Type checker for system F

Mudathir Mahgoub

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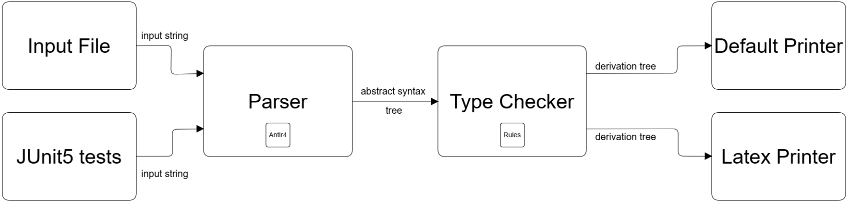
Type checking problem

Given a typing context Γ and a term t and a type T, can the term t be assigned the type T under the typing context Γ?
Is the judgment $\Gamma \vdash t : T$ derivable?

Software components



Execution

Input: test.txt

```
SubBase(bool, int);
. |- \lambda x. \lambda y. (x y)[bool]: (int ->T) -> (bool -> T);
```

Output: java -jar TypeChecker.jar -i test.txt

```
Yes
                                              SubBase(bool, int)
--------------------(var)                     --------------------(subBase)
x: bool |- x : bool                                  bool <: int
------------------------------------------------------(subsumption)
x: bool |- x : int
```

Execution

Input: test.txt

```
SubBase(bool, int);
. |- \lambda x. \lambda y. (x y)[bool]: (int ->T) -> (bool -> T);
```

Output: java -jar TypeChecker.jar -i test.txt -latex

```
Yes
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize var}
\UnaryInfC{$x: bool \vdash x : bool$}
\AxiomC{\scriptsize $SubBase(bool,int)$}
\RightLabel{\scriptsize subBase}
\UnaryInfC{$bool <: int$}
\RightLabel{\scriptsize subsumption}
\BinaryInfC{$x: bool \vdash x : int$}
\end{prooftree}
```

Demo

Demo

Simple types

Terms

$$t ::= x \mid (t_1 t_2)[T] \mid \lambda x.t$$

Typing rules

1. Variable rule

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T}\ \text{var}$$

2. Application rule

$$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2)[T_1] : T_2}\ \text{app}$$

3. λ rule

$$\frac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2}\ \lambda$$

Simple types

Subtyping rules

$$\frac{\Gamma \vdash t : T_1 \qquad T_1 <: T_2}{\Gamma \vdash t : T_2}\ \text{subsumption}$$

$$\frac{}{T <: T}\ \text{reflexive}$$

$$\frac{SubBase(b_1, b_2)}{b_1 <: b_2}\ \text{subBase}$$

$$\frac{T_1' <: T_1 \qquad T_2 <: T_2'}{T_1 \rightarrow T_2 <: T2_1 \rightarrow T_2'}\ \text{arrow}$$

$$\frac{T_1 <: T_2 \qquad T_2 <: T_3}{T_1 <: T_3}\ \text{transitive}$$

Simple types

Examples

$$\dfrac{}{x : T \vdash x : T}\ \text{var}$$

$$\dfrac{}{\color{red}{\cdot \vdash x : T}}\ \color{red}{\text{invalid var}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : (T_1 \to T_2), y : T_1 \vdash x : (T_1 \to T_2)}\ \text{var} \qquad \dfrac{}{x : (T_1 \to T_2), y : T_1 \vdash y : T_1}\ \text{var}}{x : (T_1 \to T_2), y : T_1 \vdash (x\ y)[T_1] : T_2}\ \text{app}}{x : (T_1 \to T_2) \vdash \lambda y.(x\ y)[T_1] : (T_1 \to T_2)}\ \lambda}{\cdot \vdash \lambda x.\lambda y.(x\ y)[T_1] : ((T_1 \to T_2) \to (T_1 \to T_2))}\ \lambda$$

Subtyping

Examples

$$\dfrac{\dfrac{}{x : bool \vdash x : bool}\ \text{var} \qquad \dfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase}}{x : bool \vdash x : int}\ \text{subsumption}$$

$$\dfrac{\dfrac{}{x : int \vdash x : int}\ \text{var} \qquad \color{red}{\dfrac{\bot}{int <: bool}\ \text{invalid}}}{x : int \vdash x : bool}\ \text{subsumption}$$

$$\dfrac{\dfrac{}{x : bool \vdash x : bool}\ \text{var} \qquad \dfrac{\dfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase} \qquad \dfrac{\text{SubBase}(int, double)}{int <: double}\ \text{subBase}}{bool <: double}\ \text{transitive}}{x : bool \vdash x : double}\ \text{subsumption}$$

Subtyping

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : (int \to T), y : bool \vdash x : (int \to T)}\,\text{var} \qquad \dfrac{\dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase} \qquad \dfrac{}{T <: T}\,\text{reflexive}}{(int \to T) <: (bool \to T)}\,\text{arrow}}{x : (int \to T), y : bool \vdash x : (bool \to T)}\,\text{subsumption} \qquad \dfrac{}{x : (int \to T), y : bool \vdash y : bool}\,\text{var}}{x : (int \to T), y : bool \vdash (x\ y)[bool] : T}\,\text{app}}{x : (int \to T) \vdash \lambda y.(x\ y)[bool] : (bool \to T)}\,\lambda}{\cdot \vdash \lambda x.\lambda y.(x\ y)[bool] : ((int \to T) \to (bool \to T))}\,\lambda
$$

Delaying the subsumption rule

Remark

The sumbumption rule can be delayed until the term in the judgment is a variable.

$$\frac{\Gamma \vdash t : T \qquad T <: T'}{\Gamma \vdash t : T'}\ \text{subsumption}$$

Delaying the subsumption rule

Case $t = x$: trivial since t is a variable.

$$\frac{\Gamma \vdash x : T \qquad T <: T'}{\Gamma \vdash x : T'}\,\text{subsumption}$$

Delaying the subsumption rule

Case $t = t_1 t_2$: assume $T_2 <: T_2'$ and the following derivation:

$$\dfrac{\dfrac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2)[T_1] : T_2}\,\text{app} \qquad T_2 <: T_2'}{\Gamma \vdash (t_1 t_2)[T_1] : T_2'}\,\text{subsumption}$$

Alternatively

$$\dfrac{\dfrac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \dfrac{\dfrac{}{T_1 <: T_1}\,\text{reflexive} \qquad T_2 <: T_2'}{T_1 \to T_2 <: T_1 \to T_2'}\,\text{arrow}}{\Gamma \vdash t_1 : T_1 \to T_2'}\,\text{subsumption} \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2)[T_1] : T_2'}$$

Delaying the subsumption rule

Case $t = \lambda x.t'$: assume $T_1 \to T_2 <: T_1' \to T_2'$ and the following derivation:

$$\dfrac{\dfrac{\Gamma, x : T_1 \vdash t' : T_2}{\Gamma \vdash \lambda x.t' : T_1 \to T_2}\lambda \qquad \dfrac{T_1' <: T_1 \qquad T_2 <: T_2'}{T_1 \to T_2 <: T_1' \to T_2'}\text{arrow}}{\Gamma \vdash \lambda x.t' : T_1' \to T_2'}\text{subsumption}$$

Alternatively

$$\dfrac{\dfrac{\Gamma, x : T_1' \vdash t' : T_2 \qquad T_2 <: T_2'}{\Gamma, x : T_1' \vdash t' : T_2'}\text{subsumption}}{\Gamma \vdash \lambda x.t' : T_1' \to T_2'}\lambda$$

Assuming $T_1' <: T_1$ is derivable, then by the subsumption rule:

$$\dfrac{\dfrac{}{\Gamma, x : T_1' \vdash x : T_1'}\text{var} \qquad T_1' <: T_1}{\Gamma, x : T_1' \vdash x : T_1}\text{subsumption}$$

Delaying the subsumption rule

Remark

If $T_1' <: T_1$ and $\Gamma, x : T_1' \vdash t' : T_2$, it is not always true that $\Gamma, x : T_1 \vdash t' : T_2$.

Counter example

$$x : bool \vdash x : int \not\Rightarrow x : double \vdash x : int, \quad bool <: int <: double$$

$$x : int \vdash x : double \Rightarrow x : bool \vdash x : double, \quad bool <: int <: double$$

Delaying the subsumption rule

$$\dfrac{\dfrac{\color{red}{x : double \vdash x : int}}{\Gamma \vdash \lambda x.x : double \to int}\,\lambda \qquad \dfrac{\dfrac{SubBase(bool, double)}{bool <: double}\,\text{subBase} \quad \dfrac{}{int <: int}\,\text{reflexive}}{double \to int <: bool \to int}\,\text{arrow}}{\Gamma \vdash \lambda x.x : bool \to int}\,\text{subsumption}$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x : bool \vdash x : bool}\,\text{var} \qquad \dfrac{SubBase(bool, int)}{bool <: int}\,\text{suBase}}{\Gamma, x : bool \vdash x : int}\,\text{subsumption}}{\Gamma \vdash \lambda x.x : bool \to int}\,\lambda$$

System F

Terms

$$t ::= x \mid (t_1 t_2)[T] \mid \lambda x.t \mid t\ [[T]]$$

Types

$$T ::= X \mid T_1 \to T_2 \mid \forall X.T$$

System F additional rules

1. If X is not free in Γ

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash t : \forall X.T} \text{ introduction}$$

2. If X is free in Γ, choose X_i such that X_i is not free in Γ

$$\cfrac{\cfrac{\Gamma \vdash t : [X_i/X]T}{\Gamma \vdash t : \forall X_i.[X_i/X]T} \text{ introduction}}{\Gamma \vdash t : \forall X.T} \text{ renaming}$$

3. Elimination rule with annotation

$$\frac{\Gamma \vdash t : \forall X.\ [X/T']T}{\Gamma \vdash t\ [[T']] : T} \text{ elimination}$$

System F examples

- Different variables

$$\frac{}{x : \forall X.(X \to X) \vdash x : \forall Y.(Y \to Y)} \text{ var}$$

- Y is free in the typing context and the term type

$$\frac{}{x : \forall X.(X \to Y) \vdash x : \forall Z.(Z \to Y)} \text{ var}$$

- Y is free in the typing context

$$\frac{}{x : \forall X.(X \to Y) \vdash x : \forall Y.(Y \to Y)} \text{ invalid var}$$

- Y is free in the typing context, and Z is free in the term type

$$\frac{}{x : \forall X.(X \to Y) \vdash x : \forall Y.(Y \to Z)} \text{ invalid var}$$

System F examples

- Elimination annotation

$$\dfrac{\dfrac{}{x : \forall X.X \vdash x : \forall X1.\ X1}\,\text{var}}{x : \forall X.X \vdash x[Y] : Y}\,\text{elimination}$$

- Elimination annotation with arrow

$$\dfrac{\dfrac{}{x : \forall X.X \vdash x : \forall X1.\ X1}\,\text{var}}{x : \forall X.X \vdash x[(Y \to Y)] : (Y \to Y)}\,\text{elimination}$$

System F examples

- Zero

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z : X, s : (X \to X) \vdash z : X}\text{var}}{s : (X \to X) \vdash (\lambda z.z) : (X \to X)}\lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : ((X \to X) \to (X \to X))}\lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \to X) \to (X \to X))}\text{introduction}$$

- Zero with free variable X

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y : X, z : X_2, s : (X_2 \to X_2) \vdash z : X_2}\text{var}}{y : X, s : (X_2 \to X_2) \vdash (\lambda z.z) : (X_2 \to X_2)}\lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : ((X_2 \to X_2) \to (X_2 \to X_2))}\lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X_2.((X_2 \to X_2) \to (X_2 \to X_2))}\text{introduction}}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \to X) \to (X \to X))}\text{renaming}$$