Topic: Numerical Modelling

Concepts: Initial Value Problem (IVP), time stepping; Boundary Value Problem (BVP), solving a matrix equation.

Ref:
https://mude.citg.tudelft.nl/book/2024/numerical_methods/6-initial-value-problem-singlestep.html

https://mude.citg.tudelft.nl/book/2024/numerical_methods/8-boundary-value-problem.html

(1) Initial Value Problem (time stepping)

Consider the following problem from Structural Dynamics.

Damped Free Vibration of a Mass-Spring-Damper System (Overdamped or Critically Damped)

Consider a single-degree-of-freedom (SDOF) system with mass (m), damping coefficient (c), and stiffness (k). The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = 0$$

To reduce this to a first-order ODE, consider the velocity

$$v(t) = \dot{x}(t)$$

as the primary variable, and assume displacement is negligible or constant (e.g., ($x(t) \approx 0$)) for simplicity. Then the equation simplifies to:

Simplified Model: Damped Velocity Decay

$$m\,\frac{dv}{dt} + c\,v = 0$$

or

$$\frac{dv}{dt} = -\frac{c}{m}\,v(t)$$

Initial Condition

Let:

$$v(0) = v_0$$

Analytical Solution

This has the solution:

$$v(t) = v_0 e^{-\frac{c}{m}t}$$

Numerical Schemes

Let (Δt) be the time step.

- **Euler Forward (Explicit):**

$$v_{n+1} = v_n - \frac{c}{m}\,\Delta t \cdot v_n$$

- **Euler Backward (Implicit):**

$$v_{n+1} = \frac{v_n}{1 + \frac{c}{m}\Delta t}$$

Use the following values of parameters for the simulation

- m = 1 kg
- c = 0.5 Ns/m
- v_0 = 1 m/s
- \Delta t = 0.1 s
- Simulate for (t \in [0, 5])

(i) Verify that the analytical and numerical solution scheme equations given above are correct for the problem under consideration. Show the derivation and calculation checks manually.

(ii) Find the velocity variation as a function of time for a duration of 5 seconds using both the methods (explicit & implicit) through a computer program (you may use the script available in the teachbook examples in the link provided at the beginning of this document).

(iii) Plot the velocity solutions obtained using both the methods and compare it with the analytical solution on the same plot.

(iv) Identify the value of the time step at which the methods become unstable and also identify the range of time step values in which the methods (both of them) are stable across the iterations.

(v) Find the solutions for a decreasing set of step sizes and show that the schemes are converging to the exact solution. Also, find out the rate of convergence from the log-log plot of error vs. \Delta t. (Note: error can be found as root mean square error or the maximum absolute error across all values of velocity in the time domain under consideration.)

(2) Boundary Value Problem (matrix equation)

Consider the following problem from Structural Mechanics.

Axial Deformation of a Uniform Elastic Bar

Consider a uniform elastic bar of length (L), fixed at both ends, subjected to a uniform axial load. The governing equation for axial displacement (u(x)) is:

$$\frac{d^2u}{dx^2} = -\frac{f}{EA}$$

Where:

- (u(x)): axial displacement
- (f): uniform axial force per unit length
- (E): Young's modulus
- (A): cross-sectional area

Boundary Conditions

Assume the bar is fixed at both ends: u(0) = 0, u(L) = 0

Analytical Solution

Integrating twice:

$$u(x) = \frac{f}{2EA}x(L-x)$$

This is a parabolic displacement profile.

Numerical Solution: Central Finite Difference

Discretize the domain ($x \in [0, L]$) into (N+1) nodes with spacing (h = L/N). For interior nodes (i = 1, 2, …, N-1), the second derivative is approximated as:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = -\frac{f}{EA}$$

This leads to a **linear system of equations**:

$$\mathbf{Au} = \mathbf{b}$$

Where:

- (**A**) is a tridiagonal matrix
- (**u**) is the vector of unknown displacements at interior nodes
- (**b**) is a constant vector with entries (-f/EA)

Use the following parameters

- (L = 1.0 m)
- (E = 200 \times 10^9 Pa)
- (A = 0.01 sq. m)
- (f = 1000 N/m)
- (N = 10) (number of intervals) (i.e. h=0.1m)

(i) Verify that the analytical and numerical solution scheme equations given above are correct for the problem under consideration. Show the derivation and calculation checks manually.
(ii) Find the displacement profile in the given domain using the central finite difference method scheme for discretization through a computer program (you may use the script available in the teachbook examples in the link provided at the beginning of this document). Clearly write down the tridiagonal matrix that is obtained.
(iii) Plot the displacement solution obtained numerically and compare it with the analytical solution on the same plot.
(iv) Find the solutions for a decreasing set of step sizes and show that the numerical method converges to the exact solution. Also, find out the rate of convergence from the log-log plot of error vs. *h*. (Note: error can be found as root mean square error or the maximum absolute error across all values of displacement in the domain under consideration.)