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Project 3

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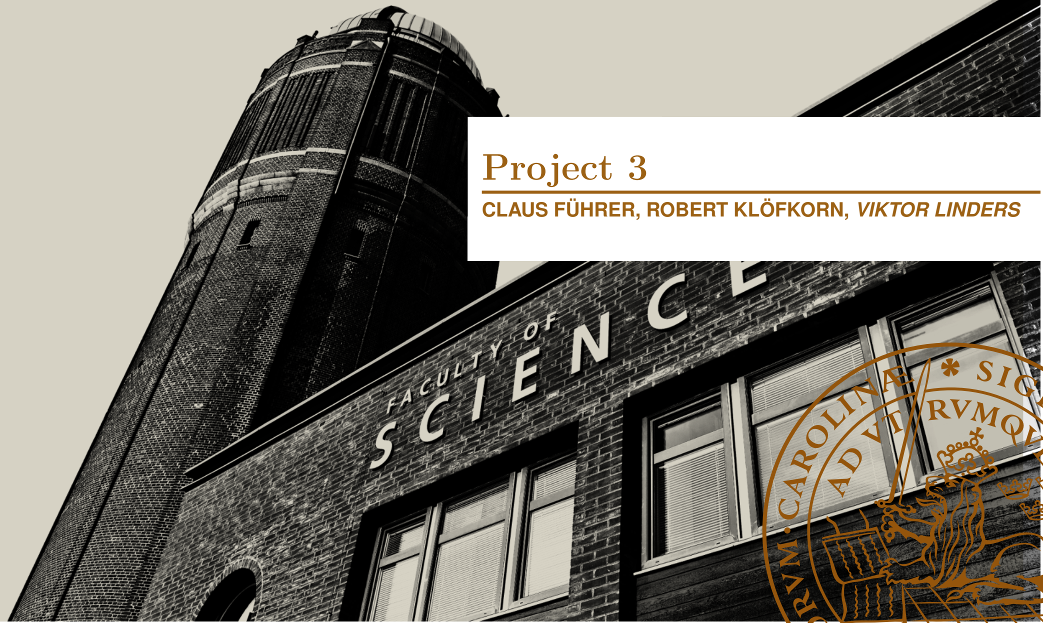


Project 3: Goals

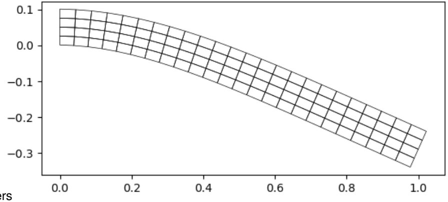
- Learn about the Newmark-β and the HHT-α methods
- Extend the functionality of Assimulo
- Solve the equations of linear elastodynamics
- Have fun with DUNE-FEM

Recall: Elastodynamics

$$\rho\ddot{u} = \nabla \cdot \sigma + f(t) \tag{1}$$

- ρ: Density
- u: Displacement field
- $\ddot{u}$: Acceleration
- σ: Stress tensor
- f : External forces

$$\sigma = \lambda \mathrm{tr}(\epsilon)1 + 2\mu\epsilon, \qquad \epsilon = \frac{\nabla u + (\nabla u)^\top}{2}, \qquad \lambda, \mu \text{ Lamé coefficients}$$



Discretization

Dune-FEM provides us with a finite element discretization of the spatial terms:

$$M\ddot{\underline{u}} + K\underline{u} = \underline{f}(t), \qquad \underline{f}, \underline{u} \in \mathbb{R}^p, \quad M, K \in \mathbb{R}^{p\times p}$$

To model friction we may add some artificial damping:

$$M\ddot{\underline{u}} + C\dot{\underline{u}} + K\underline{u} = \underline{f}(t), \qquad \underline{u}(0) = \underline{u}_0, \quad \dot{\underline{u}}(0) = \dot{\underline{u}}_0 \tag{2}$$

- M: Mass matrix
- C: Damping matrix
- K: Stiffness matrix

We choose $C = \eta_M M + \eta_K K$ (Rayleigh damping) for some positive scalars η_M and η_K.

Newmark I

$$M\ddot{\underline{u}} + C\dot{\underline{u}} + K\underline{u} = \underline{f}(t)$$

We must handle both a first and a second derivative in time.

Option 1: Introduce an auxiliary variable $\underline{v} = \dot{\underline{u}}$ and solve the system

$$\begin{bmatrix} \dot{\underline{u}} \\ \dot{\underline{v}} \end{bmatrix} + \begin{pmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{pmatrix} \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{f} \end{bmatrix}$$

using e.g. Runge-Kutta.

But doubling the system size may not be an option if the spatial dimension p is large.

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Newmark II

$$M\underline{\ddot{u}} + C\underline{\dot{u}} + K\underline{u} = \underline{f}(t)$$

We must handle both a first and a second derivative in time.

Option 2: Do something clever.

Newmark's method attempts to solve the problem without increasing the system size.

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Newmark III - Taylor's theorem

For any function $f(t)$, $k+1$ times differentiable in $(t, t+\Delta t)$, there is a $\delta \in (0,1)$ such that,

$$f(t+\Delta t) = \frac{f(t)\Delta t^0}{0!} + \frac{f'(t)\Delta t^1}{1!} + \cdots + \frac{f^{(k)}(t)\Delta t^k}{k!} + \underbrace{\frac{f^{(k+1)}(t+\delta\Delta t)\Delta t^{k+1}}{(k+1)!}}_{\text{Error term}} \tag{3}$$

Newmark IV - Taylor's theorem

Apply (3) to $\underline{u}(t+\Delta t)$ using $k=1$:

$$\underline{u}(t+\Delta t) = \underline{u}(t) + \dot{\underline{u}}(t)\Delta t + \ddot{\underline{u}}(t+\delta\Delta t)\frac{\Delta t^2}{2}$$

Trick 1: Taylor expand the error term and introduce $\beta = \delta/2 \in (0, 1/2)$:

$$\underline{u}(t+\Delta t) = \underline{u}(t) + \dot{\underline{u}}(t)\Delta t + \ddot{\underline{u}}(t)\frac{\Delta t^2}{2} + \beta\dddot{\underline{u}}(t)\Delta t^3 + \ldots \tag{4}$$

Similarly, Taylor expand $\dot{\underline{u}}(t+\Delta t)$ using $k=0$ and use Trick 1 with $\gamma = \delta \in (0, 1)$:

$$\dot{\underline{u}}(t+\Delta t) = \dot{\underline{u}}(t) + \ddot{\underline{u}}(t)\Delta t + \gamma\dddot{\underline{u}}(t)\Delta t^2 + \ldots \tag{5}$$

Newmark V - Taylor's theorem

Trick 2: Truncate (4) and (5) and observe that

$$\underline{\dddot{u}}(t) \approx \frac{\underline{\ddot{u}}(t+\Delta t) - \underline{\ddot{u}}(t)}{\Delta t}$$

$$\underline{u}(t+\Delta t) \approx \underline{u}(t) + \underline{\dot{u}}(t)\Delta t + (1-2\beta)\underline{\ddot{u}}(t)\frac{\Delta t^2}{2} + \beta\underline{\ddot{u}}(t+\Delta t)\Delta t^2$$

$$\underline{\dot{u}}(t+\Delta t) \approx \underline{\dot{u}}(t) + (1-\gamma)\underline{\ddot{u}}(t)\Delta t + \gamma\underline{\ddot{u}}(t+\Delta t)\Delta t$$

Newmark VI - The scheme

For $\beta \in (0, 1/2)$ and $\gamma \in (0, 1)$, the Newmark-β scheme reads

$$\underline{u}_n = \underline{u}_{n-1} + \underline{\dot{u}}_{n-1}\Delta t + (1 - 2\beta)\underline{\ddot{u}}_{n-1}\frac{\Delta t^2}{2} + \beta\underline{\ddot{u}}_n\Delta t^2 \tag{6}$$

$$\underline{\dot{u}}_n = \underline{\dot{u}}_{n-1} + (1 - \gamma)\underline{\ddot{u}}_{n-1}\Delta t + \gamma\underline{\ddot{u}}_n\Delta t \tag{7}$$

$$\underline{\ddot{u}}_n = M^{-1}\left(\underline{f}_n - C\underline{\dot{u}}_n - K\underline{u}_n\right) \tag{8}$$

But isn't this a $3p \times 3p$ implicit system?

Looks worse than Runge-Kutta!

Newmark VII - Explicit version

Note that if $\beta = 0$ and if there is no damping ($C = 0$), then the scheme is explicit. In this case we use $\gamma = 1/2$:

$$\underline{u}_n = \underline{u}_{n-1} + \underline{\dot{u}}_{n-1}\Delta t + \underline{\ddot{u}}_{n-1}\frac{\Delta t^2}{2} \tag{9}$$

$$\underline{\dot{u}}_n = \underline{\dot{u}}_{n-1} + \frac{1}{2}\underline{\ddot{u}}_{n-1}\Delta t + \frac{1}{2}\underline{\ddot{u}}_n\Delta t \tag{10}$$

$$\underline{\ddot{u}}_n = M^{-1}\left(\underline{f}_n - K\underline{u}_n\right) \tag{11}$$

<u>OBS</u>: First solve (11) for $\underline{\ddot{u}}_0$, then (9), (11), (10) for $\underline{u}_1$, $\underline{\ddot{u}}_1$, $\underline{\dot{u}}_1$ etc.

Try this on the pendum in Project 1!

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Newmark VIII - Implicit version

$$\underline{u}_n = \underline{u}_{n-1} + \underline{\dot{u}}_{n-1}\Delta t + (1-2\beta)\underline{\ddot{u}}_{n-1}\frac{\Delta t^2}{2} + \beta\underline{\ddot{u}}_n\Delta t^2 \tag{6}$$

$$\underline{\dot{u}}_n = \underline{\dot{u}}_{n-1} + (1-\gamma)\underline{\ddot{u}}_{n-1}\Delta t + \gamma\underline{\ddot{u}}_n\Delta t \tag{7}$$

$$\underline{\ddot{u}}_n = M^{-1}\left(\underline{f}_n - C\underline{\dot{u}}_n - K\underline{u}_n\right) \tag{8}$$

Newmark VIII - Implicit version

$$\underline{u}_n = \underline{u}_{n-1} + \dot{\underline{u}}_{n-1}\Delta t + (1-2\beta)\ddot{\underline{u}}_{n-1}\frac{\Delta t^2}{2} + \beta\ddot{\underline{u}}_n\Delta t^2 \tag{6}$$

$$\dot{\underline{u}}_n = \dot{\underline{u}}_{n-1} + (1-\gamma)\ddot{\underline{u}}_{n-1}\Delta t + \gamma\ddot{\underline{u}}_n\Delta t \tag{7}$$

$$\ddot{\underline{u}}_n = M^{-1}\left(\underline{f}_n - C\dot{\underline{u}}_n - K\underline{u}_n\right) \tag{8}$$

In (6), solve for $\ddot{\underline{u}}_n$:

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Newmark VIII - Implicit version

$$\underline{\ddot{u}}_n = \frac{\underline{u}_n - \underline{u}_{n-1}}{\beta \Delta t^2} - \frac{\underline{\dot{u}}_n}{\beta \Delta t} - \left(\frac{1}{2\beta} - 1\right) \underline{\ddot{u}}_n \tag{6'}$$

$$\underline{\dot{u}}_n = \underline{\dot{u}}_{n-1} + (1-\gamma)\underline{\ddot{u}}_{n-1}\Delta t + \gamma \underline{\ddot{u}}_n \Delta t \tag{7}$$

$$\underline{\ddot{u}}_n = M^{-1}\left(\underline{f}_n - C\underline{\dot{u}}_n - K\underline{u}_n\right) \tag{8}$$

Newmark VIII - Implicit version

$$\underline{\ddot{u}}_n = \frac{\underline{u}_n - \underline{u}_{n-1}}{\beta \Delta t^2} - \frac{\underline{\dot{u}}_{n-1}}{\beta \Delta t} - \left(\frac{1}{2\beta} - 1\right) \underline{\ddot{u}}_{n-1} \tag{6'}$$

$$\underline{\dot{u}}_n = \underline{\dot{u}}_{n-1} + (1-\gamma)\underline{\ddot{u}}_{n-1}\Delta t + \gamma \underline{\ddot{u}}_n \Delta t \tag{7}$$

$$\underline{\ddot{u}}_n = M^{-1}\left(\underline{f}_n - C\underline{\dot{u}}_n - K\underline{u}_n\right) \tag{8}$$

Insert $\underline{\ddot{u}}_n$ into (7):

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Newmark VIII - Implicit version

$$\underline{\ddot{u}}_n = \frac{\underline{u}_n - \underline{u}_{n-1}}{\beta \Delta t^2} - \frac{\underline{\dot{u}}_{n-1}}{\beta \Delta t} - \left(\frac{1}{2\beta} - 1 \right) \underline{\ddot{u}}_{n-1} \tag{6'}$$

$$\underline{\dot{u}}_n = \frac{\gamma}{\beta} \left(\frac{\underline{u}_n - \underline{u}_{n-1}}{\Delta t} \right) + \left(1 - \frac{\gamma}{\beta} \right) \underline{\dot{u}}_{n-1} + \left(1 - \frac{\gamma}{2\beta} \right) \Delta t \underline{\ddot{u}}_{n-1} \tag{7'}$$

$$\underline{\ddot{u}}_n = M^{-1} \left(\underline{f}_n - C \underline{\dot{u}}_n - K \underline{u}_n \right) \tag{8}$$

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Newmark VIII - Implicit version

$$\ddot{\underline{u}}_n = \frac{\underline{u}_n - \underline{u}_{n-1}}{\beta \Delta t^2} - \frac{\dot{\underline{u}}_{n-1}}{\beta \Delta t} - \left(\frac{1}{2\beta} - 1\right) \ddot{\underline{u}}_{n-1} \tag{6'}$$

$$\dot{\underline{u}}_n = \frac{\gamma}{\beta}\left(\frac{\underline{u}_n - \underline{u}_{n-1}}{\Delta t}\right) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\underline{u}}_{n-1} + \left(1 - \frac{\gamma}{2\beta}\right)\Delta t \ddot{\underline{u}}_{n-1} \tag{7'}$$

$$\ddot{\underline{u}}_n = M^{-1}\left(\underline{f}_n - C\dot{\underline{u}}_n - K\underline{u}_n\right) \tag{8}$$

Insert $\ddot{\underline{u}}_n$ and $\dot{\underline{u}}_n$ into (8) and solve for $\underline{u}_n$:

Newmark VIII - Implicit version

$$\ddot{\underline{u}}_n = \frac{\underline{u}_n - \underline{u}_{n-1}}{\beta \Delta t^2} - \frac{\dot{\underline{u}}_{n-1}}{\beta \Delta t} - \left(\frac{1}{2\beta} - 1\right) \ddot{\underline{u}}_{n-1} \tag{6'}$$

$$\dot{\underline{u}}_n = \frac{\gamma}{\beta} \left(\frac{\underline{u}_n - \underline{u}_{n-1}}{\Delta t}\right) + \left(1 - \frac{\gamma}{\beta}\right) \dot{\underline{u}}_{n-1} + \left(1 - \frac{\gamma}{2\beta}\right) \Delta t \ddot{\underline{u}}_{n-1} \tag{7'}$$

$$\begin{aligned}
&\left[\frac{M}{\beta \Delta t^2} + \frac{\gamma C}{\beta \Delta t} + K\right] \underline{u}_n = f_n \\
&+ M \left[\frac{\underline{u}_{n-1}}{\beta \Delta t^2} + \frac{\dot{\underline{u}}_{n-1}}{\beta \Delta t} + \left(\frac{1}{2\beta} - 1\right) \ddot{\underline{u}}_{n-1}\right] \\
&+ C \left[\frac{\gamma \underline{u}_{n-1}}{\beta \Delta t} - \left(1 - \frac{\gamma}{\beta}\right) \dot{\underline{u}}_{n-1} - \left(1 - \frac{\gamma}{2\beta}\right) \Delta t \ddot{\underline{u}}_{n-1}\right]
\end{aligned} \tag{8'}$$

Newmark VIII - Implicit version

The solution procedure is thus:

1. Use initial data $\underline{u}_0$ and $\underline{\dot{u}}_0$ to solve (8) with $n = 0$ and get $\underline{\ddot{u}}_0$.
2. Solve (8'), then (6') and (7') to get $\underline{u}_1$, $\underline{\dot{u}}_1$ and $\underline{\ddot{u}}_1$.
3. Keep iterating until final time.

We only need to solve a single $p \times p$ system in each iteration.

Newmark IX - Stability and accuracy

How to choose γ and β? Play around and experiment!

The implicit method is unconditionally stable if $1/2 \leq \gamma \leq 2\beta$.

If $\gamma > 1/2$, then extra artificial damping is added.

Second order accuracy if and only if $\gamma = 2\beta = 1/2$. Otherwise first order.

What do we do if we want both second order accuracy and artificial damping?

HHT-α method

The HHT-α methods tries to achieve accuracy and damping by generalizing (8'):

$$
\begin{aligned}
&\left[\frac{M}{\beta \Delta t^2} + \frac{\gamma C}{\beta \Delta t} + (1+\alpha)K\right] \underline{u}_n = f_n \\
&+ M\left[\frac{\underline{u}_{n-1}}{\beta \Delta t^2} + \frac{\dot{\underline{u}}_{n-1}}{\beta \Delta t} + \left(\frac{1}{2\beta} - 1\right)\ddot{\underline{u}}_{n-1}\right] \\
&+ C\left[\frac{\gamma \underline{u}_{n-1}}{\beta \Delta t} - \left(1 - \frac{\gamma}{\beta}\right)\dot{\underline{u}}_{n-1} - \left(1 - \frac{\gamma}{2\beta}\right)\Delta t \ddot{\underline{u}}_{n-1}\right] \\
&+ \alpha K \underline{u}_{n-1}
\end{aligned}
\tag{8''}
$$

Here,

$$
\gamma = \frac{1}{2} - \alpha, \qquad \beta = \left(\frac{1-\alpha}{2}\right)^2, \qquad \alpha \in \left[-\frac{1}{3}, 0\right]
$$