1 Rules

1. $$\dfrac{\Gamma(x) = T}{\Gamma \vdash x : T}\ \text{var}$$

2. $$\dfrac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash (t_1 t_2){\color{blue}[T_1]} : T_2}\ \text{app}$$

3. $$\dfrac{\Gamma, x : T_1 \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2}\ \lambda$$

4. $$\dfrac{\Gamma \vdash t : T_1 \qquad T_1 <: T_2}{\Gamma \vdash t : T_2}\ \text{subsumption}$$

5. $$\dfrac{}{T <: T}\ \text{reflexive}$$

6. $$\dfrac{SubBase(b_1, b_2)}{b_1 <: b_2}\ \text{subBase}$$

7. $$\dfrac{T_1' <: T_1 \qquad T_2 <: T_2'}{T_1 \to T_2 <: T2_1 \to T_2'}\ \text{arrow}$$

8. $$\dfrac{T_1 <: T_2 \qquad T_2 <: T_3}{T_1 <: T_3}\ \text{transitive}$$

2 Variable terms

2.1 Valid

$$\dfrac{}{x : T \vdash x : T}\ \text{var}$$

2.2 Invalid

$${\color{red}\dfrac{}{\cdot \vdash x : T}\ \text{invalid var}}$$

3 Lambda & application terms

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : (T1 \to T2), y : T1 \vdash x : (T1 \to T2)}\ \text{var} \qquad \dfrac{}{x : (T1 \to T2), y : T1 \vdash y : T1}\ \text{var}}{x : (T1 \to T2), y : T1 \vdash (x\ y)[T1] : T2}\ \text{app}}{x : (T1 \to T2) \vdash \lambda y.(x\ y)[T1] : (T1 \to T2)}\ \lambda}{\cdot \vdash \lambda x.\lambda y.(x\ y)[T1] : ((T1 \to T2) \to (T1 \to T2))}\ \lambda$$

4 Direct Subtyping

4.1 Valid

$$\dfrac{\dfrac{}{x : bool \vdash x : bool}\ \text{var} \qquad \dfrac{\text{SubBase}(bool, int)}{bool <: int}\ \text{subBase}}{x : bool \vdash x : int}\ \text{subsumption}$$

4.2 Invalid

$$\dfrac{\dfrac{}{x : int \vdash x : int}\,\text{var} \qquad \dfrac{\bot}{int <: bool}\,\text{invalid}}{x : int \vdash x : bool}\,\text{subsumption}$$

5 Transitive Subtyping

$$\dfrac{\dfrac{}{x : bool \vdash x : bool}\,\text{var} \qquad \dfrac{\dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase} \qquad \dfrac{\text{SubBase}(int, double)}{int <: double}\,\text{subBase}}{bool <: double}\,\text{transitive}}{x : bool \vdash x : double}\,\text{subsumption}$$

$$\dfrac{\dfrac{}{x : bool \vdash x : bool}\,\text{var} \qquad \dfrac{\dfrac{\dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase} \qquad \dfrac{\text{SubBase}(int, quotient)}{int <: quotient}\,\text{subBase}}{bool <: quotient}\,\text{transitive} \qquad \dfrac{\text{SubBase}(quotient, double)}{quotient <: double}\,\text{subBase}}{bool <: double}\,\text{transitive}}{x : bool \vdash x : double}\,\text{subsumption}$$

6 Arrow Types

6.1 Valid

$$\dfrac{\dfrac{}{x : (int \to bool) \vdash x : (int \to bool)}\,\text{var} \qquad \dfrac{\dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase} \qquad \dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase}}{(int \to bool) <: (bool \to int)}\,\text{arrow}}{x : (int \to bool) \vdash x : (bool \to int)}\,\text{subsumption}$$

6.1.1 Reflexive

$$\dfrac{\dfrac{}{x : (int \to bool) \vdash x : (int \to bool)}\,\text{var} \qquad \dfrac{\dfrac{}{int <: int}\,\text{reflexive} \qquad \dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase}}{(int \to bool) <: (int \to int)}\,\text{arrow}}{x : (int \to bool) \vdash x : (int \to int)}\,\text{subsumption}$$

6.1.2 Nested

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : (int \to T), y : bool \vdash x : (int \to T)}\,\text{var} \qquad \dfrac{\dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase} \qquad \dfrac{}{T <: T}\,\text{reflexive}}{(int \to T) <: (bool \to T)}\,\text{arrow}}{x : (int \to T), y : bool \vdash x : (bool \to T)}\,\text{subsumption} \qquad \dfrac{}{x : (int \to T), y : bool \vdash y : bool}\,\text{var}}{x : (int \to T), y : bool \vdash (x\ y)[bool] : T}\,\text{app}}{x : (int \to T) \vdash \lambda y.(x\ y)[bool] : (bool \to T)}\,\lambda}{\cdot \vdash \lambda x.\lambda y.(x\ y)[bool] : ((int \to T) \to (bool \to T))}\,\lambda$$

6.2 Invalid

$$\dfrac{\dfrac{}{x : (bool \to bool) \vdash x : (bool \to bool)}\,\text{var} \qquad \dfrac{\dfrac{\bot}{int <: bool}\,\text{invalid} \qquad \dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase}}{(bool \to bool) <: (int \to int)}\,\text{arrow}}{x : (bool \to bool) \vdash x : (int \to int)}\,\text{subsumption}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : (bool \to T), y : bool \vdash x : (bool \to T)}\,\text{var} \qquad \dfrac{\dfrac{\bot}{int <: bool}\,\text{invalid} \qquad \dfrac{}{T <: T}\,\text{reflexive}}{(bool \to T) <: (int \to T)}\,\text{arrow}}{x : (bool \to T), y : bool \vdash x : (int \to T)}\,\text{subsumption} \qquad \dfrac{\dfrac{}{x : (bool \to T), y : bool \vdash y : bool}\,\text{var} \qquad \dfrac{\text{SubBase}(bool, int)}{bool <: int}\,\text{subBase}}{x : (bool \to T), y : bool \vdash y : int}\,\text{subsumption}}{x : (bool \to T), y : bool \vdash (x\ y)[int] : T}\,\text{app}}{x : (bool \to T) \vdash \lambda y.(x\ y)[int] : (bool \to T)}\,\lambda}{\cdot \vdash \lambda x.\lambda y.(x\ y)[int] : ((bool \to T) \to (bool \to T))}\,\lambda$$

6.3 Delaying applying the subsumption rule

The sumbumption rule can be delayed until the term in the judgment is a just a variable. This simplifies the code since there is only one rule to be applied for application and λ abstraction terms. Here is the correctness proof.

$$\frac{\Gamma \vdash t : T \qquad T <: T'}{\Gamma \vdash t : T'}\,\text{subsumption}$$

Proof:
By induction hypothesis on the structure of the term.

1. Case $t = x$: trivial since t is a variable.

$$\frac{\Gamma \vdash x : T \qquad T <: T'}{\Gamma \vdash x : T'}\,\text{subsumption}$$

2. Case $t = t_1 t_2$: assume $T_2 <: T_2'$ and the following derivation:

$$\dfrac{\dfrac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 t_2[T_1] : T_2}\,\text{app} \qquad T_2 <: T_2'}{\Gamma \vdash t_1 t_2[T_1] : T_2'}\,\text{subsumption}$$

We can get a different derivation tree where the sumbsumption rule is delayed:

$$\dfrac{\dfrac{\Gamma \vdash t_1 t_2[T_1] : T_2 \qquad \dfrac{\dfrac{}{T_1 <: T_1}\,\text{reflexive} \qquad T_2 <: T_2'}{T_1 \to T_2 <: T_1 \to T_2'}\,\text{arrow}}{\Gamma \vdash t_1 : T_1 \to T_2'}\,\text{subsumption} \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 t_2[T_1] : T_2'}\,\text{app}$$

By the induction hypothesis, the subsumption rule can be delayed until variable terms are generated in the derivation of $\Gamma \vdash t_1 : T_1 \to T_2'$ and $\Gamma \vdash t_2 : T_1$.

3. Case $t = \lambda x.t'$: assume $T_1 \to T_2 <: T_1' \to T_2'$ and the following derivation:

$$\dfrac{\dfrac{\Gamma, x : T_1 \vdash t' : T_2}{\Gamma \vdash \lambda x.t' : T_1 \to T_2}\,\lambda \qquad \dfrac{T_1' <: T_1 \qquad T_2 <: T_2'}{T_1 \to T_2 <: T_1' \to T_2'}\,\text{arrow}}{\Gamma \vdash \lambda x.t' : T_1' \to T_2'}\,\text{subsumption}$$

Alternatively we can derive:

$$\dfrac{\dfrac{\Gamma, x : T_1' \vdash t' : T_2 \qquad T_2 <: T_2'}{\Gamma, x : T_1' \vdash t' : T_2'}\,\text{subsumption}}{\Gamma \vdash \lambda x.t' : T_1' \to T_2'}\,\lambda$$

Since $T_1' <: T_1$ then by the subsumption rule:

$$\dfrac{\dfrac{}{\Gamma, x : T_1' \vdash x : T_1'}\,\text{var} \qquad T_1' <: T_1}{\Gamma, x : T_1' \vdash x : T_1}\,\text{subsumption}$$

Therefore if $\Gamma, x : T_1 \vdash t' : T_2$ then $\Gamma, x : T_1' \vdash t' : T_2$ which concludes the proof.

Note:

If $\Gamma, x : T_1' \vdash t' : T_2$, it is not always true that $\Gamma, x : T_1 \vdash t' : T_2$ where $T_1' <: T_1$. A counter example would be

$$x : bool \vdash x : int \not\Rightarrow x : double \vdash x : int, \quad bool <: int <: double$$

However

$$x : int \vdash x : double \Rightarrow x : bool \vdash x : double, \quad bool <: int <: double$$

Therefore in the following example, using the subsumption rule first would fail and slow the type checker because it needs to backtrack and check the λ rule. However using the subsumption rule would prove the type checking and is faster.

$$\dfrac{\dfrac{\color{red}{x : double \vdash x : int}}{\Gamma \vdash \lambda x.x : double \to int}\,\lambda \qquad \dfrac{\dfrac{SubBase(bool, double)}{bool <: double}\,\text{subBase} \qquad \dfrac{}{int <: int}\,\text{reflexive}}{double \to int <: bool \to int}\,\text{arrow}}{\Gamma \vdash \lambda x.x : bool \to int}\,\text{subsumption}$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x : bool \vdash x : bool}\,\text{var} \qquad \dfrac{SubBase(bool, int)}{bool <: int}\,\text{suBase}}{\Gamma, x : bool \vdash x : int}\,\text{subsumption}}{\Gamma \vdash \lambda x.x : bool \to int}\,\lambda$$

7 System F

7.1 Variables

7.1.1 Valid

$$\dfrac{}{x : \forall X.X \vdash x : \forall X.X}\,\text{var}$$

$$\dfrac{}{x : \forall X.X \vdash x : \forall Y.Y}\,\text{var}$$

$$\dfrac{}{x : \forall X.(X \to X) \vdash x : \forall Y.(Y \to Y)}\,\text{var}$$

Y is free in the context and the type:

$$\dfrac{}{x : \forall X.(X \to Y) \vdash x : \forall Z.(Z \to Y)}\,\text{var}$$

$$\dfrac{\dfrac{}{x : \forall X.X \vdash x : \forall X1.X1}\,\text{var}}{x : \forall X.X \vdash x[Y] : Y}\,\text{elimination}$$

$$\dfrac{\dfrac{}{x : \forall X.X \vdash x : \forall X1.X1}\,\text{var}}{x : \forall X.X \vdash x[(Y \to Y)] : (Y \to Y)}\,\text{elimination}$$

$$\dfrac{\dfrac{\dfrac{}{x : \forall X.X \vdash x : \forall X1.X1}\,\text{var}}{x : \forall X.X \vdash x[[(\forall X.X \to \forall X.X)]][(\forall X.X \to \forall X.X)] : (\forall X.X \to \forall X.X)}\,\text{elimination} \qquad \dfrac{}{x : \forall X.X \vdash x : \forall X.X}\,\text{var}}{x : \forall X.X \vdash (x[[(\forall X.X \to \forall X.X)]]\ x)[\forall X.X] : \forall X.X}\,\text{app}$$

7.1.2 Invalid

Y is free in the context:

$$\dfrac{}{x : \forall X.(X \to Y) \vdash x : \forall Y.(Y \to Y)}\ \text{invalid var}$$

Y is free in the context, and Z is free in the type:

$$\dfrac{}{x : \forall X.(X \to Y) \vdash x : \forall Y.(Y \to Z)}\ \text{invalid var}$$

$$\dfrac{\dfrac{}{x : \forall X.X \vdash x : \forall X1.X1}\ \text{var}}{x : \forall X.X \vdash x[Y] : Y}\ \text{elimination}$$

7.2 Numbers

Zero

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z : X, s : (X \to X) \vdash z : X}\ \text{var}}{s : (X \to X) \vdash (\lambda z.z) : (X \to X)}\ \lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : ((X \to X) \to (X \to X))}\ \lambda}{\cdot \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \to X) \to (X \to X))}\ \text{introduction}$$

Zero with free variable X

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y : X, z : X_2, s : (X_2 \to X_2) \vdash z : X_2}\ \text{var}}{y : X, s : (X_2 \to X_2) \vdash (\lambda z.z) : (X_2 \to X_2)}\ \lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : ((X_2 \to X_2) \to (X_2 \to X_2))}\ \lambda}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X_2.((X_2 \to X_2) \to (X_2 \to X_2))}\ \text{introduction}}{y : X \vdash (\lambda s.(\lambda z.z)) : \forall X.((X \to X) \to (X \to X))}\ \text{renaming}$$

Successor missing annotation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash s : (X \to X)}\,\text{var} \quad \dfrac{\dfrac{\dfrac{}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash n : ((X \to X) \to (X \to X))}\,\text{unknown} \quad \dfrac{}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash s : (X \to X)}\,\text{var}}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash (n\ s)[(X \to X)] : (X \to X)}\,\text{app} \quad \dfrac{}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash z : X}\,\text{var}}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash ((n\ s)[(X \to X)]\ z)[X] : X}\,\text{app}}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash (s\ ((n\ s)[(X \to X)]\ z)[X])[X] : X}\,\text{app}}{n : \forall X.((X \to X) \to (X \to X)), s : (X \to X) \vdash (\lambda z.(s\ ((n\ s)[(X \to X)]\ z)[X])[X]) : (X \to X)}\,\lambda}{n : \forall X.((X \to X) \to (X \to X)) \vdash (\lambda s.(\lambda z.(s\ ((n\ s)[(X \to X)]\ z)[X])[X])) : ((X \to X) \to (X \to X))}\,\lambda}{n : \forall X.((X \to X) \to (X \to X)) \vdash (\lambda s.(\lambda z.(s\ ((n\ s)[(X \to X)]\ z)[X])[X])) : \forall X.((X \to X) \to (X \to X))}\,\text{introduction}}{\cdot \vdash (\lambda n.(\lambda s.(\lambda z.(s\ ((n\ s)[(X \to X)]\ z)[X])[X]))) : (\forall X.((X \to X) \to (X \to X)) \to \forall X.((X \to X) \to (X \to X)))}\,\lambda
$$

Successor well annotated

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash s : (X \to X)}\,\text{var} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash n : \forall X1.((X1 \to X1) \to (X1 \to X1))}\,\text{var}}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash n[[X]][X] : ((X \to X) \to (X \to X))}\,\text{elimination} \quad \dfrac{}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash s : (X \to X)}\,\text{var}}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash (n[[X]]\ s)[(X \to X)] : (X \to X)}\,\text{app} \quad \dfrac{}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash z : X}\,\text{var}}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash ((n[[X]]\ s)[(X \to X)]\ z)[X] : X}\,\text{app}}{n : \forall X.((X \to X) \to (X \to X)), z : X, s : (X \to X) \vdash (s\ ((n[[X]]\ s)[(X \to X)]\ z)[X])[X] : X}\,\text{app}}{n : \forall X.((X \to X) \to (X \to X)), s : (X \to X) \vdash (\lambda z.(s\ ((n[[X]]\ s)[(X \to X)]\ z)[X])[X]) : (X \to X)}\,\lambda}{n : \forall X.((X \to X) \to (X \to X)) \vdash (\lambda s.(\lambda z.(s\ ((n[[X]]\ s)[(X \to X)]\ z)[X])[X])) : ((X \to X) \to (X \to X))}\,\lambda}{n : \forall X.((X \to X) \to (X \to X)) \vdash (\lambda s.(\lambda z.(s\ ((n[[X]]\ s)[(X \to X)]\ z)[X])[X])) : \forall X.((X \to X) \to (X \to X))}\,\text{introduction}}{\cdot \vdash (\lambda n.(\lambda s.(\lambda z.(s\ ((n[[X]]\ s)[(X \to X)]\ z)[X])[X]))) : (\forall X.((X \to X) \to (X \to X)) \to \forall X.((X \to X) \to (X \to X)))}\,\lambda
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