Zero Temperature Dynamics on Ising Systems on Hypercubes

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Ising Sytems on Graphs

Definition

An Ising System on a Graph G consists the following:

- An undirected graph $G = (V, E)$
- At each vertex $i \in V$, an associated spin variable $\sigma_i \in \{-1, 1\}$
- For a given spin configuration $\sigma \in \{-1, 1\}^{|V|}$, its Hamiltonian $\mathcal{H}(\sigma)$:

$$\mathcal{H}(\sigma) = -\sum_{\{i,j\} \in E} J_{ij}\sigma_i\sigma_j$$

  where J_{ij} is the coupling between spins at vertex i and j.
- We call this a homogeneous Ising System if $\forall\{i, j\} \in E$, $J_{ij} = 1$
- We call $\sigma^+ = \{1\}^{|V|}$ and $\sigma^- = \{-1\}^{|V|}$ the ground state configurations

Equilibrium Statistical Mechanics

Definition

A Gibbs distribution $\mathbb{P}$ is a probability measure on$\sigma \in \{-1,1\}^{|V|}$:

$$\mathbb{P}(\sigma) = \frac{1}{Z} e^{-\beta \mathcal{H}(\sigma)},$$

where $\beta = \frac{1}{T}$ the inverse temperature,
and Z the normalizing term named the partition function:

$$Z = \sum_{\sigma} e^{-\beta \mathcal{H}(\sigma)}$$

In particular:
When $\beta = 0$, $\mathbb{P}$ is uniform on $\sigma \in \{-1,1\}^{|V|}$.
When $\beta \to \infty$, $\mathbb{P}$ should concentrate on ground states.

Time Dependent Behaviour: Glauber Dynamics

Definition

A discrete version of Glauber Dynamics ($T > 0$) is implemented as the following:

- Start with a random spin configuration
- At each time step
  - Choose a vertex i from V at random
  - Compute energy change ΔE_i of flipping the spin ($\sigma_i \to -\sigma_i$)
  - If $\Delta E_i \leq 0$, flip σ_i with probability proportional to $e^{-\beta \Delta E_i}$
  - If $\Delta E_i > 0$, σ_i is not flipped
  - Update time by Δt

Fact: For $T > 0$, Glauber Dynamics converges to the Gibbs Distribution at the corresponding temperature T.

Non-equilibrium Dynamics: Deep Quench

Consider the time evolution of an Ising System following a deep quench from infinite to zero temperature.

Definition

A zero temperature dynamics is implemented as the following:

- Start with a random initial spin configuration
- At each time step
  - Choose a vertex i from V at random
  - Compute energy change ΔE_i of flipping the spin $(\sigma_i \to -\sigma_i)$
  - If $\Delta E_i < 0$, flip σ_i
  - If $\Delta E_i = 0$, flip σ_i with probability $1/2$ (A tie!)
  - If $\Delta E_i > 0$, σ_i is not flipped
  - Update time by Δt

Zero temperature dynamics generally don't converge to the Gibbs distribution.

Zero Temperature Dynamics on Euclidean Lattice: An example

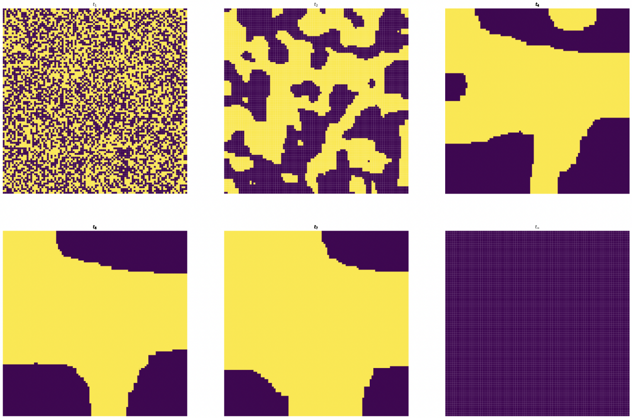

Figure 1: Zero Temperatur Dynamics on 2D Euclidean Lattice with Periodic Boundary Condition

Zero Temperature Dynamics on Ising System on Hypercubes

We are interested in the zero temperature dynamics of Ising System on Hypercubes Q_d, where d is the dimension, and:

- $V(Q_d) = \{(b_1 b_2 ... b_d) | b_i \in \{0, 1\}\}$
- $|V(Q_d)| = 2^d$
- For $i, j \in V(Q_d), \{i, j\} \in E(Q_d)$ iff $d_h(i, j) = 1$, where d_h is the hamming distance

In particular, we are interested in the asymptotic behaviors of the system as $d \to \infty$

Experiment Methods

For each d, we simulate 20000 dynamic runs on Hypercube Q_d. At each trial:

- Start with a random initial configuration with zero magnetization, i.e.:$\sum_i^{2^d} \sigma_i(0) = 0$.
- Implement a random dynamical realization.
- Let $t \to \infty$

Remark: In some trials, the system will eventually saturate as $t \to \infty$, i.e.: There are no flippable spins ($\Delta E_i \leq 0$) left. But on even-dimensional hypercubes, we can have ties! Hence we may end up with some sites that flip permanently, we call them **Blinkers**.

Distribution of the Final Magnetization

We are interested in the quantity: $M_{\infty}^{d} = \lim_{t\to\infty} \sum_{i}^{2^d} \sigma_i(t)$

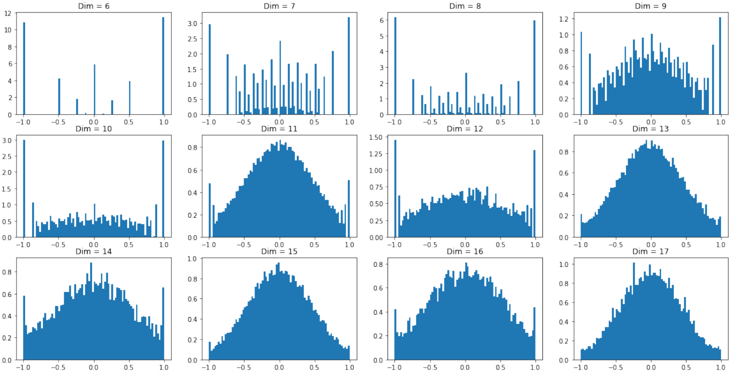

Figure 2: Sample distribution of final magnetization (normalized by $\frac{1}{N}$) at different d

Correct Normalization 1

For possible CLT, what is the correct normalization for M_∞^d?
Recall in classical CLT, the correct normalization is $N^{-\frac{1}{2}}$

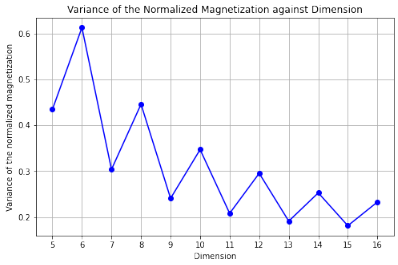

Figure 3: Sample variance of M_∞^d scaled by N^{-1}

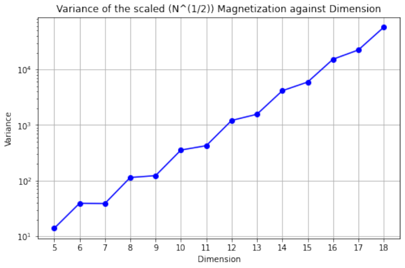

Figure 4: Sample variance of M_∞^d scaled by $N^{-\frac{1}{2}}$

Correct Normalization 2

The correct normalization should be somewhere between $N^{-\frac{1}{2}}$ and N^{-1}. Numerically, it suggests $(\frac{N}{\log\log N})^{-1}$.

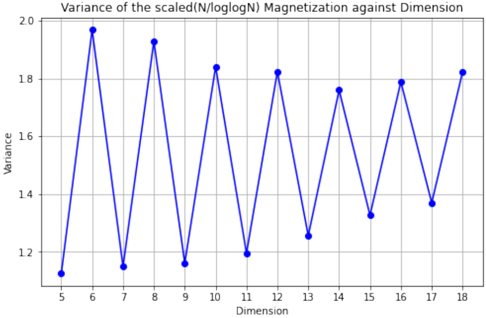

Figure 5: Sample variance of M_∞^d scaled by $(\frac{N}{\log\log N})^{-1}$

Probability of Entering Ground States

Consider event $G^d = \{\lim_{t\to\infty} \sigma^d(t) = \sigma^+ \text{ or } \sigma^-\}$, we're interested in:

$$\lim_{d\to\infty} \mathbb{P}_{\sigma^0,\omega}(G^d)$$

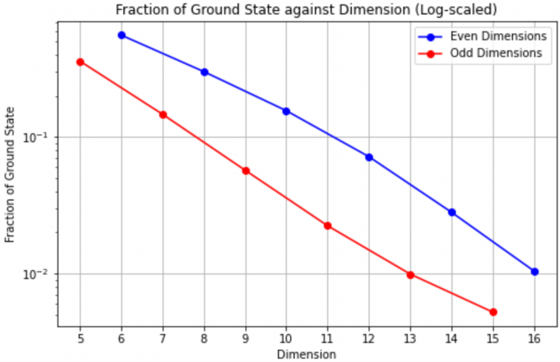

Figure 6: Fraction of samples that enter ground states against dimension

Metastable States: k-dim sub-cubes

Intuition behind the Gaussian Limit of M_∞^d and $\lim_{d\to\infty} \mathbb{P}_{\sigma^0,\omega}(G^d)$?

Definition

A **metastable state** σ^m ($\sigma_m \neq \sigma^+, \sigma^-$) is a configuration that has no flippable spins ($\Delta E_i \leq 0$).

Example

On Hypercube Q_d, the simplest construction for a metastable state would be a sub-cube Q_k with all $+1$ spins embedded inside, with $k \geq \lfloor \frac{d}{2} \rfloor + 1$, and set its complement to be all -1 spins.

As $d \to \infty$, # metastable states grows rapidly (actually lower bounded by $2^{2^{d/2-1}}$), which causes the system to be trapped in one of them and prevents it from entering the ground states.

Metastable States: largest k-core 1

Fact: A metastable state σ_m is composed by k-cores with $k \geq \lfloor \frac{d}{2} \rfloor + 1$

Definition

A **k-core** of a graph G is the maximal subgraph in which every vertex has a degree of at least k.

Example

A sub-cube Q_k is a k-core. Any union of a collection of connected sub-cubes Q_k's (not necessarily disjoint) is also a k-core

Split the final configuration of the Hypercube into two subgraphs: Q_d^+ consisting of only $+1$ spins, and Q_d^- consisting of only -1 spins, preserving the edges that connect spins with the same signs and throwing away the edges that connect spins with opposite signs.

Note that a k-core in Q_d^+ and Q_d^- with $k \geq \lfloor \frac{d}{2} \rfloor + 1$ is stable, we can therefore ask the question: What are the largest k-cores of Q_d^+ and Q_d^- ?

Metastable States: largest k-core 2

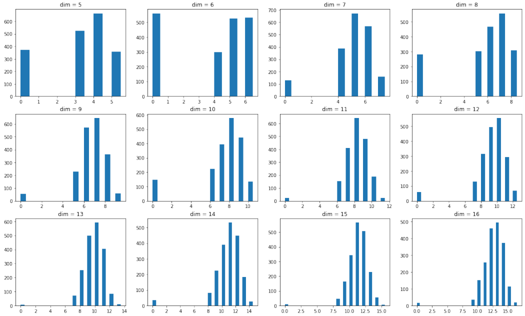

Figure 7: Distribution of Largest k-cores against dimension

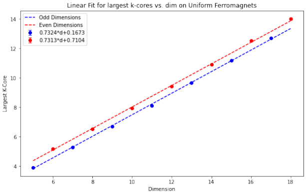

Figure 8: Linear fit for the mean of largest k-core

Q: Are stable k-cores (in the sense of configuration) always unions of sub-cubes Q_k's with $k \geq \lfloor \frac{d}{2} \rfloor + 1$?

A: No! The geometry of stable k-cores can be much more complicated.

Blinkers

On the final configuration, a single blinker would just be a spin σ_i with $\Delta E_i = 0$, i.e.: an equal number of $+1$ and -1 neighbors.
But we can have multi-site blinkers with more interesting structures (chain-like, tree-like, loop-like, etc).

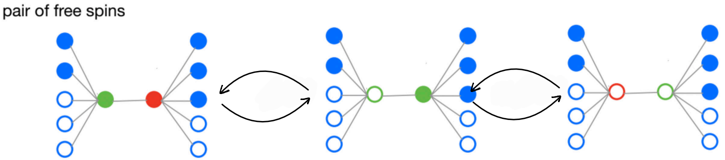

Figure 9: Filled circles are plus spins and open circles are minus spins. Green spins are currently free to flip while red spins are currently frozen. Blue spins are permanently frozen. Each state can enter the states next to them within a single time step

Statistics of Blinkers

Consider the following two events:

$$B_i^d = \{\text{Spin } \sigma_i^d \text{ ends up to be a blinker}\}$$

$$B^d = \{\text{Final configuration } \lim_{t\to\infty} \sigma^d(t) \text{ contains at least one blinker}\}$$

We are interested in: $\lim_{d\to\infty} \mathbb{P}_{\sigma^0,\omega}(B_i^d)$ and $\lim_{d\to\infty} \mathbb{P}_{\sigma^0,\omega}(B^d)$

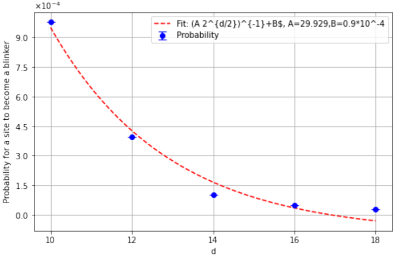

Figure 10: Probability for a site to be a blinker

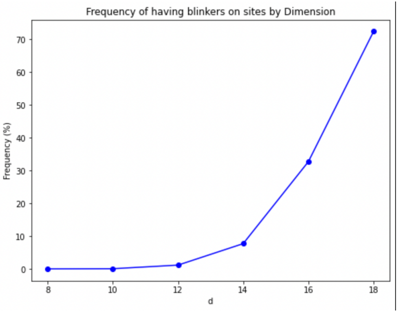

Figure 11: Probability for a configuration to contain blinkers

Theoretical Results

Things we have analytically shown so far:

- A lower bound for the dimension requirement for a Hypercube to have a single-site blinker: $d \geq 8$.
- A canonical construction of a blinker configuration at arbitrary d using unions of sub-cubes.
- A lower bound for the entropy of stable states: $\Omega(2^{2^{d/2-1}})$.
- A construction of a loop blinker at $d = 10$.
- A construction of metastable configuration using unions of sub-cubes Q_k, but with $k \leq \lfloor \frac{d}{2} \rfloor$

Possible Approach to Show Gaussian Limit of M_∞^d

- For simplification, consider $d = 2k - 1$ (No Blinkers!)
- Renormalize Q_d into $Q_{d-\tilde{k}}$, with a $\tilde{k}$-dim ($\tilde{k} \geq j$) sub-cube $Q_{\tilde{k},j}$ on each of the new vertex j.
- Intuitively, the couplings inside each sub-cube are much larger than the couplings between them for large d.
- Let the final magnetization on each sub-cube $Q_{\tilde{k},j}$ to be $\tilde{M}_\infty^{d,j}$, we should expect:

$$\forall j_1, j_2 : \lim_{d\to\infty} \mathrm{Cov}(\tilde{M}_\infty^{d,j_1}, \tilde{M}_\infty^{d,j_2}) = 0$$

- We should therefore aim to derive the site-to-site correlation function as $t \to \infty$ for any fixed d, i.e.:

$$G^d(\sigma_i, \sigma_j) = \lim_{t\to\infty} \mathrm{Cov}(\sigma_i(t), \sigma_j(t))$$

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Q&A

Any Questions?