Laplace transform table:

|  |  |  |  |
| --- | --- | --- | --- |
| 1 | 1/s |  |  |
| t |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  | F(t).G(t) | F(s)\*G(s) |

Convolution:

Fundamental theorem of calculus and product rule:

These hold for matrix functions

**Leibniz rule holds for matrix functions:**

General solution of is given by:

If asked to prove that this is solution, use Leibniz rule.

To get the state-space representation of a given differential eqn, first assume states x1, x2,…, xn. Then take derivatives of all these states to get X\_dot. Then for xn\_dot find equation in terms of all other states. Make matrix from the state equations obtained.

**Properties of LTI system:**

A system is jointly linear in the initial condition response (u=0) and the force response (x(t0)=0), if the following conditions hold:

A system is jointly time-invariant if a delay of τ in th state or input produces corresponding time delay in the output. Essentially, the behaviour of the system doesn’t change with time. A, B, C, D are not functions of time.

If the system is both jointly linear and time-invariant, then it is a Linear Time Invariant System (LTI).

**For time varying systems**, solution is given by:

**State transition matrix** is given by the Peano-Baker series:

+..

**Properties of transition matrix:**

If A(t)=A (const. matrix),

||

||

The order of an ODE is the highest derivative in it, and the degree is the power of highest derivative.

-> Order is 3 and degree is 1.

**Stability of 2nd order systems:**

A = [0 1; -a0 -a1]

The A matrix is as above, and eigen values of A are,

If a system has a pole in right hand plane, then it is unstable. Else, it can be stable or critically stable.

For critically stable system, atleast one pole has 0 real part.

To compute eigen values use: det(lambda.I – A) = 0.

To **derive the eigen value decomposition** of a matrix, consider the matrix as . Write by taylor expansion. Substitute A. Simplify to get (the state transition matrix can also be obtained).

**Eigen value stability test:**

The system is produces a bounded output for a bounded input. All the eigen values of A have negative real part.

**Diagonalization of a matrix:**

Find eigen values using . For each eigen value, find eigen vectors using , where   
 . The eigen vectors obtained are the columns of S matrix. Then using get the diagonalized matrix.

**LU Decomposition:**

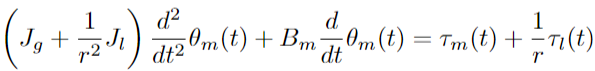
A = LU =

Use this equation to get values of L’s and U’s. Then the solution for Ax = b, for constant A and b can be found using: Ly = b and then Ux = y.

Det(A) = Det(LU) = Det(L)\*Det(U).

**Independent Joint Control:**

Mechanical dynamic equations:



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m is for the motor and l is for the load. r is the gear ratio.

Internal dynamics of the motor:





Possible choice for state: Text, whiteboard

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We can rewrite these equations as:





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For a simplified model, we can take L/R << Jm/Bm. Reobserve the equations, get ia in terms of V and dtheta\_m. You can then get the modified A & B matrices.

For the simplified model we get: Text

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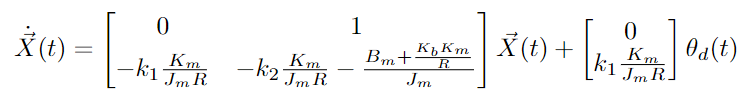
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The goal is to design the cntrl to reduce the tracking error theta\_d – theta\_m. Linear state feedback control:



The state space representation of the closed loop is:



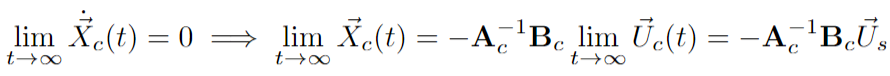
You can check stability by stab. of 2nd order systems.

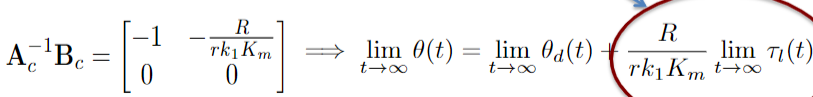
Analysis of static gain and error for closed-loop systems:

Including the disturbance as an additional input:

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The circled part is tracking error.

**General State feedback:**

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U=KX is the state feedback. K is mxn gain matrix. Bk is nxm

The closed loop state space representation is:

Stability of closed loop is found by eig vals of A+B\*K

**Controllability and Stability:**

The pair A,B is controllable if any of the following hold:

-The eig vals of A+B\*K can be placed anywhere by K.

-Given any initial state X(0) and a desired state Xd(t\*), in the absence of disturbances there is a control signal Uk(t) that takes the system’s state to the desired state at any time t\*.

-The Controllability Grammian is invertible (+ definite).

**For time invariant systems:**

You can just find the rank of nxnm controllability matrix:

**For time varying systems (when A is const. but not B):**

Find

Find the controllability Grammian by:

Diagram

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To prove this, put in the general solution of X(t\*):

Simplify this, and get

**Definition of controllability:**

The linear state equation is called controllable on finite interval [t0,tf] if given any initial state x(t0)=x0 there exists a continuous input signal u(t) such that the corresponding solution satisfies x(tf) = 0.

**Caley Hamilton Theorem:**

Using this we get the following property of matrix exp:

where alpha’s are scalar anal fns.

**Prove every SISO nth order LTI system is controllable:**

-Take general nth order linear ODE in y and u.

-Write A and B matrices of the system.

-Find controllability matrix [B AB A2B …].

-Show that all the columns are linearly independent.

-Hence, the system is controllable.

**You can also be asked to show that independent joint model is controllable.** To show this find the controllability matrix using A and B in first image and show it is full rank.

**For two mass spring damper assemblage:** take the x1, x1\_dot, x2, x2\_dot as states, use Newton’s laws of motion to get equations of motion, get state space representation and then check for controllability using ctrb matrix.

**PBH (Popov-Belevitch Houtus) test:**

for all λ, -> Controllable

for all +ve real{λ}, -> Stabilizable

**Standard form for uncontrollable systems:**

If the pair A,B is not controllable, we can separate the controllable and uncontrollable parts of the system using a similarity transformation.

where S is given by:

v1 to v\_nr columns are nr LI columns from the controllability matrix of A,B. The remaining columns are n-nr LI columns different from v’s.

**To show that A,B is uncontrollable iff there’s a 1xn vector v != 0, such that v[λI-A B]=0.**

-Take a non-zero vector v s.t. vA=λv, vB = 0.

-Multiply v with ctrb matrix.

-Substitute vA with λv, and vB=0.

-You get a zero matrix, proves that system is not ctrb.

**Lyapunov Stability for LTI systems in State-space form:**

A function V(x) is a Lyapunov fn if:

-V(x)>0 for x!=0. V(x)=0 for x=0.

-In absence of external inputs,

, means the gradient is negative & the fn will converge to a global minimum.

LTI system in state space form is stable iff it has a Lp fn.

**Lyapunov Equation:**

An LTI system specified in State-Space form is stable iff for any symmetric positive definite matrix Q there exists a symmetric positive definite P s.t. Lyapunov eqn holds:

For this case, is a **Lyapunov Function**.

If the system is stable,

And is energy of system at t=0

**The stability of a system can be cast as the LMI:**

, > is +ve def sense.

**Convex set:**

Set C is convex if for any 2 points in set x1, x2 we have:

or the line segment x1, x2 is in C.

**Convex function:**

Geometrically this means, line segment is above f.

The pair A,B is stabilizable can be cast as the following optimization problem:

**LQR: Linear Quadratic Regulator**

We loop for optimal K that minimizes the cost function:

Q & R are +d

The optimal solution: , where P is soln of

**Stationary Ricatti equation:**

The optimal cost:

Also P is soln of the following Lyapunov Equation:

To write Ricatti eqn from Lyapunov equation use:

For optimal reference tracking using LQR:

And we will get optimal control as:

Laplace transform:

Inverse laplace transform:

**LQR to get 0 state error for constant disturbance:**

Consider equilibrium point:

Start with:

Augment state to obtain the integral term in state:



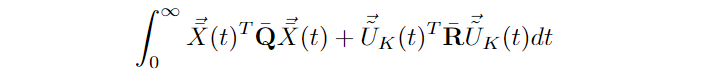
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X\_I is integral of X\_tilda.

This works because at steady state we get x(t) as x\_d:

I can apply LQR to design that minimizes the cost:



For the system:



The optimal closed-loop system describing the tracking err

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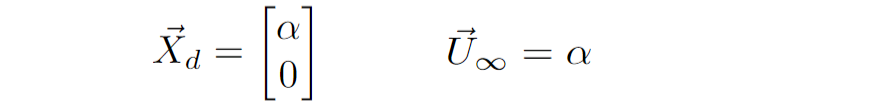
We are using the following optimal control:



If U\_D is a constant vector will have X\_1(t) bounded.

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Also augment B\_D with zero.

Luenberger Observer’63 in state-space representation is:



The estimation error state space representation is:



And estimation error is:



When can we select L, s.t. estimation error is stable?

Key observation: the matrix A-LC is stable if and only if

is stable.

If the pair is stabilizable, then (A,C) is detectable.

If the pair is controllable, then (A,C) is observable

**Definition of Observability (for the LTV system):**

The linear state equation is called observable on [t0,tf] if any initial state x(t0)=x0 is uniquely determined by the corresponding response y(t), t belongs to [t0,tf].

**Theorem:** The linear state equation is observable on [t0,tf], iff the nxn observability grammian is invertible, i.e. full rank.

**For the time invariant case:** the state is observable if & only if the npxn observability matrix has rank n:

Suppose we design K and L such that the full state feedback

system and the Luenberger observer are both stable. Will the corresponding output feedback be stable? -> Yes. This is known as **separation principle**.

The following is the ss representation of the closed-loop:

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is stable if and only if both (A + BK) and (A – LC) are stable

**Kalman-Bucy Filter (1961):**

Continuous time version of the Kalman Filter.





Where UD(t) and V(t) are independent mean white Gaussian processes with covariance respectively





A real valued continuous time process X(t) is a Gaussian process, if each finite dimensional vector (X(t1),X(t2)… X(tn))T has the multivariate normal distribution N(u(t),SIGMA(t)) for some mean vector u and some covariance matrix sigma which may depend on t=(t1..tn)T

**White Noise:** A process is said to be white noise in the strongest sense if x(t) for any t is statistically independent of its entire history before t.

White noise is rndm signal having const pwr spectral dnsity

In our context we get:





**LQG (Linear Quadratic Gaussian Method):**

Again consider the state space equations in Kalman-Bucy

We consider the case when U\_D and V are independent zero mean white gaussina processes with SIGMA\_D and SIGMA\_V covariances. We want to minimize the cost:



The structure of the optimal solution is given by the standard output feedback configuration with the Luenberger Observer and the optimal K and L are computed separately using the LQR and Kalman-Bucy methods — this is called the separation principle.

**Methods of solving ODEs:**

**-Variable Separable Equation:**

write eqn in this form and then separately integrate f(x) and g(y) to get the solution of ODE.

**-Exact Equations:**

First write eqn in for Adx + Bdy=0. Then check if it is exact, by . Then, find .

Differentiate U(x,y) w.r.t. y and equate it to B(x,y). Solve for F(y), and put it in U(x,y), to get the final solution.

**-Bernoulli’s Equation:**

, n != 0. This is a non-linear equation that can be made linear by substituting and

divide the original eqn by y^n. And solve for v. Then resubstitute v in terms of y.

**-Homogeneous Differential Equation:**

Put y = vx, dy/dx = v + x.dy/dx.

How to make inexact eqn, exact: Multiply original equation by

**-Method of Integrating Factors (IF):**

form of eqn. Integrating factor is defined as: . Multiply original eqn by IF, the LHS can is of the form P(x)y’ + P’(x)y, which can be combined to get d(P(x).y)/dx = Q(x), solve to get ODE soln.

**-Method of undetermined coefficients:**

Used to solve higher order ODEs. Involves finding the complementary solution y\_c and particular soln y\_p.

Y\_c is found by LHS by assuming and RHS = 0.

If initial conditions are given find singular soln of y\_c.

Next, find y\_p by taking y\_p as some multiple of e and sin and cos. Eg. For RHS = , we can take

Next put this in original diff equation and find yp. Final solution is yc+yp.

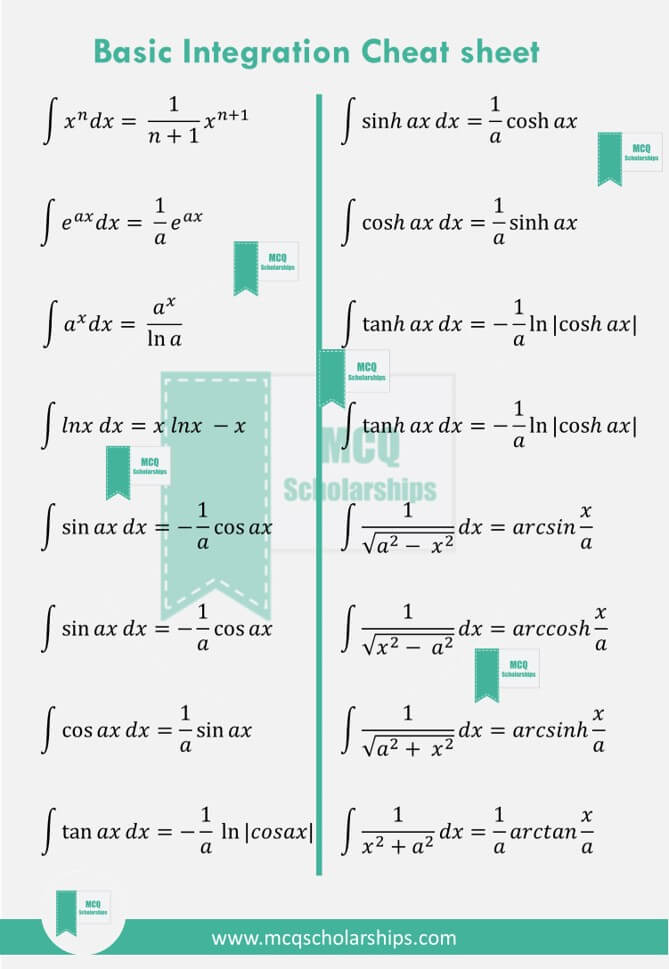
**-Method of Variation of Parameters:**

First, get yc similar to method of undeter. Coeff. Then assume by observing yc on yc. Then impose conditions, most common is yp=0. Differentiate yp once and twice, put in original equation and get values of K1(x) and K2(x). Combine yc & yp for ans.

**-Laplace Transform:**

Take Laplace transform of equation, then simplify it. Then take inverse laplace transform to get final solution.

**Integration by parts:**



Find transfer function of a system:

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