

Partial Differential Equations  
Thành Nam Phan  
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Lecture notes T<sub>E</sub>Xed by Thomas Eingartner

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Please note that I write this lecture notes for my personal use. I may write things differently than presented in the lecture. This script also contains some of my personal solutions for exercises (which may be wrong).

# Chapter 1

## Introduction

A differential equation is an equation of a function and its derivatives.

**Example 1.1** (Linear ODE) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\begin{cases} f'(t) = af(t) \text{ for all } t \geq 0, a \in \mathbb{R} \\ f(0) = a_0 \end{cases}$$

is a linear ODE (Ordinary differential equation). The solution is:  $f(t) = a_0 e^{at}$  for all  $t \geq 0$ .

**Example 1.2** (Non-Linear ODE)  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{cases} f'(t) = 1 + f^2(t) \\ f(0) = 1 \end{cases}$$

Lets consider  $f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$ . Then we have

$$f'(t) = \frac{1}{\cos^2(t)} = 1 + \tan^2(t) = 1 + f^2(t),$$

but this solution only is *good* in  $(-\pi, \pi)$ . It's a problem to extend this to  $\mathbb{R} \rightarrow \mathbb{R}$ .

A PDE (Partial Differential Equation) is an equation of a function of 2 or more variables and its derivatives.

**Remark 1.3** Recall for  $\Omega \subseteq \mathbb{R}^d$  open and  $f : \Omega \rightarrow \{\mathbb{R}, \mathbb{C}\}$  the notation of partial derivatives:

- $\partial_{x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x + h e_i) - f(x)}{h}$ , where  $e_i = (0, 0, \dots, 1, \dots, 0, 0) \in \mathbb{R}^d$
- $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x)$ , where  $\alpha \in \mathbb{N}^d$
- $Df = \nabla f = (\partial_{x_1}, \dots, \partial_{x_d})$
- $\Delta f = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 f$
- $D^k f = (D^\alpha f)_{|\alpha|=k}$ , where  $|\alpha| = \sum_{i=1}^d |\alpha_i|$
- $D^2 f = (\partial_{x_i} \partial_{x_j} f)_{1 \leq i, j \leq d}$

**Definition 1.4** Given a function  $F$ . Then the equation of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0$$

with the unknown function  $u : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$  is called a *PDE of order  $k$* .

- Equations  $\sum_d a_\alpha(x) D^\alpha u(x) = 0$ , where  $a_\alpha$  and  $u$  are unknown functions are called *Linear PDEs*.
- Equations  $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(D^{k-1} u, D^{k-2} u, \dots, Du, u, x) = 0$  are called *semi-linear PDEs*.

Goals: For *solving a PDE* we want to

- Find an explicit solution! This is in many cases impossible.
- Prove a *well-posed theory* (existence of solutions, uniqueness of solutions, continuous dependence of solutions on the data)

We have two notations of solutions:

1. Classical solution: The solution is continuous differentiable (e.g.  $\Delta u = f \rightsquigarrow u \in C^2$ )
2. Weak Solutions: The solution is not smooth/continuous

**Definition 1.5** (Spaces of continuous and differentiable functions) Let  $\Omega \subseteq \mathbb{R}^d$  be open

$$\begin{aligned} C(\Omega) &= \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ continuous}\} \\ C^k(\Omega) &= \{f : \Omega \rightarrow \mathbb{R} \mid D^\alpha f \text{ is continuous for all } |\alpha| \leq k\} \end{aligned}$$

Classical solution of a PDE of order  $k \rightsquigarrow C^k$  solutions!

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ Lebesgue measurable} \mid \int_\Omega |f|^p d\lambda < \infty, 1 \leq p < \infty \right\}$$

Sobolev Space:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \text{ exists}\}$$

In this course we will investigate

- Laplace / Poisson Equation:  $-\Delta u = f$
- Heat Equation:  $\partial_t u - \Delta u = f$
- Wave Equation:  $\partial_t^2 u - \Delta u = f$
- Schrödinger Equation:  $i\partial_t u - \Delta u = f$

## Chapter 2

# Laplace / Poisson Equation

### 2.1 Laplace Equation

$-\Delta u = 0$  (Laplace) or  $-\Delta u = f(x)$  (Poisson).

**Definition 2.1** (Harmonic Function) Let  $\Omega$  be an open set in  $\mathbb{R}^d$ . If  $u \in C^2(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ , then  $u$  is a harmonic function in  $\Omega$ .

**Theorem 2.2** (Gauss-Green Theorem) Let  $A \subseteq \mathbb{R}^d$  open,  $\vec{F} \in C^1(A, \mathbb{R}^d)$  and  $K \subseteq A$  compact with  $C^1$  boundary. Then

$$\int_{\partial K} \vec{F} \cdot \vec{\nu} \, dS(x) = \int_K \operatorname{div}(\vec{F}) \, dx$$

where  $\nu$  is the outward unit normal vector field on  $\partial K$ . Thus

$$\int_{\partial V} \nabla u \cdot \vec{\nu} \, dS(x) = \int_V \operatorname{div}(\nabla u) \, dx = \int_V \Delta u(x) \, dx$$

for any  $V \subseteq \Omega$  open.

**Theorem 2.3** (Green's Identities) Let  $A \subseteq \mathbb{R}^d$  open,  $K \subseteq A$  d-dim. compactum with  $C^1$  boundary and  $f, g \in C^2(A)$

1. Green's first identity (Integration by parts):

$$\int_K \nabla f \cdot \nabla g \, dx = \int_{\partial K} f \frac{\partial g}{\partial \nu} \, dS - \int_K f \Delta g \, dx$$

where  $\frac{\partial g}{\partial \nu} = \partial_\nu g = \nu \cdot \nabla g$

2. Green's second identity:

$$\int_K f \Delta g - (\Delta f)g \, dx = \int_{\partial K} \left( f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) \, dS$$

**Exercise 2.4** Let  $\Omega \subseteq \mathbb{R}^d$  open, let  $f : \Omega \rightarrow \mathbb{R}$  be continuous. Prove that if  $\int_B f(x) \, dx = 0$ , then  $u \equiv 0$  in  $\Omega$ .

**Theorem 2.5** (Fundamental Lemma of Calculus of Variations) Let  $\Omega \subseteq \mathbb{R}^d$  open, let  $f \in L^1(\Omega)$ . If  $\int_B f(x) \, dx = 0$  for all  $x \in B_r(x) \subseteq \Omega$ , then  $f(x) = 0$  a.e. (almost everywhere)  $x \in \Omega$ .

**Remark 2.6** (Solving Laplace Equation)  $-\Delta u = 0$  in  $\mathbb{R}^d$ . Consider the case when  $u$  is radial, i.e.  $u(x) = v(|x|)$ ,  $v : \mathbb{R} \rightarrow \mathbb{R}$ . Denote  $r = |x|$ , then

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x_i} \left( \sqrt{x_1^2 + \cdots + x_d^2} \right) = \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{x_i}{r}$$

Then

$$\begin{aligned} \partial_{x_i} u &= \partial_{x_i} v = (\partial_r v) \frac{\partial r}{\partial x_i} = v'(r) \frac{x_i}{r} \\ \partial_{x_i}^2 u &= \partial_{x_i} \left( v(r) \frac{x_i}{r} \right) = (\partial_{x_i} v(r))' \frac{x_i}{r} + v'(r) \partial_{x_i} \left( \frac{x_i}{r} \right) \\ &= (\partial_r v'(r)) \left( \frac{dr}{\partial x_i} \right) \frac{x_i}{r} + v'(r) \left( \frac{1}{r} - \frac{x_i}{r^2} (\partial_{x_i} r) \right) = v'(r) \frac{x_i^2}{r^2} + v'(r) \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right) \end{aligned}$$

So we have  $\Delta u = \left( \sum_{i=1}^d \partial_{x_i}^2 \right) u = v''(r) + v'(r) \left( \frac{d}{r} - \frac{1}{r} \right)$

Thus  $\Delta u = v'(r) + v(r) \frac{d-1}{r}$ . We consider  $d \geq 2$ . Laplace operator  $\Delta u = 0$  now becomes  $v''(r) + v'(r) \frac{d-1}{r} = 0$

$$\begin{aligned} \Rightarrow \log(v(r))' &= \frac{v'(r)}{v(r)} = -\frac{d-1}{r} = -(d-1)(\log r)' \quad (\text{recall } \log(f)' = \frac{f'}{f}) \\ \Rightarrow v'(r) &= \frac{1}{v^{d-2} + \text{const.}} \end{aligned}$$

$$\begin{cases} \frac{\text{const}}{r^{d-2}} + \text{const}x + \text{const} & , d \geq 3 \\ \text{const} \log(r) + \text{const}x + \text{const} & , d = 2 \end{cases}$$

**Definition 2.7** (Fundamental Solution of Laplace Equation)

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log(|x|), & d = 2 \\ \frac{1}{(d-2)d|B_1|} \frac{1}{|x|^{d-2}}, & d \geq 3 \end{cases}$$

Where  $|B_1|$  is the Volume of the ball  $B_1(0) = B(0, 1) \subseteq \mathbb{R}^d$ .

**Remark 2.8**  $\Delta \Phi(x) = 0$  for all  $x \in \mathbb{R}^d$  and  $x \neq 0$ .

## 2.2 Poisson-Equation

The Poisson-Equation is  $-\Delta u(x) = f(x)$  in  $\mathbb{R}^d$ . The explicit solution is given by

$$u(x) = (\Phi \star f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy$$

This can be heuristically justified with

$$-\Delta(\Phi \star f) = (-\Delta \Phi) \star f = \delta_0 \star f = f$$

**Theorem 2.9** Assume  $f \in C_c^2(\mathbb{R}^d)$ . Then  $u = \Phi \star f$  satisfies that  $u \in C^2(\mathbb{R}^d)$  and  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$

*Proof.* By definition we have

$$u(x) = \int_{\mathbb{R}^d} \Phi(y) f(x-y) dy.$$

First we check that  $u$  is continuous: Take  $x_k \rightarrow x_0$  in  $\mathbb{R}^d$ . We prove that  $u(x_n) \xrightarrow{n} u_0$ , i.e.

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \Phi(y) f(x_n - y) dy = \int_{\mathbb{R}^d} \Phi(y) f(x_0 - y) dy$$

This follows from the Dominated Convergence Theorem. More precisely:

$$\lim_{n \rightarrow \infty} \Phi(y) f(x_n - y) = \Phi(y) f(x_0 - y) \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}$$

and

$$|\Phi(y) f(x - y)| \leq \|f\|_{L^\infty} \cdot \mathbb{1}(|y| \leq R) \cdot |\Phi(y)| \in L^1(\mathbb{R}^d, dy)$$

where  $R > 0$  depends on  $\{x_n\}$  and  $\text{supp}(f)$  but independent of  $y$ . Now we compute the derivatives:

$$\begin{aligned} \partial_{x_i} u(x) &= \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) f(x - y) dy = \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \Phi(y) \frac{f(x + h e_i - y) - f(x - y)}{h} dy \\ (\text{dom. conv.}) &= \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x - y) dy \\ \Rightarrow D^\alpha u(x) &= \int_{\mathbb{R}^d} \Phi(y) D_x^\alpha f(x - y) dy \quad \text{for all } |\alpha| \leq 2 \end{aligned}$$

$D^\alpha u(x)$  is continuous, thus  $u \in C^2(\mathbb{R}^d)$ . Now we check if this solves the Poisson-Equation:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^d} \Phi(y) (-\Delta_x) f(x - y) dy = \int_{\mathbb{R}^d} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy + \int_{B(0, \epsilon)} \Phi(y) (-\Delta_x) f(x - y) dy \quad (\epsilon > 0 \text{ small}) \end{aligned}$$

Now we come to the main part. We apply integration by parts (2.3):

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus B(0, \epsilon)} \Phi(y) (-\Delta_y) f(x - y) dy \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} (\nabla_y \Phi(y)) \cdot \nabla_y f(x - y) dy - \int_{\partial B(0, \epsilon)} \Phi(y) \cdot \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \\ &= \int_{\mathbb{R}^d \setminus B(0, \epsilon)} \underbrace{(-\Delta_y \Phi(y))}_{=0} f(x - y) dy \\ &\quad + \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) - \int_{\partial B(0, \epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x - y) dS(y) \end{aligned}$$

We have that  $\nabla_y \Phi(y) = -\frac{1}{d|B_1|} \frac{y}{|y|^d}$  and  $\vec{n} = \frac{y}{|y|}$  in  $\partial B(0, \epsilon)$ . This leads to

$$\frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d|B_1|} \frac{1}{|y|^{d-1}} = \frac{1}{d|B_1| \epsilon^{d-1}} \quad \text{for } y \in \partial B(0, \epsilon)$$

Hence:

$$\begin{aligned} \int_{\partial B(0, \epsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y) f(x - y) dS(y) &= \frac{1}{d|B_1| \epsilon^{d-1}} \int_{\partial B(0, \epsilon)} f(x - y) dS(y) \\ &= \oint_{\partial B(0, \epsilon)} f(x - y) dS(y) = \oint_{\partial B(x, \epsilon)} f(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} f(x) \end{aligned}$$



We have to regard the following error terms:

$$\begin{aligned}
\bullet \left| \int_{B(0,\epsilon)} \Phi(y) (-\Delta_y) f(x-y) dy \right| &\leq \int_{B(0,\epsilon)} |\Phi(y)| \underbrace{|-\Delta_y f(x-y)|}_{\leq \|\Delta f\|_{L^\infty} \mathbb{1}(|y| \leq R)} dy \\
&\leq \|\Delta f\|_{L^\infty} \int_{\mathbb{R}^d} \underbrace{|\Phi(y)| \mathbb{1}(|y| \leq R)}_{L^1(\mathbb{R}^d)} \mathbb{1}(|y| \leq \epsilon) \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

Where  $R > 0$  depends on  $x$  and the support of  $f$  but is independent of  $y$ .

$$\begin{aligned}
\bullet \left| \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \vec{n}}(x-y) dS(y) \right| &\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)| dy \\
&\leq \begin{cases} \text{const} \cdot \epsilon |\log \epsilon| \rightarrow 0, & d = 2 \\ \text{const} \cdot \epsilon \rightarrow 0, & d \geq 3 \end{cases}
\end{aligned}$$

Conclusion:  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$  proved that  $u = \Phi \star f$  and  $f \in C_c^2(\mathbb{R}^d)$ . ■

Thus, if  $f \in C_c^2(\mathbb{R})$ , then  $u = \Phi \star f$  satisfies  $u \in C^2(\mathbb{R}^2)$  and  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$ .

**Remark 2.10** The result holds for a much bigger class of functions  $f$ . For example if  $f \in C_c^1(\mathbb{R})$  we can easily extend the previous proof:

$$\partial_{x_i} u = \int_{\mathbb{R}^d} \Phi(y) \partial_{x_i} f(x-y) dy \in C(\mathbb{R}^d) \Rightarrow u \in C^1(\mathbb{R}^d)$$

Consequently:

$$\partial_{x_i} \partial_{x_j} u = \partial_{x_i} \int_{\mathbb{R}^d} \Phi(y) \partial_{x_j} f(x-y) dy = \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_j} f(x-y) dy \in C(\mathbb{R}^d)$$

So we have  $u \in C^2(\mathbb{R}^d)$ . Now we can compute

$$\Delta u = \sum_{i=1}^d \int_{\mathbb{R}^d} \partial_{x_i} \Phi(y) \partial_{x_i} f(x-y) dy \stackrel{(IBP)}{=} f(x).$$

**Exercise 2.11** Extend this to more general functions!

## 2.3 Equations in general domains

**Theorem 2.12** (Mean Value Theorem for Harmonic Functions) Let  $\Omega \subseteq \mathbb{R}$  be open, let  $u \in C^2(\Omega)$  and  $\Delta u = 0$  in  $\Omega$ . Then

$$u(x) = \oint_{B(x,r)} u = \oint_{\partial B(x,r)} u \quad \text{for all } x \in \Omega, B(x,r) \subseteq \Omega$$

*Proof.* Consider all  $r > 0$  s.t.  $B(x,r) \subseteq \Omega$ ,

$$f(r) = \oint_{\partial B(x,r)} u$$

We need to prove that  $f(r)$  is independent of  $r$ . When it is done, then we immediately obtain

$$f(r) = \lim_{t \rightarrow 0} f(t) = u(x)$$

as  $u$  is continuous. To prove that, consider

$$\begin{aligned}
f'(r) &= \frac{d}{dr} \left( \oint_{\partial B(0,r)} u(x+y) dS(y) \right) \\
&= \frac{d}{dr} \left( \oint_{\partial B(0,1)} u(x+rz) dS(z) \right) \\
(\text{dom. convergence}) \quad &= \oint_{\partial B(0,1)} \frac{d}{dr} [u(x+rz)] dS(z) \\
&= \oint_{\partial B(0,1)} \nabla u(x+rz) z dS(z) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r} dS(y) \\
&= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{\partial B(x,r)} \nabla \cdot u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green 2.2}) \quad &= \frac{1}{|B(x,r)|_{\mathbb{R}^d}} \int_{B(x,r)} \underbrace{(\Delta u)(y)}_{=0} dy = 0 \quad \blacksquare
\end{aligned}$$

**Exercise 2.13** In 1D:  $\Delta u = 0 \Leftrightarrow u'' = 0 \Leftrightarrow u(x) = ax + b$  (Linear Equation)

**Remark 2.14** Recall the polar decomposition. Let  $x \in \mathbb{R}^d, x = (r, w), r = |x| > 0, w \in S^{d-1}$ , then

$$\int_{B(0,r)} g(y) dy = \int_0^r \left( \int_{B(0,s)} g(y) dS(y) \right) ds$$

**Remark 2.15** We already proved that for  $u$  harmonic we have  $u(x) = \oint_{\partial B(x,r)} u dy$ . Now we have

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_{B(0,r)} u(x+y) dy \\
(\text{Pol. decomposition}) \quad &= \int_0^r \left( \int_{\partial B(0,s)} u(x+y) dS(y) \right) ds \\
&= \int_0^r \left( \int_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\text{Mean value property}) \quad &= \int_0^r (|\partial B(x,s)| u(x)) ds = |B(x,r)| u(x)
\end{aligned}$$

This implies

$$\oint_{B(x,r)} u(y) dy = u(x) \quad \text{for any } B(x,r) \subseteq \Omega.$$

**Remark 2.16** The reverse direction is also correct, namely if  $u \in C^2(\Omega)$  and

$$u(x) = \oint_{B(x,r)} u(y) dy = \oint_{\partial B(x,r)} u(y) dy \quad \text{for all } B(x,r) \subseteq \Omega,$$

then  $u$  is harmonic, i.e.  $\Delta u = 0$  in  $\Omega$ . (The proof is exactly like before)

**Theorem 2.17** (Maximum Principle) Let  $\Omega \subseteq \mathbb{R}^d$  be open, let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ ,  $\Delta u = 0$  in  $\Omega$ . Then

- a)  $\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x)$
- b) Assume that  $\Omega$  is connected. Then if there is a  $x_0 \in \Omega$  s.t.  $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$ , then  $u \equiv \text{const.}$  in  $\Omega$ .

*Proof.* Given  $U \subseteq \mathbb{R}^d$  open, we can write  $U = \bigcup_i U_i$ , where  $U_i$  is open and connected.

- b) Assume that  $\Omega$  is connected and there is a  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{y \in \Omega} u(y)$ . Define  $U = \{x \in \Omega \mid u(x) = u(x_0)\} = u^{-1}(u(x_0))$ .  $U$  is closed since  $u$  is continuous. Moreover,  $U$  is open by the mean-value theorem. I.e. for all  $x \in U$  there is a  $r > 0$  s.t.  $B(x, r) \subseteq U \subseteq \Omega$ . Since  $U$  is connected we get  $U = \Omega$ , so  $u$  is constant in  $\Omega$ . On the other hand, if there is no  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{x \in \Omega} u(x)$  we have  $\forall x_0 \in \Omega : u(x) < \sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x)$
- a) Given  $\Omega \subseteq \mathbb{R}^d$  open, we can write  $\Omega = \bigcup_i \Omega_i$ , where  $\Omega_i$  is open and connected. By b) we have

$$\sup_{x \in \bar{\Omega}_i} u(x) = \sup_{x \in \partial\Omega_i} u(x), \quad \forall i$$

So we can conclude

$$\sup_{x \in \bar{\Omega}} u(x) = \sup_{x \in \partial\Omega} u(x). \quad \blacksquare$$

**Definition 2.18** • If  $\Omega \subseteq \mathbb{R}^d$  is open,  $u \in C^2(\Omega)$ , then  $u$  is called *sub-harmonic* if  $\Delta u \geq 0$  in  $\Omega$ .

- If  $\Delta u \leq 0$ , then  $u$  is called *super-harmonic*.

**Exercise 2.19** (E 1.4) Let  $\Omega \subseteq \mathbb{R}^d$  be open and  $u \in C^2(\Omega)$  be subharmonic.

- a) Prove that  $u$  satisfies the Mean Value Inequality

$$\oint_{\partial B(x, r)} u(y) dS(y) \geq \int_{B(x, r)} u(y) dy \geq u(x)$$

for all  $B(x, r) \subseteq \mathbb{R}^d$ .

- b) Assume further that  $\Omega$  is connected and  $u \in C(\bar{\Omega})$ . Prove that  $u$  satisfies the strong maximum principle, namely either

- $u$  is constant in  $\Omega$ , or
- $\sup_{y \in \partial\Omega} u(y) > u(x)$  for all  $x \in \Omega$ .

*My Solution.* a) Let  $f(r) = \oint_{\partial B(x,r)} u(y) dS(y)$ , then we have

$$\begin{aligned}
\partial_r f(r) &= \partial_r \oint_{\partial B(x,r)} u(y) dS(y) \\
(\text{Dom. Convergence}) \quad &= \oint_{\partial B(x,r)} \partial_r u(y) dS(y) \\
&= \oint_{\partial B(0,1)} \partial_r u(x + yr) dS(y) \\
&= \oint_{\partial B(0,1)} \nabla u(x + yr) \cdot y dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r} dS(y) \\
&= \oint_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}_y dS(y) \\
(\text{Gauss-Green}) \quad &= \oint_{B(x,r)} \text{div}(\nabla u(y)) dS(y) \\
&= \oint_{B(x,r)} \underbrace{\Delta u(y)}_{\geq 0} dS(y) \geq 0
\end{aligned}$$

So we can conclude that

$$\oint_{\partial B(x,r)} u(y) dS(y) = f(r) \geq \lim_{r \rightarrow 0} f(r) = u(x).$$

Now regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left( \int_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&= \int_0^r \left( |\partial B(x,r)| \oint_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&\geq \int_0^r |\partial B(x,r)| \cdot u(x) dS(y) \\
&= u(x) \int_0^r |\partial B(x,r)| dS(y) = u(x) |B(x,r)|.
\end{aligned}$$

Thus we have

$$u(x) \leq \oint_{B(x,r)} u(y) dy.$$

Finally, lets regard

$$\begin{aligned}
\int_{B(x,r)} u(y) dy &= \int_0^r \left( |\partial B(x,s)| \oint_{\partial B(x,s)} u(y) dS(y) \right) ds \\
(\partial_r f(r) \geq 0) \quad &\leq \int_0^r \left( |\partial B(x,s)| \oint_{\partial B(x,r)} u(y) dS(y) \right) ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \int_0^r |\partial B(x,s)| ds \\
&= \oint_{\partial B(x,r)} u(y) dS(y) \cdot |B(x,s)|
\end{aligned}$$

and we conclude

$$\oint_{B(x,r)} u(y) dy \leq \oint_{\partial B(x,r)} u(y) dS(y).$$

b) Let  $x_0 \in \Omega$  s.t.  $u(x_0) = \sup_{x \in \Omega} u(x)$ . Now,

$$\begin{aligned} \sup_{x \in \Omega} u(x) = u(x_0) &\leq \oint_{\partial B(x_0,r)} u(y) dy \\ &\leq \oint_{\partial B(x_0,r)} \sup_{x \in \Omega} u(x) dy = \sup_{x \in \Omega} u(x) \end{aligned}$$

Since  $u$  is continuous we get  $u(y) = u(x_0)$  for all  $y \in B(x_0, r)$ , so  $u$  is constant. ■

**Definition 2.20** The *Poisson Equation* for given  $f, g$  on a bounded set is:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

**Theorem 2.21** (Uniqueness) Let  $\Omega \subseteq \mathbb{R}^d$  be bounded, open and connected. Let  $f \in C(\Omega), g \in C(\partial\Omega)$ . Then there exists *at most* one solution  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ , s.t.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

*Proof.* Assume that we have two solutions  $u_1$  and  $u_2$ . Then  $u := u_1 - u_2$  is a solution to

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

By the maximum principle, we know that  $u = 0$  in  $\Omega$ . More precisely, by the maximum principle we have  $\forall x \in \Omega$

$$\sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} u(x) = 0 \quad \Rightarrow \quad u(x) \leq 0$$

Since  $-u$  satisfies the same property we have  $\forall x \in \Omega$ :

$$\sup_{x \in \Omega} (-u(x)) \leq \sup_{x \in \partial\Omega} (-u(x)) = 0 \quad \Rightarrow \quad -u(x) \leq 0 \quad \Rightarrow \quad u(x) \geq 0$$

So we get  $u(x) = 0$  in  $\Omega$ . ■

**Exercise 2.22** (Bonus 1) Let  $\Omega$  be open, connected and bounded in  $\mathbb{R}^d$ . Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  s.t.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Prove that

a) If  $g \geq 0$  on  $\partial\Omega$ , then  $u \geq 0$  in  $\Omega$ .

b) If  $g \geq 0$  on  $\partial\Omega$  and  $g \neq 0$ , then  $u > 0$  in  $\Omega$ .

**Lemma 2.23** (Estimates for derivatives) If  $u$  is harmonic in  $\Omega \subseteq \mathbb{R}^d$ ,  $\alpha \in \mathbb{N}_0^d$ ,  $|\alpha| = N$  and  $B(x_0, r) \subseteq \Omega$ , then

$$|D^\alpha u(x)| \leq \frac{(c_d N)^N}{r^{d+N}} \int_{B(x, r)} |u| dy$$

*Proof.* Induction: Assume  $|\alpha| = N - 1$ , Take  $|\alpha| = N$

$$|D^\alpha u(x_0)| \leq \frac{|S_1|}{|B_1| \frac{r}{N}} \|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))}, \quad D^\alpha u = \partial_{x_i}(D^\beta u)_{|\beta|=N-1}$$

Note:  $x \in B(x_0, \frac{r}{N})$ , so  $B(x, \frac{r(N-1)}{N}) \subseteq B(x_0, r)$ . By the induction hypothesis:

$$\|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))} \leq \frac{[c_d(N-1)]^{N-1}}{[r \frac{(N-1)}{N}]^{d+N-1}} \int_{B(x_0, r)} |u| dy$$

The conclusion is:

$$\begin{aligned} |D^\alpha u(x_0)| &\leq \frac{|S_1|}{|B_1| \frac{r}{N}} \frac{[c_d(N-1)]^{N-1}}{(r \frac{N-1}{N})^{d+N-1}} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} (N-1)^d} \int_{B(x_0, r)} |u| dy \\ &= \frac{|S_1|}{|\beta_1|} \frac{c_d^{N-1}}{(\frac{r}{N})^{d+N} N^d} \left(\frac{N}{N-1}\right)^d \int_{B(x_0, r)} |u| dy \\ &\leq \frac{2^d |S_1|}{|B_1|} \frac{c_d^{N-1} N^N}{r^{d+N}} \int_{B(x_0, r)} |u| dy \quad \text{if } c_d \geq \frac{2^d |S_1|}{|B_1|} \end{aligned}$$

■

**Theorem 2.24** (Regularity) Let  $\Omega$  be open in  $\mathbb{R}^d$ . Let  $u \in C(\Omega)$  satisfy  $u(x) = \int_{\partial B} u dy$  for any  $x \in B(x, r) \subseteq \Omega$ . Then  $u$  is a harmonic function in  $\Omega$ . Moreover,  $u \in C^\infty(\Omega)$  and  $u$  is analytic in  $\Omega$ .

**Exercise 2.25** (E 1.1: Proof the Gauss–Green formula) Let  $f := (f_i)_1^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ . Prove that for every open ball  $B(y, r) \subseteq \mathbb{R}^d$  we have

$$\int_{\partial B(y, r)} f(y) \cdot \nu_y dS(y) = \int_{B(y, r)} \operatorname{div} f dx.$$

Here  $\nu_y$  is the outward unit normal vector and  $dS$  is the surface measure on the sphere.

*Solution.* We proof this in  $d=3$ . Let  $f \in C^1(\mathbb{R}^3)$

$$\int_{B(0,1)} \partial_{x_3} f dx = \int_{\partial B(0,1)} f x_3 dS(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \vec{n} = \frac{x}{|x|} \text{ on } \partial B(0,1)$$

$$\begin{aligned} B(0,1) &= \{x_1^2 + x_2^2 + x_3^2 \leq 1\} \\ &= \{x_1^2 + x_2^2 \leq 1 - \sqrt{1 - x_1^2 - x_2^2} \leq x_3 \leq \sqrt{1 - x_1^2 - x_2^2}\} \end{aligned}$$

Then:

$$\begin{aligned}\int_{B(0,1)} \partial_{x_3} f \, dx &= \int_{x_1^2 + x_2^2 \leq 1} \left( \int_{-\sqrt{1-x_1^2-x_2^2} \leq x_3 \leq \sqrt{1-x_1^2-x_2^2}} \partial_{x_3} f \, dx_3 \right) dx_1 \, dx_2 \\ &= \int_{x_1^2 + x_2^2 \leq 1} \left[ f(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \right. \\ &\quad \left. - f(x_1, x_2, -\sqrt{1-x_1^2-x_2^2}) \right] dx_1 \, dx_2\end{aligned}$$

Lets take polar coordinates in 2D:

$$\begin{aligned}x_1 &= r \cos \phi & r > 0, \phi \in [0, 2\pi) \\ x_2 &= r \sin \phi & \det \frac{\partial(x_1, x_2)}{\partial(r, \phi)} = r\end{aligned}$$

$$(\star) = \int_0^1 \int_0^{2\pi} [f(r \cos \phi, r \sin \phi, r) - f(r \cos \phi, r \sin \phi, -r)] r \, dr \, d\phi$$

On the other hand:

$$\int_{\partial B(0,1)} f x_3 \, dS$$

The polar coordinates in 3D are:

$$\begin{aligned}x_1 &= r \cos \phi \sin \theta & r > 0, \phi \in (0, 2\pi), \theta \in (0, \pi) \\ x_2 &= r \sin \phi \sin \theta & \det \frac{\partial(x_1, x_2, x_3)}{\partial(r, \phi, \theta)} = r^2 \sin \theta \\ x_3 &= r \cos \theta\end{aligned}$$

Then:

$$\begin{aligned}(\star\star) &= \int_0^{2\pi} \int_0^\pi f(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \sin \theta \cos \theta \, d\theta \, d\phi \\ &= \int_0^{2\pi} \left( \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^\pi d\theta \right) d\phi \\ (r = \sin \theta) &= \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, \sqrt{1-r^2}) r \, dr \, d\phi \\ &\quad - \int_0^{2\pi} \int_0^1 f(r \cos \phi, r \sin \phi, -\sqrt{1-r^2}) r \, dr \, d\phi\end{aligned} \quad \blacksquare$$

**Exercise 2.26** (E 1.2) Let  $u \in C(\mathbb{R}^d)$  and  $\int_{B(x,r)} u \, dy = 0$  for every open ball  $B(x,r) \subseteq \mathbb{R}^d$ . Show that  $u(x) = 0$  for all  $x \in \mathbb{R}^d$ .

*My Solution.* Assume there is a  $x_0 \in \mathbb{R}^d$  s.t. w.l.o.g.  $u(x_0) > 0$ . Since  $u$  is continous there is a ball  $B(x_0, r)$  s.t.  $u(y) > \frac{u(x_0)}{2}$  for all  $y \in B(x_0, r)$ . But then we get

$$\int_{B(x_0, r)} u(y) \, dy \geq \int_{B(x_0, r)} \frac{u(x_0)}{2} \, dy = \frac{u(x_0)}{2} |B(x_0, r)| > 0. \quad \blacksquare$$

**Exercise 2.27** (E 1.3) Let  $f \in C_c^1(\mathbb{R}^d)$  with  $d \geq 2$  and  $u(x) := (\Phi \star f)(x)$ . Prove that  $u \in C^2(\mathbb{R}^2)$  and  $-\Delta u(x) = f(x)$  for all  $x \in \mathbb{R}^d$  (2.9 was the same for  $f \in C_1(\mathbb{R})$ )

**Theorem 2.28** (Liouville's Theorem) If  $u \in C^2(\mathbb{R}^d)$  is harmonic and bounded, then  $u = \text{const.}$

*Proof.* By the bound of the derivative 2.23 we have

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\leq \frac{c_d}{r^{d+1}} \int_{B(x_0, r)} |u| dy \quad \forall x_0 \in \mathbb{R}^d \quad \forall r > 0 \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r^{d+1}} |B(x_0, r)| \\ &\leq \|u\|_{L^\infty} \frac{c_d}{r} \xrightarrow{r \rightarrow \infty} 0 \end{aligned}$$

Thus  $\partial_{x_i} u = 0$  for all  $i = 1, 2, \dots, d$  and  $u = \text{const.}$  in  $\mathbb{R}^d$  ■

**Theorem 2.29** (Uniqueness of solutions to Poisson Equation in  $\mathbb{R}^d$ ) If  $u \in C^2(\mathbb{R}^d)$  is a bounded function and satisfies  $-\Delta u = f$  in  $\mathbb{R}^d$  where  $f \in C_c^2(\mathbb{R}^d)$ , then we have

$$u(x) = \Phi \star f(x) + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y) dy + C \quad \forall x \in \mathbb{R}^d$$

where  $C$  is a constant and  $\Phi$  is the fundamental solution of the Laplace equation in  $\mathbb{R}^d$ .

*Proof.* If we can prove that  $v$  is bounded, then  $v = \text{const.}$  We first need to show that  $\Phi \star f$  is bounded.

$$\begin{aligned} \Phi &= \Phi_1 + \Phi_2 = \Phi \mathbb{1}(|x| \leq 1) + \Phi(|x| \geq 1) \\ \Phi \star f &= \Phi_1 \star f + \Phi_2 \star f \end{aligned}$$

We have  $\Phi_1 \star f \in L^1(\mathbb{R}^d)$  and  $\Phi_2 \star f$  is bounded since  $\Phi \rightarrow 0$  as  $|x| \rightarrow \infty$  in  $d \geq 3$ . ■

**Exercise 2.30** (Hanack's inequality) Let  $u \in C^2(\mathbb{R}^d)$  be harmonic and non-negative. Prove that for all open, bounded and connected  $\Omega \subseteq \mathbb{R}^d$ , we have

$$\sup_{x \in \Omega} u(x) \leq C_\Omega \inf_{x \in \Omega} u(x),$$

where  $C_\Omega$  is a finite constant depending only on  $\Omega$ .

*Proof.* (Exercise) Hint:  $\Omega = B(x, r)$ . General case cover  $\Omega$  by finitely many balls, one ball is inside  $\Omega$ . ■

**Exercise 2.31** (E 3.1 Lebesgue Differentiation Theorem) Let  $f \in L^1_{loc}(\mathbb{R}^d)$ . Prove that that for almost every  $x \in \mathbb{R}^d$ :

$$\oint_{B(x, r)} |f(x) - f(y)| dy \xrightarrow{r \rightarrow 0} 0$$

*Proof.* Clearly the same result holds with  $\mathbb{R}^d \rightsquigarrow \Omega \subseteq \mathbb{R}^d$  open. Also it suffices to consider  $f \in L^1(\mathbb{R}^d)$ . From the last time discussion, by a density argument there exists  $r_n \rightarrow 0$  s.t.

$$\oint_{B(x, r_n)} |f(y) - f(x)| dy = 0$$



for a.e.  $x \in \mathbb{R}^d$ . We prove that for all  $\epsilon > 0$ , the set  $A_\epsilon = \{x \in \mathbb{R}^d \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy > \epsilon\}$  has measure 0. This will imply that

$$\bigcup_{n=1}^{\infty} A_{\frac{1}{n}} = \left\{ x \in \mathbb{R}^d \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy > 0 \right\}$$

has measure 0, which is what we want to show. First, we show that  $|A_\epsilon| = 0$ : Take  $\{f_n\} \subseteq C_c^\infty$ ,  $f_n \rightarrow f$  in  $L^1(\mathbb{R}^d)$ . By the triangle inequality:

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

So we get

$$\begin{aligned} & \int_{B(x,r)} |f(y) - f(x)| dy \\ & \leq \int_{B(x,r)} |f(y) - f_n(y)| dy + \int_{B(x,r)} |f_n(y) - f_n(x)| + |f_n(x) - f(x)| dy \\ \Rightarrow \quad \limsup_{r \rightarrow 0} \dots & \leq \limsup_{r \rightarrow 0} (\dots) + 0 + |f_n(x) - f(x)| \end{aligned}$$

Thus, for all  $x \in A_\epsilon$ , then:

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy + |f_n(x) - f(x)| > 2\epsilon$$

Observation: If  $a, b \geq 0$ ,  $a + b > 2\epsilon$  then either  $a > \epsilon$  or  $b > \epsilon$ . Therefore  $A_\epsilon \subseteq (S_{n,\epsilon} \cup \tilde{S}_{n,\epsilon})$ , where

$$\begin{aligned} S_{n,\epsilon} &= \{x \mid |f_n(x) - f(x)| > \epsilon\} \\ \tilde{S}_{n,\epsilon} &= \{x \mid \limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy > \epsilon\} \end{aligned}$$

Consequently:  $|A_\epsilon| \leq |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}|$  for all  $n \geq 1$ . By the Markov / Chebyshev inequality:

$$|S_{n,\epsilon}| \leq \int_{S_{n,\epsilon}} \frac{|f_n(x) - f(x)|}{\epsilon} dx = \int_{\mathbb{R}^d} \frac{|f_n(x) - f(x)|}{\epsilon} dx = \frac{\|f_n - f\|_{L^1}}{\epsilon}$$

We want to prove a simpler bound for  $\tilde{S}_{n,\epsilon}$ . For all  $x \in \tilde{S}_{n,\epsilon}$ :

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} |f_n(y) - f(y)| dy > \epsilon$$

So there is a  $r_x \in (0, 1)$  s.t.

$$\int_{B(x, r_x) = B_x} |f_n(y) - f(y)| dy > \epsilon$$

Thus  $\tilde{S}_{n,\epsilon} \subseteq \left( \bigcup_{x \in \tilde{S}_{n,\epsilon}} B_x \right)$ .

**Lemma 2.32** (Vitali Covering) If  $F$  is a collection of balls in  $\mathbb{R}^d$  with bounded radius, then there exists a sub-collection  $G \subseteq F$  s.t.

- $G$  has disjoint balls

- $\bigcup_{B \in F} B \subseteq \bigcup_{B \in G} 5B, 5B(x, r) = B(x, 5r)$

**Remark 2.33** The condition of the boundedness of the radius is necessary. Otherwise, consider  $\{B(0, n)\}_{n=1}^{\infty}$

Here consider  $F = \{B_x\}_{x \in \tilde{S}_{n,\epsilon}}$ . With the Vitali covering lemma there is a  $G \subseteq F$  s.t.  $G$  contains disjoint balls and:

$$\tilde{S}_{n,\epsilon} \subseteq \bigcup_{B \in F} B \subseteq \bigcup_{B \in G} 5B$$

So we get

$$|\tilde{S}_{n,\epsilon}| \leq \left| \bigcup_{B \in G} 5B \right| \leq \sum_{B \in G} |5B| = \sum_{B \in G} 5^d |B|$$

On the other hand, for all  $B \in G \subseteq F$ :

$$\int_B |f_n(y) - f(y)| dy > \epsilon \Rightarrow \int_B |f_n - f| > \epsilon |B|$$

This implies:

$$\sup_{B \in G} \int_B |f_n - f| > \epsilon \sum_{B \in G} |B|$$

Since balls in  $G$  are disjoint:

$$\int_{\mathbb{R}^d} \geq \int_{\bigcup_{B \in G}} |f_n - f| dy > \epsilon \sum_{B \in G} |B| \geq \frac{\epsilon}{5^d} |\tilde{S}_{n,\epsilon}|$$

So

$$|\tilde{S}_{n,\epsilon}| \leq \frac{5^d}{\epsilon} \|f_n - f\|_{L^1}$$

In summary:

$$|A_\epsilon| \leq |S_{n,\epsilon}| + |\tilde{S}_{n,\epsilon}| \leq \frac{5^d + 1}{\epsilon} \|f_n - f\|_{L^1} \rightarrow 0$$

as  $n \rightarrow \infty$ . So  $|A_\epsilon| = 0$  for all  $\epsilon > 0$  ■

**Remark 2.34** 1. The proof can be done by using the Besicovitch covering lemma: For all  $E \subseteq \mathbb{R}^d$  s.t.  $E$  is bounded. Let  $F =$  collection of balls s.t. for all  $x \in E$  there is a  $B_x \in F$  s.t.  $x$  is the center of  $B_x$ . There is a sub-collection  $G \subseteq F$  s.t.

- $E \subseteq \bigcup_{B \in G} B$
- Any point in  $E$  belongs to at most  $C_d$  balls in  $G$  ( $C_d$  depends only on  $\mathbb{R}^d$ ), i.e.

$$\mathbb{1}_E(x) \leq \sum_{B \in G} \mathbb{1}_B(x) \leq C_d \mathbb{1}_E(x) \forall x$$

2. By a simpler argument we can prove the weak  $L^1$ -estimate:

$$\{x \mid f^\star(x) > \epsilon\} \leq \frac{C_d}{\epsilon} \|f\|_{L^1(\mathbb{R}^d)}$$

(Hardy-Littlewood maximal function)

**Exercise 2.35** (E 3.3)  $f \in C_c^\infty(\mathbb{R}^d)$ . Prove  $|\hat{f}(k)| \leq \frac{C_N}{(1+|k|)^N}$

*Solution.* Since  $f \in C_c^\infty$  we have that  $D^\alpha f \in C_c^\infty$ . Recall

$$\widehat{D^\alpha f}(k) = (-2\pi i k)^\alpha \hat{f}(k)$$

For example

$$\begin{aligned} \widehat{-\Delta f}(k) &= |2\pi i k|^2 \hat{f}(k) \\ (\text{Induction}) \rightsquigarrow \widehat{(-\Delta)^N f}(k) &= |2\pi k|^{2N} \hat{f}(k) \end{aligned}$$

So we can conclude

$$\hat{f}(k) = \frac{\widehat{(-\Delta)^N f}(k)}{|2\pi k|^{2N}} \quad \forall k \in \mathbb{R}^d$$

1.  $f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \hat{f} \in L^\infty$
2.  $(-\Delta)^N f \in C_c^\infty \subseteq L^1(\mathbb{R}^d) \Rightarrow \widehat{(-\Delta)^N f} \in L^\infty$

Conclusion:  $\hat{f}(k) \leq \begin{cases} C & \forall k \\ \frac{C_N}{|k|^{2N}} & \forall k \end{cases}$  So  $\hat{f}(k) \leq \frac{C_N}{(1+|k|)^N}$  ■

## Chapter 3

# Convolution, Fourier Transform and Distributions

### 3.1 Convolutions

**Definition 3.1** (Convolution) Let  $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$  or  $\mathbb{C}$ .

$$(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) dy = \int_{\mathbb{R}^d} f(y)g(x-y) dy = (g \star f)(x)$$

**Remark 3.2** (Properties of the Convolution)

- $f \star g = g \star f$
- $(f \star g) \star h = f \star (g \star h)$  (by Fubini)
- $\widehat{f \star g} = \hat{f}\hat{g}$  (Proof, see exercise 3.21)

**Theorem 3.3** (Young Inequality) If  $f \in L^1(\mathbb{R}^d)$  and  $g \in L^p(\mathbb{R}^d)$ , where  $1 \leq p \leq \infty$ , then  $f \star g \in L^p(\mathbb{R}^d)$  and

$$\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

More generally, if  $f \in L^p(\mathbb{R}^d)$ ,  $g \in L^q(\mathbb{R}^d)$ , then  $f \star g \in L^r(\mathbb{R}^d)$ ,

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q},$$

where  $1 \leq p, q, r \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ .

*Proof.* Let  $f \in L^1, g \in L^p$ . With the Hölder Inequality we have:

$$\begin{aligned}
|(f \star g)(x)| &= \left| \int_{\mathbb{R}^d} f(x-y)g(y) dy \right| \\
&\leq \left( \int_{\mathbb{R}^d} |f(x-y)| dy \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^d} |f(x-y)||g(y)|^p dy \right)^{\frac{1}{p}} \\
&= \|f\|_{L^1}^{\frac{1}{q}} \left( \int_{\mathbb{R}^d} |f(x-y)||g(y)|^p dy \right)^{\frac{1}{p}} \\
\|f \star g\|_{L^p}^p &= \int_{\mathbb{R}^d} |f \star g(x)|^p dx \\
&\leq \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)||g(y)|^p dy dx \\
&= \|f\|_{L^1}^{\frac{p}{q}+1} \|g\|_{L^p}^p
\end{aligned}$$

So we have  $\|f \star g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$  ■

**Theorem 3.4** (Smoothness of the Convolution) If  $f \in C_c^\infty(\mathbb{R}^d)$ ,  $g \in L^p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ . Then  $f \star g \in C^\infty(\mathbb{R})$  and

$$D^\alpha(f \star g) = (D^\alpha f) \star g$$

for all  $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \{0, 1, 2, \dots\}$

*Proof.* First we note that  $x \mapsto (f \star g)$  is continuous as  $x_n \rightarrow x$  in  $\mathbb{R}^d$  since

$$(f \star g)(x_n) = \int_{\mathbb{R}^d} f(x_n - y)g(y) dy \xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^d} f(x - y)g(y) dy = (f \star g)(x)$$

We can apply Dominated convergence because

$$f(x_n - y)g(y) \rightarrow f(x - y)g(y) \quad \forall y \text{ as } f \text{ is continuous and } x_n \rightarrow x$$

and

$$|f(x_n - y)g(y)| \leq \|f\|_{L^\infty} |g(y)| \mathbf{1}(|y| \leq R) \in L^1(\mathbb{R}^d).$$

Where  $R > 0$  satisfies  $B(0, R) \supseteq \text{supp } f + \sup_n |x_n|$ . Now we can compute the derivatives:

$$\begin{aligned}
\partial_{x_i}(f \star g)(x) &= \lim_{h \rightarrow 0} \frac{(f \star g)(x + he_i) - (f \star g)(x)}{h} \\
&= \lim_{h \rightarrow 0} \int_{\mathbb{R}^d} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\
(\text{Dominated Convergence}) \quad &= \int_{\mathbb{R}^d} \lim_{h \rightarrow 0} \frac{f(x + he_i - y) - f(x - y)}{h} g(y) dy \\
&= \int_{\mathbb{R}^d} (\partial_{x_i} f)(x - y)g(y) dy
\end{aligned}$$

We could apply Dominated Convergence since

$$\begin{aligned}
&\frac{f(x + he_i - y) - f(x - y)}{h} g(y) \xrightarrow{h \rightarrow 0} (\partial_{x_i} f)(x - y)g(y) \quad \text{as } f \in C^1 \\
&\left| \frac{f(x + he_i - y) - f(x - y)}{h} g(y) \right| \leq \|\partial_{x_i} f\|_{L^\infty} |g(y)| \mathbf{1}(|y| \leq R) \in L^1(\mathbb{R}^d)
\end{aligned}$$

where  $B(0, R) \supseteq \text{supp}(f) + B(0, |x| + 1)$  and  $\partial_{x_i}(f \star g) = (\partial_{x_i} f) \star g \in C(\mathbb{R}^d)$  since  $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$ . By induction we get  $D^\alpha(f \star g) = (D^\alpha f \star g) \in C(\mathbb{R}^d)$ . ■

**Remark 3.5** Question: Is there a  $f$  s.t.  $f \star g = g$  for all  $g$ ? In fact there is no regular function  $f$  that solves this formally:

$$f \star g = g \Rightarrow \widehat{f \star g} = \widehat{g} \Rightarrow \hat{f} \hat{g} = \hat{g} \Rightarrow \hat{f} = 1 \Rightarrow f \text{ is not a regular function!}$$

However, if  $f$  is the Dirac-Delta Distribution,  $f = \delta_0$  then  $\delta_0 \star g = g$  for all  $g$ . Formally:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \\ \int \delta_0 = 1 \end{cases}$$

In fact, if  $f \in L^1(\mathbb{R}^d)$ ,  $\int f = 1$ ,  $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ , then  $f_\epsilon \rightarrow \delta_0$  in an appropriate sense and  $f_\epsilon \star g \rightarrow g$  for all  $g$  nice enough.

**Theorem 3.6** (Approximation by convolution) Let  $f \in L^1(\mathbb{R}^d)$ ,  $\int f = 1$ ,  $f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon})$ . Then for all  $g \in L^p(\mathbb{R}^d)$ , where  $1 \leq p < \infty$ , then

$$f_\epsilon \star g \rightarrow g \quad \text{in } L^p(\mathbb{R}^d)$$

*Proof.*

Step 1: Let  $f, g \in C_c(\mathbb{R}^d)$ . Then

$$\begin{aligned} (f_\epsilon \star g)(x) - g(x) &= \int_{\mathbb{R}^d} f_\epsilon(y) g(x-y) dy - g(x) \int_{\mathbb{R}^d} f_\epsilon(y) dy \\ &= \int_{\mathbb{R}^d} f_\epsilon(y) g(x-y) dy - \int_{\mathbb{R}^d} f_\epsilon(y) g(x) dy \\ &= \int_{\mathbb{R}^d} f_\epsilon(y) (g(x-y) - g(x)) dy \\ |(f_\epsilon \star g)(x) - g(x)| &= \left| \int_{\mathbb{R}^d} f_\epsilon(y) (g(x-y) - g(x)) dy \right| \\ &\leq \int_{\mathbb{R}^d} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\ &\leq \int_{|y| \leq R_\epsilon} |f_\epsilon(y)| |g(x-y) - g(x)| dy \\ &\leq \left[ \sup_{|z| \leq R} |g(x-z) - g(x)| \right] \underbrace{\int_{|y| \leq R_\epsilon} |f_\epsilon(y)| dy}_{\leq \|f_\epsilon\|_{L^1} = \|f\|_{L^1}} \xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned}$$

We have Dominated Convergence since:

$$(f_\epsilon \star g)(x) - g(x) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

and

$$|f_\epsilon \star g(x) - g(x)| \leq \|f\|_{L^1} \sup_{|z| \leq R_\epsilon} |g(x-z) - g(x)| \leq 2\|f\|_1 \|g\|_{L^\infty} \mathbf{1}(|x| \leq R_1).$$

Where  $B(0, R_1) \supseteq \text{supp}(g) + B(0, R_\epsilon)$ , thus  $f_\epsilon \star g \rightarrow g$  in  $L^p(\mathbb{R}^d)$ . To remove the technical assumptions  $f, g \in C_c(\mathbb{R}^d)$ , then we use a density argument. We use the fact that  $C_c(\mathbb{R}^d)$  is dense in  $L^p(\mathbb{R}^d)$ ,  $1 \leq p < \infty$ .

Step 2: Let  $g \in C_c(\mathbb{R}^d), g \in L^p(\mathbb{R}^d)$ . Then there is  $\{g_m\} \subseteq L^p(\mathbb{R}^d), g_m \rightarrow g$  in  $L^p(\mathbb{R}^d)$ . Then

$$\begin{aligned} \|f_\epsilon \star g - g\|_{L^p} &\leq \|f_\epsilon \star (g - g_m)\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\ (\text{Young}) \quad &\leq \|f_\epsilon\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\ &\leq \|f\|_{L^1} \|g - g_m\|_{L^p} + \|f_\epsilon \star g_m - g_m\|_{L^p} + \|g_m - g\|_{L^p} \\ &\leq (\|f\|_{L^1} + 1) \|g - g_m\|_{L^p} + \|f \star g_m - g_m\|_{L^p} \end{aligned}$$

So we get:

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq (\|f\|_{L^p} + 1) \|g - g_m\|_{L^p} + \underbrace{\limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g_m - g_m\|_{L^p}}_{0 \text{ by step 1.}} \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Step 3: Let  $f \in L^1(\mathbb{R}^d)$  and  $g \in L^p(\mathbb{R}^d)$ . Take  $\{f_m\} \subseteq C_c(\mathbb{R}^d)$ , s.t.

$$\begin{cases} F_m \rightarrow g \in L^1(\mathbb{R}) \text{ as } m \rightarrow \infty \\ \int_{\mathbb{R}^d} F_m = 1 \text{ (it is possible since } \int_{\mathbb{R}^d} f = 1) \end{cases}$$

Define  $F_{m,\epsilon}(x) = \epsilon^{-d} F_m(\epsilon^{-1}x)$  (recall  $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ ). Then:

$$\begin{aligned} f_\epsilon \star g - g &= (f_\epsilon - F_{m,\epsilon}) \star g + F_{m,\epsilon} \star g - g \\ \Rightarrow \|f_\epsilon - g\|_{L^p} &\leq \underbrace{\|f_\epsilon - F_{m,\epsilon} \star g\|_{L^p}}_{\substack{\text{Young} \\ \leq \|f_\epsilon - F_{m,\epsilon}\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p}}} + \|F_{m,\epsilon} \star g - g\|_{L^p} \\ \Rightarrow \limsup_{\epsilon \rightarrow 0} \|f_\epsilon \star g - g\|_{L^p} &\leq \|f - F_m\|_{L^1} \|g\|_{L^p} = \|f - F_m\|_{L^1} \|g\|_{L^p} \quad \blacksquare \end{aligned}$$

**Lemma 3.7**  $C_c(\mathbb{R}^d)$  is dense in  $L^p(\mathbb{R}^d)$ ,  $1 \leq p < \infty$

*Proof.* For all  $g \in L^p(\mathbb{R}^d)$  there are step functions  $(g_m)_m$  and  $g_m \rightarrow g$  in  $L^p(\mathbb{R}^d)$ ,

$$g_m(x) = \sum_{\substack{\Omega \\ \text{finite sum} \\ \Omega \subseteq \mathbb{R}^d \text{ measurable}}} \chi_\Omega(x) a_\Omega.$$

We can assume that  $\Omega$  is open and bounded and we want to approximate  $\chi_\Omega$  by  $C_c(\mathbb{R}^d)$ .  $\blacksquare$

**Lemma 3.8** (Urnson) Define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \epsilon\}$$

Then there is a  $\eta_\epsilon \in C_c(\mathbb{R}^d)$  s.t.

$$\begin{cases} 0 \leq \eta(x) \leq 1 & \forall x \in \mathbb{R}^d \\ \eta_\epsilon(x) = 1 & \text{if } x \in \Omega_\epsilon \\ \eta_\epsilon(x) = 0 & \text{if } x \notin \Omega \end{cases}$$

**Lemma 3.9** (General Version of Urnson) If  $A, B \subseteq \mathbb{R}^d$ ,  $A$  closed,  $B$  closed,  $A \cap B = \emptyset$ . Then

$$\eta(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}$$

Then  $\eta \in C(\mathbb{R}^d)$ ,  $0 \leq \eta \leq 1$  and  $\eta = 0$  if  $x \in B$ ,  $\eta = 1$  if  $x \in A$ .

For example, this lemma can be applied to  $A = \overline{\Omega_\epsilon} \subsetneq \Omega$  and  $B = \mathbb{R}^d \setminus \Omega$  for  $\Omega \subseteq \mathbb{R}^d$  open.

**Theorem 3.10** (Appendix C4 in Evans) Let  $\Omega$  be open in  $\mathbb{R}^d$  and for  $\epsilon > 0$  define

$$\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \mathbb{R}^d \setminus \Omega) > \epsilon\}$$

Let  $f \in C_c^\infty(\mathbb{R}^d)$ ,  $\int_{\mathbb{R}^d} f = 1$ ,  $\text{supp } f \subseteq B(0, 1)$ ,  $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ . Then  $\text{supp } f_\epsilon \subseteq B(0, \epsilon)$  and for all  $g \in L_{loc}^p(\Omega)$  (i.e.  $\mathbb{1}_K g \in L^p(\Omega)$  for all  $K \subseteq \Omega$  compact) we have

- a)  $g_\epsilon(x) = (f_\epsilon \star g)(x) = \int_{\mathbb{R}^d} f_\epsilon(x-y)g(y) dy - \int_\Omega f_\epsilon(x-y)g(y) dy$  is well-defined in  $\Omega_\epsilon$  and  $g_\epsilon \in C^\infty(\Omega_\epsilon)$ ,
- b)  $g_\epsilon \rightarrow g$  in  $L_{loc}^p(\Omega)$  if  $1 \leq p < \infty$  and  $g_\epsilon(x) \rightarrow g(x)$  almost everywhere  $x \in \Omega$ ,
- c) If  $g \in C(\Omega)$ , then  $g_\epsilon(x) \rightarrow g(x)$  uniformly in any compact subset of  $\Omega$ .

*Proof.* a)  $D^\alpha(g_\epsilon) = (D^\alpha f_\epsilon) \star g \in C(\Omega_\epsilon)$

b) Already proved in  $\mathbb{R}^d$  space. ■

**Corollary 3.11** (Lebesgue differentiation theorem) If  $f \in L_{loc}^p(\mathbb{R}^d)$ , then

$$\int_{B(x, \epsilon)} |f(y) - f(x)|^p dy \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0$$

**Exercise 3.12** (E 2.1) Let  $u \in C^2(\mathbb{R}^2)$  be convex, i.e.

$$tu(x) + u(y)(1-t) \geq u(tx + (1-t)y)$$

for all  $x, y \in \mathbb{R}^d$ ,  $t \in [0, 1]$ .

- a) Prove for all  $x \in \mathbb{R}^d$  that the Hessian matrix  $H$  is positive semidefinite.
- b) Prove that  $u$  is sub-harmonic in  $\mathbb{R}^d$ .

*Solution.*

- a) In 1D: If  $u$  is convex  $\Leftrightarrow u''(x) \geq 0$  for all  $x \in \mathbb{R}$ . In general: Taylor expansion for all  $x, z \in \mathbb{R}^d$ :

$$u(x) = u(z) + \nabla u(z)(x-z) + \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z)) \frac{(x-z)^\alpha}{\alpha!} ds$$

Note that we have  $x = z + s(x-z)$  if  $s = 1$ . Use  $z = tx + (1-t)y \Rightarrow x-z = (1-t)(x-y)$

$$tu(x) = tu(z) + t\nabla u(z)(1-t)(x-y) + t \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z)) \frac{[(1-t)(x-y)]^\alpha}{\alpha!} ds$$

$$(1-t)u(y) = (1-t)u(z) + (1-t)\nabla u(z)t(y-x) + (1-t) \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y-z)) \frac{[t(y-x)]^\alpha}{\alpha!} ds$$



$$\begin{aligned} &\Rightarrow tu(x) + (1-t)u(y) = u(z) + t \int_0^1 \dots + (1-t) \int_0^1 \dots \\ &\Rightarrow t \int_0^1 \dots + (1-t) \int_0^1 \dots \geq 0 \forall x, y, t, z = tx + (1-t)y \end{aligned}$$

$$t(1-t)^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(x-z)) \frac{(x-y)^\alpha}{\alpha!} ds + (1-t)t^2 \int_0^1 \sum_{|\alpha|=2} D^\alpha u(z + s(y-z)) \frac{(y-z)^\alpha}{\alpha!} ds \geq 0$$

for all  $x, y \in \mathbb{R}^d$ ,  $t \in [0, 1]$ ,  $z = tx + (1-t)y$ . Divides for  $t(1-t)$

$$(1-t) \int_0^1 \dots + \int_0^1 \dots \geq 0$$

Take  $t \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(y + s(x-y)) \frac{(x-y)^\alpha}{\alpha!} ds \geq 0 \forall x, y \in \mathbb{R}^d$$

Take  $y = x + a$ ,  $a \in \mathbb{R}^d$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x + a + sa) \frac{a^\alpha}{\alpha!} ds \geq 0 \forall \epsilon > 0, \forall x, a \in \mathbb{R}^d$$

Take  $\epsilon \rightarrow 0$

$$\int_0^1 \sum_{|\alpha|=2} D^\alpha u(x) \frac{a^\alpha}{\alpha!} \geq 0 \Rightarrow \sum_{i,j=1, i \neq j} \partial_{x_i} \partial_{x_j} u(x) a_i a_j + \sum_{i=j=1}^d \partial_{x_i}^2 u(x) \frac{a_i^2}{2}$$

We get

$$\frac{1}{2} a^T H a \geq 0 \forall a(a_i)_{i=1}^d \in \mathbb{R}^d$$

$$\text{b) } H(x) \geq 0 \Rightarrow (\partial_i \partial_j u) \geq 0 \Rightarrow \text{Tr} H(x) \geq 0 \Rightarrow \sum_{i=1}^d \partial_{x_i}^2 u(x) \geq 0 \Rightarrow \Delta u(x) \geq 0 \forall x \in \mathbb{R}^d$$

■

**Exercise 3.13** (E 2.2, Newton's Theorem) Let  $d \geq 3$ .

a) Prove that for all  $r > 0$  and  $x \in \mathbb{R}^d$ , we have

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|^{d-2}} = \frac{1}{\max(|x|, r)^{d-2}}$$

where  $dS(y)$  is the surface measure on the sphere  $\partial B(x, r) \subseteq \mathbb{R}^d$ .

b) Let  $0 \leq f_1, f_2 \in L^1(\mathbb{R}^d)$  be radial functions with  $\int_{\mathbb{R}^d} f_i = M_i$ . Prove that for all  $z_1, z_2 \in \mathbb{R}^d$  we have

$$\iint_{\mathbb{R}^d} \frac{f_1(x - z_1) f_2(y - z_2)}{|x - y|^{d-2}} dx dy \leq \frac{M_1 M_2}{|z_1 - z_2|^{d-2}}$$

Moreover, prove that we have the equality if  $f_1, f_2$  are compactly supported and  $|z_1 - z_2|$  is sufficiently large.

Hint: For a) you may use the mean-value theorem (the function  $\frac{1}{|x|^{d-2}}$  is harmonic in  $\Omega$  if  $0 \notin \Omega$ ). For b) you may use a) and polar coordinates.

*Solution.* a) Regard  $d = 3$ . The function  $\frac{1}{|x|}$  is harmonic in  $\mathbb{R}^3 \setminus \{0\}$ . We prove

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{\max(|x|, r)}$$

If  $|x| > r$ , then  $0 \notin B(x, r + \epsilon)$ . Then

$$y \mapsto \frac{1}{|y|}$$

is harmonic in  $B(x, r + \epsilon)$ . Then by the Mean Value Property:

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \frac{1}{|x|}$$

If  $|x| < r$ : Then  $\frac{1}{|y|}$  is not harmonic in  $B(x, r)$  since  $0 \in B(x, r)$ . Note

$$\oint_{\partial B(x,r)} \frac{dS(y)}{|y|} = \oint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$$

This function depends on  $x$  only via  $|x|$ .

$$\dots = \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - Ry|}$$

for all  $R$  rotation  $SO(3)$ ,  $dS(Ry) = dS(y)$

$$\begin{aligned} &= \oint_{\partial B(0,r)} \frac{dS(y)}{|Rx - y|} \\ &= \oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \\ \text{(Radial in } z) &= \oint_{\partial B(0,|x|)} \left( \oint_{\partial B(0,r)} \frac{dS(y)}{|z - y|} \right) dS(z) \\ \text{(Fubini)} &= \oint_{\partial B(0,r)} \left( \oint_{\partial B(0,|x|)} \frac{dS(z)}{|z - y|} \right) dS(y) \\ \text{(case 1 since } |y| = r > |x|) &= \oint_{\partial B(0,r)} \frac{1}{|y|} dS(y) = \frac{1}{r} \end{aligned}$$

If  $|x| = r$ : Continuity:  $x \mapsto \oint_{\partial B(0,r)} \frac{dS(y)}{|x-y|}$

b)

**Remark 3.14** For  $f \in C^{|\alpha|}, g \in C^{|\beta|}$ :

$$D^{\alpha+\beta}(f \star g) = (D^\alpha f) \star (D^\beta g)$$

**Lemma 3.15** If  $d \geq 3$  and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  radial. Then:

$$\left( \frac{1}{|x|^{d-2}} \star f \right) (x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} dy = \int_{\mathbb{R}^d} \frac{f(y)}{\max(|x|^{d-2}, |y|^{d-2})} dy$$

*Proof.* (d=3) Polar coordinates:

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy &= \int_0^\infty \left[ \int_{\partial B(0,1)} \frac{d\omega}{|x-rw|} \right] f(r) dr \\
(a) \quad &= \int_0^\infty \left[ \int_{\partial B(0,1)} \frac{d\omega}{\max(|x|, r)} \right] f(r) dr \\
&= \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|, |y|)} dy \quad \blacksquare
\end{aligned}$$

Now, for  $d = 3$ , if  $f$  radial and non-negative with the lemma we get

$$(\star) \quad \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \frac{f(y)}{\max(|x|, |y|)} dy \leq \int_{\mathbb{R}^3} \frac{f(y)}{|x|} dy = \frac{(\int_{\mathbb{R}^3} f(y) dy)}{|x|}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x-z_1)f_2(y-z_2)}{|x-y|} dx dy &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_1(x)f_2(y)}{|x+z_1-y-z_2|} dx dy \\
&= \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{f_1(x)}{|x+z_1-y-z_2|} dx \right) f_2(y) dy \\
(\star) \quad &\leq \int_{\mathbb{R}^3} \frac{(\int_{\mathbb{R}^3} f_1(x) dx) f_2(y)}{|y+z_2-z_1|} dy \\
&\leq \frac{(\int_{\mathbb{R}^3} f_1)(\int_{\mathbb{R}^3} f_2)}{|z_1-z_2|} \quad \blacksquare
\end{aligned}$$

**Exercise 3.16** (Bonus 2) a) Prove that  $u(x) = \frac{1}{|x|}$  is sub-harmonic in  $\mathbb{R}^2 \setminus \{0\}$ .

b) Prove that if  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  radial, non-negative, measurable:

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy \geq \int_{\mathbb{R}^2} \frac{f(y)}{\max(|x|, |y|)} dy$$

## 3.2 Fourier Transformation

**Definition 3.17** (Fourier Transform) For  $f \in L^1(\mathbb{R}^d)$  define

$$\mathcal{F}f(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i k \cdot x} dx, \quad k \cdot x = \sum_{i=1}^d k_i x_i$$

**Theorem 3.18** (Basic Properties) 1. If  $f \in L^1(\mathbb{R}^d)$ , then  $\hat{f} \in L^\infty(\mathbb{R}^d)$  and  $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

2. For all  $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ ,  $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ . Moreover,  $\mathcal{F}$  can be extended to be a unitary transformation  $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  s.t.

$$\|\mathcal{F}g\|_{L^2} = \|g\|_{L^2} \quad \forall g \in L^2(\mathbb{R}^d)$$

3. The inverse of  $\mathcal{F}$  can be defined as

$$(F^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(x) e^{2\pi i k x} dk$$

for all  $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

4.  $\widehat{D^\alpha f}(k) = (2\pi i k)^\alpha \hat{f}(k)$  as  $(2\pi i k)^\alpha f(k) \in L^2(\mathbb{R}^d)$  ( $k^\alpha = k_1^{\alpha_1} \dots k_d^{\alpha_d}$ )
5.  $\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k)$  if  $f, g$  are nice enough.

**Theorem 3.19** (Hausdorff-Young-Inequality) Let  $f \in L^p(\mathbb{R}^d)$ . If  $1 \leq p \leq 2$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then  $\mathcal{F} : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$  is well-defined and

$$\|\mathcal{F}f\|_{L^q(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}.$$

**Remark 3.20** If  $1 \leq p \leq 2$  and  $f \in L^p(\mathbb{R}^d)$  we can write  $f = f_1 + f_2$  when  $f_1 \in L^1$ ,  $f_2 \in L^2$ , e.g.

$$f = \underbrace{f\mathbb{1}(|f| \geq 1)}_{f_1} + \underbrace{f\mathbb{1}(|f| < 1)}_{f_2}$$

$$\begin{aligned} \int_{\mathbb{R}^d} |f_2|^2 dy &= \int_{\mathbb{R}^d} |f|^2 \mathbb{1}(|f| < 1) \leq \int_{\mathbb{R}^d} |f|^p dy < \infty \\ \int_{\mathbb{R}^d} |f_1| dy &= \int_{\mathbb{R}^d} |f| \mathbb{1}(|f| \geq 1) \leq \int_{\mathbb{R}^d} |f|^p < \infty \end{aligned}$$

thus we can define  $\hat{f} = \hat{f}_1 + \hat{f}_2$  well defined in  $L^\infty(\mathbb{R}^d) + L^2(\mathbb{R}^d)$ .

*Proof of the Hausdorff-Young-Inequality 3.19.* We need Riesz-Thorin interpolation theorem. If  $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ , and  $\Omega \subseteq \mathbb{R}^d$  open and

$$T : L^{p_0}(\Omega) + L^{p_1}(\Omega) \longrightarrow L^{q_0}(\Omega) + L^{q_1}(\Omega)$$

is a linear operator and

$$T : L^{p_0} \rightarrow L^{q_0}$$

and  $\|T\|_{L^{p_i} \rightarrow L^{q_i}} \leq 1$  for  $i = 0, 1$ . Then,

$$T : L^{p_\theta} \rightarrow L^{q_\theta} \text{ and } \|T\|_{L^{p_\theta} \rightarrow L^{q_\theta}} \leq 1$$

for any  $0 < \theta < 1$  where

$$\begin{cases} \frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1} \\ \frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1} \end{cases}.$$

Consider the Fourier Transform:

$$F : L^1 + L^1 \rightarrow L^2 + L^\infty$$

and

$$\begin{aligned} \|F\|_{L^1 \rightarrow L^\infty} &\leq 1 \text{ as } \|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1} & \forall f \in L^1 \\ \|F\|_{L^2 \rightarrow L^2} &= 1 \text{ as } \|\hat{f}\|_{L^2} = \|f\|_{L^2} & \forall f \in L^2 \\ \Rightarrow \|F\|_{L^{p_\theta} \rightarrow L^{q_\theta}} &\leq 1 & \forall \theta \in (0, 1) \end{aligned}$$

$$p_0 = 1, p_1 = 2, q_0 = \infty, q_1 = 2$$

$$\begin{aligned} \frac{1}{p_\theta} &= \frac{\theta}{p_0} + \frac{1-\theta}{p_1} = \theta + \frac{1-\theta}{2} = \frac{1+\theta}{2} \\ \frac{1}{q_\theta} &= \frac{\theta}{q_0} + \frac{1-\theta}{q_1} = \frac{1-\theta}{2} \\ \Rightarrow 1 &= \frac{1}{p_\theta} + \frac{1}{q_\theta} = \frac{1+\theta}{2} + \frac{1-\theta}{2} \end{aligned}$$

■

**Exercise 3.21** (E 3.2) Let  $1 \leq p, q, r \leq 2$ ,  $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ . Recall that if  $f \in L^p(\mathbb{R}^d)$ ,  $g \in L^q(\mathbb{R}^d)$ , then  $f \star g \in L^r(\mathbb{R}^d)$  by Young's Inequality, and its Fourier transform is well-defined by the Hausdorff-Young inequality. Prove that

$$\widehat{f \star g}(k) = \hat{f}(k)\hat{g}(k) \quad \forall k \in \mathbb{R}^d$$

*Solution.*

Step 1)  $f, g \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \widehat{f \star g}(k) &= \int_{\mathbb{R}^d} (f \star g)(x) e^{-2\pi i k x} dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) g(y) e^{-2\pi i k x} dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) e^{-2\pi i k(x-y)} g(y) e^{-2\pi i k y} dx dy \\ (z(x) := x-y) \quad &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(z) e^{-2\pi i k z} dz \right) g(y) e^{-2\pi i k y} dy \\ &= \left( \int_{\mathbb{R}^d} f(z) e^{-2\pi i k z} dz \right) \left( \int_{\mathbb{R}^d} g(y) e^{-2\pi i k y} dy \right) = \hat{f}(k)\hat{g}(k) \end{aligned}$$

Step 2)  $f \in L^p, g \in L^q$ , find  $f_n, g_n \in C_c^\infty$  s.t.  $f_n \rightarrow f$  in  $L^p$ ,  $g_n \rightarrow g$  in  $L^q$ . Then we have  $\widehat{f_n \star g_n} = \hat{f}_n \hat{g}_n$  almost everywhere. We get with the Hausdorff-Young Inequality (3.19)

$$\begin{aligned} \|\widehat{f \star g} - \widehat{f_n \star g_n}\|_{L^{r'}} &\leq \|f \star g - f_n \star g_n\|_{L^r} \\ &= \|(f - f_n) \star g_n + f_n \star (g_n - g)\|_{L^r} \\ &\leq \|(f - f_n) \star g_n\|_{L^r} + \|f_n \star (g_n - g)\|_{L^r} \\ (\text{Young 3.3}) \quad &\leq \|f - f_n\|_{L^p} \|g_n\|_{L^q} + \|f_n\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Moreover:

$$\begin{aligned} \|\hat{f}_n \hat{g}_n - \hat{f} \hat{g}\|_{L^{r'}} &= \|(\hat{f}_n \hat{f}) \hat{g}_n + \hat{f}(\hat{g}_n - \hat{g})\|_{L^{r'}} \\ (\text{H\"older}) \quad &\leq \|\hat{f}_n - \hat{f}\|_{L^{p'}} \|\hat{g}_n\|_{L^{q'}} + \|\hat{f}\|_{L^{q'}} \\ (\text{Hausdorff-Young 3.19}) \quad &\leq \|f_n - f\|_{L^p} \|g_n\|_{L^q} + \|f\|_{L^p} \|g_n - g\|_{L^q} \xrightarrow{n \rightarrow \infty} 0 \\ \text{So } \hat{f}_n \hat{g}_n &\rightarrow \hat{f} \hat{g} \text{ in } L^{r'} \quad \widehat{f \star g} = \hat{f} \hat{g} \text{ in } L^{r'} \quad \frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'} \quad \blacksquare \end{aligned}$$

**Remark 3.22** We want to apply the Fourier transform to find the solution of a PDE, e.g. the Poisson-Equation:

$$-\Delta u = f \text{ in } \mathbb{R}^d \Rightarrow |2\pi k|^2 \hat{u}(k) = \hat{f}(k) \Rightarrow \hat{u}(k) = \frac{1}{|2\pi k|^2} \hat{f}(k)$$

If we can find  $G$  s.t.  $\hat{G}(k) = \frac{1}{|2\pi k|^2}$ , then

$$\begin{aligned} \hat{u}(k) &= \hat{G}(k) \hat{f}(k) = \widehat{G \star f} \\ \Rightarrow u(x) &= (G \star f)(x) = \int_{\mathbb{R}^d} G(x-y) f(y) dy \end{aligned}$$

Thus we need to compute

$$G(x) = \left( \frac{1}{|2\pi k|^2} \right)^\vee$$

It turns out for  $d \geq 3$  that

$$G(x) = \frac{1}{d(d-2)|B_1||x|^{d-2}}.$$

In fact  $G$  is the fundamental solution of the Laplace Equation. To make it rigorous, we need to compute the Fourier transform of  $\frac{1}{|x|^\alpha}$  for  $0 \leq \alpha < d$ .

**Theorem 3.23** (Fourier Transform of  $\frac{1}{|x|^\alpha}$  for  $0 < \alpha < d$ ) We have formally

$$\widehat{\frac{c_\alpha}{|x|^\alpha}} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \quad \forall 0 < \alpha < d$$

Here

$$c_\alpha = \pi^{-\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda$$

More precisely, for all  $f \in C_c^\infty(\mathbb{R}^d)$ ,

$$\frac{c_\alpha}{|x|^\alpha} \star f = \left( \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee$$

Moreover if  $\alpha > \frac{d}{2}$ , then we also have

$$\left( \frac{c_\alpha}{|x|^\alpha} \star f \right)^\wedge = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k).$$

*Proof.*

$$\begin{aligned} \frac{c_\alpha}{|x|^\alpha} &= \frac{1}{|x|^\alpha} \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty e^{-\pi\lambda|x|^2} \lambda^{\frac{\alpha}{2}-1} d\lambda \\ \Rightarrow \frac{\hat{c}_\alpha}{|x|^\alpha}(k) &= \int_0^\infty \widehat{e^{-\pi\lambda|x|^2}}(k) \lambda^{\frac{\alpha}{2}-1} d\lambda = \int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi\frac{|k|^2}{\lambda}} \lambda^{\frac{\alpha}{2}-1} d\lambda \\ (\lambda \rightarrow \frac{1}{\lambda}) &= \int_0^\infty \lambda^{\frac{d}{2}} e^{-\pi|k|^2\lambda} \lambda^{-\frac{\alpha}{2}+1} \lambda^{-2} d\lambda = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \end{aligned}$$

Let  $f \in C_c(\mathbb{R}^d)$ . Then  $\left( \frac{1}{|x|^\alpha} \star f \right)(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^\alpha} f(y) dy$  is well defined as  $\frac{1}{|x-y|} \in L^1_{loc}(\mathbb{R}^d, dy)$ . It is bounded

$$\frac{1}{|x|^\alpha} \star f = \frac{1}{|x|^\alpha} \underbrace{\mathbb{1}(|x| \leq 1)}_{\in L^\infty(\mathbb{R}^d)} \star \underbrace{f}_{L^\infty} + \underbrace{\frac{1}{|x|^\alpha} \mathbb{1}(|x| > 1)}_{\in L^\infty} \star \underbrace{f}_{L^1} \in L^\infty(\mathbb{R}^d)$$

When  $|x| \rightarrow \infty$ :

$$\left( \frac{1}{|x|^\alpha} \star f \right)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^\alpha} dy = \int_{|y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy \sim \frac{\int_{\mathbb{R}^d} f(y) dy}{|x|^\alpha}$$

Note that  $\underbrace{\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)}_{\text{bounded}} \in L^1(\mathbb{R}^d)$ .

$$\begin{aligned} (...)\mathbb{1}(|k| \leq 1) + (...)\mathbb{1}(|k| > 1) \frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| \leq 1) &\leq \|f\|_{L^1} \frac{\mathbb{1}(|k| \leq 1)}{|k|^{d-\alpha}} \in L^1(\mathbb{R}^d, dk) \\ \frac{1}{|k|^{d-\alpha}} |\hat{f}(k)| \mathbb{1}(|k| > 1) &\leq |\hat{f}(k)| \in L^2(\mathbb{R}^d, dK) \text{ as } f \in L^2(\mathbb{R}^d) \end{aligned}$$

**Lemma 3.24** If  $f \in C_c^\infty(\mathbb{R}^d)$ , then  $\hat{f} \in L^1(\mathbb{R}^d)$

*Proof.* (Exercise) Hint:  $|\widehat{D^\alpha f}| = |2\pi k|^{|\alpha|} |\hat{f}(k)| \rightsquigarrow |\hat{f}(k)| \leq \frac{1}{|k|^{|\alpha|}} \text{ as } |k| \rightarrow \infty.$  ■

Compute:

$$\begin{aligned}
\left( \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee(x) &= \int_{\mathbb{R}^d} \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_{\mathbb{R}^d} \left( \int_0^\infty e^{-\pi |k|^2 \lambda} \lambda^{\frac{d-\alpha}{2}-1} d\lambda \right) \hat{f}(k) e^{2\pi i k x} dk \\
&= \int_0^\infty \left( \int_{\mathbb{R}^d} e^{-\pi |k|^2 \lambda} \hat{f}(k) e^{2\pi i k x} dk \right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left( e^{-\pi k^2 \lambda} \hat{f}(x) \right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left( \lambda^{-\frac{d}{2}} e^{-\pi \frac{x^2}{\lambda}}(k) \hat{f}(k) \right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \int_0^\infty \left( \lambda^{-\frac{d}{2}} e^{-\pi \frac{x^2}{\lambda}} \star f \right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda \\
&= \left( \int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi \frac{x^2}{\lambda}} \lambda^{\frac{d-\alpha}{2}-1} d\lambda \right) \star f
\end{aligned}$$

Assume  $d > \alpha > \frac{d}{2}$ . Then  $\frac{c_\alpha}{|x|^\alpha} \star f \in L^\infty$  and behaves  $\frac{c_\alpha(f)}{|x|^\alpha}$  as  $|x| \rightarrow \infty$ . This implies:

$$\int_{\mathbb{R}^d} \left| \frac{c_\alpha}{|x|^\alpha} \star f \right|^2 \leq c + \int_{|x| \geq R} \frac{c}{|x|^{2d}} dx < \infty$$

Thus the Fourier Transform  $\widehat{\frac{c_\alpha}{|x|^\alpha} \star f}$  exists. Combining with

$$\begin{aligned}
\frac{c_\alpha}{|x|^\alpha} \star f &= \left( \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee \\
\Rightarrow \widehat{\frac{c_\alpha}{|x|^\alpha} \star f} &= \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)
\end{aligned}$$

**Remark 3.25** If  $f \in L^p$ ,  $1 \leq p \leq 2$  is such that  $\tilde{f}_1 + \tilde{f}_2 = f = f_1 + f_2$  with  $f_1, \tilde{f}_1 \in L^1, f_2, \tilde{f}_2 \in L^2$ . Do we have that  $\hat{f}_1 + \hat{f}_2 = \hat{\tilde{f}}_1 + \hat{\tilde{f}}_2 \in L^1 \cap L^2$ ? In fact, from  $f_1 + f_2 = \tilde{f}_1 + \tilde{f}_2$  we get

$$\begin{aligned}
\underbrace{f_1 - \tilde{f}_1}_{\in L^1} &= \underbrace{\tilde{f}_2 - f_2}_{\in L^2} \in L^1 \cap L^2 \\
\Rightarrow \hat{f}_1 - \hat{\tilde{f}}_1 &= \hat{\tilde{f}}_2 - \hat{f}_2 \quad \Rightarrow \quad \hat{f}_1 + \hat{f}_2 = \hat{\tilde{f}}_1 + \hat{\tilde{f}}_2.
\end{aligned}$$

**Lemma 3.26** (Fourier Transform of Gaussians) In  $\mathbb{R}^d$ ,

$$\widehat{e^{-\pi |x|^2}} = e^{-\pi |k|^2}$$

More generally for all  $\lambda > 0$ :

$$\widehat{e^{-\pi \lambda^2 |x|^2}} = \lambda^{-d} e^{-\pi \frac{|k|^2}{\lambda^2}}$$

(exercise)

**Remark 3.27** If  $d \geq 3$

$$\begin{aligned}\hat{G}(k) &= \frac{1}{|2\pi k|^2} \\ \Rightarrow G(x) &= \left( \frac{1}{|2\pi k|^2} \right)^\vee = \frac{1}{d(d-2)(k)|x|^{d-2}} = \Phi(x)\end{aligned}$$

### 3.3 Theory of Distribution

In the following let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . Let  $\Omega \subseteq \mathbb{R}^d$  be open.

- $D(\Omega) = C_c^\infty(\Omega)$  the space of test functions.
- We say that  $\phi_n \rightarrow \phi$  in  $D(\Omega)$  if for all  $K \subseteq \Omega$  compact we have that
 
$$\begin{cases} \text{supp}(\phi_n - \phi) \subseteq K \text{ for all } n \\ \|D^\alpha(\phi_n - \phi)\|_{L^\infty(K)} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } \alpha \end{cases}$$
- $D'(\Omega) = \{T : D(\Omega) \rightarrow \mathbb{K} \text{ linear and continuous}\}$  the space of distributions.

Motivation:  $L^2(\Omega)' = L^2(\Omega)$ ,  $(L^p(\Omega))' = (L^q(\Omega))$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Example 3.28** ("normal functions" are distributions) If  $f \in L^1_{loc}(\Omega)$ , then  $T = T_f$  defined by:

$$T(\phi) = \int_{\Omega} f(x)\phi(x) dx$$

is a distribution for all  $\phi \in D(\Omega)$ , i.e.  $T \in D'(\Omega)$ . Indeed, it is clear that  $T(\phi)$  is well-defined for all  $\phi \in D(\Omega)$  and  $\phi \mapsto T(\phi)$  is linear. Let us check that  $\phi \mapsto T(\phi)$  is continuous. Take  $\phi_n \rightarrow \phi$  in  $D(\Omega)$  and prove that  $T(\phi_n) \rightarrow T(\phi)$ . Since  $\phi_n \rightarrow \phi$  in  $D(\Omega)$ , there is a compact  $K$  s.t.  $\text{supp}(\phi_n), \text{supp}(\phi) \subseteq K \subseteq \Omega$ .

Question: Why is  $f \mapsto T_f$  injective?

**Theorem 3.29** (Fundamental theorem of calculus of variants) Let  $\Omega \subseteq \mathbb{R}^d$  be open. If  $f, g \in L^1_{loc}(\Omega)$  and  $\int_{\Omega} f\phi dy = \int_{\Omega} g\phi dy$  for all  $\phi \in D(\Omega)$ , then  $f = g$  in  $L^1_{loc}(\Omega)$

**Example 3.30** (Dirac delta function) Let  $\Omega \subseteq \mathbb{R}^d$  open and let  $x_0 \in \Omega$ . Define  $T : D(\Omega) \rightarrow \mathbb{K}$  by  $T(\phi) = \phi(x_0)$ . Then  $T \in D'(\Omega)$  and we denote it by  $\delta_{x_0}$ . It is clear that  $\phi \mapsto T(\phi) = \phi(x_0)$  is well-defined and linear for all  $\phi \in D(\Omega)$ . Take  $\phi_n \rightarrow \phi$  in  $D(\Omega)$  and prove  $T(\phi_n) \rightarrow T(\phi)$ , i.e.  $\phi_n(x_0) \rightarrow \phi(x_0)$  (obvious.)

**Example 3.31** (Principle Value) The function  $f(x) = \frac{1}{x}$  is not in  $L^1_{loc}(\mathbb{R})$ , but we can still define

$$\int_{\mathbb{R}} f(x)\phi(x) dx = \int_{\mathbb{R}} \frac{\phi(x)}{x} dx$$

for all  $\phi \in D(\mathbb{R})$  s.t.  $\phi(0) = 0$ . In fact,

$$\phi(x) = |\phi(x) - \phi(0)| \leq x \sup |\phi'|,$$

so  $\frac{|\phi(x)|}{|x|} \in L^\infty(\mathbb{R})$  and compactly supported. So  $\frac{\phi(x)}{x} \in L^1(\mathbb{R})$ . Define  $T : D(\mathbb{R}) \rightarrow \mathbb{K}$  by

$$T(\phi) = \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx \quad \forall \phi \in D(\mathbb{R}) \text{ s.t. } \phi(0) = 0$$



We write  $T = \frac{1}{x}$  and check that  $T \in D'(\mathbb{R})$ : For all  $\epsilon > 0$  we have

$$\left| \frac{\phi(x)}{x} \right| \leq \frac{\|\phi\|_{L^\infty}}{\epsilon}$$

for all  $|x| \geq \epsilon$  and  $\phi$  is compactly supported. So we get for all  $\epsilon > 0$ :

$$\mathbb{1}(|x| \geq \epsilon) \frac{\phi(x)}{x} \in L^1(\mathbb{R}) \rightsquigarrow \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx < \infty$$

We can write:

$$\int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx = \int_{|x| \geq 1} \frac{\phi(x)}{x} dx + \int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx$$

The second part can be written as:

$$\int_{\epsilon \leq |x| \leq 1} \frac{\phi(x)}{x} dx = \int_{\epsilon}^1 \frac{\phi(x)}{x} dx + \int_{-1}^{-\epsilon} \frac{\phi(x)}{x} dx = \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx$$

Since  $\phi \in C_c^\infty(\mathbb{R})$  it holds that  $|\phi(x) - \phi(-x)| \leq 2\|\phi'\|_{L^\infty}(x)$ .

$$\begin{aligned} \Rightarrow \frac{\phi(x) - \phi(-x)}{x} &\in L^\infty(\mathbb{R}) \Rightarrow \frac{\phi(x) - \phi(-x)}{x} \in L^1((0, 1)) \\ &\Rightarrow \int_0^1 \frac{\phi(x) - \phi(-x)}{x} dx = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 \frac{\phi(x) - \phi(-x)}{x} dx \end{aligned}$$

**Remark 3.32** The function  $\frac{1}{|x|^d}$  is not in  $L_{loc}^1(\mathbb{R}^d)$  but  $\exists T \in D'(\mathbb{R}^d)$  s.t.  $T(\phi) = \int_{\mathbb{R}^d} \frac{\phi(x)}{|x|^d} dx$  for all  $\phi \in C_c^\infty(\mathbb{R}^d)$  s.t.  $\phi(0) = 0$

**Definition 3.33** (Derivatives of distributions) Let  $\Omega \subseteq \mathbb{R}^d$  and  $T \in D'(\Omega)$ . Define for  $\alpha \in \mathbb{N}^d$ :

$$\begin{aligned} D^\alpha T : D(\Omega) &\longrightarrow \mathbb{K} \\ \phi &\longmapsto (-1)^{|\alpha|} T(D^\alpha \phi) \end{aligned}$$

Motivation:  $f \in C_c^\infty(\Omega)$

$$\int_{\Omega} (D^\alpha f) \phi = (-1)^{|\alpha|} \int_{\Omega} f (D^\alpha \phi)$$

„If the classical derivative exists, then it is the same as the distributional derivative.“  
We write

$$(D^\alpha T)(\phi) = T_{D^\alpha f}(\phi) = (-1)^{|\alpha|} T_f(D^\alpha \phi).$$

**Remark 3.34** For all  $T \in D'(\Omega)$  it holds  $D^\alpha T \in D'(\Omega)$  for all  $\alpha \in \mathbb{N}^d$ . Clearly

$$\phi \longmapsto (D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

is linear. Moreover, if  $\phi_n \rightarrow \phi$  in  $D(\Omega)$ , then  $D^\alpha \phi_n \rightarrow D^\alpha \phi$  in  $D(\Omega)$ , so

$$(D^\alpha T)(\phi_n) = (-1)^{|\alpha|} T(D^\alpha \phi_n) \xrightarrow{n \rightarrow \infty} (-1)^{|\alpha|} T(D^\alpha \phi) = (D^\alpha T)(\phi)$$

**Example 3.35** Consider  $f : x \mapsto |x|$ , then  $f \in C(\mathbb{R})$  but  $f \notin C^1(\mathbb{R})$ . However,

$$f'(x) = g(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \in L^1_{loc} \quad \text{in } D'(\mathbb{R})$$

Lets check  $f' = g$ , i.e.  $-f(\phi') = f'(\phi) \stackrel{!}{=} g(\phi)$  for all  $\phi \in D(\mathbb{R})$ . Thus we need to prove:

$$-\int_{\mathbb{R}} f(x)\phi'(x) dx = \int_{\mathbb{R}} g(x)\phi(x) dx \quad \forall \phi \in D(\mathbb{R})$$

namely we have to show

$$\underbrace{-\int_{\mathbb{R}} |x|\phi'(x) dx}_{:= (\star)} = \int_0^{\infty} \phi(x) dx - \int_{-\infty}^0 \phi(x) dx.$$

Now we have

$$(\star) = -\int_0^{\infty} x\phi'(x) dx + \int_{-\infty}^0 x\phi'(x) dx.$$

By integration by parts we have

$$\int_0^{\infty} x\phi'(x) dx = \underbrace{[x\phi(x)]_0^{\infty}}_{=0} - \int_0^{\infty} \phi(x) dx = -\int_0^{\infty} \phi(x) dx$$

and similary

$$\int_{-\infty}^0 x\phi'(x) dx = -\int_{-\infty}^0 \phi(x) dx.$$

Thus  $f' = g$  in  $D'(\Omega)$ . We claim that  $g' = 2\delta_0$  in  $D'(\mathbb{R})$ . In fact, for all  $\phi \in D(\mathbb{R})$ , then:

$$\begin{aligned} g'(\phi) &= -g(\phi') = -\int_{\mathbb{R}} g\phi' dx = -\int_{-\infty}^0 (-1)\phi' dx - \int_0^{\infty} (1)\phi' dx \\ &= -\int_0^{\infty} \phi' dx + \int_{-\infty}^0 \phi' dx = [\phi(0) - \underbrace{\phi(\infty)}_{=0}] + [\phi(0) - \underbrace{\phi(-\infty)}_{=0}] \\ &= 2\phi(0) = 2\delta_0(\phi) \end{aligned}$$

So  $g' = 2\delta_0$  in  $D'(\mathbb{R})$ .

**Exercise 3.36** Prove that  $(D^\alpha \delta_x)(\phi) = (-1)^{|\alpha|}(D^\alpha \phi)(x)$  for all  $\phi \in D(\mathbb{R})$  for all  $x \in \mathbb{R}$ .

**Definition 3.37** (Convergence of distributions) Let  $\Omega \subseteq \mathbb{R}^d$  be open, then

$$T_n \xrightarrow{n \rightarrow \infty} T$$

in  $D'(\Omega)$  if  $T_n(\phi) \xrightarrow{n \rightarrow \infty} T(\phi)$  for all  $\phi \in D(\Omega)$ .

**Exercise 3.38** Let  $f \in L^1(\mathbb{R}^d)$ ,  $\int f = 1$  For  $\epsilon > 0$ , define  $f_\epsilon(x) = \epsilon^{-d}f(\epsilon^{-1}x)$ . Then:  $f_\epsilon \rightarrow \delta_0$  in  $D'(\Omega)$ .

**Exercise 3.39** Let  $\Omega \subseteq \mathbb{R}^d$  be open and  $T_n \rightarrow T$  in  $D'(\Omega)$ . Then:  $D^\alpha T_n \rightarrow D^\alpha T$  in  $D'(\Omega)$  for all  $\alpha = (\alpha_1, \dots, \alpha_d)$

**Definition 3.40** (Convolution of distributions) Let  $T \in D'(\mathbb{R})$  and  $f \in L_c^\infty(\mathbb{R}^d)$ . Define

$$(T \star f)(y) = T(f_y)$$

We write  $f_y(x) = f(x - y)$  and  $\tilde{f}(x) = f(-x)$ .

**Theorem 3.41** Let  $T \in D'(\mathbb{R})$ . Then for all  $f \in D(\mathbb{R})$ :

1.  $y \mapsto T(f_y)$  is  $C^\infty(\mathbb{R}^d)$  and

$$D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D^\alpha f_y)$$

2. If  $g \in L^1(\mathbb{R}^d)$  and  $g$  is compactly supported, then

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(\underbrace{f \star g}_{\in C_c^\infty(\mathbb{R})})$$

*Proof.* 1. We prove that  $y \mapsto T(f_y)$  is continuous. Take  $y_n \rightarrow y$  in  $\mathbb{R}^d$ , then:

$$T(f_{y_n}) \rightarrow T(f_y)$$

since  $f_{y_n} \rightarrow f_y$  in  $D(\mathbb{R}^d)$ . We check this: Since  $f \in C_c^\infty(\mathbb{R}^d)$ , it holds that  $\text{supp } f \subseteq B(0, R) \subseteq \mathbb{R}^d$ . Since  $y_n \rightarrow y$  in  $\mathbb{R}^d$ . We have  $\sup_n |y_n| < \infty$ . Thus  $f_{y_n}, f_y$  are supported in  $\overline{B(0, R + \sup_n |y_n|)} = K$  compact. Moreover

$$|f_{y_n}(x) - f_y(x)| = |f(x - y_n) - f(x - y)| \leq \|\nabla f\|_{L^\infty} \|y_n - y\| \rightarrow 0$$

So we get  $\|f_{y_n} - f_y\|_{L^\infty} \rightarrow 0$  Similary:

$$\|D^\alpha f_{y_n} - D^\alpha f_y\|_{L^\infty} \rightarrow 0 \quad \blacksquare$$

**Exercise 3.42** (E 3.4) Compute the Fourier Transform of the Gaussian.

**Exercise 3.43** (Bonus 3) Let  $f \in L^1(\mathbb{R}^d)$  such that

$$|\hat{f}(k)| \leq \frac{C_N}{(1 + |k|)^N}$$

for all  $k \in \mathbb{R}^d$ , for all  $N \geq 1$ . ( $C_N$  is independent of  $k$ ). Prove that  $f \in C^\infty(\mathbb{R}^d)$

( $f \in C^\infty$ ) i.e.  $\exists \tilde{f} \in C^\infty$  s.t.  $f = \tilde{f}$  a.e.

**Theorem 3.44** Take  $T \in D'(\mathbb{R})$ ,  $f \in C_c^\infty(\mathbb{R}^d) = D(\mathbb{R}^d)$ ,  $f_y(x) = f(x - y)$

- a)  $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$  and  $D_y^\alpha(T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T(D_x^\alpha f_y)$
- b)  $\forall g \in L^1(\mathbb{R}^d)$  and compactly supported

$$\int_{\mathbb{R}^d} g(y) T(f_y) dy = T(\underbrace{f \star g}_{\in C_c^\infty})$$

*Proof.* a)  $y \mapsto T(f_y)$  is continuous since  $y_n \rightarrow y$  in  $\mathbb{R}^d$ , then  $f_{y_n} \rightarrow f_y$  implies  $T(f_{y_n}) \rightarrow T(f_y)$ . Let's check that  $y \mapsto T(f_y) \in C^1$ :

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} = \lim_{h \rightarrow 0} T\left(\frac{f_{y-he_i} - f_y}{h}\right)$$

We have  $\frac{f_{y-he_i} - f_y}{h} \xrightarrow{h \rightarrow 0} (\partial_i f)_y$  in  $D(\mathbb{R}^d)$

- $\exists K$  compact set such that  $\text{supp}(f_{y-he_i} - f_y), \text{supp } \partial_i f \subseteq K$  as  $|h|$  small.

$$\begin{aligned} & \bullet \frac{f_{y-he_i}(x) - f_y(x)}{h} - (\partial_i f)_y(x) \\ &= \frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y) \end{aligned}$$

$$\left| \int_0^1 \partial_i f(x-y+the_i) dt - \partial_i f(x-y) \right| \xrightarrow{h \rightarrow 0} 0 \text{ uniformly in } x$$

Similary:

$$\begin{aligned} & \left| D_x^\alpha \left( \frac{f(x-y+he_i) - f(x-y)}{h} - (\partial_i f)(x-y) \right) \right| \\ &= \left| \frac{D^\alpha f(x-y+he_i) - D^\alpha f(x-y)}{h} - \partial_i(D^\alpha f)(x-y) \right| \xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

uniformly in  $x$ . Conclude:

$$\lim_{h \rightarrow 0} \frac{T(f_{y-he_i}) - T(f_y)}{h} \xrightarrow{h \rightarrow 0} T((\partial_i f)_y) \in C(\mathbb{R}^d)$$

So we get that  $y \mapsto T(f_y) \in C^1$  and  $-\partial_{y_i} T(f_y) = T((\partial_i f)_y)$

By induction:

$$D_y^\alpha T(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y) = (D^\alpha T)(f_y) \quad \forall \alpha \in \mathbb{N}^d$$

b) Heuristic:  $T = T(x)$

$$\begin{aligned} \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \int_{\mathbb{R}^d} g(y) \left( \int_{\mathbb{R}^d} T(x) f(x-y) dx \right) dy \\ &= \int_{\mathbb{R}^d} T(x) \left( \int_{\mathbb{R}^d} g(y) f(x-y) dy \right) dx \\ &= \int_{\mathbb{R}^d} T(x) (f \star g)(x) dx = T(f \star g) \end{aligned}$$

Step 1:  $g \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \text{(Riemann Sum)} \quad \int_{\mathbb{R}^d} g(y) T(f_y) dy &= \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) T(f_{y_j}) \\ &= \lim_{\Delta_N \rightarrow 0} T \left( \Delta_N \sum_{j=1}^N g(y_j) f_{y_j} \right) \\ &= T(f \star g) \end{aligned}$$

because

$$\begin{aligned} \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f_{y_j}(x) &\rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d) \\ \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x-y_j) &\xrightarrow{\text{Riemann}} \int_{\mathbb{R}^d} g(y) f(x-y) dy = (f \star g)(x) \end{aligned}$$

Proof of:

$$\lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \rightarrow (f \star g)(x) \text{ in } D(\mathbb{R}^d)$$

1) Since  $f, g \in C_c^\infty$  we have  $f \star g \in C_c^\infty$ . And we have

$$x \mapsto \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) \in C^\infty$$

since  $f \in C^\infty$  supported in  $(\text{supp } g + \text{supp } f)$ . So all functions are  $C_c^\infty$  and supported in  $(\text{supp } g + \text{supp } f)$ .

2)

$$\left| \lim_{\Delta_N \rightarrow 0} \Delta_N \sum_{j=1}^N g(y_j) f(x - y_j) - \int_{\mathbb{R}^d} g(y) f(x - y) dy \right| \xrightarrow{\Delta_N \rightarrow 0} 0$$

uniformly in  $x$ . (Result from the Riemann-Sum)

3)

$$\begin{aligned} & \left| D_x^\alpha (\Delta_N \sum_{j=1}^N g(y_j) f(x - y) - (f \star g)(x)) \right| \\ &= \left| \Delta_N \sum_{j=1}^N g(y_j) D^\alpha f(x - y) - (D^\alpha f) \star g(x) \right| \xrightarrow{\Delta_N \rightarrow 0} 0 \end{aligned}$$

uniformly in  $x$  for all  $\alpha$ .

Step 2: Take  $g \in L^1(\mathbb{R}^d)$  and compactly supported. Then  $\exists \{g_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ ,  $\text{supp } g_n \subseteq \text{supp } g + B(0, 1)$  such that  $g_n \rightarrow g$  in  $L^1(\mathbb{R}^d)$ . By Step 1:

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy = T(g_n \star f)$$

Take  $n \rightarrow \infty$ :

$$\int_{\mathbb{R}^d} g_n(y) T(f_y) dy \rightarrow \int_{\mathbb{R}^d} g(y) T(f_y) dy$$

since  $g_n \rightarrow g$  in  $L^1$  compactly supported and  $y \mapsto T(f_y) \in C^\infty \subseteq L^\infty(K)$ . Moreover (exercise):

$$\underbrace{g_n \star f}_{\in C_c^\infty} \rightarrow g \star f \quad \text{in } D(\mathbb{R}^d)$$

So  $T(g_n \star f) \xrightarrow{n \rightarrow \infty} T(g \star f)$ . Finally we obtain:

$$\int g(y) T(f_y) dy = T(g \star f) \quad \blacksquare$$

**Theorem 3.45** Let  $\Omega \subseteq \mathbb{R}^d$  be open. Let  $T \in D'(\Omega)$  and  $f \in C_c^\infty(\Omega)$ . Denote

$$\Omega_f = \{y \in \mathbb{R}^d \mid \text{supp } f_y = y + \text{supp } f \subseteq \Omega\}$$

a)  $y \mapsto T(f_y) \in C^\infty(\Omega_f)$  and  $D_y^\alpha (T(f_y)) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y)$

b) For all  $g \in L^1(\Omega_g)$  compactly supported in  $\Omega_f$  and it holds:

$$\int_{\Omega} g(y)T(f_y) dy = T(f \star g).$$

**Theorem 3.46** Let  $T \in D'(\Omega)$  s.t.  $\nabla T = 0$  in  $D'(\Omega)$ . Then:  $T = \text{const.}$  in  $\Omega$ .

*Proof.* ( $\Omega = \mathbb{R}^d$ ) We show for all  $f \in C_c^\infty$  that  $y \mapsto T(f_y) \in C^\infty(\mathbb{R}^d)$  and  $\partial_{y_i} T(f_y) = (\partial_j T)(f_y) = 0$  for all  $i = 1, \dots, d$ . Then by the result of the theorem for  $C^\infty$  functions,  $y \mapsto T(f_y) = \text{const}$  independent of  $y$ . Consequently:

$$T(f_y) = T(f_0) = T(f) \quad \forall y \in \mathbb{R}^d \quad \forall f \in C_c^\infty(\mathbb{R}^d)$$

For any  $g \in C^\infty(\mathbb{R}^d)$ :

$$\left( \int_{\mathbb{R}^d} g dy \right) T(f) = \int_{\mathbb{R}^d} g(y)T(f_y) dy = T(f \star g) = T(g \star f) = \left( \int_{\mathbb{R}^d} f dy \right) T(g)$$

So  $\frac{T(f)}{\int_{\mathbb{R}^d} f}$  is independent of  $f$  (as soon as  $\int f \neq 0$ ). So we get that  $T(f) = \text{const} \int_{\mathbb{R}^d} f$ , where const is independent of  $f$ . ■

**Remark 3.47** If  $u \in C^1(\mathbb{R}^d)$ , then:

$$u(x+y) - u(x) = \int_0^1 \sum_{j=1}^d y_j (\partial_j u)(x + ty_j) dt = \int_0^1 y \nabla u(x + ty) dt$$

So we get that if  $\nabla u = 0$ , then  $u(x+y) - u(x) = 0$  for all  $x, y$ , so  $u = \text{const.}$

**Theorem 3.48** (Taylor expansion for distributions) Let  $T \in D'(\mathbb{R}^d)$  and  $f \in C_c^\infty(\mathbb{R}^d)$ . Then  $y \mapsto T(f_y) \in C^\infty$  and

$$T(f_y) - T(f) = \int_0^1 \sum_{j=1}^d y_j (\partial_j T)(f_{ty}) dt.$$

In particular, if  $g \in L_{loc}^1$  and  $\nabla g \in L_{loc}^1$ , then  $\forall y \in \mathbb{R}^d$ :

$$g(x+y) - g(x) = \int_0^1 g(x+ty)y dt$$

for a.e.  $x \in \mathbb{R}^d$ .

*Proof.*  $y \mapsto T(f_y)$  is  $C^\infty$  and  $\frac{d}{dt}[T(f_{ty})] = (\nabla T)(f_{ty})y$  So we get

$$\begin{aligned} T(f_y) - T(f) &= \int_0^1 \frac{d}{dt}(T(f_{ty})) dt \\ &= \int_0^1 (\nabla T)(f_{ty})y dt \\ &= \int_0^1 \sum_{j=1}^d (\partial_j T)(f_{ty})y_j dt \end{aligned} \quad \blacksquare$$

**Corrolary 3.49** Let  $g \in L^1_{loc}(\mathbb{R}^d)$  s.t.  $\partial_j g \in L^1_{loc}(\mathbb{R}^d)$  for all  $j = 1, 2, \dots, d$  (i.e.  $g \in W^{1,1}_{loc}(\mathbb{R}^d)$ ). Then for all  $y \in \mathbb{R}^d$ :

$$\begin{aligned} g(x+y) - g(x) &= \int_0^1 y \cdot \nabla g(x+ty) dt \\ &= \int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \end{aligned}$$

for a.e.  $x$ .

*Proof.* For all  $f \in C_c^\infty$  we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)[g(x+y) - g(x)] dx &= \int_{\mathbb{R}^d} g(x)[f(x-y) - f(x)] dx \\ &= g(f_y) - g(f) \\ &= \int_0^1 \sum_{j=1}^d y_j (\partial_j g)(f_{ty}) dt \\ &= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \sum_{j=1}^d y_j \left[ \int_{\mathbb{R}^d} (\partial_j g)(x) f_{ty}(x) dx \right] \\ &= \int_0^1 \sum_{j=1}^d y_i \left[ \int_{\mathbb{R}^d} (\partial_j g)(x+ty) f(x) dx \right] dt \\ &= \int_{\mathbb{R}^d} f(x) \left[ \int_0^1 \sum_{j=1}^d y_j \partial_j g(x+ty) dt \right] dx \end{aligned}$$

For all  $\phi \in C_c^\infty$ :  $= g(x+y) - g(x)$  a.e.  $x \in \mathbb{R}^d$ . ■

**Remark 3.50** If  $T \in D'(\Omega)$ ,  $\Omega \subseteq \mathbb{R}^d$  open, if  $y \nabla T = 0$ , then  $T = \text{const}$ .

**Theorem 3.51** (Equivalence of the classical and distributional derivatives) Let  $\Omega \subseteq \mathbb{R}^d$ . Then the following are equivalent:

1.  $T \in D'(\Omega)$  s.t.  $\partial_{x_i} T = g_i \in C(\Omega)$  for all  $i = 1, \dots, d$ .
2.  $T = f \in C^1(\Omega)$  and  $g_i = \partial_{x_i} f$

*Proof.*

(2)  $\Rightarrow$  (1): If  $T = f \in C^1(\Omega)$ , then:  $\partial_{x_i} f \in C(\Omega)$ .

$$\partial_{x_i} T(\phi) = -T(\partial_{x_i} \phi) = - \int_{\Omega} f(\partial_{x_i} \phi) = \int_{\Omega} (\partial_{x_i} f) \phi$$

for all  $\phi \in D(\Omega)$ , so  $\partial_{x_i} T = \partial_{x_i} f$ .

(1)  $\Rightarrow$  (2): Why is  $T = f \in C^1(\Omega)$ ? As  $\partial_{x_i} f = g_i$ :

$$f(x+y) - f(x) = \int_0^1 \nabla f(x+ty) y dt = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i dt$$

So we get

$$f(y) = f(0) + \int_0^1 \sum_{i=1}^d g_i(ty) g_i dt.$$

We expect that  $f \in C^1$  and  $\partial_{x_i} f = g_i$ . But this is not trivial to prove.

$$\begin{aligned} \frac{f(y + he_i) - f(y)}{h} &= \int_0^1 \sum_{i=1}^d [g_i(ty + the_i)(y_i + h\delta_{ij})] dt \\ &= \int_0^1 g_i(ty + the_i) dt + \int_0^1 \sum_{j \neq i} \frac{[g_i(ty + the_i) - g_i(ty)]}{h} y_j dt \\ &\xrightarrow{h \rightarrow 0} \int_0^1 g_i(ty) dt + \text{is difficult ...} \end{aligned}$$

Lets take  $\phi \in C_c^\infty$ , then:

$$\begin{aligned} T(\phi_y) - T(\phi) &= \int_0^1 \underbrace{\nabla T}_{(g_i)_{i=1}^d}(\phi_{ty}) y dt \\ &= \int_0^1 \sum_{i=1}^d \left( \int_{\Omega} g_i(x) \underbrace{\phi_{ty}}_{=\phi(x-ty)} dx \right) dt \\ &= \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_0^1 g_i(x) \phi(x - ty) y_i dt \right) dx \\ &= \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \int_0^1 g_i(x + ty) \phi(x) y_i dt \right) dx \\ &= \int_{\mathbb{R}^d} \left( \sum_i \int_0^1 g_i(x + ty) y_i dt \right) \phi(x) dx \end{aligned}$$

Integrating against  $\psi(y)$  with  $\psi \in C_c^\infty$ :

$$\begin{aligned} &\int_{\mathbb{R}^d} T(\phi_y) \psi(y) dy - T(\phi) \int_{\mathbb{R}^d} \psi(y) dy \\ &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \sum_i \int_0^1 g_i(x + ty) y_i \psi(y) dt dy \right) \phi(x) dx \\ &\Rightarrow T(\phi \star \psi) - T(\phi) \int \psi = \dots \\ &\Rightarrow \int_{\mathbb{R}^d} T(\psi_y) \phi(y) dy - T(\phi) \int \psi = \dots \end{aligned}$$

Take  $\psi \in C_c^\infty(\mathbb{R}^d)$  such that  $\int \psi = 1$ . Then:

$$T(\phi) = \underbrace{\int_{\mathbb{R}^d} T(\psi_x) - \left( \int_{\mathbb{R}^d} \sum_{i=1}^d \int_0^1 g_i(x + ty) y_i \psi(y) dt dy \right)}_{f(x)} \phi(x) dx$$

for all  $\phi \in C_c^\infty$ , so  $T = f \in C(\Omega)$ . Thus  $T = f \in C(\Omega)$  and  $\partial_{x_i} T = g_i \in C(\Omega)$ . Then we need to prove that  $f \in C^1(\Omega)$  and  $\partial_{x_i} f = g_i$  (classical derivative). Since



$f \in W_{loc}^{1,1}$ :

$$f(x+y) - f(x) = \int_0^1 \sum_{i=1}^d g_i(x+ty) y_i dt \quad \forall x, y$$

In particular:

$$\begin{aligned} \frac{f(x+he_i) - f(x)}{h} &= \int_0^1 \frac{1}{h} \sum_{i=1}^d g_i(x+the_i) h \delta_{ij} dt \\ &= \int_0^1 g_i(x+the_i) dt \xrightarrow{h \rightarrow 0} g_i(x) \end{aligned}$$

So we get  $\partial_{x_i} f(x) = g_i(x) \in C(\Omega)$  in the classical sense. So  $f \in C^1(\Omega)$ .  $\blacksquare$

**Definition 3.52** (Sobolev Spaces) Let  $\Omega \subseteq \mathbb{R}^d$  be open. We define for  $1 \leq p \leq \infty$ :

$$\begin{aligned} W^{1,p}(\Omega) &= \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega) \ \forall i = 1, \dots, d\} \\ W^{k,p}(\Omega) &= \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leq k\} \\ W_{loc}^{k,p}(\Omega) &= \{f \in L_{loc}^p(\Omega) \mid D^\alpha f \in L_{loc}^p(\Omega) \ \forall |\alpha| \leq k\} \end{aligned}$$

**Theorem 3.53** (Approximation of  $W_{loc}^{1,p}(\Omega)$  by  $C^\infty(\Omega)$ ) Let  $\Omega \subseteq \mathbb{R}^d$  be open, let  $f \in W_{loc}^{1,p}(\Omega)$ . Then there exists  $\{f_n\} \subseteq C^\infty(\Omega)$  such that  $f_n \rightarrow f$  in  $W_{loc}^{1,p}(\Omega)$ , i.e. for all  $K \subseteq \Omega$  compact:  $\|f_n - f\|_{L^p(K)} + \sum_{i=1}^d \|\partial_{x_i}(f_n - f)\|_{L^p(K)} \rightarrow 0$ .

*Proof.* Case  $\Omega = \mathbb{R}^d$ : Take  $g \in C_c^\infty$ ,  $\int g = 1$ ,  $g_\epsilon(x) = \epsilon^{-d} g(\epsilon^{-1}x)$ . Then  $g_\epsilon \star f \in C_c^\infty$ . Since  $f \in L_{loc}^p(\Omega)$  we have  $g_\epsilon \star f \rightarrow f$  in  $L_{loc}^p$  as  $\epsilon \rightarrow 0$ . Moreover  $\partial_{x_i}(g_\epsilon \star f) = (g_\epsilon \star \partial_{x_i} f) \xrightarrow{\epsilon \rightarrow 0} \partial_{x_i} f$  in  $L_{loc}^p$ . Then we can take  $f_n = g_{\frac{1}{n}} \star f$ .  $\blacksquare$

**Remark 3.54** In general, if we want to compute the distributional derivative  $D^\alpha f$ , then we can find  $f_n \rightarrow f$  in  $D'(\Omega)$  and compute  $D^\alpha f_n$ . Then  $D^\alpha f_n \rightarrow D^\alpha f$  in  $D'(\Omega)$ . As an example we can compute  $\nabla|f|$  with  $f \in W_{loc}^{1,p}(\Omega)$ .

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

**Theorem 3.55** (Chain Rule) Let  $G \in C^1(\mathbb{R}^d)$  with  $|\nabla G|$  is bounded. Let  $f = (f_i)_{i=1}^d \in W_{loc}^{1,p}(\Omega)$ . Then  $x \mapsto G(f(x)) \in W_{loc}^{1,p}(\Omega)$  and

$$\partial_{x_i} G(f) = \sum_{k=1}^d (\partial_k G)(f) \cdot \partial_{x_i} f_k \quad \text{in } D'(\Omega).$$

Moreover, if  $G(0) \in L^p(\Omega)$  (i.e. either  $|\Omega| < \infty$  or  $G(0) = 0$ ), then if  $f = (f_i)_{i=1}^d \in W^{1,p}(\Omega)$ , then  $G(f) \in W^{1,p}(\Omega)$ .

*Proof.* Since  $G \in C^1$  we have that  $G$  is bounded in any compact set. Moreover  $\|\nabla G\|_{L^\infty} < \infty$  implies:

$$|G(f) - G(0)| \leq \|\nabla G\|_{L^\infty} |f| \in L_{loc}^p$$

So  $G(f) \in L^p_{loc}$ . Let us compute  $\partial_{x_i} G(f)$ . Let  $\{f^{(n)}\}_{n=1}^\infty \subseteq C^\infty$  such that  $f^{(n)} \rightarrow f$  in  $W^{1,p}_{loc}$ , then:

$$|G(f^{(n)}) - G(f)| \leq \|\nabla G\|_{L^\infty} |f^{(n)} - f| \rightarrow 0 \text{ in } L^p_{loc}$$

So  $G(f^{(n)}) \rightarrow G(f)$  in  $L^p_{loc}$ , thus  $\partial_{x_i} G(f^{(n)}) \rightarrow \partial_{x_i} G(f)$  in  $D'(\Omega)$ . On the other hand, by the standard Chain-Rule for  $C^1$ -functions:

$$\partial_{x_i} G(f^{(k)}) = \sum_{k=1}^d \underbrace{\partial_k G(f^{(k)})}_{(\text{b.d.} \rightarrow \partial_k G(f))} \underbrace{\partial_i f_k^{(n)}}_{(\rightarrow \partial_i f_k \text{ in } L^p(\Omega))} \rightarrow \sum_{k=1}^d \partial_k G(f) \partial_i f_k \text{ in } L^p_{loc}(\Omega)$$

Thus

$$\partial_{x_i} G(f) = \sum_{k=1}^d \underbrace{\partial_k G(f)}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p_{loc}} \in L^p_{loc} \text{ in } D'(\Omega)$$

So  $G(f) \in W^{1,p}_{loc}(\Omega)$ . Assume that  $G(0) \in L^p(\Omega)$  (i.e.  $|\Omega| < \infty$  or  $G(0) = 0$ ). If  $f \in W^{1,p}(\Omega)$ , then  $G(f) \in W^{1,p}(\Omega)$  since

$$|G(f) - G(0)| \leq \|\nabla G\|_{L^\infty} |f| \in L^p \Rightarrow G(f) \in L^p$$

and

$$\partial_{x_i} G(f) = \sum_k \underbrace{\partial_k G}_{\in L^\infty} \underbrace{\partial_i f_k}_{\in L^p} \in L^p \Rightarrow G(f) \in W^{1,p}(\Omega) \quad \blacksquare$$

**Theorem 3.56** (Derivative of absolute value) Let  $\Omega \subseteq \mathbb{R}^d$  be open. Let  $f \in W^{1,p}(\Omega)$ . Then  $|f| \in W^{1,p}(\Omega)$  and if  $f$  is real-valued:

$$(\nabla |f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

*Proof.* Exercise. Hint: Use the Chain-Rule for  $G_\epsilon(x) = \sqrt{\epsilon^2 + x^2} - \epsilon \rightarrow |x|$  as  $\epsilon \rightarrow 0$   $\blacksquare$

### 3.4 Distribution vs. measures

Let  $\mu$  be a Borel measure in  $\mathbb{R}^d$  s.t.  $\mu(K) < \infty$  for all compact  $K \subseteq \mathbb{R}^d$ . Then define

$$T : D(\mathbb{R}^d) \longrightarrow \mathbb{C} \\ \phi \longmapsto \int_{\mathbb{R}^d} \phi(x) d\mu(x) \quad \forall \phi \in C_c^\infty$$

$\leadsto$  T is a distribution since if  $\phi_n \rightarrow \phi$  in  $D(\Omega)$ , then

$$|T(\phi_n) - T(\phi)| \leq \int_{\mathbb{R}^d} |\phi_n - \phi| d\mu(x) \leq \|\phi_n - \phi\|_{L^\infty} \left( \int_K d\mu \right) \xrightarrow{n \rightarrow \infty} 0$$

**Example 3.57**  $\partial_0$  in  $D'(\mathbb{R}^d)$  is a Borel probability measure.

**Theorem 3.58** (Positive distributions are measures) Let  $\Omega \subseteq \mathbb{R}^d$  be open, let  $T \in D'(\Omega)$ . Assume  $T \geq 0$ , i.e.  $T(\phi) \geq 0$  for all  $\phi \in D(\Omega)$  satisfying  $\phi(x) \geq 0$  for all  $x$ . Then there is a Borel positive measure  $\mu$  on  $\Omega$  such that  $\mu(K) < \infty$  for all  $K \subseteq \Omega$  compact and:

$$T(\phi) = \int_{\Omega} \phi(x) d\mu(x) \quad \forall \phi \in D(\Omega)$$

*Proof.* See Lieb-Loss Analysis. Sketch: If  $O \subseteq \mathbb{R}^d$  is open, then

$$\mu(O) = \sup\{T(\phi) \mid \phi \in D(\Omega), 0 \leq \phi \leq 1, \text{supp } \phi \subseteq O\}$$

For all  $A \subseteq \Omega$  (not necessarily open),

$$\mu(A) = \inf\{\mu(O) \mid O \text{ open}, A \subseteq O\}$$

The mapping  $\mu : 2^{\Omega} \rightarrow [0, \infty]$  is an outer measure, i.e.

1.  $\mu(\emptyset) = 0$
2.  $\mu(A) \leq \mu(B)$  if  $A \subseteq B$
3.  $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$

From the outer measure we can find a  $\sigma$ -algebra  $\Sigma$  and  $\mu$  is a measure on  $\Sigma$  s.t.  $E$  is measurable iff

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c).$$

So all open sets are measurable, thus outer regularity (by def  $\mu(A) = \inf\{\mu(O) \mid O \text{ open } \supseteq A\}$ ), so inner regularity  $\mu(A) = \sup\{\mu(K) \mid K \text{ compact } \subseteq A\}$ . ■

**Exercise 3.59** (E 4.1) Prove that if  $T_n \rightarrow T$  in  $D'(\mathbb{R}^d)$ , then  $D^{\alpha}T_n \rightarrow D^{\alpha}T$  in  $D'(\mathbb{R}^d)$  for all  $\alpha \in \mathbb{N}^d$ .

**Exercise 3.60** (E 4.2)

**Exercise 3.61** (E 4.3)  $f \in L^1(\mathbb{R}^d)$ ,  $\int f = 1$   $f_{\epsilon}(x) = \epsilon^{-d}f(\epsilon^{-1}x)$ . Then  $f_{\epsilon} \rightarrow \delta_0$  in  $D'(\mathbb{R}^d)$ .

**Exercise 3.62** (E 4.4) Let  $\{f_n\} \subseteq L^1$ ,  $\text{supp } f \subseteq B(0, 1)$ ,  $f_n \rightarrow f$  in  $L^1$ . Prove for all  $g \in C_c^{\infty}$  that  $f_n \star g \rightarrow f \star g$  in  $D(\mathbb{R}^d)$ .

*Solution.* Since  $f_n \in L^1$ ,  $\text{supp } f \subseteq B(0, 1)$  and  $g \in C_c^{\infty}$  we have  $f_n \star g \in C_c^{\infty}$  and

$$\text{supp}(f_n \star g) \subseteq (\text{supp } g) + \overline{B(0, 1)} = K.$$

Since  $f_n \rightarrow f$  in  $L^1$  there is a subsequence  $f_{n_k} \rightarrow f$  almost everywhere, so  $f$   $\text{supp}$  in  $\overline{B(0, 1)}$ , so  $f \star g \in C_c^{\infty}$ ,  $\text{supp}(f \star g) \subseteq K$ . We have:

$$\begin{aligned} |f_n \star g(x) - f \star g(x)| &= \left| \int_{\mathbb{R}^d} (f_n(y) - f(y))g(x - y) dy \right| \\ &\leq \int_{\mathbb{R}^d} |f_n(y) - f(y)||g(x - y)| dy \\ &\leq \|g\|_{L^{\infty}} \|f_n - f\|_{L^1} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

thus  $\|f_n \star g - f \star g\|_{L^\infty} \rightarrow 0$ . Similary:

$$\|D^\alpha(f_n \star g) - D^\alpha(f \star g)\|_{L^\infty} = \|f_n \star \underbrace{(D^\alpha g)}_{\in C_c^\infty} - f \star (D^\alpha g)\|_{L^\infty} \xrightarrow{n \rightarrow \infty} 0$$

for all  $\alpha \in \mathbb{N}^d$ , so  $f_n \star g \rightarrow f \star g$  in  $D(\mathbb{R}^d)$ . ■

**Exercise 3.63** (E 4.5) Compute distributional derivatives  $f', f''$  of  $f(x) = x|x-1|$ .

*Solution.* We prove  $f'(x) = g(x) := \begin{cases} 2x-1 & x > 1 \\ 1-2x & x < 1 \end{cases}$ . Take  $\phi \in C_c^\infty(\mathbb{R}^d)$ .

$$\begin{aligned} -f'(\phi) &= \int_{\mathbb{R}^d} f \phi' dy \\ &= \int_{-\infty}^1 f \phi' dy + \int_1^\infty f \phi' dy \\ &= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 f' \phi dy + [f\phi]_1^\infty - \int_1^\infty f' \phi dy \\ &= [f\phi]_{-\infty}^1 - \int_{-\infty}^1 g \phi dy + [f\phi]_1^\infty - \int_1^\infty g \phi dy \\ &= f(1-)\phi(1) - f(1+)\phi(1) - \int_{\mathbb{R}^d} g \phi dy \\ &= 0 - \int_{\mathbb{R}^d} g \phi dy \end{aligned}$$

Now we compute  $f'' = g'$ . Take  $\phi \in C_c^\infty(\mathbb{R}^d)$ :

$$\begin{aligned} -(g')(\phi) &= \int_{\mathbb{R}^d} g \phi' dy \\ &= \int_{-\infty}^1 g \phi' dy + \int_1^\infty g \phi' dy \\ &= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 g' \phi dy - \int_1^\infty g' \phi dy \\ &= [g(1-) - g(1+)]\phi(1) - \int_{-\infty}^1 (-2)\phi dy - \int_1^\infty 2\phi dy \\ &= -2\phi(1) + \int_{-\infty}^\infty [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x) dx \\ &= -2\delta_1(\phi) + \int_{-\infty}^\infty [2\mathbb{1}_{(-\infty,1)}(x) - 2\mathbb{1}_{(1,\infty)}(x)]\phi(x) dx \\ &\Rightarrow g' = \underbrace{2\delta_1}_{\notin L_{loc}^1} - \underbrace{2\mathbb{1}_{(-\infty,1)} + 2\mathbb{1}_{(1,\infty)}}_{\in L_{loc}^1} \end{aligned} \quad \blacksquare$$

## Chapter 4

# Weak Solutions and Regularity

**Definition 4.1** Consider the linear PDE:

$$\sum_{\alpha} c_{\alpha} D^{\alpha} u(x) = F(x), \quad c_{\alpha} \text{ constant, } F \text{ given}$$

A function  $u$  is called a weak solution (a distributional solution) if

$$\sum_{\alpha} c_{\alpha} D^{\alpha} u = F \quad \text{in } D'(\Omega).$$

Namely,

$$\sum_{\alpha} (-1)^{|\alpha|} c_{\alpha} \int_{\Omega} u D^{\alpha} \phi = \int_{\Omega} F \phi, \quad \forall \phi \in D(\Omega)$$

Regularity: Given some condition on the data  $F$ , what can we say about the smoothness of  $u$ ? Can we say that the equation holds in the classical sense? We derived  $G$  (the solution of the Laplace Equation) before in two ways:

1.  $\Delta G(x) = 0$  for all  $x \neq 0$ , assuming  $G(x) = G(|x|)$  and  $d \geq 2$
2.  $\hat{G}(k) = \frac{1}{|2\pi k|^2}$  for  $d \geq 3$

**Theorem 4.2** For all  $d \geq 1$  we have  $G \in L^1_{loc}(\mathbb{R}^d)$  and  $-\Delta G = \delta_0$  in  $D'(\mathbb{R}^d)$ .

*Proof.* Take  $\phi \in D(\mathbb{R}^d)$ . Then:

$$\begin{aligned} (-\Delta G_y)(\phi) &= G_y(-\Delta \phi) = \int_{\mathbb{R}^d} G_y(x) (-\Delta \phi)(x) dx \\ &= \int_{\mathbb{R}^d} G(y-x) (-\Delta \phi)(x) dx \\ &= [G \star (-\Delta \phi)](y) = (-\Delta)(G \star \phi)(y) \end{aligned}$$

Recall for all  $f \in C^2$ ,  $-\Delta(G \star f) = f$  pointwise. So we can conclude  $-\Delta G_y = \delta_y$  in  $D'(\mathbb{R}^d)$ . ■

**Remark 4.3** In  $d = 1$ ,  $G(x) = -\frac{1}{2}|x|$ , so  $-G'(x) = \text{sgn}(x)/2$ , so  $-G''(x) = \delta_0$ .

**Remark 4.4** Formally:

$$-\Delta(G_y \star \phi) = (-\Delta G_y) \star \phi(x) = (\delta_0 \star \phi)(x) = \int_{\mathbb{R}^d} \delta_0(y) \phi(x-y) dy = \delta_0(\phi(x - \bullet))$$

**Theorem 4.5** (Poisson's equation with  $L^1_{loc}$  data) Let  $f \in L^1_{loc}(\mathbb{R}^d)$  s.t.  $\omega_d f \in L^1(\mathbb{R}^d)$  where

$$\omega_d(x) = \begin{cases} 1 + |x| & d = 1 \\ \log(1 + |x|) & d = 2 \\ \frac{1}{1+|x|^{d-2}} & d \geq 3, \end{cases}$$

then  $u(x) = (G \star f)(x) \in L^1_{loc}(\mathbb{R}^d)$ . Moreover  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ . In fact,  $u \in W^{1,1}_{loc}(\mathbb{R}^d)$  and:

$$\partial_{x_i} u(x) = (\partial_{x_i} G) \star f(x) = \int_{\mathbb{R}^d} (\partial_{x_i} G)(x-y) f(y) dy$$

**Remark 4.6** We can also replace  $\mathbb{R}^d$  by  $\Omega$  and get  $-\Delta u = f$  in  $D'(\Omega)$ .

*Proof of Theorem 4.5.* First we check that  $u \in L^1_{loc}$ . Take any Ball  $B(0, R) \subseteq \mathbb{R}^d$ , prove  $\int_B |u| dy < \infty$ . We have

$$\begin{aligned} \int_B |u| dy &= \int_B \left| \int_{\mathbb{R}^d} G(x-y) f(y) dy \right| dx \\ &\leq \int_B \int_{\mathbb{R}^d} |G(x-y)| |f(y)| dy dx \\ &= \int_{\mathbb{R}^d} \left( \int_B |G(x-y)| dx \right) |f(y)| dy \end{aligned}$$

If  $y \notin B = B(0, R)$ , then by Newtons's theorem (Mean-value theorem):

$$\int_{B(0,R)} |G(x-y)| dx = |B(0, R)| |G(y)| \leq C |B| \omega_d(y)$$

If  $y \in B$ , then  $|y| \leq R$ , so  $|x-y| \leq 2R$  if  $x \in B$ .

$$\int_{B(0,R)} |G(x-y)| dx \leq \int_{|x-y| \leq 2R} |G(x-y)| dx = \int_{|z| \leq 2R} |G(z)| dz \leq c_R$$

as  $G \in L^1_{loc}$ . Thus

$$\int_B |u| dy \leq c_B \int_{|y| \geq R} \omega_d(y) |f(y)| dy + c_B \int_{|y| \leq R} |f(y)| dy < \infty$$

Let us prove  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ . Take  $\phi \in D(\mathbb{R}^d)$ . Then:

$$\begin{aligned}
(-\Delta u)(\phi) &= u(-\Delta \phi) \\
&= \int_{\mathbb{R}^d} u(x)(-\Delta \phi)(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)f(y)(-\Delta \phi)(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y-x)f(y)(-\Delta \phi)(x) dx dy \\
&= \int_{\mathbb{R}^d} [G \star (-\Delta \phi)](y)f(y) dy \\
&= \int_{\mathbb{R}^d} -\Delta(G \star \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} \phi(y)f(y) dy
\end{aligned}$$

So  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ . We check that  $\partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$ . Note that

$$|\partial_i G(x)| \leq c \frac{1}{|x|^{d-1}} \in L^1_{loc}(\mathbb{R}^d)$$

and

$$\int_{B(0,R)} |\partial_i G(x-y)| dx \leq \begin{cases} C_r \omega_d(y) & |y| \geq R \\ C_r & |y| \leq R \end{cases}$$

So  $\int_{B(0,R)} |(\partial_i G \star f)(y)| dy < \infty$  for all  $R > 0$ . For all  $\phi \in D(\mathbb{R}^d)$ :

$$\begin{aligned}
-(\partial_i u)(\phi) &= u(\partial_i \phi) = \int_{\mathbb{R}^d} u(x) \partial_i \phi(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)f(y) \partial_i \phi(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y-x)f(y) \partial_i \phi(x) dx dy \\
&= \int_{\mathbb{R}^d} (G \star \partial_i^y \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} (\partial_i^y G \star \phi)(y)f(y) dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i^y G(y-x)f(y)\phi(x) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} -(\partial_i G)(x-y)f(y)\phi(x) dx dy \\
&= - \int_{\mathbb{R}^d} (\partial_i G \star f)(x)\phi(x) dx
\end{aligned}$$

So  $\partial_i u = \partial_i G \star f \in L^1_{loc}(\mathbb{R}^d)$ . Thus  $u \in L^1_{loc}$ ,  $\partial_i u \in L^1_{loc}$  for all  $i$ . So  $u \in W^{1,1}_{loc}(\mathbb{R}^d)$ .  $\blacksquare$

Regularity: We consider the Laplace Equation  $\Delta u = 0$  in  $\mathbb{R}^d$ .

**Lemma 4.7** (Weyl) If  $\Omega \subseteq \mathbb{R}^d$  open and  $T \in D'(\Omega)$  s.t.  $\Delta T = 0$  in  $D'(\Omega)$ , then:  $T = f \in C^\infty(\Omega)$  and  $f$  is a harmonic function.

*Proof.* ( $\Omega = \mathbb{R}^d$ ). Take  $\phi \in C_c^\infty$ , then  $y \mapsto T(\phi_y) = T(\phi(-y))$  is  $C^\infty$  and  $\Delta_y T(\phi_y) = T((\Delta\phi)_y) = (\Delta T)(\phi_y) = 0$ . Take  $g \in C_c^\infty$ ,  $g$  is radial. Then:

$$\int_{\mathbb{R}^d} T(\phi_y)g(y) dy \stackrel{(\text{exercise})}{=} \int_{\mathbb{R}^d} T(\phi)g(y) dy = T(\phi) \left( \int_{\mathbb{R}^d} g dy \right)$$

**Exercise 4.8** Let  $f \in C^\infty(\mathbb{R}^d)$  be a harmonic function and  $g \in C_c^\infty$ ,  $g$  is radial. Then:

$$\int_{\mathbb{R}^d} f(x)g(x) dx = f(0) \left( \int_{\mathbb{R}^d} g(x) dx \right)$$

On the other hand:

$$\int_{\mathbb{R}^d} T(\phi_y)g(y) dy = T(\phi \star g) = T(g \star \phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y) dy$$

Take  $\int_{\mathbb{R}^d} g dy = 1$ , then:

$$T(\phi) = \int_{\mathbb{R}^d} T(g_y)\phi(y) dy$$

For all  $\phi \in C_c^\infty$ . Then  $T = T(g_y) \in C^\infty$  ■

Now let's regard the Poisson Equation  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ .

**Remark 4.9** Any solution has the form  $u = G \star g + h$  where  $\Delta h = 0$  in  $D'(\mathbb{R}^d)$ . By Weyl's Lemma (4.7),  $h \in C^\infty$ , then we only need to consider the regularity of  $G \star f$ .

**Remark 4.10** The regularity is a *local question*, namely if we write

$$f = f_1 + f_2 = f\phi + f(1 - \phi),$$

where  $\phi = 1$  in a ball  $B$  and  $\phi \in C_c^\infty$ .

Then  $G \star f = G \star f_1 + G \star f_2$ . Here  $f_2 = f(1 - \phi) = 0$  in  $B$ . With Weyl's Lemma (4.7),  $G \star f_2 \in C^\infty$ .

**Theorem 4.11** (Low Regularity of Poisson Equation) Let  $f \in L^p(\mathbb{R}^d)$  and compactly supported. Then

a) If  $p \geq 1$ , then

- $G \star f \in C^1(\mathbb{R}^d)$  if  $d = 1$ .
- $G \star f \in L_{loc}^q(\mathbb{R}^d)$  for any  $q < \infty$  if  $d = 2$ .
- $G \star f \in L_{loc}^q(\mathbb{R}^d)$  for  $q < \frac{d}{d-2}$  if  $d \geq 3$ .

b) If  $\frac{d}{2} < p \leq d$ , then  $G \star f \in C_{loc}^{0,\alpha}(\mathbb{R}^d)$  for all  $0 < \alpha < 2 - \frac{d}{p}$ , i.e.

$$|(G \star f)(x) - (G \star f)(y)| \leq C_k |x - y|^\alpha \quad \forall x, y \in K$$

with  $K$  compact in  $\mathbb{R}^d$ .

c) If  $p > d$ , then  $G \star f \in C_{loc}^{1,\alpha}(\mathbb{R}^d)$  for all  $0 < \alpha < 1 - \frac{d}{p}$ .



where  $G$  is den fundamental solution of the laplace equation.

**Example 4.12** Let  $r = |x|$

$$u(x) = \omega(r) = \log(|\log(r)|)$$

if  $0 < r < \frac{1}{2}$ , so  $u$  is well-defined in  $B = B(0, \frac{1}{2})$ . We conclude:

$$-\Delta_{\mathbb{R}^3} u(x) = -\omega''(r) - \frac{2\omega'(r)}{r} = f(x) \in L^{\frac{3}{2}}(B)$$

But the Theorem (b) tells us that if  $f \in L^{\frac{3}{2}}$  then  $u$  is continuous but  $u \notin C(B)$ .

*Proof of theorem 4.11.* a) ( $p = 1$ ) Why is  $G \star f \in L_{loc}^q$ ? Recall from the proof of Youngs inequality:

$$\begin{aligned} |(G \star f)(x)| &= \left| \int_{\mathbb{R}^d} G(x-y) f(y) dy \right| \\ (\text{H\"older}) &= \left( \int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^d} |f(y)| dy \right)^{\frac{1}{q'}} \end{aligned}$$

Where  $\frac{1}{q} + \frac{1}{q'} = 1$ . Then:

$$|(G \star f)(x)|^q \leq C \int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy$$

For any Ball  $B = B(0, R) \subseteq \mathbb{R}^d$ :

$$\begin{aligned} \int_B |G \star f(x)|^q dx &\leq C \int_B \left( \int_{\mathbb{R}^d} |G(x-y)|^q |f(y)| dy \right) dx \\ &= C \int_{\mathbb{R}^d} \left( \int_B |G(x-y)|^q dx \right) |f(y)| dy \end{aligned}$$

$G(x) \sim \frac{1}{|x|^{\frac{1}{d-2}}} \rightsquigarrow |G|^q = \frac{1}{|x|^{\frac{1}{(d-2)q}}} \in L_{loc}^1(\mathbb{R}^d)$  if  $(d-2)q < 2 \Leftrightarrow q < \frac{d}{d-2}$ . Here,  $y \in \text{supp } f$ , so  $|y| \leq R_1$ , then  $|x-y| \leq R+R_1$  if  $|x| \leq R$ . With  $y \in \text{supp } f$ , this implies:

$$\int_{B(0,R)} |G(x-y)|^q dx \leq \int_{|z| \leq R+R_1} |G(z)|^q dz < \infty$$

b)

$$(G \star f)(x) - (G \star f)(y) = \int_{\mathbb{R}^d} (G(x-z) - G(y-z)) f(z) dz$$

So

$$|G \star f(x) - (G \star f)(y)| \leq C \int_{\mathbb{R}^d} \left| \frac{1}{|x-z|^{d-2}} - \frac{1}{|y-z|^{d-2}} \right| |f(z)| dz$$

for all  $x, y \in \mathbb{R}^d$ :

$$\begin{aligned} \left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| &= \left| \left( \frac{1}{|x|} - \frac{1}{|y|} \right) \left( \frac{1}{|x|^{d-3}} + \dots + \frac{1}{|y|^{d-3}} \right) \right| \\ &\leq C \frac{||x| - |y||}{|x||y|} \max \left( \frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &= C \frac{|x-y|}{|x||y|} \max \left( \frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \\ &\leq C \max(|x|, |y|)^{1-\alpha} \frac{|x-y|^\alpha}{|x||y|} \max \left( \frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right) \end{aligned}$$

as

$$||x| - |y|| \leq \min(|x - y|, \max(|x|, |y|)) \leq |x - y|^\alpha \max(|x|, |y|)^{1-\alpha}$$

Thus, for all  $x, y \in \mathbb{R}^d$ :

$$\begin{aligned} \left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| &\leq C|x - y|^\alpha \frac{\max(|x|, |y|)^{1-\alpha}}{|x||y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}}\right) \\ &\leq C|x - y|^\alpha \max\left(\frac{1}{|x|^{d-2+\alpha}}, \frac{1}{|y|^{d-2+\alpha}}\right) \end{aligned}$$

So we get

$$\left| \frac{1}{|x - y|^{d-2}} - \frac{1}{|y - z|^{d-2}} \right| \leq C|x - y|^\alpha \max\left(\frac{1}{|x - z|^{d-2+\alpha}}, \frac{1}{|y - z|^{d-2+\alpha}}\right)$$

Therefore:

$$\begin{aligned} &|G \star f(x) - G \star f(y)| \\ &\leq C \int_{\mathbb{R}^d} |x - y|^\alpha \max\left(\frac{1}{|x - z|^{d-2+\alpha}}, \frac{1}{|y - z|^{d-2+\alpha}}\right) |f(z)| dz \\ &\leq C|x - y|^\alpha \left( \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \right) \end{aligned}$$

Claim: If  $f \in L^p(\mathbb{R}^d)$  is compactly supported,  $d \geq p > \frac{d}{2}$ , then:

$$\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz < \infty$$

for all  $0 < \alpha < 2 - \frac{d}{p}$ . Assume  $\text{supp } f \subseteq \overline{B(0, R_1)}$ . Consider 2 cases:

- If  $|\xi| > 2R_1$ , then:  $|\xi - z| \geq R_1$  for all  $z \in B(0, R_1)$ . Hence:

$$\int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \leq \frac{1}{R_1^{d-2+\alpha}} \|f\|_{L^1} < \infty$$

- If  $|\xi| \leq 2R_1$ , then:  $|\xi - z| \leq 3R_1$  for all  $z \in B(0, R_1)$ :

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz &\leq \int_{|\xi - z| \leq 3R_1} \frac{1}{|\xi - z|^{d-2+\alpha}} |f(z)| dz \\ \text{(H\"older)}, \left(\frac{1}{p} + \frac{1}{q} = 1\right) &\leq \left( \int_{\mathbb{R}^d} |f(z)|^p dz \right)^{\frac{1}{p}} \\ &\quad \cdot \left( \int_{|\xi - z| \leq 3R_1} \frac{1}{|\xi - z|^{(d-2+\alpha)q}} dz \right)^{\frac{1}{q}} \\ &= \|f\|_{L^p} \left( \int_{|z| \leq 3R_1} \frac{1}{|z|^{(d-2+\alpha)q}} dz \right)^{\frac{1}{q}} < \infty \end{aligned}$$

c) ( $d \geq 3$ ) We already know:

$$\partial_i(G \star f) = (\partial_i G \star f) \in L_{loc}^1(\mathbb{R}^d)$$

as  $\omega_d f \in L^1(\mathbb{R}^d)$ . We claim that  $\partial_i G \star f \in C^{0,\alpha}(\mathbb{R}^d)$ . So  $G \star f \in C^{1,\alpha}(\mathbb{R}^d)$  by the equivalence between the classical and the distributional derivatives. Exercise. Hint:

$$|\partial_i G \star f(x) - \partial_i G \star f(y)| \leq \int_{\mathbb{R}^d} |\partial_i G(x-z) - \partial_i G(y-z)| |f(z)| dz,$$

$$\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d} \rightsquigarrow \text{Need to estimate } |\partial_i G(x) - \partial_i G(y)| \leq C|x-y|^\alpha. \quad \blacksquare$$

**Theorem 4.13** (High regularity for Poisson's equation) Let  $f \in C^{0,\alpha}(\mathbb{R}^d)$ ,  $0 < \alpha < 1$  be compactly supported. Then  $G \star f \in C^{2,\alpha}(\mathbb{R}^d)$ .

**Remark 4.14**  $(-\Delta u = f)$  and  $f \in C(\mathbb{R}^d)$  does not imply that  $u \in C^2(\mathbb{R}^d)$ . (exercise)

**Remark 4.15** If  $f \in C^{k,\alpha}(\mathbb{R}^d)$ ,  $k \in \{0, 1, \dots\}$ ,  $0 < \alpha < 1$  is compactly supported, then  $G \star f \in C^{k+2,\alpha}(\mathbb{R}^d)$ . This more general statement is a consequence of the theorem since

$$D^\beta(G \star f) = G \star \underbrace{(D^\beta f)}_{\in C^{0,\alpha}}$$

for all  $\beta = (\beta_1, \dots, \beta_d)$ ,  $|\beta| \leq k$ .

*Proof of theorem 4.13.* Since  $f \in L^p$  for all  $p \leq \infty$  by the low regularity (4.11) we have  $G \star f \in C^{1,\alpha}$  and  $\partial_i(G \star f) = \partial_i G \star f$  in the classical sense. We will compute the distributional derivatives  $\partial_i \partial_j(G \star f)$  and prove that they are Hölder continuous. Compute  $\partial_j \partial_i(G \star f)$ : For all  $\phi \in C_c^\infty(\mathbb{R}^d)$  we have

$$\begin{aligned} -(\partial_j \partial_i G \star f)(\phi) &= \underbrace{(\partial_i(G \star f))}_{\in C}(\partial_j \phi) \\ &= \int_{\mathbb{R}^d} ((\partial_i G) \star f)(x) \partial_j \phi(x) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_i G(x-y) f(y) \partial_j \phi(x) dx dy \\ &= \int_{\mathbb{R}^d} f(y) \left[ \int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx \right] dy \\ &\stackrel{?}{=} \int_{\mathbb{R}^d} \square \phi(y) dy \end{aligned}$$

Recall:  $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$ ,  $\partial_i \partial_j G(x) = \frac{1}{|B_1|} \left[ \frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{d} \right] \frac{1}{|x|^d}$ . We have:

$$\int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_{|x-y| \geq \epsilon} \partial_i G(x-y) \partial_j \phi(x) dx$$

By dominated convergence we have  $|\partial_i G(x-y) \partial_j \phi(x)| \in L^1(dx)$ . By the Gauss-Green-Theorem (2.2) for all  $\epsilon > 0$ :

$$\begin{aligned} &\int_{|x-y| \geq \epsilon} \partial_i G(x-y) \partial_j \phi(x) dx \\ &= \int_{\partial B(y,\epsilon)} \partial_i G(x-y) \phi(x) \omega_j dS(x) - \int_{|x-y| \geq \epsilon} \partial_j \partial_i G(x-y) \phi(x) dx \end{aligned}$$

Where  $\omega = \frac{x-y}{|x-y|}$ . For the boundary term:

$$\begin{aligned}
- \int_{\partial B(y, \epsilon)} \partial_i G(x-y) \phi(x) \omega_j dS(x) &= -\epsilon^{d-1} \int_{\partial B(0,1)} \partial_i G(\epsilon \omega) \phi(y + \epsilon \omega) \omega_j d\omega \\
(\star) \quad &= \int_{\partial B(0,1)} \frac{1}{d|B_1|} \omega_i \omega_j \phi(y + \epsilon \omega) d\omega \\
&\xrightarrow{\epsilon \rightarrow 0} \int_{\partial B(0,1)} \frac{1}{d|B_1|} \omega_i \omega_j \phi(y) d\omega \\
&= \frac{1}{d} \delta_{i,j} \phi(y)
\end{aligned}$$

( $\star$ )  $\partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$ , so  $\partial_i G(\epsilon \omega) = -\frac{-\omega_i}{d|B_1|} \frac{1}{\epsilon^{d-1}}$ . for all  $|\omega| = 1$ .

Now we split:

$$\begin{aligned}
& - \int_{|x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx \\
&= - \int_{|x-y| \geq 1} \partial_i \partial_j G(x-y) \phi(x) dx - \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx
\end{aligned}$$

The key observation is:  $\int_{\partial B(0,r)} \partial_i \partial_j G(x) dx = 0$  since

$$\partial_i \partial_j G(x) = \frac{1}{|B_1|} \left( \omega_i \omega_j - \frac{\partial_{ij}}{d} \right) \frac{1}{|x|^d},$$

$\omega = \frac{x}{|x|}$ . For example if  $i = 1, j = 2, r = 1$ :

$$\int_{\partial B(0,1)} \partial_1 \partial_2 G(x) dS(x) = \frac{1}{|B_1|} \int_{\partial B(0,1)} \omega_1 \omega_2 d\omega,$$

$\partial B(0,1) = \{\omega \mid |\omega| = 1\}$ . Consider:  $\omega \mapsto R\omega, (\omega_1, \dots, \omega_d) \mapsto (-\omega_1, \omega_2, \dots, \omega_d)$ . Then

$$- \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(y) dx = 0.$$

So

$$- \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) \phi(x) dx = - \int_{1 \geq |x-y| \geq \epsilon} \partial_i \partial_j G(x-y) (\phi(x) - \phi(y)) dx$$

In summary:

$$\begin{aligned}
\partial_i \partial_j (G \star f)(\phi) &= \int_{\mathbb{R}^d} f(y) \left( \int_{\mathbb{R}^d} \partial_i G(x-y) \partial_j \phi(x) dx \right) dy \\
&= \int_{\mathbb{R}^d} f(y) \frac{1}{d} \partial_{ij} \phi(y) dy \\
&\quad - \int_{\mathbb{R}^d} f(y) \left( \int_{|x-y| > 1} \partial_i \partial_j G(x-y) \phi(x) dx \right) \\
&\quad - \int_{\mathbb{R}^d} \left[ \lim_{\epsilon \rightarrow 0} \int_{1 \geq |x-y| \geq \epsilon} \underbrace{\partial_i \partial_j G(x-y) (\phi(x) - \phi(y)) dx}_{\leq \frac{C}{|x-y|^d} |x-y| \|\nabla \phi\|_{L^\infty} \leq \frac{C}{|x-y|^{d-1}} \in L^1_{loc}(dx) \forall y} \right] dy \\
&= \int_{\mathbb{R}^d} \frac{\delta_{ij}}{d} f(x) \phi(x) dx - \int_{\mathbb{R}^d} \phi(x) \left( \int_{|x-y| > 1} \partial_i \partial_j G(x-y) f(y) dy \right) dx \\
&\quad - \int_{\mathbb{R}^d} \phi(x) \left[ \int_{|x-y| \leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy \right] dx
\end{aligned}$$

Conclusion:

$$\begin{aligned}\partial_i \partial_j (G \star f)(x) &= -\frac{\delta_{ij}}{d} f(x) + \int_{|x-y|>1} \partial_i \partial_j G(x-y) f(y) dy \\ &\quad + \int_{|x-y|\leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy\end{aligned}$$

The first term  $f \in C^{0,\alpha}$ . The second term is also at least  $C^{0,\alpha}$  since  $\partial_i \partial_j G(x)$  is smooth as  $|x| > 1$ . We need to prove that the third term

$$W_{ij}(x) = \int_{|x-y|\leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy$$

is Hölder-continuous,  $|W_{ij}(x) - W_{ij}(y)| \leq C|x-y|^\alpha$ . Recall:

$$|\partial_i \partial_j G(x-y) (f(y) - f(x))| \leq C \frac{1}{|x-y|^d} |x-y|^\alpha = \frac{C}{|x-y|^{d-\alpha}} \in L^1_{loc}(dy)$$

We write

$$\begin{aligned}W_{ij}(x) &= \int_{|x-y|\leq 1} \partial_i \partial_j G(x-y) (f(y) - f(x)) dy \\ &= \int_{|z|\leq 1} \partial_i \partial_j G(z) (f(x+z) - f(x)) dz\end{aligned}$$

So we get:

$$W_{ij} - W_{ij}(y) = \int_{|z|\leq 1} \partial_i \partial_j G(z) (f(x+z) - f(y+z) - f(x) + f(y)) dz$$

Easy thought: Use  $|\partial_i \partial_j G(z)| \leq \frac{C}{|z|^d}$  and

$$\begin{aligned}&|f(x+z) - f(y+z) - f(x) + f(y)| \\ &\leq \begin{cases} |f(x+z) - f(x)| + |f(y+z) - f(y)| \leq C|z|^\alpha \\ |f(x+z) - f(y+z)| + |f(x) - f(y)| \leq C|x-y|^\alpha \end{cases}\end{aligned}$$

Thus:

$$\begin{aligned}|W_{ij}(x) - W_{ij}(y)| &\leq C \int_{|z|\leq 1} \frac{1}{|z|^d} \min(|z|^\alpha, |x-y|^\alpha) dz \\ &\leq C \int_{|z|\leq 1} \frac{1}{|z|^d} (|z|^\alpha)^\epsilon (|x-y|^\alpha)^{1-\epsilon}, \quad 0 < \epsilon < 1 \\ &\leq C \left( \int_{|z|\leq 1} \frac{1}{|z|^{d-\alpha\epsilon}} \right) |x-y|^{\alpha(1-\epsilon)} \\ &\leq C_\epsilon |x-y|^{\alpha(1-\epsilon)}\end{aligned}$$

thus it is easy to prove  $|W_{ij}(x) - W_{ij}(y)| \leq C_\alpha |x-y|^\alpha$  for all  $\alpha' \leq \alpha$ . However, to get  $\alpha' = \alpha$  we need a more precise estimate. We split:

$$W_{ij}(x) - W_{ij}(y) = \int_{|z|\leq 1} \dots = \int_{|z|\leq \min(4|x-y|, 1)} + \int_{4|x-y| < |z|\leq 1}$$

For the first domain:

$$\begin{aligned} & \int_{|z| \leq 4|x-y|} |\partial_{ij}G(z)| |f(x+z) - f(y+z) - f(y) + f(x)| dz \\ & \leq C \int_{|z| \leq 4|x-y|} \frac{1}{|z|^d} |z|^\alpha dz = \text{const} \cdot |x-y|^\alpha \end{aligned}$$

For the second domain:

$$\begin{aligned} & \int_{4|x-y| < |z| \leq 1} \partial_{ij}G(z)(f(x+z) - f(y+z) + f(y)f(x)) dz \\ & = \int_{4|x-y| < |z| \leq 1} \partial_{ij}G(z)(f(x+z) - f(y+z)) dz = (\dots) \end{aligned}$$

since  $\int_{4|x-y| < |z| \leq 1} \partial_{ij}G(z) dz = 0$ . Then

$$(\dots) = \int_{4|x-y| < |z-x| \leq 1} \partial_{ij}G(z-x)f(z) dz - \int_{4|x-y| < |z-y| \leq 1} \partial_{ij}G(z-y)f(z) dz.$$

Denote  $A = \{z \mid 4|x-y| < |z-x| \leq 1\}$ ,  $B = \{z \mid 4|x-y| < |z-y| \leq 1\}$ . Consider

$$\begin{aligned} & \int_A \partial_{ij}G(z-x)f(z) dz - \int_B \partial_{ij}G(z-y)f(z) dz \\ & = \int_{A \setminus B} + \int_{B \setminus A} + \int_{A \cap B} (\partial_{ij}G(z-x) - \partial_{ij}G(z-y))f(z) dz \end{aligned}$$

Lets regard the intersection. We have

$$\begin{aligned} \partial_{ij}G(x) &= \frac{1}{|B_1|} \frac{1}{|x|^d} (\omega_i \omega_j - \frac{1}{d} \delta_{ij}) \\ |\partial_{ij}G(x) - \partial_{ij}G(y)| &\leq C|x-y| \left( \frac{1}{|x|^{d+1}} + \frac{1}{|y|^{d+1}} \right) \end{aligned}$$

Now,

$$|\partial_{ij}G(z-x) - \partial_{ij}G(z-y)| \leq C|x-y| \left( \frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right)$$

So we have

$$\begin{aligned} & \left| \int_{A \cap B} (\partial_{ij}G(z-x) - \partial_{ij}G(z-y))f(z) dz \right| \\ & \leq C \int_{A \cap B} |x-y| \left( \frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right) |f(z)| dz = (\dots) \end{aligned}$$

Now we replace  $f(z)$  by  $f(z) - f(x)$ , then:

$$\begin{aligned} & \left| \int_{A \cap B} (\partial_{ij}G(z-x) - \partial_{ij}G(z-y))(f(z) - f(x)) dz \right| \\ & \leq C \int_{A \cap B} |x-y| \left( \frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}} \right) |z-x|^\alpha dz \\ & = C \underbrace{\int_{A \cap B} |x-y| \frac{1}{|z-x|^{d+1-\alpha}} dz}_{(I)} + C \underbrace{\int_{A \cap B} |x-y| \frac{1}{|z-y|^{d+1}} |z-x|^\alpha dz}_{(II)} \end{aligned}$$

Now,

$$\begin{aligned}
(I) &\leq C|x-y| \int_{4|x-y| < |z-x| \leq 1} \frac{1}{|z-x|^{d+1-\alpha}} dz \\
&= C|x-y| \int_{4|x-y| < |z| \leq 1} \frac{1}{|z|^{d+1-\alpha}} dz \\
&\leq C|x-y| \int_{4|x-y|}^1 \frac{1}{r^{d+1-\alpha}} r^{d-1} dr \\
&= C|x-y| \int_{4|x-y|}^1 \frac{1}{r^{2-\alpha}} dr \\
&\leq C|x-y| \left[ -1 + \frac{1}{(4|x-y|)^{1-\alpha}} \right] \\
&\leq C|x-y|^\alpha
\end{aligned}$$

$$\begin{aligned}
(II) &\leq C|x-y| \int_{A \cap B} \frac{1}{|z-y|^{d+1}} |z-x|^\alpha dz \\
&\leq C|x-y| \int_{A \cap B} \frac{1}{|z-y|^{d+1}} (|z-y|^\alpha + |x-y|^\alpha) dz \\
&\leq C|x-y| \underbrace{\int_B \frac{1}{|z-y|^{d+1-\alpha}} dz}_{\text{similar to (I)}} + C|x-y|^{1+\alpha} \int_B \frac{1}{|z-y|^{d+1}} dz
\end{aligned}$$

and

$$C|x-y|^{1+\alpha} \int_B \frac{1}{|z-y|^{d+1}} dz \leq \int_{4|x-y|}^1 \frac{1}{r^{d+1}} r^{d-1} dr \leq \frac{C}{|x-y|}$$

Consider  $A \setminus B$ :

$$\left| \int_{A \setminus B} \right| \leq C \|f\|_{L^\infty} \int_{A \setminus B} \frac{1}{|z-x|^d} dz$$

where

$$\begin{aligned}
A &= \{z \mid 4|x-y| < |z-x| \leq 1\} \\
B &= \{z \mid 4|x-y| < |z-y| \leq 1\} \\
A \setminus B &= \{z \in A \mid |z-y| \leq 4|x-y|\} \cup \{z \in A \mid |z-y| > 1\} = E_1 \cup E_2
\end{aligned}$$

for

$$\begin{aligned}
E_1 &= \{z \mid |z-y| \leq 4|x-y| < |z-x| \leq 1\} \\
&\subseteq \{z \mid 4|x-y| \leq |x-z| \leq 5|x-y|\}.
\end{aligned}$$

$|x - z| \leq |x - y| + |y - z| \leq 5|x - y|$  in  $E_1$ . We have

$$\begin{aligned}
\int_{E_1} \frac{1}{|z - x|^d} dz &\leq \int_{4|x-y| \leq |x-z| \leq 5|x-y|} \frac{1}{|z - x|^{d-\alpha}} dz \\
&= \int_{4|x-y| \leq |z| \leq 5|x-y|} \frac{1}{|z|^{d-\alpha}} dz \\
&= \int_{4|x-y|} \frac{1}{r^d} r^{d-1} dr \\
&= \int_{4|x-y|} \frac{1}{r^{1-\alpha}} dr \\
&\leq C|x - y|^\alpha
\end{aligned}$$

Now in  $E_2$ :  $|z - x| \geq |z - y| - |y - x| \geq 1 - |y - x|$ .

$$\begin{aligned}
\int_{E_2} \frac{1}{|z - x|^{d-\alpha}} dz &\leq \int \frac{1}{|z - x|^{d-\alpha}} dz = \int_{1-|x-y|}^1 \frac{1}{r^{d-\alpha}} r^{d-1} dr \\
&\leq \text{const.} \left| 1 - \frac{1}{(1 - |x - y|)^\alpha} \right| \leq C|x - y|^\alpha \quad \blacksquare
\end{aligned}$$

**Exercise 4.16** (E 5.1) Prove that if  $f$  is a harmonic function in  $\mathbb{R}^d$  and  $g \in C_c(\mathbb{R}^d)$  is radial, then

$$\int_{\mathbb{R}^d} f(x)g(x) dx = f(0) \int_{\mathbb{R}^d} g(x) dx$$

*Solution.*  $x = r\omega, r > 0, |\omega| = 1$

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)g(x) dx &\stackrel{(\text{Polar})}{=} \int_0^\infty \left( \int_{\partial B(0,1)} f(r\omega)g(r\omega) d\omega \right) dr \\
&= \int_0^\infty \left( g_0(r) \int_{\partial B(0,1)} f(r\omega) d\omega \right) dr \\
(\text{Mean value theorem (2.12)}) \quad &= \int_0^\infty \left( g_0(r) f(0) \int_{\partial B(0,1)} d\omega \right) dr \\
&= f(0) \int_0^\infty \left( \int_{\partial B(0,1)} g(r\omega) d\omega \right) dr \\
&= f(0) \int_{\mathbb{R}^d} g(x) dx \quad \blacksquare
\end{aligned}$$

**Remark 4.17** Let  $g \in C_c(\mathbb{R}^d)$  be radial. Why is  $\int_{\mathbb{R}^3} \frac{g(x)}{|x|} dx \neq \infty$ ? Because  $f(x) = \frac{1}{|x|}$  is harmonic in  $\mathbb{R}^d \setminus \{0\}$  and sub-harmonic in  $\mathbb{R}^d$ ,  $-\Delta f = c\delta_0$ .

**Exercise 4.18** (E 5.2) Let  $1 \leq p < \infty$ . Let  $\Omega \subseteq \mathbb{R}^d$  be open. Consider the Sobolev Space

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) \mid \partial_{x_i} f \in L^p(\Omega), \forall i = 1, 2, \dots, d\}$$

with the norm

$$\|f\|_{W^{1,p}} = \|f\| + \sum_{i=1}^d \|\partial_{x_i} f\|_{L^p(\Omega)}.$$



Prove that  $W^{1,p}(\Omega)$  is a Banach space. Here  $x = (x_i)_{i=1}^d \in \mathbb{R}^d$ . Hint: You can use the fact that  $L^p(\Omega)$  is a Banach Space.

*Solution.*  $W^{1,p}(\Omega) \subseteq L^p(\Omega) \times L^p(\Omega) \cdots \times L^p(\Omega) = (L^p(\Omega))^{d+1}$ . For an element  $f \in W^{1,p}(\Omega)$  we can think of it as  $f \mapsto (f, \partial_1 f, \partial_2 f, \dots, \partial_d f)$ , so  $W^{1,p}(\Omega)$  is a subspace of  $(L^p(\Omega))^{d+1}$ , which is a norm-space. Why is  $W^{1,p}(\Omega)$  closed in  $(L^p(\Omega))^{d+1}$ ? Take  $\{f_n\}_{n=1}^\infty \subseteq W^{1,p}(\Omega)$  such that  $f_n \rightarrow f$  in  $L^p$  and  $\partial_i f_n \rightarrow g_i$  in  $L^p$  for all  $i = 1, \dots, d$ . We prove that  $(f, g_1, \dots, g_d) \in W^{1,p}(\Omega)$ , i.e.  $f \in W^{1,p}$  and  $g_i = \partial_i f$  for all  $i = 1, \dots, d$ . We know that  $f_n \rightarrow f$  in  $L^p(\Omega)$ , so  $f_n \rightarrow f$  in  $D'(\Omega)$  and  $\partial_i f_n \rightarrow \partial_i f$  in  $D'(\Omega)$ . On the other hand we have  $\partial_i f_n \rightarrow g_i$  in  $L^p(\Omega)$ , so  $\partial_i f_n \rightarrow g_i$  in  $D'(\Omega)$ . So we get  $\partial_i f = g_i \in L^p(\Omega)$  for all  $i = 1, \dots, d$  in  $D'(\Omega)$ . So we can conclude  $f \in W^{1,p}(\Omega)$  and  $\partial_i f = g_i$  for all  $i = 1, \dots, d$ . ■

**Exercise 4.19** (E 5.3) Let  $f$  be a real-valued function in  $W^{1,p}(\mathbb{R}^d)$  for some  $1 \leq p < \infty$ . Prove that  $|f| \in W^{1,p}(\mathbb{R}^d)$  and

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) & f(x) > 0 \\ -\nabla f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}.$$

*Solution.* Consider  $G_\epsilon(t) = \sqrt{\epsilon^2 + t^2} - \epsilon$  for  $\epsilon > 0, t \in \mathbb{R}$ . Clearly we have  $G_\epsilon(t) \rightarrow |t|$  as  $\epsilon \rightarrow 0$  and

$$G'_\epsilon(t) = \frac{2t}{2\sqrt{\epsilon^2 + t^2}} = \frac{t}{\sqrt{\epsilon^2 + t^2}},$$

so  $|G'_\epsilon(t)| \leq 1, G_\epsilon(0) = 0$ . By the chain rule,  $G_\epsilon(f) \in W^{1,p}(\mathbb{R}^d)$  and

$$(\partial_i G_\epsilon(f))(x) = G'_\epsilon(f) \partial_i f(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \partial_i f(x) \in L^p(\mathbb{R}^d)$$

for all  $i = 1, \dots, d$ . Note then when  $\epsilon \rightarrow 0$  that  $G_\epsilon(f)(x) \rightarrow |f(x)|$  pointwise, so  $G_\epsilon(f) \rightarrow |f|$  in  $L^p(\mathbb{R}^d)$ .  $|G_\epsilon(f)(x) - G_\epsilon(0)| \leq |f(x)| \in L^p(\mathbb{R}^d)$  by dominated convergence.

$$\partial_i G_\epsilon(f)(x) = \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \partial_i f(x) \xrightarrow{\epsilon \rightarrow 0} g_i(x) := \begin{cases} \partial_i f(x) & f(x) > 0 \\ -\partial_i f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

$$|\partial_i G_\epsilon(f)(x)| \leq \left| \frac{f(x)}{\sqrt{\epsilon^2 + f^2(x)}} \right| |\partial_i f(x)| \leq |\partial_i f(x)| \in L^p(\mathbb{R}^d)$$

So we get  $\partial_i G_\epsilon(f) \xrightarrow{\epsilon \rightarrow 0} g_i$  in  $L^p(\mathbb{R}^d)$  by Dominated Convergence. So we conclude:  $\partial_i(|f|) = g_i \in L^p(\mathbb{R}^d)$  for all  $i = 1, \dots, d$ , so  $|f| \in W^{1,p}(\mathbb{R}^d), |f| \in L^p$ . ■

**Exercise 4.20** (E 5.4) Let  $\Omega \subseteq \mathbb{R}^d$  be open and bounded,  $f \in L^1(\Omega)$ ,

$$u(x) = \int_{\Omega} G(x-y)f(y) dy$$

Let  $-\Delta u = f$  in  $D'(\Omega)$ ,  $u \in L^1_{loc}(\Omega)$ ,  $f \in L^1_{loc}(\mathbb{R}^d)$  and  $\omega_d f \in L^1(\mathbb{R}^d)$ , where

$$\omega_d(x) = \begin{cases} 1 + |x| & d = 1 \\ \log(1 + |x|) & d = 2 \\ \frac{1}{(1+|x|)^{d-2}} & d \geq 3 \end{cases}.$$

Prove that

$$G \star f = \int_{\mathbb{R}^d} G(x-y)f(y) dy \in L_{loc}^1(\mathbb{R}^d)$$

and  $-\Delta(G \star f) = f$  in  $D'(\mathbb{R}^d)$ .

*Solution.* Define  $\tilde{f} = \mathbb{1}_\Omega(x)f(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$ . Then

$$u(x) = \int_{\Omega} G(x-y)f(y) dy = \int_{\mathbb{R}^d} G(x-y)\tilde{f}(y) dy = (G \star \tilde{f})(x)$$

We have  $u \in L_{loc}^1(\mathbb{R}^d)$ , so  $u \in L^1(\Omega)$ . Then  $-\Delta u = \tilde{f}$  in  $D'(\mathbb{R}^d)$ , so  $-\Delta u = f$  in  $D'(\Omega)$ . Claim:  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ , so  $-\Delta u = f$  in  $D'(\Omega)$  if  $\Omega \subseteq \mathbb{R}^d$ ,  $\tilde{f}|_\Omega = f$ . Take  $\phi \in C_c^\infty(\Omega)$ . We need:  $(-\Delta u)(\phi) \stackrel{?}{=} \int_{\Omega} f\phi$ . We have  $\phi \in C_c^\infty(\Omega)$ , so  $\phi C_c^\infty(\mathbb{R}^d)$ . This implies:

$$(-\Delta u)(\phi) = \int_{\mathbb{R}^d} \tilde{f}\phi = \int_{\substack{\Omega, \\ \text{supp } \phi \subseteq \Omega}} \tilde{f}\phi = \int_{\Omega} f\phi \quad \blacksquare$$

**Exercise 4.21** (E 5.5) Let  $B = B(0, \frac{1}{2}) \subseteq \mathbb{R}^3$ . Consider  $u : B \rightarrow \mathbb{R}$ , defined by

$$u(x) = \log |\log |x||.$$

Prove that the distributional derivative  $f = -\Delta u$  is a function in  $L^{\frac{3}{2}}(B)$ .

*Solution.*

$$\begin{aligned} \omega(r) &= \log(-\log(r)), \quad \text{for } r \in \left(0, \frac{1}{2}\right) \\ \omega'(r) &= \frac{1}{-\log(r)} \left(-\frac{1}{r}\right) = \frac{1}{r \log r} \\ \omega''(r) &= -\frac{1}{(r \log(r))^2} (r \log(r))' = -\frac{\log(r) + 1}{(r \log r)^2} \end{aligned}$$

So we have

$$-\Delta u = \omega''(r) = \frac{1}{(r \log r)^2} - \frac{1}{r^2 \log(r)} = f(r)$$

We show that  $f \in L^{\frac{3}{2}}$ :

$$\begin{aligned} \int_B |f(x)|^{\frac{3}{2}} dx &= \text{const} \int_0^{\frac{1}{2}} \left| \frac{1}{r^2(\log r)^2} - \frac{1}{r^2 \log r} \right|^{\frac{3}{2}} r^2 dr \\ &\lesssim \int_0^{\frac{1}{2}} \frac{1}{r} \left| \frac{1}{(\log(r))^2} - \frac{1}{(\log(r))} \right|^{\frac{3}{2}} dr \\ \left( \begin{array}{l} r = e^{-x}, \\ x \in (\log(2), \infty), \\ dr = -e^{-x} dx \end{array} \right) &\lesssim \int_{\log(2)}^{\infty} e^x \left( \frac{1}{x^2} + \frac{1}{x} \right)^{\frac{3}{2}} e^{-x} dx \\ &\lesssim \int_{\log(2)}^{\infty} \frac{1}{x^{\frac{3}{2}}} dx < \infty \end{aligned}$$

Where  $\lesssim$  means *up to a constant*. Now,  $u(x) = \omega(r) = \log(-\log(r))$ .

$$-\Delta u(x) = f(r) = \frac{1}{r^2(\log(r))^2} - \frac{1}{r^2 \log(r)}$$

for all  $x \neq 0, |x| = r < \frac{1}{2}$ . Why is  $-\Delta u(x) = f$  in  $D'(B)$ ? Take  $\phi \in C_c^\infty(B)$ , check:  $\int_B u(-\Delta\phi) = \int_B f\phi$ .

$$\int_{|x| < \frac{1}{2}} u(-\Delta\phi) dx = \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |x| < \frac{1}{2}} u(x)(-\Delta\phi)(x) dx$$

by Dominated convergence.  $u \in L^1(B)$ . For all  $\epsilon > 0$ :

$$\begin{aligned} \int_{\epsilon < |x| < \frac{1}{2}} u(x)(-\Delta\phi)(x) dx &= \int_{|x| > \epsilon} u(x)(-\Delta\phi)(x) dx \\ &= \int_{\partial B(0, \epsilon)} u(x) \nabla \phi(x) \frac{x}{|x|} dS(x) + \int_{|x| > \epsilon} \nabla u(x) \nabla \phi(x) dx \end{aligned}$$

The boundary term vanishes as  $\epsilon \rightarrow 0$  since

$$\left| u(x) \nabla \phi(x) \frac{x}{|x|} \right| \leq \|\nabla \phi\|_{L^\infty} |u(x)| = C \log |\log(r)|$$

$$\begin{aligned} \left| \int_{\partial B(0, \epsilon)} u(x) \nabla \phi(x) \frac{x}{|x|} dS(x) \right| &\leq C \int_{\partial B(0, \epsilon)} \log |\log(\epsilon)| dS(x) \\ &= C \log |\log \epsilon| \underbrace{|\partial B(0, \epsilon)|}_{\sim \epsilon^2} \xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned}$$

$$\begin{aligned} \int_{|x| > \epsilon} \nabla u(x) \nabla \phi(x) dx &= \sum_{i=1}^d \int_{|x| > \epsilon} \partial_i u(x) \partial_i \phi(x) dx \\ &= \sum_{i=1}^d \left( - \int_{\partial B(0, \epsilon)} \partial_i u(x) \phi(x) \frac{x_i}{|x|} dS(x) - \int_{|x| > \epsilon} \underbrace{\partial_i \partial_i u(x)}_{f(x)} \phi(x) dx \right) \end{aligned}$$

The boundary term vanishes as  $\epsilon \rightarrow 0$  as

$$\begin{aligned} \left| \int_{\partial B(0, \epsilon)} \partial_i u(x) \phi(x) \frac{x_i}{|x|} dS(x) \right| &\leq \|\phi\|_{L^\infty} \int_{\partial B(0, \epsilon)} |\partial_i u(x)| dS(x) \\ (\star) \quad &\leq C \frac{1}{|\epsilon \log(r)|} |\partial B(0, \epsilon)| \rightarrow 0 \end{aligned}$$

as  $\epsilon \rightarrow 0$ .  $(\star)u = u(r), u(x) = \omega(|x|), \partial_i u(x) = \omega'(|x|) \frac{x_i}{|x|}, |\partial_i u(x)| \leq |\omega'(|x|)| = \left| \frac{1}{r \log(r)} \right|$ . Finally:

$$\int_{|x| > \epsilon} f(x) \phi(x) dx \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} f(x) \phi(x) dx$$

Since  $f\phi \in L^1$  and Dominated Convergence. ■

**Exercise 4.22** (Bonus 5) Construct  $u \in L^1(\mathbb{R}^3)$  compactly supported s.t.  $-\Delta u \in L^{\frac{3}{2}}(\mathbb{R}^3)$  and  $u$  is not continuous at 0.

Hint: Related to E 5.5.  $u_0(x) = \omega(r) = \log(|\log(r)|)$  if  $0 < r = |x| < \frac{1}{2}$ . Consider  $\chi u_0$  where  $\chi \in C_c^\infty$ ,  $\chi = 0$  if  $|x| > \frac{1}{2}$ ,  $\chi = 1$  if  $|x| < \frac{1}{4}$ . You can prove that  $\Delta(\chi u_0) = (\Delta\chi)u_0 + 2\nabla\chi\nabla u_0 + \chi(\underbrace{\Delta u_0}_{\in L^{\frac{3}{2}}})$  in  $D'(\mathbb{R}^3)$ . (almost everywhere, in distributional sense, integration by parts)

**Theorem 4.23** (Regularity on Domains) Let  $\Omega \subseteq \mathbb{R}^d$  be open. Assume  $u, f \in D'(\Omega)$  such that  $-\Delta u = f$  in  $D'(\Omega)$ .

- a) If  $f \in L_{loc}^1(\Omega)$ , then
  - $u \in C^1(\Omega)$  if  $d = 1$
  - $u \in L_{loc}^q(\Omega)$  for all  $q < \infty$  if  $d = 2$
  - $u \in L_{loc}^q(\Omega)$  for all  $q < \frac{d}{d-2}$  if  $d \geq 3$
- b) If  $f \in L_{loc}^q(\Omega)$ ,  $d \geq p < \frac{d}{2}$ , then  $u \in C_{loc}^{0,\alpha}(\Omega)$ , where  $0 < \alpha < 2 - \frac{d}{p}$
- c) If  $f \in L_{loc}^p(\Omega)$ ,  $p > df$ , then  $u \in C_{loc}^{1,\alpha}(\Omega)$ , where  $0 \leq \alpha < 1 - \frac{d}{p}$
- d) If  $f \in C_{loc}^{0,\alpha}(\Omega)$  for some  $0 < \alpha < 1$ , then  $u \in C_{loc}^{2,\alpha}(\Omega)$
- e) If  $f \in C_{loc}^{m,\alpha}(\Omega)$ , then  $u \in C_{loc}^{m+2,\alpha}(\Omega)$

*Proof.* Let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . Take a ball  $\bar{B} \subseteq \Omega$ . Define  $f_B : \mathbb{R}^d \rightarrow \mathbb{K}$ ,

$$f_B(x) = (\mathbb{1}_B f)(x) = \begin{cases} f(x) & x \in B \\ 0 & x \notin B \end{cases}$$

Then if  $f \in L_{loc}^1(\Omega)$ ,  $f_B$  is compactly supported. From the previous theorems:  $G \star f_B \in L_{loc}^1(\mathbb{R}^d)$  and  $-\Delta(G \star f_B) = f_B$  in  $D'(\mathbb{R}^d)$ . On the other hand,  $-\Delta u = f$  in  $D'(\Omega)$ , so  $-\Delta(u - G \star f_B) = 0$  in  $D'(B)$ . Indeed, for all  $\phi \in C_c^\infty(B)$ , then:

$$(-\Delta u)(\phi) = \int_{\Omega} f\phi = \int_B f_B\phi = - \int_{\mathbb{R}^d} f_B\phi = (-\Delta)(G \star f_B)(\phi)$$

Then  $-\Delta u = -\Delta(G \star f_B)$  in  $D'(B)$ . Then  $u - G \star f_B$  is harmonic in  $B$  and by Weyl's lemma we have  $u - G \star f_B \in C^\infty(B)$ . So the smoothness of  $u$  in  $B$  is the same to that of  $G \star f$ . ■

**Exercise 4.24** (E 6.1) Show that If  $\chi \in C^\infty(\mathbb{R}^d)$ , then  $f \in W^{1,p}(\mathbb{R}^d)$ ,  $1 \leq p < \infty$ , then  $\chi f \in W_{loc}^{1,p}(\mathbb{R}^d)$  and

$$\partial_i(\chi f) = (\partial_i \chi)f + \chi(\partial_i f) \quad \text{in } D'(\mathbb{R}^d)$$

*Solution.*  $\chi f \in L_{loc}^p(\mathbb{R}^d)$  obvious.  $\partial(\chi f) \in L_{loc}^p(\mathbb{R}^d)$  is nontrivial but follows from  $\partial_i(\chi f) = \underbrace{(\partial_i \chi)f + \chi(\partial_i f)}_{\in L_{loc}^p}$  in  $D'(\mathbb{R}^d)$ . To compute the distributional derivative

$\partial_i(\chi f)$ , then: Take  $\phi \in C_c^\infty(\mathbb{R}^d)$ :

$$- \int_{\mathbb{R}^d} \chi f(\partial_i \phi) = \int_{\mathbb{R}^d} (?)\phi$$

We have

$$\begin{aligned}
-\int_{\mathbb{R}^d} \chi f(\partial_i \phi) &= -\int_{\mathbb{R}^d} f(\chi \partial_i \phi) \\
(\chi \partial_i \phi = (\partial_i \chi) \phi + \chi(\partial_i \phi)) &= -\int_{\mathbb{R}^d} f(\partial_i(\chi \phi) - (\partial_i \chi) \phi) \\
&= -\int_{\mathbb{R}^d} f \underbrace{\partial_i(\chi \phi)}_{\in C_c^\infty} + \int_{\mathbb{R}^d} f(\partial_i \chi) \phi \\
&= \int_{\mathbb{R}^d} (\partial_i f) \chi \phi + \int_{\mathbb{R}^d} f(\partial_i \chi) \phi \\
&= \int_{\mathbb{R}^d} ((\partial_i f) \chi + f(\partial_i \chi)) \phi
\end{aligned}$$

So  $\partial_i(\chi f) = (\partial_i f) \chi + f(\partial_i \chi)$  in  $D'(\mathbb{R}^d)$ . ■

**Remark 4.25** Question: If  $\chi \in C^1(\mathbb{R}^d)$ ,  $f \in W^{1,p}(\mathbb{R}^d)$ . Is this it still correct that  $\partial_i(\chi f) = (\partial_i \chi) f + \chi(\partial_i f)$  in  $D'(\mathbb{R}^d)$ ?

*Proof.* It suffices to show that we still can apply intergration by parts.

$$(\star) \quad -\int f \partial_i g \stackrel{?}{=} \int (\partial_i f) g$$

Approximation:  $(\star)$  is correct if  $g \in C_c^\infty$

- If  $g \in C_c^1$ , there is  $\{g_n\} \subseteq C_c^\infty$  s.t.  $g_n \rightarrow g$  in  $W_{loc}^{1,p}$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

$$\int (\partial_i g) f \xrightarrow{n \rightarrow \infty} - \int \underbrace{f}_{L^p} \underbrace{\partial_i g_n}_{\rightarrow \partial_i g \text{ in } L^q} = \int \underbrace{(\partial_i f)}_{\in L^p} \underbrace{g_n}_{\rightarrow g \text{ in } L^q} \xrightarrow{n \rightarrow \infty} \int (\partial_i f) g$$

■

**Exercise 4.26** (E 6.2)  $\mathbb{R}^2$ ,  $G(x) = -\frac{1}{2\pi} \log |x|$ . Let  $f \in L^p(\mathbb{R}^d)$ , compactly supported. Define  $u(x) = (G \star f)(x) = \int_{\mathbb{R}^2} G(x-y) f(y) dy$

1. If  $p = 1$ , then  $u \in L_{loc}^q(\mathbb{R}^2)$  for all  $q < \infty$ .
2. If  $p > 2$ , then  $u \in C^{1,\alpha}$  with  $0 < \alpha < 1 - \frac{2}{p}$ .

*Solution.* 1. Take any ball  $B = B(0, R)$  and:

$$\begin{aligned}
\int_B |u(x)|^q dx &= \int_B \left( \int_{\mathbb{R}^d} |G(x-y)| |f(y)| dy \right)^q dx \\
&\leq C \int_B \left( \int_{\mathbb{R}^2} |G(x-y)|^q |f(y)| dy \right) dx \\
&= C \int_{\mathbb{R}^2} \left( \int_B |G(x-y)|^q dx \right) |f(y)| dy
\end{aligned}$$

Recall from the proof of Youngs inequality:

$$\begin{aligned}
|u(x)| &= \left| \int_{\mathbb{R}^2} G(x-y) f(y) dy \right| \\
&\leq \int_{\mathbb{R}^2} |G(x-y)| |f(y)| dy \\
&\leq \left( \int_{\mathbb{R}^2} |G(x-y)|^q |f(y)| dy \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^2} |f(y)| dy \right)^{\frac{1}{q'}}, \quad \frac{1}{q} + \frac{1}{q'} = 1
\end{aligned}$$

Assume  $\text{supp } f \subseteq \overline{B(0, R)}$ . Then if  $y \in \text{supp } f$  and  $x \in B(0, R)$ , then  $|x-y| \leq |x| + |y| \leq R + R_1$ . For all  $y \in \text{supp } f$ :

$$\begin{aligned}
\int_{B(0, R)} |G(x-y)|^q dx &\leq \int_{|x-y| \leq R+R_1} |G(x-y)|^q dx \\
&= \int_{|z| \leq R+R_1} |G(z)|^q dz < \infty
\end{aligned}$$

as  $G \in L_{loc}^q$  ( $|G(z)| = \frac{1}{2\pi} |\log(z)| \leq \frac{C_{R+R_1, \epsilon}}{|z|^\epsilon}$  for all  $|z| \leq R + R_1$ ), so

$$\int_{|z| \leq R+R_1} |G(z)|^q \leq C_{R+R_1, \epsilon} \int_{|z| \leq R+R_1} \frac{1}{|z|^{\epsilon q}} dz < \infty$$

if  $\epsilon q < 2$ .

2. Recall  $\partial_i u \in L_{loc}^1(\mathbb{R}^2)$  and:

$$\partial_i u(x) = (\partial_i G \star f)(x) = c \int_{\mathbb{R}^2} \frac{x_i - y_i}{|x - y|^2} f(y) dy$$

First we show  $\partial_i u \in C^{0, \alpha}$ :

$$\begin{aligned}
|\partial_i u(x) - \partial_i u(z)| &= \left| C \int_{\mathbb{R}^2} \left( \frac{x_i - y_i}{|x - y|^2} - \frac{z_i - y_i}{|z - y|^2} \right) f(y) dy \right| \\
&\leq C \int_{\mathbb{R}^2} \left| \frac{x_i y_i}{|x - y|^2} - \frac{z_i y_i}{|z - y|^2} \right| |f(y)| dy \\
&\stackrel{?}{\leq} C |x - z|^\alpha
\end{aligned}$$

Note that

$$\begin{aligned}
\left| \frac{x_i - y_i}{|x - y|^2} - \frac{z_i - y_i}{|z - y|^2} \right| &= \left| (x_i - y_i) \left( \frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right) + \frac{x_i - z_i}{|z - y|^2} \right| \\
&\leq |x_i - y_i| \left| \frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right| + \frac{|x_i - z_i|}{|z - y|^2} \\
&\leq C |z - x|^\alpha \left( \frac{1}{|x - y|^{1+\alpha}} + \frac{1}{|z - y|^{1+\alpha}} + \frac{|x - z|}{|z - y|^2} \right)
\end{aligned}$$

Here  $|x_i - z_i| \leq |x - z|$  and  $|x_i - y_i| \leq |x - y|$  and:

$$\begin{aligned}
\underbrace{\left| \frac{1}{|x - y|^2} - \frac{1}{|z - y|^2} \right|}_{\text{sym } x \leftrightarrow z} &= \left| \frac{1}{|x - y|} - \frac{1}{|z - y|} \right| \left| \frac{1}{|x - y|} + \frac{1}{|z - y|} \right| \\
&= \frac{||z - y| - |x - y||}{|x - y| |z - y|} \left| \frac{1}{|x - y|} + \frac{1}{|z - y|} \right| \\
&\leq |z - x|^\alpha \frac{\max(|z - y|, |x - y|)^{1-\alpha}}{|x - y| |z - y|} \left( \frac{1}{|x - y|} + \frac{1}{|z - y|} \right) \\
&\leq C |z - x|^\alpha \left( \frac{1}{|x - y|^{2+\alpha}} + \frac{1}{|z - y|^{2+\alpha}} \right)
\end{aligned}$$

By the symmetrie  $x \leftrightarrow z$ :

$$\begin{aligned}
LHS &\leq C|z-x|^\alpha \left( \frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} \right) + \frac{|x-y|}{|x-y|^2} \\
&\Rightarrow LHS \leq C \cdots + |x-z| \min \left( \frac{1}{|z-y|^2}, \frac{1}{|x-y|^2} \right) \\
&\leq (|x-y| + |z-y|)^{1-\alpha} \\
&C|z-x|^\alpha \left( \frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} \right)
\end{aligned}$$

In summary:

$$\begin{aligned}
|\partial_i u(x) - \partial_i u(z)| &\leq C \int_{\mathbb{R}^2} \left| \frac{x_i - y_i}{|x-y|^2} - \frac{z_i - y_i}{|z-y|^2} \right| |f(y)| dy \\
&= C|x-y|^\alpha \int_{\mathbb{R}^2} \left( \frac{1}{|x-y|^{1+\alpha}} + \frac{1}{|z-y|^{1+\alpha}} \right) |f(y)| dy
\end{aligned}$$

Consider if  $|x| > 2R_1$ :

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|^{1+\alpha}} |f(y)| dy \leq \int_{\mathbb{R}^2} \frac{1}{R_1^{1+\alpha}} |f(y)| dy \leq C$$

$\text{supp } f \subseteq B(0, R_1)$ . If  $|x| < 2R_1$ , then  $|x-y| \leq 3R$  if  $y \in B(0, R_1)$ . Hence:

$$\begin{aligned}
&\int_{|x-y| \leq 3R_1} \frac{1}{|x-y|^{1+\alpha}} |f(y)| dy \\
&\leq \left( \int_{|x-y| \leq 3R_1} \frac{1}{|x-y|^{(1+\alpha)p'}} \right)^{\frac{1}{p'}} \left( \int |f(y)|^p dy \right)^{\frac{1}{p}} \\
&= \int_{|z| \leq 3R_1} \frac{1}{|z|^{(1+\alpha)p'}} dz < \infty
\end{aligned}$$

So  $\alpha < 1 - \frac{2}{p}$ . ■

**Exercise 4.27** (E 6.3) Let  $f \in C_{loc}^{0,\alpha}$  and  $-\Delta u = f$  in  $D'(\Omega)$ . Prove  $u \in C_{loc}^{2,\alpha}(\Omega)$ .

*Solution.* Take an open ball  $B \subseteq \bar{B} \subseteq \Omega$ . We prove  $u \in C^{2,\alpha}(B)$ . There is an open  $\Omega_B$  s.t.  $\bar{B} \subseteq \bar{\Omega}_B \subseteq \Omega$ . Then there is a  $\chi_B \in C_c^\infty(\mathbb{R}^d)$  s.t.  $\chi_B(x) = 1$  if  $x \in B$  and  $\chi_B(x) = 0$  if  $x \notin \Omega_B$ . Define

$$f_B(x) = \chi_B(x)f(x) : \mathbb{R}^d \rightarrow \mathbb{R}$$

We prove that  $f_B \in C^{0,\alpha}(\mathbb{R}^d)$ . Since  $f \in C_{loc}^{0,\alpha}(\Omega)$  we have  $f \in C^{0,\alpha}(\Omega)$ , so  $|f(x) - f(y)| \leq C|x-y|^\alpha$  for all  $x, y \in \Omega_B$ . Then:

$$\begin{aligned}
|f_B(x) - f_B(y)| &= |\chi_B(x)f(x) - \chi_B(y)f(y)| \\
&\leq |(\chi_B(x) - \chi_B(y))f(x) + \chi_B(y)(f(x) - f(y))| \\
&\leq C|x-y|^\alpha \|f\|_{L^\infty} + C\|\chi\|_{L^\infty(\Omega_B)} |x-y|^\alpha \leq C_{\Omega_B} |x-y|^\alpha
\end{aligned}$$

What about other cases? If  $x, y$  are bot not in  $\Omega_B$ , then  $|f_B(x) - f_B(y)| = 0$ , then if  $x \in \Omega_B$  and  $y \notin \Omega_B$ :  $|f_B(x) - f_B(y)| = |f_B(x)| = |\chi_B(x)||f(x)| = |\chi_B(x) - \chi_B(y)||f(x)| \leq C|x-y|^\alpha$ . Conclusion:  $|f_B(x) - f_B(y)| \leq C|x-y|^\alpha$  for all  $x, y \in \mathbb{R}^d$ , i.e.  $f_B \in C^{0,\alpha}(\mathbb{R}^d)$ . Also  $f_B$  is compactly supported. By a theorem in the lecture:  $G \star f_B \in C^{2,\alpha}(\mathbb{R}^d)$ . Finally:  $-\Delta u = f$  in  $D'(\Omega)$ ,  $-\Delta(G \star f_B) = f_B$  in  $D'(\mathbb{R}^d)$ . So we conclude  $-\Delta u = f = f_B = -\Delta(G \star f_B)$  in  $D'(B)$ .  $-\Delta(u - G \star f_B) = 0$  in  $D'(B)$ , so  $u - G \star f_B \in C^\infty(B)$ , so  $u \in C^{2,\alpha}(B)$ . ■

**Exercise 4.28** (E 6.4)  $u, f \in L^2(\mathbb{R}^d)$ ,  $-\Delta u = f$  in  $D'(\mathbb{R}^d)$ . Prove  $u \in W^{2,2}(\mathbb{R}^d)$ ,  $\|u\|_{W^{2,2}(\mathbb{R}^d)} \leq C(\|u\|_{L^2} + \|f\|_{L^2})$ .

$$\begin{aligned} W^{2,2}(\mathbb{R}^d) &= \{g \in L^2(\mathbb{R}^d) \mid D^\alpha g \in L^2 \text{ for all } |\alpha| \leq 2\} \\ &= \{g \in L^2(\mathbb{R}^d) \mid \widehat{D^\alpha g}(k) = (-2\pi i k)^\alpha \hat{g}(k) \in L^2(\mathbb{R}^d) \text{ for all } |\alpha| \leq 2\} \\ &= \{g \in L^2(\mathbb{R}^d) \mid (1 + |k|^2)\hat{g}(k) \in L^2(\mathbb{R}^d)\} \end{aligned}$$

$\|u\|_{W^{2,2}(\mathbb{R}^d)}$  is comparable  $\int_{\mathbb{R}^d} (1 + |k|^2)^2 |\hat{g}(k)|^2 dk$ . If  $D^\alpha g \in L^2$ , then  $\widehat{D^\alpha g}(k) = (-2\pi i k)^\alpha \hat{g}(k)$ . For any  $\phi \in C_c^\infty(\mathbb{R}^d)$ :

$$\begin{aligned} \int \widehat{D^\alpha g}(k) \hat{\phi}(k) dk &= \int (D^\alpha g) \phi = (-1)^{|\alpha|} \int g (D^\alpha \phi) \\ &= (-1)^{|\alpha|} \int \tilde{\hat{g}}(k) \widehat{D^\alpha \phi}(k) \\ &= (-1)^{|\alpha|} \int \tilde{\hat{g}}(k) (-2\pi i k)^\alpha \hat{\phi}(k) dk \end{aligned}$$

so  $\hat{D^\alpha g}(k) = (-1)^{|\alpha|} \hat{g}(k) \overline{(-2\pi i k)^\alpha} = \hat{g}(k) (-2\pi i k)^\alpha$ . This implies:

$$\begin{aligned} \|u\|_{W^{2,2}(\mathbb{R}^d)} &\leq C \int_{\mathbb{R}^d} (1 + |k|^2)^2 |\hat{u}(k)|^2 dk \\ &= C \left( \|u\|_{L^2}^2 + \int_{\mathbb{R}^d} |k|^4 |\hat{u}(k)|^2 dk \right) \\ &\leq C (\|u\|_{L^2}^2 + \|f\|_{L^2}^2) \\ &\leq C (\|u\|_{L^2} + \|f\|_{L^2})^2 \end{aligned}$$

**Remark 4.29** (Bonus 6) Let  $f, g \in W^{1,2}(\mathbb{R}^d)$ . Prove that  $fg \in W^{1,1}(\mathbb{R}^d)$  and

$$\partial_i(fg) = (\partial_i f)g + f(\partial_i g) \quad \text{in } D'(\mathbb{R}^d)$$



## Chapter 5

# Existence for Poisson's Equation on Domains

Let  $\Omega \subseteq \mathbb{R}^d$  be open. Consider Poisson's equation.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for given data  $(f, g)$  and  $u$  the unknown function.

- Classical solutions:  $f \in C^2(\bar{\Omega}) \rightsquigarrow$  explicit representation formula.
- Weak solution:  $f \in L^p(\Omega)$ ,  $g \in L^p(\partial\Omega) \rightsquigarrow u \in W^{2,p}(\Omega)$ . We are going to establish the existence by *Energy Methods*. (Calculus of variations)

**Definition 5.1** ( $C^1$ -Domains) Let  $\Omega \subseteq \mathbb{R}^d$  be open. We say that  $\Omega$  is of class  $C^1$  (i.e.  $\partial\Omega \in C^1$ ) if for all  $x_0 \in \partial\Omega$  there is a bijective function  $h : U \rightarrow Q$ , where

- $x_0 \in U$  open in  $\mathbb{R}^d$
- $Q = \{x = (x_1, \dots, x_d) = (x', x_d)\} \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, |x_d| < 1\}$
- $h \in C^1(\bar{U})$  and  $h^{-1} \in C^1(\bar{Q})$  ( $C^1$ -diffeomorphism)
- $h(U) = Q$

$$\begin{aligned} h(U \cap \Omega) &= Q_+ = Q \cap \mathbb{R}_+^d = \{x = (x', x_d) \in Q \mid x_d > 0\} \\ h(U \cap \partial\Omega) &= Q_0 = Q \cap \partial\mathbb{R}_+^d = \{x = (x', x_d) \in Q \mid x_d = 0\} \\ h(U \setminus \bar{\Omega}) &= Q_- = Q \cap \mathbb{R}_-^d = \{x = (x', x_d) \in Q \mid x_d < 0\} \end{aligned}$$

(From Brezis' book)

**Remark 5.2** The set  $Q$  can be replaced by a ball, i.e.  $\Omega$  is of  $C^1$  if for all  $x_0 \in \partial\Omega$  there is a function  $U \rightarrow B(0, 1) \subseteq \mathbb{R}^d$ .

- $x_0 \in U$  with  $U \subseteq \mathbb{R}^d$  open.
- $h \in C^1(\bar{U})$ ,  $h^{-1} \in C^1(\overline{B(0, 1)})$
- $h(U \cap \Omega) = B(0, 1) \cap \mathbb{R}_+^d$ ,  $h(U \cap \partial\Omega) = B(0, 1) \cap \mathbb{R}^d$ .

**Remark 5.3** (An equivalent definition from Evan's book App. C) Let  $\Omega \subseteq \mathbb{R}^d$  be open. Then  $\Omega$  is  $C^1$  if for all  $x_0 \in \partial\Omega$  there is a  $r > 0$  and a  $C^1$ -function  $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  s.t. (upon relabeling and reorienting the axes if necessary) such that:

$$\Omega \cap B(x_0, r) = \{x = (x', x_d) \in B(x_0, r) \mid x_d < \gamma(x')\}$$

*Proof of the equivalence of the two definitions.*

Def. 2  $\Rightarrow$  Def. 1: In fact, given  $x_0 \in \partial\Omega$  and  $\gamma$  we can define

$$\begin{aligned} h(x', x_d) &= (x', x_d - \gamma(x')) \in C^1(\mathbb{R}^d, \mathbb{R}^d) \\ h^{-1}(x', x_d) &= (x', x_d + \gamma(x')) \in C^1(\mathbb{R}^d, \mathbb{R}^d) \end{aligned}$$

Def. 1  $\Rightarrow$  Def. 2: We need the inverse function theorem and the implicit function theorem. Let  $x_0 \in \partial\Omega$ , let  $h : U \rightarrow B(0, 1)$  as in Def. 1. Denote  $h = (h_1, h_2, \dots, h_d)$ . Since  $h$  is invertible near  $x_0$ , by the inverse function theorem we have for the Jacobi matrix  $Jh(x_0) = (\partial_j h_i(x_0))_{1 \leq i, j \leq d}$  is invertible. So we have  $\nabla h_d(x_0) = (\partial_j h_d(x_0))_{1 \leq j \leq d} \neq \vec{0}^{\mathbb{R}^d}$ , so there is a  $j \in \{1, 2, \dots, d\}$  s.t.  $\partial_j h_d(x_0) \neq 0$ . By relabeling and reorienting the axes, we can assume that  $\partial_d h_d(x_0) > 0$ . By continuity there is a  $r > 0$  such that  $\partial_d h_d(x) > 0$  for all  $x \in B(x_0, r)$ . Define  $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  s.t. in  $B(x_0, r)$ :

$$x = (x', x_d) \in \partial\Omega \iff h_d(x', x_d) = 0 \iff x_d = \gamma(x'),$$

$h_d : \mathbb{R}^d \rightarrow \mathbb{R}$ . This gives a solution  $\gamma$  if  $\partial_d h_d > 0$  in  $B(x_0, r)$ . (For implicit function theorem,  $\partial_d h_d(x_0) \neq 0$ ) Question: Why in  $B(x_0, r)$ ?

$$x = (x', x_d) \in \Omega \iff x_d > \gamma(x')$$

Since  $\partial_d h_d(x) > 0$  for all  $x \in B(x_0, r)$  we have that  $x_d \mapsto h_d(x', x_d)$  is strictly increasing, hence

$$\begin{aligned} x &= (x', x_d) \in \Omega \\ \iff h(x', x_d) &\in \mathbb{R}_+^d \\ \iff h_d(x', x_d) &> 0 = h_d(x', \gamma(x')) \\ \iff x_d &> \gamma(x') \end{aligned} \quad \blacksquare$$

**Theorem 5.4** (Gauss-Green formula / Integration by parts) Let  $\Omega \subseteq \mathbb{R}^d$  be open and bounded. Then

1. For all  $u, v \in C^1(\bar{\Omega})$ :

$$\int_{\Omega} (\partial_i u) v = - \int_{\Omega} u (\partial_i v) + \int_{\partial\Omega} u v n_i dS,$$

where  $\vec{n} = (n_i)_{i=1}^d$  is the outwards unit normal vector.

2. For all  $u, v \in C^2(\bar{\Omega})$ :

$$\int_{\Omega} u (-\Delta v) = \int_{\Omega} \nabla u \nabla v - \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}} dS$$

where  $\frac{\partial v}{\partial \vec{n}} = \nabla v \vec{n} = \sum_{i=1}^d \partial_i v n_i$ .

Classical solutions via Green's function:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Let  $\Omega \subseteq \mathbb{R}^d$  be open, bounded,  $\partial\Omega \in C^1$ . Assume there exists a  $u \in C^2(\bar{\Omega})$ ,  $f \in C(\bar{\Omega})$ ,  $g \in C(\partial\Omega)$ . Let  $G$  be the fundamental solution of the Laplace Equation in  $\mathbb{R}^d$ . We use integration by parts in  $\Omega \setminus B(x, \epsilon)$ :

$$\begin{aligned} & \int_{\Omega \setminus B(x, \epsilon)} u(y)(-\Delta G)(y-x) dy \\ &= \int_{\Omega \setminus B(x, \epsilon)} \nabla u(y) \nabla G(y-x) dy - \int_{\partial\Omega \cup \partial B(x, \epsilon)} u(y) \frac{\partial G}{\partial \vec{n}}(y-x) dS(y) \\ & \int_{\Omega \setminus B(x, \epsilon)} G(y-x)(-\Delta u)(y) dy \\ &= \int_{\Omega \setminus B(x, \epsilon)} \nabla G(y-x) \nabla u(y) dy - \int_{\partial\Omega \cup \partial B(x, \epsilon)} G(y-x) \frac{\partial u}{\partial \vec{n}}(y) dS(y) \end{aligned}$$

This implies:

$$\begin{aligned} & \int_{\Omega \setminus B(x, \epsilon)} [u(y)(-\Delta G(y-x)) - G(y-x)(-\Delta u)(y)] dy \\ &= - \int_{\partial\Omega \cup \partial B(x, \epsilon)} \left[ u(y) \frac{\partial G}{\partial \vec{n}}(y-x) - G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right] dS(y) \end{aligned}$$

for all  $x \in \Omega$ ,  $x \in B(x, \epsilon) \subseteq \Omega$ . When  $\epsilon \rightarrow 0$ , then the left hand side converges to  $-\int_{\Omega} G(y-x)f(y) dy$  and the right hand side (for  $d \geq 2$ ) we have  $\partial_j G(y) = \frac{-y_j}{d|B_1||y|^d}$ , so

$$\frac{\partial G}{\partial \vec{n}} = \nabla G \vec{n} = \nabla G(y) \left( \frac{-y}{|y|} \right) = \sum_{j=1}^d \frac{-y_i}{d|B_1||y|^d} \frac{-y_j}{|y|} = \frac{1}{d|B_1||y|^{d-1}} \text{ on } \partial B(0, \epsilon)$$

so we have

$$\frac{\partial G}{\partial \vec{n}}(y-x) = \frac{1}{d|B_1|\epsilon^{d-1}}$$

on  $\partial B(x, \epsilon)$ . Hence

$$\begin{aligned} \int_{\partial B(x, \epsilon)} u(y) \frac{\partial G}{\partial \vec{n}}(y-x) dS(y) &= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(x, \epsilon)} u(y) dS(y) \\ &= \oint_{\partial B(x, \epsilon)} u(y) dS(y) \xrightarrow{\epsilon \rightarrow 0} u(x) \end{aligned}$$

On the other hand:

$$\left| \int_{\partial B(x, \epsilon)} G(y-x) \frac{\partial u(y)}{\partial \vec{n}} dS(y) \right| \leq C\epsilon^{d-1} \sup_{|z|=\epsilon} |G(z)| \xrightarrow{\epsilon \rightarrow 0} 0$$

since  $|G(z)| \leq \frac{C}{|z|^{d-2}}$  if  $d \geq 3$ ,  $|G(z)| \leq C|\log(z)|$  if  $d = 2$  and  $|G(z)| \leq C|z|$  if  $d = 1$ .

In summary:

$$\begin{aligned} & - \int_{\Omega} G(y-x)f(y) dy = - \int_{\partial\Omega} \left[ u(y) \frac{\partial G}{\partial \vec{n}}(y-x) - G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right] dS(y) - u(x) \\ & \Leftrightarrow u(x) = \int_{\Omega} G(y-x)f(y) dy + \int_{\partial\Omega} \left[ G(y-x) \frac{\partial u}{\partial \vec{n}}(y) - g(y) \frac{\partial G}{\partial \vec{n}}(y-x) \right] dS(y) \end{aligned}$$

Problem: We don't know anything about  $\frac{\partial u}{\partial \vec{n}}$  on  $\partial\Omega$ . Trick: We can resolve that by using the *corrector* function:  $\Phi_x = \Phi_x(y)$  which solves:

$$\begin{cases} -\Delta \Phi_x = 0 & \text{in } \Omega \\ \Phi_x(y) = G(y-x) & \text{on } \partial\Omega \end{cases}$$

We assume that  $\Phi_x$  exists.

**Definition 5.5** (Green's function)  $\tilde{G}(x-y) = G(y-x) - \Phi_x(y)$  for all  $x, y \in \Omega$ ,  $x \neq y$ .

**Exercise 5.6** (E 7.1) Let  $\Omega \subseteq \mathbb{R}^d$  be open and bounded with  $C^1$  boundary. For  $x \in \Omega$ , assume there exist  $\Phi_x(y)$ ,  $y \in \bar{\Omega}$ , s.t.

$$\begin{cases} \Delta_y \Phi_x(y) = 0 \\ \Phi_x(y) = G(y-x) \end{cases},$$

$G(z) = \frac{1}{d(d-2)|B_1||z|^{d-2}}$ ,  $d \geq 3$ . Prove that  $\Phi_x(y) = \Phi_y(x)$  for all  $x, y \in \Omega$ . Then  $\tilde{G}(x, y) = G(y-x) - \Phi_x(y)$  is symmetric, i.e.  $\tilde{G}(x, y) = \tilde{G}(y, x)$ .

*Solution.* Assume  $x \neq y$ . Define

$$\begin{aligned} f(z) &= \tilde{G}(x, z) = G(z-x) - \Phi_x(z) \\ g(z) &= \tilde{G}(y, z) = G(z-y) - \Phi_y(z) \end{aligned}$$

Integration by parts:

$$\begin{aligned} \int_{\Omega \setminus (B(x, \epsilon) \cup B(y, \epsilon))} (f \Delta g - g \Delta f) &= \int_{\partial\Omega \cup \partial B(x, \epsilon) \cup \partial B(y, \epsilon)} \left( f \frac{\partial g}{\partial \vec{n}_z} - g \frac{\partial f}{\partial \vec{n}_z} \right) dS(z) \\ &= \int_{\partial B(x, \epsilon) \cup \partial B(y, \epsilon)} \left( f \frac{\partial g}{\partial \vec{n}_z} - g \frac{\partial f}{\partial \vec{n}_z} \right) dS(z) \end{aligned}$$

Consider  $f \frac{\partial g}{\partial \vec{n}_z}$  on  $\partial B(x, \epsilon)$ . Since  $g$  is only singular at  $y$ , so  $\left| \frac{\partial g}{\partial \vec{n}} \right| \leq C$  on  $\partial B(x, \epsilon)$ . This implies:

$$\begin{aligned} \int_{\partial B(x, \epsilon)} \left| f \frac{\partial g}{\partial \vec{n}_z} \right| dS(z) &\leq C \int_{\partial B(x, \epsilon)} |f| dS(z) \\ &\leq C \int_{\partial B(x, \epsilon)} \left( \frac{1}{|x-z|^{d-2}} + \|\Phi_x\|_{L^\infty(\Omega)} \right) dS(z) \\ &\leq C \epsilon^{d-1} \left( \frac{1}{\epsilon^{d-2}} + 1 \right) \leq C \epsilon \xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned}$$

Consider  $f \frac{\partial g}{\partial \vec{n}_z}$  on  $\partial B(y, \epsilon)$ . Decompose  $\frac{\partial g}{\partial \vec{n}} = [\nabla_z G(z-y) - \nabla_z \Phi_y(z)] \frac{(z-y)}{|z-y|}$ . Since  $\Phi_y(z)$  is harmonic in  $\Omega$ , we have that

$$\int_{\partial B(y, \epsilon)} \left| \nabla_z \Phi_y(z) \frac{(z-y)}{|z-y|} \right| \leq C \int_{\partial B(y, \epsilon)} |f| \leq C \epsilon^{d-1} \xrightarrow{\epsilon \rightarrow 0} 0$$

Thus the main contribution from  $f \frac{\partial g}{\partial \vec{n}}$  is

$$\begin{aligned}
& \int_{\partial B(y, \epsilon)} f(z) \nabla_z G(z-y) \frac{-(z-y)}{|z-y|} dS(z) \\
&= \int_{\partial B(y, \epsilon)} f(z) \frac{-(z-y)}{d|B_1||z-y|^d} \frac{-(z-y)}{|z-y|} dS(z) \\
&= \frac{1}{d|B_1|\epsilon^{d-1}} \int_{\partial B(y, \epsilon)} f(z) dS(z) \\
&= \oint_{\partial B(y, \epsilon)} f(z) dS(z) = f(y)
\end{aligned}$$

In summary:

$$\int_{\partial B(x, \epsilon) \cup \partial B(y, \epsilon)} f \frac{\partial g}{\partial \vec{n}_z} dS(z) \xrightarrow{\epsilon \rightarrow 0} f(y)$$

Similary:

$$\int_{\partial B(x, \epsilon) \cup \partial B(y, \epsilon)} g \frac{\partial f}{\partial \vec{n}_z} dS(z) \xrightarrow{\epsilon \rightarrow 0} g(x)$$

So we have that  $f(y) = g(x)$ , so

$$\begin{aligned}
f(y) &= G(y-x) - \Phi_x(y) \\
g(x) &= G(x-y) - \Phi_y(x).
\end{aligned}$$

So  $\Phi_x(y) = \Phi_y(x)$  for all  $x \neq y \in \Omega$ . This implies  $\Phi_x(y) = \Phi_y(x)$  for all  $x, y \in \Omega$ . ■

**Theorem 5.7** Let  $\Omega \subseteq \mathbb{R}^d$  be open, bounded and  $C^1$ . If  $u \in C^2(\Omega)$  solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases},$$

then

$$u(x) = - \int_{\partial\Omega} g(y) \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) dS(y) + \int_{\Omega} \tilde{G}(x, y) dy$$

*Proof.* We need to prove:

$$\int_{\Omega} \Phi_x(y) f(y) dy + \int_{\partial\Omega} \left( -g(y) \frac{\partial \Phi_x(y)}{\partial \vec{n}_y} + G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \right) = 0$$

By integration by parts:

$$\begin{aligned}
\int_{\Omega} \Phi_x(y) f(y) dy &= \int_{\Omega} \Phi_x(y) (-\Delta u(y)) dy \\
&= \int_{\Omega} [\Phi_x(y) (-\Delta u(y)) + (\Delta \Phi_x(y)) u(y)] dy \\
(\Delta \Phi_x(y) = 0) &= \int_{\partial\Omega} \left( -\Phi_x(y) \frac{\partial u}{\partial \vec{n}} + \frac{\partial \Phi_x(y)}{\partial \vec{n}} \underbrace{u(y)}_{g(y)} \right) dS(y) \quad \blacksquare
\end{aligned}$$

How can we compute  $\Phi_x(y)$ ? It is not easy for general domains. But let us prove on two cases:

- $\Omega = \mathbb{R}_+^d$  (half-space)
- $\Omega = B(0, r)$  (a ball)

## 5.1 Green's function on the upper half plane

We use the following notation:

$$\begin{aligned}\mathbb{R}_+^d &= \{x = (x_1, x_2, \dots, x_d) = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid x_d > 0\} \\ \partial\mathbb{R}_+^d &= \{x = (x', x_d) \mid x_d = 0\} = \mathbb{R}^{d-1} \times \{0\}\end{aligned}$$

For all  $x \in \mathbb{R}^d$  we want to find the correction function  $\Phi_x(y)$  with  $y \in \overline{\mathbb{R}_+^d}$  s.t.

$$\begin{cases} +\Delta_y \Phi_x(y) = 0 & \text{in } \mathbb{R}_+^d \\ \Phi_x(y) = G(y - x) & \text{in } \partial\mathbb{R}_+^d \end{cases}$$

**Definition 5.8** (Reflection for  $\mathbb{R}_+^d$ ) For all  $x = (x', x_d) \in \mathbb{R}^d$ ,  $\tilde{x} = (x', -x_d) \in \mathbb{R}^d$ , (if  $x \in \mathbb{R}_+^d \Rightarrow \tilde{x} \in \mathbb{R}_-^d$ )

Claim:  $\Delta_y \Phi_x(y) = G(y - \tilde{x})$  is a corrector function.

- $\Delta_y \Phi_x(y) = \Delta_y G(y - \tilde{x}) = 0$  for all  $y \in \mathbb{R}_+^d$  for all  $x \in \mathbb{R}_+^d$  (as  $\tilde{x} \in \mathbb{R}_-^d = \mathbb{R}^d \setminus \overline{\mathbb{R}_+^d}$ )
- $\Phi_x(y) = G(y - \tilde{x}) = G(y - x)$  on  $y \in \partial\mathbb{R}_+^d$ . In fact,  $y \in \partial\mathbb{R}_+^d$ , so  $y_d = 0$ , so

$$G(y - \tilde{x}) = G_0(|y - \tilde{x}|) = G_0\left(\sqrt{\sum_{i=1}^{d-1} |x_i - y_i|^2 + |x_d|^2}\right) = G_0(|y - x|)$$

Consider  $f = 0$  and

$$\begin{cases} -\Delta = 0 & \text{in } \mathbb{R}_+^d \\ u = g & \text{on } \partial\mathbb{R}_+^d \end{cases}$$

Then we expect

$$u(x) = - \int_{\partial\Omega} g(y) \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) dS(y)$$

We compute

$$\frac{\partial \tilde{G}}{\partial \vec{n}_y}(x - y) = \sum_{j=1}^d \frac{\partial \tilde{G}}{\partial y_j}(x, y) \vec{n}_j = - \frac{\partial \tilde{G}}{\partial y_d}(x, y) = \frac{\partial}{\partial y_d} (G(y - \tilde{x}) - G(y - x)) = \dots$$

because  $\tilde{G}(x, y) = G(y - x) - \Phi_x(y) = G(y - x) - G(y - \tilde{x})$ .

$$\begin{aligned} \dots &= \frac{1}{d|B_1|} \left[ \frac{-(y_d - \tilde{x}_d)}{|y - \tilde{x}|^d} - \frac{-(y_d - x_d)}{|y - x|^d} \right] \\ (y \in \partial\mathbb{R}_+^d) &= \frac{1}{d|B_1|} \left[ \frac{\tilde{x}_d}{|y - x|} - \frac{x_d}{|y - x|^d} \right] = \frac{-2x_d}{d|B_1||y - x|^d} \end{aligned}$$

We expect

$$u(x) = - \int_{\partial\mathbb{R}_+^d} g(y) \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) dS(y) = \int_{\partial\mathbb{R}_+^d} g(y) \frac{2x_d}{d|B_1||y - x|^d} dS(y)$$

**Theorem 5.9** Assume  $g \in C(\mathbb{R}^{d-1}) \cap L^\infty(\mathbb{R}^{d-1})$  Then

$$u(x) = \int_{\partial\mathbb{R}_+^d} g(y) K(x, y) dS(y)$$

and

$$K(x, y) = \frac{2x_d}{d|B_1||y-x|^d} \quad \text{for all } x \in \mathbb{R}_+^d.$$

satisfies that  $u \in C^\infty(\mathbb{R}_+^d) \cap L^\infty(\mathbb{R}_+^d)$  and

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^d \\ \lim_{\substack{x \rightarrow 0 \\ x \in \mathbb{R}_+^d}} u(x) = g(x_0) & \forall x_0 \in \partial\mathbb{R}_+^d \end{cases}$$

*Proof.* Claim: For all  $y \in \partial\mathbb{R}_+^d$ ,  $x \mapsto K(x, y)$  is harmonic in  $\mathbb{R}_+^d$  (i.e.  $\Delta_x K(x, y) = 0$  in  $\mathbb{R}_+^d$ )

- Argument from Evans:

$$K(x, y) = -\frac{\partial}{\partial y_d}, \quad \tilde{G}(y-x) = -\frac{\partial}{\partial y_d}(G(y-x) - G(y-\tilde{x}))$$

We know that for all  $x \in \mathbb{R}_+^d$ ,  $y \mapsto \tilde{G}(y, x)$  is harmonic in  $\mathbb{R}_+^d \setminus \{x\}$ . By symmetry we have  $\tilde{G}(y, x) = \tilde{G}(x, y)$  for all  $x, y \in \mathbb{R}_+^d$ . So for all  $y \in \mathbb{R}_+^d$ ,  $x \mapsto \tilde{G}(y, x)$  is harmonic in  $\mathbb{R}_+^d \setminus \{y\}$ . Then for all  $y \in \mathbb{R}_+^d$ :  $-\frac{\partial}{\partial y_d} \tilde{G}(y, x) = K(x, y)$  is harmonic  $x \in \mathbb{R}_+^d \setminus \{y\}$ . By a limit argument, for all  $y \in \partial\mathbb{R}_+^d$ ,  $x \mapsto K(x, y)$  is harmonic for all  $x \in \mathbb{R}_+^d$ .

- A direct proof:

$$K(x, y) = \frac{2x_d}{d|B_1|} \frac{1}{|x-y|^d}$$

for all  $x \in \mathbb{R}_+^d$ ,  $y \in \partial\mathbb{R}_+^d$ . For  $i \neq d$ ,  $x = (x_1, \dots, x_d)$ ,

$$\begin{aligned} \partial_{x_i} K(x, y) &= \frac{2x_d}{d|B_1|} \frac{(-d)}{|x-y|^{d+1}} \frac{x_i - y_i}{|x-y|} = \frac{-2x_d}{|B_1|} \frac{x_i - y_i}{|x-y|^{d+2}} \\ \partial_{x_i}^2 K(x, y) &= -\frac{2x_d}{|B_1|} \left[ \frac{1}{|x-y|^{d+1}} - \frac{(d+2)}{|x-y|^{d+3}} (x_i - y_i) \frac{(x_i - y_i)}{|x-y|} \right] \\ &= -\frac{2x_d}{|B_1|} \left[ \frac{1}{|x-y|^{d+1}} - \frac{(d+2)}{|x-y|^{d+4}} (x_i - y_i)^2 \right] \end{aligned}$$

Moreover:

$$\begin{aligned} \partial_{x_d} K(x, y) &= \frac{2}{d|B_1|} \frac{1}{|x-y|^d} + \frac{2x_d}{d|B_1|} (-d) \frac{(x_d - y_d)}{|x-y|^{d+2}} \\ (y_d = 0) &= \frac{2}{d|B_1|} \frac{1}{|x-y|^d} + \frac{2x_d^2}{|B_1||x-y|^{d+2}} \\ \partial_{x_d}^2 K(x, y) &= \frac{-2}{|B_1|} \frac{(x_d - y_d)}{|x-y|^{d+2}} + \frac{4x_d}{|B_1||x-y|^{d+2}} - \frac{2(d+2)|B_1|^2}{x} \frac{(x_d - y_d)}{d|x-y|^{d+4}} \end{aligned}$$

Then:

$$\begin{aligned}
\Delta_x K(x, y) &= \sum_{i=1}^{d-1} \partial_{x_i}^2 K(x, y) + \partial_{x_d}^2 K(x, y) \\
&= -\frac{2x_d}{|B_1|} \left[ \frac{d-1}{|x-y|^{d+2}} - (d+2) \sum_{i=1}^{d-1} \frac{(x_i - y_i)^2}{|x-y|^{d+4}} \right. \\
&\quad \left. + \frac{1+2}{|x-y|^{d+2}} - \frac{(d+2)x_d(x_d - y_d)}{|x-y|^{d+4}} \right] \\
&= -\frac{2x_d}{|B_1|} \left[ \frac{d+2}{|x-y|^{d+2}} - (d+2) \frac{1}{|x-y|^{d+4}} \left( \underbrace{\sum_{i=1}^d |x_i - y_i|^2}_{|x-y|^2} \right) \right] = 0
\end{aligned}$$

for all  $x \in \mathbb{R}_+^d$ ,  $y \in \partial\mathbb{R}_+^d$ . Claim (exercise) for all  $x \in \mathbb{R}_+^d$ ,

$$\int_{\partial\mathbb{R}_+^d} K(x, y) dy = 1$$

Consider

$$u(x) = \int_{\partial\mathbb{R}_+^d} K(x, y) g(y) dy, \quad x \in \mathbb{R}_+^d$$

Since  $g \in L^\infty(\mathbb{R}^{d-1}) = L^\infty(\partial\mathbb{R}_+^d)$  and  $K(x, y) \geq 0$ , hence

$$|u(x)| \leq \left( \int_{\partial\mathbb{R}_+^d} K(x, y) dy \right) \|g\|_{L^\infty}$$

Thus  $\|u\|_{L^\infty} \leq \|g\|_{L^\infty}$ . Moreover

$$D_x^\alpha u(x) = \int_{\partial\mathbb{R}_+^d} D_x^\alpha K(x, y) g(y) dy$$

bounded, so  $u \in C^\infty(\mathbb{R}_+^d)$ ,  $x \mapsto K(x, y)$  is smooth as  $x \neq y$ .

$$\Delta_x u(x) = \int_{\partial\mathbb{R}_+^d} \underbrace{\Delta_x K(x, y)}_{=0} g(y) dy = 0$$

So  $u$  is harmonic in  $\mathbb{R}_+^d$ . ( $\Rightarrow u \in C^\infty$  by Weyl's lemma). Take  $x_0 \in \partial\mathbb{R}_+^d$  and  $x \in \mathbb{R}_+^d$ . Then:

$$\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial\mathbb{R}_+^d} K(x, y) (g(y) - g(x_0)) dy \right| \\
&\leq \int_{\partial\mathbb{R}_+^d} K(x, y) |g(y) - g(x_0)| dy \\
&= \underbrace{\int_{|y-x_0| \leq L|x-x_0|}}_{(I)} + \underbrace{\int_{|y-x_0| > L|x-x_0|}}_{(II)}
\end{aligned}$$

$$\begin{aligned}
(I) &= \int_{|y-x_0| \leq L|x-x_0|} K(x, y) |g(y) - g(x_0)| dy \\
&= \sup_{|y-x_0| \leq L|x-x_0|} |g(y) - g(x_0)| \xrightarrow{x \rightarrow x_0} 0 \quad \forall L > 0
\end{aligned}$$



(II): If  $|y - x_0| > L|x - x_0|$ , then  $|y - x| > \frac{1}{2}|y - x_0| > \frac{L}{2}|x - x_0|$  if  $L \geq 2$ .

$$\begin{aligned} \int_{|y-x_0| > L|x-x_0|} K(x, y) |g(y) - g(x_0)| dy &\leq C \int_{y \in \partial \mathbb{R}_+^d} \frac{x_d}{|x_0 - y|} dy \\ Cx_d \int_{\substack{z \in \mathbb{R}^{d-1} \\ |z| > L|x-x_0|}} \frac{1}{|z|^d} dz &= \text{const.} \frac{x_d}{L|x-x_0|} \leq \frac{\text{const.}}{L} \xrightarrow{L \rightarrow \infty} 0 \end{aligned}$$

$$x_d = |x_d - (x_0)_d| \leq |x - x_0|$$

■

## 5.2 Green's function for a ball

Let  $B = B(0, 1)$ . For all  $x \in B$ , for all  $y \in \bar{B}$  we want to find the corrector function  $\Phi_x(y)$  s.t.

$$\begin{cases} \Delta_y \Phi_x(y) = 0 & \text{in } B \\ \Phi_x(y) = G(y - x) & \text{on } \partial B \end{cases}$$

where for  $d \geq 3$ :  $G(z) = \frac{1}{d(d-2)|B_1||z|^{d-2}}$ .

**Definition 5.10** (Reflection / Duality through the sphere  $\partial B$ ) For all  $x \in \mathbb{R}^d \setminus \{0\}$  we define  $\tilde{x} = \frac{x}{|x|^2}$ . Clearly we have for all  $x \in B$  that if  $|x| < 1$ , then  $|\tilde{x}| = \left| \frac{x}{|x|^2} \right| = \frac{1}{|x|} > 1$ , so  $\tilde{x} \notin \bar{B}$

**Lemma 5.11** For  $d \geq 3$  the function  $\Phi_x(y) = G(|x|(y - \tilde{x}))$  is a corrector function.

*Proof.*

$$\Phi_x(y) = \frac{1}{d(d-2)|B_1||x|^{d-2}|y - \tilde{x}|^{d-2}}$$

for all  $x \in B, x \neq 0$ , for all  $y \in \bar{B}$ . Then clearly  $y \mapsto \Phi_x(y)$  is harmonic in  $B$  (Since  $\frac{1}{|z|^{d-2}}$  is harmonic in  $\mathbb{R} \setminus \{0\}$ ). Let's check the boundary: Let  $y \in \partial B$ , i.e.  $|y| = 1$ . Then

$$\begin{aligned} ||x|(y - \tilde{x})| &= |x| \left| y - \frac{x}{|x|^2} \right| \\ &= |x| \sqrt{|y|^2 - 2 \frac{xy}{|x|^2} + \left| \frac{x}{|x|^2} \right|^2} \\ &= \sqrt{|x|^2 |y|^2 - 2xy + 1} \\ (|y| = 1) \quad &= \sqrt{|x|^2 - 2xy + |y|^2} = |x - y| \end{aligned}$$

Thus  $\Phi_x(y) = G(|x||y - \tilde{x}|) = G(y - x)$  for all  $0 \neq x \in B$ , for all  $y \in \partial B$ . Let's compute the Poisson kernel: If want to solve

$$\begin{cases} -\Delta u = 0 & \text{in } B \\ u = g & \text{on } \partial B \end{cases}$$

then

$$u(x) = - \int_{\partial B} \frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) g(y) dS(y).$$

$\tilde{G}(x, y) = G(y - x) - \Phi_x(y) = G(y - x) - G(|x|(y - \tilde{x}))$  for all  $x \in B \setminus \{0\}$ ,  $y \in \bar{B}$ .

$$\frac{\partial \tilde{G}}{\partial \vec{n}_y} = \sum_{i=1}^d \partial_{y_i} \tilde{G} y_i$$

Here

$$\begin{aligned} \partial_{y_i} \tilde{G} &= \partial_{y_i} G(y - x) - \partial_{y_i} [G(|x|(y - \tilde{x}))] \\ &= \frac{-(y_i - x_i)}{d|B_1||y - x|^d} + \frac{y_i - \tilde{x}_i}{d|B_1||x|^{d-2}|y - \tilde{x}|^d} \\ \Rightarrow \frac{\partial \tilde{G}}{\partial \vec{n}_y} &= \sum_{i=1}^d [\dots] y_i \\ &= \frac{-y(y - x)}{d|B_1||y - x|^d} + \frac{y(y - \tilde{x})}{d|B_1||x|^{d-2}|y - \tilde{x}|^d} \\ &= \frac{1}{d|B_1||y - x|^d} (-y(y - x) + y(y - \tilde{x})|x|^2) \\ &= \frac{1}{d|B_1||y - x|^d} [-|y|^2 + xy + |y|^2|x|^2 - xy] \\ &= \frac{-1 + |x|^2}{d|B_1||y - x|^d} \end{aligned}$$

as  $y \in \partial B$ . ■

**Theorem 5.12** (Poisson Formula for a Ball) Let  $B = B(0, 1)$ ,  $g \in C(\partial B)$ . Define for all  $x \in B$ :

$$u(x) = \int_{\partial B} K(x, y) g(y) dS(y),$$

$K(x, y) = -\frac{\partial \tilde{G}}{\partial \vec{n}_y}(x, y) = \frac{1 - |x|^2}{d|B_1||y - x|^d}$  for all  $x \in B$ , for all  $y \in \partial B$ . Then  $u \in C^\infty(B)$ ,  $\Delta u = 0$  and for all  $x_0 \in \partial B$  we have  $\lim_{x \rightarrow x_0} u(x) = g(x_0)$ . This holds for all  $d \geq 2$ .

*Proof.* We need to check:

1. For all  $y \in \partial B$ ,  $x \mapsto K(x, y)$  is harmonic in  $B$ .
2.  $\int_{\partial B} K(x, y) dS(y) = 1$  for all  $x \in B$  (exercise)

Now for all  $x \in B$ , for all  $y \in \partial B$ :

$$\begin{aligned}
K(x, y) &= \frac{1 - |x|^2}{d|B_1||y - x|^d} \\
\partial_{x_i} K(x, y) &= \frac{-2x_i}{d|B_1|} \frac{1}{|x - y|^d} - \frac{1 - |x|^2}{|B_1|} \frac{x_i - y_i}{|x - y|^{d+2}} \\
\partial_{x_i}^2 K(x, y) &= -\frac{2}{d|B_1|} \frac{1}{|x - y|^d} + \frac{2x_i}{|B_1|} \frac{x_i - y_i}{|x - y|^{d+2}} + \frac{2x_i}{|B_1|} \frac{x_i - y_i}{|x - y|^{d+2}} \\
&\quad - \frac{1 - |x|^2}{|B_1|} \frac{1}{|x - y|^{d+2}} + \frac{1 + |x|^2}{|B_1|} (d + 2) \frac{(x_i - y_i)^2}{|x - y|^{d+4}} \\
\Delta_x K &= \sum_{i=1}^d \partial_{x_i}^2 K = -\frac{2}{|B_1|} \frac{1}{|x - y|^d} + \frac{4x(x - y)}{|B_1||x - y|^{d+2}} \\
&\quad - \frac{d(1 - |x|^2)}{|B_1|} \frac{1}{|x - y|^{d+2}} + (d + 2) \frac{1 - |x|^2}{|B_1|} \frac{1}{|x - y|^{d+2}} \\
&= \frac{2}{|B_1||x - y|^{d+2}} [-|x|^2 + 2xy - |y|^2 + 2|x|^2 - 2xy + 1 - |x|^2] \\
&= \frac{2}{|B_1||x - y|^{d+2}} [-|x|^2 + 2xy - |y|^2 + 2|x|^2 - 2xy + 1 - |x|^2]
\end{aligned}$$

$1 - |y|^2 = 0$  as  $y \in \partial B$ . Thus  $\Delta_x K(x, y) = 0$ , for all  $x \in B$ , for all  $y \in \partial B$ .

$$|u(x)| = \left| \int_{\partial B} K(x, y) g(y) dS(y) \right| \leq \|g\|_{L^\infty(\partial B)}$$

$$\int_{\partial B} K(x, y) dS(y) = \|g\|_{L^\infty},$$

$$\Delta_x u(x) = \int_{\partial B} \underbrace{\Delta_x K(x, y)}_{=0} g(y) dS(y) = 0$$

Take  $x \in B$ ,  $x \rightarrow x_0 \in \partial B$ .

$$\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial B} K(x, y) (g(y) - g(x_0)) dS(y) \right| \\
&\leq \int_{A_1} + \int_{A_2} K(x, y) |g(y) - g(x_0)| dS(y),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \{y \in \partial B \mid |y - x_0| \leq |x - x_0|^\alpha\} \\
A_2 &= \{y \in \partial B \mid |y - x_0| > |x - x_0|^\alpha\}
\end{aligned}$$

On  $A_1$  we have:

$$\int_{A_1} \dots \leq \sup_{\substack{|z - x_0| \leq |x - x_0|^\alpha \\ z \in \partial B}} \int_{\partial B} K(x, y) dS(y) \xrightarrow{x \rightarrow x_0} 0$$

since  $G \in C(\partial B)$ . On  $A_2$ :

$$\begin{aligned}
&|y - x_0| > |x - x_0|^\alpha \\
\Rightarrow |y - x| &\geq |y - x_0| - |x - x_0| \geq |x - x_0|^\alpha - |x - x_0| \geq \frac{1}{2} |x - x_0|^\alpha
\end{aligned}$$

if  $\alpha < 1$  and  $|x - x_0|$  small. So we get

$$K(x, y) = \frac{1 - |x|^2}{d|B_1||x - y|^d} \leq C \frac{1 - |x|^2}{|x - x_0|^{d\alpha}} \leq C |x - x_0|^{1-d\alpha}$$

Thus

$$\int_{A_2} K(x, y) |g(y) - g(x_0)| dS(y) \leq C \|g\|_{L^\infty} |x - x_0|^{1-d\alpha} \xrightarrow{x \rightarrow x_0} 0$$

if  $1 - d\alpha > 0 \Leftrightarrow \alpha < \frac{1}{d}$ . ■

**Exercise 5.13** (E 7.2) Define  $\mathbb{R}_+^d = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid x_d > 0\}$ . Let  $K(x, y) = \frac{2x_d}{d|B_1||x-y|^d}$  for all  $x \in \mathbb{R}_+^d, y \in \partial\mathbb{R}_+^d = \{(y', 0) \mid y' \in \mathbb{R}^{d-1}\} \simeq \mathbb{R}^{d-1}$ . Prove

$$\int_{\partial\mathbb{R}_+^d} K(x, y) dS(y) = 1 \quad \forall x \in \mathbb{R}_+^d$$

*Solution.* Denote  $x = (x', x_d), y = (y', 0), x', y' \in \mathbb{R}^{d-1}, x_d > 0$ .

$$\int_{\partial\mathbb{R}_+^d} K(x, y) dS(y) = \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|(|x' - y'|^2 + x_d^2)^{\frac{d}{2}}} dy' = \dots$$

as  $|x - y| = |(x' - y', x_d)| = \sqrt{|x' - y'|^2 + x_d^2}$ .

$$\begin{aligned} (y' - x' \mapsto y') \quad \dots &= \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|(|y'|^2 + x_d^2)^{\frac{d}{2}}} dy' \\ (y' = x_d z) &= \int_{\mathbb{R}^{d-1}} \frac{2x_d}{d|B_1|(x_d^2(|z|^2 + 1))^{\frac{d}{2}}} (x_d^{d-1}) dz \\ &= \int_{\mathbb{R}^{d-1}} \frac{2}{d|B_1|(|z|^2 + 1)^{\frac{d}{2}}} dz \\ &= \int_0^\infty \frac{2\omega_{d-1}}{d|B_1|} \frac{1}{(r^2 + 1)^{\frac{d}{2}}} r^{d-2} dr \\ &= \frac{2\omega_{d-1}}{\omega_d} \int_0^\infty \frac{1}{(r^2 + 1)^{\frac{d}{2}}} r^{d-2} dr \end{aligned}$$

Set  $d = 2$ :  $\omega_1 = 1, |\omega_2| = 2\pi$

$$\frac{2}{\pi} \int_0^\infty \frac{1}{r^2 + 1} dr = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{(\tan t)^2 + 1} [(\tan t)^2 + 1] dt = 1$$

we we set  $r = \tan t, t \in (0, \frac{\pi}{2}), \frac{dr}{dt} = (\tan t)' = 1 + (\tan t)^2$  ■

For  $d = 3$ :

$$\frac{2 \cdot 2\pi}{4\pi} \int_0^\infty \frac{1}{(r^2 + 1)^{\frac{3}{2}}} r dr = \int_0^\infty \frac{d}{dr} \left[ \frac{-1}{(r^2 + 1)^{\frac{1}{2}}} \right] dr = \frac{-1}{(r^2 + 1)^{\frac{1}{2}}} \Big|_0^\infty = 1$$

**Exercise 5.14** (7.3) Let  $g \in C(\partial\mathbb{R}_+^d) \cap L^\infty(\partial\mathbb{R}_+^d)$  ( $\partial\mathbb{R}_+^d \simeq \mathbb{R}^{d-1}$ ).

$$u(x) = \int_{\partial\mathbb{R}_+^d} K(x, y) g(y) dS(y) \quad K(x, y) = \frac{2x_d}{d|B_1||x - y|^d}, x \in \mathbb{R}_+^d$$

Prove that if  $g(y) = |y|$ , if  $|y| \leq 1$ , then  $|\nabla u|$  is unbounded in  $B(0, r) \cap \mathbb{R}_+^d$  for all  $r > 0$ .

*Solution.*

$$\begin{aligned}
\partial_{x_d} u(x) &= \int_{\partial \mathbb{R}_+^d} \partial_{x_d} K(x, y) g(y) dy \quad \forall x \in \mathbb{R}_+^d \\
&= \frac{2}{d|B_1|} \int_{\partial \mathbb{R}_+^d} \left[ \frac{1}{|x-y|^d} - \frac{dx_d^2}{|x-y|^{d+2}} \right] g(y) dy \\
&= \frac{2}{d|B_1|} \int_{\partial \mathbb{R}_+^d} \frac{1}{|x-y|^{d+2}} [|x-y|^2 - dx_d^2] g(y) dy \\
&= \frac{2}{d|B_1|} \int_{\partial \mathbb{R}_+^d} \frac{1}{(|x'-y'| + x_d^2)^{\frac{d+2}{2}}} [|y'|^2 - (d-1)x_d^2] g(y) dy
\end{aligned}$$

Assume that  $\partial_d u$  is bounded in  $B(0, r) \cap \mathbb{R}_+^d$ . Then:

$$|u(0, x_d) - \underbrace{u(0, 0)}_{g(0)=0}| \leq C|x_d|$$

if  $x_d$  small. Consider:

$$\begin{aligned}
\limsup_{x_d \rightarrow 0^+} \frac{u(0, x_d)}{x_d} &= \limsup_{x_d \rightarrow 0^+} c \int_{\mathbb{R}^{d-1}} \frac{1}{(|y'|^2 + x_d^2)^{\frac{d}{2}}} g(y) dy' \\
&\geq \int_{\mathbb{R}^{d-1}} \frac{1}{|y'|^d} g(y) dy = \int_{|y'| \leq 1} + \int_{|y'| > 1} \\
&\text{to } \int_{\mathbb{R}^{d-1}} \frac{1}{|y'|^{d-1}} dy' = \infty
\end{aligned}$$

■

**Exercise 5.15** (Bonus 7) Recall the Poisson kernel on a ball  $B(0, r) \subseteq \mathbb{R}^d$ :

$$K(x, y) = \frac{r^2 - |x|^2}{d|B_1|r} \frac{1}{|x-y|^d}$$

for all  $x \in B(0, r)$ ,  $y \in \partial B(0, r)$ . Prove:

$$\int_{\partial B(0, r)} K(x, y) dS(y) = 1$$

for all  $x \in B(0, r)$ . (It suffices if you can prove  $d = 2$  and  $d = 3$ )

### 5.3 Energy Method

Consider  $u \in C^2(\Omega)$  for  $\Omega \subseteq \mathbb{R}^d$  open, bounded and with  $C^1$  boundary and

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Take  $\phi \in C_c^\infty(\Omega)$ , then by integration by parts:

$$0 = \int_{\Omega} (-\Delta u - f) \phi = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi$$

Key observation: This is the *derivative* of the energy functional

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u$$

If  $u$  is a minimizer of  $E$ , then it solves the equation  $-\Delta u = f$  in  $\Omega$ . The boundary condition  $u = g$  does not appear on  $E$ , but this is encoded in the set of *admissible functions*. (The set of candidates of solutions). For the classical solutions, we have

**Theorem 5.16** (Dirichlet's principle) Let  $\Omega \subseteq \mathbb{R}^d$  be open, bounded with  $C^1$ -boundary. Let  $f \in C(\bar{\Omega})$  and  $g \in C(\partial\Omega)$ . Then the following statements are equivalent:

1.  $u \in C^2(\bar{\Omega})$  solves 
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$
2.  $u$  is a minimizer of the variational problem  $E = \inf_{v \in A} E(v)$ , where

$$E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v,$$

$$A = \{v \in C^2(\bar{\Omega}) \mid v = g \text{ on } \partial\Omega\}.$$

Moreover there is at most a solution / minimizer (uniqueness).

*Proof.* The result holds even for complex-valued functions. Let us write the proof for real-valued functions.

1.  $\Rightarrow$  2.: Let  $u \in C^2(\bar{\Omega})$  be a solution of 
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}.$$
 Then we prove  $E(u) \leq E(v)$  for all  $v \in A$ . If  $v \in A$ , then  $u - v = 0$  on  $\partial\Omega$ . Using this and  $-\Delta u = f$  in  $\Omega$ , we have:

$$\begin{aligned} 0 &= \int_{\Omega} (-\Delta u - f) \cdot (u - v) dy \\ (\text{Part. Int.}) &= \int_{\Omega} \nabla u (\nabla u - \nabla v) dy - \int_{\Omega} f(u - v) dy \\ &= \left[ \frac{1}{2} \int_{\Omega} |\nabla u|^2 dy - \int_{\Omega} f u dy \right] - \left[ \frac{1}{2} \int_{\Omega} |\nabla v|^2 dy - \int_{\Omega} f v dy \right] \\ &\quad + \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \\ &= E(u) - E(v) + \underbrace{\frac{1}{2} \int_{\Omega} |\nabla u - \nabla v|^2}_{\geq 0} \end{aligned}$$

$E(u) \leq E(v)$ , so  $u$  is a minimizer of  $\inf_{v \in A} E(v)$ . Moreover  $u$  is the unique minimizer on  $A$ . Since  $E(u) = E(v)$  we have  $\int_{\Omega} |\nabla(u - v)|^2 = 0$ , so  $u - v = \text{const.}$ , so  $u - v = 0$  in  $\bar{\Omega}$ .

2.  $\Rightarrow$  1.: Assume that  $u$  is a minimizer of  $\inf_{v \in A} E(v)$ . Then  $E(u) \leq E(v)$  for all  $v \in A$ . Take  $\phi \in C_c^\infty(\Omega)$ , then  $u + t\phi \in A$  for all  $t \in \mathbb{R}$ .

$$\Rightarrow E(u) \leq E(u + t\phi) \text{ for all } t \in \mathbb{R}$$

$$\Rightarrow t \mapsto E(u + t\phi) \text{ has a minimizer at } t = 0$$

$$\Rightarrow 0 = \frac{d}{dt} E(u + t\phi)|_{t=0}$$

$$= \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} |\nabla u + t \nabla \phi|^2 - \int_{\Omega} f(u + t\phi) \right) \Big|_{t=0}$$

$$= \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} |\nabla u|^2 + t^2 |\nabla \phi|^2 + 2t \nabla u \nabla \phi - \int_{\Omega} f(u + t\phi) \right) \Big|_{t=0}$$

$$\int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi = \int_{\Omega} (-\Delta u - f) \phi$$

for all  $\phi \in C_c^\infty(\Omega)$ . So  $-\Delta u - f = 0$  in  $\Omega$  and  $u = g$  since  $u \in A$ .

Direct method of calculus of variations. Think  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C(\mathbb{R})$ ,  $f(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ . There is a  $x_0 \in \mathbb{R}$  s.t.  $f(x_0) = \inf_{x \in \mathbb{R}} f(x)$ .

Step 1:  $E = \inf_{x \in \mathbb{R}} f(x) > -\infty$

Step 2: Take a minimizing sequence  $\{x_n\} \subseteq \mathbb{R}$ ,  $f(x_n) \rightarrow E$ . Up to a subsequence  $x_n \rightarrow x_0$  in  $\mathbb{R}$  (compactness)

Step 3: Lower semicontinuity  $E = \liminf_{n \rightarrow \infty} f(x_n) \geq f(x_0)$

If we apply the direct method to  $\inf_{v \in A} E(v)$ ,

$$E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v,$$

$$A = \{v \in C^2(\bar{\Omega}), v = g \text{ on } \partial\Omega\}$$

Step 1: Easy  $E = \inf_{v \in A} E(v) > -\infty$

Step 2: There is a minimizing sequence  $\{v_n\} \subseteq A$  s.t.  $E(v_n) \rightarrow E$ . We don't know if there is a subsequence of  $\{v_n\}$  that converges to  $u \in A$ . The lack of compactness is a serious problem! We need to find the right set  $A$ ! Consider again

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Consider the simple case  $g = 0$ .  $\Delta u = f$  in  $\Omega \Leftrightarrow \nabla u \nabla \phi \dots$  The right set  $A$  should be  $A = \{v \mid \int_{\Omega} |\nabla v|^2 < \infty, v = 0 \text{ on } \partial\Omega\}$ . Rigorously we take  $W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,2}(\Omega)}$  (Notation:  $H_0^1 = W_0^{1,2}$ ,  $H^1 = W^{1,2}$ ) Recall that  $W^{1,p}$  is a Banach space with norm  $\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}$ . We know that  $C_c^\infty(\Omega)$  is dense in  $W_{loc}^{1,p}(\Omega)$ , i.e. for all  $u \in W_{loc}^{1,p}(\Omega)$  there is  $\{u_n\} \subseteq C_c^\infty$  s.t.  $u_n \rightarrow u$  in  $W^{1,p}(K)$  for all  $K \subseteq \Omega$  compact. However in general  $C_c^\infty(\Omega)$  is not dense in  $W^{1,p}(\Omega)$ , i.e.  $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,p}(\Omega)} \subsetneq W^{1,p}(\Omega)$ . Clearly  $W_0^{1,p}$  is a closed subspace of  $W^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)$  is a Banach space with  $\|\cdot\|_{W^{1,p}(\Omega)}$ . Why does  $W_0^{1,p}(\Omega)$  encode the 0-boundary condition? Note that by definition for all  $u \in W_0^{1,p}(\Omega)$  there is a sequence  $\{u_n\} \subseteq C_c^\infty(\Omega)$ ,  $u_n \rightarrow u$  in  $W^{1,p}(\Omega)$  up to a subsequence  $u_n(x) \rightarrow u(x)$  for almost every  $x \in \Omega$ . Note  $u_n|_{\partial\Omega} = 0 \rightarrow u|_{\partial\Omega} = 0$  since  $\partial\Omega$  must be of 0-measure. ■

**Theorem 5.17** (Characterization for  $W_0^{1,p}$ ) Let  $\Omega$  be open, bounded with  $C^1$ -boundary. Let  $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ . Then the following statements are equivalent:

- a)  $u = 0$  on  $\partial\Omega$
- b)  $u \in W_0^{1,p}(\Omega)$

(Later we will remove the condition  $C(\bar{\Omega})$  by introducing the *Trace operator*.)

**Remark 5.18** If  $d = 1$ , it holds that  $W^{1,p} \subseteq C(\bar{\Omega})$ . Then the theorem gives a full characterization for  $W_0^{1,p}$ , but if  $d \geq 2$ , then in general  $W^{1,p} \not\subseteq C(\Omega)$ . (later)

*Proof of theorem 5.17.*

a)  $\Rightarrow$  b):

**Lemma 5.19** If  $u \in W^{1,p}(\Omega)$  and  $\text{supp } u \subseteq \Omega$ , then  $u \in W_0^{1,p}(\Omega)$ .

*Proof.* Since  $K := \text{supp } u$  is a compact subset in  $\Omega$ , we can find a function  $\chi \in C_c^\infty(\Omega)$ ,  $\chi = 1$  on  $K$ . Moreover since  $u \in W^{1,p}(\Omega)$ , there is a sequence  $\{u_n\} \subseteq C_c^\infty(\Omega)$  s.t.  $u_n \rightarrow u$  in  $W_{loc}^{1,p}(\Omega)$ . We claim that  $\chi u_n \rightarrow \chi u$  in  $W_{loc}^{1,p}(\Omega)$ . (exercise,  $\nabla(\chi u) = \nabla \chi u + \chi \nabla u$ ). This implies  $\chi u_n \rightarrow u$  in  $W^{1,p}(\text{supp } \chi)$ , thus  $\chi u_n \rightarrow u$  in  $W^{1,p}(\Omega)$ , so  $u \in W_0^{1,p}(\Omega)$ . ■

Assume  $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$  and  $u = 0$  on  $\partial\Omega$ . Take  $G \in C^1(\mathbb{R})$  s.t.  $|G(t)| \leq t$  for all  $t$ ,  $G(t) = t$  if  $t \geq 2$  and  $G(t) = 0$  if  $t \leq 1$ . Then let

$$\begin{aligned} u_n(x) &:= \frac{1}{n} G(nu(x)) \in W^{1,p}(\Omega) \\ \stackrel{(\text{Chain-rule})}{\Rightarrow} \nabla u_n(x) &= \frac{1}{n} G'(nu(x)) n \nabla u(x) = G'(nu(x)) \nabla u(x) \end{aligned}$$

Moreover,  $u_n$  is compactly supported in  $\Omega$ , so  $u_n \in W_0^{1,p}(\Omega)$  by the lemma and  $u_n \rightarrow u$  in  $W^{1,p}(\Omega)$ , so  $u \in W_0^{1,p}(\Omega)$  since  $W_0^{1,p}$  is a closed space. Recall that  $u \in C(\bar{\Omega})$  and  $u = 0$  on  $\partial\Omega$ . Thus for all  $\epsilon > 0$  there is a compact  $K_\epsilon \subseteq \Omega$  s.t.  $\sup_{x \in \Omega \setminus K_\epsilon} |u(x)| \leq \epsilon$ . For any given  $n \in \mathbb{N}$ ,  $u_n(x) \neq 0$ , so  $G(nu(x)) \neq 0$ . This implies  $n|u(x)| > 1$ , hence  $|u(x)| > \frac{1}{n}$ . Thus  $u_n(x) = 0$  for all  $x$  such that  $|u(x)| \leq \frac{1}{n}$ , so  $\text{supp } u_n \subseteq K_{\frac{1}{n}}$  compact in  $\Omega$ . Next, let us check  $u_n \rightarrow u$  in  $W^{1,p}(\Omega)$ .

$$\int_{\Omega} |u_n(x) - u(x)|^p dx \rightarrow 0$$

since  $u_n(x) = \frac{1}{n} G(nu(x)) \xrightarrow{n \rightarrow \infty} u(x)$  for all  $x \in \Omega$  and  $|u_n(x)| \leq \frac{1}{n} |G(nu(x))| \leq \frac{1}{n} |nu(x)| \leq |u(x)| \in L^p(\Omega)$ .

$$\int_{\Omega} |\nabla u_n(x) - \nabla u(x)|^p dx = \int_{\Omega} |G'(nu(x)) - 1|^p |\nabla u(x)|^p dx \rightarrow 0$$

as  $|G'(v(x)) - 1| \rightarrow 0$  for all  $x$  s.t.  $u(x) \neq 0$  and  $\nabla u(x) = 0$  on  $\{x \mid u(x) = 0\}$ . (exercise)

(b)  $\Rightarrow$  (a): Let  $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$  and  $u \in W_0^{1,p}(\Omega)$ . Then we prove  $u = 0$  on  $\partial\Omega$ . Lets regard the case  $\Omega = Q_+ = \{(x', x_d) \mid \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, 0 < x_d < 1\}$ . We prove that if  $u \in W_0^{1,p}(Q_+) \cap C(\bar{Q}_+)$ , then  $u = 0$  on  $Q_0 = \{(x', 0) \mid x' \in \mathbb{R}^{d-1}, |x'| < 1\}$ . Since  $u \in W_0^{1,p}(Q_+)$  there is  $\{u_n\} \subseteq C_c^\infty(Q_+)$  s.t.  $u_n \rightarrow u$  in  $W^{1,p}(Q_+)$  for all  $x = (x', x_d) \in Q_+$ , then:

$$u_n(x', x_d) = \underbrace{u_n(x', 0)}_{=0} + \int_0^{x_d} \partial_d u_n(x', t) dt$$

Hence

$$|u_n(x', x_d)| \leq \int_0^{x_d} |\partial_d u_n(x', t)| dt$$

This implies:

$$\begin{aligned} & \int_{0 < x_d < \epsilon} \int_{|x'| \leq 1} |u_n(x', x_d)| dx' dx_d \\ & \leq \int_{0 < x_d < \epsilon} \int_{|x'| < 1} \left( \int_0^{x_d} |\partial_d u_n(x', t)| dt \right) dx' dx_d \\ & \leq \epsilon \int_{|x'| < 1} \int_0^\epsilon |\partial_d u_n(x', t)| dx' dt \end{aligned}$$



$$\Rightarrow \frac{1}{\epsilon} \int_0^\epsilon \int_{|x'| \leq 1} |u_n(x', x_d)| dx' dx_d \leq \int_0^\epsilon \int_{|x'| < 1} |\partial_d u_n(x', x_d)| dx' dx_d$$

for all  $n \in \mathbb{N}$ ,  $\epsilon > 0$ . Take now  $n \rightarrow \infty$ , use  $u_n \rightarrow u$  in  $W^{1,p}(\Omega)$ . Then:

$$\frac{1}{\epsilon} \int_0^\epsilon \int_{|x'| \leq 1} |u(x', x_d)| dx' dx_d \leq \int_0^\epsilon \int_{|x'| < 1} |\partial_d u(x', x_d)| dx' dy$$

for all  $\epsilon > 0$ . Take  $\epsilon \rightarrow 0$ :

$$\int_{|x'| \leq 1} |u(x', 0)| dx' \leq 0$$

here we use  $u \in C(\bar{\Omega})$  for the left side and Dominated Convergence for the right side. Thus  $u(x', 0) = 0$  for all  $|x'| \leq 1$ , i.e.  $u = 0$  on  $\partial\Omega$ . Let's regard the general case: Let  $\Omega$  be open, bounded and with  $C^1$ -boundary. Lets define *local charts* By definition for all  $x \in \partial\Omega$ , there is a  $U_x$  open, such there is a bijective map  $h : U_x \rightarrow Q$ , and  $h, h^{-1}$  are  $C^1$ . Then clearly  $\partial\Omega \subseteq \bigcup_{x \in \partial\Omega} U_x$ . Since  $\partial\Omega$  is compact, there is a finite subcover  $\{U_i\}_{i=1}^N$  s.t.  $\partial\Omega \subseteq \bigcup_{i=1}^N U_i$ . We can find  $U_0$  open s.t.  $\bar{U}_0 \subseteq \Omega$  and  $\Omega \subseteq \bigcup_{i=0}^N U_i$ .

**Lemma 5.20** There is a sequence  $\{\chi_i\}_{i=0}^N \subseteq C^\infty(\mathbb{R}^d)$  s.t.

1.  $\chi_i \geq 0$ ,  $\sum_{i=0}^N \chi_i = 1$  in  $\mathbb{R}^d$  ( $\{\chi_i\}$  is a partition of unity)
2. For all  $i = 1, \dots, N$ ,  $\text{supp } \chi_i$  is in  $U_i$ , i.e.  $\chi_i \in C_c^\infty(U_i)$ .
3.  $i = 0$ ,  $\text{supp } \chi_0 \subseteq \mathbb{R}^d \setminus \partial\Omega$  and  $\chi_0|_\Omega \in C_c^\infty(\Omega)$ . (exercise)

Given  $u \in W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$ . Then  $u = \sum_{i=0}^N \chi_i u$ , where  $\chi_i \geq 0$ ,  $\chi_0 \in C_c^\infty(\Omega)$ ,  $\chi_i \in C_c^\infty(U_i)$ . Since  $\chi_0 u$  is supported in a compact set inside  $\Omega$ ,  $\chi_0 u = 0$  on  $\partial\Omega$ . It remains to show that for all  $i = 1, \dots, N$ ,  $\chi_i u = 0$  on  $U_i \cap \partial\Omega$ . Then  $\chi_i u(h^{-1}x) \in W_0^{1,p}(Q) \cap C(\bar{Q})$ . This implies  $\chi_i u(h^{-1}x) = 0$  on  $Q_0$ , so  $\chi_i u(x) = 0$  on  $U_i \cap \partial\Omega$ . Why  $W_0^{1,p}(U_i \cap \Omega) \rightarrow W_0^{1,p}(Q_+)$ . If  $v \in W_0^{1,p}(U_i \cap \Omega)$ , then  $v_n \rightarrow v$ ,  $v_n \in C_c^\infty$ .  $v_n \circ h^{-1} \rightarrow v \circ h^{-1} \Rightarrow v \circ h^{-1} \in W_0^{1,p}(Q_+)$

■

**Exercise 5.21** (E 8.1) Let  $u \in W_{loc}^{1,1}(\mathbb{R}^d)$ . Let  $B = u^{-1}(\{0\})$ . Prove that  $\nabla u(x) = 0$  for a.e.  $x \in B$ .

*Solution.* We have already seen that if  $f, g \in W_{loc}^{1,1}(\mathbb{R}^d)$ , then  $\max(f, g) \in W_{loc}^{1,1}$ . This implies that if  $u = u^+ - u^- \in W_{loc}^{1,1}$ , then  $u^+, u^- \in W_{loc}^{1,1}$  since  $u^+ = \max(u, 0)$  and  $u^- = \max(-u, 0)$ . We have that  $\nabla u = \nabla u^+ - \nabla u^-$ . Claim:

$$\nabla u^+ = \begin{cases} 0 & u(x) \leq 0 \\ \nabla u & u(x) > 0 \end{cases} \quad \nabla u^- = \begin{cases} 0 & u(x) \geq 0 \\ \nabla u & u(x) < 0 \end{cases}$$

$$\begin{aligned} \int_{\mathbb{R}^d} (\partial_i u^+) \phi &= - \int_{\mathbb{R}^d} u^+ \partial_i \phi = - \int_{\{u(x) \leq 0\}} 0 \partial_i \phi - \int_{\{u(x) > 0\}} u \partial_i \phi \\ &= \int_{\{u(x) \leq 0\}} 0 \phi + \int_{\{u(x) > 0\}} \partial_i u \phi \end{aligned}$$

Alternative way: We showed for  $f \in W^{1,p}(\mathbb{R}^d)$ , that

$$\nabla|f|(x) = \begin{cases} (\nabla f)(x) & f(x) > 0 \\ -(\nabla f)(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

$u_+ = \frac{1}{2}(u + |u|)$ . Hence  $\nabla u_+ = \frac{1}{2}(\nabla u + \nabla|u|)$ . Remark: If  $A \subseteq \mathbb{R}$  has measure zero, then  $\nabla u 1_{\{u(x) \in A\}} = 0$  a.e. (Th. 6.19 Lieb-Loss Analysis) ■

**Exercise 5.22** (E 8.2) Let  $\Omega, U \subseteq \mathbb{R}^d$  be open,  $U \cap \Omega \neq \emptyset$ ,  $u \in W_0^{1,p}(\Omega)$ ,  $1 \leq p < \infty$ ,  $\chi \in C_c^\infty(U)$ . Prove:  $\chi u \in W_0^{1,p}(\Omega \cap U)$  Hint: Recall  $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}}}$

*Solution.* By definition there is a sequence  $(u_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\Omega)$  s.t.  $u_n \xrightarrow[n \rightarrow \infty]{\|\cdot\|_{W^{1,p}}} u$ , i.e.

$$\|u_n - u\|_p + \|\nabla u_n - \nabla u\|_p \xrightarrow{n \rightarrow \infty} 0.$$

Define  $f_n : \mathbb{R}^d \rightarrow \mathbb{C}$ ,  $f_n(x) := u_n(x)\chi(x)$ . Note  $f_n \in C_c^\infty(\Omega \cap U)$  for all  $n \in \mathbb{N}$ . Claim:  $(f_n)_{n \in \mathbb{N}}$  is Cauchy with respect to  $\|\cdot\|_{W^{1,p}}$ . Proof:

$$\|f_n - f_m\|_p = \|\chi(u_n - u_m)\|_p \leq \|\chi\|_\infty \underbrace{\|u_n - u_m\|_p}_{\xrightarrow[n, m \rightarrow \infty]{} 0} \xrightarrow{n, m \rightarrow \infty} 0$$

$$\nabla f_n = \nabla(\chi u_n) = (\nabla \chi)u_n + \chi \nabla u_n$$

$$\begin{aligned} \|\nabla f_n - \nabla f_m\|_p &\leq \|\nabla \chi(u_n - u_m)\|_p + \|\chi(\nabla u_n - \nabla u_m)\|_p \\ &\leq \|\nabla \chi\|_\infty \underbrace{\|u_n - u_m\|_p}_{\xrightarrow[n, m \rightarrow \infty]{} 0} + \underbrace{\|\chi\|}_{< \infty} \underbrace{\|\nabla u_n - \nabla u_m\|_p}_{\xrightarrow[n, m \rightarrow \infty]{} 0} \xrightarrow{n, m \rightarrow \infty} 0 \end{aligned}$$

Thus, there is a  $f \in W_0^{1,p}(\Omega \cap U)$  s.t.  $\|f_n - f\|_{W^{1,p}} \xrightarrow{n \rightarrow \infty} 0$ . We know:

$$\begin{aligned} \|f_n - \chi u\|_{L^p} &= \|\chi u_n - \chi u\|_p \\ &\leq \|\chi\|_\infty \underbrace{\|u_n - u\|_p}_{\rightarrow 0} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Since limits in  $L^p$  are unique, we get  $\chi u = f \in W_0^{1,p}(\Omega \cap U)$ . ■

**Exercise 5.23** (E 8.3) Let  $\Omega, U \subseteq \mathbb{R}^d$  open and bounded,  $h : \bar{U} \rightarrow \bar{\Omega}$   $C^1$ -diffeomorphisms,  $u \in W_0^{1,p}(\Omega)$ ,  $1 \leq p < \infty$ . Prove  $(x \mapsto u(h(x))) \in W_0^{1,p}(U)$ .

*Solution.* Since  $u \in W_0^{1,p}(\Omega)$  there is a sequence  $(u_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\Omega)$  s.t.

$$\|u - u_n\|_p + \|\nabla u - \nabla u_n\|_p \xrightarrow{n \rightarrow \infty} 0$$

Define for all  $n \in \mathbb{N}$   $f_n : U \rightarrow \mathbb{C}$ ,  $f_n(x) = u_n(h(x))$ . Note  $f_n \in C_c^1(U)$ . Claim 1:  $(f_n)_{n \in \mathbb{N}}$  is Cauchy wrt.  $\|\cdot\|_{W^{1,p}}$ .

$$\begin{aligned} \|f_n - f_m\|_p^p &= \int_U |u_n(h(x)) - u_m(h(x))|^p dx \\ &= \int_\Omega |u_n(y) - u_m(y)|^p dy \underbrace{|\det(Dh^{-1})(y)|}_{\leq C < \infty} \xrightarrow{n, m \rightarrow \infty} 0 \end{aligned}$$

$$(\nabla f_n)(x) = \nabla(u_n(h(x))) = (\nabla u_n)(h(x))(Dh)(x)$$

$$\begin{aligned}
\|\nabla f_n - \nabla f_m\|_p^p &= \int_U |[(\nabla u_n)(h(x)) - (\nabla u_m)(h(x))](Dh(x))|^p dx \\
&\leq C \int_U |(\nabla u_n)(h(x)) - (\nabla u_m)(h(x))|^p dx \\
&= C \int_\Omega |(\nabla u_n)(y) - (\nabla u_m)(y)|^p \underbrace{|\det Dh^{-1}(y)|}_{\leq \tilde{C}} dy \xrightarrow{n,m \rightarrow 0} 0
\end{aligned}$$

Claim 2:  $\|f_n - u \circ h\|_p \xrightarrow{n \rightarrow \infty} 0$ .

$$\begin{aligned}
\|f_n - u \circ h\|_p &= \int_U |u_n(h(x)) - u(h(x))|^p dx \\
&= \int_\Omega |u_n(y) - u(y)|^p \underbrace{|\det Dh^{-1}(y)|}_{\leq C} dy \xrightarrow{n \rightarrow \infty} 0
\end{aligned}$$

Conclusion: Since  $(f_n)_{n \in \mathbb{N}} \subseteq C_c^1(U)$  is Cauchy with respect to  $\|\cdot\|_{W^{1,p}}$ , there is a  $f \in W_0^{1,p}(U)$  s.t.  $f_n \xrightarrow{n \rightarrow \infty} f$ . Since limits in  $L^p$  are unique by claim 2 we get  $u \circ h = f \in W_0^{1,p}(U)$ .  $\|\cdot\|_{W^{1,p}}$  ■

**Exercise 5.24** (E 8.4) Let  $\Gamma \subseteq \mathbb{R}^d$  be compact,  $\{U_i\}_{i=1}^N$  open s.t.  $\Gamma \subseteq \bigcup_{i=1}^N U_i$ . Prove: There exists  $\{\chi_i\}_{i=0}^N \subseteq C^\infty(\mathbb{R}^d)$  s.t.

1.  $\chi_i \geq 0$  for all  $i$ ,  $\sum_{i=0}^N \chi_i = 1$
2.  $\text{supp}(\chi_i) \subseteq U_i$  for all  $i \in \{1, \dots, N\}$
3.  $\text{supp}(\chi_0) \subseteq \mathbb{R}^d \setminus \Gamma$

*Solution.* WLOG assume that  $U_i \neq \emptyset$  for all  $i$ . If  $\Gamma \neq \emptyset$ , then  $\chi_0 = 1$  does the job. Now suppose  $\Gamma \neq \emptyset$ . Let  $\psi \in C_c^\infty(B_1(0))$ ,  $\psi \geq 0$ ,  $\int \psi = 1$ ,  $\psi|_{B_{\frac{1}{2}}(0)} > 0$  and for  $\epsilon > 0$  let  $\psi_\epsilon(x) = \frac{1}{\epsilon^d} \psi\left(\frac{x}{\epsilon}\right)$ , so  $\int \psi_\epsilon = 1$ . Define

$$\tilde{d} := \sup\{\tilde{d} > 0 \mid \forall x \in \Gamma \exists i \in \{1, \dots, N\} \text{ s.t. } \text{dist}(x, U_i^c) \geq \tilde{d}\}$$

Claim 1:  $\tilde{d} > 0$  Suppose this was not true. Then there is a sequence  $(x_n)_{n \in \mathbb{N}} \subseteq \Gamma$  s.t. for all  $i \in \{1, \dots, N\}$ ,

$$\text{dist}(x_n, U_i^c) < \frac{1}{n}$$

Since  $\Gamma$  is compact, there is a subsequence, which we call  $x_n$  again, s.t.  $x_n \xrightarrow{n \rightarrow \infty} \bar{x}$  for some  $\bar{x} \in \Gamma$ . By  $\Gamma \subseteq \bigcup_{i=1}^N U_i$  there is a  $\bar{i} \in \{1, \dots, N\}$  s.t.  $B_{\epsilon_{\bar{x}}}(\bar{x}) \subseteq U_{\bar{i}}$ . Define  $d := \min\{\tilde{d}, 1\} > 0$ . For all  $\epsilon > 0$ , for all  $A \subseteq \mathbb{R}^d$ :  $(A)_\epsilon := \{x \in A \mid \text{dist}(x, A^c) \geq \epsilon\}$ . for every  $i \in \{1, \dots, N\}$  define  $\phi_i : U_i \rightarrow [0, \infty)$  by

$$\phi_i(x) := \mathbb{1}_{(U_i \cap B_R(0))_{\frac{d}{4}}} \star \phi_{\frac{d}{4}}$$

Note  $\phi_i \in C_c^\infty(U_i)$  and  $(U_i \cap B_R(0))_{\frac{d}{4}} \subseteq (\text{supp}(\phi_i))^0$ . Define  $\phi_0 : \mathbb{R}^d \setminus \Gamma \rightarrow [0, \infty)$  by  $\phi_0(x) = \mathbb{1}_{(\mathbb{R}^d \setminus \Gamma)_{\frac{d}{4}}} \star \psi_{\frac{d}{4}}$ . Again,  $\phi_0 \in C^\infty(\mathbb{R}^d \setminus \Gamma)$ ,  $\text{supp}(\phi_0)^0 \supseteq (\mathbb{R}^d \setminus \Gamma)_{\frac{d}{4}}$ ,  $\text{supp}(\phi_0) \subseteq \mathbb{R}^d \setminus \Gamma$ . Claim 2: For all  $x \in \mathbb{R}^d$  there is a  $i \in \{0, 1, \dots, N\}$  :  $\phi_i(x) > 0$ . Proof: By construction, we know for  $i \in \{1, \dots, N\}$  that  $\phi_i$  is  $> 0$  on  $(U_i \cap B_R(0))_{\frac{d}{4}}$ . Moreover  $\phi_0 > 0$  on  $(\mathbb{R}^d \setminus \Gamma)_{\frac{d}{4}}$ . thus, we are done if we can show that  $\bigcup_{i=1}^N (U_i \cap B_R(0))_{\frac{d}{4}} \cup$

$(\mathbb{R}^d \setminus \Gamma)_{\frac{d}{4}} = \mathbb{R}^d$ . Suppose there is a  $x \in \mathbb{R}^d \setminus A$ . Then  $\text{dist}(x, \Gamma) < \frac{d}{4}$ . Since  $\Gamma \subseteq B_{\frac{R}{2}}(0)$  and  $R > 2$  and  $d \leq 1$ .

$$|x - 0| \leq \text{dist}(x, \Gamma) + \frac{R}{2} < \frac{d}{4} + \frac{R}{2} = R - \frac{d}{4} - \frac{R}{2} + \frac{d}{2} < R - \frac{d}{4} - \frac{2}{2} + \frac{1}{2} < R - \frac{d}{4}$$

Thus  $x \in (B_R(c))_{\frac{d}{4}}$ . Thus, we are done if we can show that  $x \in (U_i)_{\frac{d}{4}}$  for some  $i \in \{1, \dots, N\}$ . Since  $\text{dist}(x, \Gamma) < \frac{d}{4}$ , there is a  $y \in \Gamma$  s.t.  $|x - y| < \frac{d}{4}$ . By definition of  $\tilde{d}$  there is a  $i \in \{1, \dots, N\}$  s.t.  $\text{dist}(y, U_i^c) \geq \tilde{d} \geq d$ , i.e. for all  $z \in U_i^c$  we have  $|y - z| \geq d$ . We get

$$|x - z| \geq \underbrace{|x - y|}_{< \frac{d}{4}} - \underbrace{|y - z|}_{\geq d} \geq \frac{3d}{4} < \frac{d}{4}$$

This implies  $\text{dist}(x, U_i^c) > \frac{d}{4}$ , so  $x \in (U_i)_{\frac{d}{4}} \not\subset$ . Define for all  $i \in \{0, \dots, N\}$  :  $\chi_i : \mathbb{R}^d \rightarrow [0, \infty)$  by

$$\chi_i(x) = \frac{\phi_i(x)}{\sum_{j=0}^N \phi_j(x)}$$

$\chi_i$  is well-defined by Claim 2 and  $\chi_i \in C^\infty(\mathbb{R}^d)$ . Also note that  $\sum \chi_i = 1$ ,  $\chi_i \geq 0$ , which implies 1. Furthermore, since  $\text{supp}(\phi_i) \subseteq U_i$ , we have  $\text{supp}(\chi_i) \subseteq U_i$  for all  $i \in \{1, \dots, N\}$ , which implies 2. Finally, since  $\text{supp}(\phi_0) \subseteq \mathbb{R}^d \setminus \Gamma$ , we get  $\text{supp}(\chi_0) \subseteq \mathbb{R}^d \setminus \Gamma$ . This implies 3. ■

## 5.4 Variational problem for weak solutions

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

("formally") for all  $\phi \in C_c^\infty(\Omega)$ , then

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi$$

if  $\nabla u \in L^2$ ,  $f \in L^2$ . By a density argument:

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi$$

for all  $\phi \in \overline{C_c^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$ .

**Theorem 5.25** (Poincare inequality) There is a  $C > 0$  s.t.

$$C \int_{\Omega} |\nabla v|^2 \geq \int_{\Omega} |v|^2$$

for all  $v \in H_0^1(\Omega)$ .

**Remark 5.26**  $H^1(\Omega)$  with  $\|v\|_{H^1(\Omega)} = (\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2)^{\frac{1}{2}}$  is a Hilbert-Space. This implies that  $H_0^1(\Omega) \stackrel{(\text{closed})}{\subseteq} H^1(\Omega)$  is also a Hilbert space. By the Poincare inequality (5.25) we have for all  $v \in H_0^1(\Omega)$ :

$$\|v\|_{H^1(\Omega)} \geq \|\nabla v\|_{L^2} \geq \frac{1}{2C} \|v\|_{L^2} + \frac{1}{2} \|\nabla v\|_{L^2} \geq \frac{1}{C^1} \|v\|_{H^1(\Omega)}$$

We can think of  $H_0^1(\Omega)$  as a Hilbert space with  $\|v\|_{H_0^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)}$ .

*Proof.* (Of the Poincare inequality (5.25)) We need to prove:

$$\begin{aligned} \exists C > 0 : \quad C \int_{\Omega} |\nabla v|^2 &\geq \int_{\Omega} |v|^2 \quad \forall v \in H_0^1(\Omega) \\ \Leftrightarrow \quad \exists C > 0 : \quad C \int_{\Omega} |\nabla v|^2 &\geq \int_{\Omega} |v|^2 \quad \forall v \in C_c^\infty(\Omega) \end{aligned}$$

Assume by contradiction that this does not hold, i.e. there is no  $C > 0$  s.t. the statement holds. Thus there is a sequence  $\{v_n\} \subseteq C_c^\infty(\Omega)$  s.t.

$$\int_{\Omega} |v_n|^2 = 1, \quad \int_{\Omega} |\nabla v_n|^2 \xrightarrow{n \rightarrow \infty} 0$$

Since  $v_n \in C_c^2(\Omega)$  we can extend  $v_n$  by 0 outside  $\Omega$ , so  $v_n \in C_c^\infty(\mathbb{R}^d)$ . Then:

$$\int_{\mathbb{R}^d} |v_n|^2 = 1, \quad \int_{\mathbb{R}^d} |\nabla v_n|^2 \rightarrow 0, \quad \text{supp } v_n \subseteq \Omega$$

By the Fourier transform:

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk = 1, \quad \int_{\mathbb{R}^d} |2\pi k|^2 |\hat{v}_n(k)|^2 dk \rightarrow 0, \quad \text{supp } v_n \subseteq \Omega$$

We prove that

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk \rightarrow 0$$

We write

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk = \int_{|k| \leq \epsilon} + \int_{|k| > \epsilon}$$

First, for all  $\epsilon > 0$ :

$$\int_{|k| > \epsilon} |\hat{v}_n(k)|^2 \leq \int_{\mathbb{R}^d} \frac{|k|^2}{\epsilon^2} |\hat{v}_n(k)|^2 dk \xrightarrow{n \rightarrow \infty} 0$$

Second:

$$\begin{aligned} \int_{|k| \leq \epsilon} |\hat{v}_n(k)|^2 dk &\leq \left( \int_{|k| \leq \epsilon} 1 dk \right)^{\frac{1}{q}} \left( \int_{|k| \leq \epsilon} |\hat{v}_n(k)|^{2p} dk \right)^{\frac{1}{p}}, \quad 1 < p, q < \infty \\ &\leq C \epsilon^{\frac{d}{q}} \|\hat{v}_n\|_{L^{2p}}^2, \quad \frac{1}{p} + \frac{1}{q} = 1 \text{ and } 1 \leq r \leq 2 \end{aligned}$$

Moreover, since  $\Omega$  is bounded,

$$\|v_n\|_{L^r} \leq \left( \int_{\Omega} |v_n|^r \right)^{\frac{1}{r}} \leq \|1_{\Omega}\|_{L^s} \|v_n\|_{L^2}^{1-\theta} \leq C_{\Omega} \quad \forall 1 \leq r \leq 2.$$

Thus we can take  $r < 1$  but close to 1. Then  $p$  is sufficiently large, so  $q$  is close to 1. Then

$$\int_{|k| \leq \epsilon} |\hat{v}_n(k)|^2 \leq C \epsilon^{\frac{d}{q}} \|\hat{v}_n\|_{L^{2p}}^2 \leq C \epsilon^{\frac{d}{q}} \|v_n\|_{L^r}^2 \leq C \epsilon^{\frac{d}{q}}$$

Conclusion:

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 = \int_{|k| \leq \epsilon} + \int_{|k| > \epsilon} \leq C \epsilon^{\frac{d}{q}} + \int_{|k| > \epsilon} \xrightarrow{n \rightarrow \infty} C \epsilon^{\frac{d}{q}} \xrightarrow{\epsilon \rightarrow 0} 0$$

which contradicts to the assumption  $\|\hat{v}\|_{L^2} = \|v\|_{L^2} = 1$ . ■

**Exercise 5.27** Let  $\Omega \subseteq \mathbb{R}^d$  be open, bounded with  $C^1$ -boundary. Let  $u \in W^{1,p}(\Omega)$ , for some  $1 \leq p < \infty$ . Then the following is equivalent:

a)  $u \in W_0^{1,p}(\Omega)$

b)  $\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^d \setminus \Omega \end{cases} \in W^{1,p}(\mathbb{R}^d)$

**Theorem 5.28** (Dirichlet, Riemann, Poincare, Hilbert) Let  $\Omega \subseteq \mathbb{R}^d$  be open, bounded with  $C^1$ -boundary. Let  $f \in L^2(\Omega)$ . Then there exists a unique solution  $u \in H_0^1(\Omega)$  of the variational problem

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi$$

for all  $\phi \in H_0^1(\Omega)$ . ( $\Rightarrow -\Delta u = f$  in  $D'(\Omega)$ ). Moreover,  $u$  is the unique minimizer of

$$\inf_{v \in H_0^1(\Omega)} \left( \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \right)$$

*Proof.* Let us prove that there is a solution  $u \in H_0^1(\Omega)$  for  $\inf_{v \in H_0^1(\Omega)} E(v)$ ,  $E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v$ .

Step 1: We prove  $E > -\infty$ . Take  $v \in H_0^1(\Omega)$ . By the Poincare and Hölder inequalities:

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \\ &\geq \frac{1}{2C} \|v\|_{L^2(\Omega)} - \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\geq \frac{1}{2C} \|v\|_{L^2(\Omega)}^2 - \left( \frac{1}{4C} \|v\|_{L^2(\Omega)}^2 + C \|f\|_{L^2(\Omega)}^2 \right) \\ &\geq -C \|f\|_{L^2(\Omega)}^2 > -\infty \end{aligned}$$

We can also bound:

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \\ &\geq \frac{1}{4} \int_{\Omega} |\nabla v|^2 - \frac{1}{4C} \int_{\Omega} |v|^2 - \|f\|_{L^2} \|v\|_{L^2} \\ &\geq \frac{1}{4} \int_{\Omega} |\nabla v|^2 - C \|f\|_{L^2}^2 \end{aligned}$$

Step 2: We can take a minimizing sequence  $\{v_n\} \subseteq H_0^1(\Omega)$  s.t.  $E(v_n) \xrightarrow{n \rightarrow \infty} E$ . Then:

$$\frac{1}{4} \int_{\Omega} |\nabla v_n|^2 \leq E(v_n) + C \|f\|_{L^2}^2 \longrightarrow \text{const.}$$

So  $|\nabla v_n|$  is bounded in  $L^2(\Omega)$ . We know that  $H_0^1(\Omega)$  is a Hilbert space with norm  $\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$  (and the norm is equivalent to the  $H^1$ -norm). Thus  $\{v_n\}$  is bounded in  $H_0^1(\Omega)$ .

**Remark 5.29** (Reminder from functional analysis) Let  $H$  be a Hilbert space. We say that  $v_n \rightarrow v$  if  $\|v_n - v\| \rightarrow 0$  and  $v_n \rightarrow v$  weakly in  $H$  if  $\langle v_n, \phi \rangle \rightarrow \langle v, \phi \rangle$  for all  $\phi \in H$ .

**Theorem 5.30** (Banach-Alaoglu) If  $H$  is a Hilbert space and  $\{v_n\}$  is a bounded sequence, then there is a subsequence  $\{v_{n_k}\}$  s.t.  $v_{n_k} \rightarrow v$  weakly in  $H$ .

**Remark 5.31** –  $v_n \rightarrow v$  in  $H$  iff  $f(v_n) \rightarrow f(v)$  for all  $f \in H^* = \mathcal{L}(H, \mathbb{R})$ .

– If  $v_n \rightarrow v$  in  $H$ , then:  $\liminf_{n \rightarrow \infty} \|v_n\| \geq \|v\|$  (Fatous Lemma)

In fact, for all  $\phi \in H$   $\langle v_n, \phi \rangle \rightarrow \langle v, \phi \rangle$  and  $|\langle v_n, \phi \rangle| \leq \|v_n\| \|\phi\|$ . This implies

$$\frac{|\langle v, \phi \rangle|}{\|\phi\|} \leq \liminf_{n \rightarrow \infty} \|v_n\|.$$

So we get

$$\|v\| = \sup_{\phi \neq 0} \frac{|\langle v, \phi \rangle|}{\|\phi\|} \leq \liminf_{n \rightarrow \infty} \|v_n\|$$

By the Banach-Alaoglu theorem, up to a subsequence,  $v_n \rightarrow u$  weakly in  $H_0^1(\Omega)$ . We prove that  $u$  is a minimizer for  $\mathcal{E}$

$$E \leftarrow \mathcal{E}(v_n) = \frac{1}{2} \int |\nabla v_n|^2 - \int f v_n$$

– Since  $v_n \rightarrow u$  in  $H_0^1(\Omega)$  we have that

$$\liminf_{n \rightarrow \infty} \|v_n\|_{H_0^1(\Omega)}^2 \geq \|u\|_{H_0^1(\Omega)}^2$$

So we have

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^2 \geq \int_{\Omega} |\nabla u|^2.$$

– Consider the functional  $\mathcal{L} : \phi \in H_0^1(\Omega) \rightarrow \int_{\Omega} f \phi$ . We claim that  $\mathcal{L}$  is continuous. In fact:

$$|\mathcal{L}| = \left| \int_{\Omega} f \phi \right| \leq \|f\|_{L^2} \|\phi\|_{L^2} \leq C \|f\|_{L^2} \|\nabla f\|_{L^2} = C \|f\|_{L^2} \|\phi\|_{H_0^1(\Omega)}$$

Thus from  $v_n \rightarrow v$  in  $H_0^1(\Omega)$  we get  $\mathcal{L}(v_n) \rightarrow \mathcal{L}(u)$ , thus  $\int_{\Omega} f v_n \rightarrow \int_{\Omega} f u$ .

Conclusion:  $E = \liminf \mathcal{E}(v_n) \geq \mathcal{E}(u)$ , so  $u$  is a minimizer for  $\mathcal{E}$ .

Step 3: Uniqueness. If  $E$  has 2 minimizers  $u_1, u_2$  we can prove that  $u_1 = u_2$ . This is because of the convexity:

$$\begin{aligned} 0 &\geq \frac{\mathcal{E}(u_1) + \mathcal{E}(u_2)}{2} - \mathcal{E}\left(\frac{u_1 + u_2}{2}\right) \\ &= \frac{1}{8} \left[ 2 \int_{\Omega} |\nabla u_1|^2 + 2 \int_{\Omega} |\nabla u_2|^2 - \int_{\Omega} |\nabla(u_1 + u_2)|^2 \right] \\ &= \frac{1}{8} \int_{\Omega} |\nabla u_1 - \nabla u_2|^2 \geq 0 \end{aligned}$$

This implies that  $\nabla(u_1 - u_2) = 0$ , so  $u_1 - u_2 = \text{const} = c_0$ . Since  $u_1, u_2 \in H_0^1(\Omega)$ , we have that  $u_1 - u_2 \in H_0^1(\Omega)$  and  $c_0 \in C(\bar{\Omega})$ . Hence  $c_0 = 0$  on  $\partial\Omega$ , so  $c_0 = 0$ .  $\blacksquare$

**Remark 5.32** We can also prove directly that there is a unique  $u \in H_0^1(\Omega)$  s.t.

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \quad \forall \phi \in H_0^1(\Omega)$$

by Riesz theorem. So we get  $\langle u, \phi \rangle_{H_0^1(\Omega)} = \mathcal{L}(\phi)$ .

Recall the corrector function for the unit ball:

$$\phi_x(y) = G(|x||y - \tilde{x}|), \quad \tilde{x} = \frac{x}{|x|^2}$$

This is ok if  $x \neq 0$ . When  $x \rightarrow 0$ :

$$G(|x|(y - \tilde{x})) = G(\underbrace{|x|y - \frac{x}{|x|}}_{|\cdot| \rightarrow 1})G(z), \quad |z| = 1$$

is well-defined as  $G$  is radial.

Question: If  $u \in H^1(\Omega)$ , then how can we define  $u|_{\partial\Omega}$ ?

## 5.5 Theory of Trace

**Theorem 5.33** (Trace Operator) Let  $\Omega \subseteq \mathbb{R}^d$  be open, bounded with  $C^1$  boundary. Then there is a unique linear bounded operator  $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$  such that

- If  $u \in H^1(\Omega) \cap C(\bar{\Omega})$ , then  $Tu = u|_{\partial\Omega}$  in the usual restriction sense.
- There is a  $C > 0$  s.t.  $\|Tu\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}$  for all  $u \in H^1(\Omega)$

**Theorem 5.34** If  $u \in H^1(\Omega)$ , then  $u \in H_0^1(\Omega)$  is equivalent to  $Tu = 0$  in  $L^2(\partial\Omega)$ . ( $H_0^1(\Omega) = T^{-1}(\{0\})$ ). Then we can discuss about

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega \end{cases}$$

**Lemma 5.35** (Trace inequality on  $\mathbb{R}_+^d$ ) if  $u \in C_c^\infty(\mathbb{R}^d)$ , then:

$$\|u|_{\partial\mathbb{R}_+^d}\|_{L^2(\partial\mathbb{R}_+^d)} \leq C\|u\|_{H^1(\mathbb{R}^d)} \quad \text{with } C > 0 \text{ independent of } u.$$

*Proof.*  $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ .

$$\begin{aligned} |u(x', 0)|^2 &= - \int_0^\infty \partial_d (|u(x', x_d)|^2) dx_d \\ &= - \int_0^\infty 2\partial_d u(x', x_d) u(x', x_d) dx_d \\ &\leq \int_0^\infty [|\partial_d u(x', x_d)|^2 + |u(x', x_d)|^2] dx_d \end{aligned}$$

This implies:

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} |u(x', 0)|^2 dx' &\leq \int_{\mathbb{R}^{d-1}} \left( \int_0^\infty [\dots] dx_d \right) dx' \\ &= \int_{\mathbb{R}_+^d} [|\partial_d u|^2 + |u|^2] = \|u\|_{H^1(\mathbb{R}_+^d)}^2 \end{aligned}$$

■



**Corollary 5.36** If  $u \in H^1(Q)$  and  $u$  is compactly supported, then:

$$\|u\|_{L^2(Q_0)} \leq \|u\|_{H^1(Q_+)}$$

Here

$$\begin{aligned} Q &= \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |x'| < 1, |x_d| < 1\} \\ Q_+ &= \{x = (x', x_d) \in Q \mid x_d > 0\} \\ Q_0 &= \{x = (x', x_d) \in Q \mid x_d = 0\}. \end{aligned}$$

*Proof.* We extend  $u$  by 0 outside of  $Q$ , so  $u \in H^1(\mathbb{R}^d)$ . ■

**Theorem 5.37** (Extension) If  $\Omega \subseteq \mathbb{R}^d$  is open, bounded with  $C^1$ -boundary, then there is a bounded linear operator  $B : H^1(\Omega) \rightarrow H^1(\mathbb{R}^d)$  s.t.

- $Bu|_\Omega = u$  for all  $u \in H^1(\Omega)$
- $\|Bu\|_{H^1(\mathbb{R}^d)} \leq C\|u\|_{H^1(\Omega)}$  and  $\|Bu\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{L^2(\Omega)}$ .

*Proof of Theorem 5.37.* Since  $\partial\Omega$  is  $C^1$  there are open sets  $\{U_i\}_{i=1}^N \subseteq \mathbb{R}^d$  such that  $\partial\Omega \subseteq \bigcup_{i=1}^N U_i$  and for all  $i$  there is a  $C^1$ -diffeomorphism  $h_i : U_i \rightarrow Q$  s.t.  $h_i(U_i) = Q$ ,  $h_i(U_i \cap \Omega) = Q_+$ ,  $h_i(U_i \cap \partial\Omega) = Q_0$ . Then there exists a partition of unity  $\{\theta_i\}_{i=1}^N \subseteq C^\infty(\mathbb{R}^d)$  s.t.

1.  $\sum_{i=1}^N \theta_i = 1$  for all  $x \in \mathbb{R}^d$
2. For all  $i = 1, \dots, N$ :  $\theta_i \in C_c^\infty(U_i)$
3.  $\text{supp } \theta_0 \subseteq \mathbb{R}^d \setminus \partial\Omega$  (in particular  $\theta_0|_\Omega \in C_c^\infty(\Omega)$ )

Then given  $u \in H^1(\Omega)$ , we can write  $u = \sum_{i=1}^N \theta_i u$ , where  $u_i = \theta_i u$ . By the extension theorem (5.37),  $u \rightarrow$  extended to  $Bu \in H^1(\mathbb{R}^d)$ , thus

$$Bu = \sum_{i=1}^N \theta_i(Bu) = \sum_{i=1}^N v_i, \quad v_i = \theta_i(Bu)$$

Then  $v_i \in H^1(\mathbb{R}^d)$  and  $v_i$  is compactly supported in  $U_i$  for all  $i = 1, 2, \dots, N$  and  $\text{supp } v_0 \subseteq \mathbb{R}^d \setminus \partial\Omega$ ,  $v_i \in H^1(\mathbb{R}^d)$  and compactly supported inside  $U_i$ . This implies  $\tilde{v}_i(y) = v_i(h_i^{-1}(y)) \in H^1(Q)$  and compactly supported inside  $Q$ ,  $y \in Q$ . Thus  $\|\tilde{v}_i\|_{L^2(Q_0)} \leq C\|\tilde{v}_i\|_{H^1(Q_+)}$ . So we have  $\|v_i\|_{L^2(\partial\Omega)} \leq C\|\tilde{v}_i\|_{L^2(Q_0)} \leq C'\|\tilde{v}_i\|_{H^1(Q_+)} \leq C''\|v_i\|_{H^1(U_i \cap \Omega)}$ . Thus:

$$\begin{aligned} \|u\|_{L^2(\partial\Omega)} &= \left\| \sum_{i=1}^N v_i \right\|_{L^2(\partial\Omega)} \leq \sum_{i=1}^N \|v_i\|_{L^2(\partial\Omega)} \leq \sum_{i=1}^N C'' \|v_i\|_{H^1(U_i \cap \Omega)} \\ &= C'' \sum_{i=1}^N \|\theta_i u\|_{H^1(\Omega)} \leq C'' \sum_{i=1}^N C \|u\|_{H^1(\Omega)} \end{aligned}$$

This proof works for  $u \in C(\bar{\Omega})$ . This implies

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega) \cap C(\bar{\Omega}).$$

This allows us to define

$$\begin{aligned} T : H^1(\Omega) &\longrightarrow L^2(\partial\Omega) \\ u &\longmapsto u|_{\partial\Omega} \end{aligned}$$

by continuity. I.e. for all  $u \in H^1(\Omega)$  there is  $\{u_n\} \subseteq H^1(\Omega) \cap C(\bar{\Omega})$  s.t.  $u_n \rightarrow u$  in  $H_0^1$ . Then  $Tu_n \rightarrow Tu$  in  $L^2(\partial\Omega)$ . ■

**Lemma 5.38** (Extension for  $Q$ ) Let  $u \in H^1(Q_+)$ . Then we define  $Bu : Q \rightarrow \mathbb{R}$  by

$$Bu(x) = \begin{cases} u(x) & x \in Q_+ \\ -u(x', -x_d) & x \in Q_- \end{cases},$$

$x = (x, x_d)$ . Then  $Bu \in H^1(Q)$  and  $Bu|_{Q_+} = u$ ,  $\|Bu\|_{L^2(Q)}^2 = 2\|u\|_{L^2(Q_+)}^2$ ,  $\|\nabla(Bu)\|_{L^2(Q)}^2 = \|\nabla u\|_{L^2(Q_+)}^2$

*Proof.* It is obvious  $Bu|_{Q_+} = u$  and

$$\begin{aligned} \int_Q |Bu|^2 &= \int_{Q_+} |Bu|^2 + \int_{Q_-} |Bu|^2 \\ &= \int_{Q_+} |u|^2 + \int_{Q_- = \{(x, -x_d) | (x, x_d) \in Q_+\}} |u(x, -x_d)|^2 \\ &= 2 \int_{Q_+} |u|^2 \end{aligned}$$

We prove:

$$\nabla(Bu)(x) = \begin{cases} \nabla u(x) & u \in Q_+ \\ \nabla u(x', -x_d) & u \in Q_- \end{cases}$$

First,  $\partial_d Bu(x) = \partial_d u(x', -x_d)$  if  $x \in Q_-$ . Take  $\phi \in C_c^\infty(Q)$ , then:

$$\begin{aligned} \int_Q (Bu(x))(\partial_d \phi)(x) dx &= \int_{Q_+} u \partial_d \phi + \int_{Q_-} -u(x', -x_d) \partial_d [\phi(x', x_d)] dx \\ (x \rightarrow -x_d) &= \int_{Q_+} u \partial_d \phi + \int_{Q_+} [u(x', x_d)(\partial_d \phi)(x', -x_d)] dx \\ &\stackrel{(\phi \notin C_c^\infty(Q_+))}{\approx} \int_{Q_+} (\partial_d u) \phi(x) + \int_{Q_+} (\partial_d u(x', x_d)) \phi(x', -x_d) dx \\ &= - \int_{Q_+} (\partial_d u) \phi(x) + \int_{Q_-} \partial_d u(x', -x_d) \phi(x', x_d) dx \\ &= - \int_Q f \phi, \quad \text{where } f(x) = \begin{cases} \partial_d u & x \in Q_+ \\ -\partial_d u(x', -x_d) & x \in Q_- \end{cases} \end{aligned}$$

We prove  $\int_{Q_+} u \partial_d \tilde{\phi} = - \int_{Q_+} (\partial_d u) \tilde{\phi}$  where  $\tilde{\phi}(x, x_d) = \phi(x, x_d) - \phi(x, -x_d)$ ,  $\tilde{\phi} \notin C_c^\infty(Q_+)$ . Define  $\eta_\epsilon = 0$  when  $|x_d| \leq \epsilon$ ,  $\eta_\epsilon = 1$  if  $|x_d| \geq 2\epsilon$ ,  $\eta_\epsilon \in C^\infty$ ,  $\eta_\epsilon(x', x_d) = \eta_0(x', \frac{x_d}{\epsilon})$ ,  $\eta_0 = \begin{cases} 1 & |x_d| \geq 2 \\ 0 & |x_d| \leq 1 \end{cases}$ . We have

$$\int_{Q_+} u \partial_d (\eta_\epsilon \tilde{\phi}) = - \int_{Q_+} \partial_d u (\eta_\epsilon \tilde{\phi})$$

We take  $\epsilon \rightarrow 0$ ,

$$\int_{Q_+} (\partial_d u) (\eta_\epsilon \tilde{\phi}) \rightarrow \int_{Q_+} (\partial_d u) \tilde{\phi}$$

by dominated convergence.

$$\begin{aligned} \int_{Q_+} u \partial_d (\eta_\epsilon \tilde{\phi}) &= \int_{Q_+} u (\partial_d \eta_\epsilon) \tilde{\phi} + \int_{Q_+} u \eta_\epsilon \partial_d \tilde{\phi} \\ &\xrightarrow{\epsilon \rightarrow 0} \int_{Q_+} u \eta_\epsilon \partial_d \tilde{\phi} \end{aligned}$$

by dominated convergence.

$$\begin{aligned}
& \left| \int_{Q_+} u(\partial_d \eta_\epsilon) \tilde{\phi} \right| = \left| \int_Q u \frac{1}{\epsilon} (\partial_d \eta_0) \left( x, \frac{x_d}{\epsilon} \right) \tilde{\phi} \right| \\
& \begin{pmatrix} |\tilde{\phi}(x', x_d)| \\ = |\phi(x, x_d) - \phi(x, x_d)| \\ \leq \|\partial_d \phi\|_{L^\infty} |x_d| \end{pmatrix} \leq \frac{1}{\epsilon} \|\partial_d \eta_0\|_{L^\infty} \int_{Q_+ \cap \{x_d \leq 2\epsilon\}} |u| \underbrace{|\tilde{\phi}|}_{\leq C|x_d| \leq C\epsilon} \\
& (\text{Dominated cv } u \in L^1(Q_+)) \leq C \int_{Q_+ \cap \{0 < x_d \leq 2\epsilon\}} |u| \xrightarrow{\epsilon \rightarrow 0} 0
\end{aligned}$$

where  $u \in L^2(Q_+)$  because  $u \in H^1(Q_+)$ . ■

**Exercise 5.39** (E. 9.1) Let  $\Omega$  be open, bounded with  $C^1$ -boundary. Let  $u \in H_0^1(\Omega)$ ,  $f \in L^2(\Omega)$ . Show that the following statements are equivalent:

- 1)  $-\Delta u = f$  in  $D'(\Omega)$
- 2)  $\int \nabla u \nabla \phi = \int f \phi$  for all  $\phi \in H_0^1$
- 3)  $E = \inf_v \left( \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega f v \right)$

*Solution.*

1)  $\Rightarrow$  2) From  $-\Delta u = f$  in  $D'(\Omega)$  we get that

$$\int_\Omega u(-\Delta \phi) = \int_\Omega f \phi$$

for all  $\phi \in C_c^\infty(\Omega)$ . Claim: If  $u \in H_0^1$ ,  $\phi \in C_c^\infty$ , then

$$\int_\Omega (-\Delta \phi) = \int_\Omega \nabla u \nabla \phi$$

Density argument:  $u \in H_0^1 = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_H}$ , so there is a sequence  $\{u_n\} \subseteq C_c^\infty(\Omega)$  s.t.  $u_n \rightarrow u$  in  $H^1(\Omega)$ . Since  $u_n, \phi \in C_c^\infty(\Omega)$ , then by the integration by parts:

$$\int_\Omega u_n(-\Delta \phi) = \int_\Omega (\nabla u_n) \nabla \phi \forall n$$

Take  $n \rightarrow \infty$ , then,

$$\int_\Omega u(-\Delta \phi) = \int_\Omega (\nabla u) \nabla \phi$$

as  $u_n \rightarrow u$  and  $\nabla u_n \rightarrow \nabla u$  in  $L^2$ . Claim: If  $\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi$  for all  $\phi \in C_c^\infty(\Omega)$ , then

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi$$

for all  $\phi \in H_0^1$ . (Given  $\nabla u, f \in L^2$ ). With density argument: For all  $\phi \in H_0^1$  there is a sequence  $\{\phi_n\} \subseteq C_c^\infty(\Omega)$  s.t.  $\phi_n \rightarrow \phi$  in  $H^1$ . Then:

$$\int_\Omega \nabla u \nabla \phi_n = \int_\Omega f \phi_n$$

for all  $n$ . Take  $n \rightarrow \infty$ :

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi$$

as  $\phi_n \rightarrow \phi$ ,  $\nabla \phi_n \rightarrow \nabla \phi$  in  $L^2$ .

2)  $\Rightarrow$  3) We show  $E(u) \leq E(v)$  for all  $v \in H_0^1$ , i.e.

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u \leq \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v$$

for all  $v \in H_0^1$ . Write  $v = u + w$ , then:

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \\ &= \frac{1}{2} \int_{\Omega} |\nabla(u+w)|^2 - \int_{\Omega} f(u+w) \\ &= \frac{1}{2} \int_{\Omega} [|\nabla u|^2 + |\nabla w|^2 + 2\nabla u \nabla w] \int_{\Omega} (f u + f w) \\ &= E(u) + \frac{1}{2} \int_{\Omega} |\nabla w|^2 + \underbrace{\left( \int_{\Omega} \nabla u \nabla w - \int_{\Omega} f w \right)}_{=0} \end{aligned}$$

as  $w = v - u \in H_0^1$  (by (2))

3)  $\Rightarrow$  1)

$$E(u) \leq E(u + t\phi)$$

for all  $\phi \in H_0^1$  (or  $C_c^\infty$ ) for all  $t \in \mathbb{R}$ . This implies:

$$\frac{d}{dt} E(u + t\phi)|_{t=0} = 0$$

Here

$$\begin{aligned} E(u + t\phi) &= \frac{1}{2} \int_{\Omega} \underbrace{|\nabla(u + t\phi)|^2}_{|\nabla u|^2 + t^2 |\nabla \phi|^2 + 2t \nabla u \nabla \phi} - \int_{\Omega} f(u + t\phi) \\ &= E(u) + t \left[ \int_{\Omega} \nabla u \nabla \phi \int_{\Omega} f \phi \right] + t^2 \int_{\Omega} |\nabla \phi|^2 \end{aligned}$$

This implies

$$\frac{d}{dt} E(u + t\phi)|_{t=0} = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi$$

Conclude:

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi$$

for all  $\phi \in H_0^1$  or  $C_c^\infty$ . So we get

$$\int_{\Omega} u(-\Delta \phi) = \int_{\Omega} f \phi$$

for all  $\phi \in C_c^\infty$ . so we can conclude:

$$-\Delta u = f$$

in  $D'(\Omega) \Rightarrow 1)$

■

**Exercise 5.40** (E 9.2)

$$Q = \{(x', x_d) \mid |x'| < 1, |x_d| < 1\}$$

Given  $u \in H^1(Q_+)$ , define  $Bu : Q \rightarrow \mathbb{R}$  as

$$Bu(x) = \begin{cases} u(x) & x \in Q_+ \\ u(\tilde{x}) & x \in Q_- \end{cases},$$

$x = (x', x_d) \Leftrightarrow \tilde{x} = (x', -x_d)$ ,  $x \in Q_- \Leftrightarrow \tilde{x} \in Q_+$ . In the lectures:

$$\partial_d(Bu)(x) = \begin{cases} \partial_d u(x) & x \in Q_+ \\ -(\partial_d u)(\tilde{x}) & x \in Q_- \end{cases}$$

This implies  $\partial_d(Bu) \in L^2(Q)$ .

1. For all  $i = 1, \dots, d-1$ , then:

$$\partial_i(Bu)(x) = \begin{cases} \partial_i u(x) & x \in Q_+ \\ \partial_i u(\tilde{x}) & x \in Q_- \end{cases}$$

2. Example  $u \in H^2(Q_+)$  but  $Bu \notin H^2(Q)$ .

*Solution.* 1. For all  $\phi \in C_c^\infty(Q)$ :

$$\int_Q Bu(x) \partial_i \phi(x) dx = \int_{Q_+} u(x) \partial_i \phi(x) dx + \int_{Q_-} u(\tilde{x}) \partial_i \phi(x) dx$$

Write  $\vec{n} = (n_1, \dots, n_d)$ . Here:

$$\begin{aligned} \int_{Q_+} u(x) \partial_i \phi(x) dx &= \int_{Q_+} -\partial_i u(x) \phi(x) dx + \int_{\partial Q_+} u(x) \phi(x) n_i dS \\ \int_{Q_-} u(x', -x_d) \partial_i \phi(x', x_d) dx' dx_d &= - \int_{Q_+} u(x', x_d) \partial_i \phi(x', -x_d) dx' dx_d \\ &= \int_{Q_+} \partial_i u(x) \phi(\tilde{x}) - \int_{\partial Q_+} u \phi n_i dS \\ &= \int_{Q_-} -\partial_i u(\tilde{x}) \phi(x) - \int_{\partial Q_+} u \phi n_i dS \end{aligned}$$

with  $d(-x_d) = d(x_d)$ . Conclude:

$$\begin{aligned} \int_Q (Bu)(x) \partial_i \phi(x) dx &= \int_{Q_+} (-\partial_i u)(x) \phi(x) + \int_{Q_-} (-\partial_i u)(\tilde{x}) \phi(x) \\ &= \int_Q -h(x) \phi(x) dx, \quad h(x) = \begin{cases} \partial_i u(x), & x \in Q_+ \\ \partial_i u(\tilde{x}), & x \in Q_- \end{cases} \end{aligned}$$

for all  $\phi \in C_c^\infty(Q)$ , so  $\partial_i(Bu) \in L^2$  for all  $i = 1, 2, \dots, d-1$ . Thus  $Bu \in H^1(Q)$ .

2. 1D: Take  $Q_+(0, 1)$ ,  $Q_- = (-1, 0)$ ,  $Q_0 = \{0\}$ ,  $Q = (-1, 1)$ ,  $u(x) = x$  in  $Q_+ = (0, 1)$ ,  $Bu(x) = u(x) = -x$  if  $x \in Q_- = (-1, 0)$ , i.e.  $Bu(x) = |x|$  if  $x \in Q = (-1, 1)$ . We know

$$(Bu)'(x) = \begin{cases} 1 & x \in (0, 1) \\ -1 & x \in (-1, 0) \end{cases} \in L^2(-1, 1)$$

i.e.  $Bu \in H^1(Q)$ .

$$(Bu)''(x) = 2\delta_0(x)$$

in  $D'(Q)$  but  $\notin L^2(-1, 1)$ , i.e.  $Bu \notin H^2(Q)$ . Question: Given  $u \in H^2(Q_+)$ , can we find an extension  $Bu \in H^2(Q)$  Yes! E.g.  $u(x) = x$  in  $(0, 1)$ , so  $Bu(x) = x$  in  $(-1, 1)$ . In general:  $u \in H^2(Q) \rightsquigarrow \tilde{u} \in H^2(Q)$  but  $\nabla u = 0$  on  $\partial Q_+$ . ■

**Exercise 5.41** (Bonus 8) Assume  $u \in H^2(Q_+)$  and  $\begin{cases} u = 0 \\ \nabla u = 0 \end{cases}$  on  $\partial Q_+$ . Prove that  $Bu \in H^2(Q)$ . (Reflection extension) (Ok in 1D)

**Remark 5.42** If  $u \in H^2(Q_+)$ , then  $\nabla u \in H^1(Q_+)$ , so  $\nabla u|_{\partial Q_+}$  by trace theory. In general:  $\Omega \subseteq \mathbb{R}^d$ ,  $C^2$ -boundary condition, then the same result holds.

**Remark 5.43** In 1D:  $\begin{cases} u \in H^2(0, 1) \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$ ,  $u|_{Q_0} \in L^2(Q_0)$ , 1D:  $Q_0 = \{0\}$ . In general:

If  $u \in H^1(0, 1)$ , then  $u(0)$  is determined by trace theory. If  $u \in H^2(0, 1)$ ,  $u'(0)$  is determined. Sobolev:

$$\begin{aligned} H^1(0, 1) &\subseteq C([0, 1]) \\ H^2(0, 1) &\subseteq C^1([0, 1]) \end{aligned}$$

**Lemma 5.44** (Poincare inequality) Let  $\Omega$  be open, bounded connected with  $C^1$ -boundary. Then for all  $g \in L^2(\partial\Omega)$  s.t.  $g \neq \text{constant}$  there is a  $C > 0$  s.t.

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$$

for all  $u \in M$ , where

$$M = \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = g\}.$$

*Proof.* We assume that the statement does not hold true. Then there is a sequence  $\{u_n\} \subseteq H^1(\Omega)$ ,  $u_n|_{\partial\Omega} = g$  s.t.

$$\|\nabla u_n\|_{L^2(\Omega)} \rightarrow 0, \quad \|u_n\|_{L^2(\Omega)} = 1.$$

Since  $\{u_n\}$  is bounded in  $H^1(\Omega)$ , by the Banach-Alaoglu theorem (5.30), up to a subsequence

$$u_n \rightarrow u_0 \quad \text{weakly in } H^1(\Omega)$$

Since  $\nabla u_n \rightarrow 0$  strongly in  $L^2$  and  $\nabla u_n \rightarrow \nabla u_0$  weakly in  $L^2$ , we have  $\nabla u_0 = 0$ , so  $u_0|_{\partial\Omega} = \text{const.}$  (here we need  $\Omega$  to be connected), so  $u_0|_{\partial\Omega} = \text{const.}$  On the other hand, note that  $M$  is convex and closed in  $H^1(\Omega)$  since the trace operator  $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$  is continuous. Therefore,  $M$  is also weakly closed in  $H^1(\Omega)$  by the Hahn-Banach theorem. Thus from  $\{u_n\} \subseteq M$ ,  $u_n \rightarrow u_0$  weakly in  $H^1(\Omega)$  we get that  $u_0 \in M$ , so  $u_0|_{\partial\Omega} = g$ . We get a contradiction since  $g \neq \text{const}$  ■

**Theorem 5.45** (Solution for Poisson Equation with inhomogeneous boundary condition) Let  $\Omega$  be open, bounded with  $C^1$ -boundary. Let  $f \in L^2(\Omega)$ ,  $g \in L^2(\partial\Omega)$ . There there is a unique  $u \in H^1(\Omega)$  s.t.

$$\begin{cases} -\Delta u = f & \text{in } D'(\Omega) \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega \end{cases}$$

Here  $u|_{\partial\Omega} = T(u) \in L^2(\partial\Omega)$  is defined by the trace operator. Moreover if  $\Omega$  is connected and  $g \neq \text{constant}$ , then  $u$  is the unique minimizer for the variational problem

$$E = \inf_{v \in M} \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v,$$

where  $M = \{v \in H^1(\Omega), v|_{\partial\Omega} = g \text{ on } \partial\Omega\}$

*Proof.* First let us assume that  $\Omega$  is connected and  $g \neq \text{const.}$

Step 1: We prove that  $E = \inf_{v \in M} E(v)$  has a minimizer. By Poincaré's Inequality (5.44), for all  $v \in M$ :

$$\begin{aligned} E(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \\ (\text{Hölder}) &\geq \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 - \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ (\text{Poincaré 5.44}) &\geq \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 - C \|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\geq \frac{1}{4} \|\nabla v\|_{L^2(\Omega)}^2 - C \|f\|_{L^2(\Omega)} \end{aligned}$$

Thus  $E = \inf_{v \in M} E(v) > -\infty$ . Moreover, taking a minimizing sequence  $\{v_n\} \subseteq M$ ,  $E(v_n) \rightarrow E$ , we find that  $\|\nabla v_n\|_{L^2(\Omega)}$  is bounded, and hence  $\|v_n\|_{H^1(\Omega)}$  is bounded (by Poincaré inequality) again. By Banach-Alaoglu (5.30), up to a subsequence we have  $v_n \rightarrow u$  weakly in  $H^1(\Omega)$ . Hence

$$\begin{cases} \limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^2 \geq \int_{\Omega} |\nabla u|^2 & \text{as } \nabla v_n \rightarrow \nabla u \text{ in } L^2 \\ \int_{\Omega} v_n f \rightarrow \int_{\Omega} u f & \text{as } v_n \rightarrow u \text{ in } L^2 \end{cases}$$

Note that  $\{v_n\} \subseteq M$ ,  $v_n \rightarrow u$  in  $H^1(\Omega)$  and  $M$  is weakly closed in  $H^1(\Omega)$  (as argued in the proof of Poincaré inequality), therefore  $u \in M$ . This means that  $u$  is a minimizer for  $E = \inf_{v \in M} E(v)$ .

Step 2: Now we prove that if  $u$  is a minimizer for  $E$ , then  $-\Delta u = f$  in  $D'(\Omega)$ . In fact, for all  $\phi \in C_c^\infty(\Omega)$  we have

$$E(u) \leq E(u + t\phi) \quad \forall t \in \mathbb{R}$$

because  $u + t\phi \in M$ . So we get that

$$0 = \frac{d}{dt} E(u + t\phi)|_{t=0} = \int_{\Omega} \nabla u \nabla \phi - \int_{\Omega} f \phi$$

Thus

$$\int_{\Omega} u(-\Delta \phi) = \int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \quad \forall \phi \in C_c^\infty(\Omega).$$

So  $-\Delta u = f$  in  $D'(\Omega)$ .

Step 3: We prove that Poisson's equation has at most one solution. Assume that  $u_1, u_2$  are 2 solutions. Then  $u = u_1 - u_2$  solves

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \Omega \end{cases}$$

so  $u = 0$ .

Step 4: If  $g = c_0$  is a constant, then Poisson's equation can be rewritten with  $\tilde{u} = u - c_0$ :

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = c_0 & \text{on } \Omega \end{cases} \Leftrightarrow \begin{cases} -\Delta \tilde{u} = f & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \Omega \end{cases}$$

If  $\Omega$  is not connected, then by considering connected components of  $\Omega$  we can prove that Poisson's equation always has a unique solution (for all  $f \in L^2(\Omega), g \in L^2(\partial\Omega)$ ).

■

## 5.6 Final Remarks

We can describe  $H_0^1(\Omega)$  as the kernel of the trace operator  $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$

**Theorem 5.46** Let  $\Omega \subseteq \mathbb{R}^d$  be open, bounded with  $C^1$ -boundary. Then:

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid T(u) = 0 \text{ on } \partial\Omega\}$$

Recall that if  $u \in H^1(\Omega) \cap C(\bar{\Omega})$ , then  $T(u) = u|_{\partial\Omega}$  is the usual restriction. In this case we recover a result proved before.

*Proof.*

■

Recall that the variational characterization of the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \quad \forall \phi \in M$$

where  $M = \{v \in H^1(\Omega) \mid v = g \text{ on } \partial\Omega\}$ . In fact, if  $u \in H^1(\Omega)$  and

$$\int_{\Omega} \nabla u \nabla \phi = \int_{\Omega} f \phi \quad \forall \phi \in H^1(\Omega)$$

Then  $u$  satisfies the Neumann condition:

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n} = 0 \text{ on } \partial\Omega$$

(justification ...)

For the exercises of sheet 10: Let  $\Omega = (a, b) \subseteq \mathbb{R}$  be an open bounded interval. For every  $u \in H^1(\Omega)$  the values  $u(a)$  and  $u(b)$  are determined uniquely by trace theory, or by Sobolev's embedding theorem. Recall: If  $u \in H^1((a, b)) \rightsquigarrow \partial\Omega = \{a, b\}$  counting measure iff  $g \in L^2(\partial\Omega)$  i.e.  $g(a) = g(b)$  are *well-defined*.

**Exercise 5.47** (E 10.1) a) Prove  $H^1(\mathbb{R}) \subseteq (C(\mathbb{R}) \cap L^\infty(\mathbb{R}))$

Hint: You can use Fourier Transform

b)  $H^1(\Omega) \subseteq C(\Omega)$



*Solution.* a) Let  $u \in H^1(\mathbb{R})$ . Then  $u, u' \in L^2(\mathbb{R}) \Leftrightarrow \hat{u}(k)(1 + |2\pi k|) \in L^2(\mathbb{R})$ .  
Thus:

$$u(x) = \int_{\mathbb{R}} \hat{u}(k) e^{2\pi i k x} dk \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

if  $\hat{u} \in L^1(\mathbb{R})$ . So we have to show  $\hat{u} \in L^1(\mathbb{R})$ .

$$\begin{aligned} \int_{\mathbb{R}} |\hat{u}(k)| dk &= \int_{\mathbb{R}} \frac{|g(k)|}{1 + |2\pi k|} dk \\ &\leq \left( \int_{\mathbb{R}} |g(k)|^2 dk \right) \left[ \int_{\mathbb{R}} \left( \frac{1}{1 + |2\pi k|} \right)^2 dk \right]^{\frac{1}{2}} < \infty \end{aligned}$$

b) Given  $u \in H^1(\Omega)$ , then there is an extension  $\tilde{u} \in H^1(\mathbb{R})$ . By a)  $\tilde{u} \in C(\mathbb{R})$ , so  $u = \tilde{u}|_{\bar{\Omega}} \in C(\bar{\Omega})$ . Remak: We have  $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{H^1(\Omega)}$ , where  $\Omega = (a, b)$  or  $\mathbb{R}$  (but only in 1D) ■

Recall: If  $\Omega \subseteq \mathbb{R}^d (d \geq 1)$  open, bounded with  $C^1$ -boundary. Then

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega)$$

Actually the same bound holds if  $u \in H^1(\Omega)$  and  $u|_\Gamma = 0$  for an open subset  $\Gamma \subseteq \partial\Omega$ . In 1D we have:

**Exercise 5.48** (E 10.2 (Poincare inequality)) Let  $u \in H^1(\Omega)$ ,  $u(a) = 0$ . Prove that there exists a constant  $C > 0$  such that

$$\|u\|_{L^2(\Omega)} \leq C \|u'\|_{L^2(\Omega)}$$

*Solution.* Let  $u \in C^1(\bar{\Omega})$  and  $u(a) = 0$ . Then:

$$\begin{aligned} u(x) &= u(a) + \int_a^x u'(t) dt \quad \forall x \in (a, b) \\ \Rightarrow |u(x)| &\leq \int_a^x |u'(t)| dt \leq \int_a^b |u'(t)| dt = \|u'\|_{L^1(\Omega)} \leq C \|u'\|_{L^2(\Omega)} \end{aligned}$$

as  $\Omega$  is bounded. This implies:

$$\frac{1}{C} \|u\|_{L^2(\Omega)} \leq \|u\|_{L^\infty(\Omega)} \leq C \|u'\|_{L^2(\Omega)}$$

To extend this for  $u \in H^1(\Omega)$ , we can use a density argument. More precisely, for all  $u \in H^1(\Omega)$  there is a sequence  $\{u_n\} \subseteq C^1(\bar{\Omega})$  s.t  $u_n \rightarrow u$  in  $H^1(\Omega)$ . Then:

$$\|u\|_{L^2(\Omega)} = \lim_{n \rightarrow \infty} \|u_n\|_{L^2(\Omega)} \leq C \lim_{n \rightarrow \infty} \|u'_n\|_{L^2(\Omega)} = C \|u'\|_{L^2(\Omega)}$$

Recall: For all  $f \in W_{loc}^{1,1}(O)$  with  $O$  in  $\mathbb{R}^d$  we have

$$f(x) - f(y) = \int_0^1 \nabla f(y + t(x - y))(x - y) dt$$

if  $x, y \in O$ ,  $y + t(x - y) \in O$  for all  $t \in [0, 1]$ . For 1D: If  $u \in H^1(a, b)$ :

$$u(x) - u(y) = \int_y^x u'(t) dt \quad \forall x, y \in (a, b) \quad \blacksquare$$

**Exercise 5.49** (E 10.3 (Poincare inequality)) Let  $u \in H^2(\Omega)$  and  $f \in L^2(\Omega)$ . Prove that the following statements are equivalent:

a)  $u$  solves the equation:

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u'(0) = u'(1) = 0 \end{cases}$$

b)

$$\int_{\Omega} u' \phi' = \int_{\Omega} f \phi$$

for all  $\phi \in H^1(\Omega)$ .

Here  $u \in H^2(\Omega) \Rightarrow u' \in H^1(\Omega) \Rightarrow u'(0), u'(1)$  determined uniquely by trace theorem / Sobolev inequality  $H^1(\Omega) \subseteq C(\bar{\Omega})$

*Solution.*

b)  $\Rightarrow$  a) For all  $\phi \in C_c^\infty(\Omega)$ :

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' = - \int_{\Omega} u \phi''$$

This implies  $-u'' = f$  in  $D'(\Omega)$  a.e. Thus for all  $\phi \in H^1(\Omega)$ :

$$\int_{\Omega} f \phi = \int_{\Omega} -u'' \phi = \int_{\Omega} u' \phi' - [u' \phi]_a^b$$

By b) we conclude  $0 = [u' \phi]_a^b = u'(b)\phi(b) - u'(a)\phi(a)$  for all  $\phi \in H^1(\Omega)$ . We can choose  $\phi \in H^1(\Omega)$  s.t.  $\phi(a) = 0, \phi(b) = 1$ . This implies  $\phi'(b) = 0$ . Similarly, we can choose  $\phi \in H^1(\Omega)$  s.t.  $\phi(a) = 1, \phi(b) = 0$ . This implies  $u'(a) = 0$ .

a)  $\Rightarrow$  b) From a) and Integration by parts:

$$\int_{\Omega} f \phi = \int_{\Omega} -u'' \phi = \int_{\Omega} u' \phi' - \underbrace{[u' \phi]_a^b}_{=0 \text{ as } u'(a)=u'(b)=0}$$

This implies:

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in H^1(\Omega)$$

■

**Exercise 5.50** (E 10.4 (Robin boundary condition)) Let  $f \in L^2(\Omega)$ .

a) Prove that there exists a unique  $u \in M := \{\phi \in H^1(\Omega), u(a) = 0\}$  s.t.

$$\int_{\Omega} u' \phi' = \int_{\Omega} f \phi \quad \forall \phi \in M$$

b) Prove that the above function  $u$  is the unique solution to the equation

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = 0 & u'(b) = 0 \end{cases}$$

*Solution.* a) By 10.2 we have

$$\|\phi\|_{L^2(\Omega)} \leq C \|\phi'\|_{L^2(\Omega)} \quad \forall \phi \in M$$

Thus:  $(M, \|\phi\|_M := \|\phi'\|_{L^2(\Omega)})$  is a Hilbert space. More precisely, we know  $(M, \|\cdot\|_M)$  is a closed subspace of  $H^1 \rightsquigarrow$  a Hilbert space. And  $\|\cdot\|_M$  is comparable to  $\|\cdot\|_{H^1}$ . By Riesz representation theorem there is a unique  $u \in M$  s.t.  $\langle \phi, u \rangle_M = F(\phi)$  for all  $\phi \in M$ . We use this for

$$F(\phi) = \int_{\Omega} f \phi \quad \forall \phi \in M$$

Here  $|F(\phi)| \leq \|f\|_{L^2} \|\phi\|_{L^2}$ .

b) Let  $u \in M$  be the solution in (a) i.e.

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in M$$

Then we prove that  $u$  solves

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = u'(b) = 0 \end{cases}$$

Since  $u \in M$  we have  $u \in H^1(\Omega)$  and  $u(a) = 0$ . From

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in M$$

we get for all  $\phi \in C_c^\infty(\Omega)$ :

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' = \int_{\Omega} -u \phi''$$

So we get  $-u'' = f$  in  $D'(\Omega)$ . Since  $f \in L^2(\Omega) \Rightarrow u'' \in L^2(\Omega) \Rightarrow u \in H^2(\Omega) \Rightarrow u' \in H^1(\Omega) \Rightarrow u'(b)$  is uniquely determined. For all  $\phi \in M$ :

$$\int_{\Omega} f \phi = \int_{\Omega} -u'' \phi = \int_{\Omega} u' \phi' - (u'(b)\phi(b) - u'(a)\phi(a)) \quad \text{as } \phi \in M$$

and  $\int_{\Omega} f \phi = \int_{\Omega} u' \phi'$ . This implies:

$$u'(b)\phi(b) = 0 \quad \forall \phi \in M$$

Take  $\phi(x) = \frac{x-a}{b-a} \in M$ ,  $\phi(b) = 1$ . Uniqueness of the solution: Take  $u$  s.t.

$$\begin{cases} -u'' = f & \text{in } D'(\Omega) \\ u(a) = u'(b) = 0 \end{cases}$$

This implies  $u \in H^2(\Omega)$ . By integration by parts: For all  $\phi \in H^1(\Omega)$ ,  $\phi(a) = 0$ .

$$\int_{\Omega} f \phi = \int_{\Omega} -u'' \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in M$$

Thus  $u \in M$  and

$$\int_{\Omega} f \phi = \int_{\Omega} u' \phi' \quad \forall \phi \in M. \quad \blacksquare$$

**Exercise 5.51** (Bonus 9) Prove that the solution  $u$  in Problem E 10.4 is the unique minimizer for the minimization problem:

$$E = \inf_{v \in M} \left( \int_{\Omega} |v'|^2 - \int_{\Omega} f v \right)$$

## Chapter 6

# Heat Equation

### 6.1 Fundamental Solution

$$\begin{cases} \partial_t u = \Delta u & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u = g & (x, t) \in \mathbb{R}^d \times \{0\} \end{cases}$$

The fundamental solution is:

$$\Phi(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^d, t > 0$$

We have:

$$\begin{cases} \partial_t \Phi = \Delta \Phi & (x, t) \in \mathbb{R}^d \times (0, \infty) \\ \int_{\mathbb{R}^d} \Phi(x, t) dx = 1 & \forall t > 0 \\ \lim_{t \rightarrow 0} \Phi(x, t) = \delta_0(x) & \text{in } D'(\mathbb{R}^d) \end{cases}$$

**Theorem 6.1** If  $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , then

$$u(x, t) := \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy$$

satisfies

- (i)  $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$
- (ii)  $\partial_t u = \Delta u$  for all  $(x, t) \in \mathbb{R}^d \times (0, \infty)$
- (iii)  $\lim_{t \rightarrow 0} u(x, t) = g(x)$  for all  $x \in \mathbb{R}^d$

**Notation 6.2** For functions of  $(x, t)$  we introduce the following notation for different regularity in  $x$  and  $t$ .

$$f \in C_1^2 \Leftrightarrow f, D_x f, D_x^2 f, \partial_t f \in C$$

**Theorem 6.3** (Nonhomogeneous problem) Let  $f \in C_1^2(\mathbb{R}^d, [0, \infty))$  be compactly supported. Define

$$u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(x - y, t - s) f(y, s) dy ds$$

Then

- (i)  $u \in C_1^2(\mathbb{R}^d \times (0, \infty))$
- (ii)  $\partial_t u = \Delta u + f$  for all  $x \in \mathbb{R}^d, t > 0$
- (iii)  $\lim_{t \rightarrow 0} u(x, t) = 0$  for all  $x \in \mathbb{R}^d$ .

*Proof.* We write

$$u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) f(x - y, t - s) dy ds$$

With the Leibniz integral rule we get

$$\partial_t u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) \partial_t f(x - y, t - s) dy ds + \int_{\mathbb{R}^d} \Phi(y, s) f(x - y, 0) dy$$

and

$$\partial_{ij} u(x, t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) \partial_{ij} f(x - y, t - s) dy.$$

This shows that  $\partial_t u, \partial_{ij} u$  are in  $C(\mathbb{R}^d \times (0, \infty))$ . Next we calculate:

$$\begin{aligned} \partial_t u - \Delta u &= \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) (\partial_t - \Delta_x) f(x - y, t - s) dy ds + \int_{\mathbb{R}^d} \Phi(y, s) f(x - y, 0) dy \\ &= \underbrace{\int_\epsilon^t \int_{\mathbb{R}^d} \Phi(y, s) (\partial_t - \Delta_x) f(x - y, t - s) dy ds}_{=: I_\epsilon} \\ &\quad + \underbrace{\int_0^\epsilon \int_{\mathbb{R}^d} \Phi(y, s) (\partial_t - \Delta_x) f(x - y, t - s) dy ds}_{J_\epsilon} \\ &\quad + \underbrace{\int_{\mathbb{R}^d} \Phi(y, s) f(x - y, 0) dy}_K \end{aligned}$$

Then

$$\begin{aligned} |J_\epsilon| &\leq \|(\partial_t - \Delta_x) f\|_{L^\infty} \int_0^\epsilon \int_{\mathbb{R}^d} \Phi(y, s) dy ds \leq C\epsilon \xrightarrow{\epsilon \rightarrow 0} 0 \\ I_\epsilon &= \int_\epsilon^t \int_{\mathbb{R}^d} \Phi(y, s) (-\partial_s - \Delta_y) f(x - y, t - s) dy ds \\ \text{(Green (2.3))} \quad &= \int_\epsilon^t \int_{\mathbb{R}^d} \underbrace{(\partial_s - \Delta_y) \Phi(y, s)}_{=0} f(x - y, t - s) dy ds \\ &\quad - \left[ \int_{\mathbb{R}^d} \Phi(y, s) f(x - y, t - s) dy \right]_{s=\epsilon}^{s=t} \end{aligned}$$

This implies:

$$\begin{aligned} I_\epsilon + K &= \int_{\mathbb{R}^d} \Phi(y, \epsilon) f(x - y, t - \epsilon) dy \\ &\xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}^d} \delta_0(y) f(x - y, t) dy = f(x, t) \end{aligned}$$

Thus

$$\partial_t u - \Delta u = f(x, t) \quad \forall (x, t) \in \mathbb{R}^d \times (0, \infty)$$

Finally:

$$\|u(\cdot, t)\|_{L^\infty} \leq \|f\|_{L^\infty} \int_0^t \int_{\mathbb{R}^d} \Phi(y, s) dy ds = \|f\|_{L^\infty} t \xrightarrow{t \rightarrow 0} 0$$

■

**Exercise 6.4** If  $f, g$  are given as above, then

$$u(x, t) = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy + \int_0^t \int_{\mathbb{R}^d} \Phi(x - y, t - s) f(y, s) ds$$

solves

$$\begin{cases} \partial_t u - \Delta u = f \\ u(\cdot, 0) = g \end{cases}$$

**Remark 6.5** (Duhamel formula) Consider the ODE  $\partial_t w(t) = Aw(t)$  for all  $A \in \mathbb{R}$ . Then the solution is

$$w(t) = e^{tA} w(0).$$

More generally: If  $\partial_t w(t) = Aw(t) + f(t)$ , then

$$\begin{aligned} \partial_t(e^{-tA} w(t)) &= e^{-tA} (\partial_t w(t) - Aw(t)) = e^{-tA} f(t) = e^{-tA} f(t) \\ \Rightarrow e^{-tA} w(t) &= w(0) + \int_0^t e^{-sA} f(s) ds \\ \Rightarrow w(t) &= e^{tA} w(0) + \int_0^t e^{(t-s)A} f(s) ds \end{aligned}$$

More generally, if  $A$  is an operator (independent of time) then:

$$\begin{aligned} \partial_t w(t) &= Aw(t) + f(t) \\ \Rightarrow w(t) &= e^{tA} w(0) + \int_0^t e^{(t-s)A} f(s) ds \end{aligned}$$

Application: If  $A = \Delta$ , then the operator  $e^{t\Delta}$  has kernel

$$e^{t\Delta}(x, y) = \Phi(x - y, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}.$$

This is called the *heat kernel*.

**Theorem 6.6** ( $L^2$ -data) For every  $g \in L^2(\mathbb{R}^d)$ , define

$$u(t, x) = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy$$

Then  $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$  and it solves the heat equation

$$\begin{cases} \partial_t u = \Delta_x u & \mathbb{R}^d \times (0, \infty) \\ \lim_{t \rightarrow 0} u(\cdot, t) = g & \text{in } L^2(\mathbb{R}^d) \end{cases}$$

*Proof.* Recall the heuristic computation from the heat equation using the Fourier transform

$$\begin{aligned}
& \partial_t u(x, t) = \Delta_x u(x, t) \\
\Leftrightarrow & \partial_t \hat{u}(k, t) = -|2\pi k|^2 \hat{u}(k, t) \\
\Leftrightarrow & \partial_t (e^{t|2\pi k|^2} \hat{u}(k, t)) = 0 \\
\Leftrightarrow & e^{t|2\pi k|^2} \hat{u}(k, t) = \hat{u}(k, 0) = \hat{g}(k) \\
\Leftrightarrow & \hat{u}(k, t) = e^{-t|2\pi k|^2} \hat{g}(k) = \hat{\Phi}(k, t) \hat{g}(k) = \widehat{\Phi \star g} \quad \blacksquare \\
\Leftrightarrow & u(x, t) = \Phi \star g = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy
\end{aligned}$$

Here we only need the direction  $\Leftarrow$  which is rigorous if  $g \in L^2(\mathbb{R}^d)$ . From the Fourier transform, it is also easy to check that  $u(\cdot, t) \rightarrow g$  in  $L^2$  as  $t \rightarrow 0$  (exercise). To see the smoothness, note that for all  $t > 0$ , and for all  $m \in \mathbb{N}$ :

$$(1 + |2\pi k|^m) \hat{u}(k, t) = \underbrace{(1 + |2\pi k|^m) e^{-t|2\pi k|^2}}_{\in L^\infty} \underbrace{\hat{g}(k)}_{\in L^2} \in L^2$$

This implies  $u(\cdot, t) \in H^m(\mathbb{R}^d)$  for all  $m \geq 1$ , so  $u(\cdot, t) \in C^\infty(\mathbb{R}^d)$  by Sobolev embedding (see below). This argument can also be used to show that  $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$  (exercise)

**Theorem 6.7** (Sobolev embedding) If  $m > \frac{d}{2}$ , then  $H^m(\mathbb{R}^d) \subseteq (C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ .

*Proof.* We write for all  $u \in H^m(\mathbb{R}^d)$ :

$$\hat{u}(k) = \underbrace{\hat{u}(k)(1 + |2\pi k|^m)}_{\in L^2 \text{ as } u \in H^m} \underbrace{\frac{1}{1 + |2\pi k|^m}}_{\in L^2 \text{ as } m > \frac{d}{2}}$$

This implies  $\hat{u}(k) \in L^1(\mathbb{R}^d)$  and finally  $u = (\hat{u})^\vee \in (C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ . ■

**Exercise 6.8** (E 11.1) Let  $g \in L^2(\mathbb{R}^d)$ ,

$$u(x, t) = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy, \quad \Phi(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$$

be the fundamental solution of the heat equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \forall (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, t) \rightarrow g(x) & \text{as } t \rightarrow 0. \end{cases}$$

Prove that

- a)  $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$ .
- b)  $\|u(\cdot, t) - g\|_{L^2(\mathbb{R}^d)} \xrightarrow{t \rightarrow 0^+} 0$
- c) If  $g \in H^1(\mathbb{R}^d)$ , then  $\|u(\cdot, t) - g\|_{L^2(\mathbb{R}^d)} \leq C\sqrt{t}$  as  $t \rightarrow 0^+$ .

*Solution.* a) We prove for all  $t > 0$ :

$$u(x, t) \in \bigcap_{m \geq 1} H^m(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$$

We use the Fourier transform:

$$\hat{\Phi}(k, t) = e^{-t|2\pi k|^2}$$

Recall  $\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$ . From this we get  $\widehat{e^{-\pi\lambda|x|^2}} = \lambda^{-\frac{d}{2}} e^{-\frac{\pi|k|^2}{\lambda}}$ . Then:

$$\widehat{e^{-\frac{|x|^2}{4t}}} = e^{-\pi\frac{1}{4\pi t}|x|^2} = \left(\frac{1}{4\pi t}\right)^{-\frac{d}{2}} e^{-\pi|k|^2 4\pi t} = (4\pi t)^{\frac{d}{2}} e^{-t|2\pi k|^2}$$

Hence:

$$\hat{u}(k, t) = \hat{\Phi}(k, t)\hat{g}(k) = e^{-t|2\pi k|^2}\hat{g}(k) \in L^1(\mathbb{R}^d, dk) \quad \forall t > 0$$

This implies:

$$u(x, t) = \int_{\mathbb{R}^d} e^{-t|2\pi k|^2} \hat{g}(k) e^{2\pi i k x} dk \quad \forall (x, t) \in \mathbb{R}^d \times (0, \infty)$$

Consequently:

$$D_x^\alpha u(x, t) = \int_{\mathbb{R}^d} \underbrace{e^{-t|2\pi k|^2} \hat{g}(k) (2\pi i k)^\alpha}_{L^1(\mathbb{R}^d, dk)} e^{2\pi i k x} dk \in C(\mathbb{R}^d, (0, \infty))$$

$$D_t^\alpha u(x, t) = \int_{\mathbb{R}^d} (-|2\pi k|^2)^\alpha e^{-t|2\pi k|^2} \hat{g}(k) e^{2\pi i k x} dk \in C(\mathbb{R}^d, (0, \infty))$$

Also:

$$\begin{aligned} & \partial_t u - \Delta_x u \\ &= \int_{\mathbb{R}^d} -|2\pi k|^2 e^{-t|2\pi k|^2} \hat{g}(k) e^{2\pi i k x} dk + \int_{\mathbb{R}^d} e^{-t|2\pi k|^2} \hat{g}(k) |2\pi i k|^2 e^{2\pi i k x} dk \\ &= 0 \end{aligned}$$

b) Finally:

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x, t) - g(x)|^2 dx &= \int_{\mathbb{R}^d} |\hat{u}(k, t) - \hat{g}(k)|^2 dk \\ &= \int_{\mathbb{R}^d} \underbrace{|e^{-t|2\pi k|^2} - 1|^2}_{\in [0, 1]} \underbrace{|\hat{g}(k)|^2}_{\in L^1(\mathbb{R}^d)} dk \xrightarrow{t \rightarrow 0} 0 \end{aligned}$$

by dominated convergence. Now,

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x, t)|^2 dx &= \int_{\mathbb{R}^d} |\hat{u}(k, t)|^2 dk \\ &= \int_{\mathbb{R}^d} \underbrace{e^{-2t|2\pi k|^2}}_{\in [0, 1] \text{ and } \xrightarrow{t \rightarrow 0} 0} |\hat{g}(k)|^2 dk \xrightarrow{t \rightarrow \infty} 0 \end{aligned}$$

c) Assume  $g \in H^1(\mathbb{R}^d) \Leftrightarrow \int_{\mathbb{R}^d} (1 + |2\pi k|^2) |\hat{g}(k)|^2 dk < \infty$ . We claim for all  $s \geq 0$  that  $|1 - e^{-s}| \leq \min(1, Cs) \leq C\sqrt{s}$ : We have that  $s \mapsto \left| \frac{1 - e^{-s}}{s} \right|$  is bounded and continuous in  $[0, 1]$  as  $\left| \frac{1 - e^{-s}}{s} \right| \rightarrow 1$ , so  $\frac{1 - e^{-s}}{s} \leq C$  for all  $s \in [0, 1]$ .

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x, t) - g(x)|^2 dx &= \int_{\mathbb{R}^d} \underbrace{\left| 1 - e^{-t|2\pi k|^2} \right|^2}_{\leq C(t|2\pi k|^2)} |\hat{g}(k)|^2 dk \\ &\leq C \int_{\mathbb{R}^d} t|2\pi k|^2 |\hat{g}(k)|^2 dk \\ &\leq Ct \|g\|_{H^1}^2 \quad \forall t > 0 \end{aligned}$$

■



Step 1: Spectral problem:

$$\begin{cases} -\Delta u_n = \lambda_n u_n & \text{in } \Omega \\ u_n|_{\partial\Omega} = 0 \end{cases}$$

**Lemma 6.9** There is a  $\lambda_n > 0$ ,  $\lambda_n \xrightarrow{n \rightarrow \infty} \infty$  and an orthonormal family  $\{u_n\} \subseteq L^2(\Omega)$  s.t.  $u_n \in H_0^1(\Omega) \cap C^\infty(\Omega)$  solving this eigenvalue equation.

Step 2:

$$\begin{cases} \partial_t - \Delta_x u = 0 \\ u(x, 0) = g(x) \end{cases} \Rightarrow \begin{cases} \partial_t \langle u_n, u \rangle_{L^2(\Omega)} = \langle u_n, \Delta_x u \rangle = \langle \Delta_x u_n, u \rangle = -\lambda_n \langle u_n, u \rangle \\ \langle u_n, u \rangle_{t=0} = \langle u_n, g \rangle \end{cases}$$

$$\Rightarrow \langle u_n, u \rangle = e^{-t\lambda_n} \langle u_n, g \rangle \quad \forall t > 0, \forall n = 1, 2, \dots$$

$$\Rightarrow u = \sum_{n=0}^{\infty} \langle u_n, u \rangle u_n = - \sum e^{-t\lambda_n} \langle u_n, g \rangle u_n$$

**Example 6.10**  $\Omega = (0, 1)$ ,

$$\begin{cases} -u_n'' = \lambda_n u_n & \text{in } (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

has solution

$$\begin{cases} u_n(x) = \sqrt{2} \sin(\pi n x) & n = 1, 2, \dots \\ \lambda_n = (\pi n)^2 \end{cases}$$

has a solution:

$$u(x, t) = \sum_{n=1}^{\infty} e^{-t\lambda_n} \underbrace{\langle u_n, g \rangle}_{g_n} u_n(x) = \sum_{n=1}^{\infty} e^{-t\pi^2 n^2} g_n \sin(\pi n x),$$

$$\int_0^1 \sin(n\pi x)^2 dx = \frac{1}{2} \quad \forall n > 1$$

$$g_n = \sqrt{2} \langle u_n, g \rangle = 2 \int_0^1 \sin(\pi n x) g(x) dx$$

**Exercise 6.11** (E 11.2) Consider the heat equation in a bounded domain

$$\begin{cases} \partial_t u(x, t) = \Delta_x u(x, t) & \forall x \in \Omega, t > 0 \\ u(x, t) = 0 & \forall x \in \partial\Omega, t > 0 \\ u(x, 0) = g(x) & \forall x \in \Omega \end{cases}$$

Let us focus on the simplest case  $\Omega = (0, 1)$ . Prove that for every  $g \in C_c^1(0, 1)$ , the function

$$u(x, t) = \sum_{n=1}^{\infty} g_n e^{-t\pi^2 n^2} \sin(n\pi x), \quad g_n = 2 \int_0^1 g(y) \sin(n\pi y) dy$$

is a classical solution to the above heat equation.

*Solution.* Direct proof of heat equation.  $g \in C_c^1(0, 1) \subseteq H_0^1(0, 1)$ ,  $\Rightarrow \sum_n \pi^2 n^2 |g_n|^2 = c \|g'\|_{L^2(0,1)}^2 < \infty$ , so  $\sum_n |g_n| < \infty$ .

$$u(x, 0) = \underbrace{\sum_{n=1}^{\infty} g_n \sin(\pi n x)}_{\in C[0,1]} = g(x) \quad \forall x \in [0, 1]$$

From  $u(x, t) = \sum_{n=1}^{\infty} e^{-t\pi^2 n^2} g_n \sin(\pi n x)$  we get

$$\begin{cases} \partial_t u(x, t) = \sum_{n=1}^{\infty} (-n^2 \pi^2) e^{-t\pi^2 n^2} g_n \sin(\pi n x) & \forall t > 0, \forall x \in (0, 1) \\ \Delta_x u(x, t) = \sum_{n=1}^{\infty} e^{-t\pi^2 n^2} g_n [-(\pi n)^2] \sin(\pi n x) & \forall t > 0, \forall x \in (0, 1) \end{cases}$$

So  $\partial_t u - \Delta_x u = 0$  for all  $t > 0, x \in (0, 1)$  ■

**Exercise 6.12** (E 11.3) Let  $g(t) = e^{-\frac{1}{t^2}}$  and denote  $g^{(n)}(t)$  the  $n$ -th derivative of  $g$ . Define

$$u(x, t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \quad \forall x \in \mathbb{R}, t > 0$$

Prove that  $u$  is a classical solution to the heat equation

$$\begin{cases} \partial_t u(x, t) = \Delta_x u(x, t) & \forall x \in \mathbb{R}, t > 0 \\ \lim_{t \rightarrow 0} u(x, t) = 0 & \forall x \in \mathbb{R} \end{cases}$$

*Solution.* Formally:

$$\begin{cases} \partial_t u = \sum_{n=0}^{\infty} \frac{g^{(n+1)}(t)}{(2n)!} x^{2n} \\ -\Delta_x u = \sum_{n=1}^{\infty} \frac{g^{(n)}(t)}{(2n)!} (2n)(2n-1) x^{2n-2} = \sum_{n=1}^{\infty} \frac{g^{(n)}(t)}{(2n-2)!} x^{2n-2} = \sum_{m=0}^{\infty} \frac{g^{(m+1)}(t)}{(2m)!} x^{2m} \end{cases}$$

This implies  $\partial_t u = \Delta_x u$  (if the series are convergent)  $(x, t) \in B \times [\epsilon, \frac{1}{\epsilon}]$  for  $B \subset \mathbb{R}$  bounded,  $\epsilon > 0$ . Also

$$\begin{aligned} g(t) &= e^{-\frac{1}{t^2}} \xrightarrow{t \rightarrow 0^+} e^{-\infty} = 0 \\ g'(t) &= e^{-\frac{1}{t^2}} \left( \frac{2}{t^3} \right) \xrightarrow{t \rightarrow 0^+} 0 \\ g''(t) &= e^{-\frac{1}{t^2}} \left( -\frac{3!}{t^4} + \frac{2}{t^3} \right) \xrightarrow{t \rightarrow 0^+} 0 \\ g'''(t) &= e^{-\frac{1}{t^2}} \left( \frac{4!}{t^5} - \frac{3!}{t^4} + \frac{2}{t^3} \right) \end{aligned}$$

Let's prove the convergence of the series:

$$u(x, t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}$$

converges absolutely for  $|x| \leq C, t \in [\epsilon, \frac{1}{\epsilon}], \epsilon > 0$ . By induction,

$$g^{(n)}(t) = e^{-\frac{1}{t^2}} \underbrace{\left( \frac{(n+1)!}{t^{n+2}} - \frac{n!}{t^{n+1}} + \frac{(n+1)!}{t^n} - \dots \right)}_{\text{pol in } (\frac{1}{t}), \text{ all cos bounded by } (n+1)} (-1)^{n-1}$$

This implies

$$|g^{(n)}(t)| \leq e^{-\frac{1}{t^2}} [(n+2)!] \left( \frac{1}{t^{n+2}} + 1 \right), \quad \frac{1}{t^s} \leq \left( \frac{1}{t^{n+2}} + 1 \right) \forall s = 0, 1, \dots, n+2$$

Thus

$$\sum_{n \geq 0} \left| \frac{g^{(n)}}{(2n)!} x^{2n} \right| \leq \sum_{n \geq 0} e^{-\frac{1}{t^2}} \frac{(n+2)!}{(2n)!} \left( \frac{1}{t^{n+2}} + 1 \right) x^{2n} \quad (1)$$

$$\begin{aligned} \sum_{n \geq 0} \frac{(n+2)!}{(2n)!} x^{2n} &= \sum_{n \geq 0} \frac{1}{(n+3)(n+4) \cdots (2n)} \\ &\leq \sum_{n \geq 0} \frac{1}{n^{n-2}} x^{2n} \\ &\leq \sum_{n \geq M} + \sum_{n \geq M} \frac{1}{M^{n-2}} x^{2n} \\ &= M^2 \sum_n \left( \frac{x^2}{M} \right)^n \\ &\leq m^2 \frac{1}{1 - \left( \frac{x^2}{M} \right)} \end{aligned}$$

$$(2) \quad t \in [\epsilon, \frac{1}{\epsilon}], \text{ so } \frac{1}{t} \leq \frac{1}{\epsilon}, \text{ so } \frac{1}{t^{n+2}} \leq \frac{1}{\epsilon^{n+2}} \longrightarrow \sum_{n \geq 0} \frac{(n+2)!}{(2n)!} \frac{1}{t^{n+2}} x^{2n} \leq \sum_{n \geq 0} \frac{1}{n^{n-2}} \frac{1}{\epsilon^{n-2}} x^{2n}$$

■

**Remark 6.13**  $|u(x, t)| \leq \exp\left(\frac{cx^2}{t}\right) \rightsquigarrow$  unphysical solution. Violates  $|u(x, t)| \leq Ce^{C|x|^2}$  for all  $\forall(x, t) \in \mathbb{R} \times [0, T]$

**Exercise 6.14** (Bonus 10) Consider

$$u(x, t) = \int_{\mathbb{R}^d} \Phi(x - y, t) g(y) dy$$

where  $\Phi(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{x^2}{4t}}$ . Assume  $g \in C_c^\infty(\mathbb{R}^d)$ . Prove or disprove that

$$\|u(\cdot, t) - g\|_{L^2(\mathbb{R}^d)} \leq C_n t^n$$

as  $t \rightarrow 0^+$  for all  $n = 1, 2, \dots$

## 6.2 Maximum Principle

Recall the Poisson equation  $-\Delta u \leq 0$  in  $\Omega \subseteq \mathbb{R}^d$  open, bounded. Then

$$\sup_{\bar{\Omega}} u(x) = \sup_{\partial\Omega} u(x).$$

**Theorem 6.15** (Maximum principle for bounded sets) Let  $\Omega \subseteq \mathbb{R}^d$  be open and bounded. Let  $T > 0$  and define

$$\begin{aligned} \Omega_T &= \Omega \times (0, T), \\ \partial^* \Omega_T &= (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times [0, T]) \end{aligned}$$

If  $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$  solves  $\partial_t u - \Delta_x u \leq 0$  in  $\Omega_T$ , then

$$\max_{\bar{\Omega}_T} u = \max_{\partial^* \Omega_T} u.$$

*Proof.* We will use Hopf's argument which is simpler than the mean-value theorem (there exists a mean-value theorem for heat equation, but it is complicated and we will not discuss it). Firstly, to illustrate the principle, we prove the maximum principle for the Poisson Equation: Assume  $u \in C^2(\Omega) \cap C(\bar{\Omega})$

Step 1) Assume  $\Delta u > 0$  in  $\Omega$ . Since  $\bar{\Omega}$  is compact, there is a  $x_0 \in \bar{\Omega}$  s.t.  $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$ . We prove that  $x_0 \in \partial\Omega$ . In fact, if  $x_0 \in \Omega$ , then since  $x_0$  is a (local) maximizer of  $u$  in  $\Omega$ , we have  $\Delta u(x_0) \leq 0$ , which contradicts to the assumption that  $\Delta u > 0$  in  $\Omega$ . Thus  $x_0 \in \partial\Omega$ , and hence

$$\max_{x \in \bar{\Omega}} u(x) = u(x_0) \leq \max_{x \in \partial\Omega} u(x).$$

Step 2) Now assume  $\Delta u \geq 0$  in  $\Omega$ . Define

$$u_\epsilon(x) = u(x) + \epsilon|x|^2, \quad \epsilon > 0.$$

Then,  $\Delta u_\epsilon > 0$  in  $\Omega$ , hence by Step 1 and

$$u \leq u_\epsilon \leq u + \epsilon \sup_{x \in \bar{\Omega}} |x|^2$$

we have

$$\begin{aligned} \max_{x \in \bar{\Omega}} u(x) &\leq \max_{x \in \bar{\Omega}} u_\epsilon(x) \leq \max_{x \in \partial\Omega} u_\epsilon(x) \\ &\leq \max_{x \in \partial\Omega} u(x) + \epsilon \left( \sup_{x \in \bar{\Omega}} |x|^2 \right) \xrightarrow{\epsilon \rightarrow 0} \max_{x \in \partial\Omega} u(x) \end{aligned}$$

Proof for the heat equation:

Step 1) Assume  $u \in C_1^2(\Omega \times (0, T]) \cap C(\bar{\Omega} \times [0, T])$  and

$$\partial_t u - \Delta_x u < 0$$

in  $\Omega \times (0, T]$ . Then, because of compactness, there is  $(x_0, t_0) \in \bar{\Omega} \times [0, T]$  s.t.

$$u(x_0, t_0) = \max_{(x, t) \in \bar{\Omega} \times [0, T]} u(x, t).$$

We prove that  $(x_0, t_0) \in \partial^* \Omega_T$ . Assume by contradiction that  $(x_0, t_0) \notin \partial^* \Omega_T$ , then  $x_0 \in \Omega$  and  $t_0 \in (0, T]$ . Since  $x \mapsto u(x, t_0)$  has a (local) maximizer  $x_0 \in \Omega$  we have that  $\Delta_x u(x_0, t_0) \leq 0$ . Since  $t \mapsto u(x_0, t)$  has a (local) maximizer  $t_0 \in (0, T]$  we have that  $\partial_t u(x_0, t_0) \geq 0$ . This implies:

$$(\partial_t u - \Delta_x u)(x_0, t_0) \geq 0$$

which is a contradiction to the assumption. Thus  $(x_0, t_0) \in \partial^* \Omega_T$ , i.e.  $\max_{\bar{\Omega}_T} u = \max_{\partial^* \Omega_T} u$ .

Step 2) Assume  $u \in C_1^2(\Omega \times (0, T)) \cap C(\bar{\Omega} \times [0, T])$  and

$$\partial_t u - \Delta_x u \leq 0 \quad \text{in } \Omega \times (0, T).$$

Let  $\tilde{T} \in (0, T)$  and for  $\epsilon > 0$  :

$$u_\epsilon(x, t) = u(x, t) + \epsilon|x|^2.$$

Then:  $u_\epsilon \in C_1^2(\Omega \times (0, T']) \cap C(\bar{\Omega} \times [0, \tilde{T}])$  and  $\partial_t u_\epsilon - \Delta_x u_\epsilon < 0$  in  $\Omega \times (0, \tilde{T}]$ .  
By Step 1:

$$\begin{array}{ccc} & \max_{\Omega_{\tilde{T}}} u_\epsilon \leq \max_{\partial^* \Omega_{\tilde{T}}} u_\epsilon & \\ \xRightarrow{\epsilon \rightarrow 0} & \max_{\Omega_{\tilde{T}}} u \leq \max_{\partial^* \Omega_{\tilde{T}}} u & \\ \xRightarrow{\tilde{T} \rightarrow T} & \max_{\Omega_T} u \leq \max_{\partial^* \Omega_T} u & \blacksquare \end{array}$$

**Theorem 6.16** (Maximum principle for  $\Omega = \mathbb{R}^d$ ) Let  $\Omega_T = \mathbb{R}^d \times (0, T)$ ,  $\bar{\Omega}_T = \mathbb{R}^d \times [0, T]$ . Let  $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$  such that

- $\partial_t u - \Delta_x u \leq 0$  in  $\Omega_T$
- $u(x, t) \leq M e^{M|x|^2}$  for all  $(x, t) \in \bar{\Omega}_T$

Then

$$\sup_{(x,t) \in \bar{\Omega}_T} u(x, t) = \sup_{x \in \mathbb{R}^d} u(x, 0).$$

*Proof.*

Step 1: For all  $y \in \mathbb{R}^d$  and  $\epsilon > 0$  define

$$v(x, t) = u(x, t) - \frac{\epsilon}{(T + \epsilon - t)^{\frac{d}{2}}} \exp\left(\frac{|x - y|^2}{4(T + \epsilon - t)}\right)$$

This implies

$$\partial_t v - \Delta_x v = \partial_t u - \Delta_x u \leq 0$$

in  $\Omega_T$ . For  $U = B(y, r)$ ,  $U_T = U \times (0, T)$ ,  $\bar{U}_T = \bar{U} \times [0, T]$ ,  $\partial^* U_T = (U \times \{0\}) \cup (\partial U \times [0, T])$ , by the maximum principle for  $U$  bounded we have

$$\max_{\bar{U}_T} v \leq \max_{\partial^* U_T} v.$$

Let us bound  $\max_{\partial^* U_T} v$ .

- On  $U \times \{0\}$  we use  $v \leq u$  and hence

$$\max_{x \in \bar{U}} v(x, 0) \leq \max_{x \in \bar{U}} u(x, 0) \leq \max_{x \in \mathbb{R}^d} u(x, 0).$$

- On  $\partial U \times [0, T]$  we use  $|x - y| = r \Rightarrow |x| \leq |y| + r$ .

$$\begin{aligned} v(x, t) &= u(x, t) - \frac{\epsilon}{(T + \epsilon - t)^{\frac{d}{2}}} \exp\left(\frac{|x - y|^2}{4(T + \epsilon - t)}\right) \\ &\leq M e^{M(|y|+r)^2} - \frac{\epsilon}{(T + \epsilon)^{\frac{d}{2}}} \exp\left(\frac{r^2}{4(T + \epsilon)}\right) \xrightarrow{r \rightarrow \infty} -\infty \end{aligned}$$

if  $M < \frac{1}{4(T+\epsilon)}$ . In particular, we can choose  $r$  large s.t.

$$\max_{\substack{x \in \partial U \\ t \in [0, T]}} v(x, t) \leq \max_{x \in \mathbb{R}^d} u(x, 0).$$

In summary, if  $M < \frac{1}{4(T+\epsilon)}$ , then:

$$u(y, t) - \frac{\epsilon}{(T + \epsilon - t)^{\frac{d}{2}}} = v(y, t) \leq \max_{\bar{U}_T} v \leq \max_{x \in \mathbb{R}^d} u(x, 0)$$

This holds for all  $(y, t) \in \mathbb{R}^d \times [0, T]$ . Thus,

$$\max_{\mathbb{R}^d \times [0, T]} u \leq \frac{\epsilon}{(T + \epsilon - t)^{\frac{d}{2}}} + \max_{x \in \mathbb{R}^d} u(x, 0)$$

Taking  $\epsilon \rightarrow 0$  we conclude that if  $M < \frac{1}{4T}$ ,

$$\max_{\mathbb{R}^d \times [0, T]} u \leq \max_{x \in \mathbb{R}^d} u(x, 0)$$

Step 2: For general  $T$ , we denote  $T_1 = \frac{T}{N}$ ,  $N \in \mathbb{N}$  s.t.  $M < \frac{4}{T_1}$ . Then by step 1:

$$\begin{aligned} \max_{\mathbb{R}^d \times [0, T_1]} u &\leq \max_{x \in \mathbb{R}^d} u(x, 0) \\ \max_{\mathbb{R}^d \times [T_1, 2T_1]} u &\leq \max_{x \in \mathbb{R}^d} u(x, T_1) \leq \max_{x \in \mathbb{R}^d} u(x, 0) \\ &\vdots \\ \max_{\mathbb{R}^d \times [(N-1)T_1, NT_1]} u &\leq \max_{x \in \mathbb{R}^d} u(x, (N-1)T_1) \leq \max_{x \in \mathbb{R}^d} u(x, 0) \\ \rightsquigarrow \max_{\mathbb{R}^d \times [0, T]} u &\leq \max_{x \in \mathbb{R}^d} u(x, 0) \end{aligned} \quad \blacksquare$$

**Remark 6.17** The condition  $u \leq Me^{M|x|^2}$  is necessary, otherwise there are solutions  $u \neq 0$  s.t.  $u(x, 0) = 0$

**Theorem 6.18** (Uniqueness) If  $u \in C_1^2(\mathbb{R}^d \times (0, T)) \cap C(\mathbb{R}^d \times [0, T])$  and

$$\begin{aligned} u(x, t) &\leq Me^{M|x|^2} && \text{in } \mathbb{R}^d \times [0, T], \\ \partial_t u - \Delta_x u &= 0 && \text{in } \mathbb{R}^d \times (0, T), \\ u(x, 0) &= 0 && \text{in } \mathbb{R}^d \end{aligned}$$

Then  $u = 0$  in  $\mathbb{R}^d \times [0, T]$ .

*Proof.* Use the maximum principle for  $u$  and  $-u$ .  $\blacksquare$

**Remark 6.19** If  $u(\cdot, t) \in L^2(\mathbb{R}^d)$ , the proof of uniqueness can be done without the maximum principle. Heuristically:

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(x, t)|^2 dt = 2 \int_{\mathbb{R}^d} (\partial_t u) u dx = 2 \int_{\mathbb{R}^d} \Delta_x u u dx = -2 \int_{\mathbb{R}^d} |\nabla_x u|^2 dx \leq 0$$

This implies

$$e(t) := \int_{\mathbb{R}^d} |u(x, t)|^2 dx$$

is decreasing. Hence, if  $e(0) = 0$ , then  $e(t) = 0$  for all  $t \geq 0$ . This argument will be helpful below for the heat backward equation.

**Remark 6.20** The heat equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(t=0) = g \end{cases}$$

is a well-posed problem:

- Existence
- Uniqueness
- Stability (solution depends continuously on data)

For the latter issue, by the maximum principle we have

$$\|u(\cdot, t)\|_{L^\infty} \leq \|u(\cdot, 0)\|_{L^\infty} \quad \forall t$$

or in the  $L^2$ -situation:

$$\|u(\cdot, t)\|_{L^2} \leq \|u(\cdot, 0)\|_{L^2} \quad \forall t$$

On the other hand, the heat backward equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(t=T) = g \end{cases}$$

is *not* well-posed.

- Non-Existence: In general, the existence requires some special property on  $g$ , e.g.  $g$  is very smooth (only  $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  or  $g \in L^2(\mathbb{R}^d)$  is not enough)
- Uniqueness: On the other hand, the uniqueness still holds.

**Lemma 6.21** If  $e \in C^2(0, T)$ ,  $e(t) \geq 0$ ,  $e'(t) \leq 0$ ,  $e''(t) \geq 0$  and  $|e'(t)|^2 \leq e(t)e''(t)$  for  $t \in [0, T]$  and  $e(T) = 0$ , then  $e \equiv 0$ .

*Proof.* Since  $e$  is monotonically decreasing and  $e(T) = 0$  there is a  $t_0 \in [0, T]$  s.t.  $e(t_0) = 0$  and  $e(t) > 0$  if  $t \leq t_0$ . We need to prove that  $t_0 = 0$ . Assume by contradiction  $0 < t_0 \leq T$ , then for  $t \in (0, t_0)$  define  $f(t) := \log e(t)$ . Then

$$\begin{aligned} f'(t) &= \frac{e'(t)}{e(t)} \\ \Rightarrow f''(t) &= \frac{e''(t)e(t) - |e'(t)|^2}{e(t)^2} \geq 0 \end{aligned}$$

This means that  $f$  is convex, so for all  $t_1, t_2 \in (0, t_0)$  and  $\tau \in (0, 1)$ :

$$\begin{aligned} f(\tau t_1 + (1-\tau)t_2) &\leq \tau f(t_1) + (1-\tau)f(t_2) \\ \Rightarrow e(\tau t_1 + (1-\tau)t_2) &\leq e(t_1)^\tau e(t_2)^{1-\tau} \end{aligned}$$

Now,  $e(\tau t_1 + (1-\tau)t_2) \xrightarrow{t_2 \rightarrow t_0} 0$  and  $\tau \rightarrow 1$  implies  $e(t_1) = 0$  for all  $t_1 \in (0, t_0)$  which is a contradiction. ■

**Theorem 6.22** If  $u \in C_1^2(\mathbb{R}^d \times [0, T]) \cap C^1(H^1(\mathbb{R}^d) \times [0, T])$  and

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ u(x, T) = 0 \end{cases}$$

Then  $u = 0$  in  $\mathbb{R}^d \times [0, T]$ .

*Proof.* Recall

$$e(t) = \int_{\mathbb{R}^d} |u(x, t)|^2 dx.$$

Then,

$$\begin{aligned} e'(t) &= 2 \int_{\mathbb{R}^d} u \partial_t u dx = 2 \int_{\mathbb{R}^d} u \Delta_x u dx = -2 \int_{\mathbb{R}^d} |\nabla_x u|^2 dx \\ e''(t) &= -4 \int_{\mathbb{R}^d} \nabla_x u \nabla_x (\partial_t u) = 4 \int_{\mathbb{R}^d} \Delta_x u \partial_t u dx = 4 \int_{\mathbb{R}^d} |\Delta_x u|^2 dx \geq 0 \end{aligned}$$

and hence

$$|e'(t)|^2 = 4 \left| \int_{\mathbb{R}^d} u \Delta_x u dx \right|^2 \leq 4 \left( \int_{\mathbb{R}^d} |u|^2 dx \right) \left( \int_{\mathbb{R}^d} |\Delta_x u|^2 dx \right) = e(t) e''(t)$$

Then the statement follows with lemma 6.21. ■

Some remarks about the eat equation in unbounded domains:

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = 0 & \text{(i.e. } \lim_{t \rightarrow 0} u(x, t) = 0 \forall x \in \mathbb{R}^d) \end{cases}$$

There is a classical solution  $0 \neq u \in C^1(\mathbb{R}^d \times (0, \infty))$ . An example is

$$u(x, t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \quad g(t) = e^{-\frac{1}{t^2}}$$

(s.t.  $g \rightarrow 0$  as  $t \rightarrow 0$ ). Note

$$\begin{aligned} g(t) &= e^{-\frac{1}{t^2}}, \\ g'(t) &= \frac{2}{t^3} g(t) \\ g''(t) &= \left( \frac{2}{t^3} \right)' g(t) + \frac{2}{t^3} \frac{2}{t^3} g(t) \\ g^{(n)}(t) &= P_n \left( \frac{1}{t} \right) g(t) \end{aligned}$$

where

$$\begin{cases} P_0 = 1 \\ P_{n+1} \left( \frac{1}{t} \right) = \left( P_n \left( \frac{1}{t} \right) \right)' + \left( \frac{2}{t^3} \right) P_n \left( \frac{1}{t} \right) = A_1 P_n + A_2 P_n, \\ P_{n+1} = (A_1 + A_2) P_n = (A_1 + A_2)(A_1 + A_2) P_{n-1} = \end{cases} \begin{cases} A_1 = \partial_t \\ A_2 = \frac{2}{t^3} \end{cases}$$

This implies:

$$P_n = (A_1 + A_2)^n P_0 = \sum_{\sigma \in \{1, 2\}^n} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)} P_0$$

$$A_1 \left( \frac{\alpha}{t^s} \right) = \frac{-s\alpha}{t^{s+1}} \rightarrow A_1$$

Multiple coefficients by a factors and + power by 1

$$A_2 \left( \frac{\alpha}{t^s} \right) = \frac{2\alpha}{t^{s+3}} \rightarrow A_2$$



Mul Cof by a factor 2 and + power by 3

$$\left| \underbrace{A_{\sigma(1)} \cdots A_{\sigma(n)}, 1}_{k \text{ times } A_2, n-k \text{ times } A_1} \right| \leq \frac{2^k}{t^{3k}} \leq \frac{2^k}{t^{3k}} \frac{(3n)^{n-k}}{t^{n-k}} = \frac{2^k (3n)^{n-k}}{t^{n+2k}}$$

This implies

$$\left| P_n \left( \frac{1}{t} \right) \right| \leq \max_{0 \leq k \leq n} \frac{2^n 2^k (3n)^{n-k}}{t^{n+2k}}$$

Thus:

$$\begin{aligned} \sum_n \left| \frac{g^{(n)}(t)}{(2n)!} x^{2n} \right| &\leq \sum_n \max_{0 \leq k \leq n} \frac{2^n 2^k (3n)^{n-k}}{t^{n+2k} (2n)!} \frac{e^{-\frac{1}{t^2}}}{1} x^{2n} \\ &\leq \sum_n \max \frac{2^n 2^k (3n)^{n-k}}{t^{n+2k} (2n)!} (k!) (2t^2)^k e^{-\frac{1}{2t^2}} x^{2n} \\ &= \sum_n \frac{2^n 2^k 2^k (3n)^{n-k} (k!)}{(2n)! t^n} e^{-\frac{1}{2t^2}} x^{2n} \\ &\leq \sum_n \frac{(c_n)^n}{(2n)! t^n} e^{-\frac{1}{2t^2}} x^{2n} \\ &\leq \sum_n \frac{c^n}{n! t^n} e^{-\frac{1}{2t^2}} x^{2n} \\ &\leq \sum_n e^{\frac{cx^2}{t} - \frac{1}{2t^2}} \end{aligned}$$

Where we used that

$$e^s = \sum_k \frac{s^k}{k!} \geq \frac{s^k}{k!}$$

for all  $s \geq 0$  implies

$$e^{-\frac{1}{2t^2}} = \frac{1}{e^{\frac{1}{2t^2}}} \leq \frac{1}{\left(\frac{1}{2t^2}\right)^{\frac{1}{k!}}} = k! (2t^2)^k.$$

We conclude:

- $u(x, t)$  is well-defined,  $x \in \mathbb{R}^d$ ,  $t > 0$  real? to heat equation.
- $u(x, t) \rightarrow 0$  as  $t \rightarrow 0$  for all  $x \in \mathbb{R}^d$ .

**Exercise 6.23** (E 12.1) Let  $\Omega \subseteq \mathbb{R}^d$  be open and  $u \in C^2(\Omega)$ . Assume that  $x_0 \in \Omega$  is a local maximizer of  $u$ , namely there exists some  $r > 0$  such that  $u(x_0) \geq u(x)$  for all  $x \in B_r(x_0) \subseteq \Omega$ .

- (a) Prove that the Hessian matrix  $H = (D^\alpha u(x_0))_{|\alpha|=2}$  is negative semi-definite, namely

$$yHy \leq 0$$

for all  $y \in \mathbb{R}^d$ .

- (b) Prove that  $\Delta u(x_0) \leq 0$

Hint: Recall that we used (b) for the maximum principle by Hopf's method.

*Solution.* (a) In 1D this is obvious. If  $x_0$  is a local minimizer of  $u$ , then  $u'(x_0) = 0, u''(x_0) \leq 0$  (Taylor expansion).

In  $d$  dimensions:

$$\phi(t) = u(x_0 + t\xi) \quad \xi \in \mathbb{R}^d, t \in \mathbb{R}, |t| \text{ small}$$

So 0 is a local maximizer of  $\phi$ . This implies

$$0 = \phi'(0) = \nabla u(x_0)\xi \quad \forall \xi \in \mathbb{R}^d \Rightarrow H \leq 0$$

$$\begin{aligned} \phi''(0) &= \lim_{t \rightarrow 0} \frac{\phi'(t) - \phi'(0)}{t} = \lim_{t \rightarrow 0} \frac{(\nabla u(x_0 + t\xi) - \nabla u(x_0))\xi}{t} \\ &= \lim_{t \rightarrow 0} \sum_{i=1}^d \frac{(\partial_i u(x_0 + t\xi) - \partial_i u(x_0))\xi_i}{t} = \sum_{i=1}^d \sum_{j=1}^d \partial_j \partial_i u(x_0) \xi_j \xi_i = \langle \xi, H\xi \rangle, \end{aligned}$$

$$H = (\partial_i \partial_j u(x_0))_{i,j=1}^d.$$

(b) Consequently

$$\Delta u(x_0) = \sum_{i=1}^d \partial_i \partial_i u(x_0) = \text{Tr}(H) \leq 0 \quad \blacksquare$$

**Exercise 6.24** (E 12.2) Let  $\Omega \subseteq \mathbb{R}^d$  be open and bounded. We prove the maximum principle for a general elliptic operator

$$Lu(x) = \sum_{i,j=1}^d a_{ij}(x) \partial_i \partial_j u(x) + \sum_{i=1}^d b_i(x) \partial_i u(x),$$

$a_{ij}, b_i \in C(\bar{\Omega})$ ,  $A(x) = (a_{ij}(x))_{i,j=1}^d \geq \mathbb{1}$  (as matrices). Prove that if  $Lu(x) \geq 0$  for all  $x \in \Omega$  and  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ , then

$$\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

*Solution.*

Step 1: Assume  $Lu(x) > 0$  for all  $x \in \Omega$ : Since  $u \in C(\bar{\Omega})$  there is a  $x_0 \in \bar{\Omega}$  s.t.

$$u(x_0) = \max_{x \in \bar{\Omega}} u(x).$$

We prove  $x_0 \in \partial\Omega$ . Assume by contradiction that  $x_0 \notin \partial\Omega$ , so  $x_0 \in \Omega$  is a local maximizer. We prove  $Lu(x_0) \leq 0$ . Note:

$$\begin{aligned} Lu(x_0) &= \sum_{i,j=1}^d a_{ij}(x_0) \partial_i \partial_j u(x_0) + \sum_{i=1}^d b_i(x_0) \partial_i u(x_0) \\ &= \text{Tr}[A(x_0)H(x_0)] + B(x_0) \underbrace{\nabla u(x_0)}_{=0} \leq 0 \quad \nexists \end{aligned}$$

$$A(x_0) = (a_{ij}(x_0))_{i,j=1}^d, B(x_0) = (b_i(x_0))_{i=1}^d, \text{ where } \text{Tr}[AH] = \sum_i (AH)_{ii} = \sum_i \sum_j A_{ij} H_{ji}$$

General fact: If  $A \geq 0, B \geq 0$  (matrices), then  $\text{Tr}(AB) \geq 0$ .

$$\bullet A = (\sqrt{A})^2 \Rightarrow \text{Tr}(AB) = \text{Tr}((\sqrt{A})^2 B) = \text{Tr}(\underbrace{\sqrt{A} B \sqrt{A}}_{\geq 0}) \geq 0$$

- Spectral theorem:  $A \geq 0$ , then there are eigenvectors  $(\alpha_i)$  and eigenvalues  $\lambda_i \geq 0$  s.t.

$$\text{Tr}(AB) = \sum_i \langle \alpha_i, AB\alpha_i \rangle = \sum_i \underbrace{\lambda_i}_{\geq 0} \underbrace{\langle \alpha_i, B\alpha_i \rangle}_{\geq 0} \geq 0$$

- General Case:  $Lu(x) \geq 0$  for all  $x \in \Omega$ . Assume that there is a  $v \in C^2(\Omega) \cap C(\bar{\Omega})$  s.t.  $Lv(x) > 0$  for all  $x \in \Omega$ . Define for all  $\epsilon > 0$   $u_\epsilon = u + \epsilon v$ . Then  $Lu_\epsilon(x) = Lu(x) + \epsilon Lv(x) > 0$  for all  $x \in \Omega$ . By Step 1,

$$\begin{aligned} \max_{x \in \bar{\Omega}} u_\epsilon(x) &\leq \max_{x \in \partial\Omega} u_\epsilon(x) \\ \xrightarrow{\epsilon \rightarrow 0} \max_{x \in \bar{\Omega}} u(x) &\leq \max_{x \in \partial\Omega} u(x) \end{aligned}$$

What  $v$ ? First  $v(x) = x^2 = x_1^2 + \dots + x_d^2$ ,

$$Lv(x) = \sum_{ij} a_{ij}(x) 2\delta_{ij} + \sum_i b_i(x) 2x_i$$

not clear to be  $\geq 0$ .

$$\begin{aligned} v(x) &= x^{2n} \quad n \text{ large} \\ v(x) &= x_1^{2n} \longrightarrow Lv(x) = a_{11}(x) 2n(2n+1)x_1^{2n-2} + b_1(x) 2nx_1^{2n-1} \\ &\geq 2nx_1^{2n-2} [(2n-1) + \underbrace{b_1(x)x_1}_{\substack{\text{b.d. in } \bar{\Omega} \\ > 0}}] \geq 0 \quad \forall x \in \bar{\Omega} \end{aligned}$$

if  $n$  is large enough.

$$v(x) = (x_1 + R)^{2n}$$

where  $R > 0$  large s.t.  $x_1 + R \geq 1$  for all  $\forall x \in \bar{\Omega}$ . This implies

$$Lv(x) \geq 2n \underbrace{(x_1 + R)^{2n-2}}_{> 0} \underbrace{[2n-1 + b_1(x)(x_1 + R)]}_{> 0} > 0$$

for all  $x \in \bar{\Omega}$  if  $n$  is large. ■

**Exercise 6.25** (E 12.3) Consider the inhomogeneous heat equation

$$\begin{cases} \partial_t u - \Delta_x u = f(x, t) & \text{in } \mathbb{R}^d \times (0, T) \\ u(t=0) = g & \text{in } \mathbb{R}^d \end{cases},$$

$f \in C_1^2(\mathbb{R}^d \times (0, T))$  and compactly supported and  $g \in C(\mathbb{R}^d \times [0, T]) \cap L^\infty(\mathbb{R}^d \times [0, T])$ . Assume that there exists a solution  $u \in C_1^2(\mathbb{R}^d \times (0, T)) \cap C(\mathbb{R}^d \times [0, T])$  satisfying

$$u(x, t) \leq Me^{M|x|^2}, \quad (x, t) \in \mathbb{R}^d \times [0, T].$$

Prove that

$$\max_{(x,t) \in \mathbb{R}^d \times [0,T]} |u(x, t)| \leq \|g\|_{L^\infty} + T\|f\|_{L^\infty}.$$

*Solution.*

Step 1: There is at most one solution  $u$ .

Step 2:

$$u(x, t) = \int_{\mathbb{R}^d} \phi(x - y, t) g(y) dy + \int_0^t \int_{\mathbb{R}^d} \phi(x - y, t - s) f(y, s) dy ds$$

This implies:

$$\begin{aligned} \|u\|_{L^\infty} &\leq \int_{\mathbb{R}^d} \phi(x - y, t) \|g\|_{L^\infty} dy + \int_0^t \int_{\mathbb{R}^d} \phi(x - y, t - s) \|f\|_{L^\infty} dy ds \\ \Rightarrow \|u\|_{L_{x,t}^\infty} &\leq \int_{\mathbb{R}^d} \phi(x - y, t) \|g\|_{L^\infty} dy + \int_0^T \int_{\mathbb{R}^d} \phi(x - y, t - s) \|f\|_{L^\infty} dy ds \\ &= \|g\|_{L_x^\infty} + T \|f\|_{L_{x,t}^\infty} \end{aligned}$$

This is optimal! E.g.  $g = 0, f = 1, u(x, t) = u(t)$ .

$$\begin{cases} u' = 1 \\ u(0) = 0 \end{cases} \Rightarrow u(t) = t$$

■

**Exercise 6.26** (Bonus 11) Denote for all  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ :

$$Lu(x) = \sum_{i,j=1}^d a_{ij}(x) \partial_i \partial_j u(x)$$

where  $a_{ij} \in C(\bar{\Omega})$  s.t.  $A(x) = (a_{ij}(x)) \geq 1$ . Prove that if  $\Omega \subseteq \mathbb{R}^d$  is open and bounded,  $u \in C_1^2(\bar{\Omega} \times [0, T])$  and

$$\begin{cases} \partial_t u - Lu \leq 0 & \text{in } \Omega \times (0, T) \\ u(t = 0) = 0 \\ u(x \in \partial\Omega) = 0 \end{cases}$$

Prove that  $u(x, t) \leq 0$  for all  $(x, t) \in \bar{\Omega} \times [0, T]$ .

## 6.3 Backward heat equation

**Theorem 6.27** (Instability)

There exist functions  $u_\epsilon \in C_1^2(\mathbb{R}^d \times (0, T)) \cap C^1(H^1(\mathbb{R}^d) \times [0, T])$  s.t.

$$\partial_t u_\epsilon - \Delta_x u_\epsilon = 0 \quad \text{in } \mathbb{R}^d \times [0, T]$$

with:

$$\|u_\epsilon(\bullet, T)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \rightarrow 0^+} 0, \quad \|u_\epsilon(\bullet, 0)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \rightarrow 0^+} \infty.$$

*Proof.* Recall by Fourier Transform

$$\begin{aligned} &\partial_t \hat{u}(k, t) + |2\pi k|^2 \hat{u}(k, t) = 0 \\ \Leftrightarrow &\partial_t (e^{|2\pi k|^2 t} \hat{u}(k, t)) = 0 \\ \Rightarrow &e^{|2\pi k|^2 t} \hat{u}(k, t) = u(k, 0) \\ \Rightarrow &\hat{u}(k, t) = e^{-t|2\pi k|^2} \hat{u}(k, 0) \\ \Rightarrow &\hat{u}(k, 0) = e^{T|2\pi k|^2} \hat{u}(k, T). \end{aligned}$$

Now we can take

$$\hat{u}_\epsilon(k, t) = \mathbb{1}\left(|k| \leq \frac{1}{\epsilon}\right) \epsilon^{d+1} dk$$

Then,

$$\begin{aligned} \|u(\bullet, T)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} \hat{u}_\epsilon(k, t) dk = \lambda^d(\{|k| \leq \epsilon^{-1}\}) \epsilon^{d+1} \sim \epsilon \xrightarrow{\epsilon \rightarrow 0} 0 \\ \|u(\bullet, 0)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} e^{2T|2\pi k|^2} \mathbb{1}(|k| \leq \epsilon^{-1}) \epsilon^{d+1} dk \\ &\geq \int_{\frac{\epsilon}{2} \leq |k| \leq \frac{\epsilon}{2}} e^{2T|2\pi k|^2} \mathbb{1}(|k| \leq \epsilon^{-1}) \epsilon^{d+1} dk \gtrsim e^{2T\epsilon^{-2}} \epsilon \xrightarrow{\epsilon \rightarrow 0} \infty \end{aligned}$$

■

**Remark 6.28** This means that a small error of the data at  $t = T$  may cause a large error of the output  $t = 0$ .

**Theorem 6.29** (Regularized solution)

Assume that  $u \in C_1^2(\mathbb{R}^d \times (0, T)) \cap C^1(H^1(\mathbb{R}^d), [0, T])$

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ u(x, T) = g(x) & \text{in } \mathbb{R}^d \end{cases}$$

Then from given data  $g_\epsilon \in L^2(\mathbb{R}^d)$  s.t.

$$\|g_\epsilon - g\|_{L^2(\mathbb{R}^d)} \leq \epsilon$$

we construct a solution  $\tilde{u}_\epsilon$  s.t.

$$\sup_{t \in [0, T]} \|\tilde{u}_\epsilon(\bullet, t) - u(\bullet, t)\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \rightarrow 0} 0$$

*Proof.* Clearly we should not choose  $\tilde{u}_\epsilon$  to solve

$$\begin{cases} \partial_t u_\epsilon - \Delta_x u_\epsilon = 0 \\ u_\epsilon(t = T) = g_\epsilon \end{cases},$$

i.e.

$$\hat{u}_\epsilon(k, t) = e^{(T-t)|2\pi k|^2} \hat{g}_\epsilon(k).$$

Rather we take

$$\hat{u}_\epsilon(k, t) = e^{(T-t)|2\pi k|^2} \hat{g}_\epsilon(k) \mathbb{1}(|k| \leq \delta_\epsilon^{-1})$$

Where  $\delta_\epsilon \rightarrow 0$  (chosen later). Then we have for all  $t \in [0, T]$ :

$$\begin{aligned} \|u_\epsilon(\bullet, t) - u(\bullet, t)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} e^{2(T-t)|2\pi k|^2} |\hat{g}_\epsilon(k) \mathbb{1}(|k| \leq \delta_\epsilon^{-1}) - \hat{g}(k)|^2 dk \\ &\leq 2 \int_{\mathbb{R}^d} e^{2T|2\pi k|^2} |\hat{g}_\epsilon(k) - \hat{g}(k)|^2 \mathbb{1}(|k| \leq \delta_\epsilon^{-1}) dk \\ &\quad + 2 \int_{\mathbb{R}^d} \underbrace{e^{2T|2\pi k|^2} |\hat{g}(k)|^2}_{|\hat{u}(k, 0)|^2} \mathbb{1}(|k| > \delta_\epsilon^{-1}) dk = \text{(I)} + \text{(II)} \end{aligned}$$

We have

$$\begin{aligned} \text{(I)} &\leq 2 \int_{\mathbb{R}^d} e^{c\delta_\epsilon^{-2}} |\hat{g}_\epsilon(k) - \hat{g}(k)|^2 dk = 2e^{c\delta_\epsilon^{-2}} \epsilon^{-2} \longrightarrow 0 && \text{if } \delta_\epsilon \gg \frac{1}{\sqrt{|\log \epsilon|}} \\ \text{(II)} &= 2 \int_{\mathbb{R}^d} |\hat{u}(k, 0)|^2 \mathbb{1}(|k| \geq \delta_\epsilon^{-1}) dk \leq 2 \int_{\mathbb{R}^d} |k|^2 \delta_\epsilon^2 |\hat{u}(k, 0)|^2 dk \end{aligned}$$

Thus choosing  $\frac{1}{\sqrt{|\log \epsilon|}} \ll \delta_\epsilon \ll 1$ , e.g.  $\delta_\epsilon = (|\log \epsilon|)^{-\frac{1}{4}}$ .

$$\sup_{t \in [0, T]} \|u_\epsilon(\bullet, t) - u(\bullet, t)\|_{L^2(\mathbb{R}^d)} \leq \text{(I)} + \text{(II)} \xrightarrow{\epsilon \rightarrow 0} 0 \quad \blacksquare$$

**Remark 6.30** In application, both  $u$  and  $g$  are unknown. Only  $g_\epsilon$  is given. So we have to construct  $\tilde{u}_\epsilon$  using only information from  $g_\epsilon$ .

# Chapter 7

## Wave Equation

### 7.1 d'Alembert

Wave equation:

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & x \in \mathbb{R}^d, t > 0 \\ u = g, \partial_t u = h & x \in \mathbb{R}^d, t = 0 \end{cases}$$

In  $d = 1$ :

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & (x, t) \in \mathbb{R} \times (0, \infty) \\ u = g, \partial_t u = h, & x \in \mathbb{R}, t = 0 \end{cases}$$

Key idea: Factorization:

$$\partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x).$$

Then, if we denote  $v = (\partial_t - \partial_x)u$ , we get the transport equation

$$(\partial_t + \partial_x)v = 0.$$

This implies

$$v(x, t) = a(x - t), \quad a(x) = v(x, 0)$$

From this we get the inhomogeneous transport equation

$$(\partial_t - \partial_x)u = a(x - t).$$

Now we decompose  $u = u_1 + u_2$ , where

$$\begin{cases} (\partial_t - \partial_x)u_1 = 0 \\ (\partial_t - \partial_x)u_2 = a(x - t) \end{cases}.$$

Like above, we get  $u_1 = b(x + t)$  and an explicit choice of  $u_2$  is

$$u_2(x, t) = \frac{1}{2} \int_{x-t}^{x+t} a(y) dy$$

Thus,

$$u(x, t) = b(x + t) + \frac{1}{2} \int_{x-t}^{x+t} a(y) dy$$

Let's compute  $a$  and  $b$ :

$$\begin{aligned} b(x) &= u(x, 0) = g(x) \\ a(x) &= v(x, 0) = (\partial_t u - \partial_x u)_{t=0} = h - g'. \end{aligned}$$

**Exercise 7.1** (E 13.1, d'Alembert formula) For  $d = 1$  let  $g \in C^2(\mathbb{R}^d)$ ,  $h \in C^1(\mathbb{R})$  and define  $u$  by the *d'Alembert formula*

$$\begin{aligned} u(x, t) &= \int_{x-t}^{x+t} (h(y) - g'(y)) dy + g(x+t) \\ &= \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy. \end{aligned}$$

Then:

- $u \in C^2(\mathbb{R} \times (0, \infty))$
- $\partial_t^2 u - \partial_x^2 u = 0$
- $u = g, \partial_t u = h$  when  $t \rightarrow 0$

*Solution.* We can compute the derivative by regarding  $\int_x^{x+t} + \int_{x-t}^x$  and  $\int_t^{x+t} + \int_{x-t}^t$ :

$$\begin{aligned} \partial_t u &= \frac{1}{2} (g'(x+t) - g'(x-t)) + \frac{1}{2} (h(x+t) + h(x-t)) \\ \partial_t^2 u &= \frac{1}{2} (g''(x+t) + g''(x-t)) + \frac{1}{2} (h'(x+t) - h'(x-t)) \\ \partial_x u &= \frac{1}{2} (g'(x+t) + g'(x-t)) + \frac{1}{2} (h(x+t) - h(x-t)) \\ \partial_x^2 u &= \frac{1}{2} (g''(x+t) + g''(x-t)) + \frac{1}{2} (h'(x+t) - h'(x-t)) = \partial_t^2 u \end{aligned}$$

Now,

$$\begin{aligned} \lim_{t \rightarrow 0} u &= \frac{1}{2} (g(x) + g(x)) + \frac{1}{2} \int_x^x h = g(x) \\ \lim_{t \rightarrow 0} \partial_t u &= \frac{1}{2} (g'(x) - g'(x)) + \frac{1}{2} (h(x) + h(x)) = h(x). \end{aligned}$$

■

**Remark 7.2** If  $g \in C^k$  and  $h \in C^{k-1}$ , then  $u \in C^k$  (but not better).

Now, let's apply the *Reflection Method*. Replace  $\mathbb{R}$  by  $\mathbb{R}_+ = (0, \infty)$  and assume

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \mathbb{R}_+ \times (0, \infty) \\ u = g, \partial_t u = h & \text{on } \mathbb{R}_+ \times \{t = 0\}, g(0) = h(0) = 0 \\ u = 0 & \text{on } \{x = 0\} \times \{t > 0\} \end{cases}$$

Define

$$\begin{aligned} \tilde{u}(x, t) &= \begin{cases} u(x, t), & x \geq 0, t \geq 0 \\ -u(-x, t), & x \leq 0, t \geq 0 \end{cases} \\ \tilde{g}(x) &= \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x \leq 0 \end{cases} \\ \tilde{h}(x) &= \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x \leq 0 \end{cases} \end{aligned}$$



Then

$$\begin{cases} \partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ \tilde{u} = \tilde{g}, \partial_t \tilde{u} = \tilde{h} & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}.$$

By d'Alembert formula

$$\tilde{u}(x, t) = \frac{1}{2} [\tilde{g}(x+t) + \tilde{g}(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{h}(y) dy$$

This implies

$$u(x, t) = \begin{cases} \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy & x \geq t \geq 0 \\ \frac{1}{2} [g(x+t) - g(t-x)] + \frac{1}{2} \int_{t-x}^{x+t} h(y) dy & t \geq x \geq 0 \end{cases}.$$

This is the solution of the heat equation in  $\mathbb{R}_+ \times (0, \infty)$ .

## 7.2 Euler-Poisson-Darboux

$$(\star) \quad \begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ u = g, \partial_t u = h & \mathbb{R}^d \times \{t = 0\} \end{cases}$$

Idea: Averaging of  $u$  over sphere  $\rightsquigarrow$  1D problem. Define for  $x \in \mathbb{R}^d$ ,  $t > 0$ ,  $r > 0$ ,

$$\begin{aligned} U_r(x, t) &:= \oint_{\partial B(x, r)} u(y, t) dS(y) \\ G_r(x) &:= \oint_{\partial B(x, r)} g(y) dS(y) \\ H_r(x) &:= \oint_{\partial B(x, r)} h(y) dS(y) \end{aligned}$$

**Lemma 7.3** (Euler-Poisson-Darboux equation) If  $u \in C^2(\mathbb{R}^d \times [0, \infty))$  solves  $(\star)$ , then for all  $x \in \mathbb{R}^d$ :

- $(r, t) \mapsto U \in C^2([0, \infty) \times [0, \infty))$
- $\begin{cases} \partial_t^2 U - \partial_r^2 U - \frac{d-1}{r} \partial_r U = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\ U = G, \partial_t U = H & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$

Note that  $\partial_r^2 + \frac{d-1}{r} \partial_r$  is the radial part of  $\Delta$ .

*Proof.* We compute for  $r > 0$ :

$$\begin{aligned}
\partial_r U_r(x, t) &= \partial_r \oint_{\partial B(x, r)} u(y, t) dS(y) \\
&= \partial_r \oint_{\partial B(0, 1)} u(x + rz, t) dS(z) \\
&= \oint_{\partial B(0, 1)} \nabla u(x + rz, t) z dS(z) \\
&= \oint_{\partial B(x, r)} \nabla u(y, t) \frac{y - x}{r} dS(y) \\
&= \oint_{\partial B(x, r)} \frac{\partial u(y, t)}{\partial \vec{n}} dS(y) \\
(\text{Green 2.3}) \quad &= \frac{1}{|\partial B(0, r)|} \int_{B(x, r)} \Delta_x u(y, t) dy \\
\left( |B(0, r)| = \frac{r}{d} |\partial B(0, r)| \right) &= \frac{r}{d} \oint_{B(x, r)} \Delta_x u(y, t) dy
\end{aligned}$$

(The computation is similar to the proof of the mean-value theorem for the Poisson equation.) We compute the second derivative

$$\begin{aligned}
\partial_r^2 U_r(x, t) &= \partial_r \left[ \frac{r}{d} \oint_{B(x, r)} \Delta_x u(y, t) dy \right] \\
(|B(0, r)| = r^d |B(0, 1)|) \quad &= \partial_r \left[ \frac{1}{d |B_1| r^{d-1}} \int_{B(x, r)} \Delta_x u(y, t) dy \right]
\end{aligned}$$

Now,  $\partial_r \frac{1}{d |B_1| r^{d-1}} = \frac{-d+1}{d |B_1| r^d} = -\frac{d-1}{d |B(0, r)|}$  and

$$\begin{aligned}
\partial_r \int_{B(x, r)} \Delta_x u(y, t) dS(y) &= \partial_r \int_{B(0, r)} \Delta_x u(x + ry, t) dy \\
(\text{Green 2.3}) \quad &= \partial_r \int_{\partial B(0, 1)} \nabla_x u(x + ry, t) \frac{y}{|y|} dS(y) \\
(|y| = 1) \quad &= \int_{\partial B(0, 1)} \partial_r \nabla_x u(x + ry, t) y dS(y) \\
&= \int_{\partial B(0, 1)} \Delta_x u(x + ry, t) y \cdot y dS(y) \\
(y \cdot y = |y|^2 = 1) \quad &= \int_{\partial B(0, 1)} \Delta_x u(x + ry, t) dS(y) \\
&= \int_{\partial B(x, r)} \Delta_x u(y, t) dS(y)
\end{aligned}$$

Now with the product rule we get

$$\begin{aligned}
\partial_r^2 U_r(x, t) &= - \left( \frac{d-1}{d} \right) \oint_{B(x, r)} \Delta_x u(y, t) dy \\
&\quad + \frac{1}{d |B_1| r^{d-1}} \int_{\partial B(x, r)} \Delta_x u(y, t) dS(y) \\
&= - \left( \frac{d-1}{d} \right) \oint_{B(x, r)} \Delta_x u(y, t) dy \\
&\quad + \oint_{\partial B(x, r)} \Delta_x u dS(y)
\end{aligned}$$

And, since  $u$  is a solution,

$$\partial_t^2 U = \partial_t^2 \int_{\partial B(x,r)} u \, dS(y) = \int_{\partial B(x,r)} (\partial_t^2 u) \, dS(y) = \int_{\partial B(x,r)} (\Delta_x u) \, dS(y).$$

So we can conclude

$$\partial_t^2 U - \partial_r^2 U - \frac{d-1}{d} U = 0$$

the above computation also shows that  $U \in C^2(\mathbb{R}_+ \times [0, \infty))$ . Moreover

$$\begin{aligned} \partial_r U_r(x, t) &\xrightarrow{r \rightarrow 0^+} 0 \\ \partial_r^2 U_r(x, t) &\xrightarrow{r \rightarrow 0^+} \left(\frac{1}{d} - 1\right) \Delta_x u + \Delta_x u = \frac{1}{d} \Delta_x u \end{aligned}$$

This implies that  $U \in C^2([0, \infty) \times [0, \infty))$ . Finally, when  $t = 0$ ,

$$\begin{cases} u = g \\ \partial_t = h \end{cases} \Rightarrow \begin{cases} U = G \\ \partial_t U = H \end{cases} \quad \blacksquare$$

We showed that it is a necessary condition for  $u$  to solve the Euler-Poisson-Darboux equation. Now we try to actually solve the equation. In general, this is easier for odd  $d$  than for even  $d$ . We will consider the cases  $d = 2, 3$ .

### 7.2.1 Solution in three dimensions

Now, for  $r > 0$  let  $\tilde{U} = rU, \tilde{G} = rG, \tilde{H} = rH$ . Then

$$\begin{cases} \partial_t^2 \tilde{U} - \partial_r^2 \tilde{U} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \tilde{U} = \tilde{G}, \partial_t \tilde{U} = \tilde{H} & \text{when } t = 0 \\ \tilde{U} = 0 & \text{when } r = 0 \end{cases}$$

Then, by d'Alembert's formula, for  $0 \leq r \leq t$  we have

$$\begin{aligned} \tilde{U}_r(x, t) &= \frac{1}{2} \left[ \tilde{G}(r+t) - \tilde{G}(t-r) \right] + \frac{1}{2} \int_{t-r}^{t+r} \tilde{H}(y) \, dy. \\ \Rightarrow U_r(x, t) &= \frac{1}{2} \left[ \frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{r} \right] + \frac{1}{2r} \int_{t-r}^{t+r} \tilde{H}(y) \, dy \end{aligned}$$

Now, taking  $r \rightarrow 0$  we get

$$\begin{aligned} u(x, t) &= \tilde{G}'(t) + \tilde{H}(t) \\ &= \partial_t \left( t \int_{\partial B(x,t)} g(y) \, dS(y) \right) + t \int_{\partial B(x,t)} h(y) \, dS(y) \end{aligned}$$

Using

$$\int_{\partial B(x,t)} g(y) \, dS(y) = \int_{\partial B(0,1)} g(x + tz) \, dS(z)$$

we get

$$\begin{aligned} \partial_t \int_{\partial B(x,t)} g(y) \, dS(y) &= \int_{\partial B(0,1)} \nabla g(x + tz) \cdot z \, dz \\ &= \int_{\partial B(x,t)} \nabla g(y) \left( \frac{y-x}{t} \right) \, dS(y) \\ \Rightarrow \partial_t \left( t \int_{\partial B(x,t)} g(y) \, dS(y) \right) &= \int_{\partial B(x,t)} (g + \nabla g(y-x)) \, dS(y) \end{aligned}$$

From that we get:

**Remark 7.4** (Kirchhoff's formula in 3D) For all  $x \in \mathbb{R}^3$ ,  $t > 0$ :

$$u(x, t) = \oint_{\partial B(x, t)} (g(y) + \nabla g(y - x) + th(y)) dS(y)$$

**Exercise 7.5** (E 13.3) Let  $g \in C_c^2(\mathbb{R}^3)$ ,  $h \in C_c^1(\mathbb{R}^3)$ . Assume that  $u \in C^2(\mathbb{R}^3 \times [0, \infty))$  satisfies the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \forall x \in \mathbb{R}^3, t > 0 \\ u(x, 0) = g(x), \partial_t u(x, 0) = h(x) & \forall x \in \mathbb{R}^3 \end{cases}$$

Prove that there exists a constant  $C > 0$  such that

$$|u(x, t)| \leq \frac{C}{t}, \quad \forall x \in \mathbb{R}^3, t > 0.$$

*Solution.*

Step 1: Assume

$$(\star) \quad u(x, t) = \oint_{\partial B_{\mathbb{R}^3}} (g(y) + \nabla g(y)(y - x) + th(y)) dS(y)$$

Assume  $\text{supp } g, h \subseteq B(0, R)$ . Then

$$\begin{aligned} |u(x, t)| &\leq \frac{1}{|\partial B(x, t)|} \int_{\partial B_{\mathbb{R}^3}(x, t)} (\|g\|_{L^\infty} + \|\nabla g\|_{L^\infty} t + t\|h\|_{L^\infty}) \mathbb{1}(|y| \leq R) dS(y) \\ &= \frac{C(1+t)}{t^2} \leq \frac{C}{t} \quad \text{as } t \geq 1 \end{aligned}$$

and

$$|u(x, t)| \leq C \oint_{\partial B(x, t)} \leq C$$

if  $t \leq 1$ .

Step 2: Why is  $u$  given by  $(\star)$ ? This follows from the uniqueness of the solution of the wave equation. In fact, if  $g, h$  are compactly supported, then for all  $t \in [0, T]$ ,  $u(x, t)$  supported in  $B(0, R_T)$  a finite ball.

■

## 7.2.2 Solution in two dimensions

The transformation  $\tilde{U} = rU$  does not work! The idea is to think of the 2D problem as 3D problem with  $x_3$  hidden. We write  $\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$ . Then we get

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_x \bar{u} = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

With Kirchhoff's formula:

$$u(x, t) = \bar{u}(\bar{x}, t) = \partial_t \left( t \oint_{\partial \bar{B}(\bar{x}, t)} \bar{g}(y) d\bar{S}(y) \right) + t \int_{\partial \bar{B}(\bar{x}, t)} \bar{h} d\bar{S}(y)$$

**Remark 7.6** Let  $\gamma(y) = (t^2 - |y - x|^2)^{\frac{1}{2}}$ ,  $y \in B(x, t)$ , then

$$\begin{aligned} \oint_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{S} &= \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{S}(y) \\ &= \frac{1}{4\pi t^2} \int_{B(x, t)} g(y) 2(1 + |\nabla \gamma|^2)^{\frac{1}{2}} dy \\ &= \frac{1}{4\pi t^2} \int_{B(x, r)} g(y) \frac{2t}{\sqrt{t^2 - |y - x|^2}} dy \\ &= \frac{t}{2} \oint_{B(x, t)} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}} dy \end{aligned}$$

Similarly:

$$\oint_{\partial B(\bar{x}, t)} \bar{h} d\bar{S}(y) = \frac{t}{2} \oint_{B(x, t)} \frac{h(y)}{\sqrt{t^2 - |y - x|^2}} dy$$

This implies:

$$\begin{aligned} u(x, t) &= \partial_t \left( \frac{t^2}{2} \oint_{B(x, t)} \frac{g(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy \right) + \frac{t^2}{2} \oint_{B(x, r)} \frac{h(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy \\ &= (I) + (II) \\ (I) &= \partial_t \left( \frac{1}{2} t \oint_{B(0, 1)} \frac{g(x + tz)}{(1 - |z|^2)^{\frac{1}{2}}} dz \right) \\ &= \oint_{B(0, 1)} \frac{g(x + tz)}{(1 - |z|^2)^{\frac{1}{2}}} dz + t \oint_{B(0, 1)} \frac{\nabla g(x + tz) z}{(1 - |z|^2)^{\frac{1}{2}}} dz \\ &= t \int_{B(x, t)} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}} dy + t \oint_{B(x, r)} \frac{\nabla g(y)(y - x)}{\sqrt{t^2 - |y - x|^2}} dy \end{aligned}$$

**Exercise 7.7** (E 13.2, Poisson formula for 2D) For  $x \in \mathbb{R}^2$ ,  $t > 0$ ,  $g \in C^2(\mathbb{R}^2)$ ,  $h \in C^2(\mathbb{R}^2)$  let

$$u(x, t) = \frac{t}{2} \oint_{B(x, t)} \frac{g(y) + \nabla g(y)(y - x) + th(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy.$$

Prove that  $u \in C^2(\mathbb{R}^2 \times (0, \infty))$  and

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \forall x \in \mathbb{R}^2, t > 0 \\ \lim_{t \rightarrow 0^+} u(x, t) = g(x), \lim_{t \rightarrow 0^+} \partial_t u(x, t) = h(x), & \forall x \in \mathbb{R}^2 \end{cases}$$

*Solution.* Let  $\bar{x} = (x, x_3) \in \mathbb{R}^3$ ,  $\bar{u}(\bar{x}, t) = u(x, t)$ . We claim that

$$(\star) \quad \bar{u}(\bar{x}, t) = \partial_t \left( t \oint_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{S}(y) \right) + t \oint_{\partial \bar{B}(\bar{x}, t)} \bar{h} d\bar{S}$$

where  $\bar{B}(\bar{x}, t)$  is a ball in 3D,  $\bar{g}(\bar{x}) = g(x)$ ,  $\bar{h}(\bar{x}) = h(x)$ . From 7.6 we have

$$\oint_{\partial \bar{B}(\bar{x}, t)} \bar{g} d\bar{S} = \frac{t}{2} \oint_{B_{\mathbb{R}^2}(x, t)} \frac{g(y)}{\sqrt{t^2 - |x - y|^2}} dy$$

Now,

$$\begin{aligned} \text{RHS of } (\star) &= \partial_t \left( \frac{t^2}{2} \oint_{B_{\mathbb{R}^2}} \frac{g(y)}{\sqrt{t^2 - |x - y|^2}} dy \right) + \frac{t^2}{2} \oint_{B_{\mathbb{R}^2}(x,t)} h(y) dy \\ ?? &= \partial_t \frac{t}{2} \frac{g(y) + \nabla g(y)(y - x)}{\sqrt{t^2 - |x - y|^2}} + \frac{t^2}{2} \oint_{B_{\mathbb{R}^2}(x,t)} h(y) dy \end{aligned}$$

3D-Problem: We claim that if  $\bar{g} \in C^3(\mathbb{R}^3)$ ,  $\bar{h} \in C^2(\mathbb{R}^3)$  and

$$\bar{u}(\bar{x}, t) = \partial_t \left( t \oint_{\partial B_{\mathbb{R}^2}(\bar{x}, t)} \bar{g} \right) + t \int_{\partial B_{\mathbb{R}^2}(\bar{x}, t)} \bar{h},$$

then  $\bar{u} \in C^2(\mathbb{R}^3 \times (0, \infty))$  and

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_{\bar{x}} \bar{u} = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Proof:

Step 1: Assume  $\bar{g} = 0$ ,  $\bar{u}(\bar{x}, t) = t \int_{\partial B(\bar{x}, t)} \bar{h}$ . Lemma (from lecture):

$$\partial_r \int_{\partial B(\bar{x}, r)} \bar{h} = \frac{r}{d} \oint_{B(\bar{x}, r)} \Delta \bar{h}, \quad d = 3$$

( $\bar{U} = \oint_{B(\bar{x}, r)} u(x, t)$ ) Hence:

$$\begin{aligned} \partial_t \bar{u}(x, t) &= \partial_t \left( t \oint_{\partial B(\bar{x}, t)} \bar{h} \right) = \oint_{\partial B(\bar{x}, t)} \bar{h} + t \frac{t}{3} \oint_{B(x, t)} \Delta \bar{h} \\ \partial_t^2 \bar{u}(x, t) &= \frac{t}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} + \frac{2}{3} t \oint_{B(x, t)} \Delta \bar{h} + \frac{t^2}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} \end{aligned}$$

Where we have

$$\begin{aligned} \frac{t^2}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} &= \frac{1}{|B_1| 3t} \int_{B(\bar{x}, t)} \Delta \bar{h} \\ \partial_t(\dots) &= -\frac{1}{|B_1| 3t^2} \int_{B(\bar{x}, t)} \Delta \bar{h} + \frac{1}{|B_1| 3t} \int_{\partial B(\bar{x}, t)} \Delta \bar{h} \\ &= -\frac{t}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} + t \oint_{\partial B(\bar{x}, t)} \Delta \bar{h} \end{aligned}$$

So we get

$$\begin{aligned} \partial_t^2 \bar{u}(x, t) &= \frac{t}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} + \frac{2}{3} t \oint_{B(x, t)} \Delta \bar{h} - \frac{t}{3} \oint_{B(\bar{x}, t)} \Delta \bar{h} + t \oint_{\partial B(\bar{x}, t)} \Delta \bar{h} \\ &= t \oint_{\partial B(\bar{x}, t)} \Delta \bar{h} \end{aligned}$$

This implies

$$\begin{aligned} \Delta_{\bar{x}} \bar{u}(x, t) &= \Delta_{\bar{x}} \left( t \oint_{\partial B(\bar{x}, t)} \bar{h} \right) = \Delta_x \left( t \oint_{\partial B(0, t)} \bar{h}(\bar{x} + y) dS(y) \right) \\ &= t \oint_{\partial B(0, t)} \Delta \bar{h}(\bar{x} + y) dS(y) = t \oint_{\partial B(\bar{x}, t)} \Delta \bar{h} = \partial_t^2 \bar{u} \end{aligned}$$

Thus,  $\partial_t \bar{u} - \Delta_{\bar{x}} \bar{u} = 0$  in  $\mathbb{R}^3 \times (0, \infty)$ . Moreover,

$$\bar{u}(x, t) = t \fint_{\partial B(\bar{x}, t)} \bar{h} \xrightarrow{t \rightarrow 0} 0$$

and

$$\partial_t \bar{u}(x, t) = \fint_{\partial B(\bar{x}, t)} \bar{h} + \frac{t^2}{3} \fint_{B(x, t)} \Delta \bar{h} \xrightarrow{t \rightarrow 0} \bar{h}(\bar{x}) \quad \checkmark$$

Step 2: General Case:  $\bar{g} \neq 0$ . Assume  $h = 0$ , then

$$\bar{u}(\bar{x}, t) = \partial_t \left( t \fint_{\partial B_{\mathbb{R}^3}} \bar{g} \right)$$

By Step 1,  $v(\bar{x}, t) := t \fint_{\partial B_{\mathbb{R}^3}(0, \infty)} \bar{g}$  satisfies

$$\begin{cases} \partial_t^2 - \Delta_{\bar{x}} v = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ v = 0 \text{ and } \partial_t v = \bar{g} & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Then  $\bar{u} = \partial_t \bar{v}$ . This implies

$$\partial_t^2 \bar{u} - \Delta_{\bar{x}} \bar{u} = \partial_t^3 \bar{v} - \Delta_{\bar{x}} \partial_t \bar{v} = \partial_t (\underbrace{\partial_t^2 \bar{v} - \Delta_{\bar{x}} \bar{v}}_{=0}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty)$$

and

$$\begin{aligned} \bar{u}(t = 0) &= \partial_t \bar{v}(t = 0) = \bar{g}, \\ \partial_t \bar{u}(t = 0) &= \partial_t^2 \bar{v}(t = 0) = t \fint_{\partial B(\bar{x}, t)} \Delta \bar{g} \Big|_{t=0} = 0 \end{aligned}$$

Step 3: Consider the case  $g \neq 0$ ,  $h \neq 0$ :

$$\bar{u} = \underbrace{\partial_t \left( t \fint_{\partial B(\bar{x}, t)} \bar{g} \right)}_{\bar{u}_1} + t \underbrace{\fint_{\partial B(\bar{x}, t)} h}_{\bar{u}_2}$$

Now we have

$$\begin{cases} \partial_t^2 \bar{u}_1 - \Delta_{\bar{x}} \bar{u}_1 = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u}_1 = \bar{g}, \partial_t \bar{u}_1 = 0 & \text{in } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

and

$$\begin{cases} \partial_t^2 \bar{u}_2 - \Delta_{\bar{x}} \bar{u}_2 = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u}_2 = 0, \partial_t \bar{u}_2 = h & \text{in } \mathbb{R}^3 \times (0, \infty) \end{cases}.$$

This implies

$$\begin{cases} \partial_t^2 \bar{u} - \Delta_{\bar{x}} \bar{u} = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \bar{u} = \bar{g}, \partial_t \bar{u} = \bar{h} & \text{in } \mathbb{R}^3 \times (0, \infty) \end{cases}.$$

■

### 7.3 Spectral Method

Let  $\Omega \subseteq \mathbb{R}^d$  be open and bounded.

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \Omega \times (0, \infty) \\ u = g, \partial_t u = h & \text{when } t = 0 \\ u = 0 & \text{when } x \in \partial\Omega \end{cases}$$

$-\Delta$  has eigenvecors  $(e_i)_{i=1}^\infty$  with eigenvalues  $(\lambda_i)_{i=1}^\infty$ , i.e.

$$\begin{cases} -\Delta e_i = \lambda_i e_i \\ e_i|_{\partial\Omega} = 0 \end{cases}$$

s.t.  $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_i \rightarrow \infty$  and  $(e_i)_{i \in \mathbb{N}}$  is an orthonormal basis for  $L^2(\Omega)$ . We write

$$u(x, t) = \sum_{i \in \mathbb{N}} a_i(t) e_i(x)$$

This implies:

$$\begin{aligned} a_i''(t) + \lambda_i a_i(t) &= 0 \\ \text{(ODE)} \quad \Rightarrow \quad a_i(t) &= a_i(0) \cos(\sqrt{\lambda_i} t) + \frac{a_i'(0)}{\sqrt{\lambda_i}} \sin(\sqrt{\lambda_i} t) \end{aligned}$$

Here  $a_i(0)$ ,  $a_i'(0)$  is determined by

$$\begin{cases} g = u(t=0) = \sum_{i=1}^\infty a_i(0) e_i(x) \\ h = \partial_t u(t=0) = \sum_{i=1}^\infty a_i'(0) e_i(x) \end{cases} \Rightarrow \begin{cases} a_i(0) = \langle e_i, g \rangle \\ a_i'(0) = \langle e_i, h \rangle \end{cases}$$

### 7.4 Uniqueness

For  $\Omega \subseteq \mathbb{R}^d$  open and bounded with  $C^1$ -boundary regard

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \Omega \times (0, T) \\ u = 0, \partial_t u = 0 & \text{in } \Omega \times \{t = 0\} \\ u = 0 & \partial\Omega \times [0, T] \end{cases}$$

If  $u \in C^2(\bar{\Omega} \times [0, T])$ , then  $u = 0$ .

*Proof.* Let

$$e(t) = \int_{\Omega} (|\partial_t u|^2 + |\nabla_x u|^2) dx$$

has  $e'(t) = 0$ . This implies  $e(t) = e(0) = 0$ , so  $\partial_t u = 0$  and hence  $u = 0$ . ■

The same result holds for  $\mathbb{R}^d$ , i.e.

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \mathbb{R}^d \times (0, T) \\ u = 0, \partial_t u = 0 & \mathbb{R}^d \times \{t = 0\} \end{cases}$$

and  $u \in C^2(H^2(\mathbb{R}^d), [0, T])$  (i.e.  $u(t, \bullet) \in H^2(\mathbb{R}^d)$  and  $t \mapsto u(t, \bullet)$  continuous).



## 7.5 Propagation of the wave

**Theorem 7.8** Assume that  $u \in C^2(\mathbb{R}^d \times [0, \infty))$  and

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \mathbb{R}^d \times (0, \infty) \\ u = 0, \partial_t u = 0 & B(x_0, t_0) \times \{t = 0\} \end{cases}$$

Then  $u(x, t) = 0$  for  $x \in B(x_0, t_0 - t)$ .

*Proof.* Let for  $t \in [0, t_0]$

$$e(t) = \int_{B(x_0, t_0 - t)} (|\partial_t u|^2 + |\nabla_x u|^2) dx$$

■

This implies

$$\begin{aligned} e'(t) &= \int_{B(x_0, t_0 - t)} 2(\partial_t u \partial_t^2 u + \nabla_x u \partial_t \nabla_x u) - \int_{\partial B(x_0, t_0 - t)} (|\partial_t u|^2 + |\nabla_x u|^2) dS \\ &= \int_{B(x_0, t_0 - t)} 2(\partial_t u \partial_t^2 u - \Delta_x u \partial_t u) + \int_{\partial B(x_0, t_0 - t)} [2(\nabla_x u \vec{n}) \partial_t u - |\partial_t u|^2 - |\nabla_x u|^2] dS \\ &= \int_{B(x_0, t_0 - t)} 2(\partial_t u \underbrace{(\partial_t^2 u - \Delta_x u)}_{=0}) + \int_{\partial B(x_0, t_0 - t)} \underbrace{[2(\nabla_x u \vec{n}) \partial_t u - |\partial_t u|^2 - |\nabla_x u|^2]}_{\leq 0} dS \end{aligned}$$

where we used

$$|2(\nabla_x u \vec{n}) \partial_t u| \leq 2|\nabla_x u| |\partial_t u| \leq |\nabla_x u|^2 + |\partial_t u|^2.$$

Thus  $e'(t) \leq 0$  for all  $t \in (0, t_0)$ , so  $e(t) \leq e(0) = 0$  (as  $u = 0$ ,  $\partial_t u = 0$  in  $B(x_0, t_0) \times \{t = 0\}$ ). This implies  $e(t) = 0$  for all  $t \in (0, t_0)$ , so

$$\begin{cases} \partial_t u = 0 & x \in B(x_0, t_0 - t) \\ u = 0 & x \in B(x_0, t_0 - t) \times \{t = 0\} \end{cases}.$$

So we get that  $u = 0$  for  $x \in B(x_0, t_0 - t)$  for all  $t \in (0, t_0)$  and  $u = 0$  for  $x \in B(x_0, t_0 - t)$  for all  $t \in [0, t_0]$ . (More precisely  $u(x_0, t_0) = 0$ )

## 7.6 Wave vs. Heat Equation

**Notation 7.9** For  $g \in L^2(\mathbb{R}^d)$ ,  $t > 0$  we write  $e^{t\Delta} g \in L^2(\mathbb{R}^d)$  for

$$\begin{aligned} \widehat{e^{t\Delta} g}(k) &= e^{-t|2\pi k|^2} \hat{g}(k) \\ \Leftrightarrow (e^{t\Delta} g)(x) &= \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y) dy \\ \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} &= e^{-t|2\pi k|^2} \end{aligned}$$

**Heat Equation:**

- Improves smoothness, *i.e.*  $g \in L^2(\mathbb{R})$  implies  $e^{t\Delta}g \in \bigcap_{m \geq 1} H^m(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$  for all  $t > 0$ .
- Propagation: Speed is  $\infty$ .

$t = 0$ :  $g \in C_c^\infty$

$t > 0$ :  $e^{t\Delta}g$  does not have compact support

**Wave Equation:**

- No improvement of smoothness
- Propagation: Speed is finite

## Chapter 8

# Schrödinger Equation

$$\begin{cases} -i\partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = g(x) & x \in \mathbb{R}^d \end{cases}$$

Formally  $-i\partial_t = \frac{1}{i} \frac{d}{dt} = \frac{d}{d(it)} \Rightarrow \partial_\xi u - \Delta_x u = 0$ ,  $\xi = it \rightsquigarrow$  Heat equation with *imaginary time*. From the heat equation

$$\begin{aligned} \begin{cases} \partial_t u - \Delta_x u = 0 \\ u(x, 0) = g(x) \end{cases} &\Rightarrow u(x, t) = (e^{t\Delta} g)(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y) dy \\ \rightsquigarrow \begin{cases} -i\partial_t u - \Delta_x u = 0 \\ u(x, 0) = g(x) \end{cases} &\Rightarrow u(x, t) = (e^{it\Delta} g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy \end{aligned}$$

if  $g \in L^1$ .

**Theorem 8.1** (Fourier for Schrödinger Solution) For  $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ , define

$$(e^{it\Delta} g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy$$

Then  $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)}$ . Consequently, for all  $g \in L^2(\mathbb{R}^d)$  we can define  $e^{it\Delta} g \in L^2(\mathbb{R}^d)$  by a density argument. Moreover,

$$\widehat{e^{it\Delta} g}(k) = e^{-it|2\pi k|^2} \hat{g}(k)$$

for almost every  $k \in \mathbb{R}^d$ .

*Proof.* Fourier transform of Gaussian:

$$\frac{1}{(4\pi t)^{\frac{d}{2}}} \widehat{e^{-\frac{|x|^2}{4t}}}(k) = e^{-t|2\pi k|^2}, \quad t > 0$$

Key point: This formula also holds if  $t \in \mathbb{C}$  and  $\Re(t) > 0$ . For all  $\epsilon > 0$  consider

$$(e^{(it+\epsilon)\Delta} g)(x) = \left( e^{-(it+\epsilon)|2\pi k|^2} \hat{g}(k) \right)^\vee(x) = (\hat{G}_\epsilon g)^\vee = (G_\epsilon \star g)(x)$$

where

$$G_\epsilon(x) = \frac{1}{(4\pi(it+\epsilon))^{\frac{d}{2}}} e^{-\frac{|x|^2}{4(it+\epsilon)}}.$$

Since  $g \in L^1(\mathbb{R}^d)$ :

$$\begin{aligned}(G_\epsilon \star g)(x) &= \frac{1}{(4\pi(it + \epsilon))^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4(it+\epsilon)}} g(y) dy \\ &\longrightarrow \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}} g(y) dy = (e^{it\Delta} g)(x)\end{aligned}$$

for all  $x \in \mathbb{R}^d$ . Moreover,

$$\begin{aligned}\|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} &= \|\widehat{G_\epsilon \star g}\|_{L^2(\mathbb{R}^d)} = \|\hat{G}_\epsilon \hat{g}\|_{L^2(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \left| e^{-(it+\epsilon)|2\pi k|^2} \right|^2 |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} \\ &= \left( \int_{\mathbb{R}^d} e^{-\epsilon|2\pi k|^2} |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} \xrightarrow{\epsilon \rightarrow 0^+} \left( \int_{\mathbb{R}^d} |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} = \|g\|_{L^2(\mathbb{R}^d)}\end{aligned}$$

as  $g \in L^2(\mathbb{R}^d)$ . Thus,  $G_\epsilon \star g \rightarrow e^{it\Delta} g$  pointwise. With Fatou:

$$\liminf_{\epsilon \rightarrow 0^+} \|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} \geq \|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)}$$

Thus,  $e^{it\Delta} g \in L^2(\mathbb{R}^d)$  and  $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} \leq \|g\|_{L^2(\mathbb{R}^d)}$ . To get the equality:

$$\widehat{G_\epsilon \star g}(k) = e^{-(it+\epsilon)|2\pi k|^2} \underbrace{\hat{g}(k)}_{L^2} = \underbrace{e^{-\epsilon|2\pi k|^2}}_{\in [0,1]} \left( e^{-it|2\pi k|^2} \hat{g}(k) \right) \xrightarrow{\epsilon \rightarrow 0^+} e^{it|2\pi k|^2} \hat{g}(k)$$

in  $L^2(\mathbb{R}^d, dk)$ . Thus  $G_\epsilon \star g$  converges in  $L^2(\mathbb{R}^d)$ . Then up to a subsequence  $\epsilon \rightarrow 0^+$ , we can assume that

$$(G_\epsilon \star g)(x) \xrightarrow{\epsilon \rightarrow 0^+} H(x)$$

almost everywhere. (Dominated convergence) We already proved  $G_\epsilon \star g \rightarrow e^{it\Delta} g$  pointwise, so  $e^{it\Delta} g = H$ , i.e.  $G_\epsilon \star g \rightarrow e^{it\Delta} g$  in  $L^2(\mathbb{R}^d)$ . Conclude

$$\|g\|_{L^2(\mathbb{R}^d)} \xleftarrow{\epsilon \rightarrow 0} \|G_\epsilon \star g\|_{L^2(\mathbb{R}^d)} \xrightarrow{\epsilon \rightarrow 0^+} \|e^{it\Delta} g\|_{L^2}.$$

This implies  $\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)}$  for all  $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ . For  $g \in L^2(\mathbb{R}^d)$  there is a sequence  $\{g_n\} \subseteq L^1 \cap L^2$  s.t.  $g_n \rightarrow g$  in  $L^2(\mathbb{R}^d)$ . Then

$$\|e^{it\Delta} g_n - e^{it\Delta} g_m\|_{L^2} = \|e^{it\Delta} \underbrace{(g_n - g_m)}_{L^1 \cap L^2}\|_{L^2} = \|g_n - g_m\|_{L^2} \rightarrow 0$$

as  $m, n \rightarrow \infty$ . So  $e^{it\Delta} g_n$  is a Cauchy sequence in  $L^2$ , so it has a limit which we define as  $e^{it\Delta} g$ . Why the limit  $e^{it\Delta} g$  is independent of the choice of  $g_n$ : If we have 2 different sequences  $\{g_n\}, \{\tilde{g}_n\} \subseteq L^1 \cap L^2$ , then  $g_n, \tilde{g}_n \rightarrow g$  in  $L^2(\mathbb{R}^d)$ . Then

$$\|e^{it\Delta} g_n - e^{it\Delta} \tilde{g}_n\|_{L^2(\mathbb{R}^d)} = \|e^{it\Delta} \underbrace{(g_n - \tilde{g}_n)}_{L^1 \cap L^2}\|_{L^2} = \|g_n - \tilde{g}_n\|_{L^2} \xrightarrow{n \rightarrow \infty} 0$$

$\leadsto \lim_{n \rightarrow \infty} e^{it\Delta} g_n = \lim_{n \rightarrow \infty} e^{it\Delta} \tilde{g}_n$ . Finally, we have  $g \in L^2(\mathbb{R}^d), g_n \in L^1 \cap L^2 \rightarrow g$  in  $L^2$ .

$$\widehat{e^{it\Delta} g}(k) \xleftarrow{n \rightarrow \infty} \widehat{e^{it\Delta} g_n}(k) = e^{-it|2\pi k|^2} \hat{g}_n(k) \xrightarrow{n \rightarrow \infty} e^{-it|2\pi k|^2} \hat{g}(k)$$

in  $L^2(\mathbb{R}^d)$ . ■

**Theorem 8.2** (Long term behaviour)

1. If  $g \in L^1(\mathbb{R}^d)$ , then  $(e^{it\Delta}g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy$ . This implies

$$\|e^{it\Delta}g\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \|g\|_{L^1(\mathbb{R}^d)} \xrightarrow{t \rightarrow \infty} 0.$$

On the other hand, for all  $g \in L^2(\mathbb{R}^d)$ ,  $\|e^{it\Delta}g\|_{L^2} = \|g\|_{L^2}$ .

Exercise 8.3: If  $g \in L^1 \cap L^2$  then  $\|e^{it\Delta}g\|_{L^p} \rightarrow 0$  for all  $2 < p \leq \infty$ .

2. For all bounded sets  $\Omega$ , for all  $g \in L^2(\mathbb{R}^d)$  we have  $\|\mathbb{1}_\Omega e^{it\Delta}g\|_{L^2(\mathbb{R}^d)} \rightarrow 0$  as  $t \rightarrow \infty$ . Equivalently, for all  $R > 0$ :

$$\int_{|x| \leq R} |(e^{it\Delta}g)(x)|^2 dx \xrightarrow{t \rightarrow \infty} 0$$

(RAGE theorem)

*Proof.* 1. With the Fourier transform of  $e^{it\Delta}g$  (Theorem 8.1) we get

$$\|e^{it\Delta}g - g\|_{L^2}^2 = \|\widehat{e^{it\Delta}g} - \hat{g}\|_{L^2}^2 = \int_{\mathbb{R}^d} \underbrace{(e^{-it|2\pi k|^2} - 1)}_{\substack{\xrightarrow{t \rightarrow 0} 0 \\ \text{bd. by } 2\forall t}} \underbrace{|\hat{g}(k)|^2}_{\in L^1} dk \xrightarrow{t \rightarrow 0} 0$$

by dominated convergence.

2. Decompose  $g = g_1 + g_2$ , then  $e^{it\Delta}g = e^{it\Delta}g_1 + e^{it\Delta}g_2$ . Take  $g_1 \in L^1$ , then by 1. we have

$$\begin{aligned} \|e^{it\Delta}g_1\|_{L^\infty} &\leq \frac{\|g_1\|_{L^1}}{(4\pi t)^{\frac{d}{2}}} \xrightarrow{t \rightarrow \infty} 0 \\ \Rightarrow \int_{|x| \leq R} |(e^{it\Delta}g_1)(x)|^2 &\leq \frac{CR^d \|g_1\|_{L^1}}{(4\pi t)^{\frac{d}{2}}} \xrightarrow{t \rightarrow \infty} 0 \end{aligned}$$

On the other hand, with a good choice of  $g_2$ :

$$\begin{aligned} \|e^{it\Delta}g_2\|_{L^2(\mathbb{R}^d)} &= \|g_2\|_{L^2(\mathbb{R}^d)} \rightarrow 0 \\ \Rightarrow \int_{|x| \leq R} |(e^{it\Delta}g_2)(x)|^2 dx &\leq \|e^{it\Delta}g_2\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0 \end{aligned}$$

Conclusion:

$$\int_{|x| \leq R} |(e^{it\Delta}g)(x)|^2 dx \leq 2 \int_{|x| < R} |(e^{it\Delta}g_1)(x)|^2 + 2 \int_{|x| \leq R} |(e^{it\Delta}g_2)(x)|^2 \xrightarrow{t \rightarrow \infty} 0$$

■

**Exercise 8.3** (E 13.4) Let  $g \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ . Let  $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$  be the solution of the Schrödinger equation with the initial data  $g$ , namely

$$u(x, t) = (e^{it\Delta}g)(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy.$$

Prove that for all  $2 \leq p \leq \infty$  we have

$$\lim_{t \rightarrow \infty} \|u(\bullet, t)\|_{L^p(\mathbb{R}^d)} = 0.$$

*Solution.* We already know:

$$\begin{aligned}\|e^{it\Delta}g\|_{L^\infty} &\leq \frac{\|g\|_{L^1}}{(4\pi t)^{\frac{d}{2}}} \xrightarrow{t \rightarrow 0} 0 \\ \|e^{it\Delta}g\|_{L^2} &= \|g\|_{L^2}\end{aligned}$$

Now, interpolation for all  $2 < p < \infty$  gives

$$\|e^{it\Delta}g\|_{L^p} \leq \|e^{it\Delta}g\|_{L^2}^\theta \|e^{it\Delta}g\|_{L^\infty}^{1-\theta} \xrightarrow{t \rightarrow \infty} 0,$$

for  $\theta \in (0, 1)$ . ■

**Exercise 8.4** (Bonus 12) Assume  $g \in L^p(\mathbb{R}^d)$ , for some  $1 < p < 2$ . Prove that for all  $t > 0$ ,  $e^{it\Delta}g$  is well-defined and belongs to  $L^{p'}(\mathbb{R}^d)$  where  $\frac{1}{p} + \frac{1}{p'} = 1$ . Moreover:

$$\|e^{it\Delta}g\|_{L^{p'}} \xrightarrow{t \rightarrow \infty} 0$$

Hint: You can use Riesz–Thorin interpolation theorem.

**Lemma 8.5** If  $g \in L^2(\mathbb{R}^d)$ , then for all  $\epsilon > 0$  we can write  $g = g_1 + g_2$ ,  $g, g_2 \in L^2$ ,  $g_1 \in L^1$ . s.t.  $\|g_2\|_{L^2} \leq \epsilon$  and  $\|g_1\|_{L^1} \leq C_\epsilon$ .

*Proof.*

$$g = \underbrace{g\mathbb{1}(|g| > L)}_{g_1} + \underbrace{g\mathbb{1}(|g| \leq L)}_{g_2}$$

Now  $g \in L^2$ , then  $g_2 \in L^2$  and

$$\int_{\mathbb{R}^d} |g_2|^2 = \int_{\mathbb{R}^d} |g|^2 \mathbb{1}(|g| \leq L) \xrightarrow{L \rightarrow 0} 0$$

and  $g_1 \in L^1$  since

$$\int_{\mathbb{R}^d} |g_1| = \int_{\mathbb{R}^d} |g| \mathbb{1}(|g| > L) \leq \int_{\mathbb{R}^d} |g| \frac{|g|}{L} = \frac{1}{L} \int_{\mathbb{R}^d} |g|^2 < \infty$$

Consequently:

$$\begin{aligned}\int_{|x| \leq R} |(e^{it\Delta}g)(x)|^2 dx &\leq 2 \int_{|x| \leq R} |e^{it\Delta}g_1(x)|^2 + 2 \int_{|x| \leq R} |e^{it\Delta}g_2(x)|^2 \\ &\leq CR^d \left( \frac{\|g\|_{L^1}^2}{t^{\frac{d}{2}}} \right) + \|g_2\|_{L^2}^2 \\ &\leq \frac{C_\epsilon R^d}{t^{\frac{d}{2}}} + \epsilon^2 \xrightarrow{t \rightarrow \infty} \epsilon^2 \xrightarrow{\epsilon \rightarrow 0} 0\end{aligned}$$
■

**Theorem 8.6** If  $g \in C_c^\infty(\mathbb{R}^d)$ , then

$$u(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} g(y) dy,$$

$x \in \mathbb{R}^d, t > 0$ . satisfies  $u \in C^\infty(\mathbb{R}^d \times (0, \infty))$  and

$$\begin{cases} -i\partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ \lim_{t \rightarrow 0} u(x, t) = g(x) \text{ uniformly } x \in \mathbb{R}^d \end{cases}.$$

*Proof.*

$$\begin{aligned}\hat{u}(k, t) &= e^{-it|2\pi k|^2} \hat{g}(k) \quad \forall k \in \mathbb{R}^d, \forall t > 0 \\ \Rightarrow \widehat{D^\alpha u}(k, t) &= e^{-it|2\pi k|^2} (2\pi i k)^\alpha \hat{g}(k), \quad (2\pi i k)^\alpha = \prod_{j=1}^d (2\pi i k_j)^{\alpha_j} \\ \Rightarrow \|\widehat{D^\alpha u}(k, t)\|_{L^1(\mathbb{R}^d, dk)} &= \int_{\mathbb{R}^d} |(2\pi i k)^\alpha| |\hat{g}(k)| dk \leq \int_{\mathbb{R}^d} |k|^{|\alpha|} |\hat{g}(k)| dk < \infty\end{aligned}$$

as  $|\hat{g}(k)| \leq \frac{C_N}{(1+|k|)^N}$  for all  $N \geq 1$ . This implies  $D^\alpha u(\bullet, t) \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . Moreover,

$$\partial_t \hat{u}(k, t) = e^{-it|2\pi k|^2} \hat{g}(k) \underbrace{(-i|2\pi k|^2)}_{\in L^1(\mathbb{R}^d, dk)}$$

$\leadsto t \mapsto u(x, t) \in C^1$ . Similary  $t \mapsto u(x, t) \in C^m$  for all  $m \geq 1$ .

$$\begin{aligned}-i\partial_t \widehat{u} - \widehat{\Delta_x u} &= (-i)e^{-it|2\pi k|^2} (-i|2\pi k|^2) \hat{g}(k) + e^{-it|2\pi k|^2} |2\pi k|^2 \hat{g}(k) = 0. \\ \Rightarrow -i\partial_t u - \Delta_x u &= 0 \quad \text{almost everywhere in } \mathbb{R}^d \\ \Rightarrow -i\partial_t u - \Delta_x u &= 0 \quad \text{everywhere in } \mathbb{R}^d\end{aligned}$$

Before we proved  $u(x, t) = (e^{it\Delta} g)(x) \rightarrow g(x)$  in  $L^2(\mathbb{R}^d)$ . Here we can do better:

$$\int_{\mathbb{R}^d} |\hat{u}(k, t) - \hat{g}(k)| dk = \int_{\mathbb{R}^d} \underbrace{|e^{-it|2\pi k|^2} - 1|}_{\xrightarrow{t \rightarrow 0} 0} \underbrace{|\hat{g}(k)|}_{\in L^1} \xrightarrow{t \rightarrow 0} 0$$

So we get

$$\sup_{x \in \mathbb{R}^d} |u(x, t) - g(x)| \xrightarrow{t \rightarrow 0} 0. \quad \blacksquare$$

**Theorem 8.7** 1. If  $g \in H^2(\mathbb{R}^d)$ , then  $u = e^{it\Delta} g$  satisfies

- $-i \frac{d}{dt} u(\bullet, t)$  exists as a function in  $L^2(\mathbb{R}^d)$  and  $= \Delta_x u$  for all  $t > 0$ .

$$\left( -i \frac{d}{dt} u(\bullet, t) = \frac{u(\bullet, t+h) - u(\bullet, t)}{h} \xrightarrow{h \rightarrow 0} \Delta_x u = e^{it\Delta} \underbrace{(\Delta g)}_{\in L^2} \in L^2(\mathbb{R}^d) \right)$$

- $u(\bullet, t) \xrightarrow{t \rightarrow 0} g$  in  $L^2$ .

2. If  $g \in L^2(\mathbb{R}^d)$ , then  $u = e^{it\Delta} g$  satisfies:

$$-i \frac{d}{dt} \langle \phi, u \rangle_{L^2} = \langle \Delta \phi, u \rangle_{L^2} \quad \forall \phi \in C_c^\infty(\mathbb{R}^d) \text{ (or } \phi \in H^2(\mathbb{R}^d))$$

(Here  $-i \frac{d}{dt} = \Delta_x u$  is meant in distributional sense)

*Proof.* 1.  $g \in H^2(\mathbb{R}^d)$ ,

$$\begin{aligned}
& \left\| \frac{u(\bullet, t+h) - u(\bullet, t)}{h} e^{it\Delta}(\Delta g) \right\|_{L^2(\mathbb{R}^d)}^2 = \left\| \frac{u(\bullet, t+h) - \widehat{u(\bullet, t)}}{h} e^{it\Delta}(\Delta g) \right\|_{L^2(k, dk)}^2 \\
&= \int_{\mathbb{R}^d} \left| \frac{e^{-(t+h)|2\pi k|^2} \hat{g}(k) - e^{-it|2\pi k|^2} \hat{g}(k)}{h} - e^{-it|2\pi k|^2} |2\pi k|^2 \hat{g}(k) \right|^2 dk \\
&= \int_{\mathbb{R}^d} \underbrace{\left| (-i) \frac{(e^{-ih|2\pi k|^2} - 1)}{h} + |2\pi k|^2 \right|^2 |\hat{g}(k)|^2}_{\xrightarrow{h \rightarrow 0} 0} dk \xrightarrow{h \rightarrow 0} 0
\end{aligned}$$

by Dominated Convergence,  $|\dots| \leq C|k|^2 |\hat{g}(k)|^2 \in L^1$ . We have

$$(-i) \frac{e^{-ih|2\pi k|^2} - 1}{h} \xrightarrow{h \rightarrow 0} (-i)^2 |2\pi k|^2 = -|2\pi k|^2$$

and  $|\dots| \leq C|k|^2$ . Also

$$\begin{aligned}
& \hat{u} = \widehat{e^{it\Delta} g} = e^{-it|2\pi k|^2} \hat{g}(k) \\
& \Rightarrow |2\pi k|^2 \hat{u}(k) = e^{-it|2\pi k|^2} \underbrace{|2\pi k|^2 \hat{g}(k)}_{\in L^2(dk)} \in L^2(dk) \\
& \Rightarrow \widehat{\Delta_x u} = e^{it\Delta}(\Delta g) \\
& \Rightarrow \Delta_x u = e^{it\Delta}(\Delta g)
\end{aligned}$$

2.  $g \in L^2$ :

$$\begin{aligned}
-i \frac{d}{dt} \langle \phi, u \rangle &= -i \frac{d}{dt} \int_{\mathbb{R}^d} \underbrace{\widehat{\phi(k)}}_{\leq \frac{C_N}{(1+|k|)^N} \forall N \geq 1} e^{-it|2\pi k|^2} \hat{g}(k) dk \\
&= \int_{\mathbb{R}^d} \widehat{\phi(k)} (-|2\pi k|^2) \hat{g}(k) dk \\
&= \int_{\mathbb{R}^d} \widehat{\Delta \phi} \hat{g} = \langle \Delta \phi, g \rangle_{L^2}
\end{aligned}$$

Spectral technique: If  $\mathcal{H}$  is a Hilbertspace (separable) and  $A : D(A) \rightarrow \mathcal{H}$  is a self-adjoint  $A = A^*$ , i.e.

$$\begin{cases} \langle u, Av \rangle = \langle Au, v \rangle \\ D(A) = D(A^*) \end{cases}$$

where

$$\begin{aligned}
D(A^*) &= \{u \in \mathcal{H} \mid \langle u, Av \rangle = \langle A^* u, v \rangle\} \\
&= \left\{ u \in \mathcal{H} \mid \sup_{v \in D(A)} \frac{|\langle u, Av \rangle|}{\|v\|_{\mathcal{H}}} < \infty \right\}
\end{aligned}$$

Then we can define a unitary operator  $e^{-itA} : \mathcal{H} \rightarrow \mathcal{H}$  with  $t \in \mathbb{R}$ . Moreover, for all  $g \in \mathcal{H}$ ,  $u = e^{-itA} g$  solves

$$\begin{cases} i\partial_t u = Au \\ u(t \rightarrow 0) = g \end{cases}$$

(Stone Theorem) (eg  $A = -\Delta$ )

■



**Theorem 8.8** (Spectral theorem) If  $A$  is a self-adjoint operator on a separable Hilbert space, then, there is a unitary operator  $U : \mathcal{H} \rightarrow L^2(\Omega)$  s.t.  $UAU^* = M_f$  on  $L^2(\Omega)$ , where  $M_f$  is a multiplication operator of a real-value function  $f : \Omega \rightarrow \mathbb{R}$ :

$$(M_f)(\phi)(x) = f(x)\phi(x)$$

for all  $x \in \Omega, \forall \phi \in L^2(\Omega)$ . Consequence:  $Ue^{-itA}U^* = e^{-itM_f}$  (multiplication) on  $L^2(\Omega)$  where

$$(e^{-itM_f}\phi)(x) = e^{-itf(x)}\phi(x)$$

for all  $x \in \Omega, \phi \in L^2(\Omega)$ .

**Example 8.9**  $A = -\Delta$  on  $\mathcal{H} = L^2(\mathbb{R}^d)$ ,  $U = \mathcal{F}$  (Fourier Transform),  $\mathcal{F}(-\Delta)\mathcal{F}^* = |2\pi k|^2$  multiplication.

$$\mathcal{F}(e^{-itA}\phi) = e^{-it|2\pi k|^2}\hat{\phi}(k)$$

**Example 8.10** Consider  $A = -\Delta + V(x)$  in  $L^2(\mathbb{R}^d)$ ,  $V(x) \in C_c^\infty(\mathbb{R}^d)$  or  $V(x) = |x|^2$ . If  $V$  is nice, then  $A$  can be defined  $D(A) \rightarrow L^2(\mathbb{R}^d)$  as a self-adjoint operator.  $\rightsquigarrow e^{-itA} = e^{-it(-\Delta+v)}$  is well-defined as a unitary operator for all  $t \in \mathbb{R}$ . In particular, for all  $g \in L^2$ ,  $u = e^{-it(-\Delta+v)}g$  solves

$$\begin{cases} i\partial_t u = (-\Delta + v)u \\ u(t \rightarrow 0) = g \end{cases}$$

Scatter Theory:  $e^{-it(-\Delta+v)}g$  as  $t \rightarrow \infty$ .

1. RAGE: If  $-\Delta + V$  has no eigenfunction, then

$$\frac{1}{T} \int_0^T \int_{|x| \leq R} \left| \left( e^{-it(-\Delta+v)}g \right)(x) \right|^2 dx dt \xrightarrow{T \rightarrow \infty} 0$$

2. (Asymptotic completeness) If  $0 \leq V \in C_c^\infty$  is small, then there is a unitary operator  $U$  on  $L^2$  s.t. for all  $g \in L^2$ :

$$\left\| e^{-it(-\Delta+v)}g - e^{-it\Delta}(Ug) \right\|_{L^2(\mathbb{R}^d)} \xrightarrow{t \rightarrow \infty} 0$$