Topic: Uncertainty and Estimation

Concepts: central limit theorem, confidence interval

Note: *Start each problem with the starter code shared in the repository. Feel free to use scipy reference or any other source for syntax, but invest time in following the logic of the programs.*
https://docs.scipy.org/doc/scipy/reference/stats.html

Important note for submissions of all problems and tasks:

Each time you generate a plot, save it with a filename like a6p1_task1_fig1_yourInitials.png (assignment 6 → problem 1 → task 1 → figure 1). Upload all your figures to the figs_results directory in your GitHub repo.

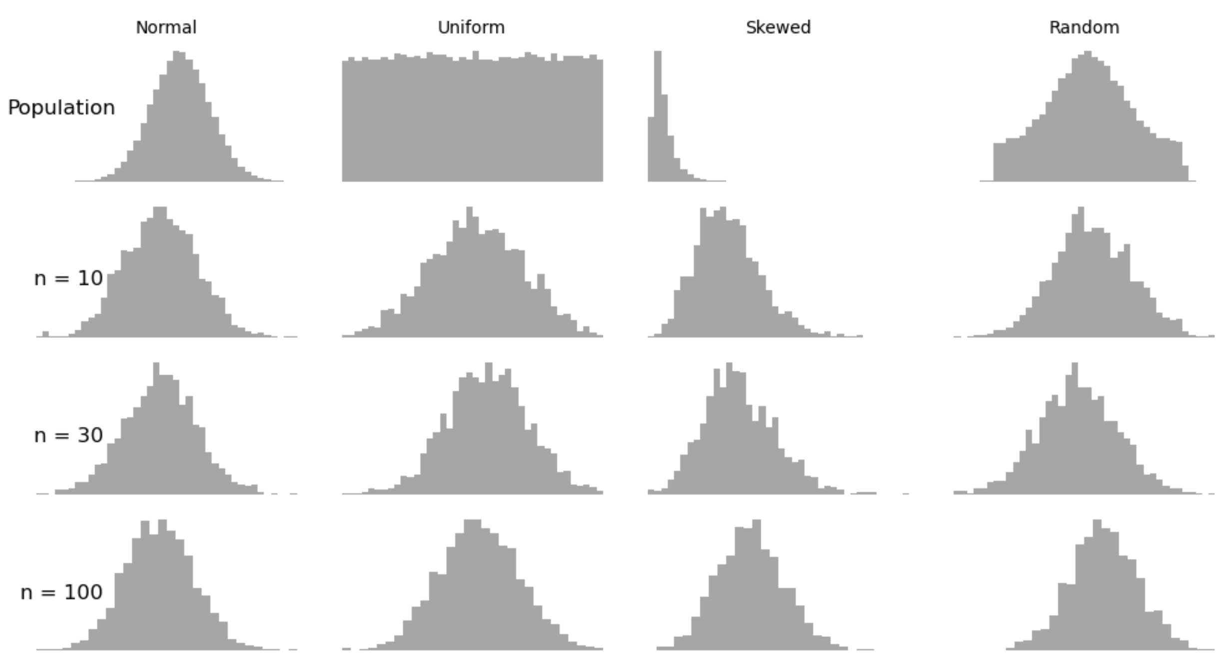
Problem-1 (Central Limit Theorem):

The **Central Limit Theorem (CLT)** says that regardless of the original distribution's shape, the sample mean is normally distributed for the sufficiently large sample (usually, $n > 30$). In other words

if you keep averaging, the averages "forget" where they came from and look Gaussian.

Discussion:

It's striking that uniform, skewed, or even heavy-tailed distributions all end up with means that follow a bell-shaped curve. This universality makes the CLT one of the most surprising and powerful ideas in probability.



Objective:

To observe and quantify how averages tend to normality; contrast the Gaussian behavior of sample means with the irregular behavior of standard deviations. Further, explore the interplay between the number of simulation (N) and the number of samples per average (m).

Step-by-step tasks:

For following tasks, use the provided starter python file named `assm6_q1_student.py` and work on each task by uncommenting the corresponding function in the `__main__`. Understand the sequence of provided code; look out for #WRITE_YOUR_CODE sections; complete these missing lines of codes. Once done, make sure to sync your complete assignment to your GitHub repo.

Task 1. Define distributions and plot PDFs: Using **`scipy.stats`**, generate $N = 1000$ random samples from four distributions as specified below:

- Uniform (`uniform`): $x \in [-2,2]$
- Normal (`norm`): $\mu = 0;\ \sigma = 1$
- Lognormal (`lognorm`): $s = 0.5;\ \text{scale} = 1$
- Gumbel-1 (`gumbel_r`): $\text{loc} = 0; \text{scale} = 1$

Plot their probability density functions (PDFs) to see densities of random variables.

Task 2. Raw sampling ($m = 1$): For each distribution, draw $N = 1000$ independent random observations and plot their histogram (use 20 bins). Note that you just plotted 1000 observations of the *average-of-1*.

Note: the histogram mimics the parent distribution.

Task 3. Averaging effect ($m > 1$): From each distributions, generate:

a) 1000 averages-of-2,
b) 1000 averages-of-10, and
c) 1000 averages-of-100.

where $N = 1000$ is the number of replicates. Plot histograms for 1, 2, 10, and 100 side by side.

Note: The curve tends toward a bell shape.

Task 4. Variance scaling: Show that the variance of the sample mean reduces with m:

$$\text{Var}(\bar{X}_m) = \frac{\sigma^2}{m}.$$

Task 5. Mean vs. standard deviations: Collect the sample means (which follow Gaussian by CLT) and the sample standard deviations (which do not). Plot histograms of both. Overlay the theoretical distribution of sample mean (i.e., normal distribution with mean same as population mean and standard deviation, $\sigma/\sqrt{n}$).

Note: The sample mean tends toward a bell shape, while sample stdev does not.

Problem-2 (Confidence intervals):

A confidence interval gives a range of plausible values for the population mean. A 95% CI means that if we repeat the experiment many times, roughly 95% of such intervals will contain the true mean.

For starter code, refer to the jupyter notebook `assm6_q2_student.py` on GitHub repository.

Step-by-step tasks:

- Generate 40 samples from a normal distribution with known mean (use the last three digits of your roll number) and variance (use 15% of the mean).
- For each sample, compute the 95% confidence interval of the sample mean.
- Plot all 40 CIs on one chart, marking the true mean as a vertical line.
- Count and highlight intervals that do not contain the true mean (expect ~2 out of 40).
- Comment on how often the true mean falls outside the CI in practice.
