

Partial Differential Equations
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Contents

1	Introduction	3
2	Laplace / Poisson Equation	5
2.1	Laplace Equation	5
2.2	Poisson-Equation	6
2.3	Equations in general domains	8
3	Convolution, Fourier Transform and Distributions	16
3.1	Convolutions	16
3.2	Fourier Transformation	24
3.3	Theory of Distribution	27

Please note that I write this lecture notes for my personal use. I may write things differently than presented in the lecture. This script also contains solutions for exercises (which may be wrong). Of course, I don't push them to GitHub while the exercises still can be handed in.

Chapter 1

Introduction

A differential equation is an equation of a function and its derivatives.

Example 1.1 (Linear ODE) Let $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{cases} f'(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is: $f(t) = a_0 e^{at}$ for all $t \geq 0$.

Example 1.2 (Non-Linear ODE) $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$. Then we have

$$f'(t) = \frac{1}{\cos^2(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in $(-\pi, \pi)$. It's a problem to extend this to $\mathbb{R} \rightarrow \mathbb{R}$.

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

Remark 1.3 Recall for $\Omega \subseteq \mathbb{R}^d$ open and $f : \Omega \rightarrow \{\mathbb{R}, \mathbb{C}\}$ the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x + he_i) - f(x)}{h}$, where $e_i = (0, 0, \dots, 1, \dots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f(x)$, where $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial_{x_1}, \dots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \dots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha|=k}$, where $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i, j \leq d}$

Definition 1.4 Given a function F . Then the equation of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function $u : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$ is called a *PDE of order k* .

- Equations $\sum_d a_\alpha(x) D^\alpha u(x) = 0$, where a_α and u are unknown functions are called *Linear PDEs*.
- Equations $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1} u, D^{k-2} u, \dots, Du, u, x) = 0$ are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explicit solution! This is in many cases impossible.
- Prove a *well-posed theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g. $\Delta u = f \rightsquigarrow u \in C^2$)
2. Weak Solutions: The solution is not smooth/continuous

Definition 1.5 (Spaces of continuous and differentiable functions) Let $\Omega \subseteq \mathbb{R}^d$ be open

$$C(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ continuous}\}$$

$$C^k(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \leq k\}$$

Classical solution of a PDE of order $k \rightsquigarrow C^k$ solutions!

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ Lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{ exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation: $-\Delta u = f$
- Heat Equation: $\partial_t u - \Delta u = f$
- Wave Equation: $\partial_t^2 u - \Delta u = f$
- Schrödinger Equation: $i\partial_t u - \Delta u = f$

Chapter 2

Laplace / Poisson Equation

2.1 Laplace Equation

$-\Delta u = 0$ (Laplace) or $-\Delta u = f(x)$ (Poisson).

Definition 2.1 (Harmonic Function) Let Ω be an open set in \mathbb{R}^d . If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω , then u is a harmonic function in Ω .

Theorem 2.2 (Gauss-Green Theorem) Let $A \subseteq \mathbb{R}^d$ open, $\vec{F} \in C^1(A, \mathbb{R}^d)$ and $K \subseteq A$ compact with C^1 boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \, dS(x) = \int_K \operatorname{div}(\vec{F}) \, dx$$

where ν is the outward unit normal vector field on ∂K . Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \, dS(x) = \int_V \operatorname{div}(\nabla u) \, dx = \int_V \Delta u(x) \, dx$$

for any $V \subseteq \Omega$ open.

Theorem 2.3 (Green's Identities) Let $A \subseteq \mathbb{R}^d$ open, $K \subseteq A$ d-dim. compactum with C^1 boundary and $f, g \in C^2(A)$

1. Green's first identity (Partial Integration):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

where $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f)g \, dx = \int_{\partial K} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) \, dS$$

Exercise 2.4 Let $\Omega \subseteq \mathbb{R}^d$ open, let $f : \Omega \rightarrow \mathbb{R}$ be continuous. Prove that if $\int_B f(x) \, dx = 0$, then $u \equiv 0$ in Ω .

Theorem 2.5 (Fundamental Lemma of Calculus of Variations) Let $\Omega \subseteq \mathbb{R}^d$ open, let $f \in L^1(\Omega)$. If $\int_B f(x) \, dx = 0$ for all $x \in B_r(x) \subseteq \Omega$, then $f(x) = 0$ a.e. (almost everywhere) $x \in \Omega$.

Remark 2.6 (Solving Laplace Equation) $-\Delta u = 0$ in \mathbb{R}^d . Consider the case when u is radial, i.e. $u(x) = v(|x|)$, $v : \mathbb{R} \rightarrow \mathbb{R}$. Denote $r = |x|$, then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i} \left(\sqrt{x_1^2 + \cdots + x_d^2} \right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned} \partial_{x_i} u &= \partial_{x_i} v = (\partial_r v) \frac{\partial r}{\partial x_i} = v'(r) \frac{x_i}{r} \\ \partial_{x_i}^2 u &= \partial_{x_i} \left(v(r) \frac{x_i}{r} \right) = (\partial_{x_i} v(r))' \frac{x_i}{r} + v'(r) \partial_{x_i} \left(\frac{x_i}{r} \right) \\ &= (\partial_r v'(r)) \left(\frac{dr}{\partial x_i} \right) \frac{x_i}{r} + v'(r) \left(\frac{1}{r} - \frac{x_i}{r^2} (\partial_{x_i} r) \right) = v'(r) \frac{x_i^2}{r^2} + v'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right) \end{aligned}$$

So we have $\Delta u = \left(\sum_{i=1}^d \partial_{x_i}^2 \right) u = v''(r) + v'(r) \left(\frac{d}{r} - \frac{1}{r} \right)$

Thus $\Delta u = v'(r) + v(r) \frac{d-1}{r}$. We consider $d \geq 2$. Laplace operator $\Delta u = 0$ now becomes $v''(r) + v'(r) \frac{d-1}{r} = 0$

$$\begin{aligned} \Rightarrow \log(v(r))' &= \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)' \quad (\text{recall } \log(f)' = \frac{f'}{f}) \\ \Rightarrow v'(r) &= \frac{1}{v^{d-2} + \text{const.}} \end{aligned}$$

$$\begin{cases} \frac{\text{const}}{r^{d-2}} + \text{const}x + \text{const} & , d \geq 3 \\ \text{const} \log(r) + \text{const}x + \text{const} & , d = 2 \end{cases}$$

Definition 2.7 (Fundamental Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|} \frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where $|B_1|$ is the Volume of the ball $B_1(0) = B(0, 1) \subseteq \mathbb{R}^d$.

Remark 2.8 $\Delta \Phi(x) = 0$ for all $x \in \mathbb{R}^d$ and $x \neq 0$.

2.2 Poisson-Equation

The Poisson-Equation is $-\Delta u(x) = f(x)$ in \mathbb{R}^d . The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy$$

This can be heuristically justified with

$$-\Delta(\Phi \star f) = (-\Delta \Phi) \star f = \delta_0 \star f = f$$

Theorem 2.9 Assume $f \in C_c^2(\mathbb{R}^d)$. Then $u = \Phi \star f$ satisfies that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$

Proof. By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy.$$

First we check that u is continuous: Take $x_k \rightarrow x_0$ in \mathbb{R}^d . We prove that $u(x_n) \xrightarrow{n} u_0$, i.e.

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y) dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n \rightarrow \infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x - y)| \leq \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \leq R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where $R > 0$ depends on $\{x_n\}$ and $\text{supp}(f)$ but independent of y . Now we compute the derivatives:

$$\begin{aligned} \partial_{x_i} u(x) &= \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x - y) dy = \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + h e_i - y) - f(x - y)}{h} dy \\ (\text{dom. conv.}) &= \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x - y) dy \\ \Rightarrow D^\alpha u(x) &= \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x - y) dy \quad \text{for all } |\alpha| \leq 2 \end{aligned}$$

$D^\alpha u(x)$ is continuous, thus $u \in C^2(\mathbb{R}^d)$. Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y) (-\Delta_x) f(x - y) dy = \int_{\mathbb{R}^d} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy + \int_{B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x - y) dy - \int_{\partial B(0, \epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x - y) dy \\ &\quad + \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) - \int_{\partial B(0, \epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \end{aligned}$$

We have that $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$ and $\vec{n} = \frac{y}{|y|}$ in $\partial B(0, \epsilon)$. This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1| \epsilon^{d-1}} \quad \text{for } y \in \partial B(0, \epsilon)$$

Hence:

$$\begin{aligned} \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) &= \frac{1}{d|B_1| \epsilon^{d-1}} \int_{\partial B(0, \epsilon)} f(x - y) dS(y) \\ &= \oint_{\partial B(0, \epsilon)} f(x - y) dS(y) = \oint_{\partial B(x, \epsilon)} f(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} f(x) \end{aligned}$$

We have to regard the following error terms:

$$\begin{aligned}
\bullet \left| \int_{B(0,\epsilon)} \Phi(y) (-\Delta_y) f(x-y) dy \right| &\leq \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leq \|\Delta f\|_{L^\infty} \mathbb{1}(|y| \leq R)} dy \\
&\leq \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)| \mathbb{1}(|y| \leq R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y| \leq \epsilon) \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

Where $R > 0$ depends on x and the support of f but is independent of y .

$$\begin{aligned}
\bullet \left| \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y) dS(y) \right| &\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)| dy \\
&\leq \begin{cases} \text{const} \cdot \epsilon |\log \epsilon| \rightarrow 0, & d = 2 \\ \text{const} \cdot \epsilon \rightarrow 0, & d \geq 3 \end{cases}
\end{aligned}$$

Conclusion: $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ proved that $u = \Phi \star f$ and $f \in C_c^2(\mathbb{R}^d)$. ■

Thus, if $f \in C_c^2(\mathbb{R})$, then $u = \Phi \star f$ satisfies $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$.

Remark 2.10 The result holds for a much bigger class of functions f . For example if $f \in C_c^1(\mathbb{R})$ we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x-y) dy \in C(\mathbb{R}^d) \Rightarrow u \in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i} \partial_{x_j} u = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) \partial_{x_j} f(x-y) dy = \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_j} f(x-y) dy \in C(\mathbb{R}^d)$$

So we have $u \in C^2(\mathbb{R}^d)$. Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_i} f(x-y) dy \stackrel{(IBP)}{=} f(x).$$

Exercise 2.11 Extend this to more general functions!

2.3 Equations in general domains

Theorem 2.12 (Mean Value Theorem for Harmonic Functions) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω . Then

$$u(x) = \oint_{B(x,r)} u = \oint_{\partial B(x,r)} u \quad \text{for all } x \in \Omega, B(x,r) \subseteq \Omega$$

Proof. Consider all $r > 0$ s.t. $B(x,r) \subseteq \Omega$,

$$f(r) = \oint_{\partial B(x,r)} u$$

We need to prove that $f(r)$ is independent of r . When it is done, then we immediately obtain

$$f(r) = \lim_{t \rightarrow 0} f(t) = u(x)$$

as u is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr} \left(\oint_{\partial B(0,r)} u(x+y) dS(y) \right) \\
&= \frac{d}{dr} \left(\oint_{\partial B(0,1)} u(x+rz) dS(z) \right) \\
(\text{dom. convergence}) \quad &= \oint_{\partial B(0,1)} \frac{d}{dr} [u(x+rz)] dS(z) \\
&= \oint_{\partial B(0,1)} \nabla u(x+rz) z dS(z) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r} dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green 2.2}) \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0} dy = 0 \quad \blacksquare
\end{aligned}$$

Exercise 2.13 In 1D: $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$ (Linear Equation)

Remark 2.14 Recall the polar decomposition. Let $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, w \in S^{d-1}$, then

$$\int_{B(0,r)} g(y) dy = \int_0^r \left(\int_{B(0,s)} g(y) dS(y) \right) ds$$

Remark 2.15 We already proved that for u harmonic we have $u(x) = \oint_{\partial B(x,r)} u dy$. Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_{B(0,r)} u(x+y) dy \\
(\text{Pol. decomposition}) \quad &= \int_0^r \left(\int_{\partial B(0,s)} u(x+y) dS(y) \right) ds \\
&= \int_0^r \left(\int_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\text{Mean value property}) \quad &= \int_0^r (|\partial B(x,s)| u(x)) ds = |B(x,r)| u(x)
\end{aligned}$$

This implies

$$\oint_{B(x,r)} u(y) dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

Remark 2.16 The reverse direction is also correct, namely if $u \in C^2(\Omega)$ and

$$u(x) = \oint_{B(x,r)} u(y) dy = \oint_{\partial B(x,r)} u(y) dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then u is harmonic, i.e. $\Delta u = 0$ in Ω . (The proof is exactly like before)

Theorem 2.17 (Maximum Principle) Let $\Omega \subseteq \mathbb{R}^d$ be open, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u = 0$ in Ω . Then

- a) $\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
- b) Assume that Ω is connected. Then if there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$, then $u \equiv \text{const.}$ in Ω .

Proof. Given $U \subseteq \mathbb{R}^d$ open, we can write $U = \bigcup_i U_i$, where U_i is open and connected.

- b) Assume that Ω is connected and there is a $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{y \in \Omega} u(y)$. Define $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$. U is closed since u is continuous. Moreover, U is open by the mean-value theorem. I.e. for all $x \in U$ there is a $r > 0$ s.t. $B(x, r) \subseteq U \subseteq \Omega$. Since U is connected we get $U = \Omega$, so u is constant in Ω . On the other hand, if there is no $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \bar{\Omega}} u(x)$ we have $\forall x_0 \in \Omega : u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
- a) Given $\Omega \subseteq \mathbb{R}^d$ open, we can write $\Omega = \bigcup_i \Omega_i$, where Ω_i is open and connected. By b) we have

$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$

So we can conclude

$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x). \quad \blacksquare$$

Definition 2.18 • If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$, then u is called *sub-harmonic* if $\Delta u \geq 0$ in Ω .

- If $\Delta u \leq 0$, then u is called *super-harmonic*.

Exercise 2.19 (E 1.4) Let $\Omega \subseteq \mathbb{R}^d$ be open and $u \in C^2(\Omega)$ be subharmonic.

- a) Prove that u satisfies the Mean Value Inequality

$$\oint_{\partial B(x, r)} u(y) dS(y) \geq \int_{B(x, r)} u(y) dy \geq u(x)$$

for all $B(x, r) \subseteq \mathbb{R}^d$.

- b) Assume further that Ω is connected and $u \in C(\bar{\Omega})$. Prove that u satisfies the strong maximum principle, namely either
 - u is constant in Ω , or
 - $\sup_{y \in \partial\Omega} u(y) > u(x)$ for all $x \in \Omega$.

My Solution. a) Let $f(r) = \oint_{\partial B(x,r)} u(y) dS(y)$, then we have

$$\begin{aligned}
\partial_r f(r) &= \partial_r \oint_{\partial B(x,r)} u(y) dS(y) \\
(\text{Dom. Convergence}) \quad &= \oint_{\partial B(x,r)} \partial_r u(y) dS(y) \\
&= \oint_{\partial B(0,1)} \partial_r u(x + yr) dS(y) \\
&= \oint_{\partial B(0,1)} \nabla u(x + yr) \cdot y dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r} dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green}) \quad &= \oint_{B(x,r)} \text{div}(\nabla u(y)) dS(y) \\
&= \oint_{B(x,r)} \underbrace{\Delta u(y)}_{\geq 0} dS(y) \geq 0
\end{aligned}$$

So we can conclude that

$$\oint_{\partial B(x,r)} u(y) dS(y) = f(r) \geq \lim_{r \rightarrow 0} f(r) = u(x).$$

Now regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left(\int_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&= \int_0^r \left(|\partial B(x,r)| \oint_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&\geq \int_0^r |\partial B(x,r)| \cdot u(x) dS(y) \\
&= u(x) \int_0^r |\partial B(x,r)| dS(y) = u(x) |B(x,r)|.
\end{aligned}$$

Thus we have

$$u(x) \leq \oint_{B(x,r)} u(y) dy.$$

Finally, lets regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left(|\partial B(x,s)| \oint_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\partial_r f(r) \geq 0) \quad &\leq \int_0^r \left(|\partial B(x,s)| \oint_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \int_0^r |\partial B(x,s)| ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \cdot |B(x,s)|
\end{aligned}$$

and we conclude

$$\oint_{B(x,r)} u(y) dy \leq \oint_{\partial B(x,r)} u(y) dS(y).$$

b) Let $x_0 \in \Omega$ s.t. $u(x_0) = \sup_{x \in \Omega} u(x)$. Now,

$$\begin{aligned} \sup_{x \in \Omega} u(x) = u(x_0) &\leq \oint_{\partial B(x_0,r)} u(y) dy \\ &\leq \oint_{\partial B(x_0,r)} \sup_{x \in \Omega} u(x) dy = \sup_{x \in \Omega} u(x) \end{aligned}$$

Since u is continuous we get $u(y) = u(x_0)$ for all $y \in B(x_0, r)$, so u is constant. ■

Definition 2.20 The *Poisson Equation* for given f, g on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Theorem 2.21 (Uniqueness) Let $\Omega \subseteq \mathbb{R}^d$ be bounded, open and connected. Let $f \in C(\Omega), g \in C(\partial\Omega)$. Then there exists *at most* one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Proof. Assume that we have two solutions u_1 and u_2 . Then $u := u_1 - u_2$ is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that $u = 0$ in Ω . More precisely, by the maximum principle we have $\forall x \in \Omega$

$$\sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leq 0$$

Since $-u$ satisfies the same property we have $\forall x \in \Omega$:

$$\sup_{x \in \Omega} (-u(x)) \leq \sup_{x \in \partial\Omega} (-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leq 0 \quad \Rightarrow \quad u(x) \geq 0$$

So we get $u(x) = 0$ in Ω . ■

Exercise 2.22 (Bonus 1) Let Ω be open, connected and bounded in \mathbb{R}^d . Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Prove that

a) If $g \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω .

b) If $g \geq 0$ on $\partial\Omega$ and $g \neq 0$, then $u > 0$ in Ω .

My Solution. a) We have that $\Delta(-u) = 0$, so $-u$ is harmonic in Ω . Since Ω is open and bounded we can apply the Maximum Principle (2.17) and get that

$$\sup_{x \in \bar{\Omega}} -u(x) \leq \sup_{x \in \partial\Omega} -g(x) \leq 0.$$

This implies $\inf_{x \in \Omega} u(x) \geq 0$, so $u \geq 0$ in Ω .

b) We prove this by contraposition. Assume there is a $x_0 \in \Omega$ s.t. $u(x_0) = 0$. Since we have $u \geq 0$ on Ω by a), it follows that

$$0 = -u(x_0) = \sup_{x \in \bar{\Omega}} -u(x) \leq \sup_{x \in \partial\Omega} -g(x) \leq 0,$$

so $-u$ attains its maximum on Ω . Hence $-u = 0 = u$ is constant by the strong maximum principle because Ω is connected, in fact $0 = u|_{\partial\Omega} = g$. ■

Lemma 2.23 (Estimates for derivatives) If u is harmonic in $\Omega \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| = N$ and $B(x_0, r) \subseteq \Omega$, then

$$|D^\alpha u(x)| \leq \frac{(c_d N)^N}{r^{d+N}} \int_{B(x, r)} |u| dy$$

Proof. Induction: Assume $|\alpha| = N - 1$, Take $|\alpha| = N$

$$|D^\alpha u(x_0)| \leq \frac{|S_1|}{|B_1| \frac{r}{N}} \|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))}, \quad D^\alpha u = \partial_{x_i}(D^\beta u)|_{|\beta|=N-1}$$

Note: $x \in B(x_0, \frac{r}{N})$, so $B(x, \frac{r(N-1)}{N}) \subseteq B(x_0, r)$. By the induction hypothesis:

$$\|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))} \leq \frac{[c_d(N-1)]^{N-1}}{[r \frac{(N-1)}{N}]^{d+N-1}} \int_{B(x_0, r)} |u| dy$$

The conclusion is:

$$\begin{aligned} |D^\alpha u(x_0)| &\leq \frac{|S_1|}{|B_1| \frac{r}{N}} \frac{[c_d(N-1)]^{N-1}}{(r \frac{N-1}{N})^{d+N-1}} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} (N-1)^d} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} N^d} \left(\frac{N}{N-1}\right)^d \int_{B(x_0, r)} |u| dy \\ &\leq \frac{2^d |S_1|}{|B_1|} \frac{c_d^{N-1} N^N}{r^{d+N}} \int_{B(x_0, r)} |u| dy \quad \text{if } c_d \geq \frac{2^d |S_1|}{|B_1|} \end{aligned}$$

■

Theorem 2.24 (Regularity) Let Ω be open in \mathbb{R}^d . Let $u \in C(\Omega)$ satisfy $u(x) = \int_{\partial B} u dy$ for any $x \in B(x, r) \subseteq \Omega$. Then u is a harmonic function in Ω . Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω .

Exercise 2.25 (E 1.1: Proof the Gauss–Green formula) Let $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that for every open ball $B(y, r) \subseteq \mathbb{R}^d$ we have

$$\int_{\partial B(y, r)} f(y) \cdot \nu_y dS(y) = \int_{B(y, r)} \operatorname{div} f dx.$$

Here ν_y is the outward unit normal vector and dS is the surface measure on the sphere.

Solution. We proof this in $d=3$. Let $f \in C^1(\mathbb{R}^3)$

$$\int_{B(0,1)} \partial_{x_3} f dx = \int_{\partial B(0,1)} f x_3 dS(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \vec{n} = \frac{x}{|x|} \text{ on } \partial B(0,1)$$

$$\begin{aligned} B(0,1) &= \{x_1^2 + x_2^2 + x_3^2 \leq 1\} \\ &= \{x_1^2 + x_2^2 \leq 1 - \sqrt{1 - x_1^2 - x_2^2} \leq x_3 \leq \sqrt{1 - x_1^2 - x_2^2}\} \end{aligned}$$

Then:

$$\begin{aligned} \int_{B(0,1)} \partial_{x_3} f dx &= \int_{x_1^2 + x_2^2 \leq 1} \left(\int_{-\sqrt{1-x_1^2-x_2^2} \leq x_3 \leq \sqrt{1-x_1^2-x_2^2}} \partial_{x_3} f dx_3 \right) dx_1 dx_2 \\ &= \int_{x_1^2 + x_2^2 \leq 1} \left[f(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \right. \\ &\quad \left. - f(x_1, x_2, -\sqrt{1-x_1^2-x_2^2}) \right] dx_1 dx_2 \end{aligned}$$

Lets take polar coordinates in 2D:

$$\begin{aligned} x_1 &= r \cos \phi & r > 0, \phi \in [0, 2\pi) \\ x_2 &= r \sin \phi & \det \frac{\partial(x_1, x_2)}{\partial(r, \phi)} = r \end{aligned}$$

$$(\star) = \int_0^1 \int_0^{2\pi} [f(r \cos \phi, r \sin \phi, r) - f(r \cos \phi, r \sin \phi, -r)] r dr d\phi$$

On the other hand:

$$\int_{\partial B(0,1)} f x_3 dS$$

The polar coordinates in 3D are:

$$\begin{aligned} x_1 &= r \cos \phi \sin \theta & r > 0, \phi \in (0, 2\pi), \theta \in (0, \pi) \\ x_2 &= r \sin \phi \sin \theta & \det \frac{\partial(x_1, x_2, x_3)}{\partial(r, \phi, \theta)} = r^2 \sin \theta \\ x_3 &= r \cos \theta \end{aligned}$$

Then:

$$\begin{aligned} (\star\star) &= \int_0^{2\pi} \int_0^\pi f(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \sin \theta \cos \theta d\theta d\phi \\ &= \int_0^{2\pi} \left(\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^\pi d\theta \right) d\phi \\ (r = \sin \theta) &= \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, \sqrt{1-r^2}) r dr d\phi \\ &\quad - \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, -\sqrt{1-r^2}) r dr d\phi \end{aligned} \quad \blacksquare$$

Exercise 2.26 (E 1.2) Let $u \in C(\mathbb{R}^d)$ and $\int_{B(x,r)} u \, dy = 0$ for every open ball $B(x,r) \subseteq \mathbb{R}^d$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

My Solution. Assume there is a $x_0 \in \mathbb{R}^d$ s.t. w.l.o.g. $u(x_0) > 0$. Since u is continuous there is a ball $B(x_0, r)$ s.t. $u(y) > \frac{u(x_0)}{2}$ for all $y \in B(x_0, r)$. But then we get

$$\int_{B(x_0, r)} u(y) \, dy \geq \int_{B(x_0, r)} \frac{u(x_0)}{2} \, dy = \frac{u(x_0)}{2} |B(x_0, r)| > 0. \quad \blacksquare$$

Exercise 2.27 (E 1.3) Let $f \in C_c^1(\mathbb{R}^d)$ with $d \geq 2$ and $u(x) := (\Phi \star f)(x)$. Prove that $u \in C^2(\mathbb{R}^2)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$ (2.9 was the same for $f \in C_1(\mathbb{R})$)

Theorem 2.28 (Liouville's Theorem) If $u \in C^2(\mathbb{R}^d)$ is harmonic and bounded, then $u = \text{const.}$

Proof. By the bound of the derivative 2.23 we have

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\leq \frac{c_d}{r^{d+1}} \int_{B(x_0, r)} |u| \, dy \quad \forall x_0 \in \mathbb{R}^d \quad \forall r > 0 \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r^{d+1}} |B(x_0, r)| \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r} \xrightarrow{r \rightarrow \infty} 0 \end{aligned}$$

Thus $\partial_{x_i} u = 0$ for all $i = 1, 2, \dots, d$ and $u = \text{const.}$ in \mathbb{R}^d ■

Theorem 2.29 (Uniqueness of solutions to Poisson Equation in \mathbb{R}^d) If $u \in C^2(\mathbb{R}^d)$ is a bounded function and satisfies $-\Delta u = f$ in \mathbb{R}^d where $f \in C_c^2(\mathbb{R}^d)$, then we have

$$u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y) \, dy + C \quad \forall x \in \mathbb{R}^d$$

where C is a constant and Φ is the fundamental solution of the Laplace equation in \mathbb{R}^d .

Proof. If we can prove that v is bounded, then $v = \text{const.}$. We first need to show that $\Phi \star f$ is bounded.

$$\begin{aligned} \Phi &= \Phi_1 + \Phi_2 = \Phi \mathbb{1}(|x| \leq 1) + \Phi(|x| \geq 1) \\ \Phi \star f &= \Phi_1 \star f + \Phi_2 \star f \end{aligned}$$

We have $\Phi_1 \star f \in L^1(\mathbb{R}^d)$ and $\Phi_2 \star f$ is bounded since $\Phi \rightarrow 0$ as $|x| \rightarrow \infty$ in $d \geq 3$. ■

Exercise 2.30 (Hanack's inequality) Let $u \in C^2(\mathbb{R}^d)$ be harmonic and non-negative. Prove that for all open, bounded and connected $\Omega \subseteq \mathbb{R}^d$, we have

$$\sup_{x \in \Omega} u(x) \leq C_\Omega \inf_{x \in \Omega} u(x),$$

where C_Ω is a finite constant depending only on Ω .

Proof. (Exercise) Hint: $\Omega = B(x, r)$. General case cover Ω by finitely many balls, one ball is inside Ω . ■

Chapter 3

Convolution, Fourier Transform and Distributions

3.1 Convolutions

Definition 3.1 (Convolution) Let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ or \mathbb{C} .

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) dy = \int_{\mathbb{R}^d} f(y)g(x-y) dy = (g \star f)(x)$$

Remark 3.2 (Properties of the Convolution) • $(f \star g)(x) = f \star (g \star h)$

$$\bullet \hat{f \star g} = \hat{f} \star \hat{g}$$

Theorem 3.3 (Young Inequality) If $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$, where $1 \leq p \leq \infty$, then $f \star g \in L^p(\mathbb{R}^d)$ and $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$. More generally, if $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, then $f \star g \in L^r(\mathbb{R}^d)$, $\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$, where $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

Proof. Let $f \in L^1, g \in L^p$. With the Hölder Inequality ??, we have:

$$\begin{aligned} \|f \star g\|_{L^p}^p &= \int_{\mathbb{R}^d} |f \star g(x)|^p dx \\ &\leq \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)| |g(y)|^p dy dx \\ &= \|f\|_{L^1}^{\frac{p}{q}+1} \|g\|_{L^p}^p \end{aligned}$$

So we have $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$ ■

Theorem 3.4 (Smoothness of the Convolution) If $f \in C_c^\infty(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. Then $f \star g \in C^\infty(\mathbb{R})$ and

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for all $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \{0, 1, 2, \dots\}$

Proof. First we note that $x \mapsto (f \star g)$ is continuous as $x_n \rightarrow x$ in \mathbb{R}^d since

$$(f \star g)(x_n) = \int_{\mathbb{R}^d} f(x_n - y)g(y) dy \xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^d} f(x - y)g(y) dy = (f \star g)(x)$$

We can apply Dominated convergence because

$$f(x_n - y)g(y) \rightarrow f(x - y)g(y) \quad \forall y \text{ as } f \text{ is continuous and } x_n \rightarrow x$$

and

$$|f(x_n - y)g(y)| \leq \|f\|_{L^\infty} |g(y)| \mathbb{1}(|y| \leq R) \in L^1(\mathbb{R}^d).$$

Where $R > 0$ satisfies $B(0, R) \supseteq \text{supp } f + \sup_n |x_n|$. Now we can compute the derivatives:

$$\begin{aligned} \partial_{x_i}(f \star g)(x) &= \lim_{h \rightarrow 0} \frac{(f \star g)(x + he_i) - (f \star g)(x)}{h} \\ &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\ (\text{Dominated Convergence}) \quad &= \int_{\mathbb{R}^d} \lim_{h \rightarrow 0} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\ &= \int_{\mathbb{R}^d} (\partial_{x_i} f)(x - y)g(y) dy \end{aligned}$$

We could apply Dominated Convergence since

$$\begin{aligned} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) &\xrightarrow{h \rightarrow 0} (\partial_{x_i} f)(x - y)g(y) \quad \text{as } f \in C^1 \\ \left| \frac{f(x + he_i - y) - f(x - y)}{h} g(y) \right| &\leq \|\partial_{x_i} f\|_{L^\infty} |g(y)| \mathbb{1}(|y| \leq R) \in L^1(\mathbb{R}^d) \end{aligned}$$

where $B(0, R) \supseteq \text{supp}(f) + B(0, |x| + 1)$ and $\partial_{x_i}(f \star g) = (\partial_{x_i} f) \star g \in C(\mathbb{R}^d)$ since $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$. By induction we get $D^\alpha(f \star g) = (D^\alpha f \star g) \in C(\mathbb{R}^d)$. ■

Remark 3.5 Question: Is there a f s.t. $f \star g = g$ for all g . In fact there is no regular function f that solves this formally:

$$f \star g = g \Rightarrow \widehat{f \star g} = \hat{g} \Rightarrow \hat{f} \hat{g} = \hat{g} \Rightarrow \hat{f} = 1 \Rightarrow f \text{ is not a regular function!}$$

However, if f is the Dirac-Delta Distribution, $f = \delta_0$ then $\delta_0 \star g = g$ for all g . Formally:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \int \delta_0 = 1$$

In fact, if $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$, then $f_\epsilon \rightarrow \delta_0$ in an appropriate sense and $f_\epsilon \star g \rightarrow g$ for all g nice enough.

Theorem 3.6 (Approximation by convolution) Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon})$. Then for all $g \in L^p(\mathbb{R}^d)$, where $1 \leq p < \infty$, then

$$f_\epsilon \star g \rightarrow g \quad \text{in } L^p(\mathbb{R}^d)$$

Proof.

Step 1: Let $f, g \in C_c(\mathbb{R}^d)$. Then

$$\begin{aligned}
(f_\epsilon \star g)(x) - g(x) &= \int_{\mathbb{R}^d} f_\epsilon(y)g(x-y) dy - \int_{\mathbb{R}^d} f_\epsilon(y)g(x) dy \\
&= \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x)) dy \\
|(f_\epsilon \star g)(x) - g(x)| &= \left| \int_{\mathbb{R}^d} f_\epsilon(y)(g(x-y) - g(x)) dy \right| \\
&\leq \int_{\mathbb{R}^d} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\
&\leq \int_{|y| \leq R_\epsilon} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\
&\leq \underbrace{\int_{|y| \leq R_\epsilon} |f_\epsilon(y)| dy}_{\leq \|f_\epsilon\|_{L^1} = \|f\|_{L^1}} \left[\sup_{|z| \leq R} |g(x-z) - g(x)| \right] \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

We have Dominated Convergence since:

$$(f_\epsilon \star g)(x) - g(x) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

and

$$|f_\epsilon \star g(x) - g(x)| \leq \|f\|_{L^1} \sup_{|z| \leq R_\epsilon} |g(x-z) - g(x)| \leq 2\|f\|_1 \|g\|_{L^\infty} \mathbf{1}(|x| \leq R_1).$$

Where $B(0, R_1) \supseteq \text{supp}(g) + B(0, R_\epsilon)$, thus $f_\epsilon \star g \rightarrow g$ in $L^p(\mathbb{R}^d)$. To remove the technical assumptions $f, g \in C_c(\mathbb{R}^d)$, then we use a density argument. We use the fact that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

Step 2: Let $g \in C_c(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$. Then there is $\{g_m\} \subseteq L^p(\mathbb{R}^d), g_m \rightarrow g$ in $L^p(\mathbb{R}^d)$. Then

$$\begin{aligned}
\|f_\epsilon \star g - g\|_{L^p} &\leq \|f_\epsilon \star (g - g_m)\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
(\text{Young}) &\leq \|f_\epsilon\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leq \|f\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\
&\leq (\|f\|_{L^1} + 1) \|g - g_m\|_{L^p} + \|f \star g_m - g_m\|_{L^p}
\end{aligned}$$

So we get:

$$\begin{aligned}
\limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq (\|f\|_{L^p} + 1) \|g - g_m\|_{L^p} + \underbrace{\limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g_m - g_m\|_{L^p}}_{\text{by step 1.}} \\
&\xrightarrow{m \rightarrow \infty} 0
\end{aligned}$$

Step 3: Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Take $\{f_m\} \subseteq C_c(\mathbb{R}^d)$, s.t.

$$\begin{cases} F_m \rightarrow g \text{ in } L^1(\mathbb{R}) \text{ as } m \rightarrow \infty \\ \int_{\mathbb{R}^d} F_m = 1 \text{ (it is possible since } \int_{\mathbb{R}^d} f = 1) \end{cases}$$

Define $F_{m,\epsilon}(x) = \epsilon^{-d} F_m(\epsilon^{-1}x)$ (recall $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$). Then:

$$\begin{aligned}
f_\epsilon \star g - g &= (f_\epsilon - F_{m,\epsilon}) \star g + F_{m,\epsilon} \star g - g \\
\Rightarrow \|f_\epsilon - g\|_{L^p} &\leq \underbrace{\|f_\epsilon - F_{m,\epsilon} \star g\|_{L^p}}_{\text{Young}} + \|F_{m,\epsilon} \star g - g\|_{L^p} \\
&\leq \|f_\epsilon - F_{m,\epsilon}\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \\
\Rightarrow \limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq \|f - F_m\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \quad \blacksquare
\end{aligned}$$

Lemma 3.7 $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$

Proof. For all $g \in L^p(\mathbb{R}^d)$ there are g_m step functions and $g_m \rightarrow g$ in $L^p(\mathbb{R}^d)$. We can assume that Ω is open and bounded and we want to approximate χ_Ω by $C_c(\mathbb{R}^d)$. ■

Lemma 3.8 (Urnson) Define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \epsilon\}$$

Then there is a $\eta_\epsilon \in C_c(\mathbb{R}^d)$ s.t.

$$\begin{cases} 0 \leq \eta(x) \leq 1 & \forall x \in \mathbb{R}^d \\ \eta_\epsilon(x) = 1 & \text{if } x \in \Omega_\epsilon \\ \eta_\epsilon(x) = 0 & \text{if } x \notin \Omega \end{cases}$$

Lemma 3.9 (General Version of Urnson) If $A, B \subseteq \mathbb{R}^d$, A closed, B closed, $A \cap B = \emptyset$. Then

$$\eta(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}$$

Then $\eta \in C(\mathbb{R}^d)$, $0 \leq \eta \leq 1$ and $\eta = 0$ if $x \in B$, $\eta = 1$ if $x \in A$. Apply to $A = \overline{\Omega_\epsilon} \subset \subset \Omega$ and $B = \mathbb{R}^d \setminus \Omega$.

Theorem 3.10 (Appendix C4 in Evans) Let Ω be open in \mathbb{R}^d and for $\epsilon > 0$ define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \mathbb{R}^d \setminus \Omega) > \epsilon\}$$

Let $f \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} f = 1$, $\text{supp } f \subseteq B(0, 1)$, $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ $\text{supp } f_\epsilon \subseteq B(0, \epsilon)$. Then for all $g \in L_{loc}^p(\Omega)$ (i.e. $\mathbb{1}_K g \in L^p(\Omega) \forall K$ compact set in Ω), then:

- a) $g_\epsilon(x) = (f_\epsilon \star g)(x) = \int_{\mathbb{R}^d} f_\epsilon(x-y)g(y) dy = \int_\Omega f_\epsilon(x-y)g(y) dy$ is well-defined in Ω_ϵ and $g_\epsilon \in C^\infty(\Omega_\epsilon)$.
- b) $g_\epsilon \rightarrow g$ in $L_{loc}^p(\Omega)$ if $1 \leq p < \infty$ and $g_\epsilon(x) \rightarrow g(x)$ almost everywhere $x \in \Omega$.
- c) If $g \in C(\Omega)$, then $g_\epsilon(x) \rightarrow g(x)$ uniformly in any compact subset of Ω .

Proof. a) $D^\alpha(g_\epsilon) = (D^\alpha f_\epsilon) \star g \in C(\Omega_\epsilon)$

b) Already proved in \mathbb{R}^d space. ■

Corollary 3.11 (Lebesgue differentiation theorem) If $f \in L_{loc}^p(\mathbb{R}^d)$, then

$$\int_{B(x, \epsilon)} |f(y) - f(x)|^p dy \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

Exercise 3.12 (E 2.1) Let $u \in C^2(\mathbb{R}^2)$ be convex. I.e.

$$tu(x) + u(y)(1-t) \geq u(tx + (1-t)y) \forall x, y \in \mathbb{R}^d \forall t \in [0, 1]$$

- a) Prove for all $x \in \mathbb{R}^d$ that $H(x) = \dots$

Solution.

- a In 1D: If u is convex $\Leftrightarrow u''(x) \geq 0$ for all $x \in \mathbb{R}$. In general: Taylor expansion for all $x, z \in \mathbb{R}^d$:

$$u(x) = u(z) + \nabla u(z)(x - z) + \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{(x - z)^\alpha}{\alpha!} ds$$

$$x = z + s(x - z), s = 1 \text{ Use } z = tx + (1 - t)y \Rightarrow x - z = (1 - t)(x - y)$$

$$tu(x) = tu(z) + t\nabla u(z)(1 - t)(x - y) + t \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{[(1 - t)(x - y)]^\alpha}{\alpha!} ds$$

$$(1 - t)u(y) = (1 - t)u(z) + (1 - t)\nabla u(z)t(y - x) + (1 - t) \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y - z)) \frac{[t(y - x)]^\alpha}{\alpha!} ds$$

$$\begin{aligned} \Rightarrow tu(x) + (1 - t)u(y) &= u(z) + t \int_0^1 \dots + (1 - t) \int_0^1 \dots \\ \Rightarrow t \int_0^1 \dots + (1 - t) \int_0^1 \dots &\geq 0 \forall x, y, t, z = tx + (1 - t)y \end{aligned}$$

$$t(1 - t)^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x - z)) \frac{(x - y)^\alpha}{\alpha!} ds + (1 - t)t^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y - z)) \frac{(y - x)^\alpha}{\alpha!} ds \geq 0$$

for all $x, y \in \mathbb{R}^d, t \in [0, 1], z = tx + (1 - t)y$. Divides for $t(1 - t)$

$$(1 - t) \int_0^1 \dots + \int_0^1 \dots \geq 0$$

Take $t \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(y + s(x - y)) \frac{(x - y)^\alpha}{\alpha!} ds \geq 0 \forall x, y \in \mathbb{R}^d$$

Take $y = x + a, a \in \mathbb{R}^d$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x + a + sa) \frac{a^\alpha}{\alpha!} ds \geq 0 \forall \epsilon > 0, \forall x, a \in \mathbb{R}^d$$

Take $\epsilon \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x) \frac{a^\alpha}{\alpha!} \geq 0 \Rightarrow \sum_{i,j=1, i \neq j} \partial_{x_i} \partial_{x_j} u(x) a_i a_j + \sum_{i=1}^d \partial_{x_i}^2 u(x) \frac{a_i^2}{2}$$

We get

$$\frac{1}{2} a^T H a \geq 0 \forall a(a_i)_{i=1}^d \in \mathbb{R}^d$$

- b $H(x) \geq 0 \Rightarrow (\partial_i \partial_j u) \geq 0 \Rightarrow \text{Tr} H(x) \geq 0 \Rightarrow \sum_{i=1}^d \partial_{x_i}^2 u(x) \geq 0 \Rightarrow \Delta u(x) \geq 0 \forall x \in \mathbb{R}^d$

■

Exercise 3.13 (E 2.2)

Solution. Regard $d = 3$. De function $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$. We prove

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{\max(|x|, r)}$$

If $|x| > r$, then $0 \notin B(x, r + \epsilon)$. Then

$$y \mapsto \frac{1}{|y|}$$

is harmonic in $B(x, r + \epsilon)$. Then by the Mean Value Property:

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{|x|}$$

If $|x| < r$: Then $\frac{1}{|y|}$ is not harmonic in $B(x, r)$ since $0 \in B(x, r)$. Note

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \oint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$$

This function depends on x only via $|x|$.

$$\dots = \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - Ry|}$$

for all R rotation $SO(3)$, $dS(Ry) = dS(y)$

$$\begin{aligned} &= \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - y|} \\ &= \oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \\ \text{(Radial in } z) &= \oint_{\partial B(0,|x|)} \left(\oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \right) dS(z) \\ \text{(Fubini)} &= \oint_{\partial B(0,r)} \left(\oint_{\partial B(0,|x|)} \frac{dS(z)}{|z - y|} \right) dS(y) \\ \text{(case 1 since } |y| = r > |x|) &= \oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) = \frac{1}{r} \end{aligned}$$

If $|x| = r$: Continuity: $x \mapsto \oint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$ ■

Remark 3.14 For $f \in C^{|\alpha|}, g \in C^{|\beta|}$:

$$D^{\alpha+\beta}(f \star g) = (D^\alpha f) \star (D^\beta g)$$

Lemma 3.15 If $d \geq 3$ and $f : \mathbb{R}^d \rightarrow \mathbb{R}$ radial. Then:

$$\begin{aligned} \left(\frac{1}{|x|^{d-2}} \star f \right) (x) &= \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} dy \\ &= \int_{\mathbb{R}^d} \frac{f(y)}{\max(|x|^{d-2}, |y|^{d-2})} dy \end{aligned}$$

Proof. (d=3) Polar coordinates:

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{1}{|x-rw|} d\omega \right] f(r) dr \\
(a) \quad &= \int_0^\infty \left[\int_{\partial B(0,1)} \frac{d\omega}{\max(|x|, r)} \right] f(r) dr \\
&= \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|, |y|)} dy
\end{aligned}$$

(b) (d=3) If f radial and non-negative

$$\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \frac{f(y)}{|x|} dy = \frac{(Sf?)}{|x|}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|} dx dy &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x)f_2(y)}{|x+z_1-y-z_2|} dx dy \\
&= \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f_1(x) dx \right) f_2(y) dy \leq \int_{\mathbb{R}^3} \frac{(\int_{\mathbb{R}^3} f_1)}{|y+z_2-z_1|} f_2(y) dy \\
&\leq \frac{(\int_{\mathbb{R}^3} f_1)(\int_{\mathbb{R}^3} f_2)}{|z_1-z_2|}
\end{aligned}$$

■

Exercise 3.16 (Bonus 2) a) Prove that $u(x) = \frac{1}{|x|}$ is sub-harmonic in $\mathbb{R}^2 \setminus \{0\}$.

b) Prove that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ radial, non-negative, measurable:

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy \geq \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)} dy$$

My Solution. a) Let $x \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned}
\partial_{x_i} u &= \partial_{x_i} |x|^{-1} = -|x|^{-2} \frac{x_i}{|x|} = -x_i |x|^{-3} \\
\Rightarrow \partial_{x_i}^2 u &= \partial_{x_i} (-x_i |x|^{-3}) = -|x|^{-3} + 3x_i^2 |x|^{-5} \\
\Rightarrow \Delta u &= -|x|^{-3} + 3x_1^2 |x|^{-5} - |x|^{-3} + 3x_2^2 |x|^{-5} \\
&= -2|x|^{-3} + 3 \underbrace{(x_2^2 + x_2^2)}_{=|x|^2} |x|^{-5} = \frac{1}{|x|^3} > 0
\end{aligned}$$

So u is sub-harmonic in $\mathbb{R}^2 \setminus \{0\}$.

b) Let $r > 0, x \in \mathbb{R}^2$ and $|x| < r$. First we show that

$$\oint_{\partial B(x,r)} \frac{1}{|y|} dS(y) \geq \frac{1}{r} \quad (\star)$$

Now,

$$\oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) = \oint_{\partial B(0,r)} \frac{1}{|x-y|} dS(y) =: \tilde{u}(x)$$

Take $z \in \mathbb{R}^2 \setminus \{0\}$ such that $z = |x|$, then $\tilde{u}(x) = \tilde{u}(z)$. Let $0 < \epsilon < r$ be small. Then we get

$$\begin{aligned}
\tilde{u}(z) &= \oint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \\
\left(\begin{array}{l} |y| = r > |x| = |z| \\ \tilde{u} \text{ radial function} \end{array} \right) &= \oint_{\partial B(0,|x|-\epsilon)} \left(\oint_{\partial B(0,r)} \frac{dS(y)}{|z-y|} \right) dS(z) \\
(\text{Fubini}) &= \oint_{\partial B(0,r)} \left(\oint_{\partial B(0,|x|-\epsilon)} \frac{dS(z)}{|z-y|} \right) dS(y) \\
&= \oint_{\partial B(0,r)} \left(\oint_{\partial B(y,|x|-\epsilon)} \frac{dS(z)}{|z|} \right) dS(y) \\
\left(\frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2 \setminus \{0\} \right) &\geq \oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) \\
&= \oint_{\partial B(0,r)} \frac{1}{r} dS(y) \\
&= \frac{1}{r}
\end{aligned}$$

Now,

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy = \int_{|x| \geq |y|} \frac{f(y)}{|x-y|} dy + \int_{|x| < |y|} \frac{f(y)}{|x-y|} dy,$$

where

$$\begin{aligned}
\int_{|x| \leq |y|} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \int_{\partial B(0,r)} \frac{f(y)}{|x-y|} \mathbb{1}(|x| \leq |y|) dS(y) dr \\
(f \text{ radial}) &= \int_0^\infty f(r) \int_{\partial B(0,r)} \frac{\mathbb{1}(|x| \leq r)}{|x-y|} dS(y) dr \\
&= \int_0^\infty f(r) \int_{\partial B(x,r)} \frac{\mathbb{1}(|x| \leq r)}{|y|} dS(y) dr \\
(\star) &\geq \int_0^\infty \frac{f(r)}{r} |\partial B(x,r)| \mathbb{1}(|x| \leq r) dr \\
&= \int_0^\infty \int_{\partial B(x,r)} \frac{f(r)}{r} \mathbb{1}(|x| \leq r) dS(y) dr \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{|y|} \mathbb{1}(|x| \leq |y|) dy \\
&= \int_{|x| \leq |y|} \frac{f(y)}{|y|} dy
\end{aligned}$$

and

$$\begin{aligned}
\int_{|x|>|y|} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \left(\int_{\partial B(0,r)} \frac{f(r)}{|x-y|} \mathbb{1}(|x|>|y|) dS(y) \right) dr \\
(f \text{ radial}) &= \int_0^\infty f(r) \mathbb{1}(|x|>r) \left(\int_{\partial B(x,r)} \frac{1}{|y|} dS(y) \right) dr \\
\left(\begin{array}{l} \frac{1}{|y|} \text{ sub-harmonic in } \mathbb{R}^2, \\ \text{MVP and } |x|>r \end{array} \right) &\geq \int_0^\infty f(r) \mathbb{1}(|x|>r) |\partial B(x,r)| \frac{1}{|x|} dr \\
&= \int_0^\infty \int_{\partial B(x,r)} f(r) \mathbb{1}(|x|>r) \frac{1}{|x|} dS(y) dr \\
&= \int_{\mathbb{R}^2} f(y) \mathbb{1}(|x|>|y|) \frac{1}{|x|} dy \\
&= \int_{|x|>|y|} f(y) \frac{1}{|x|} dy.
\end{aligned}$$

So we can conclude,

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy &= \int_{|x|>|y|} \frac{f(y)}{|x-y|} dy + \int_{|x|\leq|y|} \frac{f(y)}{|x-y|} dy \\
&\geq \int_{|x|>|y|} \frac{f(y)}{|x|} dy + \int_{|x|\leq|y|} \frac{f(y)}{|y|} dy \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)} dy
\end{aligned}$$

■

3.2 Fourier Transformation

Definition 3.17 (Fourier Transform) For $f \in L^1(\mathbb{R}^d)$ define

$$\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k \cdot x} dx, \quad k \cdot x = \sum_{i=1}^d k_i x_i$$

Theorem 3.18 (Basic Properties) 1. If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^\infty(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

2. For all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$. Moreover, \mathcal{F} can be extended to be a unitary transformation $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ s.t.

$$\|\mathcal{F}g\|_{L^2} = \|g\|_{L^2} \quad \forall g \in L^2(\mathbb{R}^d)$$

3. The inverse of \mathcal{F} can be defined as

4.

$$(F^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(k) e^{2\pi i k \cdot x} dk$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

5. $\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k)$ as $(2\pi i k)^\alpha f(k) \in L^2(\mathbb{R}^d)$ ($k^\alpha = k_1^{\alpha_1} \dots k_\alpha^{\alpha_\alpha}$)

6. $\widehat{f \star g}(k) = \hat{f}(k) \hat{g}(k)$ if f, g are nice enough.

Theorem 3.19 (Hausdorff-Young-Inequality) If $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^n)$ then

$$\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}$$

and

$$\|\hat{f}\|_{L^p} \leq \|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^d)$$

Remark 3.20 We want to apply the Fourier transform to find the solution of a PDE, e.g. the Poisson-Equation:

$$-\Delta u = f \text{ in } \mathbb{R}^d \Rightarrow |2\pi k|^2 \hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{1}{|2\pi k|^2} \hat{f}(k)$$

If we can find G s.t. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$, then

$$\begin{aligned} \hat{u}(k) &= \hat{G}(k) \hat{f}(k) = \widehat{G \star f} \\ \Rightarrow u(x) &= (G \star f)(x) = \int_{\mathbb{R}^d} G(x-y) f(y) dy \end{aligned}$$

In fact G is the fundamental solution of laplace quation.

Theorem 3.21 (Fourier Transform of $\frac{1}{|x|^\alpha}$ for $0 < \alpha < d$) We have formally

$$\frac{\widehat{c_\alpha}}{|x|^\alpha} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \quad \forall 0 < \alpha < d$$

Here

$$c_\alpha = \pi^{-\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda$$

More precisely, for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{c_\alpha}{|x|^\alpha} \star f = \left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee$$

Moreover if $\alpha > \frac{d}{2}$, then we also have

$$\left(\frac{c_\alpha}{|x|^\alpha} \star f \right)^\wedge = \frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k)$$

Lemma 3.22 (Fourier Transform of Gaussians) In \mathbb{R}^d ,

$$\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$$

More generally for all $\lambda > 0$:

$$\widehat{e^{-\pi\lambda^2|x|^2}} = \lambda^{-d} e^{-\pi \frac{|k|^2}{\lambda^2}}$$

(exercise)

Proof of Theorem. Formally:

$$\begin{aligned}
\frac{c_\alpha}{|x|^\alpha} &= \frac{1}{|x|^\alpha} \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty e^{-\pi\lambda|x|^2} \lambda^{\frac{\alpha}{2}-1} d\lambda \\
\Rightarrow \frac{\hat{c}_\alpha}{|x|^\alpha}(k) &= \int_0^\infty \widehat{e^{-\pi\lambda|x|^2}}(k) \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi\frac{|k|^2}{\lambda}} \lambda^{\frac{\alpha}{2}-1} d\lambda \\
(\lambda \rightarrow \frac{1}{\lambda}) &= \int_0^\infty \lambda^{\frac{d}{2}} e^{-\pi|k|^2\lambda} \lambda^{-\frac{\alpha}{2}+1} \lambda^{-2} d\lambda \\
&= \frac{c_{d-\alpha}}{|k|^{d-\alpha}}
\end{aligned}$$

Let $f \in C_c(\mathbb{R}^d)$. Then $\left(\frac{1}{|x|^\alpha} \star f\right)(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} f(y) dy$ is well defined as $\frac{1}{|x-y|} \in L^1_{loc}(\mathbb{R}^d, dy)$. It is bounded

$$\frac{1}{|x|^\alpha} \star f = \frac{1}{|x|^\alpha} \underbrace{\mathbb{1}(|x| \leq 1)}_{\in L^\infty(\mathbb{R}^d)} \star \underbrace{f}_{L^\infty} + \frac{1}{|x|} \underbrace{\mathbb{1}(|x| > 1)}_{\in L^\infty} \star \underbrace{f}_{\in L^1} \in L^\infty(\mathbb{R}^d)$$

When $|x| \rightarrow \infty$:

$$\left(\frac{1}{|x|^\alpha} \star f\right)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^\alpha} dy = \int_{|y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy \sim \frac{\int_{\mathbb{R}^d} f(y) dy}{|x|^\alpha}$$

Note that $\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \underbrace{\hat{f}(k)}_{\text{bounded}} \in L^1(\mathbb{R}^d)$.

$$\begin{aligned}
(\dots)\mathbb{1}(|k| \leq 1) + (\dots)\mathbb{1}(|k| > 1) \frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| \leq 1) &\leq \|f\|_{L^1} \frac{\mathbb{1}(|k| \leq 1)}{|k|^{d-\alpha}} \in L^1(\mathbb{R}^d, dk) \\
\frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| > 1) &\leq |\hat{f}(k)| \in L^2(\mathbb{R}^d, dK) \text{ as } f \in L^2(\mathbb{R}^d)
\end{aligned}$$

Lemma 3.23 If $f \in C_c^\infty(\mathbb{R}^d)$, then $\hat{f} \in L^1(\mathbb{R}^d)$

Proof. (Exercise) Hint: $|\widehat{D^\alpha f}| = |2\pi k|^\alpha |\hat{f}(k)| \rightsquigarrow |\hat{f}(k)| \leq \frac{1}{|k|^\alpha}$ as $|k| \rightarrow \infty$. ■

Compute:

$$\begin{aligned}
\left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)\right)^\vee(x) &= \int_{\mathbb{R}^d} \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_{\mathbb{R}^d} \left(\int_0^\infty e^{-\pi|k|^2\lambda} \lambda^{\frac{d-\alpha}{2}-1} d\lambda\right) \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} e^{-\pi|k|^2\lambda} \hat{f}(k) e^{2\pi i k x} dk\right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(e^{-\pi k^2\lambda} \hat{f}(x)\right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}}(k) \hat{f}(k)\right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}} \star f\right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \left(\int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi\frac{x^2}{\lambda}} \lambda^{\frac{d-\alpha}{2}-1} d\lambda\right) \star f
\end{aligned}$$

Assume $d > \alpha > \frac{d}{2}$. Then $\frac{c_\alpha}{|x|^\alpha} \star f \in L^\infty$ and behaves $\frac{c_\alpha(f)}{|x|^\alpha}$ as $|x| \rightarrow \infty$. This implies:

$$\int_{\mathbb{R}^d} \left| \frac{c_\alpha}{|x|^\alpha} \star f \right|^2 \leq c + \int_{|x| \geq R} \frac{c}{|x|^{2d}} dx < \infty$$

Thus the Fourier Transform $\widehat{\frac{c_\alpha}{|x|^\alpha} \star f}$ exists. Combining with

$$\begin{aligned} \frac{c_\alpha}{|x|^\alpha} \star f &= \left(\frac{c_{d-\alpha}}{|f|^{d-\alpha}} \hat{f}(k) \right)^\vee \\ \Rightarrow \widehat{\frac{c_\alpha}{|x|^\alpha} \star f} &= \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \end{aligned}$$

■

Remark 3.24 If $d \geq 3$

$$\begin{aligned} \hat{G}(k) &= \frac{1}{|2\pi k|^2} \\ \Rightarrow G(x) &= \left(\frac{1}{|2\pi k|^2} \right)^\vee = \frac{1}{d(d-2)|x|^{d-2}} = \Phi(x) \end{aligned}$$

3.3 Theory of Distribution

Let $\Omega \subseteq \mathbb{R}^d$ be open.

- $D(\Omega) = C_c^\infty(\Omega)$ the space of test functions.
- $\phi_n \rightarrow \phi$ in $D(\Omega)$ if $\exists K \subseteq \Omega$, $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K$ and $\|D^\alpha(\phi_n - \phi)\|_{L^\infty} \rightarrow 0$ for all $\alpha = (\alpha_1, \dots, \alpha_d)$, $d_i \in \{0, 1, 2, \dots\}$.

$$D'(\Omega) = \{T : D(\Omega) \rightarrow \mathbb{R} \text{ on } \mathbb{C} \text{ linear and continuous}\}$$

the space of distributions.

Motivation: $L^2(\Omega)' = L^2(\Omega)$, $(L^p(\Omega))' = (L^q(\Omega))$, $\frac{1}{p} + \frac{1}{q} = 1$.

Example 3.25 ("normal functions" are distributions) If $f \in L^1_{loc}(\Omega)$, then $T = T_f$ defined by:

$$T(\phi) = \int_{\Omega} f(x)\phi(x) dx \quad \forall \phi \in D(\Omega)$$

is a distribution, i.e. $T \in D'(\Omega)$. Indeed, it is clear that $T(\phi)$ is well-defined for all $\phi \in D(\Omega)$ and $\phi \mapsto T(\phi)$ is linear. Let us check that $\phi \mapsto T(\phi)$ is continuous. Take $\phi_n \rightarrow \phi$ in $D(\Omega)$ and prove that $T(\phi_n) \rightarrow T(\phi)$. Since $\phi_n \rightarrow \phi$ in $D(\Omega)$, there is a compact K s.t. $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K \subseteq \Omega$.

Question: Why is $f \mapsto T_f$ injective?

Lemma 3.26 (Fundamental lemma of calculus of variants) Let $\Omega \subseteq \mathbb{R}^d$ be open. If $f, g \in L^1_{loc}(\Omega)$ and $\int_{\Omega} f\phi = \int_{\Omega} g\phi$ for all $\phi \in D(\Omega)$, then $f = g$ in $L^1_{loc}(\Omega)$

Example 3.27 (Dirac delta function) Let $\Omega \subseteq \mathbb{R}^d$ open. Define $T : D(\Omega) \rightarrow \mathbb{R}$ or \mathbb{C} by $T(\phi) = \phi(x_0)$. Let $x_0 \in \Omega$. Then $T \in D'(\Omega)$ and we denote it by δ_{x_0} . It is clear that $\phi \mapsto T(\phi) = \phi(x_0)$ is well-defined and linear for all $\phi \in D(\Omega)$. Take $\phi_n \rightarrow \phi$ in $D(\Omega)$ and prove $T(\phi_n) \rightarrow T(\phi)$, i.e. $\phi_n(x_0) \rightarrow \phi(x_0)$ (obvious.)

Example 3.28 (Principle Value) The function $f(x) = \frac{1}{x}$ is not in $L^1_{loc}(\mathbb{R})$, but we can still define

$$\int_{\mathbb{R}} f\phi = \int_{\mathbb{R}} \frac{\phi(x)}{x} dx$$

for all $\phi \in D(\mathbb{R})$ s.t. $\phi(0) = 0$. In fact, $|\phi(x)| = |\phi(x) - \phi(0)| \leq (\sup |\phi'|)|x|$, so $\frac{|\phi(x)|}{|x|} \in L^\infty(\mathbb{R})$ and compactly supportet. So $\frac{\phi(x)}{x} \in L^1(\mathbb{R})$. Define $T : D(\mathbb{R}) \rightarrow \mathbb{R}$ or \mathbb{C} by

$$T(\phi) = \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx \quad \forall \phi \in D(\mathbb{R}) \text{ s.t. } \phi(0) = 0$$

We denote $T = p.v. \left(\frac{1}{x}\right)$. Check that $T \in D'(\mathbb{R})$: For all $\epsilon > 0$, $\left|\frac{\phi(x)}{x}\right| \leq \frac{\|\phi\|_{L^\infty}}{\epsilon}$ for all $|x| \geq \epsilon$ and ϕ is compactly supported. So we get for all $\epsilon > 0$:

$$\mathbb{1}(|x| \geq \epsilon) \frac{\phi(x)}{x} \in L^1(\mathbb{R}) \rightsquigarrow \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx \text{ finite.}$$

We can write:

$$\int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx = \int_{|x| \geq 1} \frac{\phi(x)}{x} dx + \int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx$$

The second part can be written as:

$$\int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx = \int_{\epsilon} \frac{\phi(x)}{x} dx + \int_{-1}^{-\epsilon} \frac{\phi(x)}{x} dx = \int_{\epsilon} \frac{\phi(x) - \phi(-x)}{x} dx$$

Since $\phi \in C_c^\infty(\mathbb{R})$

$$|\phi(x) - \phi(-x)| \leq 2\|\phi'\| \leq 2\|\phi'\|_{L^\infty} |x|$$

$$\begin{aligned} \Rightarrow \frac{\phi(x) - \phi(-x)}{x} &\in L^\infty(\mathbb{R}) \Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^1(0, 1) \\ \Rightarrow \int_0^1 \frac{\phi(x) - \phi(-x)}{x} dx &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx \end{aligned}$$

Remark 3.29 The function $\frac{1}{|x|^d}$ is not in $L^1_{loc}(\mathbb{R}^d)$ but $\exists T \in D'(\mathbb{R}^d)$ s.t. $T(\phi) = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^d} dx$ for all $\phi \in C_c^\infty(\mathbb{R}^d)$ s.t. $\phi(0) = 0$

Definition 3.30 (Derivatives of distributions) Let $\Omega \subseteq \mathbb{R}^d$ and $T \in D'(\Omega)$. Define

$$D^\alpha T : D(\Omega) \rightarrow \mathbb{R} \text{ or } \mathbb{C}$$

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(\underbrace{D^\alpha \phi}_{in D(\Omega)}), \quad |\alpha| = \alpha_1 + \dots + \alpha_d \text{ with } \alpha = (\alpha_1, \dots, \alpha_d)$$

Motivation: $f \in C_c^\infty(\Omega)$

$$\begin{aligned} \int_{\Omega} (D^\alpha f)\phi &= (-1)^{|\alpha|} \int_{\Omega} f(D^\alpha \phi) \\ (D^\alpha T)(\phi) &\leftarrow T_{D^\alpha f}(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi) \end{aligned}$$

"If the classical derivatives exist, then They are the same with the distributional derivatives"

Remark 3.31 For all $T \in D'(\Omega)$, then $D^\alpha T \in D'(\Omega)$ for all $\alpha = (\alpha_1, \dots, \alpha_d)$. Clearly:

$$\phi \mapsto (D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

linear. Moreover, if $\phi_n \rightarrow \phi$ in $D(\Omega)$:

$$\begin{aligned} & \Rightarrow D^\alpha \phi_n \rightarrow D^\alpha \phi \text{ in } D(\Omega) \\ \Rightarrow (D^\alpha T)(\phi_n) &= (-1)^\alpha T(D^\alpha \phi_n) \xrightarrow{n \rightarrow \infty} (-1)^\alpha T(D^\alpha \phi) = (D^\alpha T)(\phi) \end{aligned}$$

Example 3.32 Consider $f(x) = |x|$ in \mathbb{R} . Then $f \in C(\mathbb{R})$ but $f \notin C^1(\mathbb{R})$. However,

$$f'(x) = g(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \in L^1_{loc}$$

Lets check $f' = g$, i.e. $-f(\phi') = f'(\phi) = g(\phi)$ for all $\phi \in D(\mathbb{R})$. Thus we need to prove:

$$-\int_{\mathbb{R}} f(x) \phi'(x) dx = \int_{\mathbb{R}} g(x) \phi(x) dx \quad \forall \phi \in D(\mathbb{R})$$

namely:

$$-\int_{\mathbb{R}} |x| \phi'(x) dx = \int_0^\infty \phi(x) dx - \int_{-\infty}^0 \phi(x) dx$$

$$LHS = -\int_0^\infty x \phi'(x) dx + \int_{-\infty}^0 x \phi'(x) dx$$

By integration by parts:

$$\int_0^\infty x \phi'(x) dx = \underbrace{[x \phi(x)]_0^\infty}_{=0} - \int_0^\infty \phi(x) dx = -\int_0^\infty \phi(x) dx$$

and similiary:

$$\int_{-\infty}^0 x \phi'(x) dx = -\int_{-\infty}^0 \phi(x) dx$$

Thus $f' = g$ in $D'(\Omega)$.

Claim: $g' = 2\delta_0$ in $D'(\mathbb{R})$. In fact, for all $\phi \in D(\mathbb{R})$, then:

$$\begin{aligned} g'(\phi) &= -g(\phi') = -\int_{\mathbb{R}} g \phi' = -\int_{-\infty}^0 (-1) \phi' - \int_0^\infty (1) \phi' \\ &= -\int_0^\infty \phi' + \int_{-\infty}^0 \phi' = [\phi(0) - \underbrace{\phi(\infty)}_{=0}] + [\phi(0) - \underbrace{\phi(-\infty)}_{=0}] = 2\phi(0) = 2\delta_0(\phi) \end{aligned}$$

So $g' = 2\delta_0$ in $D'(\mathbb{R})$

Exercise 3.33 Prove that $(D^\alpha \delta_x)(\phi) = (-1)^{|\alpha|} (\Delta^\alpha \phi)(x)$ for all $\phi \in D(\mathbb{R})$ for all $x \in \mathbb{R}$.

Definition 3.34 (Convergence of distributions) Let $\Omega \subseteq \mathbb{R}^d$ open, then

$$T_n \xrightarrow{n \rightarrow \infty} T$$

in $D'(\Omega)$ if $T_n(\phi) \xrightarrow{n \rightarrow \infty} T(\phi)$ for all $\phi \in D(\Omega)$.

Exercise 3.35 Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$ For $\epsilon > 0$, define $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$. Then: $f_\epsilon \rightarrow \delta_0$ in $D'(\Omega)$.

Exercise 3.36 Let $\Omega \subseteq \mathbb{R}^d$ be open and $T_n \rightarrow T$ in $D'(\Omega)$. Then: $D^\alpha T_n \rightarrow D^\alpha T$ in $D'(\Omega)$ for all $\alpha = (\alpha_1, \dots, \alpha_d)$

Definition 3.37 (Convolution of distributions) Let $T \in D'(\mathbb{R})$ and $f \in L_c^\infty(\mathbb{R}^d)$. Define

$$(T \star f)(y) = T(f_y)$$

We write $f_y(x) = f(x - y)$ and $\tilde{f}(x) = f(-x)$.

Theorem 3.38 For all $T \in D'(\mathbb{R})$, for all $f \in D(\mathbb{R})$, then

1. $y \mapsto T(f_y)$ is $C^\infty(\mathbb{R}^d)$ and

$$D_y^\alpha (T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D^\alpha f_y)$$

2. For all $g \in L^1(\mathbb{R}^d)$ and compactly supported, then

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(\underbrace{f \star g}_{\in C_c^\infty(\mathbb{R})})$$

Proof. 1. We prove that $y \mapsto T(f_y)$ is continous. Take $y_n \rightarrow y$ in \mathbb{R}^d , then:

- 2.

$$T(f_{y_n}) \rightarrow T(f_y)$$

since $f_{y_n} \rightarrow f_y$ in $D(\mathbb{R}^d)$ (*)

Check (*): Since $f \in C_c^\infty(\mathbb{R}^d) \Rightarrow \text{supp } f \subseteq B(0, R) \subseteq \mathbb{R}^d$. Since $y_n \rightarrow y$ in \mathbb{R}^d . We have $\sup_n |y_n| < \infty$. Thus f_{y_n}, f_y are supported in $\overline{B(0, R + \sup_n |y_n|)} = K$ compact. Moreover

$$|f_{y_n}(x) - f_y(x)| = |f(x - y_n) - f(x - y)| \leq \|\delta f\|_{L^\infty} \|y_n - y\| \rightarrow 0$$

So we get $\|f_{y_n} - f_y\|_{L^\infty} \rightarrow 0$ Similary:

$$\|D^\alpha f_{y_n} - D^\alpha f_n\|_{L^\infty} \rightarrow 0$$

Thus $f_{y_n} \rightarrow f_y$ in $D(\mathbb{R}^d)$ i.e. (*) holds. ■

Exercise 3.39 (E 3.1 Lebesgue Differentiation Theorem) Let $f \in L_{loc}^1(\mathbb{R}^d)$. Prove that that for almost every $x \in \mathbb{R}^d$:

$$\oint_{B(x,r)} |f(y)f(x)| dy \xrightarrow{r \rightarrow 0} 0$$

Solution. If $g \in L(\mathbb{R}^d) \int g = 1, g_\epsilon(x) = \epsilon^{-d} g(\epsilon^{-1}x)$, then: $g_\epsilon \star f \rightarrow f$ in $L^1(\mathbb{R}^d)$

Lemma 3.40 If $f_n \rightarrow f$ in $L^1(\Omega)$, then there is a subsequence $\{f_{n_k}\}$ s.t. we have the Dominated Convergence

$$\begin{cases} f_{n_k}(x) \rightarrow f(x) a.e. \\ |(f_{n_k}(x))| \leq F(x) \in L^1(\Omega) \end{cases}$$

Consequence of the proof of the completeness of $L^1(\Omega)$. Consequence:

$$\begin{aligned} \int_{B(x,r)} f &= \int_{\mathbb{R}^d} \frac{\mathbb{1}(B(x,r))(y)}{|B(x,r)|} dy = \int_{\mathbb{R}^d} \frac{\mathbb{1}(B(0,r))}{|B(0,r)|} (x-y) f(y) dy \\ &= (g_r \star f)(x), \quad g_r = \frac{\mathbb{1}(B(0,r))}{|B(0,r)|} \text{ (similar to } g_\epsilon) \end{aligned}$$

We let that if $f \in L^1$, then: $g_r \star f \xrightarrow{r \rightarrow 0} f$ in L^1 . So up to a subsequence we have $(g_r \star f)(x) \rightarrow f(x)$ a.e. Here the Lebesgue differentiation then is stronger:

- $g_r \star f(x) \rightarrow f(x)$ as $r \rightarrow 0$ a.e. x .
-

$$|(g_r \star f)(x) - f(x)| = \left| \int_{B(x,r)} (f(y) - f(x)) dy \right| = \int_{B(x,r)} |f(y) - f(x)| dy$$

Step 1) Assume $f \in C(\mathbb{R}^d)$. Then:

$$\int_{B(x,r)} |f(y) - f(x)| dy \leq \sup_{|y-x| \leq r} |f(y) - f(x)| \rightarrow 0 \quad \forall x \in \mathbb{R}^d$$

since f is continuous

Step 2) Take $f \in L^1(\mathbb{R}^d)$, then $\exists \{f_n \subseteq C_c(\mathbb{R}^d)\}$ s.t. $f_n \rightarrow f$ in $L^1(\mathbb{R}^d)$ and $f_n(x) \rightarrow f(x)$ for a.e. $x \in \mathbb{R}^d$.

$$\begin{aligned} &\int_{B(x,r)} |f(y) - f(x)| dy \\ &= \int_{B(x,r)} \underbrace{|f(y) - f_n(y)|}_{|f(x) - f_n(x)|} + \int_{B(x,r)} |f_n(y) - f_n(x)| dy + \int_{B(x,r)} |f_n(x) - f(x)| dy \\ &\Rightarrow \limsup_{n \rightarrow \infty} \int_{B(x,r)} |f(y) - f(x)| dy \leq 2|f(x) - f_n(x)| \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Claim without proof: For all $g \in L^1$

$$\int_{B(x,r)} g(y) dy \xrightarrow{r \rightarrow 0} g(x)$$

■

Exercise 3.41 (E 3.2) Let $1 \leq p, q, r \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Take $f \in L^p, g \in L^q$, prove:

$$\widehat{f \star g} = \hat{f} \hat{g}$$

Aufgabenstellung ..

Solution Step 1) $f, g \in C_c^\infty(\mathbb{R}^d)$ (Fubini)

Step 2) $f \in L^p, g \in L^q$, find $f_n, g_n \in C_c^\infty$ s.t. $f_n \rightarrow f$ in L^p , $g_n \rightarrow g$ in L^q . $\widehat{f_n \star g_n} = \hat{f}_n \hat{g}_n$ pointwise a.e. We have

Step 3)

$$\begin{aligned}
& \|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^{r'}} \\
& \text{(Hausdorff-Young)} \leq \|f \star g\|_{L^r} = \|(f - f_n) \star g_n + f_n \star (g_n - g)\|_{L^r} \\
& \leq \|(f - f_n) \star g_n\|_{L^r} + \|f_n \star (g_n - g)\|_{L^r} \\
& \text{(Young)} \leq \|f - f_n\|_{L^p} \|g_n\| + \|f_n\|_{L^p} \|g_n - g\|_{L^p} \xrightarrow{n \rightarrow \infty} 0
\end{aligned}$$

Moreover:

$$\begin{aligned}
& \|\hat{f}_n \hat{g}_n - \hat{f} \hat{g}\|_{L^{r'}} = \|(\hat{f}_n \hat{f}) \hat{g}_n + \hat{f}(\hat{g}_n - \hat{g})\|_{L^{r'}} \\
& \text{(Hölder)} \leq \|\hat{f}_n - \hat{f}\|_{L^{p'}} \|\hat{g}_n\|_{L^{q'}} + \|\hat{f}\|_{L^{q'}} \\
& \text{(Hausdorff-Young)} \leq \|f_n - f\|_{L^p} \|g_n\|_{L^q} + \|f\|_{L^p} \|g_n - g\|_{L^p} \xrightarrow{n \rightarrow \infty} 0 \\
& \text{So } \hat{f}_n \hat{g}_n \rightarrow \hat{f} \hat{g} \text{ in } L^{r'} \text{ in } L^{r'} \widehat{f \star g} = \hat{f} \hat{g} \text{ in } L^{r'} \quad \frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'}
\end{aligned}$$

■

Exercise 3.42 (E 3.3) $f \in C_c^\infty(\mathbb{R}^d)$. Prove $|\hat{f}(k)| \leq \frac{C_N}{(1+|k|)^N}$

Solution. Since $f \in C_c^\infty$ we have that $D^\alpha f \in C_c^\infty$. Recall

$$\widehat{D^\alpha f}(k) = (-2\pi i k)^\alpha \hat{f}(k)$$

For example

$$\begin{aligned}
& -\Delta \hat{f}(k) = |2\pi i k|^2 \hat{f}(k) \\
& \text{(Induction)} \rightsquigarrow (-\Delta)^N \hat{f}(k) = |2\pi k|^{2N} \hat{f}(k)
\end{aligned}$$

So we can conclude

$$\hat{f}(k) = \frac{(-\Delta)^N \hat{f}(k)}{|2\pi k|^{2N}} \quad \forall k \in \mathbb{R}^d$$

1. $f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in L^\infty$
2. $(-\Delta)^N \hat{f} \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow (-\Delta)^N \hat{f} \in L^\infty$

$$\text{Conclusion: } \hat{f}(k) \leq \begin{cases} C & \forall k \\ \frac{C_N}{|k|^{2N}} & \forall k \end{cases} \text{ So } \hat{f}(k) \leq \frac{C_N}{(1+|k|)^N}$$

■

Exercise 3.43 (E 3.4)

Proof. Siehe Goodnotes

■

Exercise 3.44 (Bonus 3) Let $f \in L^1(\mathbb{R}^d)$ such that

$$|\hat{f}(k)| \leq \frac{C_N}{(1+|k|)^N}$$

for all $k \in \mathbb{R}^d$, for all $N \geq 1$. (C_N is independent of k). Prove that $f \in C^\infty(\mathbb{R}^d)$

($f \in C^\infty$) i.e. $\exists \tilde{f} \in C^\infty$ s.t. $f = \tilde{f}$ a.e.

My Solution. First we regard for $N \in \mathbb{N}$ and $|k| \geq 1$:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{(1+|k|)^N} dk &= \int_0^\infty \left(\int_{\partial B(0,r)} \frac{1}{(1+r)^N} dS(y) \right) dr \\ &= \int_0^\infty \frac{1}{(1+r)^N} |\partial B(0,r)| dr \\ &\leq \int_1^\infty \frac{1}{r^N} \underbrace{|\partial B(0,r)|}_{cr^{d-1}} dr \quad \text{for a } c \in \mathbb{R} \\ &= c \int_1^\infty \frac{1}{r^{N-d+1}} dr \end{aligned}$$

From Ana I we know that $\int_1^\infty \frac{1}{r^{N-d+1}} dr < \infty$ is equivalent to $N - d + 1 > 1 \Leftrightarrow N > d$, so for $N > d$ we have

$$\frac{1}{(1+|k|)^N} \in L^1(\mathbb{R}^d).$$

Now let $\alpha \in \mathbb{N}^d$, then we have

$$k^\alpha = k_1^{\alpha_1} \dots k_d^{\alpha_d} = |k|^{\alpha_1} \dots |k|^{\alpha_d} = |k|^{\alpha_1 + \dots + \alpha_d} = |k|^{|\alpha|} = (1+|k|)^{|\alpha|}.$$

By assumption we have for all $N \geq 1$:

$$k^\alpha \hat{f}(k) \leq k^\alpha \frac{C_n}{(1+|k|)^N} \leq (1+|k|)^{|\alpha|} \frac{C_n}{(1+|k|)^N} = \frac{C_n}{(1+|k|)^{N-|\alpha|}}$$

If we set N such that $N - |\alpha| > d$, for example $N = d + |\alpha| + 1$, then we can conclude that $k^\alpha \hat{f} \in L^1(\mathbb{R}^d)$. This implies $\widehat{k^\alpha \hat{f}} \in L^\infty(\mathbb{R}^d)$, so

$$\widehat{k^\alpha \hat{f}}(k) = \partial^\alpha \hat{f}(k) = \partial^\alpha (\hat{f})^\vee(-k) = \partial^\alpha f(-k) \in L^\infty(\mathbb{R}^d).$$

This implies $f \in C^\infty(\mathbb{R}^d)$. ■

Theorem 3.45 Take $T \in D'(\mathbb{R})$, $f \in C_c^\infty(\mathbb{R}^d) = D(\mathbb{R}^d)$, $f_y(x) = f(x - y)$

- a) $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D_x^\alpha f_y)$
- b) $\forall g \in L^1(\mathbb{R}^d)$ and compactly supported

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T\left(\underbrace{f \star g}_{\in C_c^\infty}\right)$$

Proof. a) $y \mapsto T(f_y)$ is continuous since $y_n \rightarrow y$ in \mathbb{R}^d , then $f_{y_n} \rightarrow f_y$ implies $T(f_{y_n}) \rightarrow T(f_y)$. Let's check that $y \mapsto T(f_y) \in C^1$:

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} = \lim_{h \rightarrow 0} T\left(\frac{f_{y-he_i} - f_y}{h}\right)$$

We have $\frac{f_{y-he_i} - f_y}{h} \xrightarrow{h \rightarrow 0} (\partial_i f)_y$ in $D(\mathbb{R}^d)$

- $\exists K$ compact set such that $\text{supp}(f_{y-he_i} - f_y), \text{supp } \partial_i f \subseteq K$ as $|h|$ small.

•

$$\begin{aligned} \frac{f_{y-he_i}(x) - f_y(x)}{h} - (\partial_i f)_y(x) &= \frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y) \\ \left| \int_0^1 \partial_i f(x-y+the_i) dt - \partial_i f(x-y) \right| &\xrightarrow{h \rightarrow 0} 0 \text{ uniformly in } x \end{aligned}$$

Similary:

$$\begin{aligned} &\left| D_x^\alpha \left(\frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y) \right) \right| \\ &= \left| \frac{D^\alpha f(x-y+he_i) - D^\alpha f(x-y)}{h} - \partial_i(D^\alpha f)(x-y) \right| \xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

uniformly in x . Conclude:

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} \xrightarrow{h \rightarrow 0} T((\partial_i f)_y) \in C(\mathbb{R}^d)$$

So we geht that $y \mapsto T(f_y) \in C^1$ and $-\partial_{y_i} T(f_y) = T((\partial_i f)_y)$

By induction:

$$D_y^\alpha T(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y) = (D^\alpha T)(f_y) \quad \forall \alpha$$

Heuristic: $T = T(x)$

$$\begin{aligned} \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \int_{\mathbb{R}^d} g(y) \left(\int_{\mathbb{R}^d} T(x) f(x-y) dx \right) dy \\ &= \int_{\mathbb{R}^d} T(x) \left(\int_{\mathbb{R}^d} g(y) f(x-y) dy \right) dx \\ &= \int_{\mathbb{R}^d} T(x) (f \star g)(x) dx = T(f \star g) \end{aligned}$$

Step 1: $g \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \text{(Rieman Sum)} \quad \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) T(f_{y_j}) \\ &= \lim_{\Delta_N \rightarrow 0} T \left(\Delta_N \sum_{j=1}^N g(y_j) f_{y_j} \right) \\ &= T(f \star g) \end{aligned}$$

because

$$\begin{aligned} \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f_{y_j}(x) &\rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d) \\ \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x-y_j) &\xrightarrow{\text{Riemann}} \int_{\mathbb{R}^d} g(y) f(x-y) dy = (f \star g)(x) \end{aligned}$$

Proof of:

$$\lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x-y_j) \rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d)$$

1) Since $f, g \in C_c^\infty$ we have $f \star g \in C_c^\infty$. And we have

$$x \mapsto \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \in C^\infty$$

since $f \in C^\infty$ supported in $(\text{supp } g + \text{supp } f)$. So all functions are C_c^∞ and supported in $(\text{supp } g + \text{supp } f)$.

2)

$$\left| \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) - \int_{\mathbb{R}^d} g(y) f(x - y) dy \right| \xrightarrow{\Delta_N \rightarrow 0} 0$$

uniformly in x . (Result from the Riemann-Sum)

3)

$$\begin{aligned} & \left| D_x^\alpha \left(\Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) - (f \star g)(x) \right) \right| \\ &= \left| \Delta_N \sum_{j=1}^N g(y_j) D^\alpha f(x - y_j) - (D^\alpha f) \star g(x) \right| \xrightarrow{\Delta_N \rightarrow 0} 0 \end{aligned}$$

uniformly in x for all α .

Step 2: Take $g \in L^1(\mathbb{R}^d)$ and compactly supported. Then $\exists \{g_n\} \subseteq C_c^\infty(\mathbb{R}^d)$, $\text{supp } g_n \subseteq \text{supp } g + B(0, 1)$ such that $g_n \rightarrow g$ in $L^1(\mathbb{R}^d)$. By Step 1:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy = T(g_n \star f)$$

Take $n \rightarrow \infty$:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy \rightarrow \int_{\mathbb{R}^d} g(y) T(f_y) dy$$

since $g_n \rightarrow g$ in L^1 compactly supported and $y \mapsto T(f_y) \in C^\infty \subseteq L^\infty(K)$. Moreover (exercise):

$$\underbrace{g_n \star f}_{\in C_c^\infty} \rightarrow g \star f \quad \text{in } D(\mathbb{R}^d)$$

So $T(g_n \star f) \xrightarrow{n \rightarrow \infty} T(g \star f)$. Finally we obtain:

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(g \star f)$$

■

Theorem 3.46 Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $T_1 D'(\Omega)$ and $f \in C_c^\infty(\Omega)$. Denote

$$\Omega_f = \{y \in \mathbb{R}^d \mid \text{supp } f_y = y + \text{supp } f \subseteq \Omega\}$$

a) $y \mapsto T(f_y) \in C^\infty(\Omega_f)$ and $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y)$

b) For all $g \in L^1(\Omega_g)$ and compactly supported in Ω_f , then:

$$\int_{\Omega} g(y) T(f_y) dy = T(f \star g)$$

Theorem 3.47 Let $T \in D'(\Omega)$ s.t. $\nabla T = 0$ in $D'(\Omega)$. Then: $T = \text{const.}$ in Ω .

Proof. ($\Omega = \mathbb{R}^d$) for all $f \in C_c^\infty$, $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$ and $\partial_{y_i} T(f_y) = (\partial_j T)(f_y) = 0$ for all $i = 1, \dots, d$. Then by the result of the theorem for C^∞ functions, $y \mapsto T(f_y) = \text{const}$ independent of y . Consequently:

$$T(f_y) = T(f_0) = T(f) \quad \forall y \in \mathbb{R}^d \quad \forall f \in C_c^\infty(\mathbb{R}^d)$$

For any $g \in C^\infty(\mathbb{R}^d)$:

$$\left(\int_{\mathbb{R}^d} g \, dy \right) T(f) = \int_{\mathbb{R}^d} g(y) T(f_y) \, dy = T(f \star g) = T(g \star f) = \left(\int_{\mathbb{R}^d} f \, dy \right) T(g)$$

So $\frac{T(f)}{\int_{\mathbb{R}^d} f}$ is independent of f (as soon as $\int f \neq 0$). So we get that $T(f) = \text{const} \int_{\mathbb{R}^d} f$, where const is independent of f . ■

Remark 3.48 If $u \in C^1(\mathbb{R}^d)$, then:

$$\begin{aligned} u(x+y) - u(x) &= \int_0^1 \sum_{j=1}^d y_j u(x + ty_j) \, dt \\ &= \int_0^1 y \nabla u(x + ty) \, dt \end{aligned}$$

So we get that if $\nabla u = 0$, then $u(x+y) - u(x) = 0$ for all x, y , so $u = \text{const.}$

Theorem 3.49 (Taylor expansion for distributions) Let $T \in D'(\mathbb{R}^d)$ and $f \in C_c^\infty(\mathbb{R}^d)$. Then:

$$T(f_y) - T(f) = \int_0^1 \sum_{j=1}^d y_j (\partial_j T)(f_{ty}) \, dt$$

Proof. $y \mapsto T(f_y)$ is C^∞ and $\frac{d}{dt}[T(f_{ty})] = (\nabla T)(f_{ty})y$ So we get

$$\begin{aligned} T(f_y) - T(f) &= \int_0^1 \frac{d}{dt}(T(f_{ty})) \, dt \\ &= \int_0^1 (\nabla T)(f_{ty})y \, dt \\ &= \int_0^1 \sum_{j=1}^d (\partial_j T)(f_{ty})y_j \, dt \end{aligned}$$

■

Corollary 3.50 Let $g \in L_{loc}^1(\mathbb{R}^d)$ s.t. $\partial_j g \in L_{loc}^1(\mathbb{R}^d)$ for all $j = 1, 2, \dots, d$ (i.e. $g \in W_{loc}^{1,1}(\mathbb{R}^d)$). Then for all $y \in \mathbb{R}^d$:

$$\begin{aligned} g(x+y) - g(x) &= \int_0^1 y \nabla g(x + ty) \, dt \\ &= \int_0^1 \sum_{j=1}^d y_j \partial_j g(x + ty) \, dt \end{aligned}$$

for a.e. x .

Proof. For all $f \in C_c^\infty$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)[g(x+y) - g(x)] dx &= \int_{\mathbb{R}^d} g(x)[f(x+y) - f(x)] dx \\
&= g(f_y) - g(f) \\
&= \int_0^1 \sum_{j=1}^d y_j (\partial_j g)(f_{ty}) dt \\
&= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \sum_{j=1}^d y_j \left[\int_{\mathbb{R}^d} (\partial_j g)(x) f_{ty}(x) dx \right] \\
&= \int_0^1 \sum_{j=1}^d y_j \left[\int_{\mathbb{R}^d} (\partial_j g)(x+ty) f(x) dx \right] dt \\
&= \int_{\mathbb{R}^d} f(x) \left[\int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \right] dx
\end{aligned}$$

For all $\phi \in C_c^\infty$: $= g(x+y) - g(x)$ a.e. $x \in \mathbb{R}^d$. ■