NAME – MUDIT SAND

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Tutorial – 1

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Ans 1 (1)

Ans (1) a

(i) $\vec{a}\cdot\vec{b} = a^T b = [1\ 2\ 3]\begin{bmatrix}1\\0\\1\end{bmatrix} = [4]$

(ii) $\vec{a}\cdot\vec{b} = a^T b = [1\ 1]\begin{bmatrix}1\\-1\end{bmatrix} = [0]$

Ans (1) b Norm of $\vec{a} = \|\vec{a}\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{1^2+0^2+(-1)^2}$

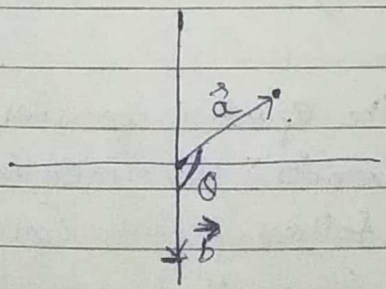
$$\boxed{\|\vec{a}\| = \sqrt{2}}$$

Ans (1) c $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$ $\left\{ \text{here } \dfrac{\vec{a}}{|\vec{a}|} = \hat{a},\ \dfrac{\vec{b}}{|\vec{b}|} = \hat{b} \right\}$

$|\vec{a}| = \sqrt{2}$ $|\vec{b}| = 1$

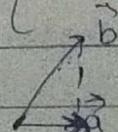
$\vec{a}\cdot\vec{b} = a^T b = [1\ 1\ 0]\begin{bmatrix}0\\-1\\0\end{bmatrix} = [-1]$

$$\boxed{\cos\theta = \frac{-1}{\sqrt{2}}}$$ {Cosine of the Angle is Asked in the Question}



Ans (1) d Projection of a on b $= \vec{a}\cdot\hat{b}$

$\hat{b} = \dfrac{\vec{b}}{|\vec{b}|} \Rightarrow \vec{a}\cdot\hat{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ $\{|\vec{b}| = \sqrt{2}\}$



$\vec{a}\cdot\vec{b} = [1\ 0]\begin{bmatrix}1\\1\end{bmatrix} = [1]$

$$\boxed{\vec{a}\cdot\hat{b} = \frac{1}{\sqrt{2}}}$$

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Ans (2)

Ans (2) a

(i) $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \end{bmatrix}_{1\times 2}$

(ii) $\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} = [\ ]_{\underset{m}{3} \times \underset{n}{2}} [\ ]_{\underset{p}{3} \times \underset{q}{1}}$

$\underline{n \neq p}$ → this Product Can't be Performed

Ans (2) b $\begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix} = 0(-3) - (1)(-2)$

$= \underline{\underline{2}}$

Ans (2) c Eigen vector Satisfies $Ax = \lambda x$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since for $\lambda = -1$ the equation Satisfies and this Can be Considered as eigen Vector Though we are following the Convention of {as discussed by Sir in the Class} that Eigen vector must have Norm 1. So we will Not Consider it as eigen Vector Although it is the one

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Ans 2 d

Given :-

$$\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

I st check $\mapsto$ Eigen Values are of Norm 1 :-

$$\|v_1\| = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = \sqrt{1} = 1 \quad ✓$$
$$\|v_2\| = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = \sqrt{1} = 1 \quad ✓$$

II nd check $\mapsto$ Last matrix is inverse of the first matrix

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since $AA^{-1} = I$

So last Matrix is inverse

III rd check $\mapsto$ $Av_i = \lambda_i v_i \rightarrow$

for $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \lambda \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \quad \{\lambda \text{ can be } 2 \text{ or } 8\}$$

$$\begin{bmatrix} 2/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix} = \lambda \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \quad \{\text{for } \lambda = 2 \text{ this is true}\}$$

for $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \lambda \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} 8/\sqrt{2} \\ 8/\sqrt{2} \end{bmatrix} = \lambda \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \{\text{for } \lambda = 8 \text{ this is true}\}$$

$\Rightarrow$ So this Given Eigen Value decomposition is valid.

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Ans ② e

(i) Rank of $\begin{bmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{bmatrix}$

Determinant of the matrix = $\begin{vmatrix} 0 & -1 & 5 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{vmatrix}$

$= -2(-5-5) + 1(6-20)$

$= 20 - 14 = 6 \neq 0$

So $\boxed{\text{Rank} = 3}$

(ii) $\begin{bmatrix} -5 & -7 \\ 5 & 7 \end{bmatrix}$

$R_2 \to R_1 + R_2$

$\begin{bmatrix} -5 & -7 \\ 0 & 0 \end{bmatrix}$

So one colourm is lenearly dependent on the another and thes matrix has the rank of one

$\boxed{\text{Rank} = 1}$

Ans ② f: Trace of $\begin{bmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{bmatrix}$

$= 0 + 4 + 5 = 9$

Trace of $\begin{bmatrix} -5 & -7 \\ 5 & 7 \end{bmatrix}$

$= -5 + 7 = 2$

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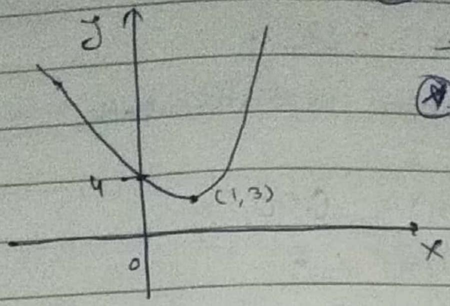
Ans ③

Ans ③ a:- $f(x) = x^2 - 2x + 4$



→ function is Continuous.

→ $f'(x) = 2x - 2$

→ function is differentiable.

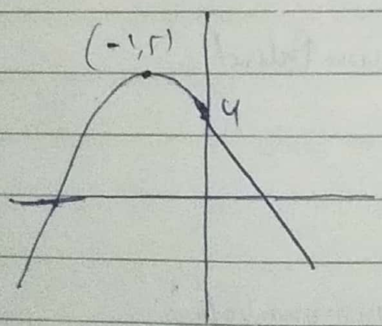
$f'(x) = 0 \Rightarrow 2x - 2 = 0$

$\boxed{x = 1}$

$f''(x)\big|_{x=1} = 2 > 0$

- function has a minima {one}
- that minima is a global minima
- $f(1) = 3$ {Minimum Value}
- Since $f''(x) > 0 \Rightarrow$ Convex function
- Since Convexity is Confirmed And its derivative exist then Subderivative is not required though it exists.

Ans ③ b:- $f(x) = -x^2 - 2x + 4$



→ function is Continuous.

→ $f'(x) = -2x - 2$

- function is differentiable

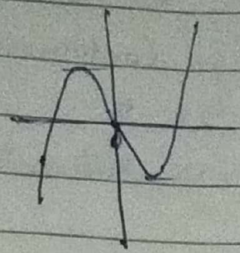
$f'(x) = 0 \Rightarrow -2x - 2 = 0$

$\boxed{x = -1}$

$f''(x) = -2 < 0$

- function has a maxima {one}
- that Maxima is a global maxima
- $f(-1) = 5$ {maximum Value}
- Since $f''(x) < 0 \Rightarrow$ Concave func.
- Since function is Concave So Subgradient doesn't exist.

Ans 3 c $f(x) = x^3 - 9x$



- function is Continuous.

$\rightarrow f'(x) = 3x^2 - 9$

- function is differentiable

$\rightarrow f''(x) = 6x$

So for $x < 0$ $f''(x) < 0$

for $x > 0$ $f''(x) > 0$

- $(-\infty, 0) \rightarrow$ function is Concave
- $(0, \infty) \rightarrow$ function is Convex
- at $0 \rightarrow$ Point of Inflexion

$\rightarrow$ $3x^2 - 9 = 0$

$x^2 = 3$

$\boxed{x = \pm\sqrt{3}}$

- function has one local minima and one Local Maxima.

- Local maxima @ $x = -\sqrt{3}$

$f(x) = (-\sqrt{3})^3 + 9\sqrt{3}$

$f(x) = 10.39$ (Maximum Value)

- Local minima @ $x = +\sqrt{3}$

$f(x) = (\sqrt{3})^3 - 9\sqrt{3}$

$f(x) = -10.39$ (Minimum Value)

Note: These maximum and minimum Value is Local

- Since it is not Convex throughout its domain so Subgradient doesn't exist

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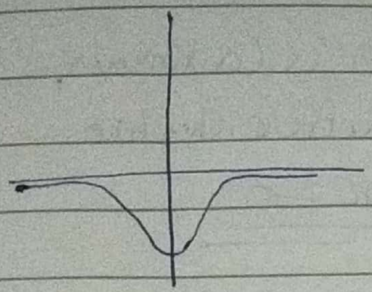
Ans ③d $f(x) = -e^{-x^2}$



* function is Continuous

$f'(x) = (e^{-x^2}) \times (2x)$

$f'(x) = 2xe^{-x^2}$

* $f(x)$ is differentiable

$f''(x) = 2[e^{-x^2} + x \times (e^{-x^2} \times -2x)]$

$f''(x) = 2[e^{-x^2} + (-2x^2 e^{-x^2})]$

$= 2e^{-x^2}[1 - 2x^2]$

$f'(x) = 0 \Rightarrow \underline{x = 0}$

$f''(x)|_{x=0} = 2 \geq 0$ {Minima}

* function has global minima
* from the graph or the equation we can se as x increases value in the Denominator will increase so $f''(x)$ will reach towards the zero.
* So it looks like a convex function
* Subgradient exists

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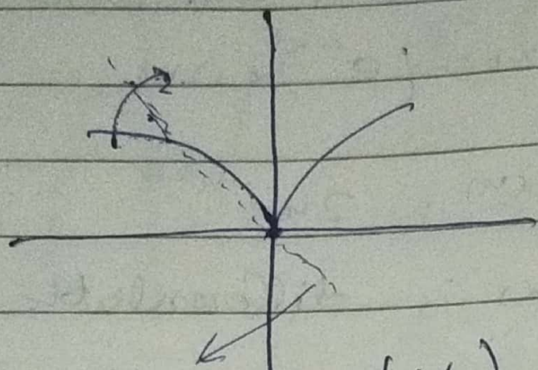
Ans 3c $f(x) = \sqrt{|x|}$



Subgradient (X)
(doesnot exists)

* function is Continuous
* Not differentiable at $x=0$
* Since it is not differentiable at $x=0$ so the question on Convex or Concave doesn't arise
* though minimum value we can see is $f(x) = 0$
* Subgradient doesn't exist

Ans 4 => [.ipynb] file is Attached in Moodle Seperately

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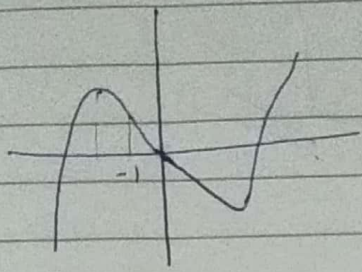
Ans → 5 (a)

$f(x) = x^3 - 9x$

$f'(x) = 3x^2 - 9$

$f'(-1) = -6 < 0$



Since $f'(-1) < 0$ So if we take a positive step then we will move towards Minima {local}

Ans (5) b $f'(-1) = -6$

So if $x \leftarrow x + \eta f'(-1)$ where $\eta = 1$

$x \leftarrow x - 6$

Since By taking this step we will fun the Critical Points and we will go over by them

So we will avoid taking $\eta = 1$ {It's very large}

Ans (5) c

Step Size $= \frac{f'(-1)}{f''(-1)}$

$f'(-1) = -6$ $f''(x) = 6x$

$f''(-1) = -6$

First Step :-

Step Size $= 1$

$x \leftarrow x - 1$

Since Maxima is at -1.732 So we will fan the maxima but Not be able to reach at maxima with optimal size given by Newton's method {Step size = 1} In first step then we move to the Second Step

B

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Second Step :-

Now $x = -2$

$f'(-2) = 3$

$f''(-2) = -12$

$$x \leftarrow x - \frac{3}{-12} = x + \frac{1}{4}$$

So we will move toward the maxima and we are moving towards maxima means we will Reach to maxima in multiple steps.

Ans → 5 d $f(x) = 3x_1^2 + 2x_2 + 5x_3^3 + 4x_1x_2$

$$\nabla f(x) = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 + 4x_2 \\ 2 + 4x_1 \\ 15x_3^2 \end{bmatrix}$$

Ans → 5 e According to Lagrange Multiplier Method,

Minimize $f(x) = x_1^2 + x_2^2$

S/t

$g(x) = (x_1-1)^2 + (x_2-1)^2 - 1 = 0$

$\nabla f(x) = \lambda \nabla g(x)$ { λ = Lagrange Multiplier }

$$\nabla f(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\nabla g(x) = \begin{bmatrix} 2(x_1-1) \\ 2(x_2-1) \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \lambda \begin{bmatrix} 2(x_1-1) \\ 2(x_2-1) \end{bmatrix}$$

$\cancel{2}x_1 = \cancel{2}\lambda(x_1-1)$ — (1)

$\cancel{2}x_2 = \cancel{2}\lambda(x_2-1)$ — (2)

$1 = x_1(\lambda - 1)$

$\lambda = \frac{1}{x_1} + 1 = \frac{1}{x_2} + 1$

$\boxed{x_1 = x_2}$

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Putting $x_1 = x_2$ in $f(x)$

$$2(x_1-1)^2 = 1$$

$$x_1 - 1 = \pm\sqrt{\frac{1}{2}}$$

$$x_1 = 1+\frac{1}{\sqrt{2}}, \quad 1-\frac{1}{\sqrt{2}}$$

for minimum we will take $x_1 = x_2 = 1-\frac{1}{\sqrt{2}}$

$$f(x) = 2 * \left(1-\frac{1}{\sqrt{2}}\right)^2$$

$$\boxed{f(x)\Big|_{Minimum} = .17157287}$$

Ans→⑥

Ans ⑥ a x = No. of Heads in a fair Coin toss.

Space = {HH, HT, TH, TT} H = Head T = Tail

x	0	1	2
Pmf(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$\boxed{Pmf(x=2) = \frac{1}{4}}$$

Ans ⑥ b: Fair Coin Has higher entropy. Since entropy Represents the randomness and fair Coin is likely to have more randomness

Ans (b)c $x \to$ No. of Heads

Space = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

$y \to$ No. of Consecutive Heads

$y = 0 \to \{TTT\}$

$y = 1 \to \{HTT, THT, TTH, HTH\}$

$y = 2 \to \{THH, HHT\}$.

$y = 3 \to \{HHH\}$

$P(y, x) \cdot \to$

$y \downarrow$; $x \to$	0	1	2	3
0	$\frac{1}{8}$	0	0	0
1	0	$3/8$	$1/8$	0
2	0	0	$2/8$	0
3	0	0	0	$1/8$

Ans (b)d $P(x=0) = \frac{1}{8}$ $P(x=1) = \frac{3}{8}$

$P(x=2) = \frac{3}{8}$ $P(x=3) = \frac{1}{8}$

$P(y=0) = \frac{1}{8}$ $P(y=1) = \frac{4}{8}$

$P(y=2) = \frac{2}{8}$ $P(y=3) = \frac{1}{8}$

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Ans (b) e

$$P(x \mid y=1) = ? = \frac{P(x)}{4/8} = 2P(x)$$

x	0	1	2	3
$P(x \mid y=1)$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{4}$

Ans (b) f

(i) ipynb file is Attached on Moodle and it is Clearly Visible the y is more likely to have the values greater than 3.

(ii) $$\text{pdf} = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

for intersection {since $\mu_x = \mu_y = 0$}

$\cancel{\frac{x^2}{2}} = \cancel{x}$ $\sigma_x = 1$, $\sigma_y = 2$

$$\frac{1}{1} \cdot \exp\left[-\frac{x^2}{2}\right] = \frac{1}{2} \exp\left[-\frac{x^2}{8}\right]$$

take log both side

$$-\frac{x^2}{2} = \ln\left(\frac{1}{2}\right) + \left(-\frac{x^2}{8}\right)$$

$$-\ln\left(\frac{1}{2}\right) = \frac{x^2}{2} - \frac{x^2}{8}$$

$$\ln(2) = \frac{3x^2}{8}$$

$$\frac{8 \times .693}{3} = x^2$$

$$\boxed{x = \pm 1.359}$$ {Here Pdf Cross each other}

Ans → 6(g) $\mu_{cat} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\Sigma_{cat} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mu_{dog} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\Sigma_{dog} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$

$$pdf = \frac{1}{(2\pi)^{k/2}} (\det(\Sigma))^{-\frac{1}{2}} \cdot \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right]$$

for Cat :-

$\det(\Sigma_{cat}) = 1$ $x - \mu_{cat} = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}$

$\Sigma_{cat}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$pdf \propto \exp\left[\frac{-1}{2}\begin{bmatrix} -0.5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -0.5 \\ 0 \end{bmatrix}\right]$$

$$pdf \propto \exp\left[\frac{-1}{2} \times \frac{1}{4}\right]$$

$$pdf_{cat} \propto e^{-\frac{1}{8}}$$

~~for dog :-~~

$$pdf \propto \exp\left[-\frac{1}{2}\begin{bmatrix} 0.5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ -1 \end{bmatrix}\right]$$

$$~~pdf \propto \exp\left[\frac{-1.25}{2}\right]~~$$

$$pdf|_{cat} > pdf|_{dog}$$

So for given slant and Nose size
It will be more likely a cat