Complete Graphs and Their Properties

Pushpendra Kumar Pateriya
School of Computing & Information Technology
Lovely Faculty of Technology & Sciences
Lovely Professional University
E-mail: pushpendra.14623@lpu.co.in

September 11, 2013

Outline I

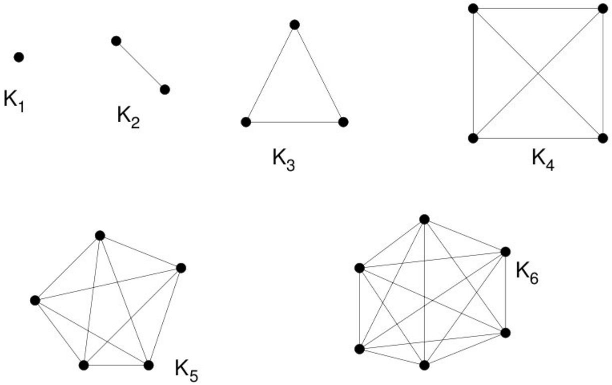
Complete Graphs I

Let N be a positive integer.
Definition: A *complete graph* is a graph with N vertices and an edge between every two vertices.

- There are no loops.
- Every two vertices share exactly one edge.

We use the symbol K_N for a complete graph with N vertices.

Complete Graphs II



How many edges does K_N have?

- K_N has N vertices.

How many edges does K_N have?

- K_N has N vertices.
- Each vertex has degree $N - 1$.

How many edges does K_N have?

- K_N has N vertices.
- Each vertex has degree $N - 1$.
- The sum of all degrees is $N(N - 1)$.

How many edges does K_N have?

- K_N has N vertices.
- Each vertex has degree $N - 1$.
- The sum of all degrees is $N(N - 1)$.
- Now, the Handshaking Theorem tells us that. . .

How many edges does K_N have?

- K_N has N vertices.
- Each vertex has degree $N - 1$.
- The sum of all degrees is $N(N - 1)$.
- Now, the Handshaking Theorem tells us that. . .
- $\boxed{\text{The number of edges in } K_N \text{ is } \frac{N(N-1)}{2}.}$

How many edges does K_N have?

- K_N has N vertices.
- Each vertex has degree $N - 1$.
- The sum of all degrees is $N(N - 1)$.
- Now, the Handshaking Theorem tells us that. . .
- $\boxed{\text{The number of edges in } K_N \text{ is } \frac{N(N-1)}{2}.}$
- This formula also counts the number of pairwise comparisons between N candidates.

How many edges does K_N have?

- K_N has N vertices.
- Each vertex has degree $N - 1$.
- The sum of all degrees is $N(N - 1)$.
- Now, the Handshaking Theorem tells us that. . .
- The number of edges in K_N is $\frac{N(N-1)}{2}$.
- This formula also counts the number of pairwise comparisons between N candidates.
- The Method of Pairwise Comparisons can be modeled by a complete graph.
  - Vertices represent candidates
  - Edges represent pairwise comparisons.
  - Each candidate is compared to each other candidate.
  - No candidate is compared to him/herself.

Handshaking Theorem

Theorem

Let $G = (V, E)$ be an undirected graph. Then $2|E| = \sum_{v \in E} deg(v)$

Proof.

Each edge contributes twice to the sum of the degrees of all vertices. □

Questions:

1. Suppose a graph has 5 vertices. Can each vertex have degree 3? degree 4?
2. Is it possible to have a graph S with 5 vertices, each with degree 4, and 8 edges?
3. A graph with 21 edges has 7 vertices of degree 1, three of degree 2, seven of degree 3, and the rest of degree 4. How

Handshaking theorem for directed graphs

The **in-degree** of a vertex v, denoted $deg^{-}(v)$ is the number of edges which terminate at v.
Similarly, the **out-degree** of v, denoted $deg^{+}(v)$, is the number of edges which initiate at v.

Theorem

For any directed graph $G = (V, E)$,
$|E| = \sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v).$

Proof.

Notice that each edge contributes one to the in-degree of some vertex and one to the out-degree of some vertex. This is essentially the proof of the Theorem. □

Hamiltonian Circuits in K_N

- A **Hamiltonian path** is a path in an undirected or directed graph that visits each vertex exactly once.
- A Hamiltonian cycle (or Hamiltonian circuit) is a Hamiltonian path that is a cycle.
- The first/last vertex is called the "**reference vertex**".
- Changing the reference vertex does not change the Hamiltonian circuit

How many different Hamiltonian circuits does K_N have?

For making a Hamiltonian circuit:

- N choices for the reference vertex.

How many different Hamiltonian circuits does K_N have?

For making a Hamiltonian circuit:

- N choices for the reference vertex.
- $N - 1$ choices for the next vertex.

How many different Hamiltonian circuits does K_N have?

For making a Hamiltonian circuit:

- N choices for the reference vertex.
- $N - 1$ choices for the next vertex.
- $N - 2$ possibilities for the vertex after that.

How many different Hamiltonian circuits does K_N have?

For making a Hamiltonian circuit:

- N choices for the reference vertex.
- $N - 1$ choices for the next vertex.
- $N - 2$ possibilities for the vertex after that.
- ...

How many different Hamiltonian circuits does K_N have?

For making a Hamiltonian circuit:

- N choices for the reference vertex.
- $N-1$ choices for the next vertex.
- $N-2$ possibilities for the vertex after that.
- ...
- 2 possibilities for the $(N-1)^{th}$ vertex.

How many different Hamiltonian circuits does K_N have?

For making a Hamiltonian circuit:

- N choices for the reference vertex.
- $N - 1$ choices for the next vertex.
- $N - 2$ possibilities for the vertex after that.
- ...
- 2 possibilities for the $(N-1)^{th}$ vertex.
- 1 possibility for the N^{th} vertex.

How many different Hamiltonian circuits does K_N have?

For making a Hamiltonian circuit:

- N choices for the reference vertex.
- $N - 1$ choices for the next vertex.
- $N - 2$ possibilities for the vertex after that.
- ...
- 2 possibilities for the $(N - 1)^{th}$ vertex.
- 1 possibility for the N^{th} vertex.
- and then the reference vertex again.

How many different Hamiltonian circuits does K_N have?

For making a Hamiltonian circuit:

- N choices for the reference vertex.
- $N - 1$ choices for the next vertex.
- $N - 2$ possibilities for the vertex after that.
- ...
- 2 possibilities for the $(N - 1)^{th}$ vertex.
- 1 possibility for the N^{th} vertex.
- and then the reference vertex again.

Note

If we are counting Hamiltonian circuits, then we don't care about the reference vertex.

How many different Hamiltonian circuits does K_N have?

Conclusion: The number of Hamiltonian circuits in K_N is

$$\boxed{(N-1) \times (N-2) \times \ldots \times 3 \times 2 \times 1 = (N-1)!}$$

Vertices(N)	Edges$\left(\frac{N(N-1)}{2}\right)$	Hamiltonian Circuits($(N-1)!$)
1	0	
2	1	
3	3	2
4	6	6
5	10	24
6	15	120
...	...	...
16	120	1307674368000

Eulerian Circuits in K_N

An **Eulerian trail (or Eulerian path)** is a trail in a graph which visits every edge exactly once. Similarly, an **Eulerian circuit** or Eulerian cycle is an Eulerian trail which starts and ends on the same vertex.

Query

For which values of N does the complete graph K_N have an Euler circuit?

Eulerian Circuits in K_N

An **Eulerian trail (or Eulerian path)** is a trail in a graph which visits every edge exactly once. Similarly, an **Eulerian circuit** or Eulerian cycle is an Eulerian trail which starts and ends on the same vertex.

Query

For which values of N does the complete graph K_N have an Euler circuit?

Answer

When N is odd. (Every vertex in K_N has degree $N - 1$, so we need $N - 1$ to be even.)

Learn with devotion...