Partial Differerential Equations

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Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} f(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geq 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives. Recall $Omega \subseteq \mathbb{R}^d$, $f : \Omega \to \{\mathbb{R}, \mathbb{C}\}$ open,

$$\partial_{x_1} f(x) = \lim \frac{f(x + he_i) - f(x)}{h}, e_i = (0, 0, 1, , 0, 0) \in \mathbb{R}^d$$

$$D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \ddots \partial_{x_d}^{\alpha_d} f(x), |\alpha| = \sum_{i=1}^{d} |\alpha_i|$$

$$Df = \nabla f = \text{ gradient of } f = (\partial x_1 \ldots \partial_{x_d})$$

$$\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$$

$$D^k f = (D^\alpha f)_{|\alpha| = k}$$

$$D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i,j \leq d}$$

Goals: For *solving a PDE* we want to

- Find an explizit solution! In many cases, it is impossible
- Prove a *well-posted theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have to notations of solution:
$\Omega \subseteq R^d$ Navier-Stokes equation:

$$\begin{cases} d_t u + u \nabla u - \Delta u = \nabla f, \quad f \text{ is known} \\ \text{div}(u) = \sum_{i=1}^d \frac{\partial}{\partial x_1} u_i(x) = 0 \end{cases}$$

open in 3D, exists smooth solution. In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

When $f = f(x)$ *leadsto* linear equation $\rightsquigarrow$ we will form on that!
$f = f(x, u, \Delta u) \rightsquigarrow$ nonlinear equation

Chapter 2

Laplace / Poisson Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Theorem 2.1 (Gauss-Green Theorem)

$$\int_{\partial V} F\vec{u}\ dS(x) = \int_V \mathrm{div}(F)\ dx$$

Thus

$$0 = \int_{\partial V} \nabla u \vec{n}\ dS(x) = \int_V \mathrm{div}(\nabla u)\ dx = \int_V \nabla u(x)\ dx$$

for any $V \subseteq \Omega$ open.

Exercise 2.2 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \to \mathbb{R}$ be continuous. Prove that if $\int_B f(x)\ dx = 0$, then $u \equiv 0$ in Ω.

Theorem 2.3 (Fundamential Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x)\ dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.4 (Solving Laplace Equation)

$$\frac{\partial r}{\partial x} = \frac{\partial_{x_i}}{\partial_{x_i}}(\sqrt{x_1^2 + \cdots + x_d^2}) = \frac{1}{2\sqrt{x_1^2 + \cdots + x_d^2}}(2x_i) = \frac{x_i}{r}$$

Then

$$\partial_{x_i} u = \partial_{x_i} v = (\partial_r v)\frac{\partial r}{\partial_{x_i}} = v'(r)\frac{x_i}{r}$$
$$\partial x_i^2 u = \partial_{x_i}(v(r)'\frac{x_i}{r}) = \partial x_i(v(r)')\frac{x_i}{r} + v'(r)\partial_{x_i}(\frac{x_i}{r}) = \partial_r(v'(r))(\frac{dr}{\partial_{x_i}})\frac{x_i}{r} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}(\partial_{x_i} r)\right) = v'(r)\frac{x_i^2}{r^2} + v'r(r)$$

So we have $\Delta u = \left(\sum_{i=1}^d d_{x_i}^2\right) u = v''(r) + v'(r)(\frac{d}{r} - \frac{1}{r})$
Thus $Deltau = v'(r) + v(r)\frac{d-1}{r}$. We consider $d \geq 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r)\frac{d-1}{r} = 0$

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|) & ,d=2 \\ \frac{1}{4\pi|x|} & ,d=3 \\ \frac{1}{(d-2)d|B_1|}\frac{1}{|x|^{d-2}} & ,d\geq 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0,1) \subseteq \mathbb{R}^d$.

Remark 2.5 $\Delta\Phi(x)) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.