1 Rules

1. $$\frac{\Gamma(x) = T}{\Gamma \vdash x : T} \text{ var}$$

2. $$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2)[T_1] : T_2} \text{ app}$$

3. $$\frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \lambda$$

4. $$\frac{\Gamma \vdash t : T_1 \qquad T_1 <: T_2}{\Gamma \vdash t : T_2} \text{ subsumption}$$

5. $$\frac{}{T <: T} \text{ reflexive}$$

6. $$\frac{SubBase(b_1, b_2)}{b_1 <: b_2} \text{ subBase}$$

7. $$\frac{T_1' <: T_1 \qquad T_2 <: T_2'}{T_1 \to T_2 <: T2_1 \to T_2'} \text{ arrow}$$

8. $$\frac{T_1 <: T_2 \qquad T_2 <: T_3}{T_1 <: T_3} \text{ transitive}$$

2 Variable terms

2.1 Valid

$$\frac{}{x : T \vdash x : T} \text{ var}$$

2.2 Invalid

$$\frac{}{\cdot \vdash x : T} \text{ invalid var}$$

3 Lambda & application terms

$$\frac{\dfrac{}{x : (T1 \to T2), y : T1 \vdash x : (T1 \to T2)} \text{ var} \qquad \dfrac{}{x : (T1 \to T2), y : T1 \vdash y : T1} \text{ var}}{\dfrac{\dfrac{x : (T1 \to T2), y : T1 \vdash (x\ y)[T1] : T2}{x : (T1 \to T2) \vdash \lambda y.(x\ y)[T1] : (T1 \to T2)} \lambda}{\cdot \vdash \lambda x.\lambda y.(x\ y)[T1] : ((T1 \to T2) \to (T1 \to T2))} \lambda} \text{ app}$$

4 Direct Subtyping

4.1 Valid

$$\frac{\dfrac{}{x : bool \vdash x : bool} \text{ var} \qquad \dfrac{SubBase(bool, int)}{bool <: int} \text{ subBase}}{x : bool \vdash x : int} \text{ subsumption}$$

4.2 Invalid

$$\cfrac{\cfrac{}{x : int \vdash x : int}\ \text{var} \quad \cfrac{\color{red}\bot}{\color{red}int <: bool}\ \color{red}\text{invalid}}{x : int \vdash x : bool}\ \text{subsumption}$$

5 Transitive Subtyping

$$\cfrac{\cfrac{}{x : bool \vdash x : bool}\ \text{var} \quad \cfrac{\cfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase} \quad \cfrac{\text{SubBase}(int, double)}{int <: double}\ \text{subBase}}{bool <: double}\ \text{transitive}}{x : bool \vdash x : double}\ \text{subsumption}$$

$$\cfrac{\cfrac{}{x : bool \vdash x : bool}\ \text{var} \quad \cfrac{\cfrac{\cfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase} \quad \cfrac{\text{SubBase}(int, quotient)}{int <: quotient}\ \text{subBase}}{bool <: quotient}\ \text{transitive} \quad \cfrac{\text{SubBase}(quotient, double)}{quotient <: double}\ \text{subBase}}{bool <: double}\ \text{transitive}}{x : bool \vdash x : double}\ \text{subsumption}$$

6 Arrow Types

6.1 Valid

$$\cfrac{\cfrac{}{x : (int \to bool) \vdash x : (int \to bool)}\ \text{var} \quad \cfrac{\cfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase} \quad \cfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase}}{(int \to bool) <: (bool \to int)}\ \text{arrow}}{x : (int \to bool) \vdash x : (bool \to int)}\ \text{subsumption}$$

6.1.1 Reflexive

$$\cfrac{\cfrac{}{x : (int \to bool) \vdash x : (int \to bool)}\ \text{var} \quad \cfrac{\cfrac{}{int <: int}\ \text{reflexive} \quad \cfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase}}{(int \to bool) <: (int \to int)}\ \text{arrow}}{x : (int \to bool) \vdash x : (int \to int)}\ \text{subsumption}$$

6.1.2 Nested

$$\cfrac{\cfrac{\cfrac{\cfrac{}{x : (int \to T), y : bool \vdash x : (int \to T)}\ \text{var} \quad \cfrac{\cfrac{\cfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase} \quad \cfrac{}{T <: T}\ \text{reflexive}}{(int \to T) <: (bool \to T)}\ \text{arrow}}{}}{x : (int \to T), y : bool \vdash x : (bool \to T)}\ \text{subsumption} \quad \cfrac{}{x : (int \to T), y : bool \vdash y : bool}\ \text{var}}{\cfrac{x : (int \to T), y : bool \vdash (x\ y)[bool] : T}{\cfrac{x : (int \to T) \vdash \lambda y.(x\ y)[bool] : (bool \to T)}{\cdot \vdash \lambda x.\lambda y.(x\ y)[bool] : ((int \to T) \to (bool \to T))}\ \lambda}\ \lambda}}{}\ \text{app}$$

6.2 Invalid

$$\cfrac{\cfrac{}{x : (bool \to bool) \vdash x : (bool \to bool)}\ \text{var} \quad \cfrac{\cfrac{\color{red}\bot}{\color{red}int <: bool}\ \color{red}\text{invalid} \quad \cfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase}}{(bool \to bool) <: (int \to int)}\ \text{arrow}}{x : (bool \to bool) \vdash x : (int \to int)}\ \text{subsumption}$$

$$\cfrac{\cfrac{\cfrac{}{x : (bool \to T), y : bool \vdash x : (bool \to T)}\ \text{var} \quad \cfrac{\cfrac{\cfrac{\color{red}\bot}{\color{red}int <: bool}\ \color{red}\text{invalid} \quad \cfrac{}{T <: T}\ \text{reflexive}}{(bool \to T) <: (int \to T)}\ \text{arrow}}{x : (bool \to T), y : bool \vdash x : (int \to T)}\ \text{subsumption}}{x : (bool \to T), y : bool \vdash x : (int \to T)} \quad \cfrac{\cfrac{}{x : (bool \to T), y : bool \vdash y : bool}\ \text{var} \quad \cfrac{\text{SubBase}(bool, in}{bool <: int}}{x : (bool \to T), y : bool \vdash y : int}\ \text{app}}{\cfrac{x : (bool \to T), y : bool \vdash (x\ y)[int] : T}{\cfrac{x : (bool \to T) \vdash \lambda y.(x\ y)[int] : (bool \to T)}{\cdot \vdash \lambda x.\lambda y.(x\ y)[int] : ((bool \to T) \to (bool \to T))}\ \lambda}\ \lambda}$$

6.3 Delaying applying the subsumption rule

The sumbumption rule can be delayed until the term in the judgment is a just a variable. This simplifies the code since there is only one rule to be applied for application and λ abstraction terms. Here is the correctness proof.

$$\frac{\Gamma \vdash t : T \qquad T <: T'}{\Gamma \vdash t : T'} \text{ subsumption}$$

Proof:
By induction hypothesis on the structure of the term.

1. Case $t = x$: trivial since t is a variable.

$$\frac{\Gamma \vdash x : T \qquad T <: T'}{\Gamma \vdash x : T'} \text{ subsumption}$$

2. Case $t = t_1 t_2$: assume $T_2 <: T_2'$ and the following derivation:

$$\frac{\dfrac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 t_2 [T_1] : T_2} \text{ app} \qquad T_2 <: T_2'}{\Gamma \vdash t_1 t_2 [T_1] : T_2'} \text{ subsumption}$$

We can get a different derivation tree where the sumbsumption rule is delayed:

$$\frac{\Gamma \vdash t_1 t_2[T_1] : T_2 \quad \dfrac{\dfrac{}{T_1 <: T_1}\text{ reflexive} \qquad T_2 <: T_2'}{\dfrac{T_1 \to T_2 <: T_1 \to T_2'}{\Gamma \vdash t_1 : T_1 \to T_2'}\text{ subsumption}}\text{ arrow} \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 t_2 [T_1] : T_2'}\text{ app}$$

By the induction hypothesis, the subsumption rule can be delayed until variable terms are generated in the derivation of $\Gamma \vdash t_1 : T_1 \to T_2'$ and $\Gamma \vdash t_2 : T_1$.

3. Case $t = \lambda x.t'$: assume $T_1 \to T_2 <: T_1' \to T_2'$ and the following derivation:

$$\frac{\dfrac{\Gamma, x : T_1 \vdash t' : T_2}{\Gamma \vdash \lambda x.t' : T_1 \to T_2}\lambda \qquad \dfrac{T_1' <: T_1 \qquad T_2 <: T_2'}{T_1 \to T_2 <: T_1' \to T_2'}\text{ arrow}}{\Gamma \vdash \lambda x.t' : T_1' \to T_2'}\text{ subsumption}$$

Alternatively we can derive:

$$\frac{\dfrac{\Gamma, x : T_1' \vdash t' : T_2 \qquad T_2 <: T_2'}{\Gamma, x : T_1' \vdash t' : T_2'}\text{ subsumption}}{\Gamma \vdash \lambda x.t' : T_1' \to T_2'}\lambda$$

Since $T_1' <: T_1$ then by the subsumption rule:

$$\frac{\dfrac{}{\Gamma, x : T_1' \vdash x : T_1'}\text{ var} \qquad T_1' <: T_1}{\Gamma, x : T_1' \vdash x : T_1}\text{ subsumption}$$

Therefore if $\Gamma, x : T_1 \vdash t' : T_2$ then $\Gamma, x : T_1' \vdash t' : T_2$ which concludes the proof.

Note:

If $\Gamma, x : T_1' \vdash t' : T_2$, it is not always true that $\Gamma, x : T_1 \vdash t' : T_2$ where $T_1' <: T_1$. A counter example would be

$$x : bool \vdash x : int \not\Rightarrow x : double \vdash x : int, \quad bool <: int <: double$$

However

$$x : int \vdash x : double \Rightarrow x : bool \vdash x : double, \quad bool <: int <: double$$

Therefore in the following example, using the subsumption rule first would fail and slow the type checker because it needs to backtrack and check the λ rule. However using the subsumption rule would prove the type checking and is faster.

$$\cfrac{\cfrac{x : double \vdash x : int}{\Gamma \vdash \lambda x.x : double \to int}\ \lambda \qquad \cfrac{\cfrac{\cfrac{SubBase(bool, double)}{bool <: double}\ \text{subBase} \qquad \cfrac{}{int <: int}\ \text{reflexive}}{double \to int <: bool \to int}\ \text{arrow}}{}}{\Gamma \vdash \lambda x.x : bool \to int}\ \text{subsumption}$$

$$\cfrac{\cfrac{\cfrac{}{\Gamma, x : bool \vdash x : bool}\ \text{var} \qquad \cfrac{\cfrac{SubBase(bool, int)}{bool <: int}\ \text{suBase}}{}}{\cfrac{\Gamma, x : bool \vdash x : int}{\Gamma \vdash \lambda x.x : bool \to int}\ \lambda}\ \text{subsumption}}{}$$

7 System F

7.1 Variables

7.1.1 Valid

$$\cfrac{}{x : \forall X.X \vdash x : \forall X.X}\ \text{var}$$

$$\cfrac{}{x : \forall X.X \vdash x : \forall Y.Y}\ \text{var}$$

$$\cfrac{}{x : \forall X.(X \to X) \vdash x : \forall Y.(Y \to Y)}\ \text{var}$$

Y is free in the context and the type:

$$\cfrac{}{x : \forall X.(X \to Y) \vdash x : \forall Z.(Z \to Y)}\ \text{var}$$

$$\cfrac{\cfrac{}{x : \forall X.X \vdash x : \forall X1.X1}\ \text{var}}{x : \forall X.X \vdash x[Y] : Y}\ \text{elimination}$$

$$\cfrac{\cfrac{}{x : \forall X.X \vdash x : \forall X1.X1}\ \text{var}}{x : \forall X.X \vdash x[(Y \to Y)] : (Y \to Y)}\ \text{elimination}$$

$$\cfrac{\cfrac{\cfrac{}{x : \forall X.X \vdash x : \forall X1.X1}\ \text{var}}{x : \forall X.X \vdash x[[(\forall X.X \to \forall X.X)]][(\forall X.X \to \forall X.X)] : (\forall X.X \to \forall X.X)}\ \text{elimination} \qquad \cfrac{\cfrac{}{x : \forall X.X \vdash x : \forall X.X}\ \text{var}}{}\ \text{app}}{x : \forall X.X \vdash (x[[(\forall X.X \to \forall X.X)]]\ x)[\forall X.X] : \forall X.X}$$

4

7.1.2 Invalid

Y is free in the context:

$$\frac{}{x : \forall X.(X \to Y) \vdash x : \forall Y.(Y \to Y)} \text{ invalid var}$$

Y is free in the context, and Z is free in the type:

$$\frac{}{x : \forall X.(X \to Y) \vdash x : \forall Y.(Y \to Z)} \text{ invalid var}$$

$$\frac{\dfrac{}{x : \forall X.X \vdash x : \forall X1.X1} \text{ var}}{x : \forall X.X \vdash x[Y] : Y} \text{ elimination}$$

7.2 Numbers

Zero

$$\frac{\dfrac{\dfrac{\dfrac{}{z : X, s : (X \to X) \vdash z : X} \text{ var}}{s : (X \to X) \vdash (\lambda z.z) : (X \to X)} \lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : ((X \to X) \to (X \to X))} \lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \to X) \to (X \to X))} \text{ introduction}$$

Zero with free variable X

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{y : X, z : X_2, s : (X_2 \to X_2) \vdash z : X_2} \text{ var}}{y : X, s : (X_2 \to X_2) \vdash (\lambda z.z) : (X_2 \to X_2)} \lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : ((X_2 \to X_2) \to (X_2 \to X_2))} \lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X_2.((X_2 \to X_2) \to (X_2 \to X_2))} \text{ introduction}}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \to X) \to (X \to X))} \text{ renaming}$$

Successor missing annotation

Successor well annotated