

Type checker for system F

Mudathir Mahgoub

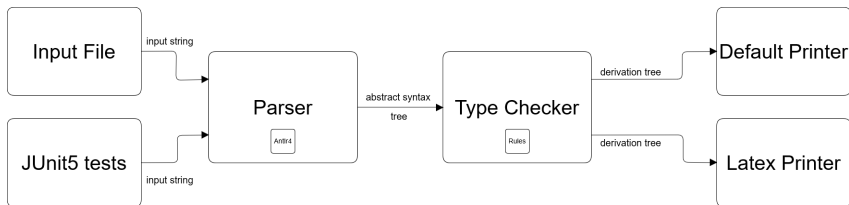
May 3, 2018

Type checking problem

Given a typing context Γ and a term t and a type T , can the term t be assigned the type T under the typing context Γ ?

Is the judgment $\Gamma \vdash t : T$ derivable?

Software components



Execution

Input: test.txt

```
SubBase(bool , int );  
. |- \lambda x. \lambda y. (x y)[bool]: (int ->T) -> (bool -> T);
```

Output: java -jar TypeChecker.jar -i test.txt

Yes

$$\frac{\frac{}{x: \text{bool} \mid - x : \text{bool}} \text{---} (\text{var}) \quad \frac{\text{SubBase}(\text{bool} , \text{int})}{\text{bool} <: \text{int}} \text{---} (\text{subBase})}{x: \text{bool} \mid - x : \text{int}} \text{---} (\text{subsumption})$$

Execution

Input: test.txt

```
SubBase(bool , int );  
. |- \lambda x. \lambda y. (x y)[bool]: (int ->T) -> (bool -> T);
```

Output: java -jar TypeChecker.jar -i test.txt -latex

```
Yes  
\begin{prooftree}  
\AxiomC{}  
\RightLabel{\scriptsize var}  
\UnaryInfC{$x: bool \vdash x : bool$}  
\AxiomC{\scriptsize $SubBase(bool, int)$}  
\RightLabel{\scriptsize subBase}  
\UnaryInfC{$bool <: int$}  
\RightLabel{\scriptsize subsumption}  
\BinaryInfC{$x: bool \vdash x : int$}  
\end{prooftree}
```

Demo

Demo

Simple types

Terms

$$t ::= x \mid (t_1 t_2)[T] \mid \lambda x. t$$

Typing rules

1 Variable rule

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T} \text{ var}$$

2 Application rule

$$\frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2)[T_1] : T_2} \text{ app}$$

3 λ rule

$$\frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x. t : T_1 \rightarrow T_2} \lambda$$

Simple types

Subtyping rules

$$\frac{\Gamma \vdash t : T_1 \quad T_1 <: T_2}{\Gamma \vdash t : T_2} \text{ subsumption}$$

$$\frac{}{T <: T} \text{ reflexive}$$

$$\frac{SubBase(b_1, b_2)}{b_1 <: b_2} \text{ subBase}$$

$$\frac{T'_1 <: T_1 \quad T_2 <: T'_2}{T_1 \rightarrow T_2 <: T'_1 \rightarrow T'_2} \text{ arrow}$$

$$\frac{T_1 <: T_2 \quad T_2 <: T_3}{T_1 <: T_3} \text{ transitive}$$

Simple types

Examples

$$\frac{}{x : T \vdash x : T} \text{ var}$$

$$\frac{}{\cdot \vdash x : T} \text{ invalid var}$$

$$\frac{\frac{\frac{}{x : (T_1 \rightarrow T_2), y : T_1 \vdash x : (T_1 \rightarrow T_2)} \text{ var} \quad \frac{}{x : (T_1 \rightarrow T_2), y : T_1 \vdash y : T_1} \text{ var}}{\frac{x : (T_1 \rightarrow T_2), y : T_1 \vdash (x y)[T_1] : T_2}{x : (T_1 \rightarrow T_2) \vdash \lambda y. (x y)[T_1] : (T_1 \rightarrow T_2)} \lambda} \text{ app}$$
$$\frac{\cdot \vdash \lambda x. \lambda y. (x y)[T_1] : ((T_1 \rightarrow T_2) \rightarrow (T_1 \rightarrow T_2))}{\cdot \vdash \lambda x. \lambda y. (x y)[T_1] : ((T_1 \rightarrow T_2) \rightarrow (T_1 \rightarrow T_2))} \lambda$$

Subtyping

Examples

$$\frac{\frac{}{x : \text{bool} \vdash x : \text{bool}} \text{var} \quad \frac{\text{SubBase}(\text{bool}, \text{int})}{\text{bool} <: \text{int}} \text{subBase}}{x : \text{bool} \vdash x : \text{int}} \text{subsumption}$$

$$\frac{\frac{}{x : \text{int} \vdash x : \text{int}} \text{var} \quad \frac{\perp}{\text{int} <: \text{bool}} \text{invalid}}{x : \text{int} \vdash x : \text{bool}} \text{subsumption}$$

$$\frac{\frac{}{x : \text{bool} \vdash x : \text{bool}} \text{var} \quad \frac{\frac{\text{SubBase}(\text{bool}, \text{int})}{\text{bool} <: \text{int}} \text{subBase} \quad \frac{\frac{\text{SubBase}(\text{int}, \text{double})}{\text{int} <: \text{double}} \text{subBase}}{\text{bool} <: \text{double}} \text{transitive}}{x : \text{bool} \vdash x : \text{double}} \text{subsumption}$$

Subtyping

$$\begin{array}{c}
 \frac{}{x : (int \rightarrow T), y : bool \vdash x : (int \rightarrow T)} \text{var} \qquad \frac{\text{SubBase}(bool, int)}{bool <: int} \text{subBase} \qquad \frac{}{T <: T} \text{reflexive} \\
 \frac{}{(int \rightarrow T) <: (bool \rightarrow T)} \text{arrow} \\
 \frac{}{x : (int \rightarrow T), y : bool \vdash x : (bool \rightarrow T)} \text{subsumption} \qquad \frac{}{x : (int \rightarrow T), y : bool \vdash y : bool} \text{var} \\
 \frac{}{x : (int \rightarrow T), y : bool \vdash x : (bool \rightarrow T)} \text{app} \\
 \frac{x : (int \rightarrow T), y : bool \vdash (x y)[bool] : T}{x : (int \rightarrow T) \vdash \lambda y. (x y)[bool] : (bool \rightarrow T)} \lambda \\
 \frac{}{\vdash \lambda x. \lambda y. (x y)[bool] : ((int \rightarrow T) \rightarrow (bool \rightarrow T))} \lambda
 \end{array}$$

Delaying the subsumption rule

Remark

The subsumption rule can be delayed until the term in the judgment is a variable.

$$\frac{\Gamma \vdash t : T \quad T <: T'}{\Gamma \vdash t : T'} \text{ subsumption}$$

Delaying the subsumption rule

Case $t = x$: trivial since t is a variable.

$$\frac{\Gamma \vdash x : T \quad T <: T'}{\Gamma \vdash x : T'} \text{ subsumption}$$

Delaying the subsumption rule

Case $t = t_1 t_2$: assume $T_2 <: T'_2$ and the following derivation:

$$\frac{\frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2)[T_1] : T_2} \text{ app} \quad T_2 <: T'_2}{\Gamma \vdash (t_1 t_2)[T_1] : T'_2} \text{ subsumption}$$

Alternatively

$$\frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \frac{\frac{}{T_1 <: T_1} \text{ reflexive} \quad T_2 <: T'_2}{T_1 \rightarrow T_2 <: T_1 \rightarrow T'_2} \text{ arrow}}{\Gamma \vdash t_1 : T_1 \rightarrow T'_2} \text{ subsumption} \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2)[T_1] : T'_2} \text{ app}$$

Delaying the subsumption rule

Case $t = \lambda x.t'$: assume $T_1 \rightarrow T_2 <: T'_1 \rightarrow T'_2$ and the following derivation:

$$\frac{\frac{\Gamma, x : T_1 \vdash t' : T_2}{\Gamma \vdash \lambda x.t' : T_1 \rightarrow T_2} \lambda \quad \frac{T'_1 <: T_1 \quad T_2 <: T'_2}{T_1 \rightarrow T_2 <: T'_1 \rightarrow T'_2} \text{arrow}}{\Gamma \vdash \lambda x.t' : T'_1 \rightarrow T'_2} \text{subsumption}$$

Alternatively

$$\frac{\frac{\Gamma, x : T'_1 \vdash t' : T_2 \quad T_2 <: T'_2}{\Gamma, x : T'_1 \vdash t' : T'_2} \text{subsumption}}{\Gamma \vdash \lambda x.t' : T'_1 \rightarrow T'_2} \lambda$$

Assuming $T'_1 <: T_1$ is derivable, then by the subsumption rule:

$$\frac{\frac{\Gamma, x : T'_1 \vdash x : T'_1}{\Gamma, x : T'_1 \vdash x : T_1} \text{var} \quad T'_1 <: T_1}{\Gamma, x : T'_1 \vdash x : T_1} \text{subsumption}$$

Delaying the subsumption rule

Remark

If $T'_1 <: T_1$ and $\Gamma, x : T'_1 \vdash t' : T_2$, it is not always true that $\Gamma, x : T_1 \vdash t' : T_2$.

Counter example

$$x : \text{bool} \vdash x : \text{int} \not\Rightarrow x : \text{double} \vdash x : \text{int}, \quad \text{bool} <: \text{int} <: \text{double}$$

$$x : \text{int} \vdash x : \text{double} \Rightarrow x : \text{bool} \vdash x : \text{double}, \quad \text{bool} <: \text{int} <: \text{double}$$

Delaying the subsumption rule

$$\frac{\frac{x : \text{double} \vdash x : \text{int}}{\Gamma \vdash \lambda x.x : \text{double} \rightarrow \text{int}} \lambda \quad \frac{\frac{\text{SubBase}(\text{bool}, \text{double})}{\text{bool} <: \text{double}} \text{subBase} \quad \frac{}{\text{int} <: \text{int}} \text{reflexive arrow}}{\text{double} \rightarrow \text{int} <: \text{bool} \rightarrow \text{int}} \text{subsumption}}{\Gamma \vdash \lambda x.x : \text{bool} \rightarrow \text{int}}$$

$$\frac{\frac{\Gamma, x : \text{bool} \vdash x : \text{bool}}{\Gamma, x : \text{bool} \vdash x : \text{int}} \text{var} \quad \frac{\text{SubBase}(\text{bool}, \text{int})}{\text{bool} <: \text{int}} \text{subBase}}{\Gamma \vdash \lambda x.x : \text{bool} \rightarrow \text{int}} \text{subsumption} \lambda$$

System F

Terms

$$t ::= x \mid (t_1 t_2) [T] \mid \lambda x. t \mid t \llbracket [T] \rrbracket$$

Types

$$T ::= X \mid T_1 \rightarrow T_2 \mid \forall X. T$$

System F additional rules

- 1 If X is not free in Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash t : \forall X. T} \text{ introduction}$$

- 2 If X is free in Γ , choose X_i such that X_i is not free in Γ

$$\frac{\frac{\Gamma \vdash t : [X_i/X]T}{\Gamma \vdash t : \forall X_i. [X_i/X]T} \text{ introduction}}{\Gamma \vdash t : \forall X. T} \text{ renaming}$$

- 3 Elimination rule with annotation

$$\frac{\Gamma \vdash t : \forall X. [X/T']T}{\Gamma \vdash t \llbracket T' \rrbracket : T} \text{ elimination}$$

System F examples

- Different variables

$$\frac{}{x : \forall X.(X \rightarrow X) \vdash x : \forall Y.(Y \rightarrow Y)} \text{var}$$

- Y is free in the typing context and the term type

$$\frac{}{x : \forall X.(X \rightarrow Y) \vdash x : \forall Z.(Z \rightarrow Y)} \text{var}$$

- Y is free in the typing context

$$\frac{}{x : \forall X.(X \rightarrow Y) \vdash x : \forall Y.(Y \rightarrow Y)} \text{invalid var}$$

- Y is free in the typing context, and Z is free in the term type

$$\frac{}{x : \forall X.(X \rightarrow Y) \vdash x : \forall Y.(Y \rightarrow Z)} \text{invalid var}$$

System F examples

- Elimination annotation

$$\frac{\frac{}{x : \forall X.X \vdash x : \forall X1. X1} \text{var}}{x : \forall X.X \vdash x[[Y]] : Y} \text{elimination}$$

- Elimination annotation with arrow

$$\frac{\frac{}{x : \forall X.X \vdash x : \forall X1. X1} \text{var}}{x : \forall X.X \vdash x[[(Y \rightarrow Y)]] : (Y \rightarrow Y)} \text{elimination}$$

System F examples

- Zero

$$\frac{\frac{\frac{\overline{z : X, s : (X \rightarrow X) \vdash z : X} \text{ var}}{s : (X \rightarrow X) \vdash (\lambda z.z) : (X \rightarrow X)} \lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : ((X \rightarrow X) \rightarrow (X \rightarrow X))} \lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \rightarrow X) \rightarrow (X \rightarrow X))} \text{introduction}$$

- Zero with free variable X

$$\frac{\frac{\frac{\overline{y : X, z : X_2, s : (X_2 \rightarrow X_2) \vdash z : X_2} \text{ var}}{y : X, s : (X_2 \rightarrow X_2) \vdash (\lambda z.z) : (X_2 \rightarrow X_2)} \lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : ((X_2 \rightarrow X_2) \rightarrow (X_2 \rightarrow X_2))} \lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X_2.((X_2 \rightarrow X_2) \rightarrow (X_2 \rightarrow X_2))} \text{introduction} \\ \frac{}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \rightarrow X) \rightarrow (X \rightarrow X))} \text{renaming}$$