Partial Differerential Equations

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Contents

Please note that I write this lecture notes for my personal use. I may write things differently than presented in the lecture. This script also contains solutions for exercises (which may be wrong). Of course, I don't push them to GitHub while the exercises still can be handed in.

Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} f(t) = af(t) \text{ for all } t \geqslant 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geqslant 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \to \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \to \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \to 0} \frac{f(x+he_i)-f(x)}{h}$, where $e_i = (0, 0, \ldots, 1, \ldots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial x_1, \ldots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha|=k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leqslant i,j \leqslant d}$

Definition 1.4 Given a function F. Then the equation of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u: \ \Omega \subseteq \mathbb{R}^d \longrightarrow \mathbb{R}$ is called a *PDE of order k*.

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1}u, D^{k-2}u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explizit solution! This is in many cases impossible.
- Prove a *well-posted theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continous and differentiable functions) Let $\Omega \subseteq R^d$ be open

$$\begin{aligned} C(\Omega) &= \{f: \ \Omega \to \mathbb{R} \mid f \text{ continuous}\} \\ C^k(\Omega) &= \{f: \ \Omega \to \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \ \leqslant k\} \end{aligned}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f: \ \Omega \to \mathbb{R} \text{ lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leqslant p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leqslant k : D^\alpha f \in L^p(\Omega) \text{exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in $\mathbb{R}^d$. If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then u is a harmonic function in Ω.

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \; dS(x) = \int_K \operatorname{div}(\vec{F}) \; dx$$

where ν is the outward unit normal vector field on ∂K. Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \; dS(x) = \int_V \operatorname{div}(\nabla u) \; dx = \int_V \Delta u(x) \; dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

   where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f) g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \to \mathbb{R}$ be continuous. Prove that if $\int_B f(x) \; dx = 0$, then $u \equiv 0$ in Ω.

Theorem 2.5 (Fundamential Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x) \; dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in $\mathbb{R}^d$. Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \to \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2 + \cdots + x_d^2}\right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned}
\partial_{x_i} u &= \partial_{x_i} v = (\partial_r v)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r} \\
\partial^2_{x_i} u &= \partial_{x_i}\left(v(r)'\frac{x_i}{r}\right) = (\partial_{x_i} v(r)')\frac{x_i}{r} + v'(r)\partial_{x_i}\left(\frac{x_i}{r}\right) \\
&= (\partial_r v'(r))\left(\frac{dr}{\partial_{x_i}}\right)\frac{x_i}{r} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}(\partial_{x_i} r)\right) = v'(r)\frac{x_i^2}{r^2} + v'r(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)
\end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d d^2_{x_i}\right) u = v''(r) + v'(r)(\frac{d}{r} - \frac{1}{r})$
Thus $\Delta u = v'(r) + v(r)\frac{d-1}{r}$. We consider $d \geqslant 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r)\frac{d-1}{r} = 0$
$\Rightarrow \log(v(r))' = \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)'$ (recall $log(f)' = \frac{f'}{f}$)
$\Rightarrow v'(r) = \frac{1}{v^{d-2} + \text{const.}}$

$$\begin{cases} \frac{const}{r^{d-2}} + const x x + const & , d \geqslant 3 \\ const \log(r) + const x x r + const & , d = 2 \end{cases}$$

Definition 2.7 (Fundamential Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|}\frac{1}{|x|^{d-2}}, & d \geqslant 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0,1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta\Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in $\mathbb{R}^d$. The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y)f(y)\ dy = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\ dy$$

This can be heuristically justifyfied with

$$-\Delta(\Phi \star f) = (-\Delta\Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C^2_c(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y)f(x-y)\,dy.$$

First we check that u is continuous: Take $x_k \to x_0$ in $\mathbb{R}^d$. We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y)\ dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y)\ dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n\to\infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{ for all } y \in \mathbb{R}^d \backslash \{0\}$$

and

$$|\Phi(y) f(x-y)| \ \leqslant \|f||_{L^\infty} \cdot \mathbb{1}(|y| \leqslant R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\operatorname{supp}(f)$ but independent of y. Now we compute the derivatives:

$$\partial_{x_i} u(x) = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x-y)\ dy = \lim_{h\to 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + he_i - y) - f(x-y)}{h}\ dy$$

$$\text{(dom. conv.)} \quad = \int \Phi(y) \partial_{x_i} f(x-y)\ dy$$

$$\Rightarrow \quad D^\alpha u(x) = \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x-y)\ dy \quad \text{for all } |\alpha| \leqslant 2$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y)(-\Delta_x) f(x-y)\, dy = \int_{\mathbb{R}^d} \Phi(y)(-\Delta_y) f(x-y)\, dy \\ &= \int_{\mathbb{R}^d \backslash B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy + \int_{B(0,\epsilon)} \Phi(y)(-\Delta_x) f(x-y)\, dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d \backslash B(0,\epsilon)} \Phi(y)(-\Delta_y) f(x-y)\, dy \\ &\quad = \int_{\mathbb{R}^d \backslash B(0,\epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x-y)\, dy - \int_{\partial B(0,\epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \\ &\quad = \int_{\mathbb{R}^d \backslash B(0,\epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x-y)\, dy \\ &\qquad + \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\, dS(y) - \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y)\, dS(y) \end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0,\epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1|\epsilon^{d-1}} \quad \text{ for } y \in \partial B(0,\epsilon)$$

Hence:

$$\begin{aligned} \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x-y)\ dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(0,\epsilon)} f(x-y)\ dS(y) \\ = ⨍_{\partial B(0,\epsilon)} f(x-y)\ dS(y) &= ⨍_{\partial B(x,\epsilon)} f(y)\ dS(y) \xrightarrow{\epsilon \to 0} f(x) \end{aligned}$$

We have to regard the following error terms:

- $\left|\int_{B(0,\epsilon)} \Phi(y)(-\Delta_y)f(x-y)\ dy\right| \leqslant \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leqslant \|\Delta f\|_{L^\infty}\mathbb{1}(|y|\leqslant R)}\ dy$

  $$\leqslant \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)|\mathbb{1}(|y|\leqslant R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y|\leqslant \epsilon) \xrightarrow{\epsilon\to 0} 0$$

  Where $R>0$ depends on x and the support of f but is independent of y.

- $\left|\int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f}{\partial \vec{n}}(x-y)\ dS(y)\right| \leqslant \|\nabla f\|_{L^\infty}\int_{\partial B(0,\epsilon)} |\Phi(y)|\ \ dy$

  $$\leqslant \begin{cases} const\cdot \epsilon|\log\epsilon| \ \to 0, & d=2 \\ const\cdot\epsilon \to 0, & d\geqslant 3\end{cases}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x\in\mathbb{R}^d$ proved that $u=\Phi\star f$ and $f\in C_c^2(\mathbb{R}^d)$. ■

Thus, if $f\in C_c^2(\mathbb{R})$, then $u=\Phi\star f$ satisfies $u\in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x\in\mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f. For example if $f\in C_c^1(\mathbb{R})$ we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y)\partial_{x_i} f(x-y)\,dy \in C(\mathbb{R}^d) \Rightarrow u\in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i}\partial_{x_j} u = \partial_{x_i}\int_{\mathbb{R}^d}\Phi(y)\partial_{x_j} f(x-y)\,dy = \int_{\mathbb{R}^d}\partial_{x_i}\Phi(y)\partial_{x_j} f(x-y)\,dy \in C(\mathbb{R}^d)$$

So we have $u\in C^2(\mathbb{R}^d)$. Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i}\Phi(y)\partial_{x_i} f(x-y)\,dy \overset{(IBP)}{=} f(x).$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega\subseteq\mathbb{R}$ be open, let $u\in C^2(\Omega)$ and $\Delta u = 0$ in Ω. Then

$$u(x) = \fint_{B(x,r)} u = \fint_{\partial B(x,r)} u \quad \text{for all } x\in\Omega, B(x,r)\subseteq\Omega$$

Proof. Consider all $r>0$ s.t. $B(x,r)\subseteq\Omega$,

$$f(r) = \fint_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r. When it is done, then we immediately obtain

$$f(r) = \lim_{t\to 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr}\left(⨍_{\partial B(0,r)} u(x+y)\,dS(y)\right) \\
&= \frac{d}{dr}\left(⨍_{\partial B(0,1)} u(x+rz)\,dS(z)\right) \\
\text{(dom. convergence)} \quad &= ⨍_{\partial B(0,1)} \frac{d}{dr}[u(x+rz)]\,dS(z) \\
&= ⨍_{\partial B(0,1)} \nabla u(x+rz)z\,dS(z) \\
&= ⨍_{\partial B(x,r)} \nabla u(y)\frac{y-x}{r}\,dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla\cdot u(y)\cdot \vec{n_y}\,dS(y) \\
\text{(Gauss-Green 2.2)} \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0}\,dy = 0
\end{aligned}$$

■

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$ (Linear Equation)

Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, \omega \in S^{d-1}$, then

$$\int_{B(0,r)} g(y)\,dy = \int_0^r \left(\int_{B(0,r)} g(y)\,dS(y)\right) dr$$

Remark 2.15 We already proved that for u harmonic we have $u(x) = ⨍_{\partial B(x,r)} u\,dy$. Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y)\,dy &= \int_{B(0,r)} u(x+y)\,dy \\
\text{(Pol. decomposition)} \quad &= \int_0^r \left(\int_{\partial B(0,s)} u(x+y)\,dS(y)\right) ds \\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y)\,dS(y)\right) ds \\
\text{(Mean value property)} \quad &= \int_0^r (|\partial B(x,s)|\,u(x))\,ds = |B(x,r)|\,u(x)
\end{aligned}$$

This implies

$$⨍_{B(x,r)} u(y)\,dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.16 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = ⨍_{B(x,r)} u(y)\,dy = ⨍_{\partial B(x,r)} u(y)\,dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then u is harmonic, i.e. $\Delta u = 0$ in Ω. (The proof is exactly like before)

Theorem 2.17 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω. Then

a) $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$

b) Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$, then $u \equiv$ *const.* in Ω.

Proof. Given $U \subseteq \mathbb{R}^d$ open, we can write $U = \bigcup_i U_i$, where U_i is open and connected.

b) Assume that Ω is connected and there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{y \in \Omega} u(x)$. Define $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$. U is closed since u is continuous. Moreover, U is open by the mean-value theorem. I.e. for all $x \in U$ there is a $r > 0$ s.t. $B(x,r) \subseteq U \subseteq \Omega$. Since U is connected we get $U = \Omega$, so u is constant in Ω. On the other hand, if there is no $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega}$ we have $\forall x_0 \in \Omega : \quad u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$

a) Given $\Omega \subseteq \mathbb{R}^d$ open, we can write $\Omega = \bigcup_i \Omega_i$, where Ω_i is open and connected. By b) we have
$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$
So we can conclude
$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x).$$
∎

Definition 2.18
- If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$, then u is called *sub-harmonic* if $\Delta u \geqslant 0$ in Ω.
- If $\Delta u \leqslant 0$, then u is called *super-harmonic*.

Exercise 2.19 (E 1.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$ be subharmonic.

a) Prove that u satisfies the Mean Value Inequality
$$⨍_{\partial B(x,r)} u(y)\, dS(y) \geqslant ⨍_{B(x,r)} u(y)\, dy \geqslant u(x)$$
for all $B(x,r) \subseteq \mathbb{R}^d$.

b) Assume further that Ω is connected and $u \in C(\bar{\Omega})$. Prove that u satisfies the strong maximum principle, namely either
   - u is constant in Ω, or
   - $\sup_{y \in \partial\Omega} u(y) > u(x)$ for all $x \in \Omega$.

My Solution. a) Let $f(r) = \fint_{\partial B(x,r)} u(y)\, dS(y)$, then we have

$$\begin{aligned}
\partial_r f(r) &= \partial_r \fint_{\partial B(x,r)} u(y)\, dS(y) \\
\text{(Dom. Convergence)} \quad &= \fint_{\partial B(x,r)} \partial_r u(y)\, dS(y) \\
&= \fint_{\partial B(0,1)} \partial_r u(x+yr)\, dS(y) \\
&= \fint_{\partial B(0,1)} \nabla u(x+yr) \cdot y\, dS(y) \\
&= \fint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\, dS(y) \\
&= \fint_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}_y\, dS(y) \\
\text{(Gauss-Green)} \quad &= \fint_{B(x,r)} \operatorname{div}(\nabla u(y))\, dS(y) \\
&= \fint_{B(x,r)} \underbrace{\Delta u(y)}_{\geqslant 0}\, dS(y) \geqslant 0
\end{aligned}$$

So we can conclude that

$$\fint_{\partial B(x,r)} u(y)\, dS(y) = f(r) \geqslant \lim_{r\to 0} f(r) = u(x).$$

Now regard

$$\begin{aligned}
\int_{B(x,r)} u(y)\, dy &= \int_0^r \left(\int_{\partial B(x,r)} u(y)\, dS(y) \right) ds \\
&= \int_0^r \left(|\partial B(x,r)| \fint_{\partial B(x,r)} u(y)\, dS(y) \right) ds \\
&\geqslant \int_0^r |\partial B(x,r)| \cdot u(x)\, dS(y) \\
&= u(x) \int_0^r |\partial B(x,r)|\, dS(y) = u(x)|B(x,r)|.
\end{aligned}$$

Thus we have

$$u(x) \leqslant \fint_{B(x,r)} u(y) dy.$$

Finally, lets regard

$$\begin{aligned}
\int_{B(x,r)} u(y)\, dy &= \int_0^r \left(|\partial B(x,s)| \fint_{\partial B(x,s)} u(y)\, dS(y) \right) ds \\
(\partial_r f(r) \geqslant 0) \quad &\leqslant \int_0^r \left(|\partial B(x,s)| \fint_{\partial B(x,r)} u(y)\, dS(y) \right) ds \\
&= \fint_{\partial B(x,r)} u(y)\, dS(y) \int_0^r |\partial B(x,s)|\, ds \\
&= \fint_{\partial B(x,r)} u(y)\, dS(y) \cdot |B(x,s)|
\end{aligned}$$

and we conclude

$$\fint_{B(x,r)} u(y)\, dy \leqslant \fint_{\partial B(x,r)} u(y)\, dS(y).$$

b) Let $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega} u(x)$. Now,

$$\begin{aligned} \sup_{x\in\Omega} u(x) = u(x_0) &\leqslant \fint_{\partial B(x_0,r)} u(y)\, dy \\ &\leqslant \fint_{\partial B(x_0,r)} \sup_{x\in\Omega} u(x)\, dy = \sup_{x\in\Omega} u(x) \end{aligned}$$

Since u is continous we get $u(y) = u(x_0)$ for all $y \in B(x_0, r)$, so u is constant. ∎

Definition 2.20 The *Poisson Equation* for given f, g on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Theorem 2.21 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open and connected. Let $f \in C(\Omega), g \in C(\partial\Omega)$. Then there exists *at most* one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof. Assume that we have two solutions u_1 and u_2. Then $u := u_1 - u_2$ is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that $u = 0$ in Ω. More precisely, by the maximum principle we have $\forall x \in \Omega$

$$\sup_{x\in\Omega} u(x) \leqslant \sup_{x\in\partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leqslant 0$$

Since $-u$ satisfies the same property we have $\forall x \in \Omega$:

$$\sup_{x\in\Omega}(-u(x)) \leqslant \sup_{x\in\partial\Omega}(-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leqslant 0 \quad \Rightarrow \quad u(x) \geqslant 0$$

So we geht $u(x) = 0$ in Ω. ∎

Exercise 2.22 (Bonus 1) Let Ω be open, connected and bounded in $\mathbb{R}^d$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Prove that

a) If $g \geqslant 0$ on $\partial\Omega$, then $u \geqslant 0$ in Ω.

b) If $g \geqslant 0$ on $\partial\Omega$ and $g \neq 0$, then $u > 0$ in Ω.

My Solution. a) We have that $\Delta(-u) = 0$, so $-u$ is harmonic in Ω. Since Ω is open and bounded we can apply the Maximum Principle (2.17) and get that

$$\sup_{x\in\bar{\Omega}} -u(x) \leqslant \sup_{x\in\partial\Omega} -g(x) \leqslant 0.$$

This implies $\inf_{x\in\Omega} u(x) \geqslant 0$, so $u \geqslant 0$ in Ω.

b) We prove this by contraposition. Assume there is a $x_0 \in \Omega$ s.t. $u(x_0) = 0$. Since we have $u \geqslant 0$ on Ω by a), it follows that

$$0 = -u(x_0) = \sup_{x\in\bar{\Omega}} -u(x) \leqslant \sup_{x\in\partial\Omega} -g(x) \leqslant 0,$$

so $-u$ attains its maximum on Ω. Hence $-u = 0 = u$ is constant by the strong maximum principle because Ω is connected, in fact $0 = u|_{\partial\Omega} = g$. ■

Lemma 2.23 (Estimates for derivatives) If u is harmonic in $\Omega \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| = N$ and $B(x_0, r) \subseteq \Omega$, then

$$|D^\alpha u(x)| \leqslant \frac{(c_d N)^N}{r^{d+N}} \int_{B(x,r)} |u|\, dy$$

Proof. Induction: Assume $|\alpha| = N-1$, Take $|\alpha| = N$

$$|D^\alpha u(x_0)| \leqslant \frac{|S_1|}{|B_1|\frac{r}{N}} \|D^\beta u\|_{L^\infty(B(x_0,\frac{r}{n}))}, \quad D^\alpha u = \partial_{x_i}(D^\beta u)_{|\beta|=N-1}$$

Note: $x \in B(x_0, \frac{r}{N})$, so $B(x, \frac{r(N-1)}{N}) \subseteq B(x_0, r)$. By the induction hypothesis:

$$\|D^\beta u\|_{L^\infty(B(x_0,\frac{r}{N}))} \leqslant \frac{[c_d(N-1)]^{N-1}}{[r\frac{(N-1)}{N}]^{d+N-1}} \int_{B(x_0,r)} |u|\, dy$$

The conclusion is:

$$\begin{aligned}
|D^\alpha u(x_0)| &\leqslant \frac{|S_1|}{|B_1|\frac{r}{N}} \frac{[c_d(N-1)]^{N-1}}{\left(r\frac{N-1}{N^d}\right)^{d+N-1}} \int_{B(x_0,r)} |u|\, dy \\
&= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{\left(\frac{r}{N}\right)^{d+N} (N-1)^d} \int_{B(x_0,r)} |u|\, dy \\
&= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{\left(\frac{r}{N}\right)^{d+N} N^d} \left(\frac{N}{N-1}\right)^d \int_{B(x_0,r)} |u|\, dy \\
&\leqslant \frac{2^d|S_1|}{|B_1|} \frac{c_d^{N-1} N^N}{r^{d+N}} \int_{B(x_0,r)} |u|\ \ dy \qquad \text{if } c_d \geqslant \frac{2^d|S_1|}{|B_1|}
\end{aligned}$$

■

Theorem 2.24 (Regularity) Let Ω be open in $\mathbb{R}^d$. Let $u \in C(\Omega)$ satisfy $u(x) = ⨍_{\partial B} u\, dy$ for any $x \in B(x,r) \subseteq \Omega$. Then u is a harmonic function in Ω. Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω.

Exercise 2.25 (E 1.1: Proof the Gauss–Green formula) Let $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that for every open ball $B(y,r) \subseteq \mathbb{R}^d$ we have

$$\int_{\partial B(y,r)} f(y) \cdot \nu_y \, dS(y) = \int_{B(y,r)} \operatorname{div} f \, dx.$$

Here ν_y is the outward unit normal vector and dS is the surface measure on the sphere.

Solution. We proof this in d=3. Let $f \in C^1(\mathbb{R}^3)$

$$\int_{B(0,1)} \partial_{x_3} f \, dx = \int_{\partial B(0,1)} f x_3 \, dS(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \vec{n} = \frac{x}{|x|} \text{ on } \partial B(0,1)$$

$$\begin{aligned} B(0,1) &= \{x_1^2 + x_2^2 + x_3^2 \leqslant 1\} \\ &= \{x_1^2 + x_2^2 \leqslant 1 - \sqrt{1 - x_1^2 - x_2^2} \leqslant x_3 \leqslant \sqrt{1 - x_1^2 - x_2^2}\} \end{aligned}$$

Then:

$$\begin{aligned} \int_{B(0,1)} \partial_{x_3} f \, dx &= \int_{x_1^2 + x_2^2 \leqslant 1} \left(\int_{-\sqrt{1-x_1^2-x_2^2} \leqslant x_3 \leqslant \sqrt{1-x_1^2-x_2^2}} \partial_{x_3} f \, dx_3 \right) dx_1 \, dx_2 \\ &= \int_{x_1^2 + x_2^2 \leqslant 1} \Big[f(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) \\ &\qquad - f(x_1, x_2, -\sqrt{1 - x_1^2 - x_2^2}) \Big] \, dx_1 \, dx_2 \end{aligned}$$

Lets take polar coordinates in 2D:

$$\begin{aligned} x_1 &= r\cos\phi & \qquad r > 0, \phi \in [0, 2\pi) \\ x_2 &= r \sin\phi & \qquad \det \frac{\partial(x_1, x_2)}{\partial(r, \phi)} = r \end{aligned}$$

$$(\star) = \int_0^1 \int_0^{2\pi} [f(r\cos\phi, r\sin\phi, r) - f(r\cos\theta, r\sin\phi, -r)] r \, dr \, d\phi$$

On the other hand:

$$\int_{\partial B(0,1)} f x_3 \, dS$$

The polar coordinates in 3D are:

$$\begin{aligned} x_1 &= r\cos\phi sin\theta & \qquad r > 0, \phi \in (0, 2\pi), \theta \in (0, \pi) \\ x_2 &= r\sin\phi\sin\theta & \qquad \det \frac{\partial x_1, x_2, x_3}{\partial(r, \phi, t)} = r^2 \sin\theta \\ x_3 &= \cos\theta \end{aligned}$$

Then:

$$\begin{aligned} (\star\star) &= \int_0^{2\pi} \int_0^{\pi} f(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \sin\theta\cos\theta \, d\theta \, d\phi \\ &= \int_0^{2\pi} \left(\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} d\theta \right) d\phi \\ (r = \sin\theta) \qquad &= \int_0^{2\pi} \int_0^1 f(r\cos\phi, r\sin\phi, \sqrt{1 - r^2}) r \, dr \, d\phi \\ &\quad - f(r\cos\phi, r\sin\phi, -\sqrt{1 - r^2}) r \, dr \, d\phi \end{aligned}$$

■

Exercise 2.26 (E 1.2) Let $u \in C(\mathbb{R}^d)$ and $\int_{B(x,r)} u\,dy = 0$ for every open ball $B(x,r) \subseteq \mathbb{R}^d$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

My Solution. Assume there is a $x_0 \in \mathbb{R}^d$ s.t. w.l.o.g. $u(x_0) > 0$. Since u is continous there is a ball $B(x_0, r)$ s.t. $u(y) > \frac{u(x_0)}{2}$ for all $y \in B(x_0, r)$. But then we get

$$\int_{B(x_0,r)} u(y)\,dy \geqslant \int_{B(x_0,r)} \frac{u(x_0)}{2}\,dy = \frac{u(x_0)}{2}\,|B(x_0,r)| > 0. \qquad \blacksquare$$

Exercise 2.27 (E 1.3) Let $f \in C_c^1(\mathbb{R}^d)$ with $d \geqslant 2$ and $u(x) := (\Phi \star f)(x)$. Prove that $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ (2.9 was the same for $f \in C_1(\mathbb{R})$)

Theorem 2.28 (Liouville's Theorem) If $u \in C^2(\mathbb{R}^d)$ is harmonic and bounded, then $u = const$.

Proof. By the bound of the derivative 2.23 we have

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\leqslant \frac{c_d}{r^{d+1}} \int_{B(x_0,r)} |u|\,dy \quad \forall x_0 \in \mathbb{R}^d\ \forall r > 0 \\ &\leqslant \|u\|_{L^\infty} \frac{c_d}{r^{d+1}} |B(x_0,r)| \\ &\leqslant \|u\|_{L^\infty} \frac{c_d}{r} \xrightarrow{r\to\infty} 0 \end{aligned}$$

Thus $\partial_{x_i} u = 0$ for all $i = 1, 2, \ldots d$ and $u = const.$ in $\mathbb{R}^d$ $\blacksquare$

Theorem 2.29 (Uniqueness of solutions to Poisson Equation in $\mathbb{R}^d$) If $u \in C^2(\mathbb{R}^d)$ is a bounded function and satisfies $-\Delta u = f$ in $\mathbb{R}^d$ where $f \in C_c^2(\mathbb{R}^d)$, then we have

$$u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y)\,dy + C \quad \forall x \in \mathbb{R}^d$$

where C is a constant and Φ is the fundamental solution of the Laplace equation in $\mathbb{R}^d$.

Proof. If we can prove that v is bounded, then $v = const.$. We first need to show that $\Phi \star f$ is bounded.

$$\begin{aligned} \Phi = \Phi_1 + \Phi_2 = \Phi\mathbb{1}(|x| \leqslant 1) + \Phi(|x| \geqslant 1) \\ \Phi \star f = \Phi_1 \star f + \Phi_2 \star f \end{aligned}$$

We have $\Phi_1 \star f \in L^1(\mathbb{R}^d)$ and $\Phi_2 \star f$ is bounded since $\Phi \to 0$ as $|x| \to \infty$ in $d \geqslant 3$. $\blacksquare$

Exercise 2.30 (Hanack's inequality) Let $u \in C^2(\mathbb{R}^d)$ be harmonic and non-negative. Prove that for all open, bounded and connected $\Omega \subseteq \mathbb{R}^d$, we have

$$\sup_{x\in\Omega} u(x) \leqslant C_\Omega \inf_{x\in\Omega} u(x),$$

where C_∞ is a finite constant depending only on Ω.

Proof. (Exercise) Hint: $\Omega = B(x,r)$. General case cover Ω by finitely many balls, one ball is inside Ω. $\blacksquare$

Chapter 3

Convolution, Fourier Transform and Distributions

Definition 3.1 (Convolution) Let $f, g : \mathbb{R}^d \to \mathbb{R}$ or $\mathbb{C}$.

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy = (g \star f)(x)$$

Remark 3.2 (Properties of the Convolution)

- $(f \star g)(x) = f \star (g \star h)$
- $\widehat{f \star g} = \hat{f} \star \hat{g}$

Theorem 3.3 (Young Inequality) If $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$, where $1 \leqslant p \leqslant \infty$, then $f \star g \in L^p(\mathbb{R}^d)$ and $\|f \star g\|_{L^p} \leqslant \|f\|_{L^1}\|g\|_{L^p}$. More generally, if $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^1(\mathbb{R}^d)$, $\|f \star g\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^q}$, where $1 \leqslant p, q, r, \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

Proof. Let $f \in L^1, g \in L^p$. With the Hölder Inequality **??**, we have:

$$\begin{aligned}
\|f \star g\|_{L^p}^p &= \int_{\mathbb{R}^d} |f \star g(x)|^p\,dx \\
&\leqslant \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)||g(y)|^p\,dy\,dx \\
&= \|f\|_{L^1}^{\frac{p}{q}+1}\|g\|_{L^p}^p
\end{aligned}$$

So we have $\|f \star g\|_{L^p} \leqslant \|f\|_{L^1}\|g\|_{L^p}$ ∎

Theorem 3.4 (Smoothness of the Convolution) If $f \in C_c^\infty(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$. Then $f \star g \in C^\infty(\mathbb{R})$ and

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for all $\alpha = (\alpha_1, \ldots, \alpha_d), \alpha_i \in \{0, 1, 2, \ldots\}$

Proof. First we note that $x \mapsto (f \star g)$ is continous as $x_n \to x$ in $\mathbb{R}^d$ since

$$(f \star g)(x_n) = \int_{\mathbb{R}^d} f(x_n - y)g(y)\,dy \xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^d} f(x-y)g(y)\,dy = (f \star g)(x)$$

We can apply Dominated convergence because

$$f(x_n - y)g(y) \to f(x-y)g(y) \quad \forall y \text{ as } f \text{ is continuous and } x_n \to x$$

and

$$|f(x_n - y)\ g(y)| \leqslant \|f\|_{L^\infty} |g(y)|\ \mathbb{1}(|y| \leqslant R) \in L^1(\mathbb{R}^d).$$

Where $R > 0$ satisfies $B(0,R) \supseteq \operatorname{supp} f + \sup_n |x_n|$. Now we can compute the derivatives:

$$\begin{aligned}
\partial_{x_i}(f \star g)(x) &= \lim_{h\to 0} \frac{(f\star g)(x+he_i) - (f \star g)(x)}{h} \\
&= \lim_{h\to 0} \int_{\mathbb{R}^d} \frac{f(x+he_i-y) - f(x-y)}{h} g(y)\, dy \\
\text{(Dominated Convergence)} \quad &= \int_{\mathbb{R}^d} \lim_{h\to 0} \frac{f(x+he_i-y) - f(x-y)}{h} g(y)\, dy \\
&= \int_{\mathbb{R}^d} (\partial_{x_i} f)(x-y) g(y)\ dy
\end{aligned}$$

We could apply Dominated Convergence since

$$\frac{f(x+he_i-y)-f(x-y)}{h} g(y) \xrightarrow{h\to 0} (\partial_{x_i} f)(x-y)g(y) \quad \text{as } f \in C^1$$
$$\left| \frac{f(x+he_i-y)-f(x-y)}{h} g(y) \right| \leqslant \|\partial_{x_i} f\|_{L^\infty} |g(y)|\ \mathbb{1}(|y| \leqslant R) \in L^1(\mathbb{R}^d)$$

where $B(0,R) \supseteq \operatorname{supp}(f) + B(0,|x|+1)$ and $\partial_{x_i}(f \star g) = (\partial_{x_i} f) \star g \in C(\mathbb{R}^d)$ since $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$. By induction we get $D^\alpha(f \star g) = (D^\alpha f \star g) \in C(\mathbb{R}^d)$. ■

Remark 3.5 Question: Is there a f s.t. $f \star g = g$ for all g. In fact there is no regular function f that solves this formally:

$$f \star g = g \Rightarrow \widehat{f \star g} = \hat{g} \Rightarrow \hat{f}\hat{g} = \hat{g} \ \Rightarrow \hat{f} = 1 \Rightarrow f \text{ is not a regular function!}$$

However, if f is the Dirac-Delta Distribution, $f = \delta_0$ then $\delta_0 \star g = g$ for all g. Formally:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \\ \int \delta_0 = 1 & \end{cases}$$

In fact, if $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$, then $f_\epsilon \to \delta_0$ in an appropriate sense and $f_\epsilon \star g \to g$ for all g nice enough.

Theorem 3.6 (Approximation by convolution) Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon})$. Then for all $g \in L^p(\mathbb{R}^d)$, where $1 \leqslant p < \infty$, then

$$f_\epsilon \star g \to g \quad \text{in } L^p(\mathbb{R}^\mathrm{d})$$

Proof.

Step 1: Let $f, g \in C_c(\mathbb{R}^d)$. Then

$$\begin{aligned}
(f_\epsilon \star g)(x) - g(x) &= \int_{\mathbb{R}^d} f_\epsilon(y) g(x-y)\, dy - \int_{\mathbb{R}^d} f_\epsilon(y) g(x)\, dy \\
&= \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x))\, dy \\
|(f_\epsilon \star g)(x) - g(x)| &= \left| \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x))\, dy \right| \\
&\leqslant \int_{\mathbb{R}^d} |f_\epsilon(y)||g(x-y) - g(x)| \;\, dy \\
&\leqslant \int_{|y| \leqslant R_\epsilon} |f_\epsilon(y)||g(x-y) - g(x)|\, dy \\
&\leqslant \underbrace{\int_{|y| \leqslant R_\epsilon} |f_\epsilon(y)| \;\, dy}_{\leqslant \|f_\epsilon\|_{L^1} = \|f\|_{L^1}} \left[\sup_{|z| \leqslant R} |g(x-z) - g(x)| \right] \xrightarrow{\epsilon \to 0} 0
\end{aligned}$$

We have Dominated Convergence since:

$$(f_\epsilon \star g)(x) - g(x) \to 0 \quad \text{as } \epsilon \to 0$$

and

$$|f_\epsilon \star g(x) - g(x)| \leqslant \|f\|_{L^1} \sup_{|z| \leqslant R_\epsilon} |g(x-z) - g(x)| \leqslant 2\|f\|_1 \|g\|_{L^\infty} \mathbb{1}(|x| \leqslant R_1).$$

Where $B(0, R_1) \supseteq \operatorname{supp}(g) + B(0, R_\epsilon)$, thus $f_\epsilon \star g \to g$ in $L^p(\mathbb{R}^d)$. To remove the technical assumptions $f, g \in C_c(\mathbb{R}^d)$, then we use a density argument. We use the fact that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$.

Step 2: Let $g \in C_c(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$. Then there is $\{g_m\} \subseteq L^p(\mathbb{R}^d), g_m \to g$ in $L^p(\mathbb{R}^d)$. Then

$$\begin{aligned}
\|f_\epsilon \star g - g\|_{L^p} &\leqslant \|f_\epsilon \star (g - g_m)\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
\text{(Young)} \quad &\leqslant \|f_\epsilon\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leqslant \|f\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leqslant (\|f\|_{L^1} + 1)\|g - g_m\|_{L^p} + \|f \star g_m - g_m\|_{L^p}
\end{aligned}$$

So we get:

$$\limsup_{\epsilon \to 0} \|f_\epsilon \star g - g\|_{L^p} \leqslant (\|f\|_{L^p} + 1)\|g - g_m\|_{L^p} + \underbrace{\limsup_{\epsilon \to 0} \|f_\epsilon \star g_m - g_m\|_{L^p}}_{0 \text{by step 1.}}$$

$$\xrightarrow{m \to \infty} 0$$

Step 3: Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Take $\{f_m\} \subseteq C_c(\mathbb{R}^d)$, s.t.

$$\begin{cases} F_m \to g in L^1(\mathbb{R}) \text{ as } m \to \infty \\ \int_{\mathbb{R}^d} F_m = 1 (\text{it is possible since } \int_{\mathbb{R}^d}) f = 1) \end{cases}$$

Define $F_{m,\epsilon}(x) = \epsilon^{-d} F_m(\epsilon^{-1} x)$ (recall $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1} x)$). Then:

$$\begin{aligned}
f_\epsilon \star g - g &= (f_\epsilon - F_{m,\epsilon}) \star g + F_{m,\epsilon} \star g - g \\
\Rightarrow \|f_\epsilon - g\|_{L^p} &\leqslant \underbrace{\|f_\epsilon - F_{m,\epsilon} \star g\|_{L^p}}_{\overset{\text{Young}}{\leqslant} \|f_\epsilon - F_{m,\epsilon}\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p}} + \|F_{m,\epsilon} \star g - g\|_{L^p}
\end{aligned}$$

$$\Rightarrow \limsup_{\epsilon \to 0} \|f_\epsilon \star g - g\|_{L^p} \leqslant \|f - F_m\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \qquad \blacksquare$$

Lemma 3.7 $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$

Proof. For all $g \in L^p(\mathbb{R}^d)$ there are g_m step functions and $g_m \to m$ in $L^p(\mathbb{R}^d)$, We can assume that Ω is open and bounded and we want to approximate χ_Ω by $C_c(\mathbb{R}^d)$. ■

Lemma 3.8 (Urnson) Define

$$\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}$$

Then there is a $\eta_\epsilon \in C_c(\mathbb{R}^d)$ s.t.

$$\begin{cases} 0 \leqslant \eta(x) \leqslant 1 & \forall x \in \mathbb{R}^d \\ \eta_\epsilon(x) = 1 & \text{if } x \in \Omega_\epsilon \\ \eta_\epsilon(x) = 0 & \text{if } x \notin \Omega \end{cases}$$

Lemma 3.9 (Gernal Version of Urnson) If $A, B \subseteq \mathbb{R}^d$, A closed, B closed, $A \cap B = \varnothing$. Then

$$\eta(x) = \frac{\operatorname{dist}(x, A)}{\operatorname{dist}(x, A) + \operatorname{dist}(x, B)}$$

Then $\eta \in C(\mathbb{R}^d)$, $0 \leqslant \eta \leqslant 1$ and $\eta = 0$if $x \in B$, $\eta = 1$ if $x \in A$. App to $A = \overline{\Omega_\epsilon} \subset\subset \Omega$ and $B = \mathbb{R}^d \backslash \Omega$.

Theorem 3.10 (Appendix C4 in Evans) Let Ω be open in $\mathbb{R}^d$ and for $\epsilon > 0$ define

$$\Omega_\epsilon = \{x \in \Omega \mid \operatorname{dist}(x, \mathbb{R}^d \backslash \Omega) > \epsilon\}$$

Let $f \in C_c^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} f = 1, \operatorname{supp} f \subseteq B(0,1), f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1} x) \operatorname{supp}$ is $B(0, \epsilon)$. Then for all $g \in L^p_{loc}(\Omega)$ (i.e. $\mathbb{1}_K g \in L^p(\Omega) \forall K$ compakt set in Ω), then:

a) $g_\epsilon(x) = (f_\epsilon \star g)(x) = \int_{\mathbb{R}^d} f_\epsilon(x-y) g(y)\,dy - \int_\Omega f_\epsilon(x-y) g(y)\,dy$ is well-defined in Ω_ϵ and $g_\epsilon \in C^\infty(\Omega_\epsilon)$.

b) $g_\epsilon \to g$ in $L^p_{loc}(\Omega)$ if $1 \leqslant p < \infty$ and $g_\epsilon(x) \to g(x)$ almost everywhere $x \in \Omega$.

c) If $g \in C(\Omega)$, then $g_\epsilon(x) \to g(x)$ uniformly in any compact subset of Ω.

Proof. a) $D^\alpha(g_\epsilon) = (D^\alpha f_\epsilon) \star g \in C(\Omega_\epsilon)$

b) Already proved in $\mathbb{R}^d$ space.

■

Corrolary 3.11 (Lebesgue differentation theorem) If $f \in L^P_{loc}(\mathbb{R}^d)$, then

$$⨍_{B(x,\epsilon)} |f(y) - f(x)|^p \,dy \to 0 \quad \text{as } \epsilon \to 0$$

Exercise 3.12 (E 2.1) Let $u \in C^2(\mathbb{R}^2)$ be convex. I.e.

$$tu(x) + u(y)(1-t) \geqslant u(tx + (1-t)y) \forall x, y \in \mathbb{R}^d \forall t \in [0,1]$$

a) Prove for all $x \in \mathbb{R}^d$ that $H(x) = \ldots$

Solution.

a In 1D: If u is convex $\Leftrightarrow u''(x) \geqslant 0$ for all $x \in \mathbb{R}$. In general: Taylor expansion for all $x, z \in \mathbb{R}^d$:

$$u(x) = u(z) + \nabla u(z)(x-y) + \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z+s(x-z))\frac{(x-z)^\alpha}{\alpha!}\, ds$$

$x = z + s(x-z), s = 1$ Use $z = tx + (t-1)y \Rightarrow x - z = (1-t)(x-y)$

$$tu(x) = tu(z) + t\nabla u(z)(1-t)(x-y) + t\int_0^1 \sum_{|\alpha|=2} D^\alpha u(z+s(x-z))\frac{[(1-t)(x-y)]^\alpha}{\alpha!}\, ds$$

$$(1-t)u(y) = (1-t)u(z) + (1-t)\nabla u(z)t(y-x) + (1-t)\int_0^r \sum_{|\alpha|=2} D^\alpha u(z+s(y-z))\frac{[t(y-x)]^\alpha}{\alpha!}\, ds$$

$$\Rightarrow tu(x) + (1-t)u(y) = u(z) + t\int_0^1 ... + (1-t)\int_0^1 ...$$
$$\Rightarrow t\int_0^1 ... + (1-t)\int_0^1 ... \geqslant 0 \forall x, y, t, z = tx + (1-t)y$$

$$t(1-t)^2\int_0^1 \sum_{|\alpha|\ =2} D^\alpha u(z+s(x-z))\frac{(x-y)}{\alpha!}\, ds + (1-t)t^2\int_0^1 \sum_{|\alpha|=2} D^\alpha u(z+s(y-z))\frac{(y-z)^\alpha}{\alpha!}\, ds \geqslant 0$$

for all $x, y \in \mathbb{R}^d, t \in [0,1], z = tx + (1-t)y$. Divides for $t(1-t)$

$$(1-t)\int_0^1 \cdots + + \int_0^1 \cdots \geqslant 0$$

Take $t \to 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(y+s(x-y))\frac{(x-y)^\alpha}{\alpha!}\, ds \geqslant 0 \forall x, y \in \mathbb{R}^d$$

Take $y = x + a, a \in \mathbb{R}^d$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x+a+sa)\frac{a^\alpha}{\alpha!}\, ds \geqslant 0 \forall \epsilon > 0, \forall x, a \in \mathbb{R}^d$$

Take $\epsilon \to 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x)\frac{a^\alpha}{\alpha!} \geqslant 0 \Rightarrow \sum_{i,j=1, i\neq j} \partial_{x_i}\partial_{x_j} u(x) a_i a_j + \sum_{i=j=1}^{d} \partial_{x_i}^2 u(x)\frac{a_i^2}{2}$$

We get

$$\frac{1}{2}a^T Ha \geqslant 0 \forall a(a_i)_{i=1}^d \in \mathbb{R}^d$$

b $H(x) \geqslant 0 \Rightarrow (\partial_i\partial_j u) \geqslant 0 \Rightarrow TrH(x) \geqslant 0 \Rightarrow \sum_{i=1}^d \partial_{x_i}^2 u(x) \geqslant 0 \Rightarrow \Delta u(x) \geqslant 0 \forall x \in \mathbb{R}^d$

■

Exercise 3.13 (E 2.2)

Solution. Regard $d = 3$. De function $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \backslash \{0\}$. We prove

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{\max(|x|, r)}$$

If $|x| > r$, then $0 \notin B(x, r+\epsilon)$. Then

$$y \mapsto \frac{1}{|y|}$$

is harmonic in $B(x, r+\epsilon)$. Then by the Mean Value Property:

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{|x|}$$

If $|x| < r$: Then $\frac{1}{|y|}$ is not harmonic in $B(x, r)$ since $0 \in B(x, r)$. Note

$$\fint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \fint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$$

This function depends on x only via $|x|$.

$$... = \fint_{\partial B(0,r)} \frac{dS(y)}{|Rx - Ry|}$$

for all R rotation $SO(3)$, $dS(R_y) = dS(y)$

$$\begin{aligned}
&= \fint_{\partial B(0,r)} \frac{dS(y)}{|Rx - y|} \\
&= \fint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \\
\text{(Radial in z)} \quad &= \fint_{\partial B(0,|x|)} \left(\fint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \right) dS(z) \\
\text{(Fubini)} \quad &= \fint_{\partial B(0,r)} \left(\fint_{\partial B(0,|x|)} \frac{dS(z)}{|z-y|} \right) dS(y) \\
\text{(case 1 since } |y| = r > |x|) \quad &= \fint_{\partial B(0,r)} \frac{1}{|y|} \, dS(y) = \frac{1}{r}
\end{aligned}$$

If $|x| = r$: Continuity: $x \mapsto \fint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$ ■

Remark 3.14 For $f \in C^{|\alpha|}, g \in C^{|\beta|}$:

$$D^{\alpha+\beta}(f \star g) = (D^\alpha f) \star (D^\beta g)$$

Lemma 3.15 If $d \geqslant 3$ and $f : \mathbb{R}^d \to \mathbb{R}$ radial. Then:

$$\begin{aligned}
\left(\frac{1}{|x|^{d-2}} \star f \right)(x) &= \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} \, dy \\
&= \int_{\mathbb{R}^d} \frac{f(y)}{\max(|x|^{d-2}, |y|^{d-2})} \, dy
\end{aligned}$$

Proof. (d=3) Polar coordinates:

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{1}{|x - rw|}\, d\omega \right] f(r)\, dr \\
(a) \qquad &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{d\omega}{\max(|x|, r)} \right] f(r)\, dr \\
&= \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|, |y|)}\, dy
\end{aligned}$$

(b) (d=3) If f radial and non-negative

$$\int_{\mathbb{R}^3} \frac{f(y))}{|x-y|} = \int_{\mathbb{R}^3} \frac{f(y)}{|x|}\, dy = \frac{(Sf?)}{|x|}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|}\, dx\, dy &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_1(x)f_2(y)}{|x+z_1-y-z_2|}\, dx\, dy \\
&= \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f_1(x)\, dx \right) f_2(y)\, dy \leqslant \int_{\mathbb{R}^3} \frac{(\int_{\mathbb{R}^3} f_1)}{|y+z_2-z_1|} f_2(y)\, dy \\
&\leqslant \frac{(\int_{\mathbb{R}^3} f_1)(\int_{\mathbb{R}^3} f_2)}{|z_1-z_2|}
\end{aligned}$$

■

Exercise 3.16 (Bonus 2) a) Prove that $u(x) = \frac{1}{|x|}$ is sub-harmonic in $\mathbb{R}^2\backslash\{0\}$.

b) Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ radial, non-negative, measurable:

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\, dy \geqslant \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)}\, dy$$

My Solution. a) Let $x \in \mathbb{R}\backslash\{0\}$.

$$\begin{aligned}
\partial_{x_i} u &= \partial_{x_i} |x|^{-1} = -|x|^{-2} \frac{x_i}{|x|} = -x_i |x|^{-3} \\
\Rightarrow \partial_{x_i}^2 u &= \partial_{x_i}(-x_i|x|^{-3}) = -|x|^{-3} + 3x_i^2 |x|^{-5} \\
\Rightarrow \Delta u &= -|x|^{-3} + 3x_1^2|x|^{-5} - |x|^{-3} + 3x_2^2|x|^{-5} \\
&= -2|x|^{-3} + 3\underbrace{(x_2^2 + x_2^2)}_{=|x|^2}|x|^{-5} = \frac{1}{|x|^3} > 0
\end{aligned}$$

So u is sub-harmonic in $\mathbb{R}^2\backslash\{0\}$.

b) Let $r > 0, x \in \mathbb{R}^2$ and $|x| < r$. First we show that

$$⨍_{\partial B(x,r)} \frac{1}{|y|}\, dS(y) \geqslant \frac{1}{r} \qquad (\star)$$

Now,

$$⨍_{\partial B(0,r)} \frac{1}{|y|}\, dS(y) = ⨍_{\partial B(0,r)} \frac{1}{|x-y|} dS(y) =: \tilde{u}(x)$$

Take $z \in \mathbb{R}^2 \backslash \{0\}$ such that $z = |x|$, then $\tilde{u}(x) = \tilde{u}(z)$. Let $0 < \epsilon < r$ be small. Then we get

$$
\begin{aligned}
\tilde{u}(z) &= \fint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \\
\begin{pmatrix} |y| = r > |x| = |z| \\ \tilde{u} \text{ radial function} \end{pmatrix} \quad &= \fint_{\partial B(0,|x|-\epsilon)} \left(\fint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \right) dS(z) \\
\text{(Fubini)} \quad &= \fint_{\partial B(0,r)} \left(\fint_{\partial B(0,|x|-\epsilon)} \frac{dS(z)}{|z-y|} \right) dS(y) \\
&= \fint_{\partial B(0,r)} \left(\fint_{\partial B(y,|x|-\epsilon)} \frac{dS(z)}{|z|} \right) dS(y) \\
\left(\frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2 \backslash \{0\} \right) \quad &\geqslant \fint_{\partial B(0,r)} \frac{1}{|y|} \, dS(y) \\
&= \fint_{\partial B(0,r)} \frac{1}{r} \, dS(y) \\
&= \frac{1}{r}
\end{aligned}
$$

Now,

$$
\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} \, dy = \int_{|x| \geqslant |y|} \frac{f(y)}{|x-y|} \, dy + \int_{|x| < |y|} \frac{f(y)}{|x-y|} \, dy,
$$

where

$$
\begin{aligned}
\int_{|x| \leqslant |y|} \frac{f(y)}{|x-y|} \, dy &= \int_0^\infty \int_{\partial B(0,r)} \frac{f(y)}{|x-y|} \mathbb{1}(|x| \leqslant |y|) \, dS(y) \, dr \\
(f \text{ radial}) \quad &= \int_0^\infty f(r) \int_{\partial B(0,r)} \frac{\mathbb{1}(|x| \leqslant r)}{|x-y|} \, dS(y) \, dr \\
&= \int_0^\infty f(r) \int_{\partial B(x,r)} \frac{\mathbb{1}(|x| \leqslant r)}{|y|} \, dS(y) \, dr \\
(\star) \quad &\geqslant \int_0^\infty \frac{f(r)}{r} |\partial B(x,r)| \mathbb{1}(|x| \leqslant r) \, dr \\
&= \int_0^\infty \int_{\partial B(x,r)} \frac{f(r)}{r} \mathbb{1}(|x| \leqslant r) \, dS(y) \, dr \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{|y|} \mathbb{1}(|x| \leqslant |y|) \, dy \\
&= \int_{|x| \leqslant |y|} \frac{f(y)}{|y|} \, dy
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{|x|>|y|} \frac{f(y)}{|x-y|}\,dy &= \int_0^\infty \left(\int_{\partial B(0,r)} \frac{f(r)}{|x-y|} \mathbb{1}(|x|>|y|)\,dS(y) \right) dr \\
(f \text{ radial}) \quad &= \int_0^\infty f(r)\mathbb{1}(|x|>r) \left(\int_{\partial B(x,r)} \frac{1}{|y|}\,dS(y) \right) dr \\
\left(\frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2, \atop \text{MVP and } |x|>r \right) \quad &\geqslant \int_0^\infty f(r)\mathbb{1}(|x|>r)|\partial B(x,r)| \frac{1}{|x|}\,dr \\
&= \int_0^\infty \int_{\partial B(x,r)} f(r)\mathbb{1}(|x|>r) \frac{1}{|x|}\,dS(y)\,dr \\
&= \int_{\mathbb{R}^2} f(y)\mathbb{1}(|x|>|y|)\frac{1}{|x|}\,dy \\
&= \int_{|x|>|y|} f(y)\frac{1}{|x|}\,dy.
\end{aligned}
$$

So we can conclude,

$$
\begin{aligned}
\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\,dy &= \int_{|x|>|y|} \frac{f(y)}{|x-y|}\,dy + \int_{|x|\leqslant|y|} \frac{f(y)}{|x-y|}\,dy \\
&\geqslant \int_{|x|>|y|} \frac{f(y)}{|x|}\,dy + \int_{|x|\leqslant|y|} \frac{f(y)}{|y|}\,dy \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|,|y|)}\,dy
\end{aligned}
$$

■

Definition 3.17 (Fourier Transform) For $f \in L^1(\mathbb{R}^d)$ define

$$
\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i k\cdot x}\,dx, \quad k\cdot x = \sum_{i=1}^d k_i x_i
$$

Theorem 3.18 (Basic Properties)

1. If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^\infty(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$
2. For all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$. Moreover, $\mathcal{F}$ can be extended to be a unitary transforamtion $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ s.t.

$$
\|\mathcal{F}g\|_{L^2} = \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^d)
$$

3. The inverse of F can be defined as
4. $$
(F^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(x)e^{2\pi i k x}\,dk
$$

   for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$
5. $\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k)$ as $(2\pi i k)^\alpha f(k) \in L^2(\mathbb{R}^d)$ $(k^\alpha = k_1^{\alpha_1}\cdots k_\alpha^{\alpha_k})$
6. (Formel) $\widehat{f+g}(k) = \hat{f}(k)\hat{g}(k)$ if f, g are nice enough.

Theorem 3.19 (Hausdorff-Young-Inequality) If $1 \leqslant p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^n)$ then

$$\|\hat{f}\|_{L^{p'}} \leqslant \|f\|_{L^p}$$

and

$$\|\hat{f}\|_{L^p} \leqslant \|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^d)$$

Remark 3.20 We want to apply the Fourier transform to find the solution of a PDE, e.g. the Poisson-Equation:

$$-\Delta u = f \text{ in } \mathbb{R}^d \Rightarrow |2\pi k|^2 \hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{1}{|2\pi k|^2} \hat{f}(k)$$

If we can find G s.t. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$, then

$$\begin{aligned} \hat{u}(k) &= \hat{G}(k)\hat{f}(k) = \widehat{G \star f} \\ \Rightarrow u(x) &= (G \star f)(x) = \int_{\mathbb{R}^d} G(x-y) f(y)\, dy \end{aligned}$$

In fact G is the fundamential solution of laplace quation.

Theorem 3.21 (Fourier Transform of $\frac{1}{|x|^\alpha}$ for $0 < \alpha < d$) We have formally

$$\widehat{\frac{c_\alpha}{|x|^\alpha}} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \quad \forall\ 0 < \alpha < d$$

Here

$$c_\alpha = \pi^{-\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1}\, d\lambda$$

More precisely, for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{c_\alpha}{|x|^\alpha} \star f = \left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)\right)^\vee$$

Moreover if $\alpha > \frac{d}{2}$, then we also have

$$\left(\frac{c_\alpha}{|x|^\alpha} \star f\right)^\wedge = \frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k)$$

Remark 3.22 If $f \in L^p$, $1 \leqslant p \leqslant 2 \Rightarrow f = \underbrace{f_1}_{\in L^1} + \underbrace{f_2}_{\in L^2} \Rightarrow \hat{f} = \hat{f}_1 + \hat{f}_2$

Lemma 3.23 (Fourier Transform of Gaussians) In $\mathbb{R}^d$,

$$\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$$

More generally for all $\lambda > 0$:

$$\widehat{e^{-\pi\lambda^2|x|^2}} = \lambda^{-d} e^{-\pi\frac{|k|^2}{\lambda^2}}$$

(exercise)

Proof of Theorem. Formally:

$$
\begin{aligned}
\frac{c_\alpha}{|x|^\alpha} &= \frac{1}{|x|^\alpha}\pi^{-\frac{\alpha}{2}}\int_0^\infty e^{-\lambda}\lambda^{\frac{\alpha}{2}-1}\,d\,\lambda = \int_0^\infty e^{-\pi\lambda|x|^2}\lambda^{\frac{\alpha}{2}-1}\,d\lambda \\
\Rightarrow \frac{\hat{c}_\alpha}{|x|^\alpha}(k) &= \int_0^\infty \widehat{e^{-\pi\lambda|x|^2}}(k)\lambda^{\frac{\alpha}{2}-1}\,d\lambda = \int_0^\infty \lambda^{-\frac{d}{2}}e^{-\pi\frac{|k|^2}{\lambda}}\lambda^{\frac{\alpha}{2}-1}\,d\lambda \\
(\lambda\to\frac{1}{\lambda}) \quad &= \int_0^\infty \lambda^{\frac{d}{2}}e^{-\pi|k|^2\lambda}\lambda^{-\frac{\alpha}{2}+1}\lambda^{-2}\,d\lambda \\
&= \frac{c_{d-\alpha}}{|k|^{d-\alpha}}
\end{aligned}
$$

Let $f\in C_c(\mathbb{R}^d)$. Then $\left(\frac{1}{|x|^\alpha}\star f\right)(x)=\int_{\mathbb{R}^d}\frac{1}{|x-y|^\alpha}f(y)\,dy$ is well defined as $\frac{1}{|x-y|}\in L^1_{loc}(\mathbb{R}^d,dy)$. It is bounded

$$
\frac{1}{|x|^\alpha}\star f = \underbrace{\frac{1}{|x|^\alpha}\mathbb{1}(|x|\leqslant 1)}_{\in L^\infty(\mathbb{R}^d)}\star\underbrace{f}_{L^\infty} + \underbrace{\frac{1}{|x|}\mathbb{1}(|x|>1)}_{\in L^\infty}\star\underbrace{f}_{\in L^1}\in L^\infty(\mathbb{R}^d)
$$

When $|x|\ \to\infty$:

$$
\left(\frac{1}{|x|^\alpha}\star f\right)(x)=\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^\alpha}\,dy=\int_{|y|\leqslant R}\frac{f(y)}{|x-y|^\alpha}\,dy\sim\frac{\int_{\mathbb{R}^d}f(y)\,dy}{|x|^\alpha}
$$

Note that $\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\underbrace{\hat{f}(k)}_{\text{bounded}}\in L^1(\mathbb{R}^d)$.

$$
\begin{aligned}
(\ldots)\mathbb{1}(|k|\leqslant 1)+(\ldots)\mathbb{1}(|k|>1)\frac{1}{|k|^{d-\alpha}}|\hat{f}(k)|\ \mathbb{1}(|k|\leqslant 1)\leqslant\|f\|_{L^1}\frac{\mathbb{1}(|k|\leqslant 1)}{|k|^{d-\alpha}}\in L^1(\mathbb{R}^d,dk) \\
\frac{1}{|k|^{d-\alpha}}|\hat{f}(k)|\mathbb{1}(k>1)\leqslant|\hat{f}(k)|\in L^2(\mathbb{R}^d,dK)\text{ as }f\in L^2(\mathbb{R}^d)
\end{aligned}
$$

Lemma 3.24 If $f\in C_c^\infty(\mathbb{R}^d)$, then $\hat{f}\in L^1(\mathbb{R}^d)$

Proof. (Exercise) Hint: $|\widehat{D^\alpha f}|=|2\pi k|^{|\alpha|}|\hat{f}(k)|\rightsquigarrow|\hat{f}(k)|\leqslant\frac{1}{|k|^{|k|}}$ as $|k|\to\infty$. ■

Compute:

$$
\begin{aligned}
\left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k)\right)^\vee(x) &= \int_{\mathbb{R}^d}\frac{c_{d-\alpha}}{|k|^{d-\alpha}}\hat{f}(k)e^{2\pi ikx}\,dk \\
&= \int_{\mathbb{R}^d}\left(\int_0^\infty e^{-\pi|k|^2\lambda}\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda\right)\hat{f}(k)e^{2\pi ikx}\,dk \\
&= \int_0^\infty\left(\int_{\mathbb{R}^d}e^{-\pi|k|^2\lambda}\hat{f}(k)e^{2\pi ikx}\,dk\right)\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty\left(e^{-\pi k^2\lambda}\hat{f}(x)\right)^\vee\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty\left(\widehat{\lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}}}(k)\hat{f}(k)\right)^\vee\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \int_0^\infty\left(\lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}}\star f\right)\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda \\
&= \left(\int_0^\infty\lambda^{-\frac{d}{2}}e^{-\pi\frac{x^2}{\lambda}}\lambda^{\frac{d-\alpha}{2}-1}\,d\lambda\right)\star f
\end{aligned}
$$

Assume $d > \alpha > \frac{d}{2}$. Then $\frac{c_\alpha}{|x|^\alpha} \star f \in L^\infty$ and behaves $\frac{c_\alpha (\int f)}{|x|^\alpha}$ as $|x| \to \infty$. This implies:

$$\int_{\mathbb{R}^d} \left| \frac{c_\alpha}{|x|^\alpha} \star f \right|^2 \leqslant c + \int_{|x| \geqslant R} \frac{c}{|x|^{2d}}\, dx < \infty$$

Thus the Fourier Transform $\widehat{\frac{c_\alpha}{|x|^\alpha}} \star f$ exists. Combining with ■

Remark 3.25 If $d \geqslant 3$

$$\begin{aligned} \hat{G}(k) &= \frac{1}{|2\pi k|^2} \\ \Rightarrow G(x) &= \left(\frac{1}{|2\pi k|^2} \right)^\vee = \frac{1}{d(d-2(k)|x|^{d-2})} = \Phi(x) \end{aligned}$$