

1 Rules

1. $\frac{\Gamma(x) = T}{\Gamma \vdash x : T} \text{ var}$
2. $\frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2)[T_1] : T_2} \text{ app}$
3. $\frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x. t : T_1 \rightarrow T_2} \lambda$
4. $\frac{\Gamma \vdash t : T_1 \quad T_1 <: T_2}{\Gamma \vdash t : T_2} \text{ subsumption}$
5. $\frac{}{T <: T} \text{ reflexive}$
6. $\frac{\text{SubBase}(b_1, b_2)}{b_1 <: b_2} \text{ subBase}$
7. $\frac{T'_1 <: T_1 \quad T_2 <: T'_2}{T_1 \rightarrow T_2 <: T'_1 \rightarrow T'_2} \text{ arrow}$
8. $\frac{T_1 <: T_2 \quad T_2 <: T_3}{T_1 <: T_3} \text{ transitive}$

2 Variable terms

2.1 Valid

$$\frac{}{x : T \vdash x : T} \text{ var}$$

2.2 Invalid

$$\frac{}{\cdot \vdash x : T} \text{ invalid var}$$

3 Lambda & application terms

$$\frac{\frac{\frac{}{x : (T1 \rightarrow T2), y : T1 \vdash x : (T1 \rightarrow T2)} \text{ var} \quad \frac{}{x : (T1 \rightarrow T2), y : T1 \vdash y : T1} \text{ var}}{x : (T1 \rightarrow T2), y : T1 \vdash (x y)[T1] : T2} \text{ app}}{\frac{x : (T1 \rightarrow T2) \vdash \lambda y. (x y)[T1] : (T1 \rightarrow T2)}{\cdot \vdash \lambda x. \lambda y. (x y)[T1] : ((T1 \rightarrow T2) \rightarrow (T1 \rightarrow T2))} \lambda} \lambda$$

4 Direct Subtyping

4.1 Valid

$$\frac{\frac{}{x : \text{bool} \vdash x : \text{bool}} \text{ var} \quad \frac{\text{SubBase}(\text{bool}, \text{int})}{\text{bool} <: \text{int}} \text{ subBase}}{x : \text{bool} \vdash x : \text{int}} \text{ subsumption}$$

4.2 Invalid

$$\frac{\frac{}{x : int \vdash x : int} \text{var} \quad \frac{\perp}{int <: bool} \text{invalid}}{x : int \vdash x : bool} \text{subsumption}$$

5 Transitive Subtyping

$$\frac{\frac{}{x : bool \vdash x : bool} \text{var} \quad \frac{\frac{\text{SubBase}(bool, int)}{bool <: int} \text{subBase} \quad \frac{\text{SubBase}(int, double)}{int <: double} \text{subBase}}{bool <: double} \text{transitive}}{x : bool \vdash x : double} \text{subsumption}$$

$$\frac{\frac{}{x : bool \vdash x : bool} \text{var} \quad \frac{\frac{\text{SubBase}(bool, int)}{bool <: int} \text{subBase} \quad \frac{\text{SubBase}(int, quotient)}{int <: quotient} \text{subBase}}{bool <: quotient} \text{transitive} \quad \frac{\text{SubBase}(quotient, double)}{quotient <: double} \text{subBase}}{x : bool \vdash x : double} \text{subsumption}$$

6 Arrow Types

6.1 Valid

$$\frac{\frac{}{x : (int \rightarrow bool) \vdash x : (int \rightarrow bool)} \text{var} \quad \frac{\frac{\text{SubBase}(bool, int)}{bool <: int} \text{subBase} \quad \frac{\text{SubBase}(bool, int)}{bool <: int} \text{subBase}}{(int \rightarrow bool) <: (bool \rightarrow int)} \text{arrow}}{x : (int \rightarrow bool) \vdash x : (bool \rightarrow int)} \text{subsumption}$$

6.1.1 Reflexive

$$\frac{\frac{}{x : (int \rightarrow bool) \vdash x : (int \rightarrow bool)} \text{var} \quad \frac{\frac{}{int <: int} \text{reflexive} \quad \frac{\text{SubBase}(bool, int)}{bool <: int} \text{subBase}}{(int \rightarrow bool) <: (int \rightarrow int)} \text{arrow}}{x : (int \rightarrow bool) \vdash x : (int \rightarrow int)} \text{subsumption}$$

6.1.2 Nested

$$\frac{\frac{\frac{}{x : (int \rightarrow T), y : bool \vdash x : (int \rightarrow T)} \text{var} \quad \frac{\frac{\text{SubBase}(bool, int)}{bool <: int} \text{subBase} \quad \frac{T <: T}{T <: T} \text{reflexive}}{(int \rightarrow T) <: (bool \rightarrow T)} \text{arrow}}{x : (int \rightarrow T), y : bool \vdash x : (bool \rightarrow T)} \text{subsumption} \quad \frac{}{x : (int \rightarrow T), y : bool \vdash y : bool} \text{var}}{x : (int \rightarrow T), y : bool \vdash (x y)[bool] : T} \text{app}$$

$$\frac{\frac{}{x : (int \rightarrow T), y : bool \vdash (x y)[bool] : (bool \rightarrow T)} \lambda}{\vdash \lambda x. \lambda y. (x y)[bool] : ((int \rightarrow T) \rightarrow (bool \rightarrow T))} \lambda$$

6.2 Invalid

$$\frac{\frac{}{x : (bool \rightarrow bool) \vdash x : (bool \rightarrow bool)} \text{var} \quad \frac{\perp}{int <: bool} \text{invalid} \quad \frac{\text{SubBase}(bool, int)}{bool <: int} \text{subBase}}{(bool \rightarrow bool) <: (int \rightarrow int)} \text{arrow}}{x : (bool \rightarrow bool) \vdash x : (int \rightarrow int)} \text{subsumption}$$

$$\frac{\frac{\frac{}{x : (bool \rightarrow T), y : bool \vdash x : (bool \rightarrow T)} \text{var} \quad \frac{\perp}{int <: bool} \text{invalid} \quad \frac{T <: T}{T <: T} \text{reflexive}}{(bool \rightarrow T) <: (int \rightarrow T)} \text{arrow}}{x : (bool \rightarrow T), y : bool \vdash x : (int \rightarrow T)} \text{subsumption} \quad \frac{\frac{}{x : (bool \rightarrow T), y : bool \vdash y : bool} \text{var} \quad \text{SubBase}(bool, int)}{x : (bool \rightarrow T), y : bool \vdash y : int} \text{app}}{x : (bool \rightarrow T), y : bool \vdash (x y)[int] : T} \text{app}$$

$$\frac{\frac{}{x : (bool \rightarrow T), y : bool \vdash (x y)[int] : (bool \rightarrow T)} \lambda}{\vdash \lambda x. \lambda y. (x y)[int] : ((bool \rightarrow T) \rightarrow (bool \rightarrow T))} \lambda$$

6.3 Delaying applying the subsumption rule

The subsumption rule can be delayed until the term in the judgment is just a variable. This simplifies the code since there is only one rule to be applied for application and λ abstraction terms. Here is the correctness proof.

$$\frac{\Gamma \vdash t : T \quad T <: T'}{\Gamma \vdash t : T'} \text{ subsumption}$$

Proof:

By induction hypothesis on the structure of the term.

1. Case $t = x$: trivial since t is a variable.

$$\frac{\Gamma \vdash x : T \quad T <: T'}{\Gamma \vdash x : T'} \text{ subsumption}$$

2. Case $t = t_1 t_2$: assume $T_2 <: T'_2$ and the following derivation:

$$\frac{\frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 t_2 [T_1] : T_2} \text{ app} \quad T_2 <: T'_2}{\Gamma \vdash t_1 t_2 [T_1] : T'_2} \text{ subsumption}$$

We can get a different derivation tree where the subsumption rule is delayed:

$$\frac{\frac{\Gamma \vdash t_1 t_2 [T_1] : T_2 \quad \frac{\frac{}{T_1 <: T_1} \text{ reflexive} \quad T_2 <: T'_2}{T_1 \rightarrow T_2 <: T_1 \rightarrow T'_2} \text{ arrow}}{\Gamma \vdash t_1 : T_1 \rightarrow T'_2} \text{ subsumption} \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 t_2 [T_1] : T'_2} \text{ app}$$

By the induction hypothesis, the subsumption rule can be delayed until variable terms are generated in the derivation of $\Gamma \vdash t_1 : T_1 \rightarrow T'_2$ and $\Gamma \vdash t_2 : T_1$.

3. Case $t = \lambda x. t'$: assume $T_1 \rightarrow T_2 <: T'_1 \rightarrow T'_2$ and the following derivation:

$$\frac{\frac{\Gamma, x : T_1 \vdash t' : T_2}{\Gamma \vdash \lambda x. t' : T_1 \rightarrow T_2} \lambda \quad \frac{T'_1 <: T_1 \quad T_2 <: T'_2}{T_1 \rightarrow T_2 <: T'_1 \rightarrow T'_2} \text{ arrow}}{\Gamma \vdash \lambda x. t' : T'_1 \rightarrow T'_2} \text{ subsumption}$$

Alternatively we can derive:

$$\frac{\frac{\Gamma, x : T'_1 \vdash t' : T_2 \quad T_2 <: T'_2}{\Gamma, x : T'_1 \vdash t' : T'_2} \text{ subsumption}}{\Gamma \vdash \lambda x. t' : T'_1 \rightarrow T'_2} \lambda$$

Since $T'_1 <: T_1$ then by the subsumption rule:

$$\frac{\frac{}{\Gamma, x : T'_1 \vdash x : T'_1} \text{ var} \quad T'_1 <: T_1}{\Gamma, x : T'_1 \vdash x : T_1} \text{ subsumption}$$

Therefore if $\Gamma, x : T_1 \vdash t' : T_2$ then $\Gamma, x : T'_1 \vdash t' : T_2$ which concludes the proof.

Note:

If $\Gamma, x : T'_1 \vdash t' : T_2$, it is not always true that $\Gamma, x : T_1 \vdash t' : T_2$ where $T'_1 <: T_1$. A counter example would be

$$x : \text{bool} \vdash x : \text{int} \not\Rightarrow x : \text{double} \vdash x : \text{int}, \quad \text{bool} <: \text{int} <: \text{double}$$

However

$$x : \text{int} \vdash x : \text{double} \Rightarrow x : \text{bool} \vdash x : \text{double}, \quad \text{bool} <: \text{int} <: \text{double}$$

Therefore in the following example, using the subsumption rule first would fail and slow the type checker because it needs to backtrack and check the λ rule. However using the subsumption rule would prove the type checking and is faster.

$$\frac{\frac{x : \text{double} \vdash x : \text{int}}{\Gamma \vdash \lambda x.x : \text{double} \rightarrow \text{int}} \lambda \quad \frac{\frac{\text{SubBase}(\text{bool}, \text{double})}{\text{bool} <: \text{double}} \text{subBase} \quad \frac{\text{int} <: \text{int}}{\text{double} \rightarrow \text{int} <: \text{bool} \rightarrow \text{int}} \begin{matrix} \text{reflexive} \\ \text{arrow} \end{matrix}}{\Gamma \vdash \lambda x.x : \text{bool} \rightarrow \text{int}} \text{subsumption}$$

$$\frac{\frac{\Gamma, x : \text{bool} \vdash x : \text{bool}}{\Gamma, x : \text{bool} \vdash x : \text{int}} \text{var} \quad \frac{\text{SubBase}(\text{bool}, \text{int})}{\text{bool} <: \text{int}} \text{subBase}}{\Gamma \vdash \lambda x.x : \text{bool} \rightarrow \text{int}} \text{subsumption} \lambda$$

7 System F

7.1 Variables

7.1.1 Valid

$$\frac{}{x : \forall X.X \vdash x : \forall X.X} \text{var}$$

$$\frac{}{x : \forall X.X \vdash x : \forall Y.Y} \text{var}$$

$$\frac{}{x : \forall X.(X \rightarrow X) \vdash x : \forall Y.(Y \rightarrow Y)} \text{var}$$

Y is free in the context and the type:

$$\frac{}{x : \forall X.(X \rightarrow Y) \vdash x : \forall Z.(Z \rightarrow Y)} \text{var}$$

$$\frac{\frac{x : \forall X.X \vdash x : \forall X1.X1}{x : \forall X.X \vdash x[Y] : Y} \text{var}}{\text{elimination}}$$

$$\frac{\frac{x : \forall X.X \vdash x : \forall X1.X1}{x : \forall X.X \vdash x[(Y \rightarrow Y)] : (Y \rightarrow Y)} \text{var}}{\text{elimination}}$$

$$\frac{\frac{\frac{x : \forall X.X \vdash x : \forall X1.X1}{x : \forall X.X \vdash x[(\forall X.X \rightarrow \forall X.X)] : (\forall X.X \rightarrow \forall X.X)} \text{var}}{\text{elimination}} \quad \frac{}{x : \forall X.X \vdash x : \forall X.X} \text{var app}}{x : \forall X.X \vdash (x[(\forall X.X \rightarrow \forall X.X)]) x : \forall X.X} \text{elimination}$$

7.1.2 Invalid

Y is free in the context:

$$\frac{}{x : \forall X.(X \rightarrow Y) \vdash x : \forall Y.(Y \rightarrow Y)} \text{invalid var}$$

Y is free in the context, and Z is free in the type:

$$\frac{}{x : \forall X.(X \rightarrow Y) \vdash x : \forall Y.(Y \rightarrow Z)} \text{invalid var}$$

$$\frac{\frac{}{x : \forall X.X \vdash x : \forall X1.X1} \text{var}}{x : \forall X.X \vdash x[Y] : Y} \text{elimination}$$

7.2 Numbers

Zero

$$\frac{\frac{\frac{}{z : X, s : (X \rightarrow X) \vdash z : X} \text{var}}{s : (X \rightarrow X) \vdash (\lambda z.z) : (X \rightarrow X)} \lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : ((X \rightarrow X) \rightarrow (X \rightarrow X))} \lambda \quad \frac{}{\cdot \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \rightarrow X) \rightarrow (X \rightarrow X))} \text{introduction}$$

Zero with free variable X

$$\frac{\frac{\frac{\frac{}{y : X, z : X2, s : (X2 \rightarrow X2) \vdash z : X2} \text{var}}{y : X, s : (X2 \rightarrow X2) \vdash (\lambda z.z) : (X2 \rightarrow X2)} \lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : ((X2 \rightarrow X2) \rightarrow (X2 \rightarrow X2))} \lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X2.((X2 \rightarrow X2) \rightarrow (X2 \rightarrow X2))} \text{introduction} \quad \frac{}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \rightarrow X) \rightarrow (X \rightarrow X))} \text{renaming}$$

