Laplace transform table:

1	$1/s$	$\delta(t-c)$	e^{-cs}
t	$1/s^2$	$e^{ct}f(t)$	$F(s-c)$
e^{at}	$1/(s-a)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\frac{dy(t)}{dt}$	$sY(s)-y(0)$	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$\frac{d^2y(t)}{dt^2}$	$s^2Y(s)-sy(0)-y'(0)$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
$F(t)*G(t)$	$F(s)G(s)$	F(t).G(t)	F(s)*G(s)

$$L\left[\int f(t)dt\right] = \frac{F(s)}{s} + \int f(t)dt\ (at\ t=0)$$

Convolution: $F(t)*G(t) = \int_0^t f(\tau)g(t-\tau)d\tau$

Fundamental theorem of calculus and product rule:

$\frac{d}{dt}\int_0^t A(x)dx = A(t)$ These hold for matrix functions

Leibniz rule holds for matrix functions:

$$\frac{d}{dt}\int_{f(t)}^{g(t)} A(t,\sigma)d\sigma = A(t,g(t))g'(t) - A(t,f(t))f'(t) + \int_{f(t)}^{g(t)} \frac{\delta}{\delta t}A(t,\sigma)d\sigma$$

General solution of $\dot{X} = AX + BU$ is given by:

$$X(t) = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + e^{A(t-t_0)}X(t_0)$$

If asked to prove that this is solution, use Leibniz rule.

To get the state-space representation of a given differential eqn, first assume states x1, x2,..., xn. Then take derivatives of all these states to get X_dot. Then for xn_dot find equation in terms of all other states. Make matrix from the state equations obtained.

Properties of LTI system:

A system is jointly linear in the initial condition response (u=0) and the force response (x(t0)=0), if the following conditions hold:

$y(t,\alpha x_1+\beta x_2,0) = \alpha y(t,x_1,0)+\beta y(t,x_2,0)$

$y(t,\alpha x,\beta u) = \alpha y(t,x,0)+\beta y(t,0,u)$

$y(t,0,c_1u_1+c_2u_2) = c_1y(t,0,u_1)+c_2y(t,0,u_2)$

A system is jointly time-invariant if a delay of τ in th state or input produces corresponding time delay in the output. Essentially, the behaviour of the system doesn't change with time. A, B, C, D are not functions of time.

If the system is both jointly linear and time-invariant, then it is a Linear Time Invariant System (LTI).

For time varying systems, solution is given by:

$x(t) = \emptyset(t,t_0)X_0 + \int_{t_0}^t \emptyset(t,\sigma)B(\sigma)u(\sigma)d\sigma$

$y(t) = C(t)\emptyset(t,t_0)X_0 + \int_{t_0}^t C(t)\emptyset(t,\sigma)B(\sigma)u(\sigma)d\sigma$

State transition matrix is given by the Peano-Baker series:

$\emptyset(t,t_0) = I + \int_{t_0}^t A(\sigma_1)d\sigma_1 + \int_{t_0}^t A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2\, d\sigma_1 + ..$

$\emptyset(t,t_0) = e^{\int_{t_0}^t A(\sigma)d\sigma}$

Properties of transition matrix:

If A(t)=A (const. matrix), $\emptyset(t,t_0) = e^{A(t-t_0)}$

$\frac{\partial}{\partial t}\emptyset(t,t_0) = A(t)\emptyset(t,t_0)\ \ ||\ \ \emptyset(t,t) = I$

$\emptyset(t,\tau) = \emptyset(t,\sigma)\emptyset(\sigma,\tau)\ \ ||\ \ \emptyset^{-1}(t,t_0) = \emptyset(t_0,t)$

$\emptyset(t,\tau)\emptyset^{-1}(t,\tau) = I$

The order of an ODE is the highest derivative in it, and the degree is the power of highest derivative.

$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 = a$ -> Order is 3 and degree is 1.

Stability of 2nd order systems: $\dot{X} = AX + BU$

$\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = u(t)$ A = [0 1; -a0 -a1]

The A matrix is as above, and eigen values of A are,

$\lambda = -\alpha \pm \sqrt{\Delta}$ $\alpha = \frac{a_1}{2}$ $\Delta = \alpha^2 - a_0$

If a system has a pole in right hand plane, then it is unstable. Else, it can be stable or critically stable.

For critically stable system, atleast one pole has 0 real part.

To compute eigen values use: det(lambda.I – A) = 0.

To **derive the eigen value decomposition** of a matrix, consider the matrix as $A = UVU^{-1}$. Write e^{At} by taylor expansion. Substitute A. Simplify to get $e^{At} = Ue^{\lambda}U^{-1}$ (the state transition matrix can also be obtained).

Eigen value stability test:

$\lim_{t\to inf}\emptyset(t) = 0$ The system is produces a bounded output for a bounded input. All the eigen values of A have negative real part.

Diagonalization of a matrix:

Find eigen values using $\det(A-\lambda I) = 0$. For each eigen value, find eigen vectors using $(A-\lambda I)X = 0$, where $X = [x_1, x_2, x_3]$. The eigen vectors obtained are the columns of S matrix. Then using $S^{-1}AS$ get the diagonalized matrix.

LU Decomposition:

A = LU = $\begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$

Use this equation to get values of L's and U's. Then the solution for Ax = b, for constant A and b can be found using: Ly = b and then Ux = y.

Det(A) = Det(LU) = Det(L)*Det(U).

Independent Joint Control:

Mechanical dynamic equations:

$$\left(J_g + \frac{1}{r^2}J_l\right)\frac{d^2}{dt^2}\theta_m(t) + B_m\frac{d}{dt}\theta_m(t) = \tau_m(t) + \frac{1}{r}\tau_l(t)$$

$$J_m\frac{d^2}{dt^2}\theta_m(t) + B_m\frac{d}{dt}\theta_m(t) = \tau_m(t) + \frac{1}{r}\tau_l(t)$$

m is for the motor and l is for the load. r is the gear ratio.

Internal dynamics of the motor:

$$L\frac{d}{dt}i_a(t) + Ri_a(t) + K_b\frac{d}{dt}\theta_m(t) = V(t)$$

$$\tau_m(t) = K_m i_a(t)$$

Possible choice for state:

$$\vec{X}(t) = \begin{bmatrix} i_a(t) \\ \theta_m(t) \\ \frac{d}{dt}\theta_m(t) \end{bmatrix}$$

We can rewrite these equations as:

$$\frac{d}{dt}i_a(t) = -\frac{R}{L}i_a(t) - \frac{K_b}{L}\frac{d}{dt}\theta_m(t) + \frac{1}{L}V(t)$$

$$\frac{d^2}{dt^2}\theta_m(t) = \frac{K_m}{J_m}i_a(t) - \frac{B_m}{J_m}\frac{d}{dt}\theta_m(t) + \frac{1}{rJ_m}\tau_l(t)$$

$$\vec{X}(t) = \begin{bmatrix} i_a(t) \\ \theta_m(t) \\ \frac{d}{dt}\theta_m(t) \end{bmatrix} \quad \vec{U}(t) = \begin{bmatrix} V(t) \\ \tau_l(t) \end{bmatrix} \quad \dot{\vec{X}}(t) = \begin{bmatrix} \frac{d}{dt}i_a(t) \\ \frac{d}{dt}\theta_m(t) \\ \frac{d^2}{dt^2}\theta_m(t) \end{bmatrix} = \mathbf{A}\vec{X}(t) + \mathbf{B}\vec{U}(t)$$

$$\mathbf{A} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_b}{L} \\ 0 & 0 & 1 \\ \frac{K_m}{J_m} & 0 & -\frac{B_m}{J_m} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & 0 \\ 0 & \frac{1}{rJ_m} \end{bmatrix}$$

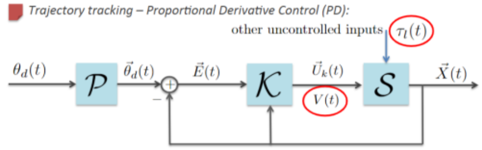
For a simplified model, we can take L/R << Jm/Bm. Reobserve the equations, get ia in terms of V and dtheta_m. You can then get the modified A & B matrices.

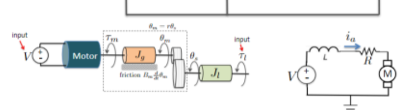
$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \frac{K_m}{J_mR} & \frac{1}{rJ_m} \end{bmatrix}$$

For the simplified model we get:

$$\vec{X}(t) = \begin{bmatrix} \theta(t) \\ \frac{d}{dt}\theta(t) \end{bmatrix}, \vec{U}(t) = \begin{bmatrix} V(t) \\ \tau_l(t) \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{B_m + \frac{K_bK_m}{R}}{J_m} \end{bmatrix}$$

Trajectory tracking – Proportional Derivative Control (PD):





The goal is to design the cntrl to reduce the tracking error theta_d – theta_m. Linear state feedback control:

$$V(t) = k_1(\theta_d(t) - \theta_m(t)) + k_2\left(\frac{d}{dt}\theta_d(t) - \frac{d}{dt}\theta_m(t)\right)$$

The state space representation of the closed loop is:

$$\dot{\vec{X}}(t) = \begin{bmatrix} 0 & 1 \\ -k_1\frac{K_m}{J_mR} & -k_2\frac{K_m}{J_mR} - \frac{B_m + \frac{K_bK_m}{R}}{J_m} \end{bmatrix}\vec{X}(t) + \begin{bmatrix} 0 \\ k_1\frac{K_m}{J_mR} \end{bmatrix}\theta_d(t)$$

You can check stability by stab. of 2nd order systems.

Analysis of static gain and error for closed-loop systems: Including the disturbance as an additional input:

$$\dot{\vec{X}}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -k_1\frac{K_m}{J_mR} & -k_2\frac{K_m}{J_mR} - \frac{B_m + \frac{K_bK_m}{R}}{J_m} \end{bmatrix}}_{\mathbf{A}_c}\vec{X}(t) + \underbrace{\begin{bmatrix} 0 & 0 \\ k_1\frac{K_m}{J_mR} & \frac{1}{J_mr} \end{bmatrix}}_{\mathbf{B}_c}\begin{bmatrix} \theta_d(t) \\ \tau_l(t) \end{bmatrix}$$

$$\lim_{t\to\infty}\dot{\vec{X}}_c(t) = 0 \implies \lim_{t\to\infty}\vec{X}_c(t) = -\mathbf{A}_c^{-1}\mathbf{B}_c\lim_{t\to\infty}\vec{U}_c(t) = -\mathbf{A}_c^{-1}\mathbf{B}_c\vec{U}_s$$

$$\mathbf{A}_c^{-1}\mathbf{B}_c = \begin{bmatrix} -1 & -\frac{R}{rk_1K_m} \\ 0 & 0 \end{bmatrix} \implies \lim_{t\to\infty}\theta(t) = \lim_{t\to\infty}\theta_d(t) + \frac{R}{rk_1K_m}\lim_{t\to\infty}\tau_l(t)$$

The circled part is tracking error.

General State feedback:

$$\dot{\vec{X}}(t) = \mathbf{A}\vec{X}(t) + \mathbf{B}_K\underbrace{\vec{U}_K(t)}_{\text{controlled input}} + \mathbf{B}_D\vec{U}_D(t)$$

$$\vec{Y}(t) = \mathbf{C}\vec{X}(t) + \mathbf{D}\underbrace{\vec{U}_K(t)}_{\text{uncontrolled input (reference, disturbance)}}$$

U=KX is the state feedback. K is mxn gain matrix. Bk is nxm

The closed loop state space representation is:

$$\dot{\vec{X}}(t) = (\mathbf{A} + \mathbf{B}_K\mathbf{K})\vec{X}(t) + \mathbf{B}_D\vec{U}_D(t)$$

$$\vec{Y}(t) = (\mathbf{C} + \mathbf{DK})\vec{X}(t)$$

Stability of closed loop is found by eig vals of A+B*K

Controllability and Stability:

The pair A,B is controllable if any of the following hold:

-The eig vals of A+B*K can be placed anywhere by K.

-Given any initial state X(0) and a desired state Xd(t*), in the absence of disturbances there is a control signal Uk(t) that takes the system's state to the desired state at any time t*.

-The Controllability Grammian is invertible (+ definite).

For time invariant systems:

You can just find the rank of nxnm controllability matrix:

$rank([B\ AB\ A^2B \ldots A^{n-1}B]) = n$

For time varying systems (when A is const. but not B):

Find $\emptyset(t) = e^{At} = L^{-1}([sI-A]^{-1})$

Find the controllability Grammian by:

$$W(t_0,t_f) = \int_{t_0}^{t_f}\emptyset(t_0,\tau)B(\tau)B^T(\tau)\emptyset^T(t_0,\tau)d\tau$$

If the system is controllable then the following control law will take the system from any initial state $\vec{X}(0)$ to $\vec{X}_d(t^*)$:

$$\vec{U}(t) = \mathbf{B}_K^T e^{-\mathbf{A}^Tt}\mathbf{W}_c(0,t^*)^{-1}(e^{-At^*}\vec{X}_d(t^*) - \vec{X}(0))$$

To prove this, put $U_k(\tau)$ in the general solution of X(t*):

$$X(t^*) = e^{At^*}X(0) + \int_0^{t^*}e^{A(t^*-\tau)}BU_k(\tau)d\tau$$

Simplify this, and get $X(t^*) = X_d(t^*)$

Definition of controllability:

The linear state equation is called controllable on finite interval [t0,tf] if given any initial state x(t0)=x0 there exists a continuous input signal u(t) such that the corresponding solution satisfies x(tf) = 0.

Caley Hamilton Theorem:

$A^n = a_0I + a_1A + a_2A^2 + \cdots + a_{n-1}A^{n-1}$

Using this we get the following property of matrix exp:

$e^{At} = \sum_{l=0}^{n-1}\alpha_l(t)A^l$ where alpha's are scalar anal fns.

Prove every SISO nth order LTI system is controllable:

-Take general nth order linear ODE in y and u.

-Write A and B matrices of the system.

-Find controllability matrix [B AB A2B ...].

-Show that all the columns are linearly independent.

-Hence, the system is controllable.

You can also be asked to show that independent joint model is controllable. To show this find the controllability matrix using A and B in first image and show it is full rank.

For two mass spring damper assemblage: take the x1, x1_dot, x2, x2_dot as states, use Newton's laws of motion to get equations of motion, get state space representation and then check for controllability using ctrb matrix.

PBH (Popov-Belevitch Houtus) test:

$Rank[(\lambda I - A)\quad B] = n$ for all λ, -> Controllable

$Rank[(\lambda I - A)\quad B] = n$ for all +ve real{λ}, -> Stabilizable

Standard form for uncontrollable systems:

If the pair A,B is not controllable, we can separate the controllable and uncontrollable parts of the system using a similarity transformation.

$\hat{A} = S^{-1}AS$ $\hat{B} = S^{-1}B$ where S is given by:

$S = [v_1\ v_2 \ldots v_{nr}\ \ S_{n-nr}]$

v1 to v_nr columns are nr LI columns from the controllability matrix of A,B. The remaining columns are n-nr LI columns different from v's.

To show that A,B is uncontrollable iff there's a 1xn vector v != 0, such that v[λI-A B]=0.

-Take a non-zero vector v s.t. vA=λv, vB = 0.

-Multiply v with ctrb matrix.
-Substitute vA with λv, and vB=0.
-You get a zero matrix, proves that system is not ctrb.

Lyapunov Stability for LTI systems in State-space form:
A function V(x) is a Lyapunov fn if:
-V(x)>0 for x!=0. V(x)=0 for x=0.
-In absence of external inputs,
$\frac{d}{dt}V(X(t)) < 0 \; when \; X(t) \neq 0$, means the gradient is negative & the fn will converge to a global minimum.
LTI system in state space form is stable iff it has a Lp fn.

Lyapunov Equation:
An LTI system specified in State-Space form is stable iff for any symmetric positive definite matrix Q there exists a symmetric positive definite P s.t. Lyapunov eqn holds:
$A^TP + AP = -Q$
For this case, $V(X) = X^TPX$ is a **Lyapunov Function**.
If the system is stable, $P = \int_o^{\infty} e^{A^Tt}Qe^{At}dt$
And $X^T(0)PX(0) = \int_0^{\infty} X^TQXdt$ is energy of system at t=0

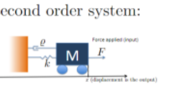
The stability of a system can be cast as the LMI:
$-A^TP - PA > 0 \; and \; P = P^T > 0, >$ is +ve def sense.

Convex set:
Set C is convex if for any 2 points in set x1, x2 we have:
$\theta x_1 + (1-\theta)x_2 \in C$ or the line segment x1, x2 is in C.

Convex function:
$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)y$
Geometrically this means, line segment is above f.
The pair A,B is stabilizable can be cast as the following optimization problem: $-(A+BK)^TP - P(A+BK) > 0$

LQR: Linear Quadratic Regulator
We loop for optimal K that minimizes the cost function:
$J(K, X(0)) = \int_0^{\infty} X^T(t)QX(t) + U^TRU(t)dt$ Q & R are +d
The optimal solution: $K = -R^{-1}B^TP$, where P is soln of

Stationary Ricatti equation:
$A^TP + AP - PBR^{-1}B^TP = -Q$
The optimal cost: $J(K, X(0) = X^T(0)PX(0)$
Also P is soln of the following Lyapunov Equation:
$(A+BK)^TP + P(A+BK) = -(Q + K^TRK)$
To write Ricatti eqn from Lyapunov equation use:
$$P^TB(R^{-1})^TRK + P^TB(R^{-1})^TRR^{-1}B^TP = 0$$
For optimal reference tracking using LQR:
$\tilde{X}(t) = X(t) - X_d \qquad \widetilde{U_k}(t) = U_k(t) - U_{\infty}$
And we will get optimal control as: $\widetilde{U_k}(t) = k\tilde{U}(t)$
Laplace transform: $L\{f(t)\} = \int_0^{\infty} e^{-st}f(t)dt$
Inverse laplace transform: $f(t) = \frac{1}{2\pi i}\int_{s'-i\infty}^{s'+i\infty} e^{st}F(s)dt$

LQR to get 0 state error for constant disturbance:
Consider equilibrium point: $AX_d + B_kU_{\infty} = 0$
Start with: $\dot{\tilde{X}} = A\tilde{X}(t) + B_k\widetilde{U_k}(t) + B_DU_D(t)$
Augment state to obtain the integral term in state:
$$\dot{\bar{\vec{X}}}(t) = \bar{\mathbf{A}}\bar{\vec{X}}(t) + \bar{\mathbf{B}}_K\tilde{\vec{U}}_K(t) + \bar{\mathbf{B}}_D\vec{U}_D(t)$$
$$\bar{\vec{X}}(t) = \begin{bmatrix} \tilde{\vec{X}}(t) \\ \vec{X}_I(t) \end{bmatrix} \quad \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{I} & 0 \end{bmatrix} \quad \bar{\mathbf{B}}_K = \begin{bmatrix} \mathbf{B}_K \\ 0 \end{bmatrix} \quad \bar{\mathbf{B}}_D = \begin{bmatrix} \mathbf{B}_D \\ 0 \end{bmatrix}$$
X_I is integral of X_tilda.
This works because at steady state we get x(t) as x_d:
$\dot{\tilde{X}}(t) = 0 \qquad \tilde{X}(t) = 0 \qquad X(t) = X_d$
I can apply LQR to design $\bar{K}$ that minimizes the cost:
$$\int_0^{\infty} \bar{\vec{X}}(t)^T\bar{\mathbf{Q}}\bar{\vec{X}}(t) + \tilde{\vec{U}}_K(t)^T\mathbf{R}\tilde{\vec{U}}_K(t)dt$$
For the system:
$$\dot{\bar{\vec{X}}}(t) = \bar{\mathbf{A}}\bar{\vec{X}}(t) + \bar{\mathbf{B}}_K\tilde{\vec{U}}_K(t) + \bar{\mathbf{B}}_D\vec{U}_D(t)$$
The optimal closed-loop system describing the tracking err
$$\dot{\bar{\vec{X}}}(t) = \underbrace{(\bar{\mathbf{A}} + \bar{\mathbf{B}}_K\bar{\mathbf{K}})}_{\text{STABLE}}\bar{\vec{X}}(t) + \bar{\mathbf{B}}_D\vec{U}_D(t)$$
We are using the following optimal control:
$$\tilde{\vec{U}}_K(t) = \bar{K}\bar{\vec{X}}(t) \;\Rightarrow\; \vec{U}_K(t) = \bar{K}\bar{\vec{X}}(t) + U_{\infty}$$
If U_D is a constant vector will have X_1(t) bounded.
Consider the following case study of a second order system:
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{B}_K = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{B}_D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$


$$\vec{X}_d = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \qquad \vec{U}_{\infty} = \alpha$$
$$\bar{\vec{X}}(t) = \begin{bmatrix} \tilde{X}(t) \\ \dot{\tilde{X}}(t) \\ X_I(t) \end{bmatrix} = \begin{bmatrix} \tilde{\vec{X}}(t) \\ X_I(t) \end{bmatrix} \bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \bar{\mathbf{B}}_K = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Also augment B_D with zero.
Luenberger Observer'63 in state-space representation is:
$$\dot{\hat{\vec{X}}}(t) = \mathbf{A}\hat{\vec{X}}(t) + \mathbf{B}_K\vec{U}_K(t) + \mathbf{L}(\vec{Y}(t) - \mathbf{C}\hat{\vec{X}}(t)), \quad \vec{X}(0) = 0$$
The estimation error state space representation is:
$$\dot{\vec{X}}_e(t) = \mathbf{A}\vec{X}_e(t) - \mathbf{L}(\vec{Y}(t) - \mathbf{C}\hat{\vec{X}}(t)) + \mathbf{B}_D\vec{U}_D(t)$$
And estimation error is: $X_e(t) = X(t) - \hat{X}(t)$
$$\dot{\vec{X}}_e(t) = (\mathbf{A} - \mathbf{LC})\vec{X}_e(t) + \mathbf{B}_D\vec{U}_D(t)$$
When can we select L, s.t. estimation error is stable?
Key observation: the matrix A-LC is stable if and only if $(A - LC)^T = A^T - C^TL^T$ is stable.
If the pair (A^T, C^T) is stabilizable, then (A,C) is detectable.
If the pair (A^T, C^T) is controllable, then (A,C) is observable

Definition of Observability (for the LTV system):
The linear state equation is called observable on [t0,tf] if any initial state x(t0)=x0 is uniquely determined by the corresponding response y(t), t belongs to [t0,tf].
Theorem: The linear state equation is observable on [t0,tf], iff the nxn observability grammian is invertible, i.e. full rank.
$M(t_0, t_f) = \int_{t_0}^{t_f} \emptyset^T(t, t_0)C^T(t)C(t)\emptyset(t,t_0)dt$
For the time invariant case: the state is observable if & only if the npxn observability matrix has rank n:
$$rank\left(\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}\right) = rank([C^T \quad A^TC^T \quad (A^T)^{\wedge}2\, C^T]) = n$$
Suppose we design K and L such that the full state feedback system and the Luenberger observer are both stable. Will the corresponding output feedback be stable? -> Yes. This is known as **separation principle**.
The following is the ss representation of the closed-loop:
$$\begin{bmatrix} \dot{\vec{X}}(t) \\ \dot{\hat{\vec{X}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_K\mathbf{K} \\ \mathbf{LC} & \mathbf{A} - \mathbf{LC} + \mathbf{B}_K\mathbf{K} \end{bmatrix} \begin{bmatrix} \vec{X}(t) \\ \hat{\vec{X}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_D \\ 0 \end{bmatrix} \vec{U}_D(t)$$
$$\begin{bmatrix} \dot{\vec{X}}(t) \\ \dot{\vec{X}}_e(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} + \mathbf{B}_K\mathbf{K} & -\mathbf{B}_K\mathbf{K} \\ 0 & \mathbf{A} - \mathbf{LC} \end{bmatrix}}_{\mathbf{A}_C} \underbrace{\begin{bmatrix} \vec{X}(t) \\ \vec{X}_e(t) \end{bmatrix}}_{\text{Closed-loop state}} + \begin{bmatrix} \mathbf{B}_D \\ \mathbf{B}_D \end{bmatrix} \vec{U}_D$$
A_C is stable if and only if both (A + BK) and (A – LC) are stable

Kalman-Bucy Filter (1961):
Continuous time version of the Kalman Filter.
$$\dot{\vec{X}}(t) = \mathbf{A}\vec{X}(t) + \mathbf{B}_K\vec{U}_K(t) + \mathbf{B}_D\vec{U}_D(t)$$
$$\vec{Y}(t) = \mathbf{C}\vec{X}(t) + \vec{V}(t)$$
Where UD(t) and V(t) are independent mean white Gaussian processes with covariance $\sum D$ & $\sum V$ respectively
The cost we want to minimize is $\lim_{t\to\infty} E[\vec{X}_e^T(t)\vec{X}_e(t)]$
Where $\vec{X}_e(t) = \vec{X}(t) - \hat{\vec{X}}(t)$
A real valued continuous time process X(t) is a Gaussian process, if each finite dimensional vector (X(t1),X(t2)… X(tn))^T has the multivariate normal distribution N(u(t),SIGMA(t)) for some mean vector u and some covariance matrix sigma which may depend on t=(t1..tn)^T
White Noise: A process is said to be white noise in the strongest sense if x(t) for any t is statistically independent of its entire history before t.
White noise is rndm signal having const pwr spectral dnsity
In our context we get:
$Cov(U_D(l), U_D(j)) = E[U_D(l)\,U_D(j)^T] = \Sigma_D\,\delta(l-j)$
$Cov(V(l), V(j)) = E[V(l)V(j)^T] = \Sigma_V\,\delta(l-j)$
$Cov(U(l), V(j)) = 0$
The optimal solution is $\mathbf{L}^* = \mathbf{PC}^T\Sigma_V^{-1}$, where $\mathbf{P}$ is the solution of:
$$\mathbf{AP} + \mathbf{PA}^T + \mathbf{B}_D\Sigma_D\mathbf{B}_D^T - \mathbf{PC}^T\Sigma_V^{-1}\mathbf{CP} = 0$$

LQG (Linear Quadratic Gaussian Method):
Again consider the state space equations in Kalman-Bucy
We consider the case when U_D and V are independent zero mean white gaussina processes with SIGMA_D and SIGMA_V covariances. We want to minimize the cost:
$$\lim_{t\to\infty} E[\vec{X}(t)^TQ\vec{X}(t) + \vec{U}(t)^TR\vec{U}(t)]$$
The structure of the optimal solution is given by the standard output feedback configuration with the Luenberger Observer and the optimal K and L are computed separately using the LQR and Kalman-Bucy methods — this is called the separation principle.

Methods of solving ODEs:

-Variable Separable Equation:
$\frac{dy}{dx} = f(x)g(y)$ write eqn in this form and then separately integrate f(x) and g(y) to get the solution of ODE.

-Exact Equations:
First write eqn in for Adx + Bdy=0. Then check if it is exact, by $\frac{\delta A}{\delta y} = \frac{\delta B}{\delta x}$. Then, find $U(x,y) = \int Adx + F(y) = c_1$.
Differentiate U(x,y) w.r.t. y and equate it to B(x,y). Solve for F(y), and put it in U(x,y), to get the final solution.

-Bernoulli's Equation:
$\frac{dy}{dx} + P(x)y = Q(x)y^n$, n != 0. This is a non-linear equation that can be made linear by substituting $v = y^{1-n}$ and $\frac{dv}{dt} = (1-n)y^{-n}\frac{dy}{dx}$ divide the original eqn by y^n. And solve for v. Then resubstitute v in terms of y.

-Homogeneous Differential Equation:
$$\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)} = F\left(\frac{y}{x}\right)$$
Put y = vx, dy/dx = v + x.dy/dx.
How to make inexact eqn, exact: Multiply original equation by $u = e^{\int \frac{1}{B}(\frac{\delta A}{\delta y} - \frac{\delta B}{\delta x})}$

-Method of Integrating Factors (IF):
$\frac{dy}{dx} + P(x)y = Q(x)$ form of eqn. Integrating factor is defined as: $\alpha = e^{\int P(x)dx}$. Multiply original eqn by IF, the LHS can is of the form P(x)y' + P'(x)y, which can be combined to get d(P(x).y)/dx = Q(x), solve to get ODE soln.

-Method of undetermined coefficients:
Used to solve higher order ODEs. Involves finding the complementary solution y_c and particular soln y_p.
Y_c is found by LHS by assuming $f = Ae^{\lambda t}$ and RHS = 0.
If initial conditions are given find singular soln of y_c.
Next, find y_p by taking y_p as some multiple of e and sin and cos. Eg. For RHS = $e^{-t}\cos(3t)$, we can take $y_p = ae^{-t}(b.\cos(3t) + c.\sin(3t))$ Next put this in original diff equation and find yp. Final solution is yc+yp.

-Method of Variation of Parameters:
First, get yc similar to method of undeter. Coeff. Then assume $y_p = K_1(x)a_1 + K2(x)a_2$ by observing yc on yc. Then impose conditions, most common is yp=0. Differentiate yp once and twice, put in original equation and get values of K1(x) and K2(x). Combine yc & yp for ans.

-Laplace Transform:
Take Laplace transform of equation, then simplify it. Then take inverse laplace transform to get final solution.

Integration by parts:
$\int u.vdx = u\int vdx - \int u'(\int vdx)dx$

$\int x^n dx = \frac{1}{n+1}x^{n+1}$	$\int \sinh ax\, dx = \frac{1}{a}\cosh ax$
$\int e^{ax} dx = \frac{1}{a}e^{ax}$	$\int \cosh ax\, dx = \frac{1}{a}\sinh ax$
$\int a^x dx = \frac{a^x}{\ln a}$	$\int \tanh ax\, dx = -\frac{1}{a}\ln\lvert\cosh ax\rvert$
$\int lnx\, dx = x\, lnx - x$	$\int \tanh ax\, dx = -\frac{1}{a}\ln\lvert\cosh ax\rvert$
$\int \sin ax\, dx = -\frac{1}{a}\cos ax$	$\int \frac{1}{\sqrt{a^2 - x^2}}dx = arcsin\frac{x}{a}$
$\int \sin ax\, dx = -\frac{1}{a}\cos ax$	$\int \frac{1}{\sqrt{x^2 - a^2}}dx = arccosh\frac{x}{a}$
$\int \cos ax\, dx = \frac{1}{a}\sin ax$	$\int \frac{1}{\sqrt{a^2 + x^2}}dx = arcsinh\frac{x}{a}$
$\int \tan ax\, dx = -\frac{1}{a}\ln\lvert cosax\rvert$	$\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}arctan\frac{x}{a}$

Find transfer function of a system: $H(s) = Y(s)/U(s)$
$$\breve{\mathbf{A}} = \begin{bmatrix} \breve{\mathbf{A}}_{r\bar{o}} & \breve{\mathbf{A}}_{12} & \breve{\mathbf{A}}_{13} & \breve{\mathbf{A}}_{14} \\ 0 & \breve{\mathbf{A}}_{ro} & 0 & \breve{\mathbf{A}}_{24} \\ 0 & 0 & \breve{\mathbf{A}}_{\bar{r}\bar{o}} & \breve{\mathbf{A}}_{34} \\ 0 & 0 & 0 & \breve{\mathbf{A}}_{\bar{r}o} \end{bmatrix} \quad \breve{\mathbf{B}} = \begin{bmatrix} \breve{\mathbf{B}}_{r\bar{o}} \\ \breve{\mathbf{B}}_{ro} \\ 0 \\ 0 \end{bmatrix}$$
$$\breve{\mathbf{C}} = [0 \quad \breve{\mathbf{C}}_{ro} \quad 0 \quad \breve{\mathbf{C}}_{\bar{r}o}]$$