E. Hairer G. Wanner

Solving Ordinary Differential Equations II

Stiff and Differential-Algebraic Problems

Second Revised Edition
With 137 Figures

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VII.7 Computation of Multibody Mechanisms

> Dynamics of multibody systems is of great importance in the fields of robotics, biomechanics, spacecraft control, road and rail vehicle design, and dynamics of machinery.
>
> (W. Schiehlen 1990)

After having seen several different approaches for the numerical solution of constrained mechanical systems, we are interested in their efficiency when applied to a concrete situation. We consider two particular multibody mechanisms with constraints, one nonstiff and one stiff. General references for the computation of mechanical systems are Haug (1989) and Roberson & Schwertassek (1988).

Description of the Model

We first consider "Andrews' squeezer mechanism", which has become prominent through the work of Giles (1978) and Manning (1981), who promoted it as a test example for numerical codes; see also Ormrod & Andrews (1986). It consists of 7 rigid bodies connected by joints without friction in plane motion. It is represented in Fig. 7.1, which we have copied (with permission) from the book of Schiehlen (1990). The numerical constants, also taken from Schiehlen (1990), are displayed in Tables 7.1 and 7.2. The arrows in the right picture of Fig. 7.1 indicate the positions of the centres of gravity $C_1, \ldots, C_7$. In Table 7.1 the spring coefficient of the spring connecting the point D with C is denoted by c_0 and the unstretched length is ℓ_0. We suppose that the mechanism is driven by a motor, located at O, whose constant drive torque is given by $mom = 0.033$. The coordinate origin is the point O in Fig. 7.1 and the coordinates of the other fixed points A, B and C are given by

$$\begin{pmatrix} xa \\ ya \end{pmatrix} = \begin{pmatrix} -0.06934 \\ -0.00227 \end{pmatrix}, \quad \begin{pmatrix} xb \\ yb \end{pmatrix} = \begin{pmatrix} -0.03635 \\ 0.03273 \end{pmatrix}, \quad \begin{pmatrix} xc \\ yc \end{pmatrix} = \begin{pmatrix} 0.014 \\ 0.072 \end{pmatrix}. \quad (7.1)$$

Table 7.1. Geometrical parameters

$d = 0.028$	$da = 0.0115$	$e = 0.02$
$ea = 0.01421$	$zf = 0.02$	$fa = 0.01421$
$rr = 0.007$	$ra = 0.00092$	$ss = 0.035$
$sa = 0.01874$	$sb = 0.01043$	$sc = 0.018$
$sd = 0.02$	$zt = 0.04$	$ta = 0.02308$
$tb = 0.00916$	$u = 0.04$	$ua = 0.01228$
$ub = 0.00449$	$c_0 = 4530$	$\ell_0 = 0.07785$

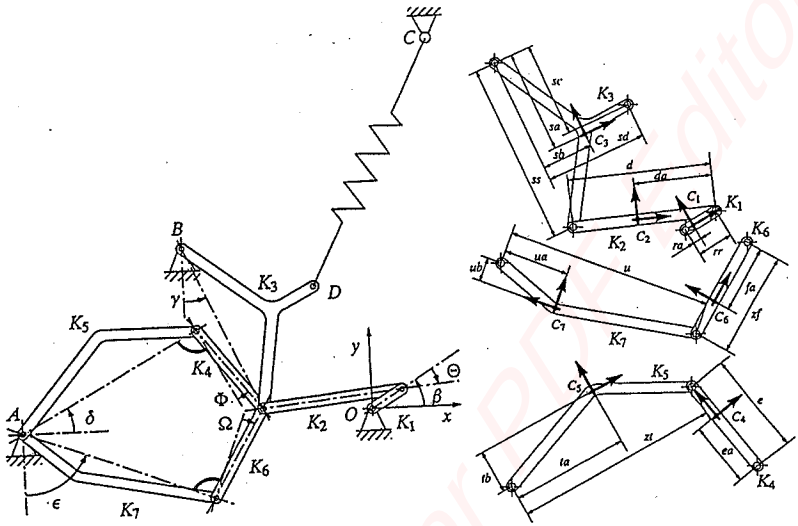

Fig. 7.1. Seven body mechanism (Schiehlen 1990, with permission)

Table 7.2. Parameters of the 7 bodies

No.	masses m_1 to m_7	inertias I_1 to I_7
1	0.04325	$2.194 \cdot 10^{-6}$
2	0.00365	$4.410 \cdot 10^{-7}$
3	0.02373	$5.255 \cdot 10^{-6}$
4	0.00706	$5.667 \cdot 10^{-7}$
5	0.07050	$1.169 \cdot 10^{-5}$
6	0.00706	$5.667 \cdot 10^{-7}$
7	0.05498	$1.912 \cdot 10^{-5}$

In order to derive the equations of motion we use the angles (see Fig. 7.1)

$$q_1 = \beta, \quad q_2 = \Theta, \quad q_3 = \gamma, \quad q_4 = \Phi, \quad q_5 = \delta, \quad q_6 = \Omega, \quad q_7 = \epsilon, \qquad (7.2)$$

as position coordinates for the mechanical system. If (x_j, y_j) are the cartesian coordinates of the centre of gravity C_j $(j = 1, \ldots, 7)$, the *kinetic energy* of the multibody system is

$$T = \sum_{j=1}^{7} m_j \frac{\dot{x}_j^2 + \dot{y}_j^2}{2} + \sum_{j=1}^{7} I_j \frac{\dot{\omega}_j^2}{2} \qquad (7.3)$$

where ω_j is the total angle of rotation of the jth body and m_j, I_j are constants given in Table 7.2. The values of x_j, y_j, $\dot{x}_j^2 + \dot{y}_j^2$ and $\dot{\omega}_j$ can be obtained in terms

of (7.2) by simple geometry (see Fig. 7.1):

$$
\begin{aligned}
C_1: \quad & x_1 = ra \cdot \cos\beta \\
& y_1 = ra \cdot \sin\beta \\
& \dot{x}_1^2 + \dot{y}_1^2 = ra^2 \cdot \dot{\beta}^2 \\
& \dot{\omega}_1 = \dot{\beta} \\
C_2: \quad & x_2 = rr \cdot \cos\beta - da \cdot \cos(\beta + \Theta) \\
& y_2 = rr \cdot \sin\beta - da \cdot \sin(\beta + \Theta) \\
& \dot{x}_2^2 + \dot{y}_2^2 = \left(rr^2 - 2 \cdot da \cdot rr \cdot \cos\Theta + da^2\right) \cdot \dot{\beta}^2 \\
& \qquad + 2 \cdot \left(-rr \cdot da \cdot \cos\Theta + da^2\right) \cdot \dot{\beta} \cdot \dot{\Theta} + da^2 \cdot \dot{\Theta}^2 \\
& \dot{\omega}_2 = \dot{\beta} + \dot{\Theta} \\
C_3: \quad & x_3 = xb + sa \cdot \sin\gamma + sb \cdot \cos\gamma \\
& y_3 = yb - sa \cdot \cos\gamma + sb \cdot \sin\gamma \\
& \dot{x}_3^2 + \dot{y}_3^2 = \left(sa^2 + sb^2\right) \cdot \dot{\gamma}^2 \\
& \dot{\omega}_3 = \dot{\gamma} \\
C_4: \quad & x_4 = xa + zt \cdot \cos\delta + (e - ea) \cdot \sin(\Phi + \delta) \\
& y_4 = ya + zt \cdot \sin\delta - (e - ea) \cdot \cos(\Phi + \delta) \\
& \dot{x}_4^2 + \dot{y}_4^2 = (e - ea)^2 \cdot \dot{\Phi}^2 + 2 \cdot \left((e - ea)^2 + zt \cdot (e - ea) \cdot \sin\Phi\right) \cdot \dot{\Phi} \cdot \dot{\delta} \\
& \qquad + \left(zt^2 + 2 \cdot zt \cdot (e - ea) \cdot \sin\Phi + (e - ea)^2\right) \cdot \dot{\delta}^2 \\
& \dot{\omega}_4 = \dot{\Phi} + \dot{\delta} \\
C_5: \quad & x_5 = xa + ta \cdot \cos\delta - tb \cdot \sin\delta \\
& y_5 = ya + ta \cdot \sin\delta + tb \cdot \cos\delta \\
& \dot{x}_5^2 + \dot{y}_5^2 = \left(ta^2 + tb^2\right) \cdot \dot{\delta}^2 \\
& \dot{\omega}_5 = \dot{\delta} \\
C_6: \quad & x_6 = xa + u \cdot \sin\varepsilon + (zf - fa) \cdot \cos(\Omega + \varepsilon) \\
& y_6 = ya - u \cdot \cos\varepsilon + (zf - fa) \cdot \sin(\Omega + \varepsilon) \\
& \dot{x}_6^2 + \dot{y}_6^2 = (zf - fa)^2 \cdot \dot{\Omega}^2 + 2 \cdot \left((zf - fa)^2 - u \cdot (zf - fa) \cdot \sin\Omega\right) \cdot \dot{\Omega} \cdot \dot{\varepsilon} \\
& \qquad + \left((zf - fa)^2 - 2 \cdot u \cdot (zf - fa) \cdot \sin\Omega + u^2\right) \cdot \dot{\varepsilon}^2 \\
& \dot{\omega}_6 = \dot{\Omega} + \dot{\varepsilon} \\
C_7: \quad & x_7 = xa + ua \cdot \sin\varepsilon - ub \cdot \cos\varepsilon \\
& y_7 = ya - ua \cdot \cos\varepsilon - ub \cdot \sin\varepsilon \\
& \dot{x}_7^2 + \dot{y}_7^2 = \left(ua^2 + ub^2\right) \cdot \dot{\varepsilon}^2 \\
& \dot{\omega}_7 = \dot{\varepsilon}
\end{aligned}
$$

The *potential energy* of the system is due to the motor at the origin and to the spring connecting the point D with C. By Hooke's law it is

$$U = -mom \cdot \beta + c_0 \frac{(\ell - \ell_0)^2}{2}, \tag{7.4}$$

where ℓ is the distance between D and C, namely

$$\begin{aligned} \ell &= \sqrt{(xd - xc)^2 + (yd - yc)^2} \\ xd &= xb + sc \cdot \sin\gamma + sd \cdot \cos\gamma \\ yd &= yb - sc \cdot \cos\gamma + sd \cdot \sin\gamma. \end{aligned}$$

Finally, we have to formulate the *algebraic constraints*. The mechanism contains three loops. The first loop connects O with B via K_1, K_2, K_3; the other two loops connect O with A, one via K_1, K_2, K_4, K_5, the other via K_1, K_2, K_6, K_7. For each loop we get two algebraic conditions:

$$\begin{aligned} &rr \cdot \cos\beta - d \cdot \cos(\beta + \Theta) - ss \cdot \sin\gamma = xb \\ &rr \cdot \sin\beta - d \cdot \sin(\beta + \Theta) + ss \cdot \cos\gamma = yb \\ &rr \cdot \cos\beta - d \cdot \cos(\beta + \Theta) - e \cdot \sin(\Phi + \delta) - zt \cdot \cos\delta = xa \\ &rr \cdot \sin\beta - d \cdot \sin(\beta + \Theta) + e \cdot \cos(\Phi + \delta) - zt \cdot \sin\delta = ya \\ &rr \cdot \cos\beta - d \cdot \cos(\beta + \Theta) - zf \cdot \cos(\Omega + \varepsilon) - u \cdot \sin\varepsilon = xa \\ &rr \cdot \sin\beta - d \cdot \sin(\beta + \Theta) - zf \cdot \sin(\Omega + \varepsilon) + u \cdot \cos\varepsilon = ya. \end{aligned} \tag{7.5}$$

With the position coordinates q from (7.2) the equations (7.5) represent the constraint $g(q) = 0$ where $g : \mathbb{R}^7 \to \mathbb{R}^6$. Together with the kinetic energy T of (7.3) the potential energy U of (7.4) and $L = T - U - \lambda_1 g_1 - \ldots - \lambda_6 g_6$ the equations of motion (1.46) are fully determined.

Fortran Subroutines

For the reader's convenience we include the essential parts of the FORTRAN subroutines describing the differential-algebraic problem. The equations of motion are of the form

$$M(q)\ddot{q} = f(q, \dot{q}) - G^T(q)\lambda \tag{7.6a}$$

$$0 = g(q) \tag{7.6b}$$

where $q \in \mathbb{R}^7$ is the vector defined in (7.2) and $\lambda \in \mathbb{R}^6$. In the following description the variables `Q(1)`, ..., `Q(7)` correspond to $\beta, \ldots, \varepsilon$ (exactly as in (7.2)) and `QP(1)`, ..., `QP(7)` to their derivatives $\dot{\beta}, \ldots, \dot{\varepsilon}$. In all subroutines we have used the abbreviations

```
SIBE = SIN (Q(1))          COBE = COS (Q(1))
SITH = SIN (Q(2))          COTH = COS (Q(2))
SIGA = SIN (Q(3))          COGA = COS (Q(3))
```

```
SIPH = SIN (Q(4))                    COPH = COS (Q(4))
SIDE = SIN (Q(5))                    CODE = COS (Q(5))
SIOM = SIN (Q(6))                    COOM = COS (Q(6))
SIEP = SIN (Q(7))                    COEP = COS (Q(7))

SIBETH = SIN (Q(1)+Q(2))             COBETH = COS (Q(1)+Q(2))
SIPHDE = SIN (Q(4)+Q(5))             COPHDE = COS (Q(4)+Q(5))
SIOMEP = SIN (Q(6)+Q(7))             COOMEP = COS (Q(6)+Q(7))

BEP = QP(1)                          THP = QP(2)
PHP = QP(4)                          DEP = QP(5)
OMP = QP(6)                          EPP = QP(7)
```

The remaining parameters XA, YA, ..., D, DA, E, EA, ..., M1, I1, M2, ... are those of (7.1) and Tables 7.1 and 7.2. They usually reside in a COMMON block. The elements of $M(q)$ in (7.6) are given by

$$m_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}.$$

This matrix is symmetric and (due to the special arrangement of the coordinates) tridiagonal. The non-zero elements (on and below the diagonal) are

```
M(1,1) = M1*RA**2 + M2*(RR**2-2*DA*RR*COTH+DA**2) + I1 + I2
M(2,1) = M2*(DA**2-DA*RR*COTH) + I2
M(2,2) = M2*DA**2 + I2
M(3,3) = M3*(SA**2+SB**2) + I3
M(4,4) = M4*(E-EA)**2 + I4
M(5,4) = M4*((E-EA)**2+ZT*(E-EA)*SIPH) + I4
M(5,5) = M4*(ZT**2+2*ZT*(E-EA)*SIPH+(E-EA)**2) + M5*(TA**2+TB**2)
+         + I4 + I5
M(6,6) = M6*(ZF-FA)**2 + I6
M(7,6) = M6*((ZF-FA)**2-U*(ZF-FA)*SIOM) + I6
M(7,7) = M6*((ZF-FA)**2-2*U*(ZF-FA)*SIOM+U**2) + M7*(UA**2+UB**2)
+         + I6 + I7
```

The ith component of the function f in (7.6) is defined by

$$f_i(q, \dot{q}) = \frac{\partial (T - U)}{\partial q_i} - \sum_{j=1}^{7} \frac{\partial^2 (T - U)}{\partial \dot{q}_i \partial q_j} \cdot \dot{q}_j.$$

Written as FORTRAN statements we have

```
XD = SD*COGA + SC*SIGA + XB
YD = SD*SIGA - SC*COGA + YB
LANG = SQRT ((XD-XC)**2 + (YD-YC)**2)
FORCE = - CO * (LANG - LO)/LANG
FX = FORCE * (XD-XC)
FY = FORCE * (YD-YC)
F(1) = MOM - M2*DA*RR*THP*(THP+2*BEP)*SITH
F(2) = M2*DA*RR*BEP**2*SITH
F(3) = FX*(SC*COGA - SD*SIGA) + FY*(SD*COGA + SC*SIGA)
F(4) = M4*ZT*(E-EA)*DEP**2*COPH
F(5) = - M4*ZT*(E-EA)*PHP*(PHP+2*DEP)*COPH
F(6) = - M6*U*(ZF-FA)*EPP**2*COOM
F(7) = M6*U*(ZF-FA)*OMP*(OMP+2*EPP)*COOM
```

The algebraic constraints $g(q) = 0$ are given by the following six equations (see (7.5))

```
G(1) = RR*COBE - D*COBETH - SS*SIGA - XB
G(2) = RR*SIBE - D*SIBETH + SS*COGA - YB
G(3) = RR*COBE - D*COBETH - E*SIPHDE - ZT*CODE - XA
G(4) = RR*SIBE - D*SIBETH + E*COPHDE - ZT*SIDE - YA
G(5) = RR*COBE - D*COBETH - ZF*COOMEP - U*SIEP - XA
G(6) = RR*SIBE - D*SIBETH - ZF*SIOMEP + U*COEP - YA
```

And here is the Jacobian matrix $G(q) = g_q(q)$. The non-zero entries of this 6×7 array are

```
GQ(1,1) = - RR*SIBE + D*SIBETH
GQ(1,2) = D*SIBETH
GQ(1,3) = - SS*COGA
GQ(2,1) = RR*COBE - D*COBETH
GQ(2,2) = - D*COBETH
GQ(2,3) = - SS*SIGA
GQ(3,1) = - RR*SIBE + D*SIBETH
GQ(3,2) = D*SIBETH
GQ(3,4) = - E*COPHDE
GQ(3,5) = - E*COPHDE + ZT*SIDE
GQ(4,1) = RR*COBE - D*COBETH
GQ(4,2) = - D*COBETH
GQ(4,4) = - E*SIPHDE
GQ(4,5) = - E*SIPHDE - ZT*CODE
GQ(5,1) = - RR*SIBE + D*SIBETH
GQ(5,2) = D*SIBETH
GQ(5,6) = ZF*SIOMEP
GQ(5,7) = ZF*SIOMEP - U*COEP
GQ(6,1) = RR*COBE - D*COBETH
GQ(6,2) = - D*COBETH
GQ(6,6) = - ZF*COOMEP
GQ(6,7) = - ZF*COOMEP - U*SIEP
```

If we apply a numerical method to the index 1 formulation of the system, we also need the expression $g_{qq}(q)(\dot{q}, \dot{q})$. It is given by

```
 GQQ(1) = - RR*COBE*V(1)**2 + D*COBETH*(V(1)+V(2))**2 +
+           SS*SIGA*V(3)**2
 GQQ(2) = - RR*SIBE*V(1)**2 + D*SIBETH*(V(1)+V(2))**2 -
+           SS*COGA*V(3)**2
 GQQ(3) = - RR*COBE*V(1)**2 + D*COBETH*(V(1)+V(2))**2 +
+           E*SIPHDE*(V(4)+V(5))**2 + ZT*CODE*V(5)**2
 GQQ(4) = - RR*SIBE*V(1)**2 + D*SIBETH*(V(1)+V(2))**2 -
+           E*COPHDE*(V(4)+V(5))**2 + ZT*SIDE*V(5)**2
 GQQ(5) = - RR*COBE*V(1)**2 + D*COBETH*(V(1)+V(2))**2 +
+           ZF*COOMEP*(V(6)+V(7))**2 + U*SIEP*V(7)**2
 GQQ(6) = - RR*SIBE*V(1)**2 + D*SIBETH*(V(1)+V(2))**2 +
+           ZF*SIOMEP*(V(6)+V(7))**2 - U*COEP*V(7)**2
```

Computation of Consistent Initial Values

We first compute a solution of $g(q) = 0$. Since g consists of 6 equations in 7 unknowns we can fix one of them arbitrarily, say $\Theta(0) = 0$, and compute the remaining coordinates by Newton iterations. This gives

$$\begin{aligned}
\beta(0) &= -0.06171389001427644965358948458001 \\
\gamma(0) &= 0.45527981916307038025591238244 9 \\
\Phi(0) &= 0.22266839016588588467447318560 9 \\
\delta(0) &= 0.48736497954384255022559895353 0 \\
\Omega(0) &= -0.22266839016588588467447318560 9 \\
\dot{e}(0) &= 1.23054744454982119249735015568.
\end{aligned} \tag{7.7}$$

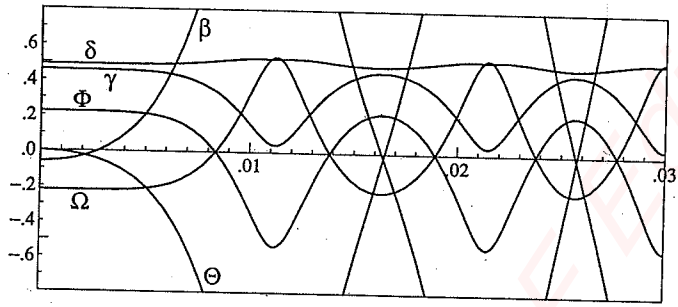

Fig. 7.2. Solution of 7 body mechanism

The condition $G(q)\dot{q} = 0$ is satisfied if we put

$$\dot{\beta}(0) = \dot{\Theta}(0) = \dot{\gamma}(0) = \dot{\Phi}(0) = \dot{\delta}(0) = \dot{\Omega}(0) = \dot{\varepsilon}(0) = 0. \tag{7.8}$$

The values of $\lambda(0)$ and $\ddot{q}(0)$ are then uniquely determined by (7.6a) and the twice differentiated constraint $0 = g_{qq}(q)(\dot{q},\dot{q}) + G(q)\ddot{q}$. We just have to solve a linear system with the matrix

$$\begin{pmatrix} M(q) & G^T(q) \\ G(q) & 0 \end{pmatrix}. \tag{7.9}$$

Observe that g_{qq} need not be evaluated, because $\dot{q}(0) = 0$. Due to the choice $\Theta(0) = 0$ most components of $\lambda(0)$ and $\ddot{q}(0)$ vanish. Only the first two of these are different from zero and given by

$$\begin{aligned} \ddot{\beta}(0) &= 14222.443919954113870591162 5887 \\ \ddot{\Theta}(0) &= -10666.832939965585402943371 9415 \\ \lambda_1(0) &= 98.566870396241089605765 4982170 \\ \lambda_2(0) &= -6.122688344255662655031143 93122. \end{aligned} \tag{7.10}$$

The solution of this seven body mechanism is plotted (mod 2π) in Fig. 7.2 for $0 \le t \le 0.03$.

Numerical Computations

We first transform (7.6) into a first order system by introducing the new variable $v = \dot{q}$. Our codes apply only to problems where the derivative is multiplied by a constant matrix. We therefore also consider $w = \ddot{q}$ as a variable so that (7.6a) becomes an algebraic relation. The various formulations of the problem, as discussed in Sect. VII.1, are now as follows:

Index 3 Formulation. With $v = \dot{q}$ and $w = \ddot{q}$ the system (7.6) can be written as

$$\dot{q} = v \tag{7.11a}$$
$$\dot{v} = w \tag{7.11b}$$
$$0 = M(q)w - f(q,v) + G^T(q)\lambda \tag{7.11c}$$
$$0 = g(q). \tag{7.11d}$$

Index 2 Formulation. If we differentiate $0 = g(q)$ once and replace (7.11d) by

$$0 = G(q)v, \tag{7.11e}$$

we get an index 2 problem which is mathematically equivalent to (7.6).

Index 1 Formulation. One more differentiation of (7.11e) yields

$$0 = g_{qq}(q)(v,v) + G(q)w, \tag{7.11f}$$

so that (7.11a,b,c,f) constitutes an index 1 problem.

We have applied several codes with many different tolerances between 10^{-2} and 10^{-10} to these formulations. The results are given in Fig. 7.3. We have plotted the computing time (on a SUN Spark 20 workstation) against the error of the (q,v)-components at $x_{\text{end}} = 0.03$ (in double logarithmic scale).

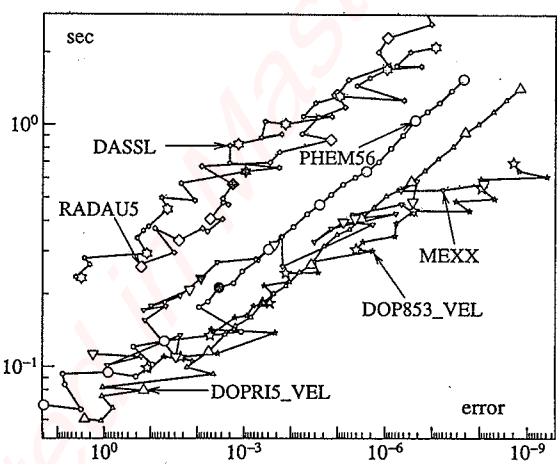

Fig. 7.3. Work-precision diagram

Explicit Runge-Kutta Methods. The index 1 formulation allows us to apply explicit methods such as DOPRI5 or DOP853 of Volume I. For this we have written a function subroutine which solves in each call the linear system (7.11c,f) for w and λ and inserts the result into (7.11a,b). Since there is no stiffness in the obtained