chi + f(\partial_i \chi)$ in $D'(\mathbb{R}^d)$. ■

Remark 4.25 Question: If $\chi \in C^1(\mathbb{R}^d)$, $f \in W^{1,p}(\mathbb{R}^d)$. Is this it still correct that $\partial_i(\chi f) = (\partial_i \chi) f + \chi(\partial_i f)$ in $D'(\mathbb{R}^d)$?

Proof. It suffices to show that we still can apply integration by parts.

$$(\star) \quad -\int f \partial_i g \stackrel{?}{=} \int (\partial_i f) g$$

Approximation: (\star) is correct if $g \in C_c^\infty$

- If $g \in C_c^1$, there is $\{g_n\} \subseteq C_c^\infty$ s.t. $g_n \rightarrow g$ in $W_{loc}^{1,p}$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\int (\partial_i g) f \xrightarrow{n \rightarrow \infty} - \int \underbrace{f}_{L^p} \underbrace{\partial_i g_n}_{\rightarrow \partial_i g \text{ in } L^q} = \int \underbrace{(\partial_i f)}_{\in L^p} \underbrace{g_n}_{\rightarrow g \text{ in } L^q} \xrightarrow{n \rightarrow \infty} \int (\partial_i f) g \quad \blacksquare$$

Exercise 4.26 (E 6.2) $G(x) = -\frac{1}{2\pi} \log|x|$. Let $f \in L^p(\mathbb{R}^2)$, compactly supported. Define $u(x) = (G \star f)(x) = \int_{\mathbb{R}^2} G(x-y) f(y) dy$

1. If $p = 1$, then $u \in L_{loc}^q(\mathbb{R}^2)$ for all $q < \infty$.
2. If $p > 2$, then $u \in C^{1,\alpha}$ with $0 < \alpha < 1 - \frac{2}{p}$.

Solution. 1. Take any ball $B = B(0, R)$ and:

$$\begin{aligned}
\int_B |u(x)|^q dx &= \int_B \left(\int_{\mathbb{R}^2} |G(x-y)| |f(y)| dy \right)^q dx \\
&\leq C \int_B \left(\int_{\mathbb{R}^2} |G(x-y)|^q |f(y)| dy \right) dx \\
&= C \int_{\mathbb{R}^2} \left(\int_B |G(x-y)|^q dx \right) |f(y)| dy
\end{aligned}$$

Recall from the proof of Youngs inequality:

$$\begin{aligned}
|u(x)| &= \left| \int_{\mathbb{R}^2} G(x-y) f(y) dy \right| \\
&\leq \int_{\mathbb{R}^2} |G(x-y)| |f(y)| dy \\
&\leq \left(\int_{\mathbb{R}^2} |G(x-y)|^q |f(y)| dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} |f(y)| dy \right)^{\frac{1}{q'}}, \quad \frac{1}{q} + \frac{1}{q'} = 1
\end{aligned}$$

Assume $\text{supp } f \subseteq \overline{B(0, R)}$. Then if $y \in \text{supp } f$ and $x \in B(0, R)$, then $|x - y| \leq |x| + |y| \leq R + R_1$. For all $y \in \text{supp } f$:

$$\begin{aligned} \int_{B(0, R)} |G(x - y)|^q dx &\leq \int_{|x - y| \leq R + R_1} |G(x - y)|^q dx \\ &= \int_{|z| \leq R + R_1} |G(z)|^q dz < \infty \end{aligned}$$

as $G \in L_{loc}^q$ ($|G(z)| = \frac{1}{2\pi} |\log(z)| \leq \frac{C_{R+R_1, \epsilon}}{|z|^\epsilon}$ for all $|z| \leq R + R_1$), so

$$\int_{|z| \leq R + R_1} |G(z)|^q \leq C_{R+R_1, \epsilon} \int_{|z| \leq R + R_1} \frac{1}{|z|^{\epsilon q}} dz < \infty$$

if $\epsilon q < 2$.

2. Recall $\partial_i u \in L_{loc}^1(\mathbb{R}^2)$ and:

$$\partial_i u(x) = (\partial_i G \star f)(x) = c \int_{\mathbb{R}^2} \frac{x_i - y_i}{|x - y|^2} f(y) dy$$

First we show $\partial_i u \in C^{0, \alpha}$:

$$\begin{aligned} |\partial_i u(x) - \partial_i u(z)| &= \left| C \int_{\mathbb{R}^2} \left(\frac{x_i - y_i}{|x - y|^2} - \frac{z_i - y_i}{|z - y|^2} \right) f(y) dy \right| \\ &\leq C \int_{\mathbb{R}^2} \left| \frac{x_i y_i}{|x - y|^2} - \frac{z_i y_i}{|z - y|^2} \right| |f(y)| dy \\ &\stackrel{?}{\leq} C |x - z|^\alpha \end{aligned}$$

Note that

$$\begin{aligned} \left| \frac{x_i - y_i}{|x - y|^2} - \frac{z_i - y_i}{|z - y|^2} \right| &= \left| (x_i - y_i) \left(\frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right) + \frac{x_i - z_i}{|z - y|^2} \right| \\ &\leq |x_i - y_i| \left| \frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right| + \frac{|x_i - z_i|}{|z - y|^2} \\ &\leq C |z - x|^\alpha \left(\frac{1}{|x - y|^{1+\alpha}} + \frac{1}{|z - y|^{1+\alpha}} + \frac{|x - z|}{|z - y|^2} \right) \end{aligned}$$

Here $|x_i - z_i| \leq |x - z|$ and $|x_i - y_i| \leq |x - y|$ and:

$$\begin{aligned} \underbrace{\left| \frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right|}_{\text{sym } x \leftrightarrow z} &= \left| \frac{1}{|x - y|} - \frac{1}{|z - y|} \right| \left| \frac{1}{|x - y|} + \frac{1}{|z - y|} \right| \\ &= \frac{||z - y| - |x - y||}{|x - y||z - y|} \left| \frac{1}{|x - y|} + \frac{1}{|z - y|} \right| \\ &\leq |z - x|^\alpha \frac{\max(|z - y|, |x - y|)^{1-\alpha}}{|x - y||z - y|} \left(\frac{1}{|x - y|} + \frac{1}{|z - y|} \right) \\ &\leq C |z - x|^\alpha \left(\frac{1}{|x - y|^{2+\alpha}} + \frac{1}{|z - y|^{2+\alpha}} \right) \end{aligned}$$

By the symmetrie $x \leftrightarrow z$:

$$\begin{aligned} LHS &\leq C |z - x|^\alpha \left(\frac{1}{|x - y|^{1+\alpha}} + \frac{1}{|z - y|^{1+\alpha}} \right) + \frac{|x - y|}{|x - y|^2} \\ \Rightarrow LHS &\leq C \cdots + |x - z| \min \left(\frac{1}{|z - y|^2}, \frac{1}{|x - y|^2} \right) \\ &\leq (|x - y| + |z - y|)^{1-\alpha} \\ &\leq C |z - x|^\alpha \left(\frac{1}{|x - y|^{1+\alpha}} + \frac{1}{|z - y|^{1+\alpha}} \right) \end{aligned}$$

In summary:

$$\begin{aligned} |\partial_i u(x) - \partial_i u(z)| &\leq C \int_{\mathbb{R}^2} \left| \frac{x_i - y_i}{|x - y|^2} - \frac{z_i - y_i}{|z - y|^2} \right| |f(y)| dy \\ &= C |x - y|^\alpha \int_{\mathbb{R}^2} \left(\frac{1}{|x - y|^{1+\alpha}} + \frac{1}{|z - y|^{1+\alpha}} \right) |f(y)| dy \end{aligned}$$

Consider if $|x| > 2R_1$:

$$\int_{\mathbb{R}^2} \frac{1}{|x - y|^{1+\alpha}} |f(y)| dy \leq \int_{\mathbb{R}^2} \frac{1}{R_1^{1+\alpha}} |f(y)| dy \leq C$$

$\text{supp } f \subseteq B(0, R_1)$. If $|x| < 2R_1$, then $|x - y| \leq 3R$ if $y \in B(0, R_1)$. Hence:

$$\begin{aligned} &\int_{|x-y| \leq 3R_1} \frac{1}{|x - y|^{1+\alpha}} |f(y)| dy \\ &\leq \left(\int_{|x-y| \leq 3R_1} \frac{1}{|x - y|^{(1+\alpha)p'}} \right)^{\frac{1}{p'}} \left(\int |f(y)|^p dy \right)^{\frac{1}{p}} \\ &= \int_{|z| \leq 3R_1} \frac{1}{|z|^{(1+\alpha)p'}} dz < \infty \end{aligned}$$

So $\alpha < 1 - \frac{2}{p}$. ■

Exercise 4.27 (E 6.3) Let $f \in C_{loc}^{0,\alpha}$ and $-\Delta u = f$ in $D'(\Omega)$. Prove $u \in C_{loc}^{2,\alpha}(\Omega)$.

Solution. Take an open ball $B \subseteq \bar{B} \subseteq \Omega$. We prove $u \in C^{2,\alpha}(B)$. There is an open Ω_B s.t. $\bar{B} \subseteq \bar{\Omega}_B \subseteq \Omega$. Then there is a $\chi_B \in C_c^\infty(\mathbb{R}^d)$ s.t. $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin \Omega_B$. Define

$$f_B(x) = \chi_B(x)f(x) : \mathbb{R}^d \rightarrow \mathbb{R}$$

We prove that $f_B \in C^{0,\alpha}(\mathbb{R}^d)$. Since $f \in C_{loc}^{0,\alpha}(\Omega)$ we have $f \in C^{0,\alpha}(\Omega_B)$, so $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all $x, y \in \Omega_B$. Then:

$$\begin{aligned} |f_B(x) - f_B(y)| &= |\chi_B(x)f(x) - \chi_B(y)f(y)| \\ &\leq |(\chi_B(x) - \chi_B(y))f(x) + \chi_B(y)(f(x) - f(y))| \\ &\leq C|x - y|^\alpha \|f\|_{L^\infty(\Omega_B)} + C\|\chi\|_{L^\infty(\Omega_B)} |x - y|^\alpha \leq C_{\Omega_B} |x - y|^\alpha \end{aligned}$$

What about other cases? If x, y are not in Ω_B , then $|f_B(x) - f_B(y)| = 0$, then if $x \in \Omega_B$ and $y \notin \Omega_B$: $|f_B(x) - f_B(y)| = |f_B(x)| = |\chi_B(x)||f(x)| = |\chi_B(x) - \chi_B(y)||f(x)| \leq C|x - y|^\alpha$. Conclusion: $|f_B(x) - f_B(y)| \leq C|x - y|^\alpha$ for all $x, y \in \mathbb{R}^d$, i.e. $f_B \in C^{0,\alpha}(\mathbb{R}^d)$. Also f_B is compactly supported. By a theorem in the lecture: $G \star f_B \in C^{2,\alpha}(\mathbb{R}^d)$. Finally: $-\Delta u = f$ in $D'(\Omega)$, $-\Delta(G \star f_B) = f_B$ in $D'(\mathbb{R}^d)$. So we conclude $-\Delta u = f = f_B = -\Delta(G \star f_B)$ in $D'(B)$. $-\Delta(u - G \star f_B) = 0$ in $D'(B)$, so $u - G \star f_B \in C^\infty(B)$, so $u \in C^{2,\alpha}(B)$. ■

Exercise 4.28 (E 6.4) $u, f \in L^2(\mathbb{R}^d)$, $-\Delta u = f$ in $D'(\mathbb{R}^d)$. Prove $u \in W^{2,2}(\mathbb{R}^d)$, $\|u\|_{W^{2,2}(\mathbb{R}^d)} \leq C(\|u\|_{L^2} + \|f\|_{L^2})$.

$$\begin{aligned} W^{2,2}(\mathbb{R}^d) &= \{g \in L^2(\mathbb{R}^d) \mid D^\alpha g \in L^2 \text{ for all } |\alpha| \leq 2\} \\ &= \{g \in L^2(\mathbb{R}^d) \mid \widehat{D^\alpha g}(k) = (-2\phi i k)^\alpha \hat{g}(k) \in L^2(\mathbb{R}^d) \text{ for all } |\alpha| \leq 2\} \\ &= \{g \in L^2(\mathbb{R}^d) \mid (1 + |k|^2)\hat{g}(k) \in L^2(\mathbb{R}^d)\} \end{aligned}$$

$\|u\|_{W^{2,2}(\mathbb{R}^d)}$ is comparable $\int_{\mathbb{R}^d} (1 + |k|^2)^2 |\hat{g}(k)|^2 dk$. If $D^\alpha g \in L^2$, then $\widehat{D^\alpha g}(k) = (-2\pi i k)^\alpha \hat{g}(k)$. For any $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned} \int \widehat{D^\alpha g}(k) \hat{\phi}(k) dk &= \int (D^\alpha g) \phi = (-1)^{|\alpha|} \int g(D^\alpha \phi) \\ &= (-1)^{|\alpha|} \int \widehat{\tilde{g}}(k) \widehat{D^\alpha \phi}(k) \\ &= (-1)^{|\alpha|} \int \widehat{\tilde{g}}(k) (-2\pi i k)^\alpha \hat{\phi}(k) dk \end{aligned}$$

so $\hat{D^\alpha g}(k) = (-1)^{|\alpha|} \hat{g}(k) \overline{(-2\pi i k)^\alpha} = \hat{g}(k) (-2\pi i k)^\alpha$. This implies:

$$\begin{aligned} \|u\|_{W^{2,2}(\mathbb{R}^d)} &\leq C \int_{\mathbb{R}^d} (1 + |k|^2)^2 |\hat{u}(k)|^2 dk \\ &= C \left(\|u\|_{L^2}^2 + \int_{\mathbb{R}^d} |k|^4 |\hat{u}(k)|^2 dk \right) \\ &\leq C (\|u\|_{L^2}^2 + \|f\|_{L^2}^2) \\ &\leq C (\|u\|_{L^2} + \|f\|_{L^2})^2 \end{aligned}$$

Remark 4.29 (Bonus 6) Let $f, g \in W^{1,2}(\mathbb{R}^d)$. Prove that $fg \in W^{1,1}(\mathbb{R}^d)$ and

$$\partial_i(fg) = (\partial_i f)g + f(\partial_i g) \quad \text{in } D'(\mathbb{R}^d)$$

Chapter 5

Existence for Poisson's Equation on Domains

Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider Poisson's equation.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for given data (f, g) and u the unknown function.

- Classical solutions: $f \in C^2(\bar{\Omega}) \rightsquigarrow$ explicit representation formula.
- Weak solution: $f \in L^p(\Omega)$, $g \in L^p(\partial\Omega) \rightsquigarrow u \in W^{2,p}(\Omega)$. We are going to establish the existence by *Energy Methods*. (Calculus of variations)

Definition 5.1 (C^1 -Domains) Let $\Omega \subseteq \mathbb{R}^d$ be open. We say that Ω is of class C^1 (i.e. $\partial\Omega \in C^1$) if for all $x_0 \in \partial\Omega$ there is a bijective function $h : U \rightarrow Q$, where

- $x_0 \in U$ open in \mathbb{R}^d
- $Q = \{x = (x_1, \dots, x_d) = (x', x_d)\} \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, |x_d| < 1\}$
- $h \in C^1(\bar{U})$ and $h^{-1} \in C^1(\bar{Q})$ (C^1 -diffeomorphism)
- $h(U) = Q$

$$\begin{aligned} h(U \cap \Omega) &= Q_+ = Q \cap \mathbb{R}_+^d = \{x = (x', x_d) \in Q \mid x_d > 0\} \\ h(U \cap \partial\Omega) &= Q_0 = Q \cap \partial\mathbb{R}_+^d = \{x = (x', x_d) \in Q \mid x_d = 0\} \\ h(U \setminus \bar{\Omega}) &= Q_- = Q \cap \mathbb{R}_-^d = \{x = (x', x_d) \in Q \mid x_d < 0\} \end{aligned}$$

(From Brezis' book)

Remark 5.2 The set Q can be replaced by a ball, i.e. Ω is of C^1 if for all $x_0 \in \partial\Omega$ there is a function $U \rightarrow B(0, 1) \subseteq \mathbb{R}^d$.

- $x_0 \in U$ with $U \subseteq \mathbb{R}^d$ open.
- $h \in C^1(\bar{U})$, $h^{-1} \in C^1(\overline{B(0, 1)})$
- $h(U \cap \Omega) = B(0, 1) \cap \mathbb{R}_+^d$, $h(U \cap \partial\Omega) = B(0, 1) \cap \mathbb{R}^d$.

Remark 5.3 (An equivalent definition from Evan's book App. C) Let $\Omega \subseteq \mathbb{R}^d$ be open. Then Ω is C^1 if for all $x_0 \in \partial\Omega$ there is a $r > 0$ and a C^1 -function $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ s.t. (upon relabeling and reorienting the axes if necessary) such that:

$$\Omega \cap B(x_0, r) = \{x = (x', x_d) \in B(x_0, r) \mid x_d < \gamma(x')\}$$

Proof of the equivalence of the two definitions.

Def. 2 \Rightarrow Def. 1: In fact, given $x_0 \in \partial\Omega$ and γ we can define

$$\begin{aligned} h(x', x_d) &= (x', x_d - \gamma(x')) \in C^1(\mathbb{R}^d, \mathbb{R}^d) \\ h^{-1}(x', x_d) &= (x', x_d + \gamma(x')) \in C^1(\mathbb{R}^d, \mathbb{R}^d) \end{aligned}$$

Def. 1 \Rightarrow Def. 2: We need the inverse function theorem and the implicit function theorem. Let $x_0 \in \partial\Omega$, let $h : U \rightarrow B(0, 1)$ as in Def. 1. Denote $h = (h_1, h_2, \dots, h_d)$. Since h is invertible near x_0 , by the inverse function theorem we have for the Jacobi matrix $Jh(x_0) = (\partial_j h_i(x_0))_{1 \leq i, j \leq d}$ is invertible. So we have $\nabla h_d(x_0) = (\partial_j h_d(x_0))_{1 \leq j \leq d} \neq \vec{0}^{\mathbb{R}^d}$, so there is a $j \in \{1, 2, \dots, d\}$ s.t. $\partial_j h_d(x_0) \neq 0$. By relabeling and reorienting the axes, we can assume that $\partial_d h_d(x_0) > 0$. By continuity there is a $r > 0$ such that $\partial_d h_d(x) > 0$ for all $x \in B(x_0, r)$. Define $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ s.t. in $B(x_0, r)$:

$$x = (x', x_d) \in \partial\Omega \iff h_d(x', x_d) = 0 \iff x_d = \gamma(x'),$$

$h_d : \mathbb{R}^d \rightarrow \mathbb{R}$. This gives a solution γ if $\partial_d h_d > 0$ in $B(x_0, r)$. (For implicit function theorem, $\partial_d h_d(x_0) \neq 0$) Question: Why in $B(x_0, r)$?

$$x = (x', x_d) \in \Omega \iff x_d > \gamma(x')$$

Since $\partial_d h_d(x) > 0$ for all $x \in B(x_0, r)$ we have that $x_d \mapsto h_d(x', x_d)$ is strictly increasing, hence

$$\begin{aligned} x &= (x', x_d) \in \Omega \\ \iff h(x', x_d) &\in \mathbb{R}_+^d \\ \iff h_d(x', x_d) &> 0 = h_d(x', \gamma(x')) \\ \iff x_d &> \gamma(x') \end{aligned} \quad \blacksquare$$

Theorem 5.4 (Gauss-Green formula / Integration by parts) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. Then

1. For all $u, v \in C^1(\bar{\Omega})$:

$$\int_{\Omega} (\partial_i u) v = - \int_{\Omega} u (\partial_i v) + \int_{\partial\Omega} u v n_i dS,$$

where $\vec{n} = (n_i)_{i=1}^d$ is the outwards unit normal vector.

2. For all $u, v \in C^2(\bar{\Omega})$:

$$\int_{\Omega} u (-\Delta v) = \int_{\Omega} \nabla u \nabla v - \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}} dS$$

where $\frac{\partial v}{\partial \vec{n}} = \nabla v \vec{n} = \sum_{i=1}^d \partial_i v n_i$.

Classical solutions via Green's function:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, $\partial\Omega \in C^1$. Assume there exists a $u \in C^2(\bar{\Omega})$, $f \in C(\bar{\Omega})$, $g \in C(\partial\Omega)$. Let G be the fundamental solution of the Laplace Equation in \mathbb{R}^d . We use integration by parts in $\Omega \setminus B(x, \epsilon)$:

$$\begin{aligned} & \int_{\Omega \setminus B(x, \epsilon)} u(y)(-\Delta G)(y-x) dy \\ &= \int_{\Omega \setminus B(x, \epsilon)} \nabla u(y) \nabla G(y-x) dy - \int_{\partial\Omega \cup \partial B(x, \epsilon)} u(y) \frac{\partial G}{\partial \vec{n}}(y-x) dS(y) \\ & \int_{\Omega \setminus B(x, \epsilon)} G(y-x)(-\Delta u)(y) dy \\ &= \int_{\Omega \setminus B(x, \epsilon)} \nabla G(y-x) \nabla u(y) dy - \int_{\partial\Omega \cup \partial B(x, \epsilon)} G(y-x) \frac{\partial u}{\partial \vec{n}}(y) dS(y) \end{aligned}$$

This implies:

$$\begin{aligned} & \int_{\Omega \setminus B(x, \epsilon)} [u(y)(-\Delta G(y-x)) - G(y-x)(-\Delta u)(y)] dy \\ &= - \int_{\partial\Omega \cup \partial B(x, \epsilon)} \left[u(y) \frac{\partial G}{\partial \vec{n}}(y-x) - G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right] dS(y) \end{aligned}$$

for all $x \in \Omega$, $x \in B(x, \epsilon) \subseteq \Omega$. When $\epsilon \rightarrow 0$, then the left hand side converges to $-\int_{\Omega} G(y-x)f(y) dy$ and the right hand side (for $d \geq 2$) we have $\partial_j G(y) = \frac{-y_j}{d|B_1||y|^d}$, so

$$\frac{\partial G}{\partial \vec{n}} = \nabla G \vec{n} = \nabla G(y) \left(\frac{-y}{|y|} \right) = \sum_{j=1}^d \frac{-y_i}{d|B_1||y|^d} \frac{-y_j}{|y|} = \frac{1}{d|B_1||y|^{d-1}} \text{ on } \partial B(0, \epsilon)$$

so we have

$$\frac{\partial G}{\partial \vec{n}}(y-x) = \frac{1}{d|B_1|\epsilon^{d-1}}$$

on $\partial B(x, \epsilon)$. Hence

$$\begin{aligned} \int_{\partial B(x, \epsilon)} u(y) \frac{\partial G}{\partial \vec{n}}(y-x) dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(x, \epsilon)} u(y) dS(y) \\ &= \oint_{\partial B(x, \epsilon)} u(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} u(x) \end{aligned}$$

On the other hand:

$$\left| \int_{\partial B(x, \epsilon)} G(y-x) \frac{\partial u(y)}{\partial \vec{n}} dS(y) \right| \leq C\epsilon^{d-1} \sup_{|z|=\epsilon} |G(z)| \xrightarrow{\epsilon \rightarrow 0} 0$$

since $|G(z)| \leq \frac{C}{|z|^{d-2}}$ if $d \geq 3$, $|G(z)| \leq C|\log(z)|$ if $d = 2$ and $|G(z)| \leq C|z|$ if $d = 1$.

In summary:

$$\begin{aligned} & - \int_{\Omega} G(y-x)f(y) dy = - \int_{\partial\Omega} \left[u(y) \frac{\partial G}{\partial \vec{n}}(y-x) - G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right] dS(y) - u(x) \\ & \Leftrightarrow u(x) = \int_{\Omega} G(y-x)f(y) dy + \int_{\partial\Omega} \left[G(y-x) \frac{\partial u}{\partial \vec{n}}(y) - g(y) \frac{\partial G}{\partial \vec{n}}(y-x) \right] dS(y) \end{aligned}$$

Problem: We don't know anything about $\frac{\partial u}{\partial \vec{n}}$ on $\partial\Omega$. Trick: We can resolve that by using the *corrector* function: $\Phi_x = \Phi_x(y)$ which solves:

$$\begin{cases} -\Delta \Phi_x = 0 & \text{in } \Omega \\ \Phi_x(y) = G(y-x) & \text{on } \partial\Omega \end{cases}$$

We assume that Φ_x exists.

Definition 5.5 (Green's function) $\tilde{G}(x-y) = G(y-x) - \Phi_x(y)$ for all $x, y \in \Omega$, $x \neq y$.

Exercise 5.6 (E 7.1) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded with C^1 boundary. For $x \in \Omega$, assume there exist $\Phi_x(y)$, $y \in \bar{\Omega}$, s.t.

$$\begin{cases} \Delta_y \Phi_x(y) = 0 \\ \Phi_x(y) = G(y-x) \end{cases},$$

$G(z) = \frac{1}{d(d-2)|B_1||z|^{d-2}}$, $d \geq 3$. Prove that $\Phi_x(y) = \Phi_y(x)$ for all $x, y \in \Omega$. Then $\tilde{G}(x, y) = G(y-x) - \Phi_x(y)$ is symmetric, i.e. $\tilde{G}(x, y) = \tilde{G}(y, x)$.

Solution. Assume $x \neq y$. Define

$$\begin{aligned} f(z) &= \tilde{G}(x, z) = G(z-x) - \Phi_x(z) \\ g(z) &= \tilde{G}(y, z) = G(z-y) - \Phi_y(z) \end{aligned}$$

Integration by parts:

$$\begin{aligned} \int_{\Omega \setminus (B(x, \epsilon) \cup B(y, \epsilon))} (f \Delta g - g \Delta f) &= \int_{\partial\Omega \cup \partial B(x, \epsilon) \cup \partial B(y, \epsilon)} \left(f \frac{\partial g}{\partial \vec{n}_z} - g \frac{\partial f}{\partial \vec{n}_z} \right) dS(z) \\ &= \int_{\partial B(x, \epsilon) \cup \partial B(y, \epsilon)} \left(f \frac{\partial g}{\partial \vec{n}_z} - g \frac{\partial f}{\partial \vec{n}_z} \right) dS(z) \end{aligned}$$

Consider $f \frac{\partial g}{\partial \vec{n}_z}$ on $\partial B(x, \epsilon)$. Since g is only singular at y , so $\left| \frac{\partial g}{\partial \vec{n}} \right| \leq C$ on $\partial B(x, \epsilon)$. This implies:

$$\begin{aligned} \int_{\partial B(x, \epsilon)} \left| f \frac{\partial g}{\partial \vec{n}_z} \right| dS(z) &\leq C \int_{\partial B(x, \epsilon)} |f| dS(z) \\ &\leq C \int_{\partial B(x, \epsilon)} \left(\frac{1}{|x-z|^{d-2}} + \|\Phi_x\|_{L^\infty(\Omega)} \right) dS(z) \\ &\leq C \epsilon^{d-1} \left(\frac{1}{\epsilon^{d-2}} + 1 \right) \leq C \epsilon \xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned}$$

Consider $f \frac{\partial g}{\partial \vec{n}_z}$ on $\partial B(y, \epsilon)$. Decompose $\frac{\partial g}{\partial \vec{n}} = [\nabla_z G(z-y) - \nabla_z \Phi_y(z)] \frac{(z-y)}{|z-y|}$. Since $\Phi_y(z)$ is harmonic in Ω , we have that

$$\int_{\partial B(y, \epsilon)} \left| \nabla_z \Phi_y(z) \frac{(z-y)}{|z-y|} \right| \leq C \int_{\partial B(y, \epsilon)} |f| \leq C \epsilon^{d-1} \xrightarrow{\epsilon \rightarrow 0} 0$$

Thus the main contribution from $f \frac{\partial g}{\partial \vec{n}}$ is

$$\begin{aligned}
& \int_{\partial B(y, \epsilon)} f(z) \nabla_z G(z-y) \frac{-(z-y)}{|z-y|} dS(z) \\
&= \int_{\partial B(y, \epsilon)} f(z) \frac{-(z-y)}{d|B_1||z-y|^d} \frac{-(z-y)}{|z-y|} dS(z) \\
&= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(y, \epsilon)} f(z) dS(z) \\
&= \oint_{\partial B(y, \epsilon)} f(z) dS(z) = f(y)
\end{aligned}$$

In summary:

$$\int_{\partial B(x, \epsilon) \cup \partial B(y, \epsilon)} f \frac{\partial g}{\partial \vec{n}_z} dS(z) \xrightarrow{\epsilon \rightarrow 0} f(y)$$

Similary:

$$\int_{\partial B(x, \epsilon) \cup \partial B(y, \epsilon)} g \frac{\partial f}{\partial \vec{n}_z} dS(z) \xrightarrow{\epsilon \rightarrow 0} g(x)$$

So we have that $f(y) = g(x)$, so

$$\begin{aligned}
f(y) &= G(y-x) - \Phi_x(y) \\
g(x) &= G(x-y) - \Phi_y(x).
\end{aligned}$$

So $\Phi_x(y) = \Phi_y(x)$ for all $x \neq y \in \Omega$. This implies $\Phi_x(y) = \Phi_y(x)$ for all $x, y \in \Omega$. ■

Theorem 5.7 Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and C^1 . If $u \in C^2(\Omega)$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases},$$

then

$$u(x) = - \int_{\partial\Omega} g(y) \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) dS(y) + \int_{\Omega} \tilde{G}(x, y) dy$$

Proof. We need to prove:

$$\int_{\Omega} \Phi_x(y) f(y) dy + \int_{\partial\Omega} \left(-g(y) \frac{\partial \Phi_x(y)}{\partial \vec{n}_y} + G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right) = 0$$

By integration by parts:

$$\begin{aligned}
\int_{\Omega} \Phi_x(y) f(y) dy &= \int_{\Omega} \Phi_x(y) (-\Delta u(y)) dy \\
&= \int_{\Omega} [\Phi_x(y) (-\Delta u(y)) + (\Delta \Phi_x(y)) u(y)] dy \\
(\Delta \Phi_x(y) = 0) &= \int_{\partial\Omega} \left(-\Phi_x(y) \frac{\partial u}{\partial \vec{n}} + \frac{\partial \Phi_x(y)}{\partial \vec{n}} \underbrace{u(y)}_{g(y)} \right) dS(y) \quad \blacksquare
\end{aligned}$$

How can we compute $\Phi_x(y)$? It is not easy for general domains. But let us prove on two cases:

- $\Omega = \mathbb{R}_+^d$ (half-space)
- $\Omega = B(0, r)$ (a ball)

5.1 Green's function on the upper half plane

We use the following notation:

$$\begin{aligned}\mathbb{R}_+^d &= \{x = (x_1, x_2, \dots, x_d) = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid x_d > 0\} \\ \partial\mathbb{R}_+^d &= \{x = (x', x_d) \mid x_d = 0\} = \mathbb{R}^{d-1} \times \{0\}\end{aligned}$$

For all $x \in \mathbb{R}^d$ we want to find the correction function $\Phi_x(y)$ with $y \in \overline{\mathbb{R}_+^d}$ s.t.

$$\begin{cases} +\Delta_y \Phi_x(y) = 0 & \text{in } \mathbb{R}_+^d \\ \Phi_x(y) = G(y - x) & \text{in } \partial\mathbb{R}_+^d \end{cases}$$

Definition 5.8 (Reflection for \mathbb{R}_+^d) For all $x = (x', x_d) \in \mathbb{R}^d$, $\tilde{x} = (x', -x_d) \in \mathbb{R}^d$, (if $x \in \mathbb{R}_+^d \Rightarrow \tilde{x} \in \mathbb{R}_-^d$)

Claim: $\Delta_y \Phi_x(y) = G(y - \tilde{x})$ is a corrector function.

- $\Delta_y \Phi_x(y) = \Delta_y G(y - \tilde{x}) = 0$ for all $y \in \mathbb{R}_+^d$ for all $x \in \mathbb{R}_+^d$ (as $\tilde{x} \in \mathbb{R}_-^d = \mathbb{R}^d \setminus \overline{\mathbb{R}_+^d}$)
- $\Phi_x(y) = G(y - \tilde{x}) = G(y - x)$ on $y \in \partial\mathbb{R}_+^d$. In fact, $y \in \partial\mathbb{R}_+^d$, so $y_d = 0$, so

$$G(y - \tilde{x}) = G_0(|y - \tilde{x}|) = G_0\left(\sqrt{\sum_{i=1}^{d-1} |x_i - y_i|^2 + |x_d|^2}\right) = G_0(|y - x|)$$

Consider $f = 0$ and

$$\begin{cases} -\Delta = 0 & \text{in } \mathbb{R}_+^d \\ u = g & \text{on } \partial\mathbb{R}_+^d \end{cases}$$

Then we expect

$$u(x) = - \int_{\partial\Omega} g(y) \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) dS(y)$$

We compute

$$\frac{\partial \tilde{G}}{\partial \vec{n}_y}(x - y) = \sum_{j=1}^d \frac{\partial \tilde{G}}{\partial y_j}(x, y) \vec{n}_j = - \frac{\partial \tilde{G}}{\partial y_d}(x, y) = \frac{\partial}{\partial y_d} (G(y - \tilde{x}) - G(y - x)) = \dots$$

because $\tilde{G}(x, y) = G(y - x) - \Phi_x(y) = G(y - x) - G(y - \tilde{x})$.

$$\begin{aligned} \dots &= \frac{1}{d|B_1|} \left[\frac{-(y_d - \tilde{x}_d)}{|y - \tilde{x}|^d} - \frac{-(y_d - x_d)}{|y - x|^d} \right] \\ (y \in \partial\mathbb{R}_+^d) &= \frac{1}{d|B_1|} \left[\frac{\tilde{x}_d}{|y - x|} - \frac{x_d}{|y - x|^d} \right] = \frac{-2x_d}{d|B_1||y - x|^d} \end{aligned}$$

We expect

$$u(x) = - \int_{\partial\mathbb{R}_+^d} g(y) \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) dS(y) = \int_{\partial\mathbb{R}_+^d} g(y) \frac{2x_d}{d|B_1||y - x|^d} dS(y)$$

Theorem 5.9 Assume $g \in C(\mathbb{R}^{d-1}) \cap L^\infty(\mathbb{R}^{d-1})$ Then

$$u(x) = \int_{\partial\mathbb{R}_+^d} g(y) K(x, y) dS(y)$$

and

$$K(x, y) = \frac{2x_d}{d|B_1||y - x|^d} \quad \text{for all } x \in \mathbb{R}_+^d.$$

satisfies that $u \in C^\infty(\mathbb{R}_+^d) \cap L^\infty(\mathbb{R}_+^d)$ and

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^d \\ \lim_{\substack{x \rightarrow 0 \\ x \in \mathbb{R}_+^d}} u(x) = g(x_0) & \forall x_0 \in \partial \mathbb{R}_+^d \end{cases}$$

Proof. Claim: For all $y \in \partial \mathbb{R}_+^d$, $x \mapsto K(x, y)$ is harmonic in \mathbb{R}_+^d (i.e. $\Delta_x K(x, y) = 0$ in \mathbb{R}_+^d)

- Argument from Evans:

$$K(x, y) = -\frac{\partial}{\partial y_d}, \quad \tilde{G}(y - x) = -\frac{\partial}{\partial y_d}(G(y - x) - G(y - \tilde{x}))$$

We know that for all $x \in \mathbb{R}_+^d$, $y \mapsto \tilde{G}(y, x)$ is harmonic in $\mathbb{R}_+^d \setminus \{x\}$. By symmetry we have $\tilde{G}(y, x) = \tilde{G}(x, y)$ for all $x, y \in \mathbb{R}_+^d$. So for all $y \in \mathbb{R}_+^d$, $x \mapsto \tilde{G}(y, x)$ is harmonic in $\mathbb{R}_+^d \setminus \{y\}$. Then for all $y \in \mathbb{R}_+^d$: $-\frac{\partial}{\partial y_d} \tilde{G}(y, x) = K(x, y)$ is harmonic $x \in \mathbb{R}_+^d \setminus \{y\}$. By a limit argument, for all $y \in \partial \mathbb{R}_+^d$, $x \mapsto K(x, y)$ is harmonic for all $x \in \mathbb{R}_+^d$.

- A direct proof:

$$K(x, y) = \frac{2x_d}{d|B_1|} \frac{1}{|x - y|^d}$$

for all $x \in \mathbb{R}_+^d$, $y \in \partial \mathbb{R}_+^d$. For $i \neq d$, $x = (x_1, \dots, x_d)$,

$$\begin{aligned} \partial_{x_i} K(x, y) &= \frac{2x_d}{d|B_1|} \frac{(-d)}{|x - y|^{d+1}} \frac{x_i - y_i}{|x - y|} = \frac{-2x_d}{|B_1|} \frac{x_i - y_i}{|x - y|^{d+2}} \\ \partial_{x_i}^2 K(x, y) &= -\frac{2x_d}{|B_1|} \left[\frac{1}{|x - y|^{d+1}} - \frac{(d+2)}{|x - y|^{d+3}} (x_i - y_i) \frac{(x_i - y_i)}{|x - y|} \right] \\ &= -\frac{2x_d}{|B_1|} \left[\frac{1}{|x - y|^{d+1}} - \frac{(d+2)}{|x - y|^{d+4}} (x_i - y_i)^2 \right] \end{aligned}$$

Moreover:

$$\begin{aligned} \partial_{x_d} K(x, y) &= \frac{2}{d|B_1|} \frac{1}{|x - y|^d} + \frac{2x_d}{d|B_1|} (-d) \frac{(x_d - y_d)}{|x - y|^{d+2}} \\ (y_d = 0) &= \frac{2}{d|B_1|} \frac{1}{|x - y|^d} + \frac{2x_d^2}{|B_1||x - y|^{d+2}} \\ \partial_{x_d}^2 K(x, y) &= \frac{-2}{|B_1|} \frac{(x_d - y_d)}{|x - y|^{d+2}} + \frac{4x_d}{|B_1||x - y|^{d+2}} - \frac{2(d+2)|B_1|^2}{x} \frac{(x_d - y_d)}{d|x - y|^{d+4}} \end{aligned}$$

Then:

$$\begin{aligned}
\Delta_x K(x, y) &= \sum_{i=1}^{d-1} \partial_{x_i}^2 K(x, y) + \partial_{x_d}^2 K(x, y) \\
&= -\frac{2x_d}{|B_1|} \left[\frac{d-1}{|x-y|^{d+2}} - (d+2) \sum_{i=1}^{d-1} \frac{(x_i - y_i)^2}{|x-y|^{d+4}} \right. \\
&\quad \left. + \frac{1+2}{|x-y|^{d+2}} - \frac{(d+2)x_d(x_d - y_d)}{|x-y|^{d+4}} \right] \\
&= -\frac{2x_d}{|B_1|} \left[\frac{d+2}{|x-y|^{d+2}} - (d+2) \frac{1}{|x-y|^{d+4}} \left(\underbrace{\sum_{i=1}^d |x_i - y_i|^2}_{|x-y|^2} \right) \right] = 0
\end{aligned}$$

for all $x \in \mathbb{R}_+^d$, $y \in \partial\mathbb{R}_+^d$. Claim (exercise) for all $x \in \mathbb{R}_+^d$,

$$\int_{\partial\mathbb{R}_+^d} K(x, y) dy = 1$$

Consider

$$u(x) = \int_{\partial\mathbb{R}_+^d} K(x, y) g(y) dy, \quad x \in \mathbb{R}_+^d$$

Since $g \in L^\infty(\mathbb{R}^{d-1}) = L^\infty(\partial\mathbb{R}_+^d)$ and $K(x, y) \geq 0$, hence

$$|u(x)| \leq \left(\int_{\partial\mathbb{R}_+^d} K(x, y) dy \right) \|g\|_{L^\infty}$$

Thus $\|u\|_{L^\infty} \leq \|g\|_{L^\infty}$. Moreover

$$D_x^\alpha u(x) = \int_{\partial\mathbb{R}_+^d} D_x^\alpha K(x, y) g(y) dy$$

bounded, so $u \in C^\infty(\mathbb{R}_+^d)$, $x \mapsto K(x, y)$ is smooth as $x \neq y$.

$$\Delta_x u(x) = \int_{\partial\mathbb{R}_+^d} \underbrace{\Delta_x K(x, y)}_{=0} g(y) dy = 0$$

So u is harmonic in \mathbb{R}_+^d . ($\Rightarrow u \in C^\infty$ by Weyl's lemma). Take $x_0 \in \partial\mathbb{R}_+^d$ and $x \in \mathbb{R}_+^d$. Then:

$$\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial\mathbb{R}_+^d} K(x, y) (g(y) - g(x_0)) dy \right| \\
&\leq \int_{\partial\mathbb{R}_+^d} K(x, y) |g(y) - g(x_0)| dy \\
&= \underbrace{\int_{|y-x_0| \leq L|x-x_0|}}_{(I)} + \underbrace{\int_{|y-x_0| > L|x-x_0|}}_{(II)}
\end{aligned}$$

$$\begin{aligned}
(I) &= \int_{|y-x_0| \leq L|x-x_0|} K(x, y) |g(y) - g(x_0)| dy \\
&= \sup_{|y-x_0| \leq L|x-x_0|} |g(y) - g(x_0)| \xrightarrow{x \rightarrow x_0} 0 \quad \forall L > 0
\end{aligned}$$

(II): If $|y - x_0| > L|x - x_0|$, then $|y - x| > \frac{1}{2}|y - x_0| > \frac{L}{2}|x - x_0|$ if $L \geq 2$.

$$\begin{aligned} \int_{|y-x_0| > L|x-x_0|} K(x, y) |g(y) - g(x_0)| dy &\leq C \int_{y \in \partial \mathbb{R}_+^d} \frac{x_d}{|x_0 - y|} dy \\ Cx_d \int_{\substack{z \in \mathbb{R}^{d-1} \\ |z| > L|x-x_0|}} \frac{1}{|z|^d} dz &= \text{const.} \frac{x_d}{L|x-x_0|} \leq \frac{\text{const.}}{L} \xrightarrow{L \rightarrow \infty} 0 \end{aligned}$$

$$x_d = |x_d - (x_0)_d| \leq |x - x_0|$$

■

5.2 Green's function for a ball

Let $B = B(0, 1)$. For all $x \in B$, for all $y \in \bar{B}$ we want to find the corrector function $\Phi_x(y)$ s.t.

$$\begin{cases} \Delta_y \Phi_x(y) = 0 & \text{in } B \\ \Phi_x(y) = G(y - x) & \text{on } \partial B \end{cases}$$

where for $d \geq 3$: $G(z) = \frac{1}{d(d-2)|B_1||z|^{d-2}}$.

Definition 5.10 (Reflection / Duality through the sphere ∂B) For all $x \in \mathbb{R}^d \setminus \{0\}$ we define $\tilde{x} = \frac{x}{|x|^2}$. Clearly we have for all $x \in B$ that if $|x| < 1$, then $|\tilde{x}| = \left| \frac{x}{|x|^2} \right| = \frac{1}{|x|} > 1$, so $\tilde{x} \notin \bar{B}$

Lemma 5.11 For $d \geq 3$ the function $\Phi_x(y) = G(|x|(y - \tilde{x}))$ is a corrector function.

Proof.

$$\Phi_x(y) = \frac{1}{d(d-2)|B_1||x|^{d-2}|y - \tilde{x}|^{d-2}}$$

for all $x \in B, x \neq 0$, for all $y \in \bar{B}$. Then clearly $y \mapsto \Phi_x(y)$ is harmonic in B (Since $\frac{1}{|z|^{d-2}}$ is harmonic in $\mathbb{R} \setminus \{0\}$). Let's check the boundary: Let $y \in \partial B$, i.e. $|y| = 1$. Then

$$\begin{aligned} ||x|(y - \tilde{x})| &= |x| \left| y - \frac{x}{|x|^2} \right| \\ &= |x| \sqrt{|y|^2 - 2 \frac{xy}{|x|^2} + \left| \frac{x}{|x|^2} \right|^2} \\ &= \sqrt{|x|^2 |y|^2 - 2xy + 1} \\ (|y| = 1) \quad &= \sqrt{|x|^2 - 2xy + |y|^2} = |x - y| \end{aligned}$$

Thus $\Phi_x(y) = G(|x||y - \tilde{x}|) = G(y - x)$ for all $0 \neq x \in B$, for all $y \in \partial B$. Let's compute the Poisson kernel: If want to solve

$$\begin{cases} -\Delta u = 0 & \text{in } B \\ u = g & \text{on } \partial B \end{cases}$$

then

$$u(x) = - \int_{\partial B} \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) g(y) dS(y).$$

$\tilde{G}(x, y) = G(y - x) - \Phi_x(y) = G(y - x) - G(|x|(y - \tilde{x}))$ for all $x \in B \setminus \{0\}$, $y \in \bar{B}$.

$$\frac{\partial \tilde{G}}{\partial \vec{n}_y} = \sum_{i=1}^d \partial_{y_i} \tilde{G} y_i$$

Here

$$\begin{aligned} \partial_{y_i} \tilde{G} &= \partial_{y_i} G(y - x) - \partial_{y_i} [G(|x|(y - \tilde{x}))] \\ &= \frac{-(y_i - x_i)}{d|B_1||y - x|^d} + \frac{y_i - \tilde{x}_i}{d|B_1||x|^{d-2}|y - \tilde{x}|^d} \\ \Rightarrow \frac{\partial \tilde{G}}{\partial \vec{n}_y} &= \sum_{i=1}^d [\dots] y_i \\ &= \frac{-y(y - x)}{d|B_1||y - x|^d} + \frac{y(y - \tilde{x})}{d|B_1||x|^{d-2}|y - \tilde{x}|^d} \\ &= \frac{1}{d|B_1||y - x|^d} (-y(y - x) + y(y - \tilde{x})|x|^2) \\ &= \frac{1}{d|B_1||y - x|^d} [-|y|^2 + xy + |y|^2|x|^2 - xy] \\ &= \frac{-1 + |x|^2}{d|B_1||y - x|^d} \end{aligned}$$

as $y \in \partial B$. ■

Theorem 5.12 (Poisson Formula for a Ball) Let $B = B(0, 1)$, $g \in C(\partial B)$. Define for all $x \in B$:

$$u(x) = \int_{\partial B} K(x, y) g(y) dS(y),$$

$K(x, y) = -\frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) = \frac{1 - |x|^2}{d|B_1||y - x|^d}$ for all $x \in B$, for all $y \in \partial B$. Then $u \in C^\infty(B)$, $\Delta u = 0$ and for all $x_0 \in \partial B$ we have $\lim_{x \rightarrow x_0, x \in B} u(x) = g(x_0)$. This holds for all $d \geq 2$.

Proof. We need to check:

1. For all $y \in \partial B$, $x \mapsto K(x, y)$ is harmonic in B .
2. $\int_{\partial B} K(x, y) dS(y) = 1$ for all $x \in B$ (exercise)

Now for all $x \in B$, for all $y \in \partial B$:

$$\begin{aligned}
K(x, y) &= \frac{1 - |x|^2}{d|B_1||y - x|^d} \\
\partial_{x_i} K(x, y) &= \frac{-2x_i}{d|B_1|} \frac{1}{|x - y|^d} - \frac{1 - |x|^2}{|B_1|} \frac{x_i - y_i}{|x - y|^{d+2}} \\
\partial_{x_i}^2 K(x, y) &= -\frac{2}{d|B_1|} \frac{1}{|x - y|^d} + \frac{2x_i}{|B_1|} \frac{x_i - y_i}{|x - y|^{d+2}} + \frac{2x_i}{|B_1|} \frac{x_i - y_i}{|x - y|^{d+2}} \\
&\quad - \frac{1 - |x|^2}{|B_1|} \frac{1}{|x - y|^{d+2}} + \frac{1 + |x|^2}{|B_1|} (d + 2) \frac{(x_i - y_i)^2}{|x - y|^{d+4}} \\
\Delta_x K &= \sum_{i=1}^d \partial_{x_i}^2 K = -\frac{2}{|B_1|} \frac{1}{|x - y|^d} + \frac{4x(x - y)}{|B_1||x - y|^{d+2}} \\
&\quad - \frac{d(1 - |x|^2)}{|B_1|} \frac{1}{|x - y|^{d+2}} + (d + 2) \frac{1 - |x|^2}{|B_1|} \frac{1}{|x - y|^{d+2}} \\
&= \frac{2}{|B_1||x - y|^{d+2}} [-|x|^2 + 2xy - |y|^2 + 2|x|^2 - 2xy + 1 - |x|^2] \\
&= \frac{2}{|B_1||x - y|^{d+2}} [-|x|^2 + 2xy - |y|^2 + 2|x|^2 - 2xy + 1 - |x|^2]
\end{aligned}$$

$1 - |y|^2 = 0$ as $y \in \partial B$. Thus $\Delta_x K(x, y) = 0$, for all $x \in B$, for all $y \in \partial B$.

$$|u(x)| = \left| \int_{\partial B} K(x, y) g(y) dS(y) \right| \leq \|g\|_{L^\infty(\partial B)}$$

$$\int_{\partial B} K(x, y) dS(y) = \|g\|_{L^\infty},$$

$$\Delta_x u(x) = \int_{\partial B} \underbrace{\Delta_x K(x, y)}_{=0} g(y) dS(y) = 0$$

Take $x \in B$, $x \rightarrow x_0 \in \partial B$.

$$\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial B} K(x, y) (g(y) - g(x_0)) dS(y) \right| \\
&\leq \int_{A_1} + \int_{A_2} K(x, y) |g(y) - g(x_0)| dS(y),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \{y \in \partial B \mid |y - x_0| \leq |x - x_0|^\alpha\} \\
A_2 &= \{y \in \partial B \mid |y - x_0| > |x - x_0|^\alpha\}
\end{aligned}$$

On A_1 we have:

$$\int_{A_1} \dots \leq \sup_{\substack{|z - x_0| \leq |x - x_0|^\alpha \\ z \in \partial B}} \int_{\partial B} K(x, y) dS(y) \xrightarrow{x \rightarrow x_0} 0$$

since $G \in C(\partial B)$. On A_2 :

$$\begin{aligned}
&|y - x_0| > |x - x_0|^\alpha \\
\Rightarrow |y - x| &\geq |y - x_0| - |x - x_0| \geq |x - x_0|^\alpha - |x - x_0| \geq \frac{1}{2} |x - x_0|^\alpha
\end{aligned}$$

if $\alpha < 1$ and $|x - x_0|$ small. So we get

$$K(x, y) = \frac{1 - |x|^2}{d|B_1||x - y|^d} \leq C \frac{1 - |x|^2}{|x - x_0|^{d\alpha}} \leq C |x - x_0|^{1-d\alpha}$$

Thus

$$\int_{A_2} K(x, y) |g(y) - g(x_0)| dS(y) \leq C \|g\|_{L^\infty} |x - x_0|^{1-d\alpha} \xrightarrow{x \rightarrow x_0} 0$$

if $1 - d\alpha > 0 \Leftrightarrow \alpha < \frac{1}{d}$. ■

Exercise 5.13 (E 7.2) Define $\mathbb{R}_+^d = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid x_d > 0\}$. Let $K(x, y) = \frac{2x_d}{d|B_1||x-y|^d}$ for all $x \in \mathbb{R}_+^d, y \in \partial\mathbb{R}_+^d = \{(y', 0) \mid y' \in \mathbb{R}^{d-1}\} \simeq \mathbb{R}^{d-1}$. Prove

$$\int_{\partial\mathbb{R}_+^d} K(x, y) dS(y) = 1 \quad \forall x \in \mathbb{R}_+^d$$

Solution. Denote $x = (x', x_d), y = (y', 0), x', y' \in \mathbb{R}^{d-1}, x_d > 0$.

$$\int_{\partial\mathbb{R}_+^d} K(x, y) dS(y) = \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|(|x' - y'|^2 + x_d^2)^{\frac{d}{2}}} dy' = \dots$$

as $|x - y| = |(x' - y', x_d)| = \sqrt{|x' - y'|^2 + x_d^2}$.

$$\begin{aligned} (y' - x' \mapsto y') \quad \dots &= \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|(|y'|^2 + x_d^2)^{\frac{d}{2}}} dy' \\ (y' = x_d z) &= \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|(x_d^2(|z|^2 + 1))^{\frac{d}{2}}} (x_d^{d-1}) dz \\ &= \int_{\mathbb{R}^{d-1}} \frac{2}{d|B_1|(|z|^2 + 1)^{\frac{d}{2}}} dz \\ &= \int_0^\infty \frac{2\omega_{d-1}}{d|B_1|} \frac{1}{(r^2 + 1)^{\frac{d}{2}}} r^{d-2} dr \\ &= \frac{2\omega_{d-1}}{\omega_d} \int_0^\infty \frac{1}{(r^2 + 1)^{\frac{d}{2}}} r^{d-2} dr \end{aligned}$$

Set $d = 2$: $\omega_1 = 1, |\omega_2| = 2\pi$

$$\frac{2}{\pi} \int_0^\infty \frac{1}{r^2 + 1} dr = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{(\tan t)^2 + 1} [(\tan t)^2 + 1] dt = 1$$

we we set $r = \tan t, t \in (0, \frac{\pi}{2}), \frac{dr}{dt} = (\tan t)' = 1 + (\tan t)^2$ ■

For $d = 3$:

$$\frac{2 \cdot 2\pi}{4\pi} \int_0^\infty \frac{1}{(r^2 + 1)^{\frac{3}{2}}} r dr = \int_0^\infty \frac{d}{dr} \left[\frac{-1}{(r^2 + 1)^{\frac{1}{2}}} \right] dr = \frac{-1}{(r^2 + 1)^{\frac{1}{2}}} \Big|_0^\infty = 1$$

Exercise 5.14 (7.3) Let $g \in C(\partial\mathbb{R}_+^d) \cap L^\infty(\partial\mathbb{R}_+^d)$ ($\partial\mathbb{R}_+^d \simeq \mathbb{R}^{d-1}$).

$$u(x) = \int_{\partial\mathbb{R}_+^d} K(x, y) g(y) dS(y) \quad K(x, y) = \frac{2x_d}{d|B_1||x - y|^d}, x \in \mathbb{R}_+^d$$

Prove that if $g(y) = |y|$, if $|y| \leq 1$, then $|\nabla u|$ is unbounded in $B(0, r) \cap \mathbb{R}_+^d$ for all $r > 0$.

Solution.

$$\begin{aligned}
\partial_{x_d} u(x) &= \int_{\partial \mathbb{R}_+^d} \partial_{x_d} K(x, y) g(y) dy \quad \forall x \in \mathbb{R}_+^d \\
&= \frac{2}{d|B_1|} \int_{\partial \mathbb{R}_+^d} \left[\frac{1}{|x-y|^d} - \frac{dx_d^2}{|x-y|^{d+2}} \right] g(y) dy \\
&= \frac{2}{d|B_1|} \int_{\partial \mathbb{R}_+^d} \frac{1}{|x-y|^{d+2}} [|x-y|^2 - dx_d^2] g(y) dy \\
&= \frac{2}{d|B_1|} \int_{\partial \mathbb{R}_+^d} \frac{1}{(|x' - y'| + x_d^2)^{\frac{d+2}{2}}} [|y'|^2 - (d-1)x_d^2] g(y) dy
\end{aligned}$$

Assume that $\partial_d u$ is bounded in $B(0, r) \cap \mathbb{R}_+^d$. Then:

$$|u(0, x_d) - \underbrace{u(0, 0)}_{g(0)=0}| \leq C|x_d|$$

if x_d small. Consider:

$$\begin{aligned}
\limsup_{x_d \rightarrow 0^+} \frac{u(0, x_d)}{x_d} &= \limsup_{x_d \rightarrow 0^+} c \int_{\mathbb{R}^{d-1}} \frac{1}{(|y'|^2 + x_d^2)^{\frac{d}{2}}} g(y) dy' \\
&\geq \int_{\mathbb{R}^{d-1}} \frac{1}{|y'|^d} g(y) dy = \int_{|y'| \leq 1} + \int_{|y'| > 1} \\
&\text{to } \int_{\mathbb{R}^{d-1}} \frac{1}{|y'|^{d-1}} dy' = \infty
\end{aligned}$$

■

Exercise 5.15 (Bonus 7) Recall the Poisson kernel on a ball $B(0, r) \subseteq \mathbb{R}^d$:

$$K(x, y) = \frac{r^2 - |x|^2}{d|B_1|r} \frac{1}{|x - y|^d}$$

for all $x \in B(0, r)$, $y \in \partial B(0, r)$. Prove:

$$\int_{\partial B(0, r)} K(x, y) dS(y) = 1$$

for all $x \in B(0, r)$. (It suffices if you can prove $d = 2$ and $d = 3$)

5.3 Energy Method

Consider $u \in C^2(\Omega)$ for $\Omega \subseteq \mathbb{R}^d$ open, bounded and with C^1 boundary and

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Take $\phi \in C_c^\infty(\Omega)$, then by integration by parts:

$$0 = \int_{\Omega} (-\Delta u - f) \phi = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi$$

Key observation: This is the *derivative* of the energy functional

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u$$

If u is a minimizer of E , then it solves the equation $-\Delta u = f$ in Ω . The boundary condition $u = g$ does not appear on E , but this is encoded in the set of *admissible functions*. (The set of candidates of solutions). For the classical solutions, we have

Theorem 5.16 (Dirichlet's principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1 -boundary. Let $f \in C(\bar{\Omega})$ and $g \in C(\partial\Omega)$. Then the following statements are equivalent:

1. $u \in C^2(\bar{\Omega})$ solves
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$
2. u is a minimizer of the variational problem $E = \inf_{v \in A} E(v)$, where

$$E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v,$$

$$A = \{v \in C^2(\bar{\Omega}) \mid v = g \text{ on } \partial\Omega\}.$$

Moreover there is at most a solution / minimizer (uniqueness).

Proof. The result holds even for complex-valued functions. Let us write the proof for real-valued functions.

1. \Rightarrow 2.: Let $u \in C^2(\bar{\Omega})$ be a solution of
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$
. Then we prove $E(u) \leq E(v)$ for all $v \in A$. If $v \in A$, then $u - v = 0$ on $\partial\Omega$. Using this and $-\Delta u = f$ in Ω , we have:

$$\begin{aligned} 0 &= \int_{\Omega} (-\Delta u - f) \cdot (u - v) dy \\ (\text{Part. Int.}) &= \int_{\Omega} \nabla u (\nabla u - \nabla v) dy - \int_{\Omega} f(u - v) dy \\ &= \left[\frac{1}{2} \int_{\Omega} |\nabla u|^2 dy - \int_{\Omega} f u dy \right] - \left[\frac{1}{2} \int_{\Omega} |\nabla v|^2 dy - \int_{\Omega} f v dy \right] \\ &\quad + \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \\ &= E(u) - E(v) + \underbrace{\frac{1}{2} \int_{\Omega} |\nabla u - \nabla v|^2}_{\geq 0} \end{aligned}$$

$E(u) \leq E(v)$, so u is a minimizer of $\inf_{v \in A} E(v)$. Moreover u is the unique minimizer on A . Since $E(u) = E(v)$ we have $\int_{\Omega} |\nabla(u - v)|^2 = 0$, so $u - v = \text{const.}$, so $u - v = 0$ in $\bar{\Omega}$.

2. \Rightarrow 1.: Assume that u is a minimizer of $\inf_{v \in A} E(v)$. Then $E(u) \leq E(v)$ for all $v \in A$. Take $\phi \in C_c^\infty(\Omega)$, then $u + t\phi \in A$ for all $t \in \mathbb{R}$.

$$\Rightarrow E(u) \leq E(u + t\phi) \text{ for all } t \in \mathbb{R}$$

$$\Rightarrow t \mapsto E(u + t\phi) \text{ has a minimizer at } t = 0$$

$$\Rightarrow 0 = \frac{d}{dt} E(u + t\phi)|_{t=0}$$

$$= \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} |\nabla u + t \nabla \phi|^2 - \int_{\Omega} f(u + t\phi) \right) \Big|_{t=0}$$

$$= \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} |\nabla u|^2 + t^2 |\nabla \phi|^2 + 2t \nabla u \nabla \phi - \int_{\Omega} f(u + t\phi) \right) \Big|_{t=0}$$

$$\int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi = \int_{\Omega} (-\Delta u - f) \phi$$

for all $\phi \in C_c^\infty(\Omega)$. So $-\Delta u - f = 0$ in Ω and $u = g$ since $u \in A$.

Direct method of calculus of variations. Think $f : \mathbb{R} \rightarrow \mathbb{R}$, $f \in C(\mathbb{R})$, $f(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. There is a $x_0 \in \mathbb{R}$ s.t. $f(x_0) = \inf_{x \in \mathbb{R}} f(x)$.

Step 1: $E = \inf_{x \in \mathbb{R}} f(x) > -\infty$

Step 2: Take a minimizing sequence $\{x_n\} \subseteq \mathbb{R}$, $f(x_n) \rightarrow E$. Up to a subsequence $x_n \rightarrow x_0$ in \mathbb{R} (compactness)

Step 3: Lower semicontinuity $E = \liminf_{n \rightarrow \infty} f(x_n) \geq f(x_0)$

If we apply the direct method to $\inf_{v \in A} E(v)$,

$$E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v,$$

$$A = \{v \in C^2(\bar{\Omega}), v = g \text{ on } \partial\Omega\}$$

Step 1: Easy $E = \inf_{v \in A} E(v) > -\infty$

Step 2: There is a minimizing sequence $\{v_n\} \subseteq A$ s.t. $E(v_n) \rightarrow E$. We don't know if there is a subsequence of $\{v_n\}$ that converges to $u \in A$. The lack of compactness is a serious problem! We need to find the right set A ! Consider again

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Consider the simple case $g = 0$. $\Delta u = f$ in $\Omega \Leftrightarrow \nabla u \cdot \nabla \phi \dots$ The right set A should be $A = \{v \mid \int_{\Omega} |\nabla v|^2 < \infty, v = 0 \text{ on } \partial\Omega\}$. Rigorously we take $W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,2}(\Omega)}$ (Notation: $H_0^1 = W_0^{1,2}$, $H^1 = W^{1,2}$) Recall that $W^{1,p}$ is a Banach space with norm $\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}$. We know that $C_c^\infty(\Omega)$ is dense in $W_{loc}^{1,p}(\Omega)$, i.e. for all $u \in W_{loc}^{1,p}(\Omega)$ there is $\|u_n\| \subseteq C_c^\infty$ s.t. $u_n \rightarrow u$ in $W^{1,p}(K)$ for all $K \subseteq \Omega$ compact. However in general $C_c^\infty(\Omega)$ is not dense in $W^{1,p}(\Omega)$, i.e. $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,p}(\Omega)} \subsetneq W^{1,p}(\Omega)$. Clearly $W_0^{1,p}$ is a closed subspace of $W^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)$ is a Banach space with $\|\cdot\|_{W^{1,p}(\Omega)}$. Why does $W_0^{1,p}(\Omega)$ encode the 0-boundary condition? Note that by definition for all $u \in W_0^{1,p}(\Omega)$ there is a sequence $\{u_n\} \subseteq C_c^\infty(\Omega)$, $u_n \rightarrow u$ in $W^{1,p}(\Omega)$ up to a subsequence $u_n(x) \rightarrow u(x)$ for almost every $x \in \Omega$. Note $u_n|_{\partial\Omega} = 0 \rightarrow u|_{\partial\Omega} = 0$ since $\partial\Omega$ must be of 0-measure. ■

Theorem 5.17 (Characterization for $W_0^{1,p}$) Let Ω be open, bounded with C^1 -boundary. Let $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. Then the following statements are equivalent:

a) $u = 0$ on $\partial\Omega$

b) $u \in W_0^{1,p}(\Omega)$

(Later we will remove the condition $C(\bar{\Omega})$ by introducing the *Trace operator*.)

Remark 5.18 If $d = 1$, it holds that $W^{1,p} \subseteq C(\bar{\Omega})$. Then the theorem gives a full characterization for $W_0^{1,p}$, but if $d \geq 2$, then in general $W^{1,p} \not\subseteq C(\Omega)$. (later)

Proof of theorem 5.17.

a) \Rightarrow b):

Lemma 5.19 If $u \in W^{1,p}(\Omega)$ and $\text{supp } u \subseteq \Omega$, then $u \in W_0^{1,p}(\Omega)$.

Proof. Since $K := \text{supp } u$ is a compact subset in Ω , we can find a function $\chi \in C_c^\infty(\Omega)$, $\chi = 1$ on K . Moreover since $u \in W^{1,p}(\Omega)$, there is a sequence $\{u_n\} \subseteq C_c^\infty(\Omega)$ s.t. $u_n \rightarrow u$ in $W_{loc}^{1,p}(\Omega)$. We claim that $\chi u_n \rightarrow \chi u$ in $W_{loc}^{1,p}(\Omega)$. (exercise, $\nabla(\chi u) = \nabla \chi u + \chi \nabla u$). This implies $\chi u_n \rightarrow u$ in $W^{1,p}(\text{supp } \chi)$, thus $\chi u_n \rightarrow u$ in $W^{1,p}(\Omega)$, so $u \in W_0^{1,p}(\Omega)$. ■

Assume $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$. Take $G \in C^1(\mathbb{R})$ s.t. $|G(t)| \leq t$ for all t , $G(t) = t$ if $t \geq 2$ and $G(t) = 0$ if $t \leq 1$. Then let

$$\begin{aligned} u_n(x) &:= \frac{1}{n} G(nu(x)) \in W^{1,p}(\Omega) \\ \stackrel{(\text{Chain-rule})}{\Rightarrow} \nabla u_n(x) &= \frac{1}{n} G'(nu(x)) n \nabla u(x) = G'(nu(x)) \nabla u(x) \end{aligned}$$

Moreover, u_n is compactly supported in Ω , so $u_n \in W_0^{1,p}(\Omega)$ by the lemma and $u_n \rightarrow u$ in $W^{1,p}(\Omega)$, so $u \in W_0^{1,p}(\Omega)$ since $W_0^{1,p}$ is a closed space. Recall that $u \in C(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$. Thus for all $\epsilon > 0$ there is a compact $K_\epsilon \subseteq \Omega$ s.t. $\sup_{x \in \Omega \setminus K_\epsilon} |u(x)| \leq \epsilon$. For any given $n \in \mathbb{N}$, $u_n(x) \neq 0$, so $G(nu(x)) \neq 0$. This implies $n|u(x)| > 1$, hence $|u(x)| > \frac{1}{n}$. Thus $u_n(x) = 0$ for all x such that $|u(x)| \leq \frac{1}{n}$, so $\text{supp } u_n \subseteq K_{\frac{1}{n}}$ compact in Ω . Next, let us check $u_n \rightarrow u$ in $W^{1,p}(\Omega)$.

$$\int_{\Omega} |u_n(x) - u(x)|^p dx \rightarrow 0$$

since $u_n(x) = \frac{1}{n} G(nu(x)) \xrightarrow{n \rightarrow \infty} u(x)$ for all $x \in \Omega$ and $|u_n(x)| \leq \frac{1}{n} |G(nu(x))| \leq \frac{1}{n} |nu(x)| \leq |u(x)| \in L^p(\Omega)$.

$$\int_{\Omega} |\nabla u_n(x) - \nabla u(x)|^p dx = \int_{\Omega} |G'(nu(x)) - 1|^p |\nabla u(x)|^p dx \rightarrow 0$$

as $|G'(v(x)) - 1| \rightarrow 0$ for all x s.t. $u(x) \neq 0$ and $\nabla u(x) = 0$ on $\{x \mid u(x) = 0\}$. (exercise)

(b) \Rightarrow (a): Let $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $u \in W_0^{1,p}(\Omega)$. Then we prove $u = 0$ on $\partial\Omega$. Lets regard the case $\Omega = Q_+ = \{(x', x_d) \mid \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, 0 < x_d < 1\}$. We prove that if $u \in W_0^{1,p}(Q_+) \cap C(\bar{Q}_+)$, then $u = 0$ on $Q_0 = \{(x', 0) \mid x' \in \mathbb{R}^{d-1}, |x'| < 1\}$. Since $u \in W_0^{1,p}(Q_+)$ there is $\{u_n\} \subseteq C_c^\infty(Q_+)$ s.t. $u_n \rightarrow u$ in $W^{1,p}(Q_+)$ for all $x = (x', x_d) \in Q_+$, then:

$$u_n(x', x_d) = \underbrace{u_n(x', 0)}_{=0} + \int_0^{x_d} \partial_d u_n(x', t) dt$$

Hence

$$|u_n(x', x_d)| \leq \int_0^{x_d} |\partial_d u_n(x', t)| dt$$

This implies:

$$\begin{aligned} & \int_{0 < x_d < \epsilon} \int_{|x'| \leq 1} |u_n(x', x_d)| dx' dx_d \\ & \leq \int_{0 < x_d < \epsilon} \int_{|x'| < 1} \left(\int_0^{x_d} |\partial_d u_n(x', t)| dt \right) dx' dx_d \\ & \leq \epsilon \int_{|x'| < 1} \int_0^\epsilon |\partial_d u_n(x', t)| dx' dt \end{aligned}$$

$$\Rightarrow \frac{1}{\epsilon} \int_0^\epsilon \int_{|x'| \leq 1} |u_n(x', x_d)| dx' dx_d \leq \int_0^\epsilon \int_{|x'| < 1} |\partial_d u_n(x', x_d)| dx' dx_d$$

for all $n \in \mathbb{N}$, $\epsilon > 0$. Take now $n \rightarrow \infty$, use $u_n \rightarrow u$ in $W^{1,p}(\Omega)$. Then:

$$\frac{1}{\epsilon} \int_0^\epsilon \int_{|x'| \leq 1} |u(x', x_d)| dx' dx_d \leq \int_0^\epsilon \int_{|x'| < 1} |\partial_d u(x', x_d)| dx' dy$$

for all $\epsilon > 0$. Take $\epsilon \rightarrow 0$:

$$\int_{|x'| \leq 1} |u(x', 0)| dx' \leq 0$$

here we use $u \in C(\bar{\Omega})$ for the left side and Dominated Convergence for the right side. Thus $u(x', 0) = 0$ for all $|x'| \leq 1$, i.e. $u = 0$ on $\partial\Omega$. Let's regard the general case: Let Ω be open, bounded and with C^1 -boundary. Let's define *local charts* By definition for all $x \in \partial\Omega$, there is a U_x open, such there is a bijective map $h : U_x \rightarrow Q$, and h, h^{-1} are C^1 . Then clearly $\partial\Omega \subseteq \bigcup_{x \in \partial\Omega} U_x$. Since $\partial\Omega$ is compact, there is a finite subcover $\{U_i\}_{i=1}^N$ s.t. $\partial\Omega \subseteq \bigcup_{i=1}^N U_i$. We can find U_0 open s.t. $\bar{U}_0 \subseteq \Omega$ and $\Omega \subseteq \bigcup_{i=0}^N U_i$.

Lemma 5.20 There is a sequence $\{\chi_i\}_{i=0}^N \subseteq C^\infty(\mathbb{R}^d)$ s.t.

1. $\chi_i \geq 0$, $\sum_{i=0}^N \chi_i = 1$ in \mathbb{R}^d ($\{\chi_i\}$ is a partition of unity)
2. For all $i = 1, \dots, N$, $\text{supp } \chi_i$ is in U_i , i.e. $\chi_i \in C_c^\infty(U_i)$.
3. $i = 0$, $\text{supp } \chi_0 \subseteq \mathbb{R}^d \setminus \partial\Omega$ and $\chi_0|_\Omega \in C_c^\infty(\Omega)$. (exercise)

Given $u \in W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$. Then $u = \sum_{i=0}^N \chi_i u$, where $\chi_i \geq 0$, $\chi_0 \in C_c^\infty(\Omega)$, $\chi_i \in C_c^\infty(U_i)$. Since $\chi_0 u$ is supported in a compact set inside Ω , $\chi_0 u = 0$ on $\partial\Omega$. It remains to show that for all $i = 1, \dots, N$, $\chi_i u = 0$ on $U_i \cap \partial\Omega$. Then $\chi_i u(h^{-1}x) \in W_0^{1,p}(Q) \cap C(\bar{Q})$. This implies $\chi_i u(h^{-1}x) = 0$ on Q_0 , so $\chi_i u(x) = 0$ on $U_i \cap \partial\Omega$. Why $W_0^{1,p}(U_i \cap \Omega) \rightarrow W_0^{1,p}(Q_+)$. If $v \in W_0^{1,p}(U_i \cap \Omega)$, then $v_n \rightarrow v$, $v_n \in C_c^\infty$. $v_n \circ h^{-1} \rightarrow v \circ h^{-1} \Rightarrow v \circ h^{-1} \in W_0^{1,p}(Q_+)$

■

Exercise 5.21 (E 8.1) Let $u \in W_{loc}^{1,1}(\mathbb{R}^d)$. Let $B = u^{-1}(\{0\})$. Prove that $\nabla u(x) = 0$ for a.e. $x \in B$.

Solution. We have already seen that if $f, g \in W_{loc}^{1,1}(\mathbb{R}^d)$, then $\max(f, g) \in W_{loc}^{1,1}$. This implies that if $u = u^+ - u^- \in W_{loc}^{1,1}$, then $u^+, u^- \in W_{loc}^{1,1}$ since $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. We have that $\nabla u = \nabla u^+ - \nabla u^-$. Claim:

$$\nabla u^+ = \begin{cases} 0 & u(x) \leq 0 \\ \nabla u & u(x) > 0 \end{cases} \quad \nabla u^- = \begin{cases} 0 & u(x) \geq 0 \\ \nabla u & u(x) < 0 \end{cases}$$

$$\begin{aligned} \int_{\mathbb{R}^d} (\partial_i u^+) \phi &= - \int_{\mathbb{R}^d} u^+ \partial_i \phi = - \int_{\{u(x) \leq 0\}} 0 \partial_i \phi - \int_{\{u(x) > 0\}} u \partial_i \phi \\ &= \int_{\{u(x) \leq 0\}} 0 \phi + \int_{\{u(x) > 0\}} \partial_i u \phi \end{aligned}$$

Alternative way: We showed for $f \in W^{1,p}(\mathbb{R}^d)$, that

$$\nabla|f|(x) = \begin{cases} (\nabla f)(x) & f(x) > 0 \\ -(\nabla f)(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

$u_+ = \frac{1}{2}(u + |u|)$. Hence $\nabla u_+ = \frac{1}{2}(\nabla u + \nabla|u|)$. Remark: If $A \subseteq \mathbb{R}$ has measure zero, then $\nabla u 1_{\{u(x) \in A\}} = 0$ a.e. (Th. 6.19 Lieb-Loss Analysis) ■

Exercise 5.22 (E 8.2) Let $\Omega, U \subseteq \mathbb{R}^d$ be open, $U \cap \Omega \neq \emptyset$, $u \in W_0^{1,p}(\Omega)$, $1 \leq p < \infty$, $\chi \in C_c^\infty(U)$. Prove: $\chi u \in W_0^{1,p}(\Omega \cap U)$ Hint: Recall $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}}}$

Solution. By definition there is a sequence $(u_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\Omega)$ s.t. $u_n \xrightarrow[n \rightarrow \infty]{\|\cdot\|_{W^{1,p}}} u$, i.e.

$$\|u_n - u\|_p + \|\nabla u_n - \nabla u\|_p \xrightarrow{n \rightarrow \infty} 0.$$

Define $f_n : \mathbb{R}^d \rightarrow \mathbb{C}$, $f_n(x) := u_n(x)\chi(x)$. Note $f_n \in C_c^\infty(\Omega \cap U)$ for all $n \in \mathbb{N}$. Claim: $(f_n)_{n \in \mathbb{N}}$ is Cauchy with respect to $\|\cdot\|_{W^{1,p}}$. Proof:

$$\|f_n - f_m\|_p = \|\chi(u_n - u_m)\|_p \leq \|\chi\|_\infty \underbrace{\|u_n - u_m\|_p}_{\xrightarrow[n, m \rightarrow \infty]{} 0} \xrightarrow{n, m \rightarrow \infty} 0$$

$$\nabla f_n = \nabla(\chi u_n) = (\nabla \chi)u_n + \chi \nabla u_n$$

$$\begin{aligned} \|\nabla f_n - \nabla f_m\|_p &\leq \|\nabla \chi(u_n - u_m)\|_p + \|\chi(\nabla u_n - \nabla u_m)\|_p \\ &\leq \underbrace{\|\nabla \chi\|_\infty \|u_n - u_m\|_p}_{\xrightarrow[n, m \rightarrow \infty]{} 0} + \underbrace{\|\chi\|_\infty}_{< \infty} \underbrace{\|\nabla u_n - \nabla u_m\|_p}_{\xrightarrow[n, m \rightarrow \infty]{} 0} \xrightarrow{n, m \rightarrow \infty} 0 \end{aligned}$$

Thus, there is a $f \in W_0^{1,p}(\Omega \cap U)$ s.t. $\|f_n - f\|_{W^{1,p}} \xrightarrow{n \rightarrow \infty} 0$. We know:

$$\begin{aligned} \|f_n - \chi u\|_{L^p} &= \|\chi u_n - \chi u\|_p \\ &\leq \|\chi\|_\infty \underbrace{\|u_n - u\|_p}_{\rightarrow 0} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Since limits in L^p are unique, we get $\chi u = f \in W_0^{1,p}(\Omega \cap U)$. ■

Exercise 5.23 (E 8.3) Let $\Omega, U \subseteq \mathbb{R}^d$ open and bounded, $h : \bar{U} \rightarrow \bar{\Omega}$ C^1 -diffeomorphisms, $u \in W_0^{1,p}(\Omega)$, $1 \leq p < \infty$. Prove $(x \mapsto u(h(x))) \in W_0^{1,p}(U)$.

Solution. Since $u \in W_0^{1,p}(\Omega)$ there is a sequence $(u_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\Omega)$ s.t.

$$\|u - u_n\|_p + \|\nabla u - \nabla u_n\|_p \xrightarrow{n \rightarrow \infty} 0$$

Define for all $n \in \mathbb{N}$ $f_n : U \rightarrow \mathbb{C}$, $f_n(x) = u_n(h(x))$. Note $f_n \in C_c^1(U)$. Claim 1: $(f_n)_{n \in \mathbb{N}}$ is Cauchy wrt. $\|\cdot\|_{W^{1,p}}$.

$$\begin{aligned} \|f_n - f_m\|_p^p &= \int_U |u_n(h(x)) - u_m(h(x))|^p dx \\ &= \int_\Omega |u_n(y) - u_m(y)|^p dy \underbrace{|\det(Dh^{-1})(y)|}_{\leq C < \infty} \xrightarrow{n, m \rightarrow \infty} 0 \end{aligned}$$

$$(\nabla f_n)(x) = \nabla(u_n(h(x))) = (\nabla u_n)(h(x))(Dh)(x)$$

$$\begin{aligned}
\|\nabla f_n - \nabla f_m\|_p^p &= \int_U |[(\nabla u_n)(h(x)) - (\nabla u_m)(h(x))](Dh(x))|^p dx \\
&\leq C \int_U |(\nabla u_n)(h(x)) - (\nabla u_m)(h(x))|^p dx \\
&= C \int_\Omega |(\nabla u_n)(y) - (\nabla u_m)(y)|^p \underbrace{|\det Dh^{-1}(y)|}_{\leq \tilde{C}} dy \xrightarrow{n,m \rightarrow 0} 0
\end{aligned}$$

Claim 2: $\|f_n - u \circ h\|_p \xrightarrow{n \rightarrow \infty} 0$.

$$\begin{aligned}
\|f_n - u \circ h\|_p &= \int_U |u_n(h(x)) - u(h(x))|^p dx \\
&= \int_\Omega |u_n(y) - u(y)|^p \underbrace{|\det Dh^{-1}(y)|}_{\leq C} dy \xrightarrow{n \rightarrow \infty} 0
\end{aligned}$$

Conclusion: Since $(f_n)_{n \in \mathbb{N}} \subseteq C_c^1(U)$ is Cauchy with respect to $\|\cdot\|_{W^{1,p}}$, there is a $f \in W_0^{1,p}(U)$ s.t. $f_n \xrightarrow{n \rightarrow \infty} f$. Since limits in L^p are unique by claim 2 we get $u \circ h = f \in W_0^{1,p}(U)$. $\|\cdot\|_{W^{1,p}}$ ■

Exercise 5.24 (E 8.4) Let $\Gamma \subseteq \mathbb{R}^d$ be compact, $\{U_i\}_{i=1}^N$ open s.t. $\Gamma \subseteq \bigcup_{i=1}^N U_i$. Prove: There exists $\{\chi_i\}_{i=1}^N \subseteq C^\infty(\mathbb{R}^d)$ s.t.

1. $\chi_i \geq 0$ for all i , $\sum_{i=1}^N \chi_i = 1$
2. $\text{supp}(\chi_i) \subseteq U_i$ for all $i \in \{1, \dots, N\}$
3. $\text{supp}(\chi_0) \subseteq \mathbb{R}^d \setminus \Gamma$

Solution. WLOG assume that $U_i \neq \emptyset$ for all i . If $\Gamma \neq \emptyset$, then $\chi_0 = 1$ does the job. Now suppose $\Gamma \neq \emptyset$. Let $\psi \in C_c^\infty(B_1(0))$, $\psi \geq 0$, $\int \psi = 1$, $\psi|_{B_{\frac{1}{2}}(0)} > 0$ and for $\epsilon > 0$ let $\psi_\epsilon(x) = \frac{1}{\epsilon^d} \psi\left(\frac{x}{\epsilon}\right)$, so $\int \psi_\epsilon = 1$. Define

$$\tilde{d} := \sup\{\tilde{d} > 0 \mid \forall x \in \Gamma \exists i \in \{1, \dots, N\} \text{ s.t. } \text{dist}(x, U_i^c) \geq \tilde{d}\}$$

Claim 1: $\tilde{d} > 0$ Suppose this was not true. Then there is a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \Gamma$ s.t. for all $i \in \{1, \dots, N\}$,

$$\text{dist}(x_n, U_i^c) < \frac{1}{n}$$

Since Γ is compact, there is a subsequence, which we call x_n again, s.t. $x_n \xrightarrow{n \rightarrow \infty} \bar{x}$ for some $\bar{x} \in \Gamma$. By $\Gamma \subseteq \bigcup_{i=1}^N U_i$ there is a $\bar{i} \in \{1, \dots, N\}$ s.t. $B_{\epsilon_{\bar{x}}}(\bar{x}) \subseteq U_{\bar{i}}$. Define $d := \min\{\tilde{d}, 1\} > 0$. For all $\epsilon > 0$, for all $A \subseteq \mathbb{R}^d$: $(A)_\epsilon := \{x \in A \mid \text{dist}(x, A^c) \geq \epsilon\}$. for every $i \in \{1, \dots, N\}$ define $\phi_i : U_i \rightarrow [0, \infty)$ by

$$\phi_i(x) := \mathbb{1}_{(U_i \cap B_R(0))_{\frac{d}{4}}} \star \phi_{\frac{d}{4}}$$

Note $\phi_i \in C_c^\infty(U_i)$ and $(U_i \cap B_R(0))_{\frac{d}{4}} \subseteq (\text{supp}(\phi_i))^0$. Define $\phi_0 : \mathbb{R}^d \setminus \Gamma \rightarrow [0, \infty)$ by $\phi_0(x) = \mathbb{1}_{(\mathbb{R}^d \setminus \Gamma)_{\frac{d}{4}}} \star \psi_{\frac{d}{4}}$. Again, $\phi_0 \in C^\infty(\mathbb{R}^d \setminus \Gamma)$, $\text{supp}(\phi_0)^0 \supseteq (\mathbb{R}^d \setminus \Gamma)_{\frac{d}{4}}$, $\text{supp}(\phi_0) \subseteq \mathbb{R}^d \setminus \Gamma$. Claim 2: For all $x \in \mathbb{R}^d$ there is a $i \in \{0, 1, \dots, N\}$: $\phi_i(x) > 0$. Proof: By construction, we know for $i \in \{1, \dots, N\}$ that ϕ_i is > 0 on $(U_i \cap B_R(0))_{\frac{d}{4}}$. Moreover $\phi_0 > 0$ on $(\mathbb{R}^d \setminus \Gamma)_{\frac{d}{4}}$. thus, we are done if we can show that $\bigcup_{i=1}^N (U_i \cap B_R(0))_{\frac{d}{4}} \cup$

$(\mathbb{R}^d \setminus \Gamma)_{\frac{d}{4}} = \mathbb{R}^d$. Suppose there is a $x \in \mathbb{R}^d \setminus A$. Then $\text{dist}(x, \Gamma) < \frac{d}{4}$. Since $\Gamma \subseteq B_{\frac{R}{2}}(0)$ and $R > 2$ and $d \leq 1$.

$$|x - 0| \leq \text{dist}(x, \Gamma) + \frac{R}{2} < \frac{d}{4} + \frac{R}{2} = R - \frac{d}{4} - \frac{R}{2} + \frac{d}{2} < R - \frac{d}{4} - \frac{2}{2} + \frac{1}{2} < R - \frac{d}{4}$$

Thus $x \in (B_R(c))_{\frac{d}{4}}$. Thus, we are done if we can show that $x \in (U_i)_{\frac{d}{4}}$ for some $i \in \{1, \dots, N\}$. Since $\text{dist}(x, \Gamma) < \frac{d}{4}$, there is a $y \in \Gamma$ s.t. $|x - y| < \frac{d}{4}$. By definition of \tilde{d} there is a $i \in \{1, \dots, N\}$ s.t. $\text{dist}(y, U_i^c) \geq \tilde{d} \geq d$, i.e. for all $z \in U_i^c$ we have $|y - z| \geq d$. We get

$$|x - z| \geq \underbrace{|x - y|}_{< \frac{d}{4}} - \underbrace{|y - z|}_{\geq d} \geq \frac{3d}{4} < \frac{d}{4}$$

This implies $\text{dist}(x, U_i^c) > \frac{d}{4}$, so $x \in (U_i)_{\frac{d}{4}} \not\subset$. Define for all $i \in \{0, \dots, N\}$: $\chi_i : \mathbb{R}^d \rightarrow [0, \infty)$ by

$$\chi_i(x) = \frac{\phi_i(x)}{\sum_{j=0}^N \phi_j(x)}$$

χ_i is well-defined by Claim 2 and $\chi_i \in C^\infty(\mathbb{R}^d)$. Also note that $\sum \chi_i = 1$, $\chi_i \geq 0$, which implies 1. Furthermore, since $\text{supp}(\phi_i) \subseteq U_i$, we have $\text{supp}(\chi_i) \subseteq U_i$ for all $i \in \{1, \dots, N\}$, which implies 2. Finally, since $\text{supp}(\phi_0) \subseteq \mathbb{R}^d \setminus \Gamma$, we get $\text{supp}(\chi_0) \subseteq \mathbb{R}^d \setminus \Gamma$. This implies 3. \blacksquare

5.4 Variational problem for weak solutions

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

("formally") for all $\phi \in C_c^\infty(\Omega)$, then

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi$$

if $\nabla u \in L^2$, $f \in L^2$. By a density argument:

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi$$

for all $\phi \in \overline{C_c^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$.

Theorem 5.25 (Poincare inequality) There is a $C > 0$ s.t.

$$C \int_{\Omega} |\nabla v|^2 \geq \int_{\Omega} |v|^2$$

for all $v \in H_0^1(\Omega)$.

Remark 5.26 $H^1(\Omega)$ with $\|v\|_{H^1(\Omega)} = (\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2)^{\frac{1}{2}}$ is a Hilbert-Space. This implies that $H_0^1(\Omega) \stackrel{(\text{closed})}{\subseteq} H^1(\Omega)$ is also a Hilbert space. By the Poincare inequality (5.25) we have for all $v \in H_0^1(\Omega)$:

$$\|v\|_{H^1(\Omega)} \geq \|\nabla v\|_{L^2} \geq \frac{1}{2C} \|v\|_{L^2} + \frac{1}{2} \|\nabla v\|_{L^2} \geq \frac{1}{C^1} \|v\|_{H^1(\Omega)}$$

We can think of $H_0^1(\Omega)$ as a Hilbert space with $\|v\|_{H_0^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)}$.

Proof. (Of the Poincare inequality (5.25)) We need to prove:

$$\begin{aligned} \exists C > 0 : \quad C \int_{\Omega} |\nabla v|^2 &\geq \int_{\Omega} |v|^2 \quad \forall v \in H_0^1(\Omega) \\ \Leftrightarrow \quad \exists C > 0 : \quad C \int_{\Omega} |\nabla v|^2 &\geq \int_{\Omega} |v|^2 \quad \forall v \in C_c^\infty(\Omega) \end{aligned}$$

Assume by contradiction that this does not hold, i.e. there is no $C > 0$ s.t. the statement holds. Thus there is a sequence $\{v_n\} \subseteq C_c^\infty(\Omega)$ s.t.

$$\int_{\Omega} |v_n|^2 = 1, \quad \int_{\Omega} |\nabla v_n|^2 \xrightarrow{n \rightarrow \infty} 0$$

Since $v_n \in C_c^2(\Omega)$ we can extend v_n by 0 outside Ω , so $v_n \in C_c^\infty(\mathbb{R}^d)$. Then:

$$\int_{\mathbb{R}^d} |v_n|^2 = 1, \quad \int_{\mathbb{R}^d} |\nabla v_n|^2 \rightarrow 0, \quad \text{supp } v_n \subseteq \Omega$$

By the Fourier transform:

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk = 1, \quad \int_{\mathbb{R}^d} |2\pi k|^2 |\hat{v}_n(k)|^2 dk \rightarrow 0, \quad \text{supp } v_n \subseteq \Omega$$

We prove that

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk \rightarrow 0$$

We write

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk = \int_{|k| \leq \epsilon} + \int_{|k| > \epsilon}$$

First, for all $\epsilon > 0$:

$$\int_{|k| > \epsilon} |\hat{v}_n(k)|^2 \leq \int_{\mathbb{R}^d} \frac{|k|^2}{\epsilon^2} |\hat{v}_n(k)|^2 dk \xrightarrow{n \rightarrow \infty} 0$$

Second:

$$\begin{aligned} \int_{|k| \leq \epsilon} |\hat{v}_n(k)|^2 dk &\leq \left(\int_{|k| \leq \epsilon} 1 dk \right)^{\frac{1}{q}} \left(\int_{|k| \leq \epsilon} |\hat{v}_n(k)|^{2p} dk \right)^{\frac{1}{p}}, \quad 1 < p, q < \infty \\ &\leq C \epsilon^{\frac{d}{q}} \|\hat{v}_n\|_{L^{2p}}^2, \quad \frac{1}{p} + \frac{1}{q} = 1 \text{ and } 1 \leq r \leq 2 \end{aligned}$$

Moreover, since Ω is bounded,

$$\|v_n\|_{L^r} \leq \left(\int_{\Omega} |v_n|^r \right)^{\frac{1}{r}} \leq \|1_{\Omega}\|_{L^s} \|v_n\|_{L^2}^{1-\theta} \leq C_{\Omega} \quad \forall 1 \leq r \leq 2.$$

Thus we can take $r < 1$ but close to 1. Then p is sufficiently large, so q is close to 1. Then

$$\int_{|k| \leq \epsilon} |\hat{v}_n(k)|^2 \leq C \epsilon^{\frac{d}{q}} \|\hat{v}_n\|_{L^{2p}}^2 \leq C \epsilon^{\frac{d}{q}} \|v_n\|_{L^r}^2 \leq C \epsilon^{\frac{d}{q}}$$

Conclusion:

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 = \int_{|k| \leq \epsilon} + \int_{|k| > \epsilon} \leq C \epsilon^{\frac{d}{q}} + \int_{|k| > \epsilon} \xrightarrow{n \rightarrow \infty} C \epsilon^{\frac{d}{q}} \xrightarrow{\epsilon \rightarrow 0} 0$$

which contradicts to the assumption $\|\hat{v}\|_{L^2} = \|v\|_{L^2} = 1$. ■

Exercise 5.27 Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1 -boundary. Let $u \in W^{1,p}(\Omega)$, for some $1 \leq p < \infty$. Then the following is equivalent:

a) $u \in W_0^{1,p}(\Omega)$

b) $\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^d \setminus \Omega \end{cases} \in W^{1,p}(\mathbb{R}^d)$

Theorem 5.28 (Dirichlet, Riemann, Poincare, Hilbert) Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1 -boundary. Let $f \in L^2(\Omega)$. Then there exists a unique solution $u \in H_0^1(\Omega)$ of the variational problem

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi$$

for all $\phi \in H_0^1(\Omega)$. ($\Rightarrow -\Delta u = f$ in $D'(\Omega)$). Moreover, u is the unique minimizer of

$$\inf_{v \in H_0^1(\Omega)} \left(\frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \right)$$

Proof. Let us prove that there is a solution $u \in H_0^1(\Omega)$ for $\inf_{v \in H_0^1(\Omega)} E(v)$, $E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v$.

Step 1: We prove $E > -\infty$. Take $v \in H_0^1(\Omega)$. By the Poincare and Hölder inequalities:

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \\ &\geq \frac{1}{2C} \|v\|_{L^2(\Omega)} - \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\geq \frac{1}{2C} \|v\|_{L^2(\Omega)}^2 - \left(\frac{1}{4C} \|v\|_{L^2(\Omega)}^2 + C \|f\|_{L^2(\Omega)}^2 \right) \\ &\geq -C \|f\|_{L^2(\Omega)}^2 > -\infty \end{aligned}$$

We can also bound:

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \\ &\geq \frac{1}{4} \int_{\Omega} |\nabla v|^2 - \frac{1}{4C} \int_{\Omega} |v|^2 - \|f\|_{L^2} \|v\|_{L^2} \\ &\geq \frac{1}{4} \int_{\Omega} |\nabla v|^2 - C \|f\|_{L^2}^2 \end{aligned}$$

Step 2: We can take a minimizing sequence $\{v_n\} \subseteq H_0^1(\Omega)$ s.t. $E(v_n) \xrightarrow{n \rightarrow \infty} E$. Then:

$$\frac{1}{4} \int_{\Omega} |\nabla v_n|^2 \leq E(v_n) + C \|f\|_{L^2}^2 \longrightarrow \text{const.}$$

So $|\nabla v_n|$ is bounded in $L^2(\Omega)$. We know that $H_0^1(\Omega)$ is a Hilbert space with norm $\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$ (and the norm is equivalent to the H^1 -norm). Thus $\{v_n\}$ is bounded in $H_0^1(\Omega)$.

Remark 5.29 (Reminder from functional analysis) Let H be a Hilbert space. We say that $v_n \rightarrow v$ if $\|v_n - v\| \rightarrow 0$ and $v_n \rightarrow v$ weakly in H if $\langle v_n, \phi \rangle \rightarrow \langle v, \phi \rangle$ for all $\phi \in H$.

Theorem 5.30 (Banach-Alaoglu) If H is a Hilbert space and $\{v_n\}$ is a bounded sequence, then there is a subsequence $\{v_{n_k}\}$ s.t. $v_{n_k} \rightarrow v$ weakly in H .

Remark 5.31 – $v_n \rightarrow v$ in H iff $f(v_n) \rightarrow f(v)$ for all $f \in H^* = \mathcal{L}(H, \mathbb{R})$.

– If $v_n \rightarrow v$ in H , then: $\liminf_{n \rightarrow \infty} \|v_n\| \geq \|v\|$ (Fatous Lemma)

In fact, for all $\phi \in H$ $\langle v_n, \phi \rangle \rightarrow \langle v, \phi \rangle$ and $|\langle v_n, \phi \rangle| \leq \|v_n\| \|\phi\|$. This implies

$$\frac{|\langle v, \phi \rangle|}{\|\phi\|} \leq \liminf_{n \rightarrow \infty} \|v_n\|.$$

So we get

$$\|v\| = \sup_{\phi \neq 0} \frac{|\langle v, \phi \rangle|}{\|\phi\|} \leq \liminf_{n \rightarrow \infty} \|v_n\|$$

By the Banach-Alaoglu theorem (5.30), up to a subsequence, $v_n \rightarrow u$ weakly in $H_0^1(\Omega)$. We prove that u is a minimizer for \mathcal{E}

$$E \leftarrow \mathcal{E}(v_n) = \frac{1}{2} \int |\nabla v_n|^2 - \int f v_n$$

– Since $v_n \rightarrow u$ in $H_0^1(\Omega)$ we have that

$$\liminf_{n \rightarrow \infty} \|v_n\|_{H_0^1(\Omega)}^2 \geq \|u\|_{H_0^1(\Omega)}^2$$

So we have

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^2 \geq \int_{\Omega} |\nabla u|^2.$$

– Consider the functional $\mathcal{L} : \phi \in H_0^1(\Omega) \rightarrow \int_{\Omega} f \phi$. We claim that \mathcal{L} is continuous. In fact:

$$|\mathcal{L}| = \left| \int_{\Omega} f \phi \right| \leq \|f\|_{L^2} \|\phi\|_{L^2} \leq C \|f\|_{L^2} \|\nabla f\|_{L^2} = C \|f\|_{L^2} \|\phi\|_{H_0^1(\Omega)}$$

Thus from $v_n \rightarrow v$ in $H_0^1(\Omega)$ we get $\mathcal{L}(v_n) \rightarrow \mathcal{L}(u)$, thus $\int_{\Omega} f v_n \rightarrow \int_{\Omega} f u$.

Conclusion: $E = \liminf \mathcal{E}(v_n) \geq \mathcal{E}(u)$, so u is a minimizer for \mathcal{E} .

Step 3: Uniqueness. If E has 2 minimizers u_1, u_2 we can prove that $u_1 = u_2$. This is because of the convexity:

$$\begin{aligned} 0 &\geq \frac{\mathcal{E}(u_1) + \mathcal{E}(u_2)}{2} - \mathcal{E}\left(\frac{u_1 + u_2}{2}\right) \\ &= \frac{1}{8} \left[2 \int_{\Omega} |\nabla u_1|^2 + 2 \int_{\Omega} |\nabla u_2|^2 - \int_{\Omega} |\nabla(u_1 + u_2)|^2 \right] \\ &= \frac{1}{8} \int_{\Omega} |\nabla u_1 - \nabla u_2|^2 \geq 0 \end{aligned}$$

This implies that $\nabla(u_1 - u_2) = 0$, so $u_1 - u_2 = \text{const} = c_0$. Since $u_1, u_2 \in H_0^1(\Omega)$, we have that $u_1 - u_2 \in H_0^1(\Omega)$ and $c_0 \in C(\bar{\Omega})$. Hence $c_0 = 0$ on $\partial\Omega$, so $c_0 = 0$. \blacksquare

Remark 5.32 We can also prove directly that there is a unique $u \in H_0^1(\Omega)$ s.t.

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \quad \forall \phi \in H_0^1(\Omega)$$

by Riesz theorem. So we get $\langle u, \phi \rangle_{H_0^1(\Omega)} = \mathcal{L}(\phi)$.

Recall the corrector function for the unit ball:

$$\phi_x(y) = G(|x||y - \tilde{x}|), \quad \tilde{x} = \frac{x}{|x|^2}$$

This is ok if $x \neq 0$. When $x \rightarrow 0$:

$$G(|x|(y - \tilde{x})) = G\left(\underbrace{|x|y - \frac{x}{|x|}}_{|\cdot| \rightarrow 1}\right) G(z), \quad |z| = 1$$

is well-defined as G is radial. Question: If $u \in H^1(\Omega)$, then how can we define $u|_{\partial\Omega}$?

5.5 Theory of Trace

Theorem 5.33 (Trace Operator) Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1 boundary. Then there is a unique linear bounded operator $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ such that

- If $u \in H^1(\Omega) \cap C(\bar{\Omega})$, then $Tu = u|_{\partial\Omega}$ in the usual restriction sense.
- There is a $C > 0$ s.t. $\|Tu\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$

Theorem 5.34 If $u \in H^1(\Omega)$, then $u \in H_0^1(\Omega)$ is equivalent to $Tu = 0$ in $L^2(\partial\Omega)$. ($H_0^1(\Omega) = T^{-1}(\{0\})$). Then we can discuss about

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega \end{cases}$$

Lemma 5.35 (Trace inequality on \mathbb{R}_+^d) if $u \in C_c^\infty(\mathbb{R}^d)$, then:

$$\|u|_{\partial\mathbb{R}_+^d}\|_{L^2(\partial\mathbb{R}_+^d)} \leq C\|u\|_{H^1(\mathbb{R}^d)} \quad \text{with } C > 0 \text{ independent of } u.$$

Proof. $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

$$\begin{aligned} |u(x', 0)|^2 &= - \int_0^\infty \partial_d(|u(x', x_d)|^2) dx_d \\ &= - \int_0^\infty 2\partial_d u(x', x_d) u(x', x_d) dx_d \\ &\leq \int_0^\infty [|\partial_d u(x', x_d)|^2 + |u(x', x_d)|^2] dx_d \end{aligned}$$

This implies:

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} |u(x', 0)|^2 dx' &\leq \int_{\mathbb{R}^{d-1}} \left(\int_0^\infty [\dots] dx_d \right) dx' \\ &= \int_{\mathbb{R}_+^d} [|\partial_d u|^2 + |u|^2] = \|u\|_{H^1(\mathbb{R}_+^d)}^2 \end{aligned} \quad \blacksquare$$

Corollary 5.36 If $u \in H^1(Q)$ and u is compactly supported, then:

$$\|u\|_{L^2(Q_0)} \leq \|u\|_{H^1(Q_+)}$$

Here

$$\begin{aligned} Q &= \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, |x_d| < 1\} \\ Q_+ &= \{x = (x', x_d) \in Q \mid x_d > 0\} \\ Q_0 &= \{x = (x', x_d) \in Q \mid x_d = 0\}. \end{aligned}$$

Proof. We extend u by 0 outside of Q , so $u \in H^1(\mathbb{R}^d)$. ■

Theorem 5.37 (Extension) If $\Omega \subseteq \mathbb{R}^d$ is open, bounded with C^1 -boundary, then there is a bounded linear operator $B : H^1(\Omega) \rightarrow H^1(\mathbb{R}^d)$ s.t.

- $Bu|_\Omega = u$ for all $u \in H^1(\Omega)$
- $\|Bu\|_{H^1(\mathbb{R}^d)} \leq C\|u\|_{H^1(\Omega)}$ and $\|Bu\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{L^2(\Omega)}$.

Proof of Theorem 5.37. Since $\partial\Omega$ is C^1 there are open sets $\{U_i\}_{i=1}^N \subseteq \mathbb{R}^d$ such that $\partial\Omega \subseteq \bigcup_{i=1}^N U_i$ and for all i there is a C^1 -diffeomorphism $h_i : U_i \rightarrow Q$ s.t. $h_i(U_i) = Q$, $h_i(U_i \cap \Omega) = Q_+$, $h_i(U_i \cap \partial\Omega) = Q_0$. Then there exists a partition of unity $\{\theta_i\}_{i=1}^N \subseteq C^\infty(\mathbb{R}^d)$ s.t.

1. $\sum_{i=1}^N \theta_i = 1$ for all $x \in \mathbb{R}^d$
2. For all $i = 1, \dots, N$: $\theta_i \in C_c^\infty(U_i)$
3. $\text{supp } \theta_0 \subseteq \mathbb{R}^d \setminus \partial\Omega$ (in particular $\theta_0|_\Omega \in C_c^\infty(\Omega)$)

Then given $u \in H^1(\Omega)$, we can write $u = \sum_{i=1}^N \theta_i u$, where $u_i = \theta_i u$. By the extension theorem (5.37), $u \rightarrow$ extended to $Bu \in H^1(\mathbb{R}^d)$, thus

$$Bu = \sum_{i=1}^N \theta_i(Bu) = \sum_{i=1}^N v_i, \quad v_i = \theta_i(Bu)$$

Then $v_i \in H^1(\mathbb{R}^d)$ and v_i is compactly supported in U_i for all $i = 1, 2, \dots, N$ and $\text{supp } v_0 \subseteq \mathbb{R}^d \setminus \partial\Omega$, $v_i \in H^1(\mathbb{R}^d)$ and compactly supported inside U_i . This implies $\tilde{v}_i(y) = v_i(h_i^{-1}(y)) \in H^1(Q)$ and compactly supported inside Q , $y \in Q$. Thus $\|\tilde{v}_i\|_{L^2(Q_0)} \leq C\|\tilde{v}_i\|_{H^1(Q_+)}$. So we have $\|v_i\|_{L^2(\partial\Omega)} \leq C\|\tilde{v}_i\|_{L^2(Q_0)} \leq C'\|\tilde{v}_i\|_{H^1(Q_+)} \leq C''\|v_i\|_{H^1(U_i \cap \Omega)}$. Thus:

$$\begin{aligned} \|u\|_{L^2(\partial\Omega)} &= \left\| \sum_{i=1}^N v_i \right\|_{L^2(\partial\Omega)} \leq \sum_{i=1}^N \|v_i\|_{L^2(\partial\Omega)} \leq \sum_{i=1}^N C'' \|v_i\|_{H^1(U_i \cap \Omega)} \\ &= C'' \sum_{i=1}^N \|\theta_i u\|_{H^1(\Omega)} \leq C'' \sum_{i=1}^N C \|u\|_{H^1(\Omega)} \end{aligned}$$

This proof works for $u \in C(\bar{\Omega})$. This implies

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega) \cap C(\bar{\Omega}).$$

This allows us to define

$$\begin{aligned} T : H^1(\Omega) &\longrightarrow L^2(\partial\Omega) \\ u &\longmapsto u|_{\partial\Omega} \end{aligned}$$

by continuity. I.e. for all $u \in H^1(\Omega)$ there is $\{u_n\} \subseteq H^1(\Omega) \cap C(\bar{\Omega})$ s.t. $u_n \rightarrow u$ in H_0^1 . Then $Tu_n \rightarrow Tu$ in $L^2(\partial\Omega)$. ■

Lemma 5.38 (Extension for Q) Let $u \in H^1(Q_+)$. Then we define $Bu : Q \rightarrow \mathbb{R}$ by

$$Bu(x) = \begin{cases} u(x) & x \in Q_+ \\ -u(x', -x_d) & x \in Q_- \end{cases},$$

$x = (x, x_d)$. Then $Bu \in H^1(Q)$ and $Bu|_{Q_+} = u$, $\|Bu\|_{L^2(Q)}^2 = 2\|u\|_{L^2(Q_+)}^2$, $\|\nabla(Bu)\|_{L^2(Q)}^2 = \|\nabla u\|_{L^2(Q_+)}^2$

Proof. It is obvious $Bu|_{Q_+} = u$ and

$$\begin{aligned} \int_Q |Bu|^2 &= \int_{Q_+} |Bu|^2 + \int_{Q_-} |Bu|^2 \\ &= \int_Q |u|^2 + \int_{Q_- = \{(x, -x_d) | (x, x_d) \in Q_+\}} |u(x, -x_d)|^2 \\ &= 2 \int_{Q_+} |u|^2 \end{aligned}$$

We prove:

$$\nabla(Bu)(x) = \begin{cases} \nabla u(x) & u \in Q_+ \\ \nabla u(x', -x_d) & u \in Q_- \end{cases}$$

First, $\partial_d Bu(x) = \partial_d u(x', -x_d)$ if $x \in Q_-$. Take $\phi \in C_c^\infty(Q)$, then:

$$\begin{aligned} \int_Q (Bu(x))(\partial_d \phi)(x) dx &= \int_{Q_+} u \partial_d \phi + \int_{Q_-} -u(x', -x_d) \partial_d [\phi(x', x_d)] dx \\ (x \rightarrow -x_d) &= \int_{Q_+} u \partial_d \phi + \int_{Q_+} [u(x', x_d)(\partial_d \phi)(x', -x_d)] dx \\ &\stackrel{(\phi \notin C_c^\infty(Q_+))}{\approx} \int_{Q_+} (\partial_d u) \phi(x) + \int_{Q_+} (\partial_d u(x', x_d)) \phi(x', -x_d) dx \\ &= - \int_{Q_+} (\partial_d u) \phi(x) + \int_{Q_-} \partial_d u(x', -x_d) \phi(x', x_d) dx \\ &= - \int_Q f \phi, \quad \text{where } f(x) = \begin{cases} \partial_d u & x \in Q_+ \\ -\partial_d u(x', -x_d) & x \in Q_- \end{cases} \end{aligned}$$

We prove $\int_{Q_+} u \partial_d \tilde{\phi} = - \int_{Q_+} (\partial_d u) \tilde{\phi}$ where $\tilde{\phi}(x, x_d) = \phi(x, x_d) - \phi(x, -x_d)$, $\tilde{\phi} \notin C_c^\infty(Q_+)$. Define $\eta_\epsilon = 0$ when $|x_d| \leq \epsilon$, $\eta_\epsilon = 1$ if $|x_d| \geq 2\epsilon$, $\eta_\epsilon \in C^\infty$, $\eta_\epsilon(x', x_d) = \eta_0(x', \frac{x_d}{\epsilon})$, $\eta_0 = \begin{cases} 1 & |x_d| \geq 2 \\ 0 & |x_d| \leq 1 \end{cases}$. We have

$$\int_{Q_+} u \partial_d (\eta_\epsilon \tilde{\phi}) = - \int_{Q_+} \partial_d u (\eta_\epsilon \tilde{\phi})$$

We take $\epsilon \rightarrow 0$,

$$\int_{Q_+} (\partial_d u) (\eta_\epsilon \tilde{\phi}) \rightarrow \int_{Q_+} (\partial_d u) \tilde{\phi}$$

by dominated convergence.

$$\begin{aligned} \int_{Q_+} u \partial_d (\eta_\epsilon \tilde{\phi}) &= \int_{Q_+} u (\partial_d \eta_\epsilon) \tilde{\phi} + \int_{Q_+} u \eta_\epsilon \partial_d \tilde{\phi} \\ &\xrightarrow{\epsilon \rightarrow 0} \int_{Q_+} u \eta_\epsilon \partial_d \tilde{\phi} \end{aligned}$$

by dominated convergence.

$$\begin{aligned}
& \left| \int_{Q_+} u(\partial_d \eta_\epsilon) \tilde{\phi} \right| = \left| \int_Q u \frac{1}{\epsilon} (\partial_d \eta_0) \left(x, \frac{x_d}{\epsilon} \right) \tilde{\phi} \right| \\
& \begin{pmatrix} |\tilde{\phi}(x', x_d)| \\ = |\phi(x, x_d) - \phi(x, x_d)| \\ \leq \|\partial_d \phi\|_{L^\infty} |x_d| \end{pmatrix} \leq \frac{1}{\epsilon} \|\partial_d \eta_0\|_{L^\infty} \int_{Q_+ \cap \{x_d \leq 2\epsilon\}} |u| \underbrace{|\tilde{\phi}|}_{\leq C|x_d| \leq C\epsilon} \\
& (\text{Dominated cv } u \in L^1(Q_+)) \leq C \int_{Q_+ \cap \{0 < x_d \leq 2\epsilon\}} |u| \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

where $u \in L^2(Q_+)$ because $u \in H^1(Q_+)$. ■

Exercise 5.39 (E. 9.1) Let Ω be open, bounded with C^1 -boundary. Let $u \in H_0^1(\Omega)$, $f \in L^2(\Omega)$. Show that the following statements are equivalent:

- 1) $-\Delta u = f$ in $D'(\Omega)$
- 2) $\int \nabla u \nabla \phi = \int f \phi$ for all $\phi \in H_0^1$
- 3) $E = \inf_v \left(\frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega f v \right)$

Solution.

1) \Rightarrow 2) From $-\Delta u = f$ in $D'(\Omega)$ we get that

$$\int_\Omega u(-\Delta \phi) = \int_\Omega f \phi$$

for all $\phi \in C_c^\infty(\Omega)$. Claim: If $u \in H_0^1$, $\phi \in C_c^\infty$, then

$$\int_\Omega (-\Delta \phi) = \int_\Omega \nabla u \nabla \phi$$

Density argument: $u \in H_0^1 = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_H}$, so there is a sequence $\{u_n\} \subseteq C_c^\infty(\Omega)$ s.t. $u_n \rightarrow u$ in $H^1(\Omega)$. Since $u_n, \phi \in C_c^\infty(\Omega)$, then by the integration by parts:

$$\int_\Omega u_n(-\Delta \phi) = \int_\Omega (\nabla u_n) \nabla \phi \forall n$$

Take $n \rightarrow \infty$, then,

$$\int_\Omega u(-\Delta \phi) = \int_\Omega (\nabla u) \nabla \phi$$

as $u_n \rightarrow u$ and $\nabla u_n \rightarrow \nabla u$ in L^2 . Claim: If $\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi$ for all $\phi \in C_c^\infty(\Omega)$, then

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi$$

for all $\phi \in H_0^1$. (Given $\nabla u, f \in L^2$). With density argument: For all $\phi \in H_0^1$ there is a sequence $\{\phi_n\} \subseteq C_c^\infty(\Omega)$ s.t. $\phi_n \rightarrow \phi$ in H^1 . Then:

$$\int_\Omega \nabla u \nabla \phi_n = \int_\Omega f \phi_n$$

for all n . Take $n \rightarrow \infty$:

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi$$

as $\phi_n \rightarrow \phi$, $\nabla \phi_n \rightarrow \nabla \phi$ in L^2 .

2) \Rightarrow 3) We show $E(u) \leq E(v)$ for all $v \in H_0^1$, i.e.

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u \leq \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v$$

for all $v \in H_0^1$. Write $v = u + w$, then:

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \\ &= \frac{1}{2} \int_{\Omega} |\nabla(u+w)|^2 - \int_{\Omega} f(u+w) \\ &= \frac{1}{2} \int_{\Omega} [|\nabla u|^2 + |\nabla w|^2 + 2\nabla u \nabla w] \int_{\Omega} (f u + f w) \\ &= E(u) + \frac{1}{2} \int_{\Omega} |\nabla w|^2 + \underbrace{\left(\int_{\Omega} \nabla u \nabla w - \int_{\Omega} f w \right)}_{=0} \end{aligned}$$

as $w = v - u \in H_0^1$ (by (2))

3) \Rightarrow 1)

$$E(u) \leq E(u + t\phi)$$

for all $\phi \in H_0^1$ (or C_c^∞) for all $t \in \mathbb{R}$. This implies:

$$\frac{d}{dt} E(u + t\phi)|_{t=0} = 0$$

Here

$$\begin{aligned} E(u + t\phi) &= \frac{1}{2} \int_{\Omega} \underbrace{|\nabla(u + t\phi)|^2}_{|\nabla u|^2 + t^2 |\nabla \phi|^2 + 2t \nabla u \nabla \phi} - \int_{\Omega} f(u + t\phi) \\ &= E(u) + t \left[\int_{\Omega} \nabla u \nabla \phi \int_{\Omega} f \phi \right] + t^2 \int_{\Omega} |\nabla \phi|^2 \end{aligned}$$

This implies

$$\frac{d}{dt} E(u + t\phi)|_{t=0} = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi$$

Conclude:

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi$$

for all $\phi \in H_0^1$ or C_c^∞ . So we get

$$\int_{\Omega} u(-\Delta \phi) = \int_{\Omega} f \phi$$

for all $\phi \in C_c^\infty$. so we can conclude:

$$-\Delta u = f$$

in $D'(\Omega) \Rightarrow 1)$

■

Exercise 5.40 (E 9.2)

$$Q = \{(x', x_d) \mid |x'| < 1, |x_d| < 1\}$$

Given $u \in H^1(Q_+)$, define $Bu : Q \rightarrow \mathbb{R}$ as

$$Bu(x) = \begin{cases} u(x) & x \in Q_+ \\ u(\tilde{x}) & x \in Q_- \end{cases},$$

$x = (x', x_d) \Leftrightarrow \tilde{x} = (x', -x_d)$, $x \in Q_- \Leftrightarrow \tilde{x} \in Q_+$. In the lectures:

$$\partial_d(Bu)(x) = \begin{cases} \partial_d u(x) & x \in Q_+ \\ -(\partial_d u)(\tilde{x}) & x \in Q_- \end{cases}$$

This implies $\partial_d(Bu) \in L^2(Q)$.

1. For all $i = 1, \dots, d-1$, then:

$$\partial_i(Bu)(x) = \begin{cases} \partial_i u(x) & x \in Q_+ \\ \partial_i u(\tilde{x}) & x \in Q_- \end{cases}$$

2. Example $u \in H^2(Q_+)$ but $Bu \notin H^2(Q)$.

Solution. 1. For all $\phi \in C_c^\infty(Q)$:

$$\int_Q Bu(x) \partial_i \phi(x) dx = \int_{Q_+} u(x) \partial_i \phi(x) dx + \int_{Q_-} u(\tilde{x}) \partial_i \phi(x) dx$$

Write $\vec{n} = (n_1, \dots, n_d)$. Here:

$$\begin{aligned} \int_{Q_+} u(x) \partial_i \phi(x) dx &= \int_{Q_+} -\partial_i u(x) \phi(x) dx + \int_{\partial Q_+} u(x) \phi(x) n_i dS \\ \int_{Q_-} u(x', -x_d) \partial_i \phi(x', x_d) dx' dx_d &= - \int_{Q_+} u(x', x_d) \partial_i \phi(x', -x_d) dx' dx_d \\ &= \int_{Q_+} \partial_i u(x) \phi(\tilde{x}) - \int_{\partial Q_+} u \phi n_i dS \\ &= \int_{Q_-} -\partial_i u(\tilde{x}) \phi(x) - \int_{\partial Q_+} u \phi n_i dS \end{aligned}$$

with $d(-x_d) = d(x_d)$. Conclude:

$$\begin{aligned} \int_Q (Bu)(x) \partial_i \phi(x) dx &= \int_{Q_+} (-\partial_i u)(x) \phi(x) + \int_{Q_-} (-\partial_i u)(\tilde{x}) \phi(x) \\ &= \int_Q -h(x) \phi(x) dx, \quad h(x) = \begin{cases} \partial_i u(x), & x \in Q_+ \\ \partial_i u(\tilde{x}), & x \in Q_- \end{cases} \end{aligned}$$

for all $\phi \in C_c^\infty(Q)$, so $\partial_i(Bu) \in L^2$ for all $i = 1, 2, \dots, d-1$. Thus $Bu \in H^1(Q)$.

2. 1D: Take $Q_+(0, 1)$, $Q_- = (-1, 0)$, $Q_0 = \{0\}$, $Q = (-1, 1)$, $u(x) = x$ in $Q_+ = (0, 1)$, $Bu(x) = u(x) = -x$ if $x \in Q_- = (-1, 0)$, i.e. $Bu(x) = |x|$ if $x \in Q = (-1, 1)$. We know

$$(Bu)'(x) = \begin{cases} 1 & x \in (0, 1) \\ -1 & x \in (-1, 0) \end{cases} \in L^2(-1, 1)$$

i.e. $Bu \in H^1(Q)$.

$$(Bu)''(x) = 2\delta_0(x)$$

in $D'(Q)$ but $\notin L^2(-1, 1)$, i.e. $Bu \notin H^2(Q)$. Question: Given $u \in H^2(Q_+)$, can we find an extension $Bu \in H^2(Q)$ Yes! E.g. $u(x) = x$ in $(0, 1)$, so $Bu(x) = x$ in $(-1, 1)$. In general: $u \in H^2(Q) \rightsquigarrow \tilde{u} \in H^2(Q)$ but $\nabla u = 0$ on ∂Q_+ . ■

Exercise 5.41 (Bonus 8) Assume $u \in H^2(Q_+)$ and $\begin{cases} u = 0 \\ \nabla u = 0 \end{cases}$ on ∂Q_+ . Prove that $Bu \in H^2(Q)$. (Reflection extension) (Ok in 1D)

Remark 5.42 If $u \in H^2(Q_+)$, then $\nabla u \in H^1(Q_+)$, so $\nabla u|_{\partial Q_+}$ by trace theory. In general: $\Omega \subseteq \mathbb{R}^d$, C^2 -boundary condition, then the same result holds.

Remark 5.43 In 1D: $\begin{cases} u \in H^2(0, 1) \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$, $u|_{Q_0} \in L^2(Q_0)$, 1D: $Q_0 = \{0\}$. In general:

If $u \in H^1(0, 1)$, then $u(0)$ is determined by trace theory. If $u \in H^2(0, 1)$, $u'(0)$ is determined. Sobolev:

$$\begin{aligned} H^1(0, 1) &\subseteq C([0, 1]) \\ H^2(0, 1) &\subseteq C^1([0, 1]) \end{aligned}$$

Lemma 5.44 (Poincare inequality) Let Ω be open, bounded connected with C^1 -boundary. Then for all $g \in L^2(\partial\Omega)$ s.t. $g \neq \text{constant}$ there is a $C > 0$ s.t.

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$$

for all $u \in M$, where

$$M = \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = g\}.$$

Proof. We assume that the statement does not hold true. Then there is a sequence $\{u_n\} \subseteq H^1(\Omega)$, $u_n|_{\partial\Omega} = g$ s.t.

$$\|\nabla u_n\|_{L^2(\Omega)} \rightarrow 0, \quad \|u_n\|_{L^2(\Omega)} = 1.$$

Since $\{u_n\}$ is bounded in $H^1(\Omega)$, by the Banach-Alaoglu theorem (5.30), up to a subsequence

$$u_n \rightarrow u_0 \quad \text{weakly in } H^1(\Omega)$$

Since $\nabla u_n \rightarrow 0$ strongly in L^2 and $\nabla u_n \rightarrow \nabla u_0$ weakly in L^2 , we have $\nabla u_0 = 0$, so $u_0|_{\partial\Omega} = \text{const.}$ (here we need Ω to be connected), so $u_0|_{\partial\Omega} = \text{const.}$ On the other hand, note that M is convex and closed in $H^1(\Omega)$ since the trace operator $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ is continuous. Therefore, M is also weakly closed in $H^1(\Omega)$ by the Hahn-Banach theorem. Thus from $\{u_n\} \subseteq M$, $u_n \rightarrow u_0$ weakly in $H^1(\Omega)$ we get that $u_0 \in M$, so $u_0|_{\partial\Omega} = g$. We get a contradiction since $g \neq \text{const}$ ■

Theorem 5.45 (Solution for Poisson Equation with inhomogeneous boundary condition) Let Ω be open, bounded with C^1 -boundary. Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. There there is a unique $u \in H^1(\Omega)$ s.t.

$$\begin{cases} -\Delta u = f & \text{in } D'(\Omega) \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega \end{cases}$$

Here $u|_{\partial\Omega} = T(u) \in L^2(\partial\Omega)$ is defined by the trace operator. Moreover if Ω is connected and $g \neq \text{constant}$, then u is the unique minimizer for the variational problem

$$E = \inf_{v \in M} \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v,$$

where $M = \{v \in H^1(\Omega), v|_{\partial\Omega} = g \text{ on } \partial\Omega\}$

Proof. First let us assume that Ω is connected and $g \neq \text{const.}$

Step 1: We prove that $E = \inf_{v \in M} E(v)$ has a minimizer. By Poincaré's Inequality (5.44), for all $v \in M$:

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \\ (\text{Hölder}) &\geq \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 - \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ (\text{Poincaré 5.44}) &\geq \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 - C \|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\geq \frac{1}{4} \|\nabla v\|_{L^2(\Omega)}^2 - C \|f\|_{L^2(\Omega)} \end{aligned}$$

Thus $E = \inf_{v \in M} E(v) > -\infty$. Moreover, taking a minimizing sequence $\{v_n\} \subseteq M$, $E(v_n) \rightarrow E$, we find that $\|\nabla v_n\|_{L^2(\Omega)}$ is bounded, and hence $\|v_n\|_{H^1(\Omega)}$ is bounded (by Poincaré inequality) again. By Banach-Alaoglu (5.30), up to a subsequence we have $v_n \rightarrow u$ weakly in $H^1(\Omega)$. Hence

$$\begin{cases} \limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^2 \geq \int_{\Omega} |\nabla u|^2 & \text{as } \nabla v_n \rightarrow \nabla u \text{ in } L^2 \\ \int_{\Omega} v_n f \rightarrow \int_{\Omega} u f & \text{as } v_n \rightarrow u \text{ in } L^2 \end{cases}$$

Note that $\{v_n\} \subseteq M$, $v_n \rightarrow u$ in $H^1(\Omega)$ and M is weakly closed in $H^1(\Omega)$ (as argued in the proof of Poincaré inequality), therefore $u \in M$. This means that u is a minimizer for $E = \inf_{v \in M} E(v)$.

Step 2: Now we prove that if u is a minimizer for E , then $-\Delta u = f$ in $D'(\Omega)$. In fact, for all $\phi \in C_c^\infty(\Omega)$ we have

$$E(u) \leq E(u + t\phi) \quad \forall t \in \mathbb{R}$$

because $u + t\phi \in M$. So we get that

$$0 = \frac{d}{dt} E(u + t\phi)|_{t=0} = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi$$

Thus

$$\int_{\Omega} u(-\Delta \phi) = \int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \quad \forall \phi \in C_c^\infty(\Omega).$$

So $-\Delta u = f$ in $D'(\Omega)$.

Step 3: We prove that Poisson's equation has at most one solution. Assume that u_1, u_2 are 2 solutions. Then $u = u_1 - u_2$ solves

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \Omega \end{cases}$$

so $u = 0$.

Step 4: If $g = c_0$ is a constant, then Poisson's equation can be rewritten with $\tilde{u} = u - c_0$:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = c_0 & \text{on } \Omega \end{cases} \Leftrightarrow \begin{cases} -\Delta \tilde{u} = f & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \Omega \end{cases}$$

If Ω is not connected, then by considering connected components of Ω we can prove that Poisson's equation always has a unique solution (for all $f \in L^2(\Omega), g \in L^2(\partial\Omega)$).

■

5.6 Final Remarks

We can describe $H_0^1(\Omega)$ as the kernel of the trace operator $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$

Theorem 5.46 Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded with C^1 -boundary. Then:

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid T(u) = 0 \text{ on } \partial\Omega\}$$

Recall that if $u \in H^1(\Omega) \cap C(\bar{\Omega})$, then $T(u) = u|_{\partial\Omega}$ is the usual restriction. In this case we recover a result proved before.

Proof.

■

Recall that the variational characterization of the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \quad \forall \phi \in M$$

where $M = \{v \in H^1(\Omega) \mid v = g \text{ on } \partial\Omega\}$. In fact, if $u \in H^1(\Omega)$ and

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \quad \forall \phi \in H^1(\Omega)$$

Then u satisfies the Neumann condition:

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n} = 0 \text{ on } \partial\Omega$$

(justification ...)

For the exercises of sheet 10: Let $\Omega = (a, b) \subseteq \mathbb{R}$ be an open bounded interval. For every $u \in H^1(\Omega)$ the values $u(a)$ and $u(b)$ are determined uniquely by trace theory, or by Sobolev's embedding theorem. Recall: If $u \in H^1((a, b)) \rightsquigarrow \partial\Omega = \{a, b\}$ counting measure iff $g \in L^2(\partial\Omega)$ i.e. $g(a) = g(b)$ are *well-defined*.

Exercise 5.47 (E 10.1) a) Prove $H^1(\mathbb{R}) \subseteq (C(\mathbb{R}) \cap L^\infty(\mathbb{R}))$

Hint: You can use Fourier Transform

b) $H^1(\Omega) \subseteq C(\Omega)$

Solution. a) Let $u \in H^1(\mathbb{R})$. Then $u, u' \in L^2(\mathbb{R}) \Leftrightarrow \hat{u}(k)(1 + |2\pi k|) \in L^2(\mathbb{R})$.

Thus:

$$u(x) = \int_{\mathbb{R}} \hat{u}(k) e^{2\pi i k x} dk \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

if $\hat{u} \in L^1(\mathbb{R})$. So we have to show $\hat{u} \in L^1(\mathbb{R})$.

$$\begin{aligned} \int_{\mathbb{R}} |\hat{u}(k)| dk &= \int_{\mathbb{R}} \frac{|g(k)|}{1 + |2\pi k|} dk \\ &\leq \left(\int_{\mathbb{R}} |g(k)|^2 dk \right)^{\frac{1}{2}} \left[\int_{\mathbb{R}} \left(\frac{1}{1 + |2\pi k|} \right)^2 dk \right]^{\frac{1}{2}} < \infty \end{aligned}$$

b) Given $u \in H^1(\Omega)$, then there is an extension $\tilde{u} \in H^1(\mathbb{R})$. By a) $\tilde{u} \in C(\mathbb{R})$, so $u = \tilde{u}|_{\bar{\Omega}} \in C(\bar{\Omega})$. Remark: We have $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{H^1(\Omega)}$, where $\Omega = (a, b)$ or \mathbb{R} (but only in 1D) ■

Recall: If $\Omega \subseteq \mathbb{R}^d (d \geq 1)$ open, bounded with C^1 -boundary. Then

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega)$$

Actually the same bound holds if $u \in H^1(\Omega)$ and $u|_\Gamma = 0$ for an open subset $\Gamma \subseteq \partial\Omega$. In 1D we have:

Exercise 5.48 (E 10.2 (Poincare inequality)) Let $u \in H^1(\Omega)$, $u(a) = 0$. Prove that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C \|u'\|_{L^2(\Omega)}$$

Solution. Let $u \in C^1(\bar{\Omega})$ and $u(a) = 0$. Then:

$$\begin{aligned} u(x) &= u(a) + \int_a^x u'(t) dt \quad \forall x \in (a, b) \\ \Rightarrow |u(x)| &\leq \int_a^x |u'(t)| dt \leq \int_a^b |u'(t)| dt = \|u'\|_{L^1(\Omega)} \leq C \|u'\|_{L^2(\Omega)} \end{aligned}$$

as Ω is bounded. This implies:

$$\frac{1}{C} \|u\|_{L^2(\Omega)} \leq \|u\|_{L^\infty(\Omega)} \leq C \|u'\|_{L^2(\Omega)}$$

To extend this for $u \in H^1(\Omega)$, we can use a density argument. More precisely, for all $u \in H^1(\Omega)$ there is a sequence $\{u_n\} \subseteq C^1(\bar{\Omega})$ s.t $u_n \rightarrow u$ in $H^1(\Omega)$. Then:

$$\|u\|_{L^2(\Omega)} = \lim_{n \rightarrow \infty} \|u_n\|_{L^2(\Omega)} \leq C \lim_{n \rightarrow \infty} \|u'_n\|_{L^2(\Omega)} = C \|u'\|_{L^2(\Omega)}$$

Recall: For all $f \in W_{loc}^{1,1}(O)$ with O in \mathbb{R}^d we have

$$f(x) - f(y) = \int_0^1 \nabla f(y + t(x - y))(x - y) dt$$

if $x, y \in O$, $y + t(x - y) \in O$ for all $t \in [0, 1]$. For 1D: If $u \in H^1(a, b)$:

$$u(x) - u(y) = \int_y^x u'(t) dt \quad \forall x, y \in (a, b) \quad \blacksquare$$

Exercise 5.49 (E 10.3 (Poincare inequality)) Let $u \in H^2(\Omega)$ and $f \in L^2(\Omega)$. Prove that the following statements are equivalent:

a) u solves the equation:

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u'(0) = u'(1) = 0 \end{cases}$$

b)

$$\int_{\Omega} u' \phi' = \int_{\Omega} f \phi$$

for all $\phi \in H^1(\Omega)$.

Here $u \in H^2(\Omega) \Rightarrow u' \in H^1(\Omega) \Rightarrow u'(0), u'(1)$ determined uniquely by trace theorem / Sobolev inequality $H^1(\Omega) \subseteq C(\bar{\Omega})$

Solution.

b) \Rightarrow a) For all $\phi \in C_c^\infty(\Omega)$:

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' = - \int_{\Omega} u \phi''$$

This implies $-u'' = f$ in $D'(\Omega)$ a.e. Thus for all $\phi \in H^1(\Omega)$:

$$\int_{\Omega} f \phi = \int_{\Omega} -u'' \phi = \int_{\Omega} u' \phi' - [u' \phi]_a^b$$

By b) we conclude $0 = [u' \phi]_a^b = u'(b)\phi(b) - u'(a)\phi(a)$ for all $\phi \in H^1(\Omega)$. We can choose $\phi \in H^1(\Omega)$ s.t. $\phi(a) = 0, \phi(b) = 1$. This implies $\phi'(b) = 0$. Similarly, we can choose $\phi \in H^1(\Omega)$ s.t. $\phi(a) = 1, \phi(b) = 0$. This implies $u'(a) = 0$.

a) \Rightarrow b) From a) and Integration by parts:

$$\int_{\Omega} f \phi = \int_{\Omega} -u'' \phi = \int_{\Omega} u' \phi' - \underbrace{[u' \phi]_a^b}_{=0 \text{ as } u'(a)=u'(b)=0}$$

This implies:

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in H^1(\Omega)$$

■

Exercise 5.50 (E 10.4 (Robin boundary condition)) Let $f \in L^2(\Omega)$.

a) Prove that there exists a unique $u \in M := \{\phi \in H^1(\Omega), u(a) = 0\}$ s.t.

$$\int_{\Omega} u' \phi' = \int_{\Omega} f \phi \quad \forall \phi \in M$$

b) Prove that the above function u is the unique solution to the equation

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = 0 & u'(b) = 0 \end{cases}$$

Solution. a) By 10.2 we have

$$\|\phi\|_{L^2(\Omega)} \leq C \|\phi'\|_{L^2(\Omega)} \quad \forall \phi \in M$$

Thus: $(M, \|\phi\|_M := \|\phi'\|_{L^2(\Omega)})$ is a Hilbert space. More precisely, we know $(M, \|\cdot\|_M)$ is a closed subspace of $H^1 \rightsquigarrow$ a Hilbert space. And $\|\cdot\|_M$ is comparable to $\|\cdot\|_{H^1}$. By Riesz representation theorem there is a unique $u \in M$ s.t. $\langle \phi, u \rangle_M = F(\phi)$ for all $\phi \in M$. We use this for

$$F(\phi) = \int_{\Omega} f \phi \quad \forall \phi \in M$$

Here $|F(\phi)| \leq \|f\|_{L^2} \|\phi\|_{L^2}$.

b) Let $u \in M$ be the solution in (a) i.e.

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in M$$

Then we prove that u solves

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = u'(b) = 0 \end{cases}$$

Since $u \in M$ we have $u \in H^1(\Omega)$ and $u(a) = 0$. From

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in M$$

we get for all $\phi \in C_c^\infty(\Omega)$:

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' = \int_{\Omega} -u \phi''$$

So we get $-u'' = f$ in $D'(\Omega)$. Since $f \in L^2(\Omega) \Rightarrow u'' \in L^2(\Omega) \Rightarrow u \in H^2(\Omega) \Rightarrow u' \in H^1(\Omega) \Rightarrow u'(b)$ is uniquely determined. For all $\phi \in M$:

$$\int_{\Omega} f \phi = \int_{\Omega} -u'' \phi = \int_{\Omega} u' \phi' - (u'(b)\phi(b) - u'(a)\phi(a)) \quad \text{as } \phi \in M$$

and $\int_{\Omega} f \phi = \int_{\Omega} u' \phi'$. This implies:

$$u'(b)\phi(b) = 0 \quad \forall \phi \in M$$

Take $\phi(x) = \frac{x-a}{b-a} \in M$, $\phi(b) = 1$. Uniqueness of the solution: Take u s.t.

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = u'(b) = 0 \end{cases}$$

This implies $u \in H^2(\Omega)$. By integration by parts: For all $\phi \in H^1(\Omega)$, $\phi(a) = 0$.

$$\int_{\Omega} f \phi = \int_{\Omega} -u'' \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in M$$

Thus $u \in M$ and

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in M. \quad \blacksquare$$

Exercise 5.51 (Bonus 9) Prove that the solution u in Problem E 10.4 is the unique minimizer for the minimization problem:

$$E = \inf_{v \in M} \left(\int_{\Omega} |v'|^2 - \int_{\Omega} f v \right)$$

Chapter 6

Heat Equation

6.1 Fundamental Solution

$$\begin{cases} \partial_t u = \Delta u & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u = g & (x, t) \in \mathbb{R}^d \times \{0\} \end{cases}$$

The fundamental solution is:

$$\Phi(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^d, t > 0$$

We have:

$$\begin{cases} \partial_t \Phi = \Delta \Phi & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ \int_{\mathbb{R}^d} \Phi(x, t) dx = 1 & \forall t > 0 \\ \lim_{t \rightarrow 0} \Phi(x, t) = \delta_0(x) & \text{in } D'(\mathbb{R}^d) \end{cases}$$

Theorem 6.1 If $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then

$$u(x, t) := \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy$$

satisfies

- (i) $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$
- (ii) $\partial_t u = \Delta u$ for all $(x, t) \in \mathbb{R}^d \times (0, \infty)$
- (iii) $\lim_{t \rightarrow 0} u(x, t) = g(x)$ for all $x \in \mathbb{R}^d$

Notation 6.2 For functions of (x, t) we introduce the following notation for different regularity in x and t .

$$f \in C_1^2 \Leftrightarrow f, D_x f, D_x^2 f, \partial_t f \in C$$

Theorem 6.3 (Nonhomogeneous problem) Let $f \in C_1^2(\mathbb{R}^d, [0, \infty))$ be compactly supported. Define

$$u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(x - y, t - s) f(y, s) dy ds$$

Then

- (i) $u \in C_1^2(\mathbb{R}^d \times (0, \infty))$
- (ii) $\partial_t u = \Delta u + f$ for all $x \in \mathbb{R}^d, t > 0$
- (iii) $\lim_{t \rightarrow 0} u(x, t) = 0$ for all $x \in \mathbb{R}^d$.

Proof. We write

$$u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) f(x - y, t - s) dy ds$$

With the Leibniz integral rule we get

$$\partial_t u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) \partial_t f(x - y, t - s) dy ds + \int_{\mathbb{R}^d} \Phi(y, s) f(x - y, 0) dy$$

and

$$\partial_{ij} u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) \partial_{ij} f(x - y, t - s) dy.$$

This shows that $\partial_t u, \partial_{ij} u$ are in $C(\mathbb{R}^d \times (0, \infty))$. Next we calculate:

$$\begin{aligned} \partial_t u - \Delta u &= \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) (\partial_t - \Delta_x) f(x - y, t - s) dy ds + \int_{\mathbb{R}^d} \Phi(y, s) f(x - y, 0) dy \\ &= \underbrace{\int_\epsilon^t \int_{\mathbb{R}^d} \Phi(y, s) (\partial_t - \Delta_x) f(x - y, t - s) dy ds}_{=: I_\epsilon} \\ &\quad + \underbrace{\int_0^\epsilon \int_{\mathbb{R}^d} \Phi(y, s) (\partial_t - \Delta_x) f(x - y, t - s) dy ds}_{J_\epsilon} \\ &\quad + \underbrace{\int_{\mathbb{R}^d} \Phi(y, s) f(x - y, 0) dy}_K \end{aligned}$$

Then

$$\begin{aligned} |J_\epsilon| &\leq \|(\partial_t - \Delta_x) f\|_{L^\infty} \int_0^\epsilon \int_{\mathbb{R}^d} \Phi(y, s) dy ds \leq C\epsilon \xrightarrow{\epsilon \rightarrow 0} 0 \\ I_\epsilon &= \int_\epsilon^t \int_{\mathbb{R}^d} \Phi(y, s) (-\partial_s - \Delta_y) f(x - y, t - s) dy ds \\ \text{(Green (2.3))} \quad &= \int_\epsilon^t \int_{\mathbb{R}^d} \underbrace{(\partial_s - \Delta_y) \Phi(y, s)}_{=0} f(x - y, t - s) dy ds \\ &\quad - \left[\int_{\mathbb{R}^d} \Phi(y, s) f(x - y, t - s) dy \right]_{s=\epsilon}^{s=t} \end{aligned}$$

This implies:

$$\begin{aligned} I_\epsilon + K &= \int_{\mathbb{R}^d} \Phi(y, \epsilon) f(x - y, t - \epsilon) dy \\ &\xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} \delta_0(y) f(x - y, t) dy = f(x, t) \end{aligned}$$

Thus

$$\partial_t u - \Delta u = f(x, t) \quad \forall (x, t) \in \mathbb{R}^d \times (0, \infty)$$

Finally:

$$\|u(\cdot, t)\|_{L^\infty} \leq \|f\|_{L^\infty} \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) dy ds = \|f\|_{L^\infty} t \xrightarrow{t \rightarrow 0} 0$$

■

Exercise 6.4 If f, g are given as above, then

$$u(x, t) = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy + \int_0^t \int_{\mathbb{R}^d} \Phi(x - y, t - s) f(y, s) ds$$

solves

$$\begin{cases} \partial_t u - \Delta u = f \\ u(\cdot, 0) = g \end{cases}$$

Remark 6.5 (Duhamel formula) Consider the ODE $\partial_t w(t) = Aw(t)$ for all $A \in \mathbb{R}$. Then the solution is

$$w(t) = e^{tA} w(0).$$

More generally: If $\partial_t w(t) = Aw(t) + f(t)$, then

$$\begin{aligned} \partial_t(e^{-tA} w(t)) &= e^{-tA} (\partial_t w(t) - Aw(t)) = e^{-tA} f(t) = e^{-tA} f(t) \\ \Rightarrow e^{-tA} w(t) &= w(0) + \int_0^t e^{-sA} f(s) ds \\ \Rightarrow w(t) &= e^{tA} w(0) + \int_0^t e^{(t-s)A} f(s) ds \end{aligned}$$

More generally, if A is an operator (independent of time) then:

$$\begin{aligned} \partial_t w(t) &= Aw(t) + f(t) \\ \Rightarrow w(t) &= e^{tA} w(0) + \int_0^t e^{(t-s)A} f(s) ds \end{aligned}$$

Application: If $A = \Delta$, then the operator $e^{t\Delta}$ has kernel

$$e^{t\Delta}(x, y) = \Phi(x - y, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}.$$

This is called the *heat kernel*.

Theorem 6.6 (L^2 -data) For every $g \in L^2(\mathbb{R}^d)$, define

$$u(t, x) = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy$$

Then $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$ and it solves the heat equation

$$\begin{cases} \partial_t u = \Delta_x u & \mathbb{R}^d \times (0, \infty) \\ \lim_{t \rightarrow 0} u(\cdot, t) = g & \text{in } L^2(\mathbb{R}^d) \end{cases}$$

Proof. Recall the heuristic computation from the heat equation using the Fourier transform

$$\begin{aligned}
& \partial_t u(x, t) = \Delta_x u(x, t) \\
\Leftrightarrow & \partial_t \hat{u}(k, t) = -|2\pi k|^2 \hat{u}(k, t) \\
\Leftrightarrow & \partial_t (e^{t|2\pi k|^2} \hat{u}(k, t)) = 0 \\
\Leftrightarrow & e^{t|2\pi k|^2} \hat{u}(k, t) = \hat{u}(k, 0) = \hat{g}(k) \\
\Leftrightarrow & \hat{u}(k, t) = e^{-t|2\pi k|^2} \hat{g}(k) = \hat{\Phi}(k, t) \hat{g}(k) = \widehat{\Phi \star g} \quad \blacksquare \\
\Leftrightarrow & u(x, t) = \Phi \star g = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy
\end{aligned}$$

Here we only need the direction \Leftarrow which is rigorous if $g \in L^2(\mathbb{R}^d)$. From the Fourier transform, it is also easy to check that $u(\cdot, t) \rightarrow g$ in L^2 as $t \rightarrow 0$ (exercise). To see the smoothness, note that for all $t > 0$, and for all $m \in \mathbb{N}$:

$$(1 + |2\pi k|^m) \hat{u}(k, t) = \underbrace{(1 + |2\pi k|^m) e^{-t|2\pi k|^2}}_{\in L^\infty} \underbrace{\hat{g}(k)}_{\in L^2} \in L^2$$

This implies $u(\cdot, t) \in H^m(\mathbb{R}^d)$ for all $m \geq 1$, so $u(\cdot, t) \in C^\infty(\mathbb{R}^d)$ by Sobolev embedding (see below). This argument can also be used to show that $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$ (exercise)

Theorem 6.7 (Sobolev embedding) If $m > \frac{d}{2}$, then $H^m(\mathbb{R}^d) \subseteq (C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$.

Proof. We write for all $u \in H^m(\mathbb{R}^d)$:

$$\hat{u}(k) = \underbrace{\hat{u}(k)(1 + |2\pi k|^m)}_{\in L^2 \text{ as } u \in H^m} \underbrace{\frac{1}{1 + |2\pi k|^m}}_{\in L^2 \text{ as } m > \frac{d}{2}}$$

This implies $\hat{u}(k) \in L^1(\mathbb{R}^d)$ and finally $u = (\hat{u})^\vee \in (C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$. ■

Exercise 6.8 (E 11.1) Let $g \in L^2(\mathbb{R}^d)$,

$$u(x, t) = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy, \quad \Phi(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$$

be the fundamental solution of the heat equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \forall (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, t) \rightarrow g(x) & \text{as } t \rightarrow 0. \end{cases}$$

Prove that

- a) $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$.
- b) $\|u(\cdot, t) - g\|_{L^2(\mathbb{R}^d)} \xrightarrow{t \rightarrow 0^+} 0$
- c) If $g \in H^1(\mathbb{R}^d)$, then $\|u(\cdot, t) - g\|_{L^2(\mathbb{R}^d)} \leq C\sqrt{t}$ as $t \rightarrow 0^+$.

Solution. a) We prove for all $t > 0$:

$$u(x, t) \in \bigcap_{m \geq 1} H^m(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$$

We use the Fourier transform:

$$\hat{\Phi}(k, t) = e^{-t|2\pi k|^2}$$

Recall $\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$. From this we get $\widehat{e^{-\pi\lambda|x|^2}} = \lambda^{-\frac{d}{2}} e^{-\frac{\pi|k|^2}{\lambda}}$. Then:

$$\widehat{e^{-\frac{|x|^2}{4t}}} = e^{-\pi\frac{1}{4\pi t}|x|^2} = \left(\frac{1}{4\pi t}\right)^{-\frac{d}{2}} e^{-\pi|k|^2 4\pi t} = (4\pi t)^{\frac{d}{2}} e^{-t|2\pi k|^2}$$

Hence:

$$\hat{u}(k, t) = \hat{\Phi}(k, t)\hat{g}(k) = e^{-t|2\pi k|^2}\hat{g}(k) \in L^1(\mathbb{R}^d, dk) \quad \forall t > 0$$

This implies:

$$u(x, t) = \int_{\mathbb{R}^d} e^{-t|2\pi k|^2} \hat{g}(k) e^{2\pi i k x} dk \quad \forall (x, t) \in \mathbb{R}^d \times (0, \infty)$$

Consequently:

$$D_x^\alpha u(x, t) = \int_{\mathbb{R}^d} \underbrace{e^{-t|2\pi k|^2} \hat{g}(k) (2\pi i k)^\alpha}_{L^1(\mathbb{R}^d, dk)} e^{2\pi i k x} dk \in C(\mathbb{R}^d, (0, \infty))$$

$$D_t^\alpha u(x, t) = \int_{\mathbb{R}^d} (-|2\pi k|^2)^\alpha e^{-t|2\pi k|^2} \hat{g}(k) e^{2\pi i k x} dk \in C(\mathbb{R}^d, (0, \infty))$$

Also:

$$\begin{aligned} & \partial_t u - \Delta_x u \\ &= \int_{\mathbb{R}^d} -|2\pi k|^2 e^{-t|2\pi k|^2} \hat{g}(k) e^{2\pi i k x} dk + \int_{\mathbb{R}^d} e^{-t|2\pi k|^2} \hat{g}(k) |2\pi i k|^2 e^{2\pi i k x} dk \\ &= 0 \end{aligned}$$

b) Finally:

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x, t) - g(x)|^2 dx &= \int_{\mathbb{R}^d} |\hat{u}(k, t) - \hat{g}(k)|^2 dk \\ &= \int_{\mathbb{R}^d} \underbrace{|e^{-t|2\pi k|^2} - 1|^2}_{\in [0, 1]} \underbrace{|\hat{g}(k)|^2}_{\in L^1(\mathbb{R}^d)} dk \xrightarrow{t \rightarrow 0} 0 \end{aligned}$$

by dominated convergence. Now,

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x, t)|^2 dx &= \int_{\mathbb{R}^d} |\hat{u}(k, t)|^2 dk \\ &= \int_{\mathbb{R}^d} \underbrace{e^{-2t|2\pi k|^2}}_{\in [0, 1] \text{ and } \xrightarrow{t \rightarrow 0} 0} |\hat{g}(k)|^2 dk \xrightarrow{t \rightarrow \infty} 0 \end{aligned}$$

c) Assume $g \in H^1(\mathbb{R}^d) \Leftrightarrow \int_{\mathbb{R}^d} (1 + |2\pi k|^2) |\hat{g}(k)|^2 dk < \infty$. We claim for all $s \geq 0$ that $|1 - e^{-s}| \leq \min(1, Cs) \leq C\sqrt{s}$: We have that $s \mapsto \left|\frac{1-e^{-s}}{s}\right|$ is bounded and continuous in $[0, 1]$ as $\left|\frac{1-e^{-s}}{s}\right| \rightarrow 1$, so $\frac{1-e^{-s}}{s} \leq C$ for all $s \in [0, 1]$.

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x, t) - g(x)|^2 dx &= \int_{\mathbb{R}^d} \underbrace{\left|1 - e^{-t|2\pi k|^2}\right|^2}_{\leq C(t|2\pi k|^2)} |\hat{g}(k)|^2 dk \\ &\leq C \int_{\mathbb{R}^d} t|2\pi k|^2 |\hat{g}(k)|^2 dk \\ &\leq Ct \|g\|_{H^1}^2 \quad \forall t > 0 \end{aligned}$$

■

Step 1: Spectral problem:

$$\begin{cases} -\Delta u_n = \lambda_n u_n & \text{in } \Omega \\ u_n|_{\partial\Omega} = 0 \end{cases}$$

Lemma 6.9 There is a $\lambda_n > 0$, $\lambda_n \xrightarrow{n \rightarrow \infty} \infty$ and an orthonormal family $\{u_n\} \subseteq L^2(\Omega)$ s.t. $u_n \in H_0^1(\Omega) \cap C^\infty(\Omega)$ solving this eigenvalue equation.

Step 2:

$$\begin{cases} \partial_t - \Delta_x u = 0 \\ u(x, 0) = g(x) \end{cases} \Rightarrow \begin{cases} \partial_t \langle u_n, u \rangle_{L^2(\Omega)} = \langle u_n, \Delta_x u \rangle = \langle \Delta_x u_n, u \rangle = -\lambda_n \langle u_n, u \rangle \\ \langle u_n, u \rangle_{t=0} = \langle u_n, g \rangle \end{cases}$$

$$\Rightarrow \langle u_n, u \rangle = e^{-t\lambda_n} \langle u_n, g \rangle \quad \forall t > 0, \forall n = 1, 2, \dots$$

$$\Rightarrow u = \sum_{n=0}^{\infty} \langle u_n, u \rangle u_n = \sum_{n=0}^{\infty} e^{-t\lambda_n} \langle u_n, g \rangle u_n$$

Example 6.10 $\Omega = (0, 1)$,

$$\begin{cases} -u_n'' = \lambda_n u_n & \text{in } (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

has solution

$$\begin{cases} u_n(x) = \sqrt{2} \sin(\pi n x) & n = 1, 2, \dots \\ \lambda_n = (\pi n)^2 \end{cases}$$

has a solution:

$$u(x, t) = \sum_{n=1}^{\infty} e^{-t\lambda_n} \underbrace{\langle u_n, g \rangle}_{g_n} u_n(x) = \sum_{n=1}^{\infty} e^{-t\pi^2 n^2} g_n \sin(\pi n x),$$

$$\int_0^1 \sin(n\pi x)^2 dx = \frac{1}{2} \quad \forall n > 1$$

$$g_n = \sqrt{2} \langle u_n, g \rangle = 2 \int_0^1 \sin(\pi n x) g(x) dx$$

Exercise 6.11 (E 11.2) Consider the heat equation in a bounded domain

$$\begin{cases} \partial_t u(x, t) = \Delta_x u(x, t) & \forall x \in \Omega, t > 0 \\ u(x, t) = 0 & \forall x \in \partial\Omega, t > 0 \\ u(x, 0) = g(x) & \forall x \in \Omega \end{cases}$$

Let us focus on the simplest case $\Omega = (0, 1)$. Prove that for every $g \in C_c^1(0, 1)$, the function

$$u(x, t) = \sum_{n=1}^{\infty} g_n e^{-t\pi^2 n^2} \sin(n\pi x), \quad g_n = 2 \int_0^1 g(y) \sin(n\pi y) dy$$

is a classical solution to the above heat equation.

Solution. Direct proof of heat equation. $g \in C_c^1(0, 1) \subseteq H_0^1(0, 1)$, $\Rightarrow \sum_n \pi^2 n^2 |g_n|^2 = c \|g'\|_{L^2(0,1)}^2 < \infty$, so $\sum_n |g_n| < \infty$.

$$u(x, 0) = \underbrace{\sum_{n=1}^{\infty} g_n \sin(\pi n x)}_{\in C[0,1]} = g(x) \quad \forall x \in [0, 1]$$

From $u(x, t) = \sum_{n=1}^{\infty} e^{-t\pi^2 n^2} g_n \sin(\pi n x)$ we get

$$\begin{cases} \partial_t u(x, t) = \sum_{n=1}^{\infty} (-n^2 \pi^2) e^{-t\pi^2 n^2} g_n \sin(\pi n x) & \forall t > 0, \forall x \in (0, 1) \\ \Delta_x u(x, t) = \sum_{n=1}^{\infty} e^{-t\pi^2 n^2} g_n [-(\pi n)^2] \sin(\pi n x) & \forall t > 0, \forall x \in (0, 1) \end{cases}$$

So $\partial_t u - \Delta_x u = 0$ for all $t > 0, x \in (0, 1)$ ■

Exercise 6.12 (E 11.3) Let $g(t) = e^{-\frac{1}{t^2}}$ and denote $g^{(n)}(t)$ the n -th derivative of g . Define

$$u(x, t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \quad \forall x \in \mathbb{R}, t > 0$$

Prove that u is a classical solution to the heat equation

$$\begin{cases} \partial_t u(x, t) = \Delta_x u(x, t) & \forall x \in \mathbb{R}, t > 0 \\ \lim_{t \rightarrow 0} u(x, t) = 0 & \forall x \in \mathbb{R} \end{cases}$$

Solution. Formally:

$$\begin{cases} \partial_t u = \sum_{n=0}^{\infty} \frac{g^{(n+1)}(t)}{(2n)!} x^{2n} \\ -\Delta_x u = \sum_{n=1}^{\infty} \frac{g^{(n)}(t)}{(2n)!} (2n)(2n-1) x^{2n-2} = \sum_{n=1}^{\infty} \frac{g^{(n)}(t)}{(2n-2)!} x^{2n-2} = \sum_{m=0}^{\infty} \frac{g^{(m+1)}(t)}{(2m)!} x^{2m} \end{cases}$$

This implies $\partial_t u = \Delta_x u$ (if the series are convergent) $(x, t) \in B \times [\epsilon, \frac{1}{\epsilon}]$ for $B \subset \mathbb{R}$ bounded, $\epsilon > 0$. Also

$$\begin{aligned} g(t) &= e^{-\frac{1}{t^2}} \xrightarrow{t \rightarrow 0^+} e^{-\infty} = 0 \\ g'(t) &= e^{-\frac{1}{t^2}} \left(\frac{2}{t^3} \right) \xrightarrow{t \rightarrow 0^+} 0 \\ g''(t) &= e^{-\frac{1}{t^2}} \left(-\frac{3!}{t^4} + \frac{2}{t^3} \right) \xrightarrow{t \rightarrow 0^+} 0 \\ g'''(t) &= e^{-\frac{1}{t^2}} \left(\frac{4!}{t^5} - \frac{3!}{t^4} + \frac{2}{t^3} \right) \end{aligned}$$

Let's prove the convergence of the series:

$$u(x, t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}$$

converges absolutely for $|x| \leq C, t \in [\epsilon, \frac{1}{\epsilon}], \epsilon > 0$. By induction,

$$g^{(n)}(t) = e^{-\frac{1}{t^2}} \underbrace{\left(\frac{(n+1)!}{t^{n+2}} - \frac{n!}{t^{n+1}} + \frac{(n+1)!}{t^n} - \dots \right)}_{\text{pol in } (\frac{1}{t}), \text{ all cos bounded by } (n+1)} (-1)^{n-1}$$

This implies

$$|g^{(n)}(t)| \leq e^{-\frac{1}{t^2}} [(n+2)!] \left(\frac{1}{t^{n+2}} + 1 \right), \quad \frac{1}{t^s} \leq \left(\frac{1}{t^{n+2}} + 1 \right) \forall s = 0, 1, \dots, n+2$$

Thus

$$\sum_{n \geq 0} \left| \frac{g^{(n)}}{(2n)!} x^{2n} \right| \leq \sum_{n \geq 0} e^{-\frac{1}{t^2}} \frac{(n+2)!}{(2n)!} \left(\frac{1}{t^{n+2}} + 1 \right) x^{2n} \quad (1)$$

$$\begin{aligned} \sum_{n \geq 0} \frac{(n+2)!}{(2n)!} x^{2n} &= \sum_{n \geq 0} \frac{1}{(n+3)(n+4) \cdots (2n)} \\ &\leq \sum_{n \geq 0} \frac{1}{n^{n-2}} x^{2n} \\ &\leq \sum_{n \geq M} + \sum_{n \geq M} \frac{1}{M^{n-2}} x^{2n} \\ &= M^2 \sum_n \left(\frac{x^2}{M} \right)^n \\ &\leq m^2 \frac{1}{1 - \left(\frac{x^2}{M} \right)} \end{aligned}$$

$$(2) \quad t \in [\epsilon, \frac{1}{\epsilon}], \text{ so } \frac{1}{t} \leq \frac{1}{\epsilon}, \text{ so } \frac{1}{t^{n+2}} \leq \frac{1}{\epsilon^{n+2}} \longrightarrow \sum_{n \geq 0} \frac{(n+2)!}{(2n)!} \frac{1}{t^{n+2}} x^{2n} \leq \sum_{n \geq 0} \frac{1}{n^{n-2}} \frac{1}{\epsilon^{n-2}} x^{2n}$$

■

Remark 6.13 $|u(x, t)| \leq \exp\left(\frac{cx^2}{t}\right) \rightsquigarrow$ unphysical solution. Violates $|u(x, t)| \leq Ce^{C|x|^2}$ for all $\forall(x, t) \in \mathbb{R} \times [0, T]$

Exercise 6.14 (Bonus 10) Consider

$$u(x, t) = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy$$

where $\Phi(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{x^2}{4t}}$. Assume $g \in C_c^\infty(\mathbb{R}^d)$. Prove or disprove that

$$\|u(\cdot, t) - g\|_{L^2(\mathbb{R}^d)} \leq C_n t^n$$

as $t \rightarrow 0^+$ for all $n = 1, 2, \dots$

6.2 Maximum Principle

Recall the Poisson equation $-\Delta u \leq 0$ in $\Omega \subseteq \mathbb{R}^d$ open, bounded. Then

$$\sup_{\bar{\Omega}} u(x) = \sup_{\partial\Omega} u(x).$$

Theorem 6.15 (Maximum principle for bounded sets) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. Let $T > 0$ and define

$$\begin{aligned} \Omega_T &= \Omega \times (0, T), \\ \partial^* \Omega_T &= (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times [0, T]) \end{aligned}$$

If $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ solves $\partial_t u - \Delta_x u \leq 0$ in Ω_T , then

$$\max_{\bar{\Omega}_T} u = \max_{\partial^* \Omega_T} u.$$

Proof. We will use Hopf's argument which is simpler than the mean-value theorem (there exists a mean-value theorem for heat equation, but it is complicated and we will not discuss it). Firstly, to illustrate the principle, we prove the maximum principle for the Poisson Equation: Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$

Step 1) Assume $\Delta u > 0$ in Ω . Since $\bar{\Omega}$ is compact, there is a $x_0 \in \bar{\Omega}$ s.t. $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$. We prove that $x_0 \in \partial\Omega$. In fact, if $x_0 \in \Omega$, then since x_0 is a (local) maximizer of u in Ω , we have $\Delta u(x_0) \leq 0$, which contradicts to the assumption that $\Delta u > 0$ in Ω . Thus $x_0 \in \partial\Omega$, and hence

$$\max_{x \in \bar{\Omega}} u(x) = u(x_0) \leq \max_{x \in \partial\Omega} u(x).$$

Step 2) Now assume $\Delta u \geq 0$ in Ω . Define

$$u_\epsilon(x) = u(x) + \epsilon|x|^2, \quad \epsilon > 0.$$

Then, $\Delta u_\epsilon > 0$ in Ω , hence by Step 1 and

$$u \leq u_\epsilon \leq u + \epsilon \sup_{x \in \bar{\Omega}} |x|^2$$

we have

$$\begin{aligned} \max_{x \in \bar{\Omega}} u(x) &\leq \max_{x \in \bar{\Omega}} u_\epsilon(x) \leq \max_{x \in \partial\Omega} u_\epsilon(x) \\ &\leq \max_{x \in \partial\Omega} u(x) + \epsilon \left(\sup_{x \in \bar{\Omega}} |x|^2 \right) \xrightarrow{\epsilon \rightarrow 0} \max_{x \in \partial\Omega} u(x) \end{aligned}$$

Proof for the heat equation:

Step 1) Assume $u \in C_1^2(\Omega \times (0, T]) \cap C(\bar{\Omega} \times [0, T])$ and

$$\partial_t u - \Delta_x u < 0$$

in $\Omega \times (0, T]$. Then, because of compactness, there is $(x_0, t_0) \in \bar{\Omega} \times [0, T]$ s.t.

$$u(x_0, t_0) = \max_{(x,t) \in \bar{\Omega} \times [0,T]} u(x, t).$$

We prove that $(x_0, t_0) \in \partial^* \Omega_T$. Assume by contradiction that $(x_0, t_0) \notin \partial^* \Omega_T$, then $x_0 \in \Omega$ and $t_0 \in (0, T]$. Since $x \mapsto u(x, t_0)$ has a (local) maximizer $x_0 \in \Omega$ we have that $\Delta_x u(x_0, t_0) \leq 0$. Since $t \mapsto u(x_0, t)$ has a (local) maximizer $t_0 \in (0, T]$ we have that $\partial_t u(x_0, t_0) \geq 0$. This implies:

$$(\partial_t u - \Delta_x u)(x_0, t_0) \geq 0$$

which is a contradiction to the assumption. Thus $(x_0, t_0) \in \partial^* \Omega_T$, i.e. $\max_{\bar{\Omega}_T} u = \max_{\partial^* \Omega_T} u$.

Step 2) Assume $u \in C_1^2(\Omega \times (0, T)) \cap C(\bar{\Omega} \times [0, T])$ and

$$\partial_t u - \Delta_x u \leq 0 \quad \text{in } \Omega \times (0, T).$$

Let $\tilde{T} \in (0, T)$ and for $\epsilon > 0$:

$$u_\epsilon(x, t) = u(x, t) + \epsilon|x|^2.$$

Then: $u_\epsilon \in C_1^2(\Omega \times (0, T']) \cap C(\bar{\Omega} \times [0, \tilde{T}])$ and $\partial_t u_\epsilon - \Delta_x u_\epsilon < 0$ in $\Omega \times (0, \tilde{T}]$.
By Step 1:

$$\begin{array}{ccc} & \max_{\Omega_{\tilde{T}}} u_\epsilon \leq \max_{\partial^* \Omega_{\tilde{T}}} u_\epsilon & \\ \xRightarrow{\epsilon \rightarrow 0} & \max_{\Omega_{\tilde{T}}} u \leq \max_{\partial^* \Omega_{\tilde{T}}} u & \\ \xRightarrow{\tilde{T} \rightarrow T} & \max_{\Omega_T} u \leq \max_{\partial^* \Omega_T} u & \blacksquare \end{array}$$

Theorem 6.16 (Maximum principle for $\Omega = \mathbb{R}^d$) Let $\Omega_T = \mathbb{R}^d \times (0, T)$, $\bar{\Omega}_T = \mathbb{R}^d \times [0, T]$. Let $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ such that

- $\partial_t u - \Delta_x u \leq 0$ in Ω_T
- $u(x, t) \leq M e^{M|x|^2}$ for all $(x, t) \in \bar{\Omega}_T$

Then

$$\sup_{(x,t) \in \bar{\Omega}_T} u(x, t) = \sup_{x \in \mathbb{R}^d} u(x, 0).$$

Proof.

Step 1: For all $y \in \mathbb{R}^d$ and $\epsilon > 0$ define

$$v(x, t) = u(x, t) - \frac{\epsilon}{(T + \epsilon - t)^{\frac{d}{2}}} \exp\left(\frac{|x - y|^2}{4(T + \epsilon - t)}\right)$$

This implies

$$\partial_t v - \Delta_x v = \partial_t u - \Delta_x u \leq 0$$

in Ω_T . For $U = B(y, r)$, $U_T = U \times (0, T)$, $\bar{U}_T = \bar{U} \times [0, T]$, $\partial^* U_T = (U \times \{0\}) \cup (\partial U \times [0, T])$, by the maximum principle for U bounded we have

$$\max_{U_T} v \leq \max_{\partial^* U_T} v.$$

Let us bound $\max_{\partial^* U_T} v$.

- On $U \times \{0\}$ we use $v \leq u$ and hence

$$\max_{x \in \bar{U}} v(x, 0) \leq \max_{x \in \bar{U}} u(x, 0) \leq \max_{x \in \mathbb{R}^d} u(x, 0).$$

- On $\partial U \times [0, T]$ we use $|x - y| = r \Rightarrow |x| \leq |y| + r$.

$$\begin{aligned} v(x, t) &= u(x, t) - \frac{\epsilon}{(T + \epsilon - t)^{\frac{d}{2}}} \exp\left(\frac{|x - y|^2}{4(T + \epsilon - t)}\right) \\ &\leq M e^{M(|y|+r)^2} - \frac{\epsilon}{(T + \epsilon)^{\frac{d}{2}}} \exp\left(\frac{r^2}{4(T + \epsilon)}\right) \xrightarrow{r \rightarrow \infty} -\infty \end{aligned}$$

if $M < \frac{1}{4(T+\epsilon)}$. In particular, we can choose r large s.t.

$$\max_{\substack{x \in \partial U \\ t \in [0, T]}} v(x, t) \leq \max_{x \in \mathbb{R}^d} u(x, 0).$$

In summary, if $M < \frac{1}{4(T+\epsilon)}$, then:

$$u(y, t) - \frac{\epsilon}{(T + \epsilon - t)^{\frac{d}{2}}} = v(y, t) \leq \max_{\bar{U}_T} v \leq \max_{x \in \mathbb{R}^d} u(x, 0)$$

This holds for all $(y, t) \in \mathbb{R}^d \times [0, T]$. Thus,

$$\max_{\mathbb{R}^d \times [0, T]} u \leq \frac{\epsilon}{(T + \epsilon - t)^{\frac{d}{2}}} + \max_{x \in \mathbb{R}^d} u(x, 0)$$

Taking $\epsilon \rightarrow 0$ we conclude that if $M < \frac{1}{4T}$,

$$\max_{\mathbb{R}^d \times [0, T]} u \leq \max_{x \in \mathbb{R}^d} u(x, 0)$$

Step 2: For general T , we denote $T_1 = \frac{T}{N}$, $N \in \mathbb{N}$ s.t. $M < \frac{4}{T_1}$. Then by step 1:

$$\begin{aligned} \max_{\mathbb{R}^d \times [0, T_1]} u &\leq \max_{x \in \mathbb{R}^d} u(x, 0) \\ \max_{\mathbb{R}^d \times [T_1, 2T_1]} u &\leq \max_{x \in \mathbb{R}^d} u(x, T_1) \leq \max_{x \in \mathbb{R}^d} u(x, 0) \\ &\vdots \\ \max_{\mathbb{R}^d \times [(N-1)T_1, NT_1]} u &\leq \max_{x \in \mathbb{R}^d} u(x, (N-1)T_1) \leq \max_{x \in \mathbb{R}^d} u(x, 0) \\ \rightsquigarrow \max_{\mathbb{R}^d \times [0, T]} u &\leq \max_{x \in \mathbb{R}^d} u(x, 0) \end{aligned} \quad \blacksquare$$

Remark 6.17 The condition $u \leq Me^{M|x|^2}$ is necessary, otherwise there are solutions $u \neq 0$ s.t. $u(x, 0) = 0$

Theorem 6.18 (Uniqueness) If $u \in C_1^2(\mathbb{R}^d \times (0, T)) \cap C(\mathbb{R}^d \times [0, T])$ and

$$\begin{aligned} u(x, t) &\leq Me^{M|x|^2} && \text{in } \mathbb{R}^d \times [0, T], \\ \partial_t u - \Delta_x u &= 0 && \text{in } \mathbb{R}^d \times (0, T), \\ u(x, 0) &= 0 && \text{in } \mathbb{R}^d \end{aligned}$$

Then $u = 0$ in $\mathbb{R}^d \times [0, T]$.

Proof. Use the maximum principle for u and $-u$. ■

Remark 6.19 If $u(\cdot, t) \in L^2(\mathbb{R}^d)$, the proof of uniqueness can be done without the maximum principle. Heuristically:

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(x, t)|^2 dt = 2 \int_{\mathbb{R}^d} (\partial_t u) u dx = 2 \int_{\mathbb{R}^d} \Delta_x u u dx = -2 \int_{\mathbb{R}^d} |\nabla_x u|^2 dx \leq 0$$

This implies

$$e(t) := \int_{\mathbb{R}^d} |u(x, t)|^2 dx$$

is decreasing. Hence, if $e(0) = 0$, then $e(t) = 0$ for all $t \geq 0$. This argument will be helpful below for the heat backward equation.

Remark 6.20 The heat equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(t=0) = g \end{cases}$$

is a well-posed problem:

- Existence
- Uniqueness
- Stability (solution depends continuously on data)

For the latter issue, by the maximum principle we have

$$\|u(\cdot, t)\|_{L^\infty} \leq \|u(\cdot, 0)\|_{L^\infty} \quad \forall t$$

or in the L^2 -situation:

$$\|u(\cdot, t)\|_{L^2} \leq \|u(\cdot, 0)\|_{L^2} \quad \forall t$$

On the other hand, the heat backward equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(t=T) = g \end{cases}$$

is *not* well-posed.

- Non-Existence: In general, the existence requires some special property on g , e.g. g is very smooth (only $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ or $g \in L^2(\mathbb{R}^d)$ is not enough)
- Uniqueness: On the other hand, the uniqueness still holds.

Lemma 6.21 If $e \in C^2(0, T)$, $e(t) \geq 0$, $e'(t) \leq 0$, $e''(t) \geq 0$ and $|e'(t)|^2 \leq e(t)e''(t)$ for $t \in [0, T]$ and $e(T) = 0$, then $e \equiv 0$.

Proof. Since e is monotonically decreasing and $e(T) = 0$ there is a $t_0 \in [0, T]$ s.t. $e(t_0) = 0$ and $e(t) > 0$ if $t \leq t_0$. We need to prove that $t_0 = 0$. Assume by contradiction $0 < t_0 \leq T$, then for $t \in (0, t_0)$ define $f(t) := \log e(t)$. Then

$$\begin{aligned} f'(t) &= \frac{e'(t)}{e(t)} \\ \Rightarrow f''(t) &= \frac{e''(t)e(t) - |e'(t)|^2}{e(t)^2} \geq 0 \end{aligned}$$

This means that f is convex, so for all $t_1, t_2 \in (0, t_0)$ and $\tau \in (0, 1)$:

$$\begin{aligned} f(\tau t_1 + (1-\tau)t_2) &\leq \tau f(t_1) + (1-\tau)f(t_2) \\ \Rightarrow e(\tau t_1 + (1-\tau)t_2) &\leq e(t_1)^\tau e(t_2)^{1-\tau} \end{aligned}$$

Now, $e(\tau t_1 + (1-\tau)t_2) \xrightarrow{t_2 \rightarrow t_0} 0$ and $\tau \rightarrow 1$ implies $e(t_1) = 0$ for all $t_1 \in (0, t_0)$ which is a contradiction. ■

Theorem 6.22 If $u \in C_1^2(\mathbb{R}^d \times [0, T]) \cap C^1(H^1(\mathbb{R}^d) \times [0, T])$ and

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ u(x, T) = 0 \end{cases}$$

Then $u = 0$ in $\mathbb{R}^d \times [0, T]$.

Proof. Recall

$$e(t) = \int_{\mathbb{R}^d} |u(x, t)|^2 dx.$$

Then,

$$\begin{aligned} e'(t) &= 2 \int_{\mathbb{R}^d} u \partial_t u dx = 2 \int_{\mathbb{R}^d} u \Delta_x u dx = -2 \int_{\mathbb{R}^d} |\nabla_x u|^2 dx \\ e''(t) &= -4 \int_{\mathbb{R}^d} \nabla_x u \nabla_x (\partial_t u) = 4 \int_{\mathbb{R}^d} \Delta_x u \partial_t u dx = 4 \int_{\mathbb{R}^d} |\Delta_x u|^2 dx \geq 0 \end{aligned}$$

and hence

$$|e'(t)|^2 = 4 \left| \int_{\mathbb{R}^d} u \Delta_x u dx \right|^2 \leq 4 \left(\int_{\mathbb{R}^d} |u|^2 dx \right) \left(\int_{\mathbb{R}^d} |\Delta_x u|^2 dx \right) = e(t) e''(t)$$

Then the statement follows with lemma 6.21. ■

Some remarks about the eat equation in unbounded domains:

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = 0 & \text{(i.e. } \lim_{t \rightarrow 0} u(x, t) = 0 \forall x \in \mathbb{R}^d) \end{cases}$$

There is a classical solution $0 \neq u \in C^1(\mathbb{R}^d \times (0, \infty))$. An example is

$$u(x, t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \quad g(t) = e^{-\frac{1}{t^2}}$$

(s.t. $g \rightarrow 0$ as $t \rightarrow 0$). Note

$$\begin{aligned} g(t) &= e^{-\frac{1}{t^2}}, \\ g'(t) &= \frac{2}{t^3} g(t) \\ g''(t) &= \left(\frac{2}{t^3} \right)' g(t) + \frac{2}{t^3} \frac{2}{t^3} g(t) \\ g^{(n)}(t) &= P_n \left(\frac{1}{t} \right) g(t) \end{aligned}$$

where

$$\begin{cases} P_0 = 1 \\ P_{n+1} \left(\frac{1}{t} \right) = \left(P_n \left(\frac{1}{t} \right) \right)' + \left(\frac{2}{t^3} \right) P_n \left(\frac{1}{t} \right) = A_1 P_n + A_2 P_n, \\ P_{n+1} = (A_1 + A_2) P_n = (A_1 + A_2)(A_1 + A_2) P_{n-1} = \end{cases} \begin{cases} A_1 = \partial_t \\ A_2 = \frac{2}{t^3} \end{cases}$$

This implies:

$$P_n = (A_1 + A_2)^n P_0 = \sum_{\sigma \in \{1, 2\}^n} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)} P_0$$

$$A_1 \left(\frac{\alpha}{t^s} \right) = \frac{-s\alpha}{t^{s+1}} \rightarrow A_1$$

Multiple coefficients by a factors and + power by 1

$$A_2 \left(\frac{\alpha}{t^s} \right) = \frac{2\alpha}{t^{s+3}} \rightarrow A_2$$

Mul Cof by a factor 2 and + power by 3

$$\left| \underbrace{A_{\sigma(1)} \cdots A_{\sigma(n)}, 1}_{k \text{ times } A_2, n-k \text{ times } A_1} \right| \leq \frac{2^k}{t^{3k}} \leq \frac{2^k}{t^{3k}} \frac{(3n)^{n-k}}{t^{n-k}} = \frac{2^k (3n)^{n-k}}{t^{n+2k}}$$

This implies

$$\left| P_n \left(\frac{1}{t} \right) \right| \leq \max_{0 \leq k \leq n} \frac{2^n 2^k (3n)^{n-k}}{t^{n+2k}}$$

Thus:

$$\begin{aligned} \sum_n \left| \frac{g^{(n)}(t)}{(2n)!} x^{2n} \right| &\leq \sum_n \max_{0 \leq k \leq n} \frac{2^n 2^k (3n)^{n-k}}{t^{n+2k} (2n)!} \frac{e^{-\frac{1}{t^2}}}{1} x^{2n} \\ &\leq \sum_n \max \frac{2^n 2^k (3n)^{n-k}}{t^{n+2k} (2n)!} (k!) (2t^2)^k e^{-\frac{1}{2t^2}} x^{2n} \\ &= \sum_n \frac{2^n 2^k 2^k (3n)^{n-k} (k!)}{(2n)! t^n} e^{-\frac{1}{2t^2}} x^{2n} \\ &\leq \sum_n \frac{(c_n)^n}{(2n)! t^n} e^{-\frac{1}{2t^2}} x^{2n} \\ &\leq \sum_n \frac{c^n}{n! t^n} e^{-\frac{1}{2t^2}} x^{2n} \\ &\leq \sum_n e^{\frac{cx^2}{t} - \frac{1}{2t^2}} \end{aligned}$$

Where we used that

$$e^s = \sum_k \frac{s^k}{k!} \geq \frac{s^k}{k!}$$

for all $s \geq 0$ implies

$$e^{-\frac{1}{2t^2}} = \frac{1}{e^{\frac{1}{2t^2}}} \leq \frac{1}{\left(\frac{1}{2t^2}\right)^{\frac{1}{k!}}} = k! (2t^2)^k.$$

We conclude:

- $u(x, t)$ is well-defined, $x \in \mathbb{R}^d$, $t > 0$ real? to heat equation.
- $u(x, t) \rightarrow 0$ as $t \rightarrow 0$ for all $x \in \mathbb{R}^d$.

Exercise 6.23 (E 12.1) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$. Assume that $x_0 \in \Omega$ is a local maximizer of u , namely there exists some $r > 0$ such that $u(x_0) \geq u(x)$ for all $x \in B_r(x_0) \subseteq \Omega$.

- (a) Prove that the Hessian matrix $H = (D^\alpha u(x_0))_{|\alpha|=2}$ is negative semi-definite, namely

$$yHy \leq 0$$

for all $y \in \mathbb{R}^d$.

- (b) Prove that $\Delta u(x_0) \leq 0$

Hint: Recall that we used (b) for the maximum principle by Hopf's method.

Solution. (a) In 1D this is obvious. If x_0 is a local minimizer of u , then $u'(x_0) = 0, u''(x_0) \leq 0$ (Taylor expansion).

In d dimensions:

$$\phi(t) = u(x_0 + t\xi) \quad \xi \in \mathbb{R}^d, t \in \mathbb{R}, |t| \text{ small}$$

So 0 is a local maximizer of ϕ . This implies

$$0 = \phi'(0) = \nabla u(x_0)\xi \quad \forall \xi \in \mathbb{R}^d \Rightarrow H \leq 0$$

$$\begin{aligned} \phi''(0) &= \lim_{t \rightarrow 0} \frac{\phi'(t) - \phi'(0)}{t} = \lim_{t \rightarrow 0} \frac{(\nabla u(x_0 + t\xi) - \nabla u(x_0))\xi}{t} \\ &= \lim_{t \rightarrow 0} \sum_{i=1}^d \frac{(\partial_i u(x_0 + t\xi) - \partial_i u(x_0))\xi_i}{t} = \sum_{i=1}^d \sum_{j=1}^d \partial_j \partial_i u(x_0) \xi_j \xi_i = \langle \xi, H\xi \rangle, \end{aligned}$$

$$H = (\partial_i \partial_j u(x_0))_{i,j=1}^d.$$

(b) Consequently

$$\Delta u(x_0) = \sum_{i=1}^d \partial_i \partial_i u(x_0) = \text{Tr}(H) \leq 0 \quad \blacksquare$$

Exercise 6.24 (E 12.2) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. We prove the maximum principle for a general elliptic operator

$$Lu(x) = \sum_{i,j=1}^d a_{ij}(x) \partial_i \partial_j u(x) + \sum_{i=1}^d b_i(x) \partial_i u(x),$$

$a_{ij}, b_i \in C(\bar{\Omega})$, $A(x) = (a_{ij}(x))_{i,j=1}^d \geq \mathbb{1}$ (as matrices). Prove that if $Lu(x) \geq 0$ for all $x \in \Omega$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then

$$\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

Solution.

Step 1: Assume $Lu(x) > 0$ for all $x \in \Omega$: Since $u \in C(\bar{\Omega})$ there is a $x_0 \in \bar{\Omega}$ s.t.

$$u(x_0) = \max_{x \in \bar{\Omega}} u(x).$$

We prove $x_0 \in \partial\Omega$. Assume by contradiction that $x_0 \notin \partial\Omega$, so $x_0 \in \Omega$ is a local maximizer. We prove $Lu(x_0) \leq 0$. Note:

$$\begin{aligned} Lu(x_0) &= \sum_{i,j=1}^d a_{ij}(x_0) \partial_i \partial_j u(x_0) + \sum_{i=1}^d b_i(x_0) \partial_i u(x_0) \\ &= \text{Tr}[A(x_0)H(x_0)] + B(x_0) \underbrace{\nabla u(x_0)}_{=0} \leq 0 \quad \nexists \end{aligned}$$

$$A(x_0) = (a_{ij}(x_0))_{i,j=1}^d, B(x_0) = (b_i(x_0))_{i=1}^d, \text{ where } \text{Tr}[AH] = \sum_i (AH)_{ii} = \sum_i \sum_j A_{ij} H_{ji}$$

General fact: If $A \geq 0, B \geq 0$ (matrices), then $\text{Tr}(AB) \geq 0$.

$$\bullet A = (\sqrt{A})^2 \Rightarrow \text{Tr}(AB) = \text{Tr}((\sqrt{A})^2 B) = \text{Tr}(\underbrace{\sqrt{A} B \sqrt{A}}_{\geq 0}) \geq 0$$

- Spectral theorem: $A \geq 0$, then there are eigenvectors (α_i) and eigenvalues $\lambda_i \geq 0$ s.t.

$$\text{Tr}(AB) = \sum_i \langle \alpha_i, AB\alpha_i \rangle = \sum_i \underbrace{\lambda_i}_{\geq 0} \underbrace{\langle \alpha_i, B\alpha_i \rangle}_{\geq 0} \geq 0$$

- General Case: $Lu(x) \geq 0$ for all $x \in \Omega$. Assume that there is a $v \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t. $Lv(x) > 0$ for all $x \in \Omega$. Define for all $\epsilon > 0$ $u_\epsilon = u + \epsilon v$. Then $Lu_\epsilon(x) = Lu(x) + \epsilon Lv(x) > 0$ for all $x \in \Omega$. By Step 1,

$$\begin{aligned} \max_{x \in \bar{\Omega}} u_\epsilon(x) &\leq \max_{x \in \partial\Omega} u_\epsilon(x) \\ \xrightarrow{\epsilon \rightarrow 0} \max_{x \in \bar{\Omega}} u(x) &\leq \max_{x \in \partial\Omega} u(x) \end{aligned}$$

What v ? First $v(x) = x^2 = x_1^2 + \dots + x_d^2$,

$$Lv(x) = \sum_{ij} a_{ij}(x) 2\delta_{ij} + \sum_i b_i(x) 2x_i$$

not clear to be ≥ 0 .

$$\begin{aligned} v(x) &= x^{2n} \quad n \text{ large} \\ v(x) &= x_1^{2n} \longrightarrow Lv(x) = a_{11}(x) 2n(2n+1)x_1^{2n-2} + b_1(x) 2nx_1^{2n-1} \\ &\geq 2nx_1^{2n-2} [(2n-1) + \underbrace{b_1(x)x_1}_{\substack{\text{b.d. in } \bar{\Omega} \\ > 0}}] \geq 0 \quad \forall x \in \bar{\Omega} \end{aligned}$$

if n is large enough.

$$v(x) = (x_1 + R)^{2n}$$

where $R > 0$ large s.t. $x_1 + R \geq 1$ for all $\forall x \in \bar{\Omega}$. This implies

$$Lv(x) \geq 2n \underbrace{(x_1 + R)^{2n-2}}_{> 0} \underbrace{[2n-1 + b_1(x)(x_1 + R)]}_{> 0} > 0$$

for all $x \in \bar{\Omega}$ if n is large. ■

Exercise 6.25 (E 12.3) Consider the inhomogeneous heat equation

$$\begin{cases} \partial_t u - \Delta_x u = f(x, t) & \text{in } \mathbb{R}^d \times (0, T) \\ u(t=0) = g & \text{in } \mathbb{R}^d \end{cases},$$

$f \in C_1^2(\mathbb{R}^d \times (0, T))$ and compactly supported and $g \in C(\mathbb{R}^d \times [0, T]) \cap L^\infty(\mathbb{R}^d \times [0, T])$. Assume that there exists a solution $u \in C_1^2(\mathbb{R}^d \times (0, T)) \cap C(\mathbb{R}^d \times [0, T])$ satisfying

$$u(x, t) \leq Me^{M|x|^2}, \quad (x, t) \in \mathbb{R}^d \times [0, T].$$

Prove that

$$\max_{(x,t) \in \mathbb{R}^d \times [0,T]} |u(x, t)| \leq \|g\|_{L^\infty} + T\|f\|_{L^\infty}.$$

Solution.

Step 1: There is at most one solution u .

Step 2:

$$u(x, t) = \int_{\mathbb{R}^d} \phi(x - y, t) g(y) dy + \int_0^t \int_{\mathbb{R}^d} \phi(x - y, t - s) f(y, s) dy ds$$

This implies:

$$\begin{aligned} \|u\|_{L^\infty} &\leq \int_{\mathbb{R}^d} \phi(x - y, t) \|g\|_{L^\infty} dy + \int_0^t \int_{\mathbb{R}^d} \phi(x - y, t - s) \|f\|_{L^\infty} dy ds \\ \Rightarrow \|u\|_{L_{x,t}^\infty} &\leq \int_{\mathbb{R}^d} \phi(x - y, t) \|g\|_{L^\infty} dy + \int_0^T \int_{\mathbb{R}^d} \phi(x - y, t - s) \|f\|_{L^\infty} dy ds \\ &= \|g\|_{L_x^\infty} + T \|f\|_{L_{x,t}^\infty} \end{aligned}$$

This is optimal! E.g. $g = 0, f = 1, u(x, t) = u(t)$.

$$\begin{cases} u' = 1 \\ u(0) = 0 \end{cases} \Rightarrow u(t) = t$$

■

Exercise 6.26 (Bonus 11) Denote for all $u \in C^2(\Omega) \cap C(\bar{\Omega})$:

$$Lu(x) = \sum_{i,j=1}^d a_{ij}(x) \partial_i \partial_j u(x)$$

where $a_{ij} \in C(\bar{\Omega})$ s.t. $A(x) = (a_{ij}(x)) \geq 1$. Prove that if $\Omega \subseteq \mathbb{R}^d$ is open and bounded, $u \in C_1^2(\bar{\Omega} \times [0, T])$ and

$$\begin{cases} \partial_t u - Lu \leq 0 & \text{in } \Omega \times (0, T) \\ u(t = 0) = 0 \\ u(x \in \partial\Omega) = 0 \end{cases}$$

Prove that $u(x, t) \leq 0$ for all $(x, t) \in \bar{\Omega} \times [0, T]$.

6.3 Backward heat equation

Theorem 6.27 (Instability)

There exist functions $u_\epsilon \in C_1^2(\mathbb{R}^d \times (0, T)) \cap C^1(H^1(\mathbb{R}^d) \times [0, T])$ s.t.

$$\partial_t u_\epsilon - \Delta_x u_\epsilon = 0 \quad \text{in } \mathbb{R}^d \times [0, T]$$

with:

$$\|u_\epsilon(\bullet, T)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \rightarrow 0^+} 0, \quad \|u_\epsilon(\bullet, 0)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \rightarrow 0^+} \infty.$$

Proof. Recall by Fourier Transform

$$\begin{aligned} &\partial_t \hat{u}(k, t) + |2\pi k|^2 \hat{u}(k, t) = 0 \\ \Leftrightarrow &\partial_t (e^{|2\pi k|^2 t} \hat{u}(k, t)) = 0 \\ \Rightarrow &e^{|2\pi k|^2 t} \hat{u}(k, t) = u(k, 0) \\ \Rightarrow &\hat{u}(k, t) = e^{-t|2\pi k|^2} \hat{u}(k, 0) \\ \Rightarrow &\hat{u}(k, 0) = e^{T|2\pi k|^2} \hat{u}(k, T). \end{aligned}$$

Now we can take

$$\hat{u}_\epsilon(k, t) = \mathbb{1}\left(|k| \leq \frac{1}{\epsilon}\right) \epsilon^{d+1} dk$$

Then,

$$\begin{aligned} \|u(\bullet, T)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} \hat{u}_\epsilon(k, t) dk = \lambda^d(\{|k| \leq \epsilon^{-1}\}) \epsilon^{d+1} \sim \epsilon \xrightarrow{\epsilon \rightarrow 0} 0 \\ \|u(\bullet, 0)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} e^{2T|2\pi k|^2} \mathbb{1}(|k| \leq \epsilon^{-1}) \epsilon^{d+1} dk \\ &\geq \int_{\frac{\epsilon}{2} \leq |k| \leq \frac{\epsilon}{2}} e^{2T|2\pi k|^2} \mathbb{1}(|k| \leq \epsilon^{-1}) \epsilon^{d+1} dk \gtrsim e^{2T\epsilon^{-2}} \epsilon \xrightarrow{\epsilon \rightarrow 0} \infty \end{aligned}$$

■

Remark 6.28 This means that a small error of the data at $t = T$ may cause a large error of the output $t = 0$.

Theorem 6.29 (Regularized solution)

Assume that $u \in C_1^2(\mathbb{R}^d \times (0, T)) \cap C^1(H^1(\mathbb{R}^d), [0, T])$

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ u(x, T) = g(x) & \text{in } \mathbb{R}^d \end{cases}$$

Then from given data $g_\epsilon \in L^2(\mathbb{R}^d)$ s.t.

$$\|g_\epsilon - g\|_{L^2(\mathbb{R}^d)} \leq \epsilon$$

we construct a solution \tilde{u}_ϵ s.t.

$$\sup_{t \in [0, T]} \|\tilde{u}_\epsilon(\bullet, t) - u(\bullet, t)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \rightarrow 0} 0$$

Proof. Clearly we should not choose \tilde{u}_ϵ to solve

$$\begin{cases} \partial_t u_\epsilon - \Delta_x u_\epsilon = 0 \\ u_\epsilon(t = T) = g_\epsilon \end{cases},$$

i.e.

$$\hat{u}_\epsilon(k, t) = e^{(T-t)|2\pi k|^2} \hat{g}_\epsilon(k).$$

Rather we take

$$\hat{u}_\epsilon(k, t) = e^{(T-t)|2\pi k|^2} \hat{g}_\epsilon(k) \mathbb{1}(|k| \leq \delta_\epsilon^{-1})$$

Where $\delta_\epsilon \rightarrow 0$ (chosen later). Then we have for all $t \in [0, T]$:

$$\begin{aligned} \|u_\epsilon(\bullet, t) - u(\bullet, t)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} e^{2(T-t)|2\pi k|^2} |\hat{g}_\epsilon(k) \mathbb{1}(|k| \leq \delta_\epsilon^{-1}) - \hat{g}(k)|^2 dk \\ &\leq 2 \int_{\mathbb{R}^d} e^{2T|2\pi k|^2} |\hat{g}_\epsilon(k) - \hat{g}(k)|^2 \mathbb{1}(|k| \leq \delta_\epsilon^{-1}) dk \\ &\quad + 2 \int_{\mathbb{R}^d} \underbrace{e^{2T|2\pi k|^2} |\hat{g}(k)|^2}_{|\hat{u}(k, 0)|^2} \mathbb{1}(|k| > \delta_\epsilon^{-1}) dk = \text{(I)} + \text{(II)} \end{aligned}$$

We have

$$\begin{aligned} \text{(I)} &\leq 2 \int_{\mathbb{R}^d} e^{c\delta_\epsilon^{-2}} |\hat{g}_\epsilon(k) - \hat{g}(k)|^2 dk = 2e^{c\delta_\epsilon^{-2}} \epsilon^{-2} \longrightarrow 0 && \text{if } \delta_\epsilon \gg \frac{1}{\sqrt{|\log \epsilon|}} \\ \text{(II)} &= 2 \int_{\mathbb{R}^d} |\hat{u}(k, 0)|^2 \mathbb{1}(|k| \geq \delta_\epsilon^{-1}) dk \leq 2 \int_{\mathbb{R}^d} |k|^2 \delta_\epsilon^2 |\hat{u}(k, 0)|^2 dk \end{aligned}$$

Thus choosing $\frac{1}{\sqrt{|\log \epsilon|}} \ll \delta_\epsilon \ll 1$, e.g. $\delta_\epsilon = (|\log \epsilon|)^{-\frac{1}{4}}$.

$$\sup_{t \in [0, T]} \|u_\epsilon(\bullet, t) - u(\bullet, t)\|_{L^2(\mathbb{R}^d)} \leq \text{(I)} + \text{(II)} \xrightarrow{\epsilon \rightarrow 0} 0 \quad \blacksquare$$

Remark 6.30 In application, both u and g are unknown. Only g_ϵ is given. So we have to construct \tilde{u}_ϵ using only information from g_ϵ .

Chapter 7

Wave Equation

7.1 d'Alembert

Wave equation:

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & x \in \mathbb{R}^d, t > 0 \\ u = g, \partial_t u = h & x \in \mathbb{R}^d, t = 0 \end{cases}$$

In $d = 1$:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & (x, t) \in \mathbb{R} \times (0, \infty) \\ u = g, \partial_t u = h, & x \in \mathbb{R}, t = 0 \end{cases}$$

Key idea: Factorization:

$$\partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x).$$

Then, if we denote $v = (\partial_t - \partial_x)u$, we get the transport equation

$$(\partial_t + \partial_x)v = 0.$$

This implies

$$v(x, t) = a(x - t), \quad a(x) = v(x, 0)$$

From this we get the inhomogeneous transport equation

$$(\partial_t - \partial_x)u = a(x - t).$$

Now we decompose $u = u_1 + u_2$, where

$$\begin{cases} (\partial_t - \partial_x)u_1 = 0 \\ (\partial_t - \partial_x)u_2 = a(x - t) \end{cases}.$$

Like above, we get $u_1 = b(x + t)$ and an explicit choice of u_2 is

$$u_2(x, t) = \frac{1}{2} \int_{x-t}^{x+t} a(y) dy$$

Thus,

$$u(x, t) = b(x + t) + \frac{1}{2} \int_{x-t}^{x+t} a(y) dy$$

Let's compute a and b :

$$\begin{aligned} b(x) &= u(x, 0) = g(x) \\ a(x) &= v(x, 0) = (\partial_t u - \partial_x u)_{t=0} = h - g'. \end{aligned}$$

Exercise 7.1 (E 13.1, d'Alembert formula) For $d = 1$ let $g \in C^2(\mathbb{R}^d)$, $h \in C^1(\mathbb{R})$ and define u by the *d'Alembert formula*

$$\begin{aligned} u(x, t) &= \int_{x-t}^{x+t} (h(y) - g'(y)) dy + g(x+t) \\ &= \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy. \end{aligned}$$

Then:

- $u \in C^2(\mathbb{R} \times (0, \infty))$
- $\partial_t^2 u - \partial_x^2 u = 0$
- $u = g, \partial_t u = h$ when $t \rightarrow 0$

Solution. We can compute the derivative by regarding $\int_x^{x+t} + \int_{x-t}^x$ and $\int_t^{x+t} + \int_{x-t}^t$:

$$\begin{aligned} \partial_t u &= \frac{1}{2} (g'(x+t) - g'(x-t)) + \frac{1}{2} (h(x+t) + h(x-t)) \\ \partial_t^2 u &= \frac{1}{2} (g''(x+t) + g''(x-t)) + \frac{1}{2} (h'(x+t) - h'(x-t)) \\ \partial_x u &= \frac{1}{2} (g'(x+t) + g'(x-t)) + \frac{1}{2} (h(x+t) - h(x-t)) \\ \partial_x^2 u &= \frac{1}{2} (g''(x+t) + g''(x-t)) + \frac{1}{2} (h'(x+t) - h'(x-t)) = \partial_t^2 u \end{aligned}$$

Now,

$$\begin{aligned} \lim_{t \rightarrow 0} u &= \frac{1}{2} (g(x) + g(x)) + \frac{1}{2} \int_x^x h = g(x) \\ \lim_{t \rightarrow 0} \partial_t u &= \frac{1}{2} (g'(x) - g'(x)) + \frac{1}{2} (h(x) + h(x)) = h(x). \end{aligned}$$

■

Remark 7.2 If $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$ (but not better).

Now, let's apply the *Reflection Method*. Replace \mathbb{R} by $\mathbb{R}_+ = (0, \infty)$ and assume

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \mathbb{R}_+ \times (0, \infty) \\ u = g, \partial_t u = h & \text{on } \mathbb{R}_+ \times \{t = 0\}, g(0) = h(0) = 0 \\ u = 0 & \text{on } \{x = 0\} \times \{t > 0\} \end{cases}$$

Define

$$\begin{aligned} \tilde{u}(x, t) &= \begin{cases} u(x, t), & x \geq 0, t \geq 0 \\ -u(-x, t), & x \leq 0, t \geq 0 \end{cases} \\ \tilde{g}(x) &= \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x \leq 0 \end{cases} \\ \tilde{h}(x) &= \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x \leq 0 \end{cases} \end{aligned}$$

Then

$$\begin{cases} \partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ \tilde{u} = \tilde{g}, \partial_t \tilde{u} = \tilde{h} & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}.$$

By d'Alembert formula

$$\tilde{u}(x, t) = \frac{1}{2} [\tilde{g}(x+t) + \tilde{g}(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{h}(y) dy$$

This implies

$$u(x, t) = \begin{cases} \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy & x \geq t \geq 0 \\ \frac{1}{2} [g(x+t) - g(t-x)] + \frac{1}{2} \int_{t-x}^{x+t} h(y) dy & t \geq x \geq 0 \end{cases}.$$

This is the solution of the heat equation in $\mathbb{R}_+ \times (0, \infty)$.

7.2 Euler-Poisson-Darboux

$$(\star) \quad \begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ u = g, \partial_t u = h & \mathbb{R}^d \times \{t = 0\} \end{cases}$$

Idea: Averaging of u over sphere \rightsquigarrow 1D problem. Define for $x \in \mathbb{R}^d$, $t > 0$, $r > 0$,

$$\begin{aligned} U_r(x, t) &:= \oint_{\partial B(x, r)} u(y, t) dS(y) \\ G_r(x) &:= \oint_{\partial B(x, r)} g(y) dS(y) \\ H_r(x) &:= \oint_{\partial B(x, r)} h(y) dS(y) \end{aligned}$$

Lemma 7.3 (Euler-Poisson-Darboux equation) If $u \in C^2(\mathbb{R}^d \times [0, \infty))$ solves (\star) , then for all $x \in \mathbb{R}^d$:

- $(r, t) \mapsto U \in C^2([0, \infty) \times [0, \infty))$
- $\begin{cases} \partial_t^2 U - \partial_r^2 U - \frac{d-1}{r} \partial_r U = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\ U = G, \partial_t U = H & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$

Note that $\partial_r^2 + \frac{d-1}{r} \partial_r$ is the radial part of Δ .

Proof. We compute for $r > 0$:

$$\begin{aligned}
\partial_r U_r(x, t) &= \partial_r \oint_{\partial B(x, r)} u(y, t) dS(y) \\
&= \partial_r \oint_{\partial B(0, 1)} u(x + rz, t) dS(z) \\
&= \oint_{\partial B(0, 1)} \nabla u(x + rz, t) z dS(z) \\
&= \oint_{\partial B(x, r)} \nabla u(y, t) \frac{y - x}{r} dS(y) \\
&= \oint_{\partial B(x, r)} \frac{\partial u(y, t)}{\partial \vec{n}} dS(y) \\
(\text{Green 2.3}) \quad &= \frac{1}{|\partial B(0, r)|} \int_{B(x, r)} \Delta_x u(y, t) dy \\
\left(|B(0, r)| = \frac{r}{d} |\partial B(0, r)| \right) &= \frac{r}{d} \oint_{B(x, r)} \Delta_x u(y, t) dy
\end{aligned}$$

(The computation is similar to the proof of the mean-value theorem for the Poisson equation.) We compute the second derivative

$$\begin{aligned}
\partial_r^2 U_r(x, t) &= \partial_r \left[\frac{r}{d} \oint_{B(x, r)} \Delta_x u(y, t) dy \right] \\
(|B(0, r)| = r^d |B(0, 1)|) \quad &= \partial_r \left[\frac{1}{d |B_1| r^{d-1}} \int_{B(x, r)} \Delta_x u(y, t) dy \right]
\end{aligned}$$

Now, $\partial_r \frac{1}{d |B_1| r^{d-1}} = \frac{-d+1}{d |B_1| r^d} = -\frac{d-1}{d |B(0, r)|}$ and

$$\begin{aligned}
\partial_r \int_{B(x, r)} \Delta_x u(y, t) dS(y) &= \partial_r \int_{B(0, r)} \Delta_x u(x + ry, t) dy \\
(\text{Green 2.3}) \quad &= \partial_r \int_{\partial B(0, 1)} \nabla_x u(x + ry, t) \frac{y}{|y|} dS(y) \\
(|y| = 1) \quad &= \int_{\partial B(0, 1)} \partial_r \nabla_x u(x + ry, t) y dS(y) \\
&= \int_{\partial B(0, 1)} \Delta_x u(x + ry, t) y \cdot y dS(y) \\
(y \cdot y = |y|^2 = 1) \quad &= \int_{\partial B(0, 1)} \Delta_x u(x + ry, t) dS(y) \\
&= \int_{\partial B(x, r)} \Delta_x u(y, t) dS(y)
\end{aligned}$$

Now with the product rule we get

$$\begin{aligned}
\partial_r^2 U_r(x, t) &= - \left(\frac{d-1}{d} \right) \oint_{B(x, r)} \Delta_x u(y, t) dy \\
&\quad + \frac{1}{d |B_1| r^{d-1}} \int_{\partial B(x, r)} \Delta_x u(y, t) dS(y) \\
&= - \left(\frac{d-1}{d} \right) \oint_{B(x, r)} \Delta_x u(y, t) dy \\
&\quad + \oint_{\partial B(x, r)} \Delta_x u dS(y)
\end{aligned}$$

And, since u is a solution,

$$\partial_t^2 U = \partial_t^2 \int_{\partial B(x,r)} u \, dS(y) = \int_{\partial B(x,r)} (\partial_t^2 u) \, dS(y) = \int_{\partial B(x,r)} (\Delta_x u) \, dS(y).$$

So we can conclude

$$\partial_t^2 U - \partial_r^2 U - \frac{d-1}{d} U = 0$$

the above computation also shows that $U \in C^2(\mathbb{R}_+ \times [0, \infty))$. Moreover

$$\begin{aligned} \partial_r U_r(x, t) &\xrightarrow{r \rightarrow 0^+} 0 \\ \partial_r^2 U_r(x, t) &\xrightarrow{r \rightarrow 0^+} \left(\frac{1}{d} - 1\right) \Delta_x u + \Delta_x u = \frac{1}{d} \Delta_x u \end{aligned}$$

This implies that $U \in C^2([0, \infty) \times [0, \infty))$. Finally, when $t = 0$,

$$\begin{cases} u = g \\ \partial_t = h \end{cases} \Rightarrow \begin{cases} U = G \\ \partial_t U = H \end{cases} \quad \blacksquare$$

We showed that it is a necessary condition for u to solve the Euler-Poisson-Darboux equation. Now we try to actually solve the equation. In general, this is easier for odd d than for even d . We will consider the cases $d = 2, 3$.

7.2.1 Solution in three dimensions

Now, for $r > 0$ let $\tilde{U} = rU, \tilde{G} = rG, \tilde{H} = rH$. Then

$$\begin{cases} \partial_t^2 \tilde{U} - \partial_r^2 \tilde{U} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \tilde{U} = \tilde{G}, \partial_t \tilde{U} = \tilde{H} & \text{when } t = 0 \\ \tilde{U} = 0 & \text{when } r = 0 \end{cases}$$

Then, by d'Alembert's formula, for $0 \leq r \leq t$ we have

$$\begin{aligned} \tilde{U}_r(x, t) &= \frac{1}{2} \left[\tilde{G}(r+t) - \tilde{G}(t-r) \right] + \frac{1}{2} \int_{t-r}^{t+r} \tilde{H}(y) \, dy. \\ \Rightarrow U_r(x, t) &= \frac{1}{2} \left[\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{r} \right] + \frac{1}{2r} \int_{t-r}^{t+r} \tilde{H}(y) \, dy \end{aligned}$$

Now, taking $r \rightarrow 0$ we get

$$\begin{aligned} u(x, t) &= \tilde{G}'(t) + \tilde{H}(t) \\ &= \partial_t \left(t \int_{\partial B(x,t)} g(y) \, dS(y) \right) + t \int_{\partial B(x,t)} h(y) \, dS(y) \end{aligned}$$

Using

$$\int_{\partial B(x,t)} g(y) \, dS(y) = \int_{\partial B(0,1)} g(x + tz) \, dS(z)$$

we get

$$\begin{aligned} \partial_t \int_{\partial B(x,t)} g(y) \, dS(y) &= \int_{\partial B(0,1)} \nabla g(x + tz) \cdot z \, dz \\ &= \int_{\partial B(x,t)} \nabla g(y) \cdot \left(\frac{y-x}{t} \right) \, dS(y) \\ \Rightarrow \partial_t \left(t \int_{\partial B(x,t)} g(y) \, dS(y) \right) &= \int_{\partial B(x,t)} (g + \nabla g(y-x)) \, dS(y) \end{aligned}$$

From that we get:

Remark 7.4 (Kirchhoff's formula in 3D) For all $x \in \mathbb{R}^3$, $t > 0$:

$$u(x, t) = \oint_{\partial B(x, t)} (g(y) + \nabla g(y - x) + th(y)) dS(y)$$

Exercise 7.5 (E 13.3) Let $g \in C_c^2(\mathbb{R}^3)$, $h \in C_c^1(\mathbb{R}^3)$. Assume that $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ satisfies the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \forall x \in \mathbb{R}^3, t > 0 \\ u(x, 0) = g(x), \partial_t u(x, 0) = h(x) & \forall x \in \mathbb{R}^3 \end{cases}$$

Prove that there exists a constant $C > 0$ such that

$$|u(x, t)| \leq \frac{C}{t}, \quad \forall x \in \mathbb{R}^3, t > 0.$$

Solution.

Step 1: Assume

$$(\star) \quad u(x, t) = \oint_{\partial B_{\mathbb{R}^3}} (g(y) + \nabla g(y)(y - x) + th(y)) dS(y)$$

Assume $\text{supp } g, h \subseteq B(0, R)$. Then

$$\begin{aligned} |u(x, t)| &\leq \frac{1}{|\partial B(x, t)|} \int_{\partial B_{\mathbb{R}^3}(x, t)} (\|g\|_{L^\infty} + \|\nabla g\|_{L^\infty} t + t\|h\|_{L^\infty}) \mathbb{1}(|y| \leq R) dS(y) \\ &= \frac{C(1+t)}{t^2} \leq \frac{C}{t} \quad \text{as } t \geq 1 \end{aligned}$$

and

$$|u(x, t)| \leq C \oint_{\partial B(x, t)} \leq C$$

if $t \leq 1$.

Step 2: Why is u given by (\star) ? This follows from the uniqueness of the solution of the wave equation. In fact, if g, h are compactly supported, then for all $t \in [0, T]$, $u(x, t)$ supported in $B(0, R_T)$ a finite ball.

■

7.2.2 Solution in two dimensions

The transformation $\tilde{U} = rU$ does not work! The idea is to think of the 2D problem as 3D problem with x_3 hidden. We write $\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$. Then we get

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_x \bar{u} = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

With Kirchhoff's formula:

$$u(x, t) = \bar{u}(\bar{x}, t) = \partial_t \left(t \oint_{\partial \bar{B}(\bar{x}, t)} \bar{g}(y) d\bar{S}(y) \right) + t \int_{\partial \bar{B}(\bar{x}, t)} \bar{h} d\bar{S}(y)$$

Remark 7.6 Let $\gamma(y) = (t^2 - |y - x|^2)^{\frac{1}{2}}$, $y \in B(x, t)$, then

$$\begin{aligned} \oint_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{S} &= \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{S}(y) \\ &= \frac{1}{4\pi t^2} \int_{B(x, t)} g(y) 2(1 + |\nabla \gamma|^2)^{\frac{1}{2}} dy \\ &= \frac{1}{4\pi t^2} \int_{B(x, r)} g(y) \frac{2t}{\sqrt{t^2 - |y - x|^2}} dy \\ &= \frac{t}{2} \oint_{B(x, t)} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}} dy \end{aligned}$$

Similarly:

$$\oint_{\partial B(\bar{x}, t)} \bar{h} d\bar{S}(y) = \frac{t}{2} \oint_{B(x, t)} \frac{h(y)}{\sqrt{t^2 - |y - x|^2}} dy$$

This implies:

$$\begin{aligned} u(x, t) &= \partial_t \left(\frac{t^2}{2} \oint_{B(x, t)} \frac{g(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy \right) + \frac{t^2}{2} \oint_{B(x, r)} \frac{h(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy \\ &= (I) + (II) \\ (I) &= \partial_t \left(\frac{1}{2} t \oint_{B(0, 1)} \frac{g(x + tz)}{(1 - |z|^2)^{\frac{1}{2}}} dz \right) \\ &= \oint_{B(0, 1)} \frac{g(x + tz)}{(1 - |z|^2)^{\frac{1}{2}}} dz + t \oint_{B(0, 1)} \frac{\nabla g(x + tz) z}{(1 - |z|^2)^{\frac{1}{2}}} dz \\ &= t \int_{B(x, t)} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}} dy + t \oint_{B(x, r)} \frac{\nabla g(y)(y - x)}{\sqrt{t^2 - |y - x|^2}} dy \end{aligned}$$

Exercise 7.7 (E 13.2, Poisson formula for 2D) For $x \in \mathbb{R}^2$, $t > 0$, $g \in C^2(\mathbb{R}^2)$, $h \in C^2(\mathbb{R}^2)$ let

$$u(x, t) = \frac{t}{2} \oint_{B(x, t)} \frac{g(y) + \nabla g(y)(y - x) + th(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy.$$

Prove that $u \in C^2(\mathbb{R}^2 \times (0, \infty))$ and

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \forall x \in \mathbb{R}^2, t > 0 \\ \lim_{t \rightarrow 0^+} u(x, t) = g(x), \lim_{t \rightarrow 0^+} \partial_t u(x, t) = h(x), & \forall x \in \mathbb{R}^2 \end{cases}$$

Solution. Let $\bar{x} = (x, x_3) \in \mathbb{R}^3$, $\bar{u}(\bar{x}, t) = u(x, t)$. We claim that

$$(\star) \quad \bar{u}(\bar{x}, t) = \partial_t \left(t \oint_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{S}(y) \right) + t \oint_{\partial \bar{B}(\bar{x}, t)} \bar{h} d\bar{S}$$

where $\bar{B}(\bar{x}, t)$ is a ball in 3D, $\bar{g}(\bar{x}) = g(x)$, $\bar{h}(\bar{x}) = h(x)$. From 7.6 we have

$$\oint_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{S} = \frac{t}{2} \oint_{B_{\mathbb{R}^2}(x, t)} \frac{g(y)}{\sqrt{t^2 - |x - y|^2}} dy$$

Now,

$$\begin{aligned} \text{RHS of } (\star) &= \partial_t \left(\frac{t^2}{2} \oint_{B_{\mathbb{R}^2}} \frac{g(y)}{\sqrt{t^2 - |x - y|^2}} dy \right) + \frac{t^2}{2} \oint_{B_{\mathbb{R}^2}(x,t)} h(y) dy \\ ?? &= \partial_t \frac{t}{2} \frac{g(y) + \nabla g(y)(y - x)}{\sqrt{t^2 - |x - y|^2}} + \frac{t^2}{2} \oint_{B_{\mathbb{R}^2}(x,t)} h(y) dy \end{aligned}$$

3D-Problem: We claim that if $\bar{g} \in C^3(\mathbb{R}^3)$, $\bar{h} \in C^2(\mathbb{R}^3)$ and

$$\bar{u}(\bar{x}, t) = \partial_t \left(t \oint_{\partial B_{\mathbb{R}^2}(\bar{x}, t)} \bar{g} \right) + t \int_{\partial B_{\mathbb{R}^2}(\bar{x}, t)} \bar{h},$$

then $\bar{u} \in C^2(\mathbb{R}^3 \times (0, \infty))$ and

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_{\bar{x}} \bar{u} = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Proof:

Step 1: Assume $\bar{g} = 0$, $\bar{u}(\bar{x}, t) = t \int_{\partial B(\bar{x}, t)} \bar{h}$. Lemma (from lecture):

$$\partial_r \int_{\partial B(\bar{x}, r)} \bar{h} = \frac{r}{d} \oint_{B(\bar{x}, r)} \Delta \bar{h}, \quad d = 3$$

($\bar{U} = \oint_{B(\bar{x}, r)} u(x, t)$) Hence:

$$\begin{aligned} \partial_t \bar{u}(x, t) &= \partial_t \left(t \oint_{\partial B(\bar{x}, t)} \bar{h} \right) = \oint_{\partial B(\bar{x}, t)} \bar{h} + t \frac{t}{3} \oint_{B(x, t)} \Delta \bar{h} \\ \partial_t^2 \bar{u}(x, t) &= \frac{t}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} + \frac{2}{3} t \oint_{B(x, t)} \Delta \bar{h} + \frac{t^2}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} \end{aligned}$$

Where we have

$$\begin{aligned} \frac{t^2}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} &= \frac{1}{|B_1| 3t} \int_{B(\bar{x}, t)} \Delta \bar{h} \\ \partial_t(\dots) &= -\frac{1}{|B_1| 3t^2} \int_{B(\bar{x}, t)} \Delta \bar{h} + \frac{1}{|B_1| 3t} \int_{\partial B(\bar{x}, t)} \Delta \bar{h} \\ &= -\frac{t}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} + t \oint_{\partial B(\bar{x}, t)} \Delta \bar{h} \end{aligned}$$

So we get

$$\begin{aligned} \partial_t^2 \bar{u}(x, t) &= \frac{t}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} + \frac{2}{3} t \oint_{B(x, t)} \Delta \bar{h} - \frac{t}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} + t \oint_{\partial B(\bar{x}, t)} \Delta \bar{h} \\ &= t \oint_{\partial B(\bar{x}, t)} \Delta \bar{h} \end{aligned}$$

This implies

$$\begin{aligned} \Delta_{\bar{x}} \bar{u}(x, t) &= \Delta_{\bar{x}} \left(t \oint_{\partial B(\bar{x}, t)} \bar{h} \right) = \Delta_x \left(t \oint_{\partial B(0, t)} \bar{h}(\bar{x} + y) dS(y) \right) \\ &= t \oint_{\partial B(0, t)} \Delta \bar{h}(\bar{x} + y) dS(y) = t \oint_{\partial B(\bar{x}, t)} \Delta \bar{h} = \partial_t^2 \bar{u} \end{aligned}$$

Thus, $\partial_t \bar{u} - \Delta_{\bar{x}} \bar{u} = 0$ in $\mathbb{R}^3 \times (0, \infty)$. Moreover,

$$\bar{u}(x, t) = t \fint_{\partial B(\bar{x}, t)} \bar{h} \xrightarrow{t \rightarrow 0} 0$$

and

$$\partial_t \bar{u}(x, t) = \fint_{\partial B(\bar{x}, t)} \bar{h} + \frac{t^2}{3} \fint_{B(x, t)} \Delta \bar{h} \xrightarrow{t \rightarrow 0} \bar{h}(\bar{x}) \quad \checkmark$$

Step 2: General Case: $\bar{g} \neq 0$. Assume $h = 0$, then

$$\bar{u}(\bar{x}, t) = \partial_t \left(t \fint_{\partial B_{\mathbb{R}^3}} \bar{g} \right)$$

By Step 1, $v(\bar{x}, t) := t \fint_{\partial B_{\mathbb{R}^3}(0, \infty)} \bar{g}$ satisfies

$$\begin{cases} \partial_t^2 - \Delta_{\bar{x}} v = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ v = 0 \text{ and } \partial_t v = \bar{g} & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Then $\bar{u} = \partial_t \bar{v}$. This implies

$$\partial_t^2 \bar{u} - \Delta_{\bar{x}} \bar{u} = \partial_t^3 \bar{v} - \Delta_{\bar{x}} \partial_t \bar{v} = \partial_t (\underbrace{\partial_t^2 \bar{v} - \Delta_{\bar{x}} \bar{v}}_{=0}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty)$$

and

$$\begin{aligned} \bar{u}(t = 0) &= \partial_t \bar{v}(t = 0) = \bar{g}, \\ \partial_t \bar{u}(t = 0) &= \partial_t^2 \bar{v}(t = 0) = t \fint_{\partial B(\bar{x}, t)} \Delta \bar{g} \Big|_{t=0} = 0 \end{aligned}$$

Step 3: Consider the case $g \neq 0$, $h \neq 0$:

$$\bar{u} = \underbrace{\partial_t \left(t \fint_{\partial B(\bar{x}, t)} \bar{g} \right)}_{\bar{u}_1} + t \underbrace{\fint_{\partial B(\bar{x}, t)} h}_{\bar{u}_2}$$

Now we have

$$\begin{cases} \partial_t^2 \bar{u}_1 - \Delta_{\bar{x}} \bar{u}_1 = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u}_1 = \bar{g}, \partial_t \bar{u}_1 = 0 & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

and

$$\begin{cases} \partial_t^2 \bar{u}_2 - \Delta_{\bar{x}} \bar{u}_2 = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u}_2 = 0, \partial_t \bar{u}_2 = h & \text{in } \mathbb{R}^3 \times (0, \infty) \end{cases}.$$

This implies

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_{\bar{x}} \bar{u} = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{in } \mathbb{R}^3 \times (0, \infty) \end{cases}.$$

■

7.3 Spectral Method

Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded.

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \Omega \times (0, \infty) \\ u = g, \partial_t u = h & \text{when } t = 0 \\ u = 0 & \text{when } x \in \partial\Omega \end{cases}$$

$-\Delta$ has eigenvecors $(e_i)_{i=1}^\infty$ with eigenvalues $(\lambda_i)_{i=1}^\infty$, i.e.

$$\begin{cases} -\Delta e_i = \lambda_i e_i \\ e_i|_{\partial\Omega} = 0 \end{cases}$$

s.t. $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_i \rightarrow \infty$ and $(e_i)_{i \in \mathbb{N}}$ is an orthonormal basis for $L^2(\Omega)$. We write

$$u(x, t) = \sum_{i \in \mathbb{N}} a_i(t) e_i(x)$$

This implies:

$$\begin{aligned} a_i''(t) + \lambda_i a_i(t) &= 0 \\ \text{(ODE)} \quad \Rightarrow \quad a_i(t) &= a_i(0) \cos(\sqrt{\lambda_i} t) + \frac{a_i'(0)}{\sqrt{\lambda_i}} \sin(\sqrt{\lambda_i} t) \end{aligned}$$

Here $a_i(0)$, $a_i'(0)$ is determined by

$$\begin{cases} g = u(t=0) = \sum_{i=1}^\infty a_i(0) e_i(x) \\ h = \partial_t u(t=0) = \sum_{i=1}^\infty a_i'(0) e_i(x) \end{cases} \Rightarrow \begin{cases} a_i(0) = \langle e_i, g \rangle \\ a_i'(0) = \langle e_i, h \rangle \end{cases}$$

7.4 Uniqueness

For $\Omega \subseteq \mathbb{R}^d$ open and bounded with C^1 -boundary regard

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \Omega \times (0, T) \\ u = 0, \partial_t u = 0 & \text{in } \Omega \times \{t = 0\} \\ u = 0 & \partial\Omega \times [0, T] \end{cases}$$

If $u \in C^2(\bar{\Omega} \times [0, T])$, then $u = 0$.

Proof. Let

$$e(t) = \int_{\Omega} (|\partial_t u|^2 + |\nabla_x u|^2) dx$$

has $e'(t) = 0$. This implies $e(t) = e(0) = 0$, so $\partial_t u = 0$ and hence $u = 0$. ■

The same result holds for \mathbb{R}^d , i.e.

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \mathbb{R}^d \times (0, T) \\ u = 0, \partial_t u = 0 & \mathbb{R}^d \times \{t = 0\} \end{cases}$$

and $u \in C^2(H^2(\mathbb{R}^d), [0, T])$ (i.e. $u(t, \bullet) \in H^2(\mathbb{R}^d)$ and $t \mapsto u(t, \bullet)$ continuous).

7.5 Propagation of the wave

Theorem 7.8 Assume that $u \in C^2(\mathbb{R}^d \times [0, \infty))$ and

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \mathbb{R}^d \times (0, \infty) \\ u = 0, \partial_t u = 0 & B(x_0, t_0) \times \{t = 0\} \end{cases}$$

Then $u(x, t) = 0$ for $x \in B(x_0, t_0 - t)$.

Proof. Let for $t \in [0, t_0]$

$$e(t) = \int_{B(x_0, t_0 - t)} (|\partial_t u|^2 + |\nabla_x u|^2) dx$$

■

This implies

$$\begin{aligned} e'(t) &= \int_{B(x_0, t_0 - t)} 2(\partial_t u \partial_t^2 u + \nabla_x u \partial_t \nabla_x u) - \int_{\partial B(x_0, t_0 - t)} (|\partial_t u|^2 + |\nabla_x u|^2) dS \\ &= \int_{B(x_0, t_0 - t)} 2(\partial_t u \partial_t^2 u - \Delta_x u \partial_t u) + \int_{\partial B(x_0, t_0 - t)} [2(\nabla_x u \vec{n}) \partial_t u - |\partial_t u|^2 - |\nabla_x u|^2] dS \\ &= \int_{B(x_0, t_0 - t)} 2(\partial_t u \underbrace{(\partial_t^2 u - \Delta_x u)}_{=0}) + \int_{\partial B(x_0, t_0 - t)} \underbrace{[2(\nabla_x u \vec{n}) \partial_t u - |\partial_t u|^2 - |\nabla_x u|^2]}_{\leq 0} dS \end{aligned}$$

where we used

$$|2(\nabla_x u \vec{n}) \partial_t u| \leq 2|\nabla_x u| |\partial_t u| \leq |\nabla_x u|^2 + |\partial_t u|^2.$$

Thus $e'(t) \leq 0$ for all $t \in (0, t_0)$, so $e(t) \leq e(0) = 0$ (as $u = 0$, $\partial_t u = 0$ in $B(x_0, t_0) \times \{t = 0\}$). This implies $e(t) = 0$ for all $t \in (0, t_0)$, so

$$\begin{cases} \partial_t u = 0 & x \in B(x_0, t_0 - t) \\ u = 0 & x \in B(x_0, t_0 - t) \times \{t = 0\} \end{cases}.$$

So we get that $u = 0$ for $x \in B(x_0, t_0 - t)$ for all $t \in (0, t_0)$ and $u = 0$ for $x \in B(x_0, t_0 - t)$ for all $t \in [0, t_0]$. (More precisely $u(x_0, t_0) = 0$)

7.6 Wave vs. Heat Equation

Notation 7.9 For $g \in L^2(\mathbb{R}^d)$, $t > 0$ we write $e^{t\Delta} g \in L^2(\mathbb{R}^d)$ for

$$\begin{aligned} \widehat{e^{t\Delta} g}(k) &= e^{-t|2\pi k|^2} \hat{g}(k) \\ \Leftrightarrow (e^{t\Delta} g)(x) &= \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y) dy \\ \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} &= e^{-t|2\pi k|^2} \end{aligned}$$

Heat Equation:

- Improves smoothness, *i.e.* $g \in L^2(\mathbb{R})$ implies $e^{t\Delta}g \in \bigcap_{m \geq 1} H^m(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$ for all $t > 0$.
- Propagation: Speed is ∞ .

$t = 0$: $g \in C_c^\infty$

$t > 0$: $e^{t\Delta}g$ does not have compact support

Wave Equation:

- No improvement of smoothness
- Propagation: Speed is finite

Chapter 8

Schrödinger Equation

$$\begin{cases} -i\partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = g(x) & x \in \mathbb{R}^d \end{cases}$$

Formally $-i\partial_t = \frac{1}{i} \frac{d}{dt} = \frac{d}{d(it)} \Rightarrow \partial_\xi u - \Delta_x u = 0$, $\xi = it \rightsquigarrow$ Heat equation with *imaginary time*. From the heat equation

$$\begin{aligned} \begin{cases} \partial_t u - \Delta_x u = 0 \\ u(x, 0) = g(x) \end{cases} &\Rightarrow u(x, t) = (e^{t\Delta} g)(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y) dy \\ \rightsquigarrow \begin{cases} -i\partial_t u - \Delta_x u = 0 \\ u(x, 0) = g(x) \end{cases} &\Rightarrow u(x, t) = (e^{it\Delta} g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy \end{aligned}$$

if $g \in L^1$.

Theorem 8.1 (Fourier for Schrödinger Solution) For $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, define

$$(e^{it\Delta} g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy$$

Then $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)}$. Consequently, for all $g \in L^2(\mathbb{R}^d)$ we can define $e^{it\Delta} g \in L^2(\mathbb{R}^d)$ by a density argument. Moreover,

$$\widehat{e^{it\Delta} g}(k) = e^{-it|2\pi k|^2} \hat{g}(k)$$

for almost every $k \in \mathbb{R}^d$.

Proof. Fourier transform of Gaussian:

$$\frac{1}{(4\pi t)^{\frac{d}{2}}} \widehat{e^{-\frac{|x|^2}{4t}}}(k) = e^{-t|2\pi k|^2}, \quad t > 0$$

Key point: This formula also holds if $t \in \mathbb{C}$ and $\Re(t) > 0$. For all $\epsilon > 0$ consider

$$(e^{(it+\epsilon)\Delta} g)(x) = \left(e^{-(it+\epsilon)|2\pi k|^2} \hat{g}(k) \right)^\vee(x) = (\hat{G}_\epsilon g)^\vee = (G_\epsilon \star g)(x)$$

where

$$G_\epsilon(x) = \frac{1}{(4\pi(it+\epsilon))^{\frac{d}{2}}} e^{-\frac{|x|^2}{4(it+\epsilon)}}.$$

Since $g \in L^1(\mathbb{R}^d)$:

$$\begin{aligned}(G_\epsilon \star g)(x) &= \frac{1}{(4\pi(it + \epsilon))^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4(it+\epsilon)}} g(y) dy \\ &\longrightarrow \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} g(y) dy = (e^{it\Delta} g)(x)\end{aligned}$$

for all $x \in \mathbb{R}^d$. Moreover,

$$\begin{aligned}\|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} &= \|\widehat{G_\epsilon \star g}\|_{L^2(\mathbb{R}^d)} = \|\hat{G}_\epsilon \hat{g}\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left| e^{-(it+\epsilon)|2\pi k|^2} \right|^2 |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^d} e^{-\epsilon|2\pi k|^2} |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} \xrightarrow{\epsilon \rightarrow 0^+} \left(\int_{\mathbb{R}^d} |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} = \|g\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

as $g \in L^2(\mathbb{R}^d)$. Thus, $G_\epsilon \star g \rightarrow e^{it\Delta} g$ pointwise. With Fatou:

$$\liminf_{\epsilon \rightarrow 0^+} \|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} \geq \|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)}$$

Thus, $e^{it\Delta} g \in L^2(\mathbb{R}^d)$ and $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} \leq \|g\|_{L^2(\mathbb{R}^d)}$. To get the equality:

$$\widehat{G_\epsilon \star g}(k) = e^{-(it+\epsilon)|2\pi k|^2} \underbrace{\hat{g}(k)}_{L^2} = \underbrace{e^{-\epsilon|2\pi k|^2}}_{\in [0,1]} \left(e^{-it|2\pi k|^2} \hat{g}(k) \right) \xrightarrow{\epsilon \rightarrow 0^+} e^{it|2\pi k|^2} \hat{g}(k)$$

in $L^2(\mathbb{R}^d, dk)$. Thus $G_\epsilon \star g$ converges in $L^2(\mathbb{R}^d)$. Then up to a subsequence $\epsilon \rightarrow 0^+$, we can assume that

$$(G_\epsilon \star g)(x) \xrightarrow{\epsilon \rightarrow 0^+} H(x)$$

almost everywhere. (Dominated convergence) We already proved $G_\epsilon \star g \rightarrow e^{it\Delta} g$ pointwise, so $e^{it\Delta} g = H$, i.e. $G_\epsilon \star g \rightarrow e^{it\Delta} g$ in $L^2(\mathbb{R}^d)$. Conclude

$$\|g\|_{L^2(\mathbb{R}^d)} \xleftarrow{\epsilon \rightarrow 0} \|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \rightarrow 0^+} \|e^{it\Delta} g\|_{L^2}.$$

This implies $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)}$ for all $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. For $g \in L^2(\mathbb{R}^d)$ there is a sequence $\{g_n\} \subseteq L^1 \cap L^2$ s.t. $g_n \rightarrow g$ in $L^2(\mathbb{R}^d)$. Then

$$\|e^{it\Delta} g_n - e^{it\Delta} g_m\|_{L^2} = \|e^{it\Delta} \underbrace{(g_n - g_m)}_{L^1 \cap L^2}\|_{L^2} = \|g_n - g_m\|_{L^2} \rightarrow 0$$

as $m, n \rightarrow \infty$. So $e^{it\Delta} g_n$ is a Cauchy sequence in L^2 , so it has a limit which we define as $e^{it\Delta} g$. Why the limit $e^{it\Delta} g$ is independent of the choice of g_n : If we have 2 different sequences $\{g_n\}, \{\tilde{g}_n\} \subseteq L^1 \cap L^2$, then $g_n, \tilde{g}_n \rightarrow g$ in $L^2(\mathbb{R}^d)$. Then

$$\|e^{it\Delta} g_n - e^{it\Delta} \tilde{g}_n\|_{L^2(\mathbb{R}^d)} = \|e^{it\Delta} \underbrace{(g_n - \tilde{g}_n)}_{L^1 \cap L^2}\|_{L^2} = \|g_n - \tilde{g}_n\|_{L^2} \xrightarrow{n \rightarrow \infty} 0$$

$\leadsto \lim_{n \rightarrow \infty} e^{it\Delta} g_n = \lim_{n \rightarrow \infty} e^{it\Delta} \tilde{g}_n$. Finally, we have $g \in L^2(\mathbb{R}^d), g_n \in L^1 \cap L^2 \rightarrow g$ in L^2 .

$$\widehat{e^{it\Delta} g}(k) \xleftarrow{n \rightarrow \infty} \widehat{e^{it\Delta} g_n}(k) = e^{-it|2\pi k|^2} \hat{g}_n(k) \xrightarrow{n \rightarrow \infty} e^{-it|2\pi k|^2} \hat{g}(k)$$

in $L^2(\mathbb{R}^d)$. ■

Theorem 8.2 (Long term behaviour)

1. If $g \in L^1(\mathbb{R}^d)$, then $(e^{it\Delta}g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy$. This implies

$$\|e^{it\Delta}g\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \|g\|_{L^1(\mathbb{R}^d)} \xrightarrow{t \rightarrow \infty} 0.$$

On the other hand, for all $g \in L^2(\mathbb{R}^d)$, $\|e^{it\Delta}g\|_{L^2} = \|g\|_{L^2}$.

Exercise 8.3: If $g \in L^1 \cap L^2$ then $\|e^{it\Delta}g\|_{L^p} \rightarrow 0$ for all $2 < p \leq \infty$.

2. For all bounded sets Ω , for all $g \in L^2(\mathbb{R}^d)$ we have $\|\mathbb{1}_\Omega e^{it\Delta}g\|_{L^2(\mathbb{R}^d)} \rightarrow 0$ as $t \rightarrow \infty$. Equivalently, for all $R > 0$:

$$\int_{|x| \leq R} |(e^{it\Delta}g)(x)|^2 dx \xrightarrow{t \rightarrow \infty} 0$$

(RAGE theorem)

Proof. 1. With the Fourier transform of $e^{it\Delta}g$ (Theorem 8.1) we get

$$\|e^{it\Delta}g - g\|_{L^2}^2 = \|\widehat{e^{it\Delta}g} - \hat{g}\|_{L^2}^2 = \int_{\mathbb{R}^d} \underbrace{(e^{-it|2\pi k|^2} - 1)}_{\substack{\xrightarrow{t \rightarrow 0} 0 \\ \text{bd. by } 2\forall t}} \underbrace{|\hat{g}(k)|^2}_{\in L^1} dk \xrightarrow{t \rightarrow 0} 0$$

by dominated convergence.

2. Decompose $g = g_1 + g_2$, then $e^{it\Delta}g = e^{it\Delta}g_1 + e^{it\Delta}g_2$. Take $g_1 \in L^1$, then by 1. we have

$$\begin{aligned} \|e^{it\Delta}g_1\|_{L^\infty} &\leq \frac{\|g_1\|_{L^1}}{|4\pi t|^{\frac{d}{2}}} \xrightarrow{t \rightarrow \infty} 0 \\ \Rightarrow \int_{|x| \leq R} |(e^{it\Delta}g_1)(x)|^2 &\leq \frac{CR^d \|g_1\|_{L^1}}{|4\pi t|^{\frac{d}{2}}} \xrightarrow{t \rightarrow \infty} 0 \end{aligned}$$

On the other hand, with a good choice of g_2 :

$$\begin{aligned} \|e^{it\Delta}g_2\|_{L^2(\mathbb{R}^d)} &= \|g_2\|_{L^2(\mathbb{R}^d)} \rightarrow 0 \\ \Rightarrow \int_{|x| \leq R} |(e^{it\Delta}g_2)(x)|^2 dx &\leq \|e^{it\Delta}g_2\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0 \end{aligned}$$

Conclusion:

$$\int_{|x| \leq R} |(e^{it\Delta}g)(x)|^2 dx \leq 2 \int_{|x| < R} |(e^{it\Delta}g_1)(x)|^2 + 2 \int_{|x| \leq R} |(e^{it\Delta}g_2)(x)|^2 \xrightarrow{t \rightarrow \infty} 0$$

■

Exercise 8.3 (E 13.4) Let $g \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Let $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ be the solution of the Schrödinger equation with the initial data g , namely

$$u(x, t) = (e^{it\Delta}g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy.$$

Prove that for all $2 \leq p \leq \infty$ we have

$$\lim_{t \rightarrow \infty} \|u(\bullet, t)\|_{L^p(\mathbb{R}^d)} = 0.$$

Solution. We already know:

$$\begin{aligned}\|e^{it\Delta}g\|_{L^\infty} &\leq \frac{\|g\|_{L^1}}{(4\pi t)^{\frac{d}{2}}} \xrightarrow{t \rightarrow 0} 0 \\ \|e^{it\Delta}g\|_{L^2} &= \|g\|_{L^2}\end{aligned}$$

Now, interpolation for all $2 < p < \infty$ gives

$$\|e^{it\Delta}g\|_{L^p} \leq \|e^{it\Delta}g\|_{L^2}^\theta \|e^{it\Delta}g\|_{L^\infty}^{1-\theta} \xrightarrow{t \rightarrow \infty} 0,$$

for $\theta \in (0, 1)$. ■

Exercise 8.4 (Bonus 12) Assume $g \in L^p(\mathbb{R}^d)$, for some $1 < p < 2$. Prove that for all $t > 0$, $e^{it\Delta}g$ is well-defined and belongs to $L^{p'}(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover:

$$\|e^{it\Delta}g\|_{L^{p'}} \xrightarrow{t \rightarrow \infty} 0$$

Hint: You can use Riesz–Thorin interpolation theorem.

Lemma 8.5 If $g \in L^2(\mathbb{R}^d)$, then for all $\epsilon > 0$ we can write $g = g_1 + g_2$, $g, g_2 \in L^2$, $g_1 \in L^1$. s.t. $\|g_2\|_{L^2} \leq \epsilon$ and $\|g_1\|_{L^1} \leq C_\epsilon$.

Proof.

$$g = \underbrace{g\mathbb{1}(|g| > L)}_{g_1} + \underbrace{g\mathbb{1}(|g| \leq L)}_{g_2}$$

Now $g \in L^2$, then $g_2 \in L^2$ and

$$\int_{\mathbb{R}^d} |g_2|^2 = \int_{\mathbb{R}^d} |g|^2 \mathbb{1}(|g| \leq L) \xrightarrow{L \rightarrow 0} 0$$

and $g_1 \in L^1$ since

$$\int_{\mathbb{R}^d} |g_1| = \int_{\mathbb{R}^d} |g| \mathbb{1}(|g| > L) \leq \int_{\mathbb{R}^d} |g| \frac{|g|}{L} = \frac{1}{L} \int_{\mathbb{R}^d} |g|^2 < \infty$$

Consequently:

$$\begin{aligned}\int_{|x| \leq R} |(e^{it\Delta}g)(x)|^2 dx &\leq 2 \int_{|x| \leq R} |e^{it\Delta}g_1(x)|^2 + 2 \int_{|x| \leq R} |e^{it\Delta}g_2(x)|^2 \\ &\leq CR^d \left(\frac{\|g\|_{L^1}^2}{t^{\frac{d}{2}}} \right) + \|g_2\|_{L^2}^2 \\ &\leq \frac{C_\epsilon R^d}{t^{\frac{d}{2}}} + \epsilon^2 \xrightarrow{t \rightarrow \infty} \epsilon^2 \xrightarrow{\epsilon \rightarrow 0} 0\end{aligned}$$
■

Theorem 8.6 If $g \in C_c^\infty(\mathbb{R}^d)$, then

$$u(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} g(y) dy,$$

$x \in \mathbb{R}^d, t > 0$. satisfies $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$ and

$$\begin{cases} -i\partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ \lim_{t \rightarrow 0} u(x, t) = g(x) \text{ uniformly } x \in \mathbb{R}^d \end{cases}.$$

Proof.

$$\begin{aligned}\hat{u}(k, t) &= e^{-it|2\pi k|^2} \hat{g}(k) \quad \forall k \in \mathbb{R}^d, \forall t > 0 \\ \Rightarrow \widehat{D^\alpha u}(k, t) &= e^{-it|2\pi k|^2} (2\pi i k)^\alpha \hat{g}(k), \quad (2\pi i k)^\alpha = \prod_{j=1}^d (2\pi i k_j)^{\alpha_j} \\ \Rightarrow \|\widehat{D^\alpha u}(k, t)\|_{L^1(\mathbb{R}^d, dk)} &= \int_{\mathbb{R}^d} |(2\pi i k)^\alpha| |\hat{g}(k)| dk \leq \int_{\mathbb{R}^d} |k|^{|\alpha|} |\hat{g}(k)| dk < \infty\end{aligned}$$

as $|\hat{g}(k)| \leq \frac{C_N}{(1+|k|)^N}$ for all $N \geq 1$. This implies $D^\alpha u(\bullet, t) \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Moreover,

$$\partial_t \hat{u}(k, t) = e^{-it|2\pi k|^2} \hat{g}(k) \underbrace{(-i|2\pi k|^2)}_{\in L^1(\mathbb{R}^d, dk)}$$

$\leadsto t \mapsto u(x, t) \in C^1$. Similary $t \mapsto u(x, t) \in C^m$ for all $m \geq 1$.

$$\begin{aligned}-i\partial_t \widehat{u} - \widehat{\Delta_x u} &= (-i)e^{-it|2\pi k|^2} (-i|2\pi k|^2) \hat{g}(k) + e^{-it|2\pi k|^2} |2\pi k|^2 \hat{g}(k) = 0. \\ \Rightarrow -i\partial_t u - \Delta_x u &= 0 \quad \text{almost everywhere in } \mathbb{R}^d \\ \Rightarrow -i\partial_t u - \Delta_x u &= 0 \quad \text{everywhere in } \mathbb{R}^d\end{aligned}$$

Before we proved $u(x, t) = (e^{it\Delta} g)(x) \rightarrow g(x)$ in $L^2(\mathbb{R}^d)$. Here we can do better:

$$\int_{\mathbb{R}^d} |\hat{u}(k, t) - \hat{g}(k)| dk = \int_{\mathbb{R}^d} \underbrace{|e^{-it|2\pi k|^2} - 1|}_{\xrightarrow{t \rightarrow 0} 0} \underbrace{|\hat{g}(k)|}_{\in L^1} \xrightarrow{t \rightarrow 0} 0$$

So we get

$$\sup_{x \in \mathbb{R}^d} |u(x, t) - g(x)| \xrightarrow{t \rightarrow 0} 0. \quad \blacksquare$$

Theorem 8.7 1. If $g \in H^2(\mathbb{R}^d)$, then $u = e^{it\Delta} g$ satisfies

- $-i \frac{d}{dt} u(\bullet, t)$ exists as a function in $L^2(\mathbb{R}^d)$ and $= \Delta_x u$ for all $t > 0$.

$$\left(-i \frac{d}{dt} u(\bullet, t) = \frac{u(\bullet, t+h) - u(\bullet, t)}{h} \xrightarrow{h \rightarrow 0} \Delta_x u = e^{it\Delta} \underbrace{(\Delta g)}_{\in L^2} \in L^2(\mathbb{R}^d) \right)$$

- $u(\bullet, t) \xrightarrow{t \rightarrow 0} g$ in L^2 .

2. If $g \in L^2(\mathbb{R}^d)$, then $u = e^{it\Delta} g$ satisfies:

$$-i \frac{d}{dt} \langle \phi, u \rangle_{L^2} = \langle \Delta \phi, u \rangle_{L^2} \quad \forall \phi \in C_c^\infty(\mathbb{R}^d) \text{ (or } \phi \in H^2(\mathbb{R}^d))$$

(Here $-i \frac{d}{dt} = \Delta_x u$ is meant in distributional sense)

Proof. 1. $g \in H^2(\mathbb{R}^d)$,

$$\begin{aligned}
& \left\| \frac{u(\bullet, t+h) - u(\bullet, t)}{h} e^{it\Delta}(\Delta g) \right\|_{L^2(\mathbb{R}^d)}^2 = \left\| \frac{\widehat{u(\bullet, t+h) - u(\bullet, t)}}{h} e^{it\Delta}(\Delta g) \right\|_{L^2(k, dk)}^2 \\
&= \int_{\mathbb{R}^d} \left| \frac{e^{-(t+h)|2\pi k|^2} \hat{g}(k) - e^{-it|2\pi k|^2} \hat{g}(k)}{h} - e^{-it|2\pi k|^2} |2\pi k|^2 \hat{g}(k) \right|^2 dk \\
&= \int_{\mathbb{R}^d} \underbrace{\left| (-i) \frac{(e^{-ih|2\pi k|^2} - 1)}{h} + |2\pi k|^2 \right|^2}_{\xrightarrow{h \rightarrow 0} 0} |\hat{g}(k)|^2 dk \xrightarrow{h \rightarrow 0} 0
\end{aligned}$$

by Dominated Convergence, $|\dots| \leq C|k|^2 |\hat{g}(k)|^2 \in L^1$. We have

$$(-i) \frac{e^{-ih|2\pi k|^2} - 1}{h} \xrightarrow{h \rightarrow 0} (-i)^2 |2\pi k|^2 = -|2\pi k|^2$$

and $|\dots| \leq C|k|^2$. Also

$$\begin{aligned}
& \hat{u} = \widehat{e^{it\Delta} g} = e^{-it|2\pi k|^2} \hat{g}(k) \\
& \Rightarrow |2\pi k|^2 \hat{u}(k) = e^{-it|2\pi k|^2} \underbrace{|2\pi k|^2 \hat{g}(k)}_{\in L^2(dk)} \in L^2(dk) \\
& \Rightarrow \widehat{\Delta_x u} = e^{it\Delta}(\Delta g) \\
& \Rightarrow \Delta_x u = e^{it\Delta}(\Delta g)
\end{aligned}$$

2. $g \in L^2$:

$$\begin{aligned}
-i \frac{d}{dt} \langle \phi, u \rangle &= -i \frac{d}{dt} \int_{\mathbb{R}^d} \underbrace{\widehat{\phi}(k)}_{\leq \frac{C_N}{(1+|k|)^N} \forall N \geq 1} e^{-it|2\pi k|^2} \hat{g}(k) dk \\
&= \int_{\mathbb{R}^d} \widehat{\phi}(k) (-|2\pi k|^2) \hat{g}(k) dk \\
&= \int_{\mathbb{R}^d} \widehat{\Delta \phi} \hat{g} = \langle \Delta \phi, g \rangle_{L^2}
\end{aligned}$$

Spectral technique: If \mathcal{H} is a Hilbertspace (separable) and $A : D(A) \rightarrow \mathcal{H}$ is a self-adjoint $A = A^*$, i.e.

$$\begin{cases} \langle u, Av \rangle = \langle Au, v \rangle \\ D(A) = D(A^*) \end{cases}$$

where

$$\begin{aligned}
D(A^*) &= \{u \in \mathcal{H} \mid \langle u, Av \rangle = \langle A^* u, v \rangle\} \\
&= \left\{ u \in \mathcal{H} \mid \sup_{v \in D(A)} \frac{|\langle u, Av \rangle|}{\|v\|_{\mathcal{H}}} < \infty \right\}
\end{aligned}$$

Then we can define a unitary operator $e^{-itA} : \mathcal{H} \rightarrow \mathcal{H}$ with $t \in \mathbb{R}$. Moreover, for all $g \in \mathcal{H}$, $u = e^{-itA} g$ solves

$$\begin{cases} i\partial_t u = Au \\ u(t \rightarrow 0) = g \end{cases}$$

(Stone Theorem) (eg $A = -\Delta$)

■

Theorem 8.8 (Spectral theorem) If A is a self-adjoint operator on a separable Hilbert space, then, there is a unitary operator $U : \mathcal{H} \rightarrow L^2(\Omega)$ s.t. $UAU^* = M_f$ on $L^2(\Omega)$, where M_f is a multiplication operator of a real-value function $f : \Omega \rightarrow \mathbb{R}$:

$$(M_f)(\phi)(x) = f(x)\phi(x)$$

for all $x \in \Omega, \forall \phi \in L^2(\Omega)$. Consequence: $Ue^{-itA}U^* = e^{-itM_f}$ (multiplication) on $L^2(\Omega)$ where

$$(e^{-itM_f}\phi)(x) = e^{-itf(x)}\phi(x)$$

for all $x \in \Omega, \phi \in L^2(\Omega)$.

Example 8.9 $A = -\Delta$ on $\mathcal{H} = L^2(\mathbb{R}^d)$, $U = \mathcal{F}$ (Fourier Transform), $\mathcal{F}(-\Delta)\mathcal{F}^* = |2\pi k|^2$ multiplication.

$$\mathcal{F}(e^{-itA}\phi) = e^{-it|2\pi k|^2}\hat{\phi}(k)$$

Example 8.10 Consider $A = -\Delta + V(x)$ in $L^2(\mathbb{R}^d)$, $V(x) \in C_c^\infty(\mathbb{R}^d)$ or $V(x) = |x|^2$. If V is nice, then A can be defined $D(A) \rightarrow L^2(\mathbb{R}^d)$ as a self-adjoint operator. $\rightsquigarrow e^{-itA} = e^{-it(-\Delta+v)}$ is well-defined as a unitary operator for all $t \in \mathbb{R}$. In particular, for all $g \in L^2$, $u = e^{-it(-\Delta+v)}g$ solves

$$\begin{cases} i\partial_t u = (-\Delta + v)u \\ u(t \rightarrow 0) = g \end{cases}$$

Scatter Theory: $e^{-it(-\Delta+v)}g$ as $t \rightarrow \infty$.

1. RAGE: If $-\Delta + V$ has no eigenfunction, then

$$\frac{1}{T} \int_0^T \int_{|x| \leq R} \left| \left(e^{-it(-\Delta+v)}g \right)(x) \right|^2 dx dt \xrightarrow{T \rightarrow \infty} 0$$

2. (Asymptotic completeness) If $0 \leq V \in C_c^\infty$ is small, then there is a unitary operator U on L^2 s.t. for all $g \in L^2$:

$$\left\| e^{-it(-\Delta+v)}g - e^{-it\Delta}(Ug) \right\|_{L^2(\mathbb{R}^d)} \xrightarrow{t \rightarrow \infty} 0$$