

Partial Differential Equations
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Please note that I write this lecture notes for my personal use. I may write things differently than presented in the lecture. This script also contains solutions for exercises (which may be wrong). Of course, I don't push them to GitHub while the exercises can be handed in.

Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{cases} f'(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geq 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos^2(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \rightarrow \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \rightarrow \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x + h e_i) - f(x)}{h}$, where $e_i = (0, 0, \dots, 1, \dots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial_{x_1}, \dots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \dots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha|=k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i, j \leq d}$

Definition 1.4 Given a function F . Then the equation of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$ is called a *PDE of order k* .

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1} u, D^{k-2} u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explicit solution! This is in many cases impossible.
- Prove a *well-posed theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continuous and differentiable functions) Let $\Omega \subseteq \mathbb{R}^d$ be open

$$C(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ continuous}\}$$

$$C^k(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \leq k\}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ Lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{ exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 u - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in \mathbb{R}^d . If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω , then u is a harmonic function in Ω .

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \, dS(x) = \int_K \operatorname{div}(\vec{F}) \, dx$$

where ν is the outward unit normal vector field on ∂K . Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \, dS(x) = \int_V \operatorname{div}(\nabla u) \, dx = \int_V \Delta u(x) \, dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f)g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) \, dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \rightarrow \mathbb{R}$ be continuous. Prove that if $\int_B f(x) \, dx = 0$, then $u \equiv 0$ in Ω .

Theorem 2.5 (Fundamental Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x) \, dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in \mathbb{R}^d . Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \rightarrow \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i} \left(\sqrt{x_1^2 + \cdots + x_d^2} \right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned} \partial_{x_i} u &= \partial_{x_i} v = (\partial_r v) \frac{\partial r}{\partial x_i} = v'(r) \frac{x_i}{r} \\ \partial_{x_i}^2 u &= \partial_{x_i} \left(v(r) \frac{x_i}{r} \right) = (\partial_{x_i} v(r))' \frac{x_i}{r} + v'(r) \partial_{x_i} \left(\frac{x_i}{r} \right) \\ &= (\partial_r v'(r)) \left(\frac{dr}{\partial x_i} \right) \frac{x_i}{r} + v'(r) \left(\frac{1}{r} - \frac{x_i}{r^2} (\partial_{x_i} r) \right) = v'(r) \frac{x_i^2}{r^2} + v'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right) \end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d \partial_{x_i}^2 \right) u = v''(r) + v'(r) \left(\frac{d}{r} - \frac{1}{r} \right)$

Thus $\Delta u = v'(r) + v(r) \frac{d-1}{r}$. We consider $d \geq 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r) \frac{d-1}{r} = 0$

$$\begin{aligned} \Rightarrow \log(v(r))' &= \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)' \quad (\text{recall } \log(f)' = \frac{f'}{f}) \\ \Rightarrow v'(r) &= \frac{1}{v^{d-2} + \text{const.}} \end{aligned}$$

$$\begin{cases} \frac{\text{const}}{r^{d-2}} + \text{const}x + \text{const} & , d \geq 3 \\ \text{const} \log(r) + \text{const}x + \text{const} & , d = 2 \end{cases}$$

Definition 2.7 (Fundamental Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|} \frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0, 1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta \Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in \mathbb{R}^d . The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy$$

This can be heuristically justified with

$$-\Delta(\Phi \star f) = (-\Delta \Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C_c^2(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy.$$

First we check that u is continuous: Take $x_k \rightarrow x_0$ in \mathbb{R}^d . We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y) dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n \rightarrow \infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x - y)| \leq \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \leq R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\text{supp}(f)$ but independent of y . Now we compute the derivatives:

$$\begin{aligned} \partial_{x_i} u(x) &= \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x - y) dy = \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + h e_i - y) - f(x - y)}{h} dy \\ (\text{dom. conv.}) &= \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x - y) dy \\ \Rightarrow D^\alpha u(x) &= \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x - y) dy \quad \text{for all } |\alpha| \leq 2 \end{aligned}$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y) (-\Delta_x) f(x - y) dy = \int_{\mathbb{R}^d} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy + \int_{B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x - y) dy - \int_{\partial B(0, \epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x - y) dy \\ &\quad + \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) - \int_{\partial B(0, \epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0, \epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1| \epsilon^{d-1}} \quad \text{for } y \in \partial B(0, \epsilon)$$

Hence:

$$\begin{aligned} \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) &= \frac{1}{d|B_1| \epsilon^{d-1}} \int_{\partial B(0, \epsilon)} f(x - y) dS(y) \\ &= \oint_{\partial B(0, \epsilon)} f(x - y) dS(y) = \oint_{\partial B(x, \epsilon)} f(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} f(x) \end{aligned}$$

We have to regard the following error terms:

$$\begin{aligned}
\bullet \left| \int_{B(0,\epsilon)} \Phi(y) (-\Delta_y) f(x-y) dy \right| &\leq \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leq \|\Delta f\|_{L^\infty} \mathbb{1}(|y| \leq R)} dy \\
&\leq \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)| \mathbb{1}(|y| \leq R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y| \leq \epsilon) \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

Where $R > 0$ depends on x and the support of f but is independent of y .

$$\begin{aligned}
\bullet \left| \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y) dS(y) \right| &\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)| dy \\
&\leq \begin{cases} \text{const} \cdot \epsilon |\log \epsilon| \rightarrow 0, & d = 2 \\ \text{const} \cdot \epsilon \rightarrow 0, & d \geq 3 \end{cases}
\end{aligned}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ proved that $u = \Phi \star f$ and $f \in C_c^2(\mathbb{R}^d)$. ■

Thus, if $f \in C_c^2(\mathbb{R})$, then $u = \Phi \star f$ satisfies $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f . For example if $f \in C_c^1(\mathbb{R})$ we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x-y) dy \in C(\mathbb{R}^d) \Rightarrow u \in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i} \partial_{x_j} u = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) \partial_{x_j} f(x-y) dy = \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_j} f(x-y) dy \in C(\mathbb{R}^d)$$

So we have $u \in C^2(\mathbb{R}^d)$. Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_i} f(x-y) dy \stackrel{(IBP)}{=} f(x).$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega \subseteq \mathbb{R}$ be open, let $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω . Then

$$u(x) = \oint_{B(x,r)} u = \oint_{\partial B(x,r)} u \quad \text{for all } x \in \Omega, B(x,r) \subseteq \Omega$$

Proof. Consider all $r > 0$ s.t. $B(x,r) \subseteq \Omega$,

$$f(r) = \oint_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r . When it is done, then we immediately obtain

$$f(r) = \lim_{t \rightarrow 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr} \left(\oint_{\partial B(0,r)} u(x+y) dS(y) \right) \\
&= \frac{d}{dr} \left(\oint_{\partial B(0,1)} u(x+rz) dS(z) \right) \\
(\text{dom. convergence}) \quad &= \oint_{\partial B(0,1)} \frac{d}{dr} [u(x+rz)] dS(z) \\
&= \oint_{\partial B(0,1)} \nabla u(x+rz) z dS(z) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r} dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green 2.2}) \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0} dy = 0 \quad \blacksquare
\end{aligned}$$

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$ (Linear Equation)

Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, w \in S^{d-1}$, then

$$\int_{B(0,r)} g(y) dy = \int_0^r \left(\int_{B(0,s)} g(y) dS(y) \right) ds$$

Remark 2.15 We already proved that for u harmonic we have $u(x) = \oint_{\partial B(x,r)} u dy$. Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_{B(0,r)} u(x+y) dy \\
(\text{Pol. decomposition}) \quad &= \int_0^r \left(\int_{\partial B(0,s)} u(x+y) dS(y) \right) ds \\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\text{Mean value property}) \quad &= \int_0^r (|\partial B(x,s)| u(x)) ds = |B(x,r)| u(x)
\end{aligned}$$

This implies

$$\oint_{B(x,r)} u(y) dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.16 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = \oint_{B(x,r)} u(y) dy = \oint_{\partial B(x,r)} u(y) dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then u is harmonic, i.e. $\Delta u = 0$ in Ω . (The proof is exactly like before)

Theorem 2.17 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω . Then

- a) $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
- b) Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$, then $u \equiv \text{const.}$ in Ω .

Proof. Given $U \subseteq \mathbb{R}^d$ open, we can write $U = \bigcup_i U_i$, where U_i is open and connected.

- b) Assume that Ω is connected and there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{y \in \Omega} u(y)$. Define $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$. U is closed since u is continuous. Moreover, U is open by the mean-value theorem. I.e. for all $x \in U$ there is a $r > 0$ s.t. $B(x, r) \subseteq U \subseteq \Omega$. Since U is connected we get $U = \Omega$, so u is constant in Ω . On the other hand, if there is no $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$ we have $\forall x_0 \in \Omega : u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
- a) Given $\Omega \subseteq \mathbb{R}^d$ open, we can write $\Omega = \bigcup_i \Omega_i$, where Ω_i is open and connected. By b) we have

$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$

So we can conclude

$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x). \quad \blacksquare$$

Definition 2.18 • If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$, then u is called *sub-harmonic* if $\Delta u \geq 0$ in Ω .

- If $\Delta u \leq 0$, then u is called *super-harmonic*.

Exercise 2.19 (E 1.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$ be subharmonic.

- a) Prove that u satisfies the Mean Value Inequality

$$\oint_{\partial B(x, r)} u(y) dS(y) \geq \int_{B(x, r)} u(y) dy \geq u(x)$$

for all $B(x, r) \subseteq \mathbb{R}^d$.

- b) Assume further that Ω is connected and $u \in C(\bar{\Omega})$. Prove that u satisfies the strong maximum principle, namely either
 - u is constant in Ω , or
 - $\sup_{y \in \partial\Omega} u(y) > u(x)$ for all $x \in \Omega$.

My Solution. a) Let $f(r) = \oint_{\partial B(x,r)} u(y) dS(y)$, then we have

$$\begin{aligned}
\partial_r f(r) &= \partial_r \oint_{\partial B(x,r)} u(y) dS(y) \\
(\text{Dom. Convergence}) \quad &= \oint_{\partial B(x,r)} \partial_r u(y) dS(y) \\
&= \oint_{\partial B(0,1)} \partial_r u(x + yr) dS(y) \\
&= \oint_{\partial B(0,1)} \nabla u(x + yr) \cdot y dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r} dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green}) \quad &= \oint_{B(x,r)} \text{div}(\nabla u(y)) dS(y) \\
&= \oint_{B(x,r)} \underbrace{\Delta u(y)}_{\geq 0} dS(y) \geq 0
\end{aligned}$$

So we can conclude that

$$\oint_{\partial B(x,r)} u(y) dS(y) = f(r) \geq \lim_{r \rightarrow 0} f(r) = u(x).$$

Now regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left(\int_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&= \int_0^r \left(|\partial B(x,r)| \oint_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&\geq \int_0^r |\partial B(x,r)| \cdot u(x) dS(y) \\
&= u(x) \int_0^r |\partial B(x,r)| dS(y) = u(x) |B(x,r)|.
\end{aligned}$$

Thus we have

$$u(x) \leq \oint_{B(x,r)} u(y) dy.$$

Finally, lets regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left(|\partial B(x,s)| \oint_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\partial_r f(r) \geq 0) \quad &\leq \int_0^r \left(|\partial B(x,s)| \oint_{\partial B(x,s)} u(y) dS(y) \right) ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \int_0^r |\partial B(x,s)| ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \cdot |B(x,s)|
\end{aligned}$$

and we conclude

$$\oint_{B(x,r)} u(y) dy \leq \oint_{\partial B(x,r)} u(y) dS(y).$$

b) Let $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega} u(x)$. Now,

$$\begin{aligned} \sup_{x \in \Omega} u(x) = u(x_0) &\leq \oint_{\partial B(x_0,r)} u(y) dy \\ &\leq \oint_{\partial B(x_0,r)} \sup_{x \in \Omega} u(x) dy = \sup_{x \in \Omega} u(x) \end{aligned}$$

Since u is continuous we get $u(y) = u(x_0)$ for all $y \in B(x_0, r)$, so u is constant. ■

Definition 2.20 The *Poisson Equation* for given f, g on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Theorem 2.21 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open and connected. Let $f \in C(\Omega), g \in C(\partial\Omega)$. Then there exists *at most* one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof. Assume that we have two solutions u_1 and u_2 . Then $u := u_1 - u_2$ is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that $u = 0$ in Ω . More precisely, by the maximum principle we have $\forall x \in \Omega$

$$\sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leq 0$$

Since $-u$ satisfies the same property we have $\forall x \in \Omega$:

$$\sup_{x \in \Omega} (-u(x)) \leq \sup_{x \in \partial\Omega} (-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leq 0 \quad \Rightarrow \quad u(x) \geq 0$$

So we get $u(x) = 0$ in Ω . ■

Exercise 2.22 (Bonus 1) Let Ω be open, connected and bounded in \mathbb{R}^d . Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Prove that

a) If $g \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω .

b) If $g \geq 0$ on $\partial\Omega$ and $g \neq 0$, then $u > 0$ in Ω .

My Solution. a) We have that $\Delta(-u) = 0$, so $-u$ is harmonic in Ω . Since Ω is open and bounded we can apply the Maximum Principle (2.17) and get that

$$\sup_{x \in \bar{\Omega}} -u(x) \leq \sup_{x \in \partial\Omega} -g(x) \leq 0.$$

This implies $\inf_{x \in \Omega} u(x) \geq 0$, so $u \geq 0$ in Ω .

b) We prove this by contraposition. Assume there is a $x_0 \in \Omega$ s.t. $u(x_0) = 0$. Since we have $u \geq 0$ on Ω by a), it follows that

$$0 = -u(x_0) = \sup_{x \in \bar{\Omega}} -u(x) \leq \sup_{x \in \partial\Omega} -g(x) \leq 0,$$

so $-u$ attains its maximum on Ω . Hence $-u = 0 = u$ is constant by the strong maximum principle because Ω is connected, in fact $0 = u|_{\partial\Omega} = g$. ■

Lemma 2.23 (Estimates for derivatives) If u is harmonic in $\Omega \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| = N$ and $B(x_0, r) \subseteq \Omega$, then

$$|D^\alpha u(x)| \leq \frac{(c_d N)^N}{r^{d+N}} \int_{B(x, r)} |u| dy$$

Proof. Induction: Assume $|\alpha| = N - 1$, Take $|\alpha| = N$

$$|D^\alpha u(x_0)| \leq \frac{|S_1|}{|B_1| \frac{r}{N}} \|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))}, \quad D^\alpha u = \partial_{x_i}(D^\beta u)|_{|\beta|=N-1}$$

Note: $x \in B(x_0, \frac{r}{N})$, so $B(x, \frac{r(N-1)}{N}) \subseteq B(x_0, r)$. By the induction hypothesis:

$$\|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))} \leq \frac{[c_d(N-1)]^{N-1}}{[r \frac{(N-1)}{N}]^{d+N-1}} \int_{B(x_0, r)} |u| dy$$

The conclusion is:

$$\begin{aligned} |D^\alpha u(x_0)| &\leq \frac{|S_1|}{|B_1| \frac{r}{N}} \frac{[c_d(N-1)]^{N-1}}{(r \frac{N-1}{N})^{d+N-1}} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} (N-1)^d} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} N^d} \left(\frac{N}{N-1}\right)^d \int_{B(x_0, r)} |u| dy \\ &\leq \frac{2^d |S_1|}{|B_1|} \frac{c_d^{N-1} N^N}{r^{d+N}} \int_{B(x_0, r)} |u| dy \quad \text{if } c_d \geq \frac{2^d |S_1|}{|B_1|} \end{aligned}$$

■

Theorem 2.24 (Regularity) Let Ω be open in \mathbb{R}^d . Let $u \in C(\Omega)$ satisfy $u(x) = \int_{\partial B} u dy$ for any $x \in B(x, r) \subseteq \Omega$. Then u is a harmonic function in Ω . Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω .

Exercise 2.25 (E 1.1: Proof the Gauss–Green formula) Let $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that for every open ball $B(y, r) \subseteq \mathbb{R}^d$ we have

$$\int_{\partial B(y, r)} f(y) \cdot \nu_y dS(y) = \int_{B(y, r)} \operatorname{div} f dx.$$

Here ν_y is the outward unit normal vector and dS is the surface measure on the sphere.

Solution. We proof this in $d=3$. Let $f \in C^1(\mathbb{R}^3)$

$$\int_{B(0,1)} \partial_{x_3} f dx = \int_{\partial B(0,1)} f x_3 dS(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \vec{n} = \frac{x}{|x|} \text{ on } \partial B(0,1)$$

$$\begin{aligned} B(0,1) &= \{x_1^2 + x_2^2 + x_3^2 \leq 1\} \\ &= \{x_1^2 + x_2^2 \leq 1 - \sqrt{1 - x_1^2 - x_2^2} \leq x_3 \leq \sqrt{1 - x_1^2 - x_2^2}\} \end{aligned}$$

Then:

$$\begin{aligned} \int_{B(0,1)} \partial_{x_3} f dx &= \int_{x_1^2 + x_2^2 \leq 1} \left(\int_{-\sqrt{1-x_1^2-x_2^2} \leq x_3 \leq \sqrt{1-x_1^2-x_2^2}} \partial_{x_3} f dx_3 \right) dx_1 dx_2 \\ &= \int_{x_1^2 + x_2^2 \leq 1} \left[f(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \right. \\ &\quad \left. - f(x_1, x_2, -\sqrt{1-x_1^2-x_2^2}) \right] dx_1 dx_2 \end{aligned}$$

Lets take polar coordinates in 2D:

$$\begin{aligned} x_1 &= r \cos \phi & r > 0, \phi \in [0, 2\pi) \\ x_2 &= r \sin \phi & \det \frac{\partial(x_1, x_2)}{\partial(r, \phi)} = r \end{aligned}$$

$$(\star) = \int_0^1 \int_0^{2\pi} [f(r \cos \phi, r \sin \phi, r) - f(r \cos \phi, r \sin \phi, -r)] r dr d\phi$$

On the other hand:

$$\int_{\partial B(0,1)} f x_3 dS$$

The polar coordinates in 3D are:

$$\begin{aligned} x_1 &= r \cos \phi \sin \theta & r > 0, \phi \in (0, 2\pi), \theta \in (0, \pi) \\ x_2 &= r \sin \phi \sin \theta & \det \frac{\partial(x_1, x_2, x_3)}{\partial(r, \phi, \theta)} = r^2 \sin \theta \\ x_3 &= r \cos \theta \end{aligned}$$

Then:

$$\begin{aligned} (\star\star) &= \int_0^{2\pi} \int_0^\pi f(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \sin \theta \cos \theta d\theta d\phi \\ &= \int_0^{2\pi} \left(\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^\pi d\theta \right) d\phi \\ (r = \sin \theta) &= \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, \sqrt{1-r^2}) r dr d\phi \\ &\quad - \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, -\sqrt{1-r^2}) r dr d\phi \end{aligned} \quad \blacksquare$$

Exercise 2.26 (E 1.2) Let $u \in C(\mathbb{R}^d)$ and $\int_{B(x,r)} u \, dy = 0$ for every open ball $B(x,r) \subseteq \mathbb{R}^d$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

My Solution. Assume there is a $x_0 \in \mathbb{R}^d$ s.t. w.l.o.g. $u(x_0) > 0$. Since u is continuous there is a ball $B(x_0, r)$ s.t. $u(y) > \frac{u(x_0)}{2}$ for all $y \in B(x_0, r)$. But then we get

$$\int_{B(x_0, r)} u(y) \, dy \geq \int_{B(x_0, r)} \frac{u(x_0)}{2} \, dy = \frac{u(x_0)}{2} |B(x_0, r)| > 0. \quad \blacksquare$$

Exercise 2.27 (E 1.3) Let $f \in C_c^1(\mathbb{R}^d)$ with $d \geq 2$ and $u(x) := (\Phi \star f)(x)$. Prove that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ (2.9 was the same for $f \in C_1(\mathbb{R})$)

Theorem 2.28 (Liouville's Theorem) If $u \in C^2(\mathbb{R}^d)$ is harmonic and bounded, then $u = \text{const.}$

Proof. By the bound of the derivative 2.23 we have

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\leq \frac{c_d}{r^{d+1}} \int_{B(x_0, r)} |u| \, dy \quad \forall x_0 \in \mathbb{R}^d \quad \forall r > 0 \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r^{d+1}} |B(x_0, r)| \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r} \xrightarrow{r \rightarrow \infty} 0 \end{aligned}$$

Thus $\partial_{x_i} u = 0$ for all $i = 1, 2, \dots, d$ and $u = \text{const.}$ in \mathbb{R}^d ■

Theorem 2.29 (Uniqueness of solutions to Poisson Equation in \mathbb{R}^d) If $u \in C^2(\mathbb{R}^d)$ is a bounded function and satisfies $-\Delta u = f$ in \mathbb{R}^d where $f \in C_c^2(\mathbb{R}^d)$, then we have

$$u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y) \, dy + C \quad \forall x \in \mathbb{R}^d$$

where C is a constant and Φ is the fundamental solution of the Laplace equation in \mathbb{R}^d .

Proof. If we can prove that v is bounded, then $v = \text{const.}$ We first need to show that $\Phi \star f$ is bounded.

$$\begin{aligned} \Phi &= \Phi_1 + \Phi_2 = \Phi \mathbb{1}(|x| \leq 1) + \Phi(|x| \geq 1) \\ \Phi \star f &= \Phi_1 \star f + \Phi_2 \star f \end{aligned}$$

We have $\Phi_1 \star f \in L^1(\mathbb{R}^d)$ and $\Phi_2 \star f$ is bounded since $\Phi \rightarrow 0$ as $|x| \rightarrow \infty$ in $d \geq 3$. ■

Exercise 2.30 (Hanack's inequality) Let $u \in C^2(\mathbb{R}^d)$ be harmonic and non-negative. Prove that for all open, bounded and connected $\Omega \subseteq \mathbb{R}^d$, we have

$$\sup_{x \in \Omega} u(x) \leq C_\Omega \inf_{x \in \Omega} u(x),$$

where C_Ω is a finite constant depending only on Ω .

Proof. (Exercise) Hint: $\Omega = B(x, r)$. General case cover Ω by finitely many balls, one ball is inside Ω . ■

Chapter 3

Convolution, Fourier Transform and Distributions

3.1 Convolutions

Definition 3.1 (Convolution) Let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ or \mathbb{C} .

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) dy = \int_{\mathbb{R}^d} f(y)g(x-y) dy = (g \star f)(x)$$

Remark 3.2 (Properties of the Convolution) • $(f \star g)(x) = f \star (g \star h)$

• $\widehat{f \star g} = \widehat{f} \star \widehat{g}$

Theorem 3.3 (Young Inequality) If $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$, where $1 \leq p \leq \infty$, then $f \star g \in L^p(\mathbb{R}^d)$ and $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$. More generally, if $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^r(\mathbb{R}^d)$, $\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$, where $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

Proof. Let $f \in L^1, g \in L^p$. With the Hölder Inequality ??, we have:

$$\begin{aligned} \|f \star g\|_{L^p}^p &= \int_{\mathbb{R}^d} |f \star g(x)|^p dx \\ &\leq \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)| |g(y)|^p dy dx \\ &= \|f\|_{L^1}^{\frac{p}{q}+1} \|g\|_{L^p}^p \end{aligned}$$

So we have $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$ ■

Theorem 3.4 (Smoothness of the Convolution) If $f \in C_c^\infty(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. Then $f \star g \in C^\infty(\mathbb{R})$ and

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for all $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \{0, 1, 2, \dots\}$

Proof. First we note that $x \mapsto (f \star g)$ is continuous as $x_n \rightarrow x$ in \mathbb{R}^d since

$$(f \star g)(x_n) = \int_{\mathbb{R}^d} f(x_n - y)g(y) dy \xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^d} f(x - y)g(y) dy = (f \star g)(x)$$

We can apply Dominated convergence because

$$f(x_n - y)g(y) \rightarrow f(x - y)g(y) \quad \forall y \text{ as } f \text{ is continuous and } x_n \rightarrow x$$

and

$$|f(x_n - y)g(y)| \leq \|f\|_{L^\infty} |g(y)| \mathbb{1}(|y| \leq R) \in L^1(\mathbb{R}^d).$$

Where $R > 0$ satisfies $B(0, R) \supseteq \text{supp } f + \sup_n |x_n|$. Now we can compute the derivatives:

$$\begin{aligned} \partial_{x_i}(f \star g)(x) &= \lim_{h \rightarrow 0} \frac{(f \star g)(x + he_i) - (f \star g)(x)}{h} \\ &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\ (\text{Dominated Convergence}) \quad &= \int_{\mathbb{R}^d} \lim_{h \rightarrow 0} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\ &= \int_{\mathbb{R}^d} (\partial_{x_i} f)(x - y)g(y) dy \end{aligned}$$

We could apply Dominated Convergence since

$$\begin{aligned} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) &\xrightarrow{h \rightarrow 0} (\partial_{x_i} f)(x - y)g(y) \quad \text{as } f \in C^1 \\ \left| \frac{f(x + he_i - y) - f(x - y)}{h} g(y) \right| &\leq \|\partial_{x_i} f\|_{L^\infty} |g(y)| \mathbb{1}(|y| \leq R) \in L^1(\mathbb{R}^d) \end{aligned}$$

where $B(0, R) \supseteq \text{supp}(f) + B(0, |x| + 1)$ and $\partial_{x_i}(f \star g) = (\partial_{x_i} f) \star g \in C(\mathbb{R}^d)$ since $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$. By induction we get $D^\alpha(f \star g) = (D^\alpha f \star g) \in C(\mathbb{R}^d)$. ■

Remark 3.5 Question: Is there a f s.t. $f \star g = g$ for all g . In fact there is no regular function f that solves this formally:

$$f \star g = g \Rightarrow \widehat{f \star g} = \hat{g} \Rightarrow \hat{f} \hat{g} = \hat{g} \Rightarrow \hat{f} = 1 \Rightarrow f \text{ is not a regular function!}$$

However, if f is the Dirac-Delta Distribution, $f = \delta_0$ then $\delta_0 \star g = g$ for all g . Formally:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \int \delta_0 = 1$$

In fact, if $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$, then $f_\epsilon \rightarrow \delta_0$ in an appropriate sense and $f_\epsilon \star g \rightarrow g$ for all g nice enough.

Theorem 3.6 (Approximation by convolution) Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon})$. Then for all $g \in L^p(\mathbb{R}^d)$, where $1 \leq p < \infty$, then

$$f_\epsilon \star g \rightarrow g \quad \text{in } L^p(\mathbb{R}^d)$$

Proof.

Step 1: Let $f, g \in C_c(\mathbb{R}^d)$. Then

$$\begin{aligned}
(f_\epsilon \star g)(x) - g(x) &= \int_{\mathbb{R}^d} f_\epsilon(y)g(x-y) dy - \int_{\mathbb{R}^d} f_\epsilon(y)g(x) dy \\
&= \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x)) dy \\
|(f_\epsilon \star g)(x) - g(x)| &= \left| \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x)) dy \right| \\
&\leq \int_{\mathbb{R}^d} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\
&\leq \int_{|y| \leq R_\epsilon} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\
&\leq \underbrace{\int_{|y| \leq R_\epsilon} |f_\epsilon(y)| dy}_{\leq \|f_\epsilon\|_{L^1} = \|f\|_{L^1}} \left[\sup_{|z| \leq R} |g(x-z) - g(x)| \right] \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

We have Dominated Convergence since:

$$(f_\epsilon \star g)(x) - g(x) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

and

$$|f_\epsilon \star g(x) - g(x)| \leq \|f\|_{L^1} \sup_{|z| \leq R_\epsilon} |g(x-z) - g(x)| \leq 2\|f\|_1 \|g\|_{L^\infty} \mathbf{1}(|x| \leq R_1).$$

Where $B(0, R_1) \supseteq \text{supp}(g) + B(0, R_\epsilon)$, thus $f_\epsilon \star g \rightarrow g$ in $L^p(\mathbb{R}^d)$. To remove the technical assumptions $f, g \in C_c(\mathbb{R}^d)$, then we use a density argument. We use the fact that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

Step 2: Let $g \in C_c(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$. Then there is $\{g_m\} \subseteq L^p(\mathbb{R}^d), g_m \rightarrow g$ in $L^p(\mathbb{R}^d)$. Then

$$\begin{aligned}
\|f_\epsilon \star g - g\|_{L^p} &\leq \|f_\epsilon \star (g - g_m)\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
(\text{Young}) &\leq \|f_\epsilon\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leq \|f\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leq (\|f\|_{L^1} + 1) \|g - g_m\|_{L^p} + \|f \star g_m - g_m\|_{L^p}
\end{aligned}$$

So we get:

$$\begin{aligned}
\limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq (\|f\|_{L^p} + 1) \|g - g_m\|_{L^p} + \underbrace{\limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g_m - g_m\|_{L^p}}_{\text{by step 1.}} \\
&\xrightarrow{m \rightarrow \infty} 0
\end{aligned}$$

Step 3: Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Take $\{f_m\} \subseteq C_c(\mathbb{R}^d)$, s.t.

$$\begin{cases} F_m \rightarrow g \text{ in } L^1(\mathbb{R}) \text{ as } m \rightarrow \infty \\ \int_{\mathbb{R}^d} F_m = 1 \text{ (it is possible since } \int_{\mathbb{R}^d} f = 1) \end{cases}$$

Define $F_{m,\epsilon}(x) = \epsilon^{-d} F_m(\epsilon^{-1}x)$ (recall $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$). Then:

$$\begin{aligned}
f_\epsilon \star g - g &= (f_\epsilon - F_{m,\epsilon}) \star g + F_{m,\epsilon} \star g - g \\
\Rightarrow \|f_\epsilon - g\|_{L^p} &\leq \underbrace{\|f_\epsilon - F_{m,\epsilon} \star g\|_{L^p}}_{\text{Young}} + \|F_{m,\epsilon} \star g - g\|_{L^p} \\
&\leq \|f_\epsilon - F_{m,\epsilon}\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \\
\Rightarrow \limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq \|f - F_m\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \quad \blacksquare
\end{aligned}$$

Lemma 3.7 $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$

Proof. For all $g \in L^p(\mathbb{R}^d)$ there are g_m step functions and $g_m \rightarrow g$ in $L^p(\mathbb{R}^d)$. We can assume that Ω is open and bounded and we want to approximate χ_Ω by $C_c(\mathbb{R}^d)$. ■

Lemma 3.8 (Urnson) Define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \epsilon\}$$

Then there is a $\eta_\epsilon \in C_c(\mathbb{R}^d)$ s.t.

$$\begin{cases} 0 \leq \eta(x) \leq 1 & \forall x \in \mathbb{R}^d \\ \eta_\epsilon(x) = 1 & \text{if } x \in \Omega_\epsilon \\ \eta_\epsilon(x) = 0 & \text{if } x \notin \Omega \end{cases}$$

Lemma 3.9 (General Version of Urnson) If $A, B \subseteq \mathbb{R}^d$, A closed, B closed, $A \cap B = \emptyset$. Then

$$\eta(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}$$

Then $\eta \in C(\mathbb{R}^d)$, $0 \leq \eta \leq 1$ and $\eta = 0$ if $x \in B$, $\eta = 1$ if $x \in A$. Apply to $A = \overline{\Omega_\epsilon} \subset \subset \Omega$ and $B = \mathbb{R}^d \setminus \Omega$.

Theorem 3.10 (Appendix C4 in Evans) Let Ω be open in \mathbb{R}^d and for $\epsilon > 0$ define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \mathbb{R}^d \setminus \Omega) > \epsilon\}$$

Let $f \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} f = 1$, $\text{supp } f \subseteq B(0, 1)$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ $\text{supp } f_\epsilon \subseteq B(0, \epsilon)$. Then for all $g \in L_{loc}^p(\Omega)$ (i.e. $\mathbb{1}_K g \in L^p(\Omega) \forall K$ compact set in Ω), then:

- a) $g_\epsilon(x) = (f_\epsilon \star g)(x) = \int_{\mathbb{R}^d} f_\epsilon(x-y)g(y) dy = \int_\Omega f_\epsilon(x-y)g(y) dy$ is well-defined in Ω_ϵ and $g_\epsilon \in C^\infty(\Omega_\epsilon)$.
- b) $g_\epsilon \rightarrow g$ in $L_{loc}^p(\Omega)$ if $1 \leq p < \infty$ and $g_\epsilon(x) \rightarrow g(x)$ almost everywhere $x \in \Omega$.
- c) If $g \in C(\Omega)$, then $g_\epsilon(x) \rightarrow g(x)$ uniformly in any compact subset of Ω .

Proof. a) $D^\alpha(g_\epsilon) = (D^\alpha f_\epsilon) \star g \in C(\Omega_\epsilon)$

b) Already proved in \mathbb{R}^d space. ■

Corollary 3.11 (Lebesgue differentiation theorem) If $f \in L_{loc}^p(\mathbb{R}^d)$, then

$$\int_{B(x, \epsilon)} |f(y) - f(x)|^p dy \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

Exercise 3.12 (E 2.1) Let $u \in C^2(\mathbb{R}^2)$ be convex. I.e.

$$tu(x) + u(y)(1-t) \geq u(tx + (1-t)y) \forall x, y \in \mathbb{R}^d \forall t \in [0, 1]$$

- a) Prove for all $x \in \mathbb{R}^d$ that $H(x) = \dots$

Solution.

- a In 1D: If u is convex $\Leftrightarrow u''(x) \geq 0$ for all $x \in \mathbb{R}$. In general: Taylor expansion for all $x, z \in \mathbb{R}^d$:

$$u(x) = u(z) + \nabla u(z)(x - z) + \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{(x - z)^\alpha}{\alpha!} ds$$

$$x = z + s(x - z), s = 1 \text{ Use } z = tx + (1 - t)y \Rightarrow x - z = (1 - t)(x - y)$$

$$tu(x) = tu(z) + t\nabla u(z)(1 - t)(x - y) + t \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{[(1 - t)(x - y)]^\alpha}{\alpha!} ds$$

$$(1 - t)u(y) = (1 - t)u(z) + (1 - t)\nabla u(z)t(y - x) + (1 - t) \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y - z)) \frac{[t(y - x)]^\alpha}{\alpha!} ds$$

$$\begin{aligned} \Rightarrow tu(x) + (1 - t)u(y) &= u(z) + t \int_0^1 \dots + (1 - t) \int_0^1 \dots \\ \Rightarrow t \int_0^1 \dots + (1 - t) \int_0^1 \dots &\geq 0 \forall x, y, t, z = tx + (1 - t)y \end{aligned}$$

$$t(1 - t)^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{(x - y)^\alpha}{\alpha!} ds + (1 - t)t^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y - z)) \frac{(y - x)^\alpha}{\alpha!} ds \geq 0$$

for all $x, y \in \mathbb{R}^d, t \in [0, 1], z = tx + (1 - t)y$. Divides for $t(1 - t)$

$$(1 - t) \int_0^1 \dots + \int_0^1 \dots \geq 0$$

Take $t \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(y + s(x - y)) \frac{(x - y)^\alpha}{\alpha!} ds \geq 0 \forall x, y \in \mathbb{R}^d$$

Take $y = x + a, a \in \mathbb{R}^d$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x + a + sa) \frac{a^\alpha}{\alpha!} ds \geq 0 \forall \epsilon > 0, \forall x, a \in \mathbb{R}^d$$

Take $\epsilon \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x) \frac{a^\alpha}{\alpha!} \geq 0 \Rightarrow \sum_{i,j=1, i \neq j} \partial_{x_i} \partial_{x_j} u(x) a_i a_j + \sum_{i=1}^d \partial_{x_i}^2 u(x) \frac{a_i^2}{2}$$

We get

$$\frac{1}{2} a^T H a \geq 0 \forall a(a_i)_{i=1}^d \in \mathbb{R}^d$$

- b $H(x) \geq 0 \Rightarrow (\partial_i \partial_j u) \geq 0 \Rightarrow \text{Tr} H(x) \geq 0 \Rightarrow \sum_{i=1}^d \partial_{x_i}^2 u(x) \geq 0 \Rightarrow \Delta u(x) \geq 0 \forall x \in \mathbb{R}^d$

■

Exercise 3.13 (E 2.2)

Solution. Regard $d = 3$. De function $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$. We prove

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{\max(|x|, r)}$$

If $|x| > r$, then $0 \notin B(x, r + \epsilon)$. Then

$$y \mapsto \frac{1}{|y|}$$

is harmonic in $B(x, r + \epsilon)$. Then by the Mean Value Property:

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{|x|}$$

If $|x| < r$: Then $\frac{1}{|y|}$ is not harmonic in $B(x, r)$ since $0 \in B(x, r)$. Note

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \oint_{\partial B(0,r)} \frac{dS(y)}{|x - y|}$$

This function depends on x only via $|x|$.

$$\dots = \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - Ry|}$$

for all R rotation $SO(3)$, $dS(Ry) = dS(y)$

$$\begin{aligned} &= \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - y|} \\ &= \oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \\ \text{(Radial in } z) \quad &= \oint_{\partial B(0,|x|)} \left(\oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \right) dS(z) \\ \text{(Fubini)} \quad &= \oint_{\partial B(0,r)} \left(\oint_{\partial B(0,|x|)} \frac{dS(z)}{|z - y|} \right) dS(y) \\ \text{(case 1 since } |y| = r > |x|) \quad &= \oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) = \frac{1}{r} \end{aligned}$$

If $|x| = r$: Continuity: $x \mapsto \oint_{\partial B(0,r)} \frac{dS(y)}{|x - y|}$ ■

Remark 3.14 For $f \in C^{|\alpha|}, g \in C^{|\beta|}$:

$$D^{\alpha+\beta}(f \star g) = (D^\alpha f) \star (D^\beta g)$$

Lemma 3.15 If $d \geq 3$ and $f : \mathbb{R}^d \rightarrow \mathbb{R}$ radial. Then:

$$\begin{aligned} \left(\frac{1}{|x|^{d-2}} \star f \right) (x) &= \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-2}} dy \\ &= \int_{\mathbb{R}^d} \frac{f(y)}{\max(|x|^{d-2}, |y|^{d-2})} dy \end{aligned}$$

Proof. (d=3) Polar coordinates:

$$\begin{aligned}
 \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{1}{|x-rw|} d\omega \right] f(r) dr \\
 (a) \quad &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{d\omega}{\max(|x|, r)} \right] f(r) dr \\
 &= \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|, |y|)} dy
 \end{aligned}$$

(b) (d=3) If f radial and non-negative

$$\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \frac{f(y)}{|x|} dy = \frac{(Sf?)}{|x|}$$

Then

$$\begin{aligned}
 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|} dx dy &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x)f_2(y)}{|x+z_1-y-z_2|} dx dy \\
 &= \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f_1(x) dx \right) f_2(y) dy \leq \int_{\mathbb{R}^3} \frac{(\int_{\mathbb{R}^3} f_1)}{|y+z_2-z_1|} f_2(y) dy \\
 &\leq \frac{(\int_{\mathbb{R}^3} f_1)(\int_{\mathbb{R}^3} f_2)}{|z_1-z_2|}
 \end{aligned}$$

■

Exercise 3.16 (Bonus 2) a) Prove that $u(x) = \frac{1}{|x|}$ is sub-harmonic in $\mathbb{R}^2 \setminus \{0\}$.

b) Prove that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ radial, non-negative, measurable:

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy \geq \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)} dy$$

My Solution. a) Let $x \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned}
 \partial_{x_i} u &= \partial_{x_i} |x|^{-1} = -|x|^{-2} \frac{x_i}{|x|} = -x_i |x|^{-3} \\
 \Rightarrow \partial_{x_i}^2 u &= \partial_{x_i} (-x_i |x|^{-3}) = -|x|^{-3} + 3x_i^2 |x|^{-5} \\
 \Rightarrow \Delta u &= -|x|^{-3} + 3x_1^2 |x|^{-5} - |x|^{-3} + 3x_2^2 |x|^{-5} \\
 &= -2|x|^{-3} + 3 \underbrace{(x_2^2 + x_2^2)}_{=|x|^2} |x|^{-5} = \frac{1}{|x|^3} > 0
 \end{aligned}$$

So u is sub-harmonic in $\mathbb{R}^2 \setminus \{0\}$.

b) Let $r > 0, x \in \mathbb{R}^2$ and $|x| < r$. First we show that

$$\oint_{\partial B(x,r)} \frac{1}{|y|} dS(y) \geq \frac{1}{r} \quad (\star)$$

Now,

$$\oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) = \oint_{\partial B(0,r)} \frac{1}{|x-y|} dS(y) =: \tilde{u}(x)$$

Take $z \in \mathbb{R}^2 \setminus \{0\}$ such that $z = |x|$, then $\tilde{u}(x) = \tilde{u}(z)$. Let $0 < \epsilon < r$ be small. Then we get

$$\begin{aligned}
\tilde{u}(z) &= \oint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \\
\left(\begin{array}{l} |y| = r > |x| = |z| \\ \tilde{u} \text{ radial function} \end{array} \right) &= \oint_{\partial B(0,|x|-\epsilon)} \left(\oint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \right) dS(z) \\
(\text{Fubini}) &= \oint_{\partial B(0,r)} \left(\oint_{\partial B(0,|x|-\epsilon)} \frac{dS(z)}{|z-y|} \right) dS(y) \\
&= \oint_{\partial B(0,r)} \left(\oint_{\partial B(y,|x|-\epsilon)} \frac{dS(z)}{|z|} \right) dS(y) \\
\left(\frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2 \setminus \{0\} \right) &\geq \oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) \\
&= \oint_{\partial B(0,r)} \frac{1}{r} dS(y) \\
&= \frac{1}{r}
\end{aligned}$$

Now,

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy = \int_{|x| \geq |y|} \frac{f(y)}{|x-y|} dy + \int_{|x| < |y|} \frac{f(y)}{|x-y|} dy,$$

where

$$\begin{aligned}
\int_{|x| \leq |y|} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \int_{\partial B(0,r)} \frac{f(y)}{|x-y|} \mathbb{1}(|x| \leq |y|) dS(y) dr \\
(f \text{ radial}) &= \int_0^\infty f(r) \int_{\partial B(0,r)} \frac{\mathbb{1}(|x| \leq r)}{|x-y|} dS(y) dr \\
&= \int_0^\infty f(r) \int_{\partial B(x,r)} \frac{\mathbb{1}(|x| \leq r)}{|y|} dS(y) dr \\
(\star) &\geq \int_0^\infty \frac{f(r)}{r} |\partial B(x,r)| \mathbb{1}(|x| \leq r) dr \\
&= \int_0^\infty \int_{\partial B(x,r)} \frac{f(r)}{r} \mathbb{1}(|x| \leq r) dS(y) dr \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{|y|} \mathbb{1}(|x| \leq |y|) dy \\
&= \int_{|x| \leq |y|} \frac{f(y)}{|y|} dy
\end{aligned}$$

and

$$\begin{aligned}
\int_{|x|>|y|} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \left(\int_{\partial B(0,r)} \frac{f(r)}{|x-y|} \mathbb{1}(|x|>|y|) dS(y) \right) dr \\
(f \text{ radial}) &= \int_0^\infty f(r) \mathbb{1}(|x|>r) \left(\int_{\partial B(x,r)} \frac{1}{|y|} dS(y) \right) dr \\
\left(\begin{array}{l} \frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2, \\ \text{MVP and } |x|>r \end{array} \right) &\geq \int_0^\infty f(r) \mathbb{1}(|x|>r) |\partial B(x,r)| \frac{1}{|x|} dr \\
&= \int_0^\infty \int_{\partial B(x,r)} f(r) \mathbb{1}(|x|>r) \frac{1}{|x|} dS(y) dr \\
&= \int_{\mathbb{R}^2} f(y) \mathbb{1}(|x|>|y|) \frac{1}{|x|} dy \\
&= \int_{|x|>|y|} f(y) \frac{1}{|x|} dy.
\end{aligned}$$

So we can conclude,

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy &= \int_{|x|>|y|} \frac{f(y)}{|x-y|} dy + \int_{|x|\leq|y|} \frac{f(y)}{|x-y|} dy \\
&\geq \int_{|x|>|y|} \frac{f(y)}{|x|} dy + \int_{|x|\leq|y|} \frac{f(y)}{|y|} dy \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)} dy
\end{aligned}$$

■

3.2 Fourier Transformation

Definition 3.17 (Fourier Transform) For $f \in L^1(\mathbb{R}^d)$ define

$$\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k \cdot x} dx, \quad k \cdot x = \sum_{i=1}^d k_i x_i$$

Theorem 3.18 (Basic Properties) 1. If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^\infty(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

2. For all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$. Moreover, \mathcal{F} can be extended to be a unitary transformation $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ s.t.

$$\|\mathcal{F}g\|_{L^2} = \|g\|_{L^2} \quad \forall g \in L^2(\mathbb{R}^d)$$

3. The inverse of F can be defined as

4.

$$(F^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(x) e^{2\pi i k x} dk$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

5. $\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k)$ as $(2\pi i k)^\alpha f(k) \in L^2(\mathbb{R}^d)$ ($k^\alpha = k_1^{\alpha_1} \dots k_\alpha^{\alpha_k}$)

6. $\widehat{f \star g}(k) = \hat{f}(k) \hat{g}(k)$ if f, g are nice enough.

Theorem 3.19 (Hausdorff-Young-Inequality) If $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^n)$ then

$$\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}$$

and

$$\|\hat{f}\|_{L^p} \leq \|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^d)$$

Remark 3.20 We want to apply the Fourier transform to find the solution of a PDE, e.g. the Poisson-Equation:

$$-\Delta u = f \text{ in } \mathbb{R}^d \Rightarrow |2\pi k|^2 \hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{1}{|2\pi k|^2} \hat{f}(k)$$

If we can find G s.t. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$, then

$$\begin{aligned} \hat{u}(k) &= \hat{G}(k) \hat{f}(k) = \widehat{G \star f} \\ \Rightarrow u(x) &= (G \star f)(x) = \int_{\mathbb{R}^d} G(x-y) f(y) dy \end{aligned}$$

In fact G is the fundamental solution of laplace quation.

Theorem 3.21 (Fourier Transform of $\frac{1}{|x|^\alpha}$ for $0 < \alpha < d$) We have formally

$$\frac{\widehat{c_\alpha}}{|x|^\alpha} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \quad \forall 0 < \alpha < d$$

Here

$$c_\alpha = \pi^{-\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda$$

More precisely, for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{c_\alpha}{|x|^\alpha} \star f = \left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee$$

Moreover if $\alpha > \frac{d}{2}$, then we also have

$$\left(\frac{c_\alpha}{|x|^\alpha} \star f \right)^\wedge = \frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k)$$

Lemma 3.22 (Fourier Transform of Gaussians) In \mathbb{R}^d ,

$$\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$$

More generally for all $\lambda > 0$:

$$\widehat{e^{-\pi\lambda^2|x|^2}} = \lambda^{-d} e^{-\pi \frac{|k|^2}{\lambda^2}}$$

(exercise)

Proof of Theorem. Formally:

$$\begin{aligned}
\frac{c_\alpha}{|x|^\alpha} &= \frac{1}{|x|^\alpha} \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty e^{-\pi\lambda|x|^2} \lambda^{\frac{\alpha}{2}-1} d\lambda \\
\Rightarrow \frac{\hat{c}_\alpha}{|x|^\alpha}(k) &= \int_0^\infty \widehat{e^{-\pi\lambda|x|^2}}(k) \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi\frac{|k|^2}{\lambda}} \lambda^{\frac{\alpha}{2}-1} d\lambda \\
(\lambda \rightarrow \frac{1}{\lambda}) &= \int_0^\infty \lambda^{\frac{d}{2}} e^{-\pi|k|^2\lambda} \lambda^{-\frac{\alpha}{2}+1} \lambda^{-2} d\lambda \\
&= \frac{c_{d-\alpha}}{|k|^{d-\alpha}}
\end{aligned}$$

Let $f \in C_c(\mathbb{R}^d)$. Then $\left(\frac{1}{|x|^\alpha} \star f\right)(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} f(y) dy$ is well defined as $\frac{1}{|x-y|} \in L^1_{loc}(\mathbb{R}^d, dy)$. It is bounded

$$\frac{1}{|x|^\alpha} \star f = \frac{1}{|x|^\alpha} \underbrace{\mathbb{1}(|x| \leq 1)}_{\in L^\infty(\mathbb{R}^d)} \star \underbrace{f}_{L^\infty} + \frac{1}{|x|} \underbrace{\mathbb{1}(|x| > 1)}_{\in L^\infty} \star \underbrace{f}_{\in L^1} \in L^\infty(\mathbb{R}^d)$$

When $|x| \rightarrow \infty$:

$$\left(\frac{1}{|x|^\alpha} \star f\right)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^\alpha} dy = \int_{|y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy \sim \frac{\int_{\mathbb{R}^d} f(y) dy}{|x|^\alpha}$$

Note that $\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \underbrace{\hat{f}(k)}_{\text{bounded}} \in L^1(\mathbb{R}^d)$.

$$\begin{aligned}
(\dots)\mathbb{1}(|k| \leq 1) + (\dots)\mathbb{1}(|k| > 1) \frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| \leq 1) &\leq \|f\|_{L^1} \frac{\mathbb{1}(|k| \leq 1)}{|k|^{d-\alpha}} \in L^1(\mathbb{R}^d, dk) \\
\frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| > 1) &\leq |\hat{f}(k)| \in L^2(\mathbb{R}^d, dK) \text{ as } f \in L^2(\mathbb{R}^d)
\end{aligned}$$

Lemma 3.23 If $f \in C_c^\infty(\mathbb{R}^d)$, then $\hat{f} \in L^1(\mathbb{R}^d)$

Proof. (Exercise) Hint: $|\widehat{D^\alpha f}| = |2\pi k|^\alpha |\hat{f}(k)| \rightsquigarrow |\hat{f}(k)| \leq \frac{1}{|k|^\alpha}$ as $|k| \rightarrow \infty$. ■

Compute:

$$\begin{aligned}
\left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)\right)^\vee(x) &= \int_{\mathbb{R}^d} \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_{\mathbb{R}^d} \left(\int_0^\infty e^{-\pi|k|^2\lambda} \lambda^{\frac{d-\alpha}{2}-1} d\lambda\right) \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} e^{-\pi|k|^2\lambda} \hat{f}(k) e^{2\pi i k x} dk\right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(e^{-\pi k^2 \lambda} \hat{f}(x)\right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}}(k) \hat{f}(k)\right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}} \star f\right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \left(\int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}} \lambda^{\frac{d-\alpha}{2}-1} d\lambda\right) \star f
\end{aligned}$$

Assume $d > \alpha > \frac{d}{2}$. Then $\frac{c_\alpha}{|x|^\alpha} \star f \in L^\infty$ and behaves $\frac{c_\alpha(\int f)}{|x|^\alpha}$ as $|x| \rightarrow \infty$. This implies:

$$\int_{\mathbb{R}^d} \left| \frac{c_\alpha}{|x|^\alpha} \star f \right|^2 \leq c + \int_{|x| \geq R} \frac{c}{|x|^{2d}} dx < \infty$$

Thus the Fourier Transform $\widehat{\frac{c_\alpha}{|x|^\alpha} \star f}$ exists. Combining with

$$\begin{aligned} \frac{c_\alpha}{|x|^\alpha} \star f &= \left(\frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k) \right)^\vee \\ \Rightarrow \widehat{\frac{c_\alpha}{|x|^\alpha} \star f} &= \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \end{aligned}$$

■

Remark 3.24 If $d \geq 3$

$$\begin{aligned} \hat{G}(k) &= \frac{1}{|2\pi k|^2} \\ \Rightarrow G(x) &= \left(\frac{1}{|2\pi k|^2} \right)^\vee = \frac{1}{d(d-2)|x|^{d-2}} = \Phi(x) \end{aligned}$$

3.3 Theory of Distribution

Let $\Omega \subseteq \mathbb{R}^d$ be open.

- $D(\Omega) = C_c^\infty(\Omega)$ the space of test functions.
- $\phi_n \rightarrow \phi$ in $D(\Omega)$ if $\exists K \subseteq \Omega$, $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K$ and $\|D^\alpha(\phi_n - \phi)\|_{L^\infty} \rightarrow 0$ for all $\alpha = (\alpha_1, \dots, \alpha_d)$, $d_i \in \{0, 1, 2, \dots\}$.

$$D'(\Omega) = \{T : D(\Omega) \rightarrow \mathbb{R} \text{ on } \mathbb{C} \text{ linear and continuous}\}$$

the space of distributions.

Motivation: $L^2(\Omega)' = L^2(\Omega)$, $(L^p(\Omega))' = (L^q(\Omega))$, $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3.25 ("normal functions" are distributions) If $f \in L^1_{loc}(\Omega)$, then $T = T_f$ defined by:

$$T(\phi) = \int_{\Omega} f(x)\phi(x) dx$$

is a distribution for all $\phi \in D(\Omega)$, i.e. $T \in D'(\Omega)$. Indeed, it is clear that $T(\phi)$ is well-defined for all $\phi \in D(\Omega)$ and $\phi \mapsto T(\phi)$ is linear. Let us check that $\phi \mapsto T(\phi)$ is continuous. Take $\phi_n \rightarrow \phi$ in $D(\Omega)$ and prove that $T(\phi_n) \rightarrow T(\phi)$. Since $\phi_n \rightarrow \phi$ in $D(\Omega)$, there is a compact K s.t. $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K \subseteq \Omega$.

Question: Why is $f \mapsto T_f$ injective?

Lemma 3.26 (Fundamental lemma of calculus of variants) Let $\Omega \subseteq \mathbb{R}^d$ be open. If $f, g \in L^1_{loc}(\Omega)$ and $\int_{\Omega} f\phi dy = \int_{\Omega} g\phi dy$ for all $\phi \in D(\Omega)$, then $f = g$ in $L^1_{loc}(\Omega)$

Example 3.27 (Dirac delta function) Let $\Omega \subseteq \mathbb{R}^d$ open. Define $T : D(\Omega) \rightarrow \mathbb{R}$ or \mathbb{C} by $T(\phi) = \phi(x_0)$. Let $x_0 \in \Omega$. Then $T \in D'(\Omega)$ and we denote it by δ_{x_0} . It is clear that $\phi \mapsto T(\phi) = \phi(x_0)$ is well-defined and linear for all $\phi \in D(\Omega)$. Take $\phi_n \rightarrow \phi$ in $D(\Omega)$ and prove $T(\phi_n) \rightarrow T(\phi)$, i.e. $\phi_n(x_0) \rightarrow \phi(x_0)$ (obvious.)

Example 3.28 (Principle Value) The function $f(x) = \frac{1}{x}$ is not in $L^1_{loc}(\mathbb{R})$, but we can still define

$$\int_{\mathbb{R}} f(x)\phi(x) dx = \int_{\mathbb{R}} \frac{\phi(x)}{x} dx$$

for all $\phi \in D(\mathbb{R})$ s.t. $\phi(0) = 0$. In fact,

$$\phi(x) = |\phi(x) - \phi(0)| \leq (\sup |\phi'|)(x),$$

so $\frac{|\phi(x)|}{|x|} \in L^\infty(\mathbb{R})$ and compactly supported. So $\frac{\phi(x)}{x} \in L^1(\mathbb{R})$. Define $T : D(\mathbb{R}) \rightarrow \mathbb{R}$ or \mathbb{C} by

$$T(\phi) = \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx \quad \forall \phi \in D(\mathbb{R}) \text{ s.t. } \phi(0) = 0$$

We denote $T = p.v. \left(\frac{1}{x}\right)$. We check that $T \in D'(\mathbb{R})$: For all $\epsilon > 0$ we have

$$\left| \frac{\phi(x)}{x} \right| \leq \frac{\|\phi\|_{L^\infty}}{\epsilon}$$

for all $|x| \geq \epsilon$ and ϕ is compactly supported. So we get for all $\epsilon > 0$:

$$\mathbb{1}(|x| \geq \epsilon) \frac{\phi(x)}{x} \in L^1(\mathbb{R}) \rightsquigarrow \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx < \infty$$

We can write:

$$\int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx = \int_{|x| \geq 1} \frac{\phi(x)}{x} dx + \int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx$$

The second part can be written as:

$$\int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx = \int_{\epsilon}^1 \frac{\phi(x)}{x} dx + \int_{-1}^{-\epsilon} \frac{\phi(x)}{x} dx = \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx$$

Since $\phi \in C_c^\infty(\mathbb{R})$ it holds that $|\phi(x) - \phi(-x)| \leq 2\|\phi'\|_{L^\infty}(x)$.

$$\begin{aligned} \Rightarrow \frac{\phi(x) - \phi(-x)}{x} &\in L^\infty(\mathbb{R}) \Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^1(0, 1) \\ \Rightarrow \int_0^1 \frac{\phi(x) - \phi(-x)}{x} dx &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx \end{aligned}$$

Remark 3.29 The function $\frac{1}{|x|^d}$ is not in $L^1_{loc}(\mathbb{R}^d)$ but $\exists T \in D'(\mathbb{R}^d)$ s.t. $T(\phi) = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^d} dx$ for all $\phi \in C_c^\infty(\mathbb{R}^d)$ s.t. $\phi(0) = 0$

Let in the following $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 3.30 (Derivatives of distributions) Let $\Omega \subseteq \mathbb{R}^d$ and $T \in D'(\Omega)$. Define for $\alpha \in \mathbb{N}^d$:

$$\begin{aligned} D^\alpha T : D(\Omega) &\longrightarrow \mathbb{K} \\ \phi &\longmapsto (-1)^{|\alpha|} T(D^\alpha \phi) \end{aligned}$$

Motivation: $f \in C_c^\infty(\Omega)$

$$\int_{\Omega} (D^\alpha f) \phi = (-1)^{|\alpha|} \int_{\Omega} f (D^\alpha \phi)$$

„If the classical derivative exists, then it is the same as the distributional derivative.“
We write

$$(D^\alpha T)(\phi) = T_{D^\alpha f}(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi).$$

Remark 3.31 For all $T \in D'(\Omega)$ it holds $D^\alpha T \in D'(\Omega)$ for all $\alpha \in \mathbb{N}^d$. Clearly

$$\phi \longmapsto (D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

is linear. Moreover, if $\phi_n \rightarrow \phi$ in $D(\Omega)$, then $D^\alpha \phi_n \rightarrow D^\alpha \phi$ in $D(\Omega)$, so

$$(D^\alpha T)(\phi_n) = (-1)^{|\alpha|} T(D^\alpha \phi_n) \xrightarrow{n \rightarrow \infty} (-1)^{|\alpha|} T(D^\alpha \phi) = (D^\alpha T)(\phi)$$

Example 3.32 Consider $f : x \mapsto |x|$, then $f \in C(\mathbb{R})$ but $f \notin C^1(\mathbb{R})$. However,

$$f'(x) = g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \in L^1_{loc}$$

Lets check $f' = g$, i.e. $-f(\phi') = f'(\phi) = g(\phi)$ for all $\phi \in D(\mathbb{R})$. Thus we need to prove:

$$-\int_{\mathbb{R}} f(x) \phi'(x) dx = \int_{\mathbb{R}} g(x) \phi(x) dx \quad \forall \phi \in D(\mathbb{R})$$

namely:

$$\underbrace{-\int_{\mathbb{R}} |x| \phi'(x) dx}_{:= (\star)} = \int_0^\infty \phi(x) dx - \int_{-\infty}^0 \phi(x) dx$$

Now we have

$$(\star) = -\int_0^\infty x \phi'(x) dx + \int_{-\infty}^0 x \phi'(x) dx.$$

By integration by parts:

$$\int_0^\infty x \phi'(x) dx = \underbrace{[x \phi(x)]_0^\infty}_{=0} - \int_0^\infty \phi(x) dx = -\int_0^\infty \phi(x) dx$$

and similary:

$$\int_{-\infty}^0 x \phi'(x) dx = -\int_{-\infty}^0 \phi(x) dx$$

Thus $f' = g$ in $D'(\Omega)$. We claim that $g' = 2\delta_0$ in $D'(\mathbb{R})$. In fact, for all $\phi \in D(\mathbb{R})$, then:

$$\begin{aligned} g'(\phi) &= -g(\phi') = -\int_{\mathbb{R}} g \phi' dx = -\int_{-\infty}^0 (-1) \phi' dx - \int_0^\infty (1) \phi' dx \\ &= -\int_0^\infty \phi' dx + \int_{-\infty}^0 \phi' dx = [\phi(0) - \underbrace{\phi(\infty)}_{=0}] + [\phi(0) - \underbrace{\phi(-\infty)}_{=0}] = 2\phi(0) = 2\delta_0(\phi) \end{aligned}$$

So $g' = 2\delta_0$ in $D'(\mathbb{R})$.

Exercise 3.33 Prove that $(D^\alpha \delta_x)(\phi) = (-1)^{|\alpha|} (D^\alpha \phi)(x)$ for all $\phi \in D(\mathbb{R})$ for all $x \in \mathbb{R}$.

Definition 3.34 (Convergence of distributions) Let $\Omega \subseteq \mathbb{R}^d$ be open, then

$$T_n \xrightarrow{n \rightarrow \infty} T$$

in $D'(\Omega)$ if $T_n(\phi) \xrightarrow{n \rightarrow \infty} T(\phi)$ for all $\phi \in D(\Omega)$.

Exercise 3.35 Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ For $\epsilon > 0$, define $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$. Then: $f_\epsilon \rightarrow \delta_0$ in $D'(\Omega)$.

Exercise 3.36 Let $\Omega \subseteq \mathbb{R}^d$ be open and $T_n \rightarrow T$ in $D'(\Omega)$. Then: $D^\alpha T_n \rightarrow D^\alpha T$ in $D'(\Omega)$ for all $\alpha = (\alpha_1, \dots, \alpha_d)$

Definition 3.37 (Convolution of distributions) Let $T \in D'(\mathbb{R})$ and $f \in L_c^\infty(\mathbb{R}^d)$. Define

$$(T \star f)(y) = T(f_y)$$

We write $f_y(x) = f(x - y)$ and $\tilde{f}(x) = f(-x)$.

Theorem 3.38 Let $T \in D'(\mathbb{R})$. Then for all $f \in D(\mathbb{R})$:

1. $y \mapsto T(f_y)$ is $C^\infty(\mathbb{R}^d)$ and

$$D_y^\alpha (T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D^\alpha f_y)$$

2. For all $g \in L^1(\mathbb{R}^d)$ and compactly supported, then

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(\underbrace{f \star g}_{\in C_c^\infty(\mathbb{R})})$$

Proof. 1. We prove that $y \mapsto T(f_y)$ is continuous. Take $y_n \rightarrow y$ in \mathbb{R}^d , then:

$$T(f_{y_n}) \rightarrow T(f_y)$$

since $f_{y_n} \rightarrow f_y$ in $D(\mathbb{R}^d)$. We check this: Since $f \in C_c^\infty(\mathbb{R}^d)$, it holds that $\text{supp } f \subseteq B(0, R) \subseteq \mathbb{R}^d$. Since $y_n \rightarrow y$ in \mathbb{R}^d . We have $\sup_n |y_n| < \infty$. Thus f_{y_n}, f_y are supported in $\overline{B(0, R + \sup_n |y_n|)} = K$ compact. Moreover

$$|f_{y_n}(x) - f_y(x)| = |f(x - y_n) - f(x - y)| \leq \|\nabla f\|_{L^\infty} \|y_n - y\| \rightarrow 0$$

So we get $\|f_{y_n} - f_y\|_{L^\infty} \rightarrow 0$ Similarly:

$$\|D^\alpha f_{y_n} - D^\alpha f_y\|_{L^\infty} \rightarrow 0$$

■

Exercise 3.39 (E 3.1 Lebesgue Differentiation Theorem) Let $f \in L_{loc}^1(\mathbb{R}^d)$. Prove that for almost every $x \in \mathbb{R}^d$:

$$\oint_{B(x,r)} |f(y)| dy \xrightarrow{r \rightarrow 0} 0$$

Proof. Clearly the same result holds with $\mathbb{R}^d \rightsquigarrow \Omega \subseteq \mathbb{R}^d$ open. Also it suffices to consider $f \in L^1(\mathbb{R}^d)$. From the last time discussion, by a density argument there exists $r_n \rightarrow 0$ s.t.

$$\oint_{B(x,r_n)} |f(y) - f(x)| dy = 0$$

for a.e. $x \in \mathbb{R}^d$. We prove that for all $\epsilon > 0$, the set $A_\epsilon = \{x \in \mathbb{R}^d \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy > \epsilon\}$ has measure 0. This will imply that

$$\bigcup_{n=1}^{\infty} A_{\frac{1}{n}} = \left\{ x \in \mathbb{R}^d \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy > 0 \right\}$$

has measure 0, which is what we want to show. First, we show that $|A_\epsilon| = 0$: Take $\{f_n\} \subseteq C_c^\infty$, $f_n \rightarrow f$ in $L^1(\mathbb{R}^d)$. By the triangle inequality:

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

So we get

$$\begin{aligned} & \int_{B(x,r)} |f(y) - f(x)| dy \\ & \leq \int_{B(x,r)} |f(y) - f_n(y)| dy + \int_{B(x,r)} |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \\ \Rightarrow \quad \limsup_{r \rightarrow 0} \dots & \leq \limsup_{r \rightarrow 0} (\dots) + 0 + |f_n(x) - f(x)| \end{aligned}$$

Thus, for all $x \in A_\epsilon$, then:

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy + |f_n(x) - f(x)| > 2\epsilon$$

Observation: If $a, b \geq 0$, $a + b > 2\epsilon$ then either $a > \epsilon$ or $b > \epsilon$. Therefore $A_\epsilon \subseteq (S_{n,\epsilon} \cup \tilde{S}_{n,\epsilon})$, where

$$\begin{aligned} S_{n,\epsilon} &= \{x \mid |f_n(x) - f(x)| > \epsilon\} \\ \tilde{S}_{n,\epsilon} &= \{x \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy > \epsilon\} \end{aligned}$$

Consequently: $|A_\epsilon| \leq |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}|$ for all $n \geq 1$. By the Markov / Chebyshev inequality:

$$|S_{n,\epsilon}| \leq \int_{S_{n,\epsilon}} \frac{|f_n(x) - f(x)|}{\epsilon} dx = \int_{\mathbb{R}^d} \frac{|f_n(x) - f(x)|}{\epsilon} dx = \frac{\|f_n - f\|_{L^1}}{\epsilon}$$

We want to prove a simpler bound for $\tilde{S}_{n,\epsilon}$. For all $x \in \tilde{S}_{n,\epsilon}$:

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy > \epsilon$$

So there is a $r_x \in (0, 1)$ s.t.

$$\int_{B(x, r_x) = B_x} |f_n(y) - f(y)| dy > \epsilon$$

Thus $\tilde{S}_{n,\epsilon} \subseteq \left(\bigcup_{x \in \tilde{S}_{n,\epsilon}} B_x \right)$.

Lemma 3.40 (Vitali Covering) If F is a collection of balls in \mathbb{R}^d with bounded radius, then there exists a sub-collection $G \subseteq F$ s.t.

- G has disjoint balls

- $\bigcup_{B \in F} B \subseteq \bigcup_{B \in G} 5B, 5B(x, r) = B(x, 5r)$

Remark 3.41 The condition of the boundedness of the radius is necessary. Otherwise, consider $\{B(0, n)\}_{n=1}^{\infty}$

Here consider $F = \{B_x\}_{x \in \tilde{S}_{n,\epsilon}}$. With the Vitali covering lemma there is a $G \subseteq F$ s.t. G contains disjoint balls and:

$$\tilde{S}_{n,\epsilon} \subseteq \bigcup_{B \in F} B \subseteq \bigcup_{B \in G} 5B$$

So we get

$$|\tilde{S}_{n,\epsilon}| \leq \left| \bigcup_{B \in G} 5B \right| \leq \sum_{B \in G} |5B| = \sum_{B \in G} 5^d |B|$$

On the other hand, for all $B \in G \subseteq F$:

$$\int_B |f_n(y) - f(y)| dy > \epsilon \Rightarrow \int_B |f_n - f| > \epsilon |B|$$

This implies:

$$\sup_{B \in G} \int_B |f_n - f| > \epsilon \sum_{B \in G} |B|$$

Since balls in G are disjoint:

$$\int_{\mathbb{R}^d} \geq \int_{\bigcup_{B \in G}} |f_n - f| dy > \epsilon \sum_{B \in G} |B| \geq \frac{\epsilon}{5^d} |\tilde{S}_{n,\epsilon}|$$

So

$$|\tilde{S}_{n,\epsilon}| \leq \frac{5^d}{\epsilon} \|f_n - f\|_{L^1}$$

In summary:

$$|A_\epsilon| \leq |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}| \leq \frac{5^d + 1}{\epsilon} \|f_n - f\|_{L^1} \rightarrow 0$$

as $n \rightarrow \infty$. So $|A_\epsilon| = 0$ for all $\epsilon > 0$ ■

Remark 3.42 1. The proof can be done by using the Besicovitch covering lemma: For all $E \subseteq \mathbb{R}^d$ s.t. E is bounded. Let $F =$ collection of balls s.t. for all $x \in E$ there is a $B_x \in F$ s.t. x is the center of B_x . There is a sub-collection $G \subseteq F$ s.t.

- $E \subseteq \bigcup_{B \in G} B$
- Any point in E belongs to at most C_d balls in G (C_d depends only on \mathbb{R}^d), i.e.

$$\mathbb{1}_E(x) \leq \sum_{B \in G} \mathbb{1}_B(x) \leq C_d \mathbb{1}_E(x) \forall x$$

2. By a simpler argument we can prove the weak L^1 -estimate:

$$\{x \mid f^\star(x) > \epsilon\} \leq \frac{C_d}{\epsilon} \|f\|_{L^1(\mathbb{R}^d)}$$

(Hardy-Littlewood maximal function)

Exercise 3.43 (E 3.2) Let $1 \leq p, q, r \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Recall that if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^r(\mathbb{R}^d)$ by Young's Inequality, and its Fourier transform is well-defined by the Hausdorff-Young inequality. Prove that

$$\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k) \quad \forall k \in \mathbb{R}^d$$

Hint: In the lecture we already discussed the case $f, g \in C_c(\mathbb{R}^d)$.

Solution.

Step 1) $f, g \in C_c^\infty(\mathbb{R}^d)$ (Fubini)

Step 2) $f \in L^p, g \in L^q$, find $f_n, g_n \in C_c^\infty$ s.t. $f_n \rightarrow f$ in L^p , $g_n \rightarrow g$ in L^q . $\widehat{f_n \star g_n} = \hat{f}_n \hat{g}_n$ pointwise a.e. we have

$$\begin{aligned} \text{(Hausdorff-Young)} \quad & \|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^{r'}} \\ & \leq \|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^r} \\ & = \|(f - f_n) \star g_n + f_n \star (g_n - g)\|_{L^r} \\ & \leq \|(f - f_n) \star g_n\|_{L^r} + \|f_n \star (g_n - g)\|_{L^r} \\ \text{(Young)} \quad & \leq \|f - f_n\|_{L^p} \|g_n\| + \|f_n\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Moreover:

$$\begin{aligned} \|\hat{f}_n \hat{g}_n - \hat{f} \hat{g}\|_{L^{r'}} & = \|(\hat{f}_n \hat{f}) \hat{g}_n + \hat{f}(\hat{g}_n - \hat{g})\|_{L^{r'}} \\ \text{(Hölder)} \quad & \leq \|\hat{f}_n - \hat{f}\|_{L^{p'}} \|\hat{g}_n\|_{L^{q'}} + \|\hat{f}\|_{L^{q'}} \\ \text{(Hausdorff-Young (3.19))} \quad & \leq \|f_n - f\|_{L^p} \|g_n\|_{L^q} + \|f\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \rightarrow \infty} 0 \\ \text{So } \hat{f}_n \hat{g}_n & \rightarrow \hat{f} \hat{g} \text{ in } L^{r'} \quad \widehat{f \star g} = \hat{f} \hat{g} \text{ in } L^{r'} \quad \frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'} \quad \blacksquare \end{aligned}$$

Exercise 3.44 (E 3.3) $f \in C_c^\infty(\mathbb{R}^d)$. Prove $|\hat{f}(k)| \leq \frac{C_N}{(1+|k|)^N}$

Solution. Since $f \in C_c^\infty$ we have that $D^\alpha f \in C_c^\infty$. Recall

$$\widehat{D^\alpha f}(k) = (-2\pi i k)^\alpha \hat{f}(k)$$

For example

$$\begin{aligned} \widehat{-\Delta f}(k) & = |2\pi i k|^2 \hat{f}(k) \\ \text{(Induction)} \rightsquigarrow \widehat{(-\Delta)^N f}(k) & = |2\pi k|^{2N} \hat{f}(k) \end{aligned}$$

So we can conclude

$$\hat{f}(k) = \frac{\widehat{(-\Delta)^N f}(k)}{|2\pi k|^{2N}} \quad \forall k \in \mathbb{R}^d$$

1. $f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in L^\infty$
2. $(-\Delta)^N f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \widehat{(-\Delta)^N f} \in L^\infty$

$$\text{Conclusion: } \hat{f}(k) \leq \begin{cases} C & \forall k \\ \frac{C_N}{|k|^{2N}} & \forall k \end{cases} \text{ So } \hat{f}(k) \leq \frac{C_N}{(1+|k|)^N} \quad \blacksquare$$

Exercise 3.45 (E 3.4)

Proof. Siehe Goodnotes ■

Exercise 3.46 (Bonus 3) Let $f \in L^1(\mathbb{R}^d)$ such that

$$|\hat{f}(k)| \leq \frac{C_N}{(1 + |k|)^N}$$

for all $k \in \mathbb{R}^d$, for all $N \geq 1$. (C_N is independent of k). Prove that $f \in C^\infty(\mathbb{R}^d)$

($f \in C^\infty$) i.e. $\exists \tilde{f} \in C^\infty$ s.t. $f = \tilde{f}$ a.e.

My Solution. First we regard for $N \in \mathbb{N}$ and $|k| \geq 1$:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{(1 + |k|)^N} dk &= \int_0^\infty \left(\int_{\partial B(0,r)} \frac{1}{(1 + r)^N} dS(y) \right) dr \\ &= \int_0^\infty \frac{1}{(1 + r)^N} |\partial B(0, r)| dr \\ &\leq \int_1^\infty \frac{1}{r^N} \underbrace{|\partial B(0, r)|}_{cr^{d-1}} dr \quad \text{for a } c \in \mathbb{R} \\ &= c \int_1^\infty \frac{1}{r^{N-d+1}} dr \end{aligned}$$

From Ana I we know that $\int_1^\infty \frac{1}{r^{N-d+1}} dr < \infty$ is equivalent to $N - d + 1 > 1 \Leftrightarrow N > d$, so for $N > d$ we have

$$\frac{1}{(1 + |k|)^N} \in L^1(\mathbb{R}^d).$$

Now let $\alpha \in \mathbb{N}^d$, then we have

$$k^\alpha = k_1^{\alpha_1} \dots k_d^{\alpha_d} = |k|^{\alpha_1} \dots |k|^{\alpha_d} = |k|^{\alpha_1 + \dots + \alpha_d} = |k|^{|\alpha|} = (1 + |k|)^{|\alpha|}.$$

By assumption we have for all $N \geq 1$:

$$k^\alpha \hat{f}(k) \leq k^\alpha \frac{C_n}{(1 + |k|)^N} \leq (1 + |k|)^{|\alpha|} \frac{C_n}{(1 + |k|)^N} = \frac{C_n}{(1 + |k|)^{N-|\alpha|}}$$

If we set N such that $N - |\alpha| > d$, for example $N = d + |\alpha| + 1$, then we can conclude that $k^\alpha \hat{f} \in L^1(\mathbb{R}^d)$. This implies $\widehat{k^\alpha \hat{f}} \in L^\infty(\mathbb{R}^d)$, so

$$\widehat{k^\alpha \hat{f}}(k) = \partial^\alpha \hat{\hat{f}}(k) = \partial^\alpha (\hat{f})^\vee(-k) = \partial^\alpha f(-k) \in L^\infty(\mathbb{R}^d).$$

This implies $f \in C^\infty(\mathbb{R}^d)$. ■

Theorem 3.47 Take $T \in D'(\mathbb{R})$, $f \in C_c^\infty(\mathbb{R}^d) = D(\mathbb{R}^d)$, $f_y(x) = f(x - y)$

a) $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D_x^\alpha f_y)$

b) $\forall g \in L^1(\mathbb{R}^d)$ and compactly supported

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T\left(\underbrace{f \star g}_{\in C_c^\infty}\right)$$

Proof. a) $y \mapsto T(f_y)$ is continuous since $y_n \rightarrow y$ in \mathbb{R}^d , then $f_{y_n} \rightarrow f_y$ implies $T(f_{y_n}) \rightarrow T(f_y)$. Let's check that $y \mapsto T(f_y) \in C^1$:

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} = \lim_{h \rightarrow 0} T\left(\frac{f_{y-he_i} - f_y}{h}\right)$$

We have $\frac{f_{y-he_i} - f_y}{h} \xrightarrow{h \rightarrow 0} (\partial_i f)_y$ in $D(\mathbb{R}^d)$

- $\exists K$ compact set such that $\text{supp}(f_{y-he_i} - f_y), \text{supp } \partial_i f \subseteq K$ as $|h|$ small.

$$\begin{aligned} & \bullet \frac{f_{y-he_i}(x) - f_y(x)}{h} - (\partial_i f)_y(x) \\ &= \frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y) \end{aligned}$$

$$\left| \int_0^1 \partial_i f(x-y+the_i) dt - \partial_i f(x-y) \right| \xrightarrow{h \rightarrow 0} 0 \text{ uniformly in } x$$

Similary:

$$\begin{aligned} & \left| D_x^\alpha \left(\frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y) \right) \right| \\ &= \left| \frac{D^\alpha f(x-y+he_i) - D^\alpha f(x-y)}{h} - \partial_i(D^\alpha f)(x-y) \right| \xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

uniformly in x . Conclude:

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} \xrightarrow{h \rightarrow 0} T((\partial_i f)_y) \in C(\mathbb{R}^d)$$

So we get that $y \mapsto T(f_y) \in C^1$ and $-\partial_{y_i} T(f_y) = T((\partial_i f)_y)$

By induction:

$$D_y^\alpha T(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y) = (D^\alpha T)(f_y) \quad \forall \alpha \in \mathbb{N}^d$$

b) Heuristic: $T = T(x)$

$$\begin{aligned} \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \int_{\mathbb{R}^d} g(y) \left(\int_{\mathbb{R}^d} T(x) f(x-y) dx \right) dy \\ &= \int_{\mathbb{R}^d} T(x) \left(\int_{\mathbb{R}^d} g(y) f(x-y) dy \right) dx \\ &= \int_{\mathbb{R}^d} T(x) (f \star g)(x) dx = T(f \star g) \end{aligned}$$

Step 1: $g \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \text{(Riemann Sum)} \quad \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) T(f_{y_j}) \\ &= \lim_{\Delta_N \rightarrow 0} T \left(\Delta_N \sum_{j=1}^N g(y_j) f_{y_j} \right) \\ &= T(f \star g) \end{aligned}$$

because

$$\begin{aligned} \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f_{y_j}(x) &\rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d) \\ \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x-y_j) &\xrightarrow{\text{Riemann}} \int_{\mathbb{R}^d} g(y) f(x-y) dy = (f \star g)(x) \end{aligned}$$

Proof of:

$$\lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d)$$

1) Since $f, g \in C_c^\infty$ we have $f \star g \in C_c^\infty$. And we have

$$x \mapsto \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \in C^\infty$$

since $f \in C^\infty$ supported in $(\text{supp } g + \text{supp } f)$. So all functions are C_c^∞ and supported in $(\text{supp } g + \text{supp } f)$.

2)

$$\left| \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) - \int_{\mathbb{R}^d} g(y) f(x - y) dy \right| \xrightarrow{\Delta_N \rightarrow 0} 0$$

uniformly in x . (Result from the Riemann-Sum)

3)

$$\begin{aligned} & \left| D_x^\alpha (\Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) - (f \star g)(x)) \right| \\ &= \left| \Delta_N \sum_{j=1}^N g(y_j) D^\alpha f(x - y_j) - (D^\alpha f) \star g(x) \right| \xrightarrow{\Delta_N \rightarrow 0} 0 \end{aligned}$$

uniformly in x for all α .

Step 2: Take $g \in L^1(\mathbb{R}^d)$ and compactly supported. Then $\exists \{g_n\} \subseteq C_c^\infty(\mathbb{R}^d)$, $\text{supp } g_n \subseteq \text{supp } g + B(0, 1)$ such that $g_n \rightarrow g$ in $L^1(\mathbb{R}^d)$. By Step 1:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy = T(g_n \star f)$$

Take $n \rightarrow \infty$:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy \rightarrow \int_{\mathbb{R}^d} g(y) T(f_y) dy$$

since $g_n \rightarrow g$ in L^1 compactly supported and $y \mapsto T(f_y) \in C^\infty \subseteq L^\infty(K)$. Moreover (exercise):

$$\underbrace{g_n \star f}_{\in C_c^\infty} \rightarrow g \star f \quad \text{in } D(\mathbb{R}^d)$$

So $T(g_n \star f) \xrightarrow{n \rightarrow \infty} T(g \star f)$. Finally we obtain:

$$\int g(y) T(f_y) dy = T(g \star f) \quad \blacksquare$$

Theorem 3.48 Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $T \in D'(\Omega)$ and $f \in C_c^\infty(\Omega)$. Denote

$$\Omega_f = \{y \in \mathbb{R}^d \mid \text{supp } f_y = y + \text{supp } f \subseteq \Omega\}$$

a) $y \mapsto T(f_y) \in C^\infty(\Omega_f)$ and $D_y^\alpha (T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y)$

b) For all $g \in L^1(\Omega_g)$ compactly supported in Ω_f and it holds:

$$\int_{\Omega} g(y)T(f_y) dy = T(f \star g).$$

Theorem 3.49 Let $T \in D'(\Omega)$ s.t. $\nabla T = 0$ in $D'(\Omega)$. Then: $T = \text{const.}$ in Ω .

Proof. ($\Omega = \mathbb{R}^d$) for all $f \in C_c^\infty$, $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $\partial_{y_i} T(f_y) = (\partial_j T)(f_y) = 0$ for all $i = 1, \dots, d$. Then by the result of the theorem for C^∞ functions, $y \mapsto T(f_y) = \text{const}$ independent of y . Consequently:

$$T(f_y) = T(f_0) = T(f) \quad \forall y \in \mathbb{R}^d \quad \forall f \in C_c^\infty(\mathbb{R}^d)$$

For any $g \in C^\infty(\mathbb{R}^d)$:

$$\left(\int_{\mathbb{R}^d} g dy \right) T(f) = \int_{\mathbb{R}^d} g(y)T(f_y) dy = T(f \star g) = T(g \star f) = \left(\int_{\mathbb{R}^d} f dy \right) T(g)$$

So $\frac{T(f)}{\int_{\mathbb{R}^d} f}$ is independent of f (as soon as $\int f \neq 0$). So we get that $T(f) = \text{const} \int_{\mathbb{R}^d} f$, where const is independent of f . \blacksquare

Remark 3.50 If $u \in C^1(\mathbb{R}^d)$, then:

$$u(x+y) - u(x) = \int_0^1 \sum_{j=1}^d y_j (\partial_j u)(x + ty_j) dt = \int_0^1 y \nabla u(x + ty) dt$$

So we get that if $\nabla u = 0$, then $u(x+y) - u(x) = 0$ for all x, y , so $u = \text{const.}$

Theorem 3.51 (Taylor expansion for distributions) Let $T \in D'(\mathbb{R}^d)$ and $f \in C_c^\infty(\mathbb{R}^d)$. Then $y \mapsto T(f_y) \in C^\infty$ and

$$T(f_y) - T(f) = \int_0^1 \sum_{j=1}^d y_j (\partial_j T)(f_{ty}) dt.$$

In particular, if $g \in L_{loc}^1$ and $\nabla g \in L_{loc}^1$, then $\forall y \in \mathbb{R}^d$:

$$g(x+y) - g(x) = \int_0^1 g(x+ty)y dt$$

for a.e. $x \in \mathbb{R}^d$.

Proof. $y \mapsto T(f_y)$ is C^∞ and $\frac{d}{dt}[T(f_{ty})] = (\nabla T)(f_{ty})y$ So we get

$$\begin{aligned} T(f_y) - T(f) &= \int_0^1 \frac{d}{dt}(T(f_{ty})) dt \\ &= \int_0^1 (\nabla T)(f_{ty})y dt \\ &= \int_0^1 \sum_{j=1}^d (\partial_j T)(f_{ty})y_j dt \end{aligned} \quad \blacksquare$$

Corrolary 3.52 Let $g \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\partial_j g \in L^1_{loc}(\mathbb{R}^d)$ for all $j = 1, 2, \dots, d$ (i.e. $g \in W^{1,1}_{loc}(\mathbb{R}^d)$). Then for all $y \in \mathbb{R}^d$:

$$\begin{aligned} g(x+y) - g(x) &= \int_0^1 y \cdot \nabla g(x+ty) dt \\ &= \int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \end{aligned}$$

for a.e. x .

Proof. For all $f \in C_c^\infty$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)[g(x+y) - g(x)] dx &= \int_{\mathbb{R}^d} g(x)[f(x-y) - f(x)] dx \\ &= g(f_y) - g(f) \\ &= \int_0^1 \sum_{j=1}^d y_j (\partial_j g)(f_{ty}) dt \\ &= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \sum_{j=1}^d y_j \left[\int_{\mathbb{R}^d} (\partial_j g)(x) f_{ty}(x) dx \right] \\ &= \int_0^1 \sum_{j=1}^d y_j \left[\int_{\mathbb{R}^d} (\partial_j g)(x+ty) f(x) dx \right] dt \\ &= \int_{\mathbb{R}^d} f(x) \left[\int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \right] dx \end{aligned}$$

For all $\phi \in C_c^\infty$: $= g(x+y) - g(x)$ a.e. $x \in \mathbb{R}^d$. ■

Remark 3.53 If $T \in D'(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open, if $y \nabla T = 0$, then $T = \text{const}$.

Theorem 3.54 (Equivalence of the classical and distributional derivatives) Let $\Omega \subseteq \mathbb{R}^d$. Then the following are equivalent:

1. $T \in D'(\Omega)$ s.t. $\partial_{x_i} T = g_i \in C(\Omega)$ for all $i = 1, \dots, d$.
2. $T = f \in C^1(\Omega)$ and $g_i = \partial_{x_i} f$

Proof.

(2) \Rightarrow (1): If $T = f \in C^1(\Omega)$, then: $\partial_{x_i} f \in C(\Omega)$.

$$\partial_{x_i} T(\phi) = -T(\partial_{x_i} \phi) = - \int_{\Omega} f(\partial_{x_i} \phi) = \int_{\Omega} (\partial_{x_i} f) \phi$$

for all $\phi \in D(\Omega)$, so $\partial_{x_i} T = \partial_{x_i} f$.

(1) \Rightarrow (2): Why is $T = f \in C^1(\Omega)$? As $\partial_{x_i} f = g_i$:

$$f(x+y) - f(x) = \int_0^1 \nabla f(x+ty) y dt = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i dt$$

So we get

$$f(y) = f(0) + \int_0^1 \sum_{i=1}^d g_i(ty) g_i dt.$$

We expect that $f \in C^1$ and $\partial_{x_i} f = g_i$. But this is not trivial to prove.

$$\begin{aligned} \frac{f(y + he_i) - f(y)}{h} &= \int_0^1 \sum_{i=1}^d [g_i(ty + the_i)(y_i + h\delta_{ij})] dt \\ &= \int_0^1 g_i(ty + the_i) dt + \int_0^1 \sum_{j \neq i} \frac{[g_i(ty + the_i) - g_i(ty)]}{h} y_j dt \\ &\xrightarrow{h \rightarrow 0} \int_0^1 g_i(ty) dt + \text{is difficult ...} \end{aligned}$$

Lets take $\phi \in C_c^\infty$, then:

$$\begin{aligned} T(\phi_y) - T(\phi) &= \int_0^1 \underbrace{\nabla T}_{(g_i)_{i=1}^d}(\phi_{ty}) y dt \\ &= \int_0^1 \sum_{i=1}^d \left(\int_{\Omega} g_i(x) \underbrace{\phi_{ty}}_{=\phi(x-ty)} dx \right) dt \\ &= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_0^1 g_i(x) \phi(x-ty) y_i dt \right) dx \\ &= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \int_0^1 g_i(x+ty) \phi(x) y_i dt \right) dx \\ &= \int_{\mathbb{R}^d} \left(\sum_i \int_0^1 g_i(x+ty) y_i dt \right) \phi(x) dx \end{aligned}$$

Integrating against $\psi(y)$ with $\psi \in C_c^\infty$:

$$\begin{aligned} &\int_{\mathbb{R}^d} T(\phi_y) \psi(y) dy - T(\phi) \int_{\mathbb{R}^d} \psi(y) dy \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \sum_i \int_0^1 g_i(x+ty) y_i \psi(y) dt dy \right) \psi(x) dx \\ &\Rightarrow T(\phi \star \psi) - T(\phi) \int \psi = \dots \\ &\Rightarrow \int_{\mathbb{R}^d} T(\psi_y) \phi(y) dy - T(\phi) \int \psi = \dots \end{aligned}$$

Take $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\int \psi = 1$. Then:

$$T(\phi) = \underbrace{\int_{\mathbb{R}^d} T(\psi_x) - \left(\int_{\mathbb{R}^d} \sum_{i=1}^d \int_0^1 g_i(x+ty) y_i \psi(y) dt dy \right)}_{f(x)} \phi(x) dx$$

for all $\phi \in C_c^\infty$, so $T = f \in C(\Omega)$. Thus $T = f \in C(\Omega)$ and $\partial_{x_i} T = g_i \in C(\Omega)$. Then we need to prove that $f \in C^1(\Omega)$ and $\partial_{x_i} f = g_i$ (classical derivative). Since

$f \in W_{loc}^{1,1}$:

$$f(x+y) - f(x) = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i dt \quad \forall x, y$$

In particular:

$$\begin{aligned} \frac{f(x+he_i) - f(x)}{h} &= \int_0^1 \frac{1}{h} \sum_{i=1}^d g_i(x+the_i) h \delta_{ij} dt \\ &= \int_0^1 g_i(x+the_i) dt \xrightarrow{h \rightarrow 0} g_i(x) \end{aligned}$$

So we get $\partial_{x_i} f(x) = g_i(x) \in C(\Omega)$ in the classical sense. So $f \in C^1(\Omega)$. ■

Definition 3.55 (Sobolev Spaces) Let $\Omega \subseteq \mathbb{R}^d$ be open. We define for $1 \leq p \leq \infty$:

$$\begin{aligned} W^{1,p}(\Omega) &= \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega) \ \forall i = 1, \dots, d\} \\ W^{k,p}(\Omega) &= \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leq k\} \\ W_{loc}^{k,p}(\Omega) &= \{f \in L_{loc}^p(\Omega) \mid D^\alpha f \in L_{loc}^p(\Omega) \ \forall |\alpha| \leq k\} \end{aligned}$$

Theorem 3.56 (Approximation of $W_{loc}^{1,p}(\Omega)$ by $C^\infty(\Omega)$) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $f \in W_{loc}^{1,p}(\Omega)$. Then there exists $\{f_n\} \subseteq C^\infty(\Omega)$ such that $f_n \rightarrow f$ in $W_{loc}^{1,p}(\Omega)$, i.e. for all $K \subseteq \Omega$ compact: $\|f_n - f\|_{L^p(K)} + \sum_{i=1}^d \|\partial_{x_i}(f_n - f)\|_{L^p(K)} \rightarrow 0$.

Proof. Case $\Omega = \mathbb{R}^d$: Take $g \in C_c^\infty$, $\int g = 1$, $g_\epsilon(x) = \epsilon^{-d} g(\epsilon^{-1}x)$. Then $g_\epsilon \star f \in C_c^\infty$. Since $f \in L_{loc}^p(\Omega)$ we have $g_\epsilon \star f \rightarrow f$ in L_{loc}^p as $\epsilon \rightarrow 0$. Moreover $\partial_{x_i}(g_\epsilon \star f) = (g_\epsilon \star \partial_{x_i} f) \xrightarrow{\epsilon \rightarrow 0} \partial_{x_i} f$ in L_{loc}^p . Then we can take $f_n = g_{\frac{1}{n}} \star f$. ■

Remark 3.57 In general, if we want to compute the distributional derivative $D^\alpha f$, then we can find $f_n \rightarrow f$ in $D'(\Omega)$ and compute $D^\alpha f_n$. Then $D^\alpha f_n \rightarrow D^\alpha f$ in $D'(\Omega)$. As an example we can compute $\nabla|f|$ with $f \in W_{loc}^{1,p}(\Omega)$.

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Theorem 3.58 (Chain Rule) Let $G \in C^1(\mathbb{R}^d)$ with $|\nabla G|$ is bounded. Let $f = (f_i)_{i=1}^d \in W_{loc}^{1,p}(\Omega)$. Then $x \mapsto G(f(x)) \in W_{loc}^{1,p}(\Omega)$ and

$$\partial_{x_i} G(f) = \sum_{k=1}^d (\partial_k G)(f) \cdot \partial_{x_i} f_k \quad \text{in } D'(\Omega)$$

Moreover, if $G(0) \in L^p(\Omega)$ (i.e. either $|\Omega| < \infty$ or $G(0) = 0$), then if $f = (f_i)_{i=1}^d \in W^{1,p}(\Omega)$, then $G(f) \in W^{1,p}(\Omega)$.

Proof. Since $G \in C^1$ we have that G is bounded in any compact set. Moreover $\|\nabla G\|_{L^\infty} < \infty$ implies:

$$|G(f) - G(0)| \leq \|\nabla G\|_{L^\infty} |f| \in L_{loc}^p$$

So $G(f) \in L^p_{loc}$. Let us compute $\partial_{x_i} G(f)$. Let $\{f^{(n)}\}_{n=1}^\infty \subseteq C^\infty$ such that $f^{(n)} \rightarrow f$ in $W^{1,p}_{loc}$, then:

$$|G(f^{(n)}) - G(f)| \leq \|\nabla G\|_{L^\infty} |f^{(n)} - f| \rightarrow 0 \text{ in } L^p_{loc}$$

So $G(f^{(n)}) \rightarrow G(f)$ in L^p_{loc} , thus $\partial_{x_i} G(f^{(n)}) \rightarrow \partial_{x_i} G(f)$ in $D'(\Omega)$. On the other hand, by the standard Chain-Rule for C^1 -functions:

$$\partial_{x_i} G(f^{(k)}) = \sum_{k=1}^d \underbrace{\partial_k G(f^{(k)})}_{(\text{b.d.} \rightarrow \partial_k G(f))} \underbrace{\partial_i f_k^{(n)}}_{(\rightarrow \partial_i f_k \text{ in } L^p(\Omega))} \rightarrow \sum_{k=1}^d \partial_k G(f) \partial_i f_k \text{ in } L^p_{loc}(\Omega)$$

Thus

$$\partial_{x_i} G(f) = \sum_{k=1}^d \underbrace{\partial_k G(f)}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p_{loc}} \in L^p_{loc} \text{ in } D'(\Omega)$$

So $G(f) \in W^{1,p}_{loc}(\Omega)$. Assume that $G(0) \in L^p(\Omega)$ (i.e. $|\Omega| < \infty$ or $G(0) = 0$). If $f \in W^{1,p}(\Omega)$, then $G(f) \in W^{1,p}(\Omega)$ since

$$|G(f) - G(0)| \leq \|\nabla G\|_{L^\infty} |f| \in L^p \Rightarrow G(f) \in L^p$$

and

$$\partial_{x_i} G(f) = \sum_k \underbrace{\partial_k G}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p} \in L^p \Rightarrow G(f) \in W^{1,p}(\Omega)$$

■

Theorem 3.59 (Derivative of absolute value) Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $f \in W^{1,p}(\Omega)$. Then $|f| \in W^{1,p}(\Omega)$ and if f is real-valued:

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Proof. Exercise. Hint: Use the Chain-Rule for $G_\epsilon(x) = \sqrt{\epsilon^2 + x^2} - \epsilon \rightarrow |x|$ as $\epsilon \rightarrow 0$ ■

3.4 Distribution vs. measures

Let μ be a Borel measure in \mathbb{R}^d s.t. $\mu(K) < \infty$ for all compact $K \subseteq \mathbb{R}^d$. Then define

$$T : D(\mathbb{R}^d) \longrightarrow \mathbb{C}$$

$$\phi \longmapsto \int_{\mathbb{R}^d} \phi(x) d\mu(x) \quad \forall \phi \in C_c^\infty$$

\leadsto T is a distribution since if $\phi_n \rightarrow \phi$ in $D(\Omega)$, then

$$|T(\phi_n) - T(\phi)| \leq \int_{\mathbb{R}^d} |\phi_n - \phi| d\mu(x) \leq \|\phi_n - \phi\|_{L^\infty} \left(\int_K d\mu \right) \xrightarrow{n \rightarrow \infty} 0$$

Example 3.60 ∂_0 in $D'(\mathbb{R}^d)$ is a Borel probability measure.

Theorem 3.61 (Positive distributions are measures) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $T \in D'(\Omega)$. Assume $T \geq 0$, i.e. $T(\phi) \geq 0$ for all $\phi \in D(\Omega)$ satisfying $\phi(x) \geq 0$ for all x . Then there is a Borel positive measure μ on Ω such that $\mu(K) < \infty$ for all $K \subseteq \Omega$ compact and:

$$T(\phi) = \int_{\Omega} \phi(x) d\mu(x) \quad \forall \phi \in D(\Omega)$$

Proof. See Lieb-Loss Analysis. Sketch: If $O \subseteq \mathbb{R}^d$ is open, then

$$\mu(O) = \sup\{T(\phi) \mid \phi \in D(\Omega), 0 \leq \phi \leq 1, \text{supp } \phi \subseteq O\}$$

For all $A \subseteq \Omega$ (not necessarily open),

$$\mu(A) = \inf\{\mu(O) \mid O \text{ open}, A \subseteq O\}$$

The mapping $\mu : 2^{\Omega} \rightarrow [0, \infty]$ is an outer measure, i.e.

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leq \mu(B)$ if $A \subseteq B$
3. $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

From the outer measure we can find a σ -algebra Σ and μ is a measure on Ω s.t. E is measurable iff

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c).$$

So all open sets are measurable, thus outer regularity (by def $\mu(A) = \inf\{\mu(O) \mid O \text{ open } \supseteq A\}$), so inner regularity $\mu(A) = \sup\{\mu(K) \mid K \text{ compact } \subseteq A\}$. ■

Exercise 3.62 (E 4.1) Prove that if $T_n \rightarrow T$ in $D'(\mathbb{R}^d)$, then $D^{\alpha}T_n \rightarrow D^{\alpha}T$ in $D'(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$.

My Solution. See Goodnotes. ■

Exercise 3.63 (E 4.2)

My Solution. See Goodnotes. ■

Exercise 3.64 (E 4.3) $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ $f_{\epsilon}(x) = \epsilon^{-d}f(\epsilon^{-1}x)$. Then $f_{\epsilon} \rightarrow \delta_0$ in $D'(\mathbb{R}^d)$.

My Solution. See Goodnotes. ■

Exercise 3.65 (E 4.4) Let $\{f_n\} \subseteq L^1$, $\text{supp } f \subseteq B(0, 1)$, $f_n \rightarrow f$ in L^1 . Prove for all $g \in C_c^{\infty}$ that $f_n \star g \rightarrow f \star g$ in $D(\mathbb{R}^d)$.

Solution. Since $f_n \in L^1$, $\text{supp } f \subseteq B(0, 1)$ and $g \in C_c^{\infty}$ we have $f_n \star g \in C_c^{\infty}$ and

$$\text{supp}(f_n \star g) \subseteq (\text{supp } g) + \overline{B(0, 1)} = K.$$

Since $f_n \rightarrow f$ in L^1 there is a subsequence $f_{n_k} \rightarrow f$ almost everywhere, so f supp in $\overline{B(0,1)}$, so $f \star g \in C_c^\infty$, $\text{supp}(f \star g) \subseteq K$. We have:

$$\begin{aligned} |f_n \star g(x) - f \star g(x)| &= \left| \int_{\mathbb{R}^d} (f_n(y) - f(y))g(x-y) dy \right| \\ &\leq \int_{\mathbb{R}^d} |f_n(y) - f(y)| |g(x-y)| dy \\ &\leq \|g\|_{L^\infty} \|f_n - f\|_{L^1} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

thus $\|f_n \star g - f \star g\|_{L^\infty} \rightarrow 0$. Similary:

$$\|D^\alpha(f_n \star g) - D^\alpha(f \star g)\|_{L^\infty} = \|f_n \star \underbrace{(D^\alpha g)}_{\in C_c^\infty} - f \star (D^\alpha g)\|_{L^\infty} \xrightarrow{n \rightarrow \infty} 0$$

for all $\alpha \in \mathbb{N}^d$, so $f_n \star g \rightarrow f \star g$ in $D(\mathbb{R}^d)$. ■

Exercise 3.66 (E 4.5) Compute distributional derivatives f', f'' of $f(x) = x|x-1|$.

Solution. We prove $f'(x) = g(x) := \begin{cases} 2x-1 & x > 1 \\ 1-2x & x < 1 \end{cases}$. Take $\phi \in C_c^\infty(\mathbb{R}^d)$.

$$\begin{aligned} -f'(\phi) &= - \int_{\mathbb{R}^d} f \phi' dy \\ &= - \int_{-\infty}^1 f \phi' dy - \int_1^\infty f \phi' dy \\ &= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 f' \phi dy + [f\phi]_1^\infty - \int_1^\infty f' \phi dy \\ &= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 g \phi dy + [f\phi]_1^\infty - \int_1^\infty g \phi dy \\ &= f(1-)\phi(1) - f(1+)\phi(1) - \int_{\mathbb{R}^d} g \phi dy \\ &= 0 - \int_{\mathbb{R}^d} g \phi dy \end{aligned}$$

Now we compute $f'' = g'$. Take $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned}
-(g')(\phi) &= \int_{\mathbb{R}^d} g\phi' dy \\
&= \int_{-\infty}^1 g\phi' dy + \int_1^\infty g\phi' dy \\
&= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 g'\phi dy - \int_1^\infty g'\phi dy \\
&= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 (-2)\phi dy - \int_1^\infty 2\phi dy \\
&= -2\phi(1) + \int_{-\infty}^\infty [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x) dx \\
&= -2\delta_1(\phi) + \int_{-\infty}^\infty [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x) dx \\
\Rightarrow g' &= \underbrace{2\delta_1}_{\notin L_{loc}^1} - \underbrace{2\mathbb{1}_{(-\infty,1)} + 2\mathbb{1}_{(1,\infty)}}_{\in L_{loc}^1}
\end{aligned}$$

■

Chapter 4

Weak Solutions and Regularity

Definition 4.1 Consider the linear PDE:

$$\sum_{\alpha} c_{\alpha} D^{\alpha} u(x) = F(x), \quad c_{\alpha} \text{ constant, } F \text{ given}$$

A function u is called a weak solution (a distributional solution) if

$$\sum_{\alpha} c_{\alpha} D^{\alpha} u = F \quad \text{in } D'(\Omega).$$

Namely,

$$\sum_{\alpha} (-1)^{|\alpha|} c_{\alpha} \int_{\Omega} u D^{\alpha} \phi = \int_{\Omega} F \phi, \quad \forall \phi \in D(\Omega)$$

Regularity: Given some condition on the data F , what can we say about the smoothness of u ? Can we say that the equation holds in the classical sense? We derived G (the solution of the Laplace Equation) before in two ways:

1. $\Delta G(x) = 0$ for all $x \neq 0$, assuming $G(x) = G(|x|)$ and $d \geq 2$
2. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$ for $d \geq 3$

Theorem 4.2 For all $d \geq 1$ we have $G \in L^1_{loc}(\mathbb{R}^d)$ and $-\Delta G = \delta_0$ in $D'(\mathbb{R}^d)$.

Proof. Take $\phi \in D(\mathbb{R}^d)$. Then:

$$\begin{aligned} (-\Delta G_y)(\phi) &= G_y(-\Delta \phi) = \int_{\mathbb{R}^d} G_y(x)(-\Delta \phi)(x) dx \\ &= \int_{\mathbb{R}^d} G(y-x)(-\Delta \phi)(x) dx \\ &= [G \star (-\Delta \phi)](y) = (-\Delta)(G \star \phi)(y) \end{aligned}$$

Recall for all $f \in C^2$, $-\Delta(G \star f) = f$ pointwise. So we can conclude $-\Delta G_y = \delta_y$ in $D'(\mathbb{R}^d)$. ■

Remark 4.3 In $d = 1$, $G(x) = -\frac{1}{2}|x|$, so $-G'(x) = \text{sgn}(x)/2$, so $-G''(x) = \delta_0$.

Remark 4.4 Formally:

$$-\Delta(G_y \star \phi) = (-\Delta G_y) \star \phi(x) = (\delta_0 \star \phi)(x) = \int \delta_0(y) \phi(x-y) dy = \delta_0(\phi(x-\bullet))$$

Theorem 4.5 (Poisson's equation with L^1_{loc} data) Let $f \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\omega_d f \in L^1(\mathbb{R}^d)$ where

$$\omega_d(x) = \begin{cases} 1 + |x| & d = 1 \\ \log(1 + |x|) & d = 2 \\ \frac{1}{1+|x|^{d-2}} & d \geq 3, \end{cases}$$

then $u(x) = (G \star f)(x) \in L^1_{loc}(\mathbb{R}^d)$. Moreover $-\Delta u = f$ in $D'(\mathbb{R}^d)$. In fact, $u \in W^{1,1}_{loc}(\mathbb{R}^d)$ and:

$$\partial_{x_i} u(x) = (\partial_{x_i} G) \star f(x) = \int_{\mathbb{R}^d} (\partial_{x_i} G)(x-y) f(y) dy$$

Remark 4.6 We can also replace \mathbb{R}^d by Ω and get $-\Delta u = f$ in $D'(\Omega)$.

Proof of Theorem 4.5. First we check that $u \in L^1_{loc}$. Take any Ball $B(0, R) \subseteq \mathbb{R}^d$, prove $\int_B |u| dy < \infty$. We have

$$\begin{aligned} \int_B |u| dy &= \int_B \left| \int_{\mathbb{R}^d} G(x-y) f(y) dy \right| dx \\ &\leq \int_B \int_{\mathbb{R}^d} |G(x-y)| |f(y)| dy dx \\ &= \int_{\mathbb{R}^d} \left(\int_B |G(x-y)| dx \right) |f(y)| dy \end{aligned}$$

If $y \notin B = B(0, R)$, then by Newtons's theorem (Mean-value theorem):

$$\int_{B(0,R)} |G(x-y)| dx = |B(0,r)| |G(y)| \leq C |B| \omega_d(y)$$

If $y \in B$, then $|y| \leq R$, so $|x-y| \leq 2R$ if $x \in B$.

$$\int_{B(0,R)} |G(x-y)| dx \leq \int_{|x-y| \leq 2R} |G(x-y)| dx = \int_{|z| \leq 2R} |G(z)| dz \leq c_R$$

as $G \in L^1_{loc}$. Thus

$$\int_B |u| dy \leq c_B \int_{|y| \geq R} \omega_d(y) |f(y)| dy + c_B \int_{|y| \leq R} |f(y)| dy < \infty$$

Let us prove $-\Delta u = f$ in $D'(\mathbb{R}^d)$. Take $\phi \in D(\mathbb{R}^d)$. Then:

$$\begin{aligned}
(-\Delta u)(\phi) &= u(-\Delta \phi) \\
&= \int_{\mathbb{R}^d} u(x)(-\Delta \phi)(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)f(y)(-\Delta \phi)(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y-x)f(y)(-\Delta \phi)(x) dx dy \\
&= \int_{\mathbb{R}^d} [G \star (-\Delta \phi)](y)f(y) dy \\
&= \int_{\mathbb{R}^d} -\Delta(G \star \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} \phi(y)f(y) dy
\end{aligned}$$

So $-\Delta u = f$ in $D'(\mathbb{R}^d)$. We check that $\partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$. Note that

$$|\partial_i G(x)| \leq c \frac{1}{|x|^{d-1}} \in L^1_{loc}(\mathbb{R}^d)$$

and

$$\int_{B(0,R)} |\partial_i G(x-y)| dx \leq \begin{cases} C_r \omega_d(y) & |y| \geq R \\ C_r & |y| \leq R \end{cases}$$

So $\int_{B(0,R)} |(\partial_i G \star f)(y)| dy < \infty$ for all $R > 0$. For all $\phi \in D(\mathbb{R}^d)$:

$$\begin{aligned}
-(\partial_i u)(\phi) &= u(\partial_i \phi) = \int_{\mathbb{R}^d} u(x) \partial_i \phi(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)f(y) \partial_i \phi(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y-x)f(y) \partial_i \phi(x) dx dy \\
&= \int_{\mathbb{R}^d} (G \star \partial_i^y \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} (\partial_i^y G \star \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i^y G(y-x)f(y)\phi(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} -(\partial_i G)(x-y)f(y)\phi(x) dx dy \\
&= - \int_{\mathbb{R}^d} (\partial_i G \star f)(x)\phi(x) dx
\end{aligned}$$

So $\partial_i u = \partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$. Thus $u \in L^1_{loc}$, $\partial_i u \in L^1_{loc}$ for all i . So $u \in W^{1,1}_{loc}(\mathbb{R}^d)$. ■

Regularity: We consider the Laplace Equation $\Delta u = 0$ in \mathbb{R}^d .

Lemma 4.7 (Weyl) If $\Omega \subseteq \mathbb{R}^d$ open and $T \in D'(\Omega)$ s.t. $\Delta T = 0$ in $D'(\Omega)$, then: $T = f \in C^\infty(\Omega)$ and f is a harmonic function.

Proof. ($\Omega = \mathbb{R}^d$). Take $\phi \in C_c^\infty$, then $y \mapsto T(\phi_y) = T(\phi(-y))$ is C^∞ and $\Delta_y T(\phi_y) = T((\Delta\phi)_y) = (\Delta T)(\phi_y) = 0$. Take $g \in C_c^\infty$, g is radial. Then:

$$\int_{\mathbb{R}^d} T(\phi_y)g(y) dy \stackrel{(\text{exercise})}{=} \int_{\mathbb{R}^d} T(\phi)g(y) dy = T(\phi) \left(\int_{\mathbb{R}^d} g dy \right)$$

Exercise 4.8 Let $f \in C^\infty(\mathbb{R}^d)$ be a harmonic function and $g \in C_c^\infty$, g is radial. Then:

$$\int_{\mathbb{R}^d} f(x)g(x) dx = f(0) \left(\int_{\mathbb{R}^d} g(x) dx \right)$$

On the other hand:

$$\int_{\mathbb{R}^d} T(\phi_y)g(y) dy = T(\phi \star g) = T(g \star \phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y) dy$$

Take $\int_{\mathbb{R}^d} g dy = 1$, then:

$$T(\phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y) dy$$

For all $\phi \in C_c^\infty$. Then $T = T(g_y) \in C^\infty$ ■

Now let's regard the Poisson Equation $-\Delta u = f$ in $D'(\mathbb{R}^d)$.

Remark 4.9 Any solution has the form $u = G \star g + h$ where $\Delta h = 0$ in $D'(\mathbb{R}^d)$. By Weyl's Lemma (4.7), $h \in C^\infty$, then we only need to consider the regularity of $G \star f$.

Remark 4.10 The regularity is a *local question*, namely if we write

$$f = f_1 + f_2 = f\phi + f(1 - \phi),$$

where $\phi = 1$ in a ball B and $\phi \in C_c^\infty$.

Then $G \star f = G \star f_1 + G \star f_2$. Here $f_2 = f(1 - \phi) = 0$ in B . With Weyl's Lemma (4.7), $G \star f_2 \in C^\infty$.

Theorem 4.11 (Low Regularity of Poisson Equation) Let $f \in L^p(\mathbb{R}^d)$ and compactly supported. Then

a) If $p \geq 1$, then

- $G \star f \in C^1(\mathbb{R}^d)$ if $d = 1$.
- $G \star f \in L_{loc}^q(\mathbb{R}^d)$ for any $q < \infty$ if $d = 2$.
- $G \star f \in L_{loc}^q(\mathbb{R}^d)$ for $q < \frac{d}{d-2}$ if $d \geq 3$.

b) If $\frac{d}{2} < p \leq d$, then $G \star f \in C_{loc}^{0,\alpha}(\mathbb{R}^d)$ for all $0 < \alpha < 2 - \frac{d}{p}$, i.e.

$$|(G \star f)(x) - (G \star f)(y)| \leq C_k |x - y|^\alpha \quad \forall x, y \in K$$

with K compact in \mathbb{R}^d .

c) If $p > d$, then $G \star f \in C_{loc}^{1,\alpha}(\mathbb{R}^d)$ for all $0 < \alpha < 1 - \frac{d}{p}$.

where G is den fundamental solution of the laplace equation.

Example 4.12 Let $r = |x|$

$$u(x) = \omega(r) = \log(|\log(r)|)$$

if $0 < r < \frac{1}{2}$, so u is well-defined in $B = B(0, \frac{1}{2})$. We conclude:

$$-\Delta_{\mathbb{R}^3} u(x) = -\omega''(r) - \frac{2\omega'(r)}{r} = f(x) \in L^{\frac{3}{2}}(B)$$

But the Theorem (b) tells us that if $f \in L^{\frac{3}{2}}$ then u is continuous but $u \notin C(B)$.

Proof of theorem 4.11. a) ($p = 1$) Why is $G \star f \in L_{loc}^q$? Recall from the proof of Youngs inequality:

$$\begin{aligned} |(G \star f)(x)| &= \left| \int_{\mathbb{R}^d} G(x-y) f(y) dy \right| \\ (\text{H\"older}) &= \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(y)| dy \right)^{\frac{1}{q'}} \end{aligned}$$

Where $\frac{1}{q} + \frac{1}{q'} = 1$. Then:

$$|(G \star f)(x)|^q \leq C \int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy$$

For any Ball $B = B(0, R) \subseteq \mathbb{R}^d$:

$$\begin{aligned} \int_B |G \star f(x)|^q dx &\leq C \int_B \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy \right) dx \\ &= C \int_{\mathbb{R}^d} \left(\int_B |G(x-y)|^q dx \right) |f(y)| dy \end{aligned}$$

$G(x) \sim \frac{1}{|x|^{\frac{1}{d-2}}} \rightsquigarrow |G|^q = \frac{1}{|x|^{\frac{1}{(d-2)q}}} \in L_{loc}^1(\mathbb{R}^d)$ if $(d-2)q < 2 \Leftrightarrow q < \frac{d}{d-2}$. Here, $y \in \text{supp } f$, so $|y| \leq R_1$, then $|x-y| \leq R+R_1$ if $|x| \leq R$. With $y \in \text{supp } f$, this implies:

$$\int_{B(0,R)} |G(x-y)|^q dx \leq \int_{|z| \leq R+R_1} |G(z)|^q dz < \infty$$

b)

$$(G \star f)(x) - (G \star f)(y) = \int_{\mathbb{R}^d} (G(x-z) - G(y-z)) f(z) dz$$

So

$$|G \star f(x) - (G \star f)(y)| \leq C \int_{\mathbb{R}^d} \left| \frac{1}{|x-z|^{d-2}} - \frac{1}{|y-z|^{d-2}} \right| |f(z)| dz$$

for all $x, y \in \mathbb{R}^d$:

$$\begin{aligned} \left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| &= \left| \left(\frac{1}{|x|} - \frac{1}{|y|} \right) \left(\frac{1}{|x|^{d-3}} + \dots + \frac{1}{|y|^{d-3}} \right) \right| \\ &\leq C \frac{|x| - |y|}{|x||y|} \max \left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &= C \frac{|x-y|}{|x||y|} \max \left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &\leq C \max(|x|, |y|)^{1-\alpha} \frac{|x-y|^\alpha}{|x||y|} \max \left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \end{aligned}$$

as

$$||x| - |y|| \leq \min(|x - y|, \max(|x|, |y|)) \leq |x - y|^\alpha \max(|x|, |y|)^{1-\alpha}$$

Thus, for all $x, y \in \mathbb{R}^d$:

$$\begin{aligned} \left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| &\leq C|x - y|^\alpha \frac{\max(|x|, |y|)^{1-\alpha}}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}}\right) \\ &\leq C|x - y|^\alpha \max\left(\frac{1}{|x|^{d-2+\alpha}}, \frac{1}{|y|^{d-2+\alpha}}\right) \end{aligned}$$

So we get

$$\left| \frac{1}{|x - y|^{d-2}} - \frac{1}{|y - z|^{d-2}} \right| \leq C|x - y|^\alpha \max\left(\frac{1}{|x - z|^{d-2+\alpha}}, \frac{1}{|y - z|^{d-2+\alpha}}\right)$$

Therefore:

$$\begin{aligned} &|G \star f(x) - G \star f(y)| \\ &\leq C \int_{\mathbb{R}^d} |x - y|^\alpha \max\left(\frac{1}{|x - z|^{d-2+\alpha}}, \frac{1}{|y - z|^{d-2+\alpha}}\right) |f(z)| dz \\ &\leq C|x - y|^\alpha \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \right) \end{aligned}$$

Claim: If $f \in L^p(\mathbb{R}^d)$ is compactly supported, $d \geq p > \frac{d}{2}$, then:

$$\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz < \infty$$

for all $0 < \alpha < 2 - \frac{d}{p}$. Assume $\text{supp } f \subseteq \overline{B(0, R_1)}$. Consider 2 cases:

- If $|\xi| > 2R_1$, then: $|\xi - z| \geq R_1$ for all $z \in B(0, R_1)$. Hence:

$$\int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \leq \frac{1}{R_1^{d-2+\alpha}} \|f\|_{L^1} < \infty$$

- If $|\xi| \leq 2R_1$, then: $|\xi - z| \leq 3R_1$ for all $z \in B(0, R_1)$:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz &\leq \int_{|\xi - z| \leq 3R_1} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \\ \text{(H\"older)}, \left(\frac{1}{p} + \frac{1}{q} = 1\right) &\leq \left(\int_{\mathbb{R}^d} |f(z)|^p dz \right)^{\frac{1}{p}} \\ &\quad \cdot \left(\int_{|\xi - z| \leq 3R_1} \frac{1}{|\xi - z|^{(d-2+\alpha)q}} dz \right)^{\frac{1}{q}} \\ &= \|f\|_{L^p} \left(\int_{|z| \leq 3R_1} \frac{1}{|z|^{(d-2+\alpha)q}} dz \right)^{\frac{1}{q}} < \infty \end{aligned}$$

c) ($d \geq 3$) We already know:

$$\partial_i(G \star f) = (\partial_i G \star f) \in L_{loc}^1(\mathbb{R}^d)$$

as $\omega_d f \in L^1(\mathbb{R}^d)$. We claim that $\partial_i G \star f \in C^{0,\alpha}(\mathbb{R}^d)$. So $G \star f \in C^{1,\alpha}(\mathbb{R}^d)$ by the equivalence between the classical and the distributional derivatives. Exercise. Hint:

$$|\partial_i G \star f(x) - \partial_i G \star f(y)| \leq \int_{\mathbb{R}^d} |\partial_i G(x-z) - \partial_i G(y-z)| |f(z)| dz,$$

$$\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d} \rightsquigarrow \text{Need to estimate } |\partial_i G(x) - \partial_i G(y)| \leq C|x-y|^\alpha. \quad \blacksquare$$

Theorem 4.13 (High regularity for Poisson's equation) Let $f \in C^{0,\alpha}(\mathbb{R}^d)$, $0 < \alpha < 1$ be compactly supported. Then $G \star f \in C^{2,\alpha}(\mathbb{R}^d)$.

Remark 4.14 $(-\Delta u = f)$ and $f \in C(\mathbb{R}^d)$ does not imply that $u \in C^2(\mathbb{R}^d)$. (exercise)

Remark 4.15 If $f \in C^{k,\alpha}(\mathbb{R}^d)$, $k \in \{0, 1, \dots\}$, $0 < \alpha < 1$ is compactly supported, then $G \star f \in C^{k+2,\alpha}(\mathbb{R}^d)$. This more general statement is a consequence of the theorem since

$$D^\beta(G \star f) = G \star \underbrace{(D^\beta f)}_{\in C^{0,\alpha}}$$

for all $\beta = (\beta_1, \dots, \beta_d)$, $|\beta| \leq k$.

Proof of theorem 4.13. Since $f \in L^p$ for all $p \leq \infty$ by the low regularity (4.11) we have $G \star f \in C^{1,\alpha}$ and $\partial_i(G \star f) = \partial_i G \star f$ in the classical sense. We will compute the distributional derivatives $\partial_i \partial_j(G \star f)$ and prove that they are Hölder continuous. Compute $\partial_j \partial_i(G \star f)$: For all $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} -(\partial_j \partial_i G \star f)(\phi) &= (\underbrace{\partial_i(G \star f)}_{\in C})(\partial_j \phi) \\ &= \int_{\mathbb{R}^d} ((\partial_i G) \star f)(x) \partial_j \phi(x) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i G(x-y) f(y) \partial_j \phi(x) dx dy \\ &= \int_{\mathbb{R}^d} f(y) \left[\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx \right] dy \\ &\stackrel{?}{=} \int_{\mathbb{R}^d} \square \phi(y) dy \end{aligned}$$

Recall: $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$, $\partial_i \partial_j G(x) = \frac{1}{|B_1|} \left[\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{d} \right] \frac{1}{|x|^d}$. We have:

$$\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_{|x-y| \geq \epsilon} \partial_i G(x-y) \partial_j \phi(x) dx$$

By dominated convergence we have $|\partial_i G(x-y) \partial_j \phi(x)| \in L^1(dx)$. By the Gauss-Green-Theorem (2.2) for all $\epsilon > 0$:

$$\begin{aligned} &\int_{|x-y| \geq \epsilon} \partial_i G(x-y) \partial_j \phi(x) dx \\ &= \int_{\partial B(y,\epsilon)} \partial_i G(x-y) \phi(x) \omega_j dS(x) - \int_{|x-y| \geq \epsilon} \partial_j \partial_i G(x-y) \phi(x) dx \end{aligned}$$

Where $\omega = \frac{x-y}{|x-y|}$. For the boundary term:

$$\begin{aligned}
- \int_{\partial B(y, \epsilon)} \partial_i G(x-y) \phi(x) \omega_j dS(x) &= -\epsilon^{d-1} \int_{\partial B(0,1)} \partial_i G(\epsilon \omega) \phi(y + \epsilon \omega) \omega_j d\omega \\
(\star) \quad &= \int_{\partial B(0,1)} \frac{1}{d|B_1|} \omega_i \omega_j \phi(y + \epsilon \omega) d\omega \\
&\xrightarrow{\epsilon \rightarrow 0} \int_{\partial B(0,1)} \frac{1}{d|B_1|} \omega_i \omega_j \phi(y) d\omega \\
&= \frac{1}{d} \delta_{i,j} \phi(y)
\end{aligned}$$

(\star) $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$, so $\partial_i G(\epsilon \omega) = -\frac{-\omega_i}{d|B_1|} \frac{1}{\epsilon^{d-1}}$. for all $|\omega| = 1$.

Now we split:

$$\begin{aligned}
&- \int_{|x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx \\
&= - \int_{|x-y| \geq 1} \partial_i \partial_j G(x-y) \phi(x) dx - \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx
\end{aligned}$$

The key observation is: $\int_{\partial B(0,r)} \partial_i \partial_j G(x) dx = 0$ since

$$\partial_i \partial_j G(x) = \frac{1}{|B_1|} \left(\omega_i \omega_j - \frac{\partial_{ij}}{d} \right) \frac{1}{|x|^d},$$

$\omega = \frac{x}{|x|}$. For example if $i = 1, j = 2, r = 1$:

$$\int_{\partial B(0,1)} \partial_1 \partial_2 G(x) dS(x) = \frac{1}{|B_1|} \int_{\partial B(0,1)} \omega_1 \omega_2 d\omega,$$

$\partial B(0,1) = \{\omega \mid |\omega| = 1\}$. Consider: $\omega \mapsto R\omega, (\omega_1, \dots, \omega_d) \mapsto (-\omega_1, \omega_2, \dots, \omega_d)$. Then

$$- \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(y) dx = 0.$$

So

$$- \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx = - \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) (\phi(x) - \phi(y)) dx$$

In summary:

$$\begin{aligned}
\partial_i \partial_j (G \star f)(\phi) &= \int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx \right) dy \\
&= \int_{\mathbb{R}^d} f(y) \frac{1}{d} \partial_{ij} \phi(y) dy \\
&\quad - \int_{\mathbb{R}^d} f(y) \left(\int_{|x-y| > 1} \partial_i \partial_j G(x-y) \phi(x) dx \right) \\
&\quad - \int_{\mathbb{R}^d} \left[\lim_{\epsilon \rightarrow 0} \int_{1 \geq |x-y| \geq \epsilon} \underbrace{\partial_i \partial_j G(x-y) (\phi(x) - \phi(y)) dx}_{\leq \frac{C}{|x-y|^d} |x-y| \|\nabla \phi\|_{L^\infty} \leq \frac{C}{|x-y|^{d-1}} \in L^1_{loc}(dx) \forall y} \right] dy \\
&= \int_{\mathbb{R}^d} \frac{\delta_{ij}}{d} f(x) \phi(x) dx - \int_{\mathbb{R}^d} \phi(x) \left(\int_{|x-y| > 1} \partial_i \partial_j G(x-y) f(y) dy \right) dx \\
&\quad - \int_{\mathbb{R}^d} \phi(x) \left[\int_{|x-y| \leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy \right] dx
\end{aligned}$$

Conclusion:

$$\begin{aligned}\partial_i \partial_j (G \star f)(x) &= -\frac{\delta_{ij}}{d} f(x) + \int_{|x-y|>1} \partial_i \partial_j G(x-y) f(y) dy \\ &\quad + \int_{|x-y| \leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy\end{aligned}$$

The first term $f \in C^{0,\alpha}$. The second term is also at least $C^{0,\alpha}$ since $\partial_i \partial_j G(x)$ is smooth as $|x| > 1$. We need to prove that the third term

$$W_{ij}(x) = \int_{|x-y| \leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy$$

is Hölder-continuous, $|W_{ij}(x) - W_{ij}(y)| \leq C|x-y|^\alpha$. Recall:

$$|\partial_i \partial_j G(x-y) (f(y) - f(x))| \leq C \frac{1}{|x-y|^d} |x-y|^\alpha = \frac{C}{|x-y|^{d-\alpha}} \in L^1_{loc}(dy)$$

We write

$$\begin{aligned}W_{ij}(x) &= \int_{|x-y| \leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy \\ &= \int_{|z| \leq 1} \partial_i \partial_j G(z) (f(x+z) - f(x)) dz\end{aligned}$$

So we get:

$$W_{ij} - W_{ij}(y) = \int_{|z| \leq 1} \partial_i \partial_j G(z) (f(x+z) - f(y+z) - f(x) + f(y)) dz$$

Easy thought: Use $|\partial_i \partial_j G(z)| \leq \frac{C}{|z|^d}$ and

$$\begin{aligned}&|f(x+z) - f(y+z) - f(x) + f(y)| \\ &\leq \begin{cases} |f(x+z) - f(x)| + |f(y+z) - f(y)| \leq C|z|^\alpha \\ |f(x+z) - f(y+z)| + |f(x) - f(y)| \leq C|x-y|^\alpha \end{cases}\end{aligned}$$

Thus:

$$\begin{aligned}|W_{ij}(x) - W_{ij}(y)| &\leq C \int_{|z| \leq 1} \frac{1}{|z|^d} \min(|z|^\alpha, |x-y|^\alpha) dz \\ &\leq C \int_{|z| \leq 1} \frac{1}{|z|^d} (|z|^\alpha)^\epsilon (|x-y|^\alpha)^{1-\epsilon}, \quad 0 < \epsilon < 1 \\ &\leq C \left(\int_{|z| \leq 1} \frac{1}{|z|^{d-\alpha\epsilon}} \right) |x-y|^{\alpha(1-\epsilon)} \\ &\leq C_\epsilon |x-y|^{\alpha(1-\epsilon)}\end{aligned}$$

thus it is easy to prove $|W_{ij}(x) - W_{ij}(y)| \leq C_\alpha |x-y|^\alpha$ for all $\alpha' \leq \alpha$. However, to get $\alpha' = \alpha$ we need a more precise estimate. We split:

$$W_{ij}(x) - W_{ij}(y) = \int_{|z| \leq 1} \dots = \int_{|z| \leq \min(4|x-y|, 1)} + \int_{4|x-y| < |z| \leq 1}$$

For the first domain:

$$\begin{aligned} & \int_{|z| \leq 4|x-y|} |\partial_{ij} G(z)| |f(x+z) - f(y+z) - f(y) + f(x)| dz \\ & \leq C \int_{|z| \leq 4|x-y|} \frac{1}{|z|^d} |z|^\alpha dz = \text{const} \cdot |x-y|^\alpha \end{aligned}$$

For the second domain:

$$\begin{aligned} & \int_{4|x-y| < |z| \leq 1} \partial_{ij} G(z) (f(x+z) - f(y+z) + f(y)f(x)) dz \\ & = \int_{4|x-y| < |z| \leq 1} \partial_{ij} G(z) (f(x+z) - f(y+z)) dz = (\dots) \end{aligned}$$

since $\int_{4|x-y| < |z| \leq 1} \partial_{ij} G(z) dz = 0$. Then

$$(\dots) = \int_{4|x-y| < |z-x| \leq 1} \partial_{ij} G(z-x) f(z) dz - \int_{4|x-y| < |z-y| \leq 1} \partial_{ij} G(z-y) f(z) dz.$$

Denote $A = \{z \mid 4|x-y| < |z-x| \leq 1\}$, $B = \{z \mid 4|x-y| < |z-y| \leq 1\}$. Consider

$$\begin{aligned} & \int_A \partial_{ij} G(z-x) f(z) dz - \int_B \partial_{ij} G(z-y) f(z) dz \\ & = \int_{A \setminus B} + \int_{B \setminus A} + \int_{A \cap B} (\partial_{ij} G(z-x) - \partial_{ij} G(z-y)) f(z) dz \end{aligned}$$

Lets regard the intersection. We have

$$\begin{aligned} \partial_{ij} G(x) &= \frac{1}{|B_1|} \frac{1}{|x|^d} (\omega_i \omega_j - \frac{1}{d} \delta_{ij}) \\ |\partial_{ij} G(x) - \partial_{ij} G(y)| &\leq C|x-y| \left(\frac{1}{|x|^{d+1}} + \frac{1}{|y|^{d+1}} \right) \end{aligned}$$

Now,

$$|\partial_{ij} G(z-x) - \partial_{ij} G(z-y)| \leq C|x-y| \left(\frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right)$$

So we have

$$\begin{aligned} & \left| \int_{A \cap B} (\partial_{ij} G(z-x) - \partial_{ij} G(z-y)) f(z) dz \right| \\ & \leq C \int_{A \cap B} |x-y| \left(\frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right) |f(z)| dz = (\dots) \end{aligned}$$

Now we replace $f(z)$ by $f(z) - f(x)$, then:

$$\begin{aligned} & \left| \int_{A \cap B} (\partial_{ij} G(z-x) - \partial_{ij} G(z-y)) (f(z) - f(x)) dz \right| \\ & \leq C \int_{A \cap B} |x-y| \left(\frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right) |z-x|^\alpha dz \\ & = C \underbrace{\int_{A \cap B} |x-y| \frac{1}{|z-x|^{d+1-\alpha}} dz}_{(I)} + C \underbrace{\int_{A \cap B} |x-y| \frac{1}{|z-y|^{d+1}} |z-x|^\alpha dz}_{(II)} \end{aligned}$$

Now,

$$\begin{aligned}
(I) &\leq C|x-y| \int_{4|x-y| < |z-x| \leq 1} \frac{1}{|z-x|^{d+1-\alpha}} dz \\
&= C|x-y| \int_{4|x-y| < |z| \leq 1} \frac{1}{|z|^{d+1-\alpha}} dz \\
&\leq C|x-y| \int_{4|x-y|}^1 \frac{1}{r^{d+1-\alpha}} r^{d-1} dr \\
&= C|x-y| \int_{4|x-y|}^1 \frac{1}{r^{2-\alpha}} dr \\
&\leq C|x-y| \left[-1 + \frac{1}{(4|x-y|)^{1-\alpha}} \right] \\
&\leq C|x-y|^\alpha
\end{aligned}$$

$$\begin{aligned}
(II) &\leq C|x-y| \int_{A \cap B} \frac{1}{|z-y|^{d+1}} |z-x|^\alpha dz \\
&\leq C|x-y| \int_{A \cap B} \frac{1}{|z-y|^{d+1}} (|z-y|^\alpha + |x-y|^\alpha) dz \\
&\leq C|x-y| \underbrace{\int_B \frac{1}{|z-y|^{d+1-\alpha}} dz}_{\text{similar to (I)}} + C|x-y|^{1+\alpha} \int_B \frac{1}{|z-y|^{d+1}} dz
\end{aligned}$$

and

$$C|x-y|^{1+\alpha} \int_B \frac{1}{|z-y|^{d+1}} dz \leq \int_{4|x-y|}^1 \frac{1}{r^{d+1}} r^{d-1} dr \leq \frac{C}{|x-y|}$$

Consider $A \setminus B$:

$$\left| \int_{A \setminus B} \right| \leq C \|f\|_{L^\infty} \int_{A \setminus B} \frac{1}{|z-x|^d} dz$$

where

$$\begin{aligned}
A &= \{z \mid 4|x-y| < |z-x| \leq 1\} \\
B &= \{z \mid 4|x-y| < |z-y| \leq 1\} \\
A \setminus B &= \{z \in A \mid |z-y| \leq 4|x-y|\} \cup \{z \in A \mid |z-y| > 1\} = E_1 \cup E_2
\end{aligned}$$

for

$$\begin{aligned}
E_1 &= \{z \mid |z-y| \leq 4|x-y| < |z-x| \leq 1\} \\
&\subseteq \{z \mid 4|x-y| \leq |x-z| \leq 5|x-y|\}.
\end{aligned}$$

$|x - z| \leq |x - y| + |y - z| \leq 5|x - y|$ in E_1 . We have

$$\begin{aligned}
\int_{E_1} \frac{1}{|z - x|^d} dz &\leq \int_{4|x-y| \leq |x-z| \leq 5|x-y|} \frac{1}{|z - x|^{d-\alpha}} dz \\
&= \int_{4|x-y| \leq |z| \leq 5|x-y|} \frac{1}{|z|^{d-\alpha}} dz \\
&= \int_{4|x-y|} \frac{1}{r^d} r^{d-1} dr \\
&= \int_{4|x-y|} \frac{1}{r^{1-\alpha}} dr \\
&\leq C|x - y|^\alpha
\end{aligned}$$

Now in E_2 : $|z - x| \geq |z - y| - |y - x| \geq 1 - |y - x|$.

$$\begin{aligned}
\int_{E_2} \frac{1}{|z - x|^{d-\alpha}} dz &\leq \int \frac{1}{|z - x|^{d-\alpha}} dz = \int_{1-|x-y|}^1 \frac{1}{r^{d-\alpha}} r^{d-1} dr \\
&\leq \text{const.} \left| 1 - \frac{1}{(1 - |x - y|)^\alpha} \right| \leq C|x - y|^\alpha
\end{aligned}$$

■

Exercise 4.16 (E 5.1) Prove that if f is a harmonic function in \mathbb{R}^d and $g \in C_c(\mathbb{R}^d)$ is radial, then

$$\int_{\mathbb{R}^d} f(x)g(x) dx = f(0) \int_{\mathbb{R}^d} g(x) dx$$

Solution. $x = r\omega, r > 0, |\omega| = 1$

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)g(x) dx &\stackrel{(\text{Polar})}{=} \int_0^\infty \left(\int_{\partial B(0,1)} f(r\omega)g(r\omega) d\omega \right) dr \\
&= \int_0^\infty \left(g_0(r) \int_{\partial B(0,1)} f(r\omega) d\omega \right) dr \\
(\text{Mean value theorem (2.12)}) \quad &= \int_0^\infty \left(g_0(r)f(0) \int_{\partial B(0,1)} d\omega \right) dr \\
&= f(0) \int_0^\infty \left(\int_{\partial B(0,1)} g(r\omega) d\omega \right) dr \\
&= f(0) \int_{\mathbb{R}^d} g(x) dx
\end{aligned}$$

■

Remark 4.17 Let $g \in C_c(\mathbb{R}^d)$ be radial. Why is $\int_{\mathbb{R}^3} \frac{g(x)}{|x|} dx \neq \infty$? Because $f(x) = \frac{1}{|x|}$ is harmonic in $\mathbb{R}^d \setminus \{0\}$ and sub-harmonic in \mathbb{R}^d , $-\Delta f = c\delta_0$.

Exercise 4.18 (E 5.2) Let $1 \leq p < \infty$. Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider the Sobolev Space

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega), \forall i = 1, 2, \dots, d\}$$

with the norm

$$\|f\|_{W^{1,p}} = \|f\| + \sum_{i=1}^d \|\partial_{x_i} f\|_{L^p(\Omega)}.$$

Prove that $W^{1,p}(\Omega)$ is a Banach space. Here $x = (x_i)_{i=1}^d \in \mathbb{R}^d$. Hint: You can use the fact that $L^p(\Omega)$ is a Banach Space.

Solution. $W^{1,p}(\Omega) \subseteq L^p(\Omega) \times L^p(\Omega) \cdots \times L^p(\Omega) = (L^p(\Omega))^{d+1}$. For an element $f \in W^{1,p}(\Omega)$ we can think of it as $f \mapsto (f, \partial_1 f, \partial_2 f, \dots, \partial_d f)$, so $W^{1,p}(\Omega)$ is a subspace of $(L^p(\Omega))^{d+1}$, which is a norm-space. Why is $W^{1,p}(\Omega)$ closed in $(L^p(\Omega))^{d+1}$? Take $\{f_n\}_{n=1}^\infty \subseteq W^{1,p}(\Omega)$ such that $f_n \rightarrow f$ in L^p and $\partial_i f_n \rightarrow g_i$ in L^p for all $i = 1, \dots, d$. We prove that $(f, g_1, \dots, g_d) \in W^{1,p}(\Omega)$, i.e. $f \in W^{1,p}$ and $g_i = \partial_i f$ for all $i = 1, \dots, d$. We know that $f_n \rightarrow f$ in $L^p(\Omega)$, so $f_n \rightarrow f$ in $D'(\Omega)$ and $\partial_i f_n \rightarrow \partial_i f$ in $D'(\Omega)$. On the other hand we have $\partial_i f_n \rightarrow g_i$ in $L^p(\Omega)$, so $\partial_i f_n \rightarrow g_i$ in $D'(\Omega)$. So we get $\partial_i f = g_i \in L^p(\Omega)$ for all $i = 1, \dots, d$ in $D'(\Omega)$. So we can conclude $f \in W^{1,p}(\Omega)$ and $\partial_i f = g_i$ for all $i = 1, \dots, d$. ■

Exercise 4.19 (E 5.3) Let f be a real-valued function in $W^{1,p}(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Prove that $|f| \in W^{1,p}(\mathbb{R}^d)$ and

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}.$$

Solution. Consider $G_\epsilon(t) = \sqrt{\epsilon^2 + t^2} - \epsilon$ for $\epsilon > 0, t \in \mathbb{R}$. Clearly we have $G_\epsilon(t) \rightarrow |t|$ as $\epsilon \rightarrow 0$ and

$$G'_\epsilon(t) = \frac{2t}{2\sqrt{\epsilon^2 + t^2}} = \frac{t}{\sqrt{\epsilon^2 + t^2}},$$

so $|G'_\epsilon(t)| \leq 1, G_\epsilon(0) = 0$. By the chain rule, $G_\epsilon(f) \in W^{1,p}(\mathbb{R}^d)$ and

$$(\partial_i G_\epsilon(f))(x) = G'_\epsilon(f) \partial_i f(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \partial_i f(x) \in L^p(\mathbb{R}^d)$$

for all $i = 1, \dots, d$. Note then when $\epsilon \rightarrow 0$ that $G_\epsilon(f)(x) \rightarrow |f(x)|$ pointwise, so $G_\epsilon(f) \rightarrow |f|$ in $L^p(\mathbb{R}^d)$. $|G_\epsilon(f)(x) - G_\epsilon(0)| \leq |f(x)| \in L^p(\mathbb{R}^d)$ by dominated convergence.

$$\partial_i G_\epsilon(f)(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \partial_i f(x) \xrightarrow{\epsilon \rightarrow 0} g_i(x) := \begin{cases} \partial_i f(x) & f(x) > 0 \\ -\partial_i f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

$$|\partial_i G_\epsilon(f)(x)| \leq \left| \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \right| |\partial_i f(x)| \leq |\partial_i f(x)| \in L^p(\mathbb{R}^d)$$

So we get $\partial_i G_\epsilon(f) \xrightarrow{\epsilon \rightarrow 0} g_i$ in $L^p(\mathbb{R}^d)$ by Dominated Convergence. So we conclude: $\partial_i(|f|) = g_i \in L^p(\mathbb{R}^d)$ for all $i = 1, \dots, d$, so $|f| \in W^{1,p}(\mathbb{R}^d), |f| \in L^p$. ■

Exercise 4.20 (E 5.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $f \in L^1(\Omega)$,

$$u(x) = \int_{\Omega} G(x-y)f(y) dy$$

Let $-\Delta u = f$ in $D'(\Omega)$, $u \in L^1_{loc}(\Omega)$, $f \in L^1_{loc}(\mathbb{R}^d)$ and $\omega_d f \in L^1(\mathbb{R}^d)$, where

$$\omega_d(x) = \begin{cases} 1 + |x| & d = 1 \\ \log(1 + |x|) & d = 2 \\ \frac{1}{(1+|x|)^{d-2}} & d \geq 3 \end{cases}.$$

Prove that

$$G \star f = \int_{\mathbb{R}^d} G(x-y)f(y) dy \in L_{loc}^1(\mathbb{R}^d)$$

and $-\Delta(G \star f) = f$ in $D'(\mathbb{R}^d)$.

Solution. Define $\tilde{f} = \mathbb{1}_\Omega(x)f(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$. Then

$$u(x) = \int_{\Omega} G(x-y)f(y) dy = \int_{\mathbb{R}^d} G(x-y)\tilde{f}(y) dy = (G \star \tilde{f})(x)$$

We have $u \in L_{loc}^1(\mathbb{R}^d)$, so $u \in L^1(\Omega)$. Then $-\Delta u = \tilde{f}$ in $D'(\mathbb{R}^d)$, so $-\Delta u = f$ in $D'(\Omega)$. Claim: $-\Delta u = f$ in $D'(\mathbb{R}^d)$, so $-\Delta u = f$ in $D'(\Omega)$ if $\Omega \subseteq \mathbb{R}^d$, $\tilde{f}|_\Omega = f$. Take $\phi \in C_c^\infty(\Omega)$. We need: $(-\Delta u)(\phi) \stackrel{?}{=} \int_{\Omega} f\phi$. We have $\phi \in C_c^\infty(\Omega)$, so $\phi C_c^\infty(\mathbb{R}^d)$. This implies:

$$(-\Delta u)(\phi) = \int_{\mathbb{R}^d} \tilde{f}\phi = \int_{\substack{\Omega, \\ \text{supp } \phi \subseteq \Omega}} \tilde{f}\phi = \int_{\Omega} f\phi \quad \blacksquare$$

Exercise 4.21 (E 5.5) Let $B = B(0, \frac{1}{2}) \subseteq \mathbb{R}^3$. Consider $u : B \rightarrow \mathbb{R}$, defined by

$$u(x) = \log |\log |x||.$$

Prove that the distributional derivative $f = -\Delta u$ is a function in $L^{\frac{3}{2}}(B)$.

Solution.

$$\begin{aligned} \omega(r) &= \log(-\log(r)), \quad \text{for } r \in \left(0, \frac{1}{2}\right) \\ \omega'(r) &= \frac{1}{-\log(r)} \left(-\frac{1}{r}\right) = \frac{1}{r \log r} \\ \omega''(r) &= -\frac{1}{(r \log(r))^2} (r \log(r))' = -\frac{\log(r) + 1}{(r \log r)^2} \end{aligned}$$

So we have

$$-\Delta u = \omega''(r) = \frac{1}{(r \log r)^2} - \frac{1}{r^2 \log(r)} = f(r)$$

We show that $f \in L^{\frac{3}{2}}$:

$$\begin{aligned} \int_B |f(x)|^{\frac{3}{2}} dx &= \text{const} \int_0^{\frac{1}{2}} \left| \frac{1}{r^2(\log r)^2} - \frac{1}{r^2 \log r} \right|^{\frac{3}{2}} r^2 dr \\ &\lesssim \int_0^{\frac{1}{2}} \frac{1}{r} \left| \frac{1}{(\log(r))^2} - \frac{1}{(\log(r))} \right|^{\frac{3}{2}} dr \\ \left(\begin{array}{l} r = e^{-x}, \\ x \in (\log(2), \infty), \\ dr = -e^{-x} dx \end{array} \right) &\lesssim \int_{\log(2)}^{\infty} e^x \left(\frac{1}{x^2} + \frac{1}{x} \right)^{\frac{3}{2}} e^{-x} dx \\ &\lesssim \int_{\log(2)}^{\infty} \frac{1}{x^{\frac{3}{2}}} dx < \infty \end{aligned}$$

Where \lesssim means *up to a constant*. Now, $u(x) = \omega(r) = \log(-\log(r))$.

$$-\Delta u(x) = f(r) = \frac{1}{r^2(\log(r))^2} - \frac{1}{r^2 \log(r)}$$

for all $x \neq 0, |x| = r < \frac{1}{2}$. Why is $-\Delta u(x) = f$ in $D'(B)$? Take $\phi \in C_c^\infty(B)$, check: $\int_B u(-\Delta \phi) = \int_B f \phi$.

$$\int_{|x| < \frac{1}{2}} u(-\Delta \phi) dx = \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |x| < \frac{1}{2}} u(x)(-\Delta \phi)(x) dx$$

by Dominated convergence. $u \in L^1(B)$. For all $\epsilon > 0$:

$$\begin{aligned} \int_{\epsilon < |x| < \frac{1}{2}} u(x)(-\Delta \phi)(x) dx &= \int_{|x| > \epsilon} u(x)(-\Delta \phi)(x) dx \\ &= \int_{\partial B(0, \epsilon)} u(x) \nabla \phi(x) \frac{x}{|x|} dS(x) + \int_{|x| > \epsilon} \nabla u(x) \nabla \phi(x) dx \end{aligned}$$

The boundary term vanishes as $\epsilon \rightarrow 0$ since

$$\left| u(x) \nabla \phi(x) \frac{x}{|x|} \right| \leq \|\nabla \phi\|_{L^\infty} |u(x)| = C \log |\log(r)|$$

$$\begin{aligned} \left| \int_{\partial B(0, \epsilon)} u(x) \nabla \phi(x) \frac{x}{|x|} dS(x) \right| &\leq C \int_{\partial B(0, \epsilon)} \log |\log(\epsilon)| dS(x) \\ &= C \log |\log \epsilon| \underbrace{|\partial B(0, \epsilon)|}_{\sim \epsilon^2} \xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned}$$

$$\begin{aligned} \int_{|x| > \epsilon} \nabla u(x) \nabla \phi(x) dx &= \sum_{i=1}^d \int_{|x| > \epsilon} \partial_i u(x) \partial_i \phi(x) dx \\ &= \sum_{i=1}^d \left(- \int_{\partial B(0, \epsilon)} \partial_i u(x) \phi(x) \frac{x_i}{|x|} dS(x) - \int_{|x| > \epsilon} \underbrace{\partial_i \partial_i u(x)}_{f(x)} \phi(x) dx \right) \end{aligned}$$

The boundary term vanishes as $\epsilon \rightarrow 0$ as

$$\begin{aligned} \left| \int_{\partial B(0, \epsilon)} \partial_i u(x) \phi(x) \frac{x_i}{|x|} dS(x) \right| &\leq \|\phi\|_{L^\infty} \int_{\partial B(0, \epsilon)} |\partial_i u(x)| dS(x) \\ (\star) \quad &\leq C \frac{1}{|\epsilon \log(r)|} |\partial B(0, \epsilon)| \rightarrow 0 \end{aligned}$$

as $\epsilon \rightarrow 0$. $(\star)u = u(r), u(x) = \omega(|x|), \partial_i u(x) = \omega(|x|) \frac{x_i}{|x|}, |\partial_i u(x)| \leq |\omega(|x|)| = \left| \frac{1}{r \log(r)} \right|$. Finally:

$$\int_{|x| > \epsilon} f(x) \phi(x) dx \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} f(x) \phi(x) dx$$

Since $f \phi \in L^1$ and Dominated Convergence. ■

Exercise 4.22 (Bonus 5) Construct $u \in L^1(\mathbb{R}^3)$ compactly supported s.t. $-\Delta u \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and u is not continuous at 0.

Hint: Related to E 5.5. $u_0(x) = \omega(r) = \log(|\log(r)|)$ if $0 < r = |x| < \frac{1}{2}$. Consider χu_0 where $\chi \in C_c^\infty$, $\chi = 0$ if $|x| > \frac{1}{2}$, $\chi = 1$ if $|x| < \frac{1}{4}$. You can prove that $\Delta(\chi u_0) = (\Delta\chi)u_0 + 2\nabla\chi\nabla u_0 + \chi(\underbrace{\Delta u_0}_{\in L^{\frac{3}{2}}})$ in $D'(\mathbb{R}^3)$. (almost everywhere, in distributional sense, integration by parts)

Theorem 4.23 (Regularity on Domains) Let $\Omega \subseteq \mathbb{R}^d$ be open. Assume $u, f \in D'(\Omega)$ such that $-\Delta u = f$ in $D'(\Omega)$.

- a) If $f \in L_{loc}^1(\Omega)$, then
 - $u \in C^1(\Omega)$ if $d = 1$
 - $u \in L_{loc}^q(\Omega)$ for all $q < \infty$ if $d = 2$
 - $u \in L_{loc}^q(\Omega)$ for all $q < \frac{d}{d-2}$ if $d \geq 3$
- b) If $f \in L_{loc}^q(\Omega)$, $d \geq p < \frac{d}{2}$, then $u \in C_{loc}^{0,\alpha}(\Omega)$, where $0 < \alpha < 2 - \frac{d}{p}$
- c) If $f \in L_{loc}^p(\Omega)$, $p > df$, then $u \in C_{loc}^{1,\alpha}(\Omega)$, where $0 \leq \alpha < 1 - \frac{d}{p}$
- d) If $f \in C_{loc}^{0,\alpha}(\Omega)$ for some $0 < \alpha < 1$, then $u \in C_{loc}^{2,\alpha}(\Omega)$
- e) If $f \in C_{loc}^{m,\alpha}(\Omega)$, then $u \in C_{loc}^{m+2,\alpha}(\Omega)$

Proof. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Take a ball $\bar{B} \subseteq \Omega$. Define $f_B : \mathbb{R}^d \rightarrow \mathbb{K}$,

$$f_B(x) = (\mathbb{1}_B f)(x) = \begin{cases} f(x) & x \in B \\ 0 & x \notin B \end{cases}$$

Then if $f \in L_{loc}^1(\Omega)$, f_B is compactly supported. From the previous theorems: $G \star f_B \in L_{loc}^1(\mathbb{R}^d)$ and $-\Delta(G \star f_B) = f_B$ in $D'(\mathbb{R}^d)$. On the other hand, $-\Delta u = f$ in $D'(\Omega)$, so $-\Delta(u - G \star f_B) = 0$ in $D'(B)$. Indeed, for all $\phi \in C_c^\infty(B)$, then:

$$(-\Delta u)(\phi) = \int_{\Omega} f\phi = \int_B f_B\phi = - \int_{\mathbb{R}^d} f_B\phi = (-\Delta)(G \star f_B)(\phi)$$

Then $-\Delta u = -\Delta(G \star f_B)$ in $D'(B)$. Then $u - G \star f_B$ is harmonic in B and by Weyl's lemma we have $u - G \star f_B \in C^\infty(B)$. So the smoothness of u in B is the same to that of $G \star f$. \blacksquare

Exercise 4.24 (E 6.1) Show that If $\chi \in C^\infty(\mathbb{R}^d)$, then $f \in W^{1,p}(\mathbb{R}^d)$, $1 \leq p < \infty$, then $\chi f \in W_{loc}^{1,p}(\mathbb{R}^d)$ and

$$\partial_i(\chi f) = (\partial_i \chi)f + \chi(\partial_i f) \quad \text{in } D'(\mathbb{R}^d)$$

Solution. $\chi f \in L_{loc}^p(\mathbb{R}^d)$ obvious. $\partial(\chi f) \in L_{loc}^p(\mathbb{R}^d)$ is nontrivial but follows from $\partial_i(\chi f) = \underbrace{(\partial_i \chi)f + \chi(\partial_i f)}_{\in L_{loc}^p}$ in $D'(\mathbb{R}^d)$. To compute the distributional derivative

$\partial_i(\chi f)$, then: Take $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$- \int_{\mathbb{R}^d} \chi f(\partial_i \phi) = \int_{\mathbb{R}^d} (?)\phi$$

We have

$$\begin{aligned}
-\int_{\mathbb{R}^d} \chi f(\partial_i \phi) &= -\int_{\mathbb{R}^d} f(\chi \partial_i \phi) \\
(\chi \partial_i \phi = (\partial_i \chi) \phi + \chi(\partial_i \phi)) &= -\int_{\mathbb{R}^d} f(\partial_i(\chi \phi) - (\partial_i \chi) \phi) \\
&= -\int_{\mathbb{R}^d} f \underbrace{\partial_i(\chi \phi)}_{\in C_c^\infty} + \int_{\mathbb{R}^d} f(\partial_i \chi) \phi \\
&= \int_{\mathbb{R}^d} (\partial_i f) \chi \phi + \int_{\mathbb{R}^d} f(\partial_i \chi) \phi \\
&= \int_{\mathbb{R}^d} ((\partial_i f) \chi + f(\partial_i \chi)) \phi
\end{aligned}$$

So $\partial_i(\chi f) = (\partial_i f) \chi + f(\partial_i \chi)$ in $D'(\mathbb{R}^d)$. ■

Remark 4.25 Question: If $\chi \in C^1(\mathbb{R}^d)$, $f \in W^{1,p}(\mathbb{R}^d)$. Is this it still correct that $\partial_i(\chi f) = (\partial_i \chi) f + \chi(\partial_i f)$ in $D'(\mathbb{R}^d)$?

Proof. It suffices to show that we still can apply intergration by parts.

$$(\star) \quad -\int f \partial_i g \stackrel{?}{=} \int (\partial_i f) g$$

Approximation: (\star) is correct if $g \in C_c^\infty$

- If $g \in C_c^1$, there is $\{g_n\} \subseteq C_c^\infty$ s.t. $g_n \rightarrow g$ in $W_{loc}^{1,p}$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\int (\partial_i g) f \xrightarrow{n \rightarrow \infty} - \int \underbrace{f}_{L^p} \underbrace{\partial_i g_n}_{\rightarrow \partial_i g \text{ in } L^q} = \int \underbrace{(\partial_i f)}_{\in L^p} \underbrace{g_n}_{\rightarrow g \text{ in } L^q} \xrightarrow{n \rightarrow \infty} \int (\partial_i f) g$$

■

Exercise 4.26 (E 6.2) \mathbb{R}^2 , $G(x) = -\frac{1}{2\pi} \log |x|$. Let $f \in L^p(\mathbb{R}^d)$, compactly supported. Define $u(x) = (G \star f)(x) = \int_{\mathbb{R}^2} G(x-y) f(y) dy$

1. If $p = 1$, then $u \in L_{loc}^q(\mathbb{R}^2)$ for all $q < \infty$.
2. If $p > 2$, then $u \in C^{1,\alpha}$ with $0 < \alpha < 1 - \frac{2}{p}$.

Solution. 1. Take any ball $B = B(0, R)$ and:

$$\begin{aligned}
\int_B |u(x)|^q dx &= \int_B \left(\int_{\mathbb{R}^d} |G(x-y)| |f(y)| dy \right)^q dx \\
&\leq C \int_B \left(\int_{\mathbb{R}^2} |G(x-y)|^q |f(y)| dy \right) dx \\
&= C \int_{\mathbb{R}^2} \left(\int_B |G(x-y)|^q dx \right) |f(y)| dy
\end{aligned}$$

Recall from the proof of Youngs inequality:

$$\begin{aligned}
|u(x)| &= \left| \int_{\mathbb{R}^2} G(x-y) f(y) dy \right| \\
&\leq \int_{\mathbb{R}^2} |G(x-y)| |f(y)| dy \\
&\leq \left(\int_{\mathbb{R}^2} |G(x-y)|^q |f(y)| dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} |f(y)| dy \right)^{\frac{1}{q'}}, \quad \frac{1}{q} + \frac{1}{q'} = 1
\end{aligned}$$

Assume $\text{supp } f \subseteq \overline{B(0, R)}$. Then if $y \in \text{supp } f$ and $x \in B(0, R)$, then $|x-y| \leq |x| + |y| \leq R + R_1$. For all $y \in \text{supp } f$:

$$\begin{aligned}
\int_{B(0, R)} |G(x-y)|^q dx &\leq \int_{|x-y| \leq R+R_1} |G(x-y)|^q dx \\
&= \int_{|z| \leq R+R_1} |G(z)|^q dz < \infty
\end{aligned}$$

as $G \in L_{loc}^q$ ($|G(z)| = \frac{1}{2\pi} |\log(z)| \leq \frac{C_{R+R_1, \epsilon}}{|z|^\epsilon}$ for all $|z| \leq R + R_1$), so

$$\int_{|z| \leq R+R_1} |G(z)|^q \leq C_{R+R_1, \epsilon} \int_{|z| \leq R+R_1} \frac{1}{|z|^{\epsilon q}} dz < \infty$$

if $\epsilon q < 2$.

2. Recall $\partial_i u \in L_{loc}^1(\mathbb{R}^2)$ and:

$$\partial_i u(x) = (\partial_i G \star f)(x) = c \int_{\mathbb{R}^2} \frac{x_i - y_i}{|x - y|^2} f(y) dy$$

First we show $\partial_i u \in C^{0, \alpha}$:

$$\begin{aligned}
|\partial_i u(x) - \partial_i u(z)| &= \left| C \int_{\mathbb{R}^2} \left(\frac{x_i - y_i}{|x - y|^2} - \frac{z_i - y_i}{|z - y|^2} \right) f(y) dy \right| \\
&\leq C \int_{\mathbb{R}^2} \left| \frac{x_i y_i}{|x - y|^2} - \frac{z_i y_i}{|z - y|^2} \right| |f(y)| dy \\
&\stackrel{?}{\leq} C |x - z|^\alpha
\end{aligned}$$

Note that

$$\begin{aligned}
\left| \frac{x_i - y_i}{|x - y|^2} - \frac{z_i - y_i}{|z - y|^2} \right| &= \left| (x_i - y_i) \left(\frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right) + \frac{x_i - z_i}{|z - y|^2} \right| \\
&\leq |x_i - y_i| \left| \frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right| + \frac{|x_i - z_i|}{|z - y|^2} \\
&\leq C |z - x|^\alpha \left(\frac{1}{|x - y|^{1+\alpha}} + \frac{1}{|z - y|^{1+\alpha}} + \frac{|x - z|}{|z - y|^2} \right)
\end{aligned}$$

Here $|x_i - z_i| \leq |x - z|$ and $|x_i - y_i| \leq |x - y|$ and:

$$\begin{aligned}
\underbrace{\left| \frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right|}_{\text{sym } x \leftrightarrow z} &= \left| \frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right| = \left| \frac{1}{|x - y|} - \frac{1}{|z - y|} \right| \left| \frac{1}{|x - y|} + \frac{1}{|z - y|} \right| \\
&= \frac{||z - y| - |x - y||}{|x - y| |z - y|} \left| \frac{1}{|x - y|} + \frac{1}{|z - y|} \right| \\
&\leq |z - x|^\alpha \frac{\max(|z - y|, |x - y|)^{1-\alpha}}{|x - y| |z - y|} \left(\frac{1}{|x - y|} + \frac{1}{|z - y|} \right) \\
&\leq C |z - x|^\alpha \left(\frac{1}{|x - y|^{2+\alpha}} + \frac{1}{|z - y|^{2+\alpha}} \right)
\end{aligned}$$

By the symmetrie $x \leftrightarrow z$:

$$\begin{aligned}
LHS &\leq C|z-x|^\alpha \left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} \right) + \frac{|x-y|}{|x-y|^2} \\
&\Rightarrow LHS \leq C \cdots + |x-z| \min \left(\frac{1}{|z-y|^2}, \frac{1}{|x-y|^2} \right) \\
&\leq (|x-y| + |z-y|)^{1-\alpha} \\
&C|z-x|^\alpha \left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} \right)
\end{aligned}$$

In summary:

$$\begin{aligned}
|\partial_i u(x) - \partial_i u(z)| &\leq C \int_{\mathbb{R}^2} \left| \frac{x_i - y_i}{|x-y|^2} - \frac{z_i - y_i}{|z-y|^2} \right| |f(y)| dy \\
&= C|x-y|^\alpha \int_{\mathbb{R}^2} \left(\frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} \right) |f(y)| dy
\end{aligned}$$

Consider if $|x| > 2R_1$:

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|^{1+\alpha}} |f(y)| dy \leq \int_{\mathbb{R}^2} \frac{1}{R_1^{1+\alpha}} |f(y)| dy \leq C$$

$\text{supp } f \subseteq B(0, R_1)$. If $|x| < 2R_1$, then $|x-y| \leq 3R$ if $y \in B(0, R_1)$. Hence:

$$\begin{aligned}
&\int_{|x-y| \leq 3R_1} \frac{1}{|x-y|^{1+\alpha}} |f(y)| dy \\
&\leq \left(\int_{|x-y| \leq 3R_1} \frac{1}{|x-y|^{(1+\alpha)p'}} \right)^{\frac{1}{p'}} \left(\int |f(y)|^p dy \right)^{\frac{1}{p}} \\
&= \int_{|z| \leq 3R_1} \frac{1}{|z|^{(1+\alpha)p'}} dz < \infty
\end{aligned}$$

So $\alpha < 1 - \frac{2}{p}$. ■

Exercise 4.27 (E 6.3) Let $f \in C_{loc}^{0,\alpha}$ and $-\Delta u = f$ in $D'(\Omega)$. Prove $u \in C_{loc}^{2,\alpha}(\Omega)$.

Solution. Take an open ball $B \subseteq \bar{B} \subseteq \Omega$. We prove $u \in C^{2,\alpha}(B)$. There is an open Ω_B s.t. $\bar{B} \subseteq \bar{\Omega}_B \subseteq \Omega$. Then there is a $\chi_B \in C_c^\infty(\mathbb{R}^d)$ s.t. $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin \Omega_B$. Define

$$f_B(x) = \chi_B(x)f(x) : \mathbb{R}^d \rightarrow \mathbb{R}$$

We prove that $f_B \in C^{0,\alpha}(\mathbb{R}^d)$. Since $f \in C_{loc}^{0,\alpha}(\Omega)$ we have $f \in C^{0,\alpha}(\Omega)$, so $|f(x) - f(y)| \leq C|x-y|^\alpha$ for all $x, y \in \Omega_B$. Then:

$$\begin{aligned}
|f_B(x) - f_B(y)| &= |\chi_B(x)f(x) - \chi_B(y)f(y)| \\
&\leq |(\chi_B(x) - \chi_B(y))f(x) + \chi_B(y)(f(x) - f(y))| \\
&\leq C|x-y|^\alpha \|f\|_{L^\infty} + C\|\chi\|_{L^\infty(\Omega_B)} |x-y|^\alpha \leq C_{\Omega_B} |x-y|^\alpha
\end{aligned}$$

What about other cases? If x, y are bot not in Ω_B , then $|f_B(x) - f_B(y)| = 0$, then if $x \in \Omega_B$ and $y \notin \Omega_B$: $|f_B(x) - f_B(y)| = |f_B(x)| = |\chi_B(x)||f(x)| = |\chi_B(x) - \chi_B(y)||f(x)| \leq C|x-y|^\alpha$. Conclusion: $|f_B(x) - f_B(y)| \leq C|x-y|^\alpha$ for all $x, y \in \mathbb{R}^d$, i.e. $f_B \in C^{0,\alpha}(\mathbb{R}^d)$. Also f_B is compactly supported. By a theorem in the lecture: $G \star f_B \in C^{2,\alpha}(\mathbb{R}^d)$. Finally: $-\Delta u = f$ in $D'(\Omega)$, $-\Delta(G \star f_B) = f_B$ in $D'(\mathbb{R}^d)$. So we conclude $-\Delta u = f = f_B = -\Delta(G \star f_B)$ in $D'(B)$. $-\Delta(u - G \star f_B) = 0$ in $D'(B)$, so $u - G \star f_B \in C^\infty(B)$, so $u \in C^{2,\alpha}(B)$. ■

Exercise 4.28 (6.4) $u, f \in L^2(\mathbb{R}^d)$, $-\Delta u = f$ in $D'(\mathbb{R}^d)$. Prove $u \in W^{2,2}(\mathbb{R}^d)$, $\|u\|_{W^{2,2}(\mathbb{R}^d)} \leq C(\|u\|_{L^2} + \|f\|_{L^2})$.

$$\begin{aligned} W^{2,2}(\mathbb{R}^d) &= \{g \in L^2(\mathbb{R}^d) \mid D^\alpha g \in L^2 \text{ for all } |\alpha| \leq 2\} \\ &= \{g \in L^2(\mathbb{R}^d) \mid \widehat{D^\alpha g}(k) = (-2\pi i k)^\alpha \hat{g}(k) \in L^2(\mathbb{R}^d) \text{ for all } |\alpha| \leq 2\} \\ &= \{g \in L^2(\mathbb{R}^d) \mid (1 + |k|^2)\hat{g}(k) \in L^2(\mathbb{R}^d)\} \end{aligned}$$

$\|u\|_{W^{2,2}(\mathbb{R}^d)}$ is comparable $\int_{\mathbb{R}^d} (1 + |k|^2)^2 |\hat{g}(k)|^2 dk$. If $D^\alpha g \in L^2$, then $\widehat{D^\alpha g}(k) = (-2\pi i k)^\alpha \hat{g}(k)$. For any $\phi \in C_c^\infty(\mathbb{R}^d)$:

$$\begin{aligned} \int \widehat{D^\alpha g}(k) \hat{\phi}(k) dk &= \int (D^\alpha g) \phi = (-1)^{|\alpha|} \int g (D^\alpha \phi) \\ &= (-1)^{|\alpha|} \int \tilde{\hat{g}}(k) \widehat{D^\alpha \phi}(k) \\ &= (-1)^{|\alpha|} \int \tilde{\hat{g}}(k) (-2\pi i k)^\alpha \hat{\phi}(k) dk \end{aligned}$$

so $\hat{D^\alpha g}(k) = (-1)^{|\alpha|} \hat{g}(k) \overline{(-2\pi i k)^\alpha} = \hat{g}(k) (-2\pi i k)^\alpha$. This implies:

$$\begin{aligned} \|u\|_{W^{2,2}(\mathbb{R}^d)} &\leq C \int_{\mathbb{R}^d} (1 + |k|^2)^2 |\hat{u}(k)|^2 dk \\ &= C \left(\|u\|_{L^2}^2 + \int_{\mathbb{R}^d} |k|^4 |\hat{u}(k)|^2 dk \right) \\ &\leq C (\|u\|_{L^2}^2 + \|f\|_{L^2}^2) \\ &\leq C (\|u\|_{L^2} + \|f\|_{L^2})^2 \end{aligned}$$

Remark 4.29 (Bonus 6) Let $f, g \in W^{1,2}(\mathbb{R}^d)$. Prove that $fg \in W^{1,1}(\mathbb{R}^d)$ and

$$\partial_i(fg) = (\partial_i f)g + f(\partial_i g) \quad \text{in } D'(\mathbb{R}^d)$$

Chapter 5

Existence for Poisson's Equation on Domains

Let $\Omega \subseteq \mathbb{R}^d$ be open. Consider Poisson's equation.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for given data (f, g) and u the unknown function.

- Classical solutions: $f \in C^2(\bar{\Omega}) \rightsquigarrow$ explicit representation formula.
- Weak solution: $f \in L^p(\Omega)$, $g \in L^p(\partial\Omega) \rightsquigarrow u \in W^{2,p}(\Omega)$. We are going to establish the existence by *Energy Methods*. (Calculus of variations)

Definition 5.1 (C^1 -Domains) Let $\Omega \subseteq \mathbb{R}^d$ be open. We say that Ω is of class C^1 (i.e. $\partial\Omega \in C^1$) if for all $x_0 \in \partial\Omega$ there is a bijective function $h : U \rightarrow Q$, where

- $x_0 \in U$ open in \mathbb{R}^d
- $Q = \{x = (x_1, \dots, x_d) = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, |x_d| < 1\}$
- $h \in C^1(\bar{U})$ and $h^{-1} \in C^1(\bar{Q})$ (C^1 -diffeomorphism)
- $h(U) = Q$

$$\begin{aligned} h(U \cap \Omega) &= Q_+ = Q \cap \mathbb{R}_+^d = \{x = (x', x_d) \in Q \mid x_d > 0\} \\ h(U \cap \partial\Omega) &= Q_0 = Q \cap \partial\mathbb{R}_+^d = \{x = (x', x_d) \in Q \mid x_d = 0\} \\ h(U \setminus \bar{\Omega}) &= Q_- = Q \cap \mathbb{R}_-^d = \{x = (x', x_d) \in Q \mid x_d < 0\} \end{aligned}$$

(From Brezis' book)

Remark 5.2 The set Q can be replaced by a ball, i.e. Ω is of C^1 if for all $x_0 \in \partial\Omega$ there is a function $U \rightarrow B(0, 1) \subseteq \mathbb{R}^d$.

- $x_0 \in U$ with $U \subseteq \mathbb{R}^d$ open.
- $h \in C^1(\bar{U})$, $h^{-1} \in C^1(\overline{B(0, 1)})$
- $h(U \cap \Omega) = B(0, 1) \cap \mathbb{R}_+^d$, $h(U \cap \partial\Omega) = B(0, 1) \cap \mathbb{R}^d$.

Remark 5.3 (An equivalent definition from Evan's book App. C) Let $\Omega \subseteq \mathbb{R}^d$ be open. Then Ω is C^1 if for all $x_0 \in \partial\Omega$ there is a $r > 0$ and a C^1 -function $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ s.t. (upon relabeling and reorienting the axes if necessary) such that:

$$\Omega \cap B(x_0, r) = \{x = (x', x_d) \in B(x_0, r) \mid x_d < \gamma(x')\}$$

Proof of the equivalence of the two definitions.

Def. 2 \Rightarrow Def. 1: In fact, given $x_0 \in \partial\Omega$ and γ we can define

$$\begin{aligned} h(x', x_d) &= (x', x_d - \gamma(x')) \in C^1(\mathbb{R}^d, \mathbb{R}^d) \\ h^{-1}(x', x_d) &= (x', x_d + \gamma(x')) \in C^1(\mathbb{R}^d, \mathbb{R}^d) \end{aligned}$$

Def. 1 \Rightarrow Def. 2: We need the inverse function theorem and the implicit function theorem. Let $x_0 \in \partial\Omega$, let $h : U \rightarrow B(0, 1)$ as in Def. 1. Denote $h = (h_1, h_2, \dots, h_d)$. Since h is invertible near x_0 , by the inverse function theorem we have for the Jacobi matrix $Jh(x_0) = (\partial_j h_i(x_0))_{1 \leq i, j \leq d}$ is invertible. So we have $\nabla h_d(x_0) = (\partial_j h_d(x_0))_{1 \leq j \leq d} \neq \vec{0}^{\mathbb{R}^d}$, so there is a $j \in \{1, 2, \dots, d\}$ s.t. $\partial_j h_d(x_0) \neq 0$. By relabeling and reorienting the axes, we can assume that $\partial_d h_d(x_0) > 0$. By continuity there is a $r > 0$ such that $\partial_d h_d(x) > 0$ for all $x \in B(x_0, r)$. Define $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ s.t. in $B(x_0, r)$:

$$x = (x', x_d) \in \partial\Omega \iff h_d(x', x_d) = 0 \iff x_d = \gamma(x'),$$

$h_d : \mathbb{R}^d \rightarrow \mathbb{R}$. This gives a solution γ if $\partial_d h_d > 0$ in $B(x_0, r)$. (For implicit function theorem, $\partial_d h_d(x_0) \neq 0$) Question: Why in $B(x_0, r)$?

$$x = (x', x_d) \in \Omega \iff x_d > \gamma(x')$$

Since $\partial_d h_d(x) > 0$ for all $x \in B(x_0, r)$ we have that $x_d \mapsto h_d(x', x_d)$ is strictly increasing, hence

$$\begin{aligned} x &= (x', x_d) \in \Omega \\ \iff h(x', x_d) &\in \mathbb{R}_+^d \\ \iff h_d(x', x_d) &> 0 = h_d(x', \gamma(x')) \\ \iff x_d &> \gamma(x') \end{aligned} \quad \blacksquare$$

Theorem 5.4 (Gauss-Green formula / Integration by parts) Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. Then

1. For all $u, v \in C^1(\bar{\Omega})$:

$$\int_{\Omega} (\partial_i u) v = - \int_{\Omega} u (\partial_i v) + \int_{\partial\Omega} u v n_i dS,$$

where $\vec{n} = (n_i)_{i=1}^d$ is the outwards unit normal vector.

2. For all $u, v \in C^2(\bar{\Omega})$:

$$\int_{\Omega} u (-\Delta v) = \int_{\Omega} \nabla u \nabla v - \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}} dS$$

where $\frac{\partial v}{\partial \vec{n}} = \nabla v \vec{n} = \sum_{i=1}^d \partial_i v n_i$.

Classical solutions via Green's function:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, $\partial\Omega \in C^1$. Assume there exists a $u \in C^2(\bar{\Omega})$, $f \in C(\bar{\Omega})$, $g \in C(\partial\Omega)$. Let G be the fundamental solution of the Laplace Equation in \mathbb{R}^d . We use integration by parts in $\Omega \setminus B(x, \epsilon)$:

$$\begin{aligned} & \int_{\Omega \setminus B(x, \epsilon)} u(y)(-\Delta G)(y-x) dy \\ &= \int_{\Omega \setminus B(x, \epsilon)} \nabla u(y) \nabla G(y-x) dy - \int_{\partial\Omega \cup \partial B(x, \epsilon)} u(y) \frac{\partial G}{\partial \vec{n}}(y-x) dS(y) \\ & \int_{\Omega \setminus B(x, \epsilon)} G(y-x)(-\Delta u)(y) dy \\ &= \int_{\Omega \setminus B(x, \epsilon)} \nabla G(y-x) \nabla u(y) dy - \int_{\partial\Omega \cup \partial B(x, \epsilon)} G(y-x) \frac{\partial u}{\partial \vec{n}}(y) dS(y) \end{aligned}$$

This implies:

$$\begin{aligned} & \int_{\Omega \setminus B(x, \epsilon)} [u(y)(-\Delta G(y-x)) - G(y-x)(-\Delta u)(y)] dy \\ &= - \int_{\partial\Omega \cup \partial B(x, \epsilon)} \left[u(y) \frac{\partial G}{\partial \vec{n}}(y-x) - G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right] dS(y) \end{aligned}$$

for all $x \in \Omega$, $x \in B(x, \epsilon) \subseteq \Omega$. When $\epsilon \rightarrow 0$, then the left hand side converges to $-\int_{\Omega} G(y-x)f(y) dy$ and the right hand side (for $d \geq 2$) we have $\partial_j G(y) = \frac{-y_j}{d|B_1||y|^d}$, so

$$\frac{\partial G}{\partial \vec{n}} = \nabla G \vec{n} = \nabla G(y) \left(\frac{-y}{|y|} \right) = \sum_{j=1}^d \frac{-y_i}{d|B_1||y|^d} \frac{-y_j}{|y|} = \frac{1}{d|B_1||y|^{d-1}} \text{ on } \partial B(0, \epsilon)$$

so we have

$$\frac{\partial G}{\partial \vec{n}}(y-x) = \frac{1}{d|B_1|\epsilon^{d-1}}$$

on $\partial B(x, \epsilon)$. Hence

$$\begin{aligned} \int_{\partial B(x, \epsilon)} u(y) \frac{\partial G}{\partial \vec{n}}(y-x) dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(x, \epsilon)} u(y) dS(y) \\ &= \oint_{\partial B(x, \epsilon)} u(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} u(x) \end{aligned}$$

On the other hand:

$$\left| \int_{\partial B(x, \epsilon)} G(y-x) \frac{\partial u(y)}{\partial \vec{n}} dS(y) \right| \leq C\epsilon^{d-1} \sup_{|z|=\epsilon} |G(z)| \xrightarrow{\epsilon \rightarrow 0} 0$$

since $|G(z)| \leq \frac{C}{|z|^{\frac{d-2}{2}}}$ if $d \geq 3$, $|G(z)| \leq C|\log(z)|$ if $d = 2$ and $|G(z)| \leq C|z|$ if $d = 1$.

In summary:

$$\begin{aligned} - \int_{\Omega} G(y-x)f(y) dy &= - \int_{\partial\Omega} \left[u(y) \frac{\partial G}{\partial \vec{n}}(y-x) - G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right] dS(y) - u(x) \\ &\Leftrightarrow u(x) = \int_{\Omega} G(y-x)f(y) dy + \int_{\partial\Omega} \left[G(y-x) \frac{\partial u}{\partial \vec{n}}(y) - g(y) \frac{\partial G}{\partial \vec{n}}(y-x) \right] dS(y) \end{aligned}$$

Problem: We don't know anything about $\frac{\partial u}{\partial \vec{n}}$ on $\partial\Omega$. Trick: We can resolve that by using the *corrector* function: $\Phi_x = \Phi_x(y)$ which solves:

$$\begin{cases} -\Delta \Phi_x = 0 & \text{in } \Omega \\ \Phi_x(y) = G(y-x) & \text{on } \partial\Omega \end{cases}$$

We assume that Φ_x exists.

Definition 5.5 (Green's function) $\tilde{G}(x-y) = G(y-x) - \Phi_x(y)$ for all $x, y \in \Omega$, $x \neq y$.

Exercise 5.6 Prove that (under the assumption that the corrector exists) that $\tilde{G}(x, y) = \tilde{G}(y, x)$ for all $x, y \in \Omega$, $x \neq y$.

Theorem 5.7 Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and C^1 . If $u \in C^2(\Omega)$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases},$$

then

$$u(x) = - \int_{\partial\Omega} g(y) \frac{\partial \tilde{G}}{\partial \vec{u}_y}(x, y) dS(y) + \int_{\Omega} \tilde{G}(x, y) dy$$

Proof. We need to prove:

$$\int_{\Omega} \Phi_x(y) f(y) dy + \int_{\partial\Omega} \left(-g(y) \frac{\partial \Phi_x(y)}{\partial \vec{n}_y} + G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right) = 0$$

By integration by parts:

$$\begin{aligned} \int_{\Omega} \Phi_x(y) f(y) dy &= \int_{\Omega} \Phi_x(y) (-\Delta u(y)) dy \\ &= \int_{\Omega} [\Phi_x(y) (-\Delta u(y)) + (\Delta \Phi_x(y)) u(y)] dy \\ (\Delta \Phi_x(y) = 0) &= \int_{\partial\Omega} \left(-\Phi_x(y) \frac{\partial u}{\partial \vec{n}} + \frac{\partial \Phi_x(y)}{\partial \vec{n}} \underbrace{u(y)}_{g(y)} \right) dS(y) \quad \blacksquare \end{aligned}$$

How can we compute $\Phi_x(y)$? It is not easy for general domains. But let us prove on two cases:

- $\Omega = \mathbb{R}_+^d$ (half-space)
- $\Omega = B(0, r)$ (a ball)

5.1 Green's function on the upper half plane

We use the following notation:

$$\begin{aligned} \mathbb{R}_+^d &= \{x = (x_1, x_2, \dots, x_d) = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid x_d > 0\} \\ \partial\mathbb{R}_+^d &= \{x = (x', x_d) \mid x_d = 0\} = \mathbb{R}^{d-1} \times \{0\} \end{aligned}$$

For all $x \in \mathbb{R}^d$ we want to find the correction function $\Phi_x(y)$ with $y \in \overline{\mathbb{R}_+^d}$ s.t.

$$\begin{cases} +\Delta_y \Phi_x(y) = 0 & \text{in } \mathbb{R}_+^d \\ \Phi_x(y) = G(y-x) & \text{in } \partial\mathbb{R}_+^d \end{cases}$$

Definition 5.8 (Reflection for \mathbb{R}_+^d) For all $x = (x', x_d) \in \mathbb{R}^d$, $\tilde{x} = (x', -x_d) \in \mathbb{R}^d$,
(if $x \in \mathbb{R}_+^d \Rightarrow \tilde{x} \in \mathbb{R}_-^d$)

Claim: $\Delta_y \Phi_x(y) = G(y - \tilde{x})$ is a corrector function.

- $\Delta_y \Phi_x(y) = \Delta_y G(y - \tilde{x}) = 0$ for all $y \in \mathbb{R}_+^d$ for all $x \in \mathbb{R}_+^d$ (as $\tilde{x} \in \mathbb{R}_-^d = \mathbb{R}^d \setminus \overline{\mathbb{R}_+^d}$)
- $\Phi_x(y) = G(y - \tilde{x}) = G(y - x)$ on $y \in \partial \mathbb{R}_+^d$. In fact, $y \in \partial \mathbb{R}_+^d$, so $y_d = 0$, so

$$G(y - \tilde{x}) = G_0(|y - \tilde{x}|) = G_0\left(\sqrt{\sum_{i=1}^{d-1} |x_i - y_i|^2 + |x_d|^2}\right) = G_0(|y - x|)$$

Consider $f = 0$ and

$$\begin{cases} -\Delta = 0 & \text{in } \mathbb{R}_+^d \\ u = g & \text{on } \partial \mathbb{R}_+^d \end{cases}$$

Then we expect

$$u(x) = - \int_{\partial \Omega} g(y) \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) dS(y)$$

We compute

$$\frac{\partial \tilde{G}}{\partial \vec{n}_y}(x - y) = \sum_{j=1}^d \frac{\partial \tilde{G}}{\partial y_j}(x, y) \vec{n}_j = - \frac{\partial \tilde{G}}{\partial y_d}(x, y) = \frac{\partial}{\partial y_d}(G(y - \tilde{x}) - G(y - x)) = \dots$$

because $\tilde{G}(x, y) = G(y - x) - \Phi_x(y) = G(y - x) - G(y - \tilde{x})$.

$$\begin{aligned} \dots &= \frac{1}{d|B_1|} \left[\frac{-(y_d - \tilde{x}_d)}{|y - \tilde{x}|^d} - \frac{-(y_d - x_d)}{|y - x|^d} \right] \\ (y \in \partial \mathbb{R}_+^d) &= \frac{1}{d|B_1|} \left[\frac{\tilde{x}_d}{|y - x|} - \frac{x_d}{|y - x|^d} \right] = \frac{-2x_d}{d|B_1||y - x|^d} \end{aligned}$$

We expect

$$u(x) = - \int_{\partial \mathbb{R}_+^d} g(y) \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) dS(y) = \int_{\partial \mathbb{R}_+^d} g(y) \frac{2x_d}{d|B_1||y - x|^d} dS(y)$$

Theorem 5.9 Assume $g \in C(\mathbb{R}^{d-1}) \cap L^\infty(\mathbb{R}^{d-1})$ Then

$$u(x) = \int_{\partial \mathbb{R}_+^d} g(y) K(x, y) dS(y)$$

and

$$K(x, y) = \frac{2x_d}{d|B_1||y - x|^d} \quad \text{for all } x \in \mathbb{R}_+^d.$$

satisfies that $u \in C^\infty(\mathbb{R}_+^d) \cap L^\infty(\mathbb{R}_+^d)$ and

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^d \\ \lim_{\substack{x \rightarrow 0 \\ x \in \mathbb{R}_+^d}} u(x) = g(x_0) & \forall x_0 \in \partial \mathbb{R}_+^d \end{cases}$$

Proof. Claim: For all $y \in \partial\mathbb{R}_+^d$, $x \mapsto K(x, y)$ is harmonic in \mathbb{R}_+^d (i.e. $\Delta_x K(x, y) = 0$ in \mathbb{R}_+^d)

- Argument from Evans:

$$K(x, y) = -\frac{\partial}{\partial y_d}, \quad \tilde{G}(y - x) = -\frac{\partial}{\partial y_d}(G(y - x) - G(y - \tilde{x}))$$

We know that for all $x \in \mathbb{R}_+^d$, $y \mapsto \tilde{G}(y, x)$ is harmonic in $\mathbb{R}_+^d \setminus \{x\}$. By symmetry we have $\tilde{G}(y, x) = \tilde{G}(x, y)$ for all $x, y \in \mathbb{R}_+^d$. So for all $y \in \mathbb{R}_+^d$, $x \mapsto \tilde{G}(y, x)$ is harmonic in $\mathbb{R}_+^d \setminus \{y\}$. Then for all $y \in \mathbb{R}_+^d$: $-\frac{\partial}{\partial y_d} \tilde{G}(y, x) = K(x, y)$ is harmonic in $\mathbb{R}_+^d \setminus \{y\}$. By a limit argument, for all $y \in \partial\mathbb{R}_+^d$, $x \mapsto K(x, y)$ is harmonic for all $x \in \mathbb{R}_+^d$.

- A direct proof:

$$K(x, y) = \frac{2x_d}{d|B_1|} \frac{1}{|x - y|^d}$$

for all $x \in \mathbb{R}_+^d$, $y \in \partial\mathbb{R}_+^d$. For $i \neq d$, $x = (x_1, \dots, x_d)$,

$$\begin{aligned} \partial_{x_i} K(x, y) &= \frac{2x_d}{d|B_1|} \frac{(-d)}{|x - y|^{d+1}} \frac{x_i - y_i}{|x - y|} = \frac{-2x_d}{|B_1|} \frac{x_i - y_i}{|x - y|^{d+2}} \\ \partial_{x_i}^2 K(x, y) &= -\frac{2x_d}{|B_1|} \left[\frac{1}{|x - y|^{d+1}} - \frac{(d+2)}{|x - y|^{d+3}} (x_i - y_i) \frac{(x_i - y_i)}{|x - y|} \right] \\ &= -\frac{2x_d}{|B_1|} \left[\frac{1}{|x - y|^{d+1}} - \frac{(d+2)}{|x - y|^{d+4}} (x_i - y_i)^2 \right] \end{aligned}$$

Moreover:

$$\begin{aligned} \partial_{x_d} K(x, y) &= \frac{2}{d|B_1|} \frac{1}{|x - y|^d} + \frac{2x_d}{d|B_1|} (-d) \frac{(x_d - y_d)}{|x - y|^{d+2}} \\ (y_d = 0) &= \frac{2}{d|B_1|} \frac{1}{|x - y|^d} + \frac{2x_d^2}{|B_1||x - y|^{d+2}} \\ \partial_{x_d}^2 K(x, y) &= \frac{-2}{|B_1|} \frac{(x_d - y_d)}{|x - y|^{d+2}} + \frac{4x_d}{|B_1||x - y|^{d+2}} - \frac{2(d+2)|B_1|^2}{x} \frac{(x_d - y_d)}{d|x - y|^{d+4}} \end{aligned}$$

Then:

$$\begin{aligned} \Delta_x K(x, y) &= \sum_{i=1}^{d-1} \partial_{x_i}^2 K(x, y) + \partial_{x_d}^2 K(x, y) \\ &= -\frac{2x_d}{|B_1|} \left[\frac{d-1}{|x - y|^{d+2}} - (d+2) \sum_{i=1}^{d-1} \frac{(x_i - y_i)^2}{|x - y|^{d+4}} \right. \\ &\quad \left. + \frac{1+2}{|x - y|^{d+2}} - \frac{(d+2)x_d(x_d - y_d)}{|x - y|^{d+4}} \right] \\ &= -\frac{2x_d}{|B_1|} \left[\frac{d+2}{|x - y|^{d+2}} - (d+2) \frac{1}{|x - y|^{d+4}} \left(\underbrace{\sum_{i=1}^d |x_i - y_i|^2}_{|x - y|^2} \right) \right] = 0 \end{aligned}$$

for all $x \in \mathbb{R}_+^d$, $y \in \partial\mathbb{R}_+^d$. Claim (exercise) for all $x \in \mathbb{R}_+^d$,

$$\int_{\partial\mathbb{R}_+^d} K(x, y) dy = 1$$

Consider

$$u(x) = \int_{\partial\mathbb{R}_+^d} K(x, y) g(y) dy, \quad x \in \mathbb{R}_+^d$$

Since $g \in L^\infty(\mathbb{R}^{d-1}) = L^\infty(\partial\mathbb{R}_+^d)$ and $K(x, y) \geq 0$, hence

$$|u(x)| \leq \left(\int_{\partial\mathbb{R}_+^d} K(x, y) dy \right) \|g\|_{L^\infty}$$

Thus $\|u\|_{L^\infty} \leq \|g\|_{L^\infty}$. Moreover

$$D_x^\alpha u(x) = \int_{\partial\mathbb{R}_+^d} D_x^\alpha K(x, y) g(y) dy$$

bounded, so $u \in C^\infty(\mathbb{R}_+^d)$, $x \mapsto K(x, y)$ is smooth as $x \neq y$.

$$\Delta_x u(x) = \int_{\partial\mathbb{R}_+^d} \underbrace{\Delta_x K(x, y)}_{=0} g(y) dy = 0$$

So u is harmonic in \mathbb{R}_+^d . ($\Rightarrow u \in C^\infty$ by Weyl's lemma). Take $x_0 \in \partial\mathbb{R}_+^d$ and $x \in \mathbb{R}_+^d$. Then:

$$\begin{aligned} |u(x) - g(x_0)| &= \left| \int_{\partial\mathbb{R}_+^d} K(x, y) (g(y) - g(x_0)) dy \right| \\ &\leq \int_{\partial\mathbb{R}_+^d} K(x, y) |g(y) - g(x_0)| dy \\ &= \underbrace{\int_{|y-x_0| \leq L|x-x_0|}}_{(I)} + \underbrace{\int_{|y-x_0| > L|x-x_0|}}_{(II)} \end{aligned}$$

$$\begin{aligned} (I) &= \int_{|y-x_0| \leq L|x-x_0|} K(x, y) |g(y) - g(x_0)| dy \\ &= \sup_{|y-x_0| \leq L|x-x_0|} |g(y) - g(x_0)| \xrightarrow{x \rightarrow x_0} 0 \quad \forall L > 0 \end{aligned}$$

(II): If $|y - x_0| > L|x - x_0|$, then $|y - x| > \frac{1}{2}|y - x_0| > \frac{L}{2}|x - x_0|$ if $L \geq 2$.

$$\begin{aligned} \int_{|y-x_0| > L|x-x_0|} K(x, y) |g(y) - g(x_0)| dy &\leq C \int_{y \in \partial\mathbb{R}_+^d} \frac{x_d}{|x_0 - y|} dy \\ C x_d \int_{\substack{z \in \mathbb{R}^{d-1} \\ |z| > L|x-x_0|}} \frac{1}{|z|^d} dz &= \text{const.} \frac{x_d}{L|x-x_0|} \leq \frac{\text{const.}}{L} \xrightarrow{L \rightarrow \infty} 0 \end{aligned}$$

$$x_d = |x_d - (x_0)_d| \leq |x - x_0|$$

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