Weak compactness of probability measures

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Abstract

We will investigate fundamental properties of finitely additive signed measures, like the Hahn-Jordan decomposition and the integration with respect to them. This notion of integral leads to a canonical representation of measures as a dual space, namely the dual of all bounded measurable functions. This approach has the advantage that it immediately gives the Banach space property of the signed measures with respect to the total variation. Further we will prove that the countably additive measures correspond exactly to the dual elements for which the desirable monotone and dominated convergence theorem hold. This characterisation naturally leads to the characterisation of dual spaces of other Banach spaces of bounded functions like spaces of continuous functions. Further we use the duality of the measures to show that the space of probability measures is compact in the weak topology. This implies that the weak topology on the probability measures over a metric space is in general not metrisable like frequently stated. Actually the metrisability turns out to be equvalent to the compactness of the space itself.

Let X be an arbitrary set and $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra. Further let $L^0_b(\mathcal{A})$ be the space of real measurable and bounded functions on X which is a Banach space with respect to the uniform norm $\|\cdot\|_\infty$. The completeness of $L^0_b(\mathcal{A})$ follows from the fact that the measurable bounded functions are closed in the larger Banach space of bounded functions as they are even closed under pointwise convergence, or more precisely the product topology. We will see that the finitely additive measures on $\mathcal{A}$ are just the dual space of $L^0_b(\mathcal{A})$, but before we can prove this we need a few basic properties of signed measures.

1 Definition (Signed measure). Let $(X, \mathcal{A})$ be a measurable space and let $\mu\colon \mathcal{A} \to \mathbb{R}$. Then we say μ is

(i) *finitely additive* if we have
$$\mu\Big(\bigcup_{k=1}^{N} A_k\Big) = \sum_{k=1}^{N} \mu(A_k)$$
for all finite collection of measurable disjoint sets $A_k \in \mathcal{A}$,

(ii) *countably additive* if the above holds for countable collections of measurable disjoint sets,

(iii) of *bounded variation* or shortly *bounded* if we have
$$\|\mu\|_{BV} := \sup_{\mathcal{E}} \sum_{E\in\mathcal{E}} |\mu(E)| < \infty$$
where the supremum is taken over all finite families of disjoint measurable sets,

(iv) *positive* if $\mu(\mathcal{A}) \subseteq \mathbb{R}_+$.

Further we denote the space of bounded and finitely additive measures by $ba(\mathcal{A})$ and the space of bounded and countably additive measures by $ca(\mathcal{A})$. The quantity $\|\mu\|_{BV}$ is called the norm of *total variation* of μ and Theorem 6 shows that it is indeed a norm and that both $ba(\mathcal{A})$ and $ca(\mathcal{A})$ are complete wrt to it.

2 Theorem (Hahn-Jordan decomposition). *For every signed measure $\mu \in ba(\mathcal{A})$ there are two positive measures $\mu_+, \mu_- \in ba(\mathcal{A})$ such that $\mu = \mu_+ - \mu_-$. Further μ is countably additive if and only if μ_+ and μ_- are countably additive.*

Proof. For $A \in \mathcal{A}$ set

$$\mu_+(A) := \sup\left\{\mu(B) \mid B \in \mathcal{A}, B \subseteq A\right\}.$$

It is clear that μ_+ is positive as we have $\mu_+(A) \geq \mu(\varnothing) = 0$. To see that μ_+ is finitely additive let A and B be disjoint measurable sets. Then we have

$$\begin{aligned}
\mu_+(A \cup B) &= \sup\left\{\mu(C) \mid C \in \mathcal{A}, C \subseteq A \cup B\right\} \\
&= \sup\left\{\mu(C \cap A) + \mu(C \cap B) \mid C \in \mathcal{A}, C \subseteq A \cup B\right\} \\
&= \sup\left\{\mu(C) + \mu(D) \mid C, D \in \mathcal{A}, C \subseteq A, D \subseteq B\right\} \\
&= \sup\left\{\mu(C) \mid C \in \mathcal{A}, C \subseteq A\right\} + \sup\left\{\mu(D) \mid D \in \mathcal{A}, D \subseteq B\right\} \\
&= \mu_+(A) + \mu_+(B).
\end{aligned}$$

If we choose $\mu_- := \mu_+ - \mu$ we get the desired decomposition. Note that μ_- is positive since we have $\mu \leq \mu_+$ via definition.

It is obvious that if μ_+ and μ_- are countably additive so is μ. Let on the other hand μ be countably additive then it suffices to show that μ_+ is countably additive so let $(A_n) \subseteq \mathcal{A}$ be a sequence of disjoint sets. Choose now N so large that

$$\sum_{i=N+1}^{\infty} \mu_+(A_i) < \varepsilon.$$

To see that such N exists, take sequences $(B_n^i)_n \subseteq \mathcal{A}$ such that $B_n^i \subseteq A_i$ and $0 \leq \mu(B_n^i) \nearrow \mu_+(A_i)$. We get now

$$\|\mu\|_{BV} \geq \sum_{i=1}^{\infty} \mu(B_n^i) \nearrow \sum_{i=1}^{\infty} \mu_+(A_i) \quad \text{for } n \to \infty$$

by monotone convergence. Let $\varepsilon > 0$ and choose $B \subseteq A := \bigcup_{i=1}^{\infty} A_i$ such that

$$\mu(B) \geq \mu_+(A) - \varepsilon.$$

With $A^N := \bigcup_{i=1}^{N} A_i$ we get

$$\mu_+(A^N) \geq \mu(B \cap A^N) = \mu(B) - \mu\big(B \cap (A \setminus A^N)\big) \geq \mu_+(A) - 2\varepsilon$$

since we can estimate

$$\mu\big(B \cap (A \setminus A^N)\big) = \sum_{i=N+1}^{\infty} \mu(B \cap A_i) \leq \sum_{i=N+1}^{\infty} \mu_+(A_i) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we get

$$\sum_{i=1}^{N} \mu_+(A_i) = \mu_+(A^N) \to \mu_+(A) \quad \text{for } N \to \infty.$$

$\square$

Integration with respect to a signed measure

The integral of a *simple function* with respect to a finitely additive measure is defined in the familiar way

$$\int \sum_{k=1}^{N} \alpha_i \chi_{A_i} \,\mathrm{d}\mu := \sum_{k=1}^{N} \alpha_i \mu(A_i).$$

It is easily seen that this definition does not depend on the representation as a finite sum. Note that the space of simple functions

$$\mathcal{E} = \left\{ \sum_{k=1}^{N} \alpha_i \chi_{A_i} \;\middle|\; \alpha_i \in \mathbb{R}, A_i \in \mathcal{A}, N \in \mathbb{N} \right\}$$

is a dense subspace of $L^0_b(\mathcal{A})$ with respect to the uniform norm. Given $f \in L^0_b(\mathcal{A})$ an approximating sequence would be given by

$$f_n = \sum_{k=-2^n}^{2^n+1} k \,\|f\|\, 2^{-n} \chi_{A_k} \quad \text{with } A_k = f^{-1}\big([k \,\|f\|\, 2^{-n}, (k+1) \,\|f\|\, 2^{-n})\big).$$

If we write $f = \sum \alpha_i \chi_{A_i} \in \mathcal{E}$ such that the sets A_i are pairwise disjoint we get

$$\left| \int f \,\mathrm{d}\mu \right| \leq \sum_{k=1}^{N} |\alpha_i| \cdot |\mu(A_i)| \leq \max_i |\alpha_i| \cdot \sum_{k=1}^{N} |\mu(A_i)| \leq \|f\|_\infty \cdot \|\mu\|_{BV} . \tag{1}$$

Therefore $\int \mathrm{d}\mu \colon \mathcal{E} \to \mathbb{R}$ is a bounded linear operator and can uniquely be extended to a bounded linear operator on the whole space $L^0_b(\mathcal{A})$ which we will denote again by $\int \mathrm{d}\mu$ in abusive notation. Further the inequality (1) carries over to all functions $f \in L^0_b(\mathcal{A})$, so we have proven the following result.

3 Theorem (Integral). *For every signed measure $\mu \in ba(\mathcal{A})$ there is a unique linear and continuous mapping*

$$\int \mathrm{d}\mu \colon L^0_b(\mathcal{A}) \to R$$

called the integral wrt to μ *such that we have*

$$\int \sum_{k=1}^{N} \alpha_i \chi_{A_i} \,\mathrm{d}\mu := \sum_{k=1}^{N} \alpha_i \mu(A_i)$$

for all simple functions. Further the estimate (1) *holds for all $f \in L^0_b(\mathcal{A})$.*

4 Remark. From the definition of the integral we immediately get

$$\int \mathrm{d}\mu = \int \mathrm{d}\mu_+ - \int \mathrm{d}\mu_-$$

for simple functions and this decomposition carries over to the general integral. Considering this identity, we see immediately that all the nice properties of the Lebesgue integral like the monotone and dominated convergence theorem carry over to the integral with respect to a signed countably additive measure.

5 Remark. The class of integrands is fairly small and it is well known from measure and integration theory that for countably additive (signed) measures the notion of integral can be extended to a much wider class of functions. However one needs the monotone convergence theorem (which only holds for countably additive measures) to show that this more general Lebesgue integral is linear.

Measures as dual spaces

Now we have set up all the preliminaries that we need to perceive $ba(\mathcal{A})$ as a dual space.

6 Theorem (Measures as a dual space). *With the above notations the linear mapping*

$$I : ba(\mathcal{A}) \to L_b^0(\mathcal{A})', \quad \mu \mapsto I\mu := \left(f \mapsto \int f \,\mathrm{d}\mu \right)$$

is an algebraic isomorphism and we have $\|I\mu\|_{L_b^0(\mathcal{A})^*} = \|\mu\|_{BV}$. In particular $\|\cdot\|_{BV}$ is a norm and $ba(\mathcal{A})$ is complete with respect to it. Further the inverse I^{-1} is given by

$$F \mapsto \Big(A \mapsto F(\chi_A) \Big).$$

For a measure $\mu \in ba(\mathcal{A})$ the following three statements are equivalent:

(*i*) *μ is countably additive.*

(*ii*) *$I\mu$ fulfills the monotone convergence theorem.*

(*iii*) *$I\mu$ fulfills the dominated convergence theorem.*

Proof. The previous theorem directly implies that I is a well defined linear contraction, i.e. $\|I\mu\| \le \|\mu\|$. To see that the mapping is an isometry we take a finite collection $\mathcal{E}$ of disjoint measurable sets. Then we get

$$\|I\mu\|_{L_b^0(\mathcal{A})'} \ge \int \sum_{E \in \mathcal{E}} \operatorname{sign}(\mu(E))\chi_E \,\mathrm{d}\mu = \sum_{E \in \mathcal{E}} |\mu(E)| \,.$$

By taking the supremum over all those collections $\mathcal{E}$ we get $\|I\mu\| = \|\mu\|$. Further the bijectivity follows from the fact $I \circ I^{-1} = \mathrm{id}_{L_b^0(\mathcal{A})'}, I^{-1} \circ I = \mathrm{id}_{ba(\mathcal{A})}$.

Let now μ be countably additive, then we know, that (*ii*) and (*iii*) hold, so we only have to show that (*ii*) and (*iii*) both imply (*i*). If we assume that the monotone convergence theorem holds for $I\mu$, we have

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n \right) = I\mu \left(\sum_{n \in \mathbb{N}} \chi_{A_n} \right) = \sum_{n \in \mathbb{N}} I\mu(\chi_{A_n}) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for disjoint $(A_n) \subseteq \mathcal{A}$. Thus μ is countably additive. The implication (*iii*)$\Rightarrow$(*i*) follows analogue. $\square$

7 Remark. From now on we will identify μ and $I\mu$ with each other and therefore we write $\mu(f)$ for the integral of f wrt to μ whenever this makes sense.

8 Corollary. *The space of countably additive signed measures is a Banach space with respect to the norm of total variation.*

Proof. Since $ba(\mathcal{A})$ is complete, we only have to prove that $ca(\mathcal{A})$ is closed. For this let $(\mu_n) \subseteq ca(\mathcal{A})$ be a sequence with $\mu_n \to \mu$. To see that μ is again countably additive we only have to show that the dominated convergence theorem holds for μ. So let $(f_n) \subseteq L_b^0(\mathcal{A})$ be a sequence of functions such that $f_n \to f$ pointwise and $\sup_n |f_n| \le K < \infty$. The computation

$$\begin{aligned} \lim_{n\to\infty} |\mu(f_n) - \mu(f)| &\le \liminf_{n\to\infty} |\mu(f_n) - \mu_N(f_n)| + |\mu_N(f_n) - \mu_N(f)| \\ &\quad + |\mu_N(f) - \mu(f)| \\ &\le 2K \cdot \|\mu - \mu_N\|_{BV} \to 0 \quad \text{for } N \to \infty \end{aligned}$$

completes the proof. $\square$

The interpretation of measures as the dual space of bounded measurable functions can be used to show that the dual space of $L^\infty(\mu)$ coincides with the absolutely continuous measures wrt μ. In the case of signed measures a measurable set $A \in \mathcal{A}$ is called a μ *Null set* if $\mu_+(A) + \mu_-(A) = 0$. A measure $\nu \in ba(\mathcal{A})$ is said to be *absolutely continuous* wrt μ if every μ Null set is also a ν Null set. Further we write $L^\infty(\mu)$ for the space of all measurable function that are bounded outside of a μ Null set which is a Banach space wrt to the usual $L^\infty(\mu)$ norm.

9 Theorem (Dual of $L^\infty(\mu)$). *Let $\mu \in ba(\mathcal{A})$ and let $ba(\mu) \subseteq ba(\mathcal{A})$ be the subspace of measure that are absolutely continuous wrt μ. Then we have*

$$ba(\mu) \cong L^\infty(\mu)'$$

isometrically in the canonical way.

Proof. Let $[f]_\mu$ denote the μ ,a.e. equivalence‘ class of $f \in L^0_b(\mathcal{A})$, then

$$\iota : L^0_b(\mathcal{A}) \to L^\infty(\mu), \quad f \mapsto [f]_\mu$$

is surjective and further we have

$$\left\| [f]_\mu \right\|_{L^\infty(\mathcal{A})} = \inf_{g \in [f]_\mu} \|g\|_\infty . \tag{2}$$

Therefore the dual operator

$$\iota' \colon L^\infty(\mu)' \to L^0_b(\mathcal{A})' \cong ba(\mathcal{A})$$

is injective and because of (3) it is isometric since it implies $\iota(B_1) = B_1$ where B_1 denotes the unit ball in the respective spaces. Further the closed range theorem implies

$$\operatorname{ran}(\iota') = \ker(\iota)^\perp = \left\{ F \in L^0_b(\mathcal{A})' \mid F(f) = 0 \text{ for all } f = 0 \ \mu \text{ a.e.} \right\} \cong ba(\mu).$$

$\square$

10 Characterisation of other dual spaces. A similar approach can be used to characterise the dual spaces of other normed spaces of bounded measurable functions, for example the dual of all bounded continuous functions $\mathcal{C}_b(X)$ if we are dealing with a topological space X. If we denote the Borel algebra by $\mathcal{B}$ we get

$$\iota \colon \mathcal{C}_b(X) \hookrightarrow L^0_b(\mathcal{B})$$

isometrically and therefore

$$\iota' \colon ba(\mathcal{B}) \cong L^0_b(\mathcal{B}) \twoheadrightarrow \mathcal{C}_b(X)', \quad \mu \mapsto \big(f \mapsto \mu(f)\big)$$

is surjective, but not necessarily isometric and not injective if $\mathcal{C}_b(X) \subsetneq L^0_b(\mathcal{B})$. Of course one could argue that we then have $\mathcal{C}_b(X)' \cong ba(\mathcal{B})/\ker(\iota')$ but that characterisation is very implicit. However one can characterise that quotient space is some cases – usually under certain regularity condition on the space – as a subspace of $ba(\mathcal{A})$. Usually one also has to work with other spaces of bounded continuous functions like the continuous functions vanishing at infinity. Since those results are somewhat technical to prove (but not necessarily hard) they will not be presented here.

elaborate and cite this!

Weak topologies on the space of probability measures

In probability theory it is common to consider the *weak topology* on the Borel probability measures $\mathcal{P}(\mathcal{A})$ over some topological space X with Borel algebra $\mathcal{A}$, which is the initial topology of the evaluations

$$f \mapsto \mu(f) = \int f \, \mathrm{d}\mu \quad \text{for } f \in \mathcal{C}_b(X).$$

Note that this corresponds exactly to the relative topology on the probability measures $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{C}_b(X)^*$ of the weak-$*$ topology, where $\mathcal{P}(\mathcal{A})$ is embedded canonically. If X is a complete metric space, then it is well known that the convergence that arrises from this topology, which is called the *weak convergence*, can be metrised by the dual bounded Lipschitz metric or the Lévy-Prokhorov metric. However this result is sometimes stated as the fact that the weak topology is metrised with by this metric (c.f. [Klenke, 2013]). Against the background of the metrisability results of the weak topologies in Banach spaces this seems very surprising and in fact it turns out to be false. More precisely the weak topology is metrisable if and only if the space X is compact, which is the case if and only if $\mathcal{C}_b(X)$ is separable. This agrees with the metrisability of the weak-$*$ topology. To prove this statement, we will need the following ingredrients.

(i) The space of probability measures over a topological space is compact in the weak topology.

(ii) If a familiity $\{\mu_i\}_{i\in I}$ of probability measures is relatively sequentially compact in the weak topology, then it is *tight*, i.e. for every $\varepsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu_i(K) > 1 - \varepsilon$ for all $i \in I$.

(iii) If X is a compact metric space, then $\mathcal{C}_b(X)$ is separable. Note that the converse statement is also true, but we will not need it here.

Assuming we have all those statements, then we can conclude as follows. If X is compact, then $\mathcal{C}_b(X)$ is seperable and thus the weak-$*$ topology on $\mathcal{C}_b(X)^*$ is metrisable. In this case also the relative topology, i.e. the weak topology on the probability measures is metrisable. If on the other hand the weak topology is metrisable, then we know because of (i) that the probability measures are sequentially compact. In particular the family of Dirac measures $\{\delta_x\}_{x\in X}$ is tight and thus there is a compact set $K \subseteq X$ such that

$$\delta_x(K) > \frac{1}{2}, \quad \text{i.e. } x \in K.$$

Hence we obtain that $X = K$ is compact. Since (ii) is well established in the literature (c.f. Prokhorov's theorem or Theorem 5.2 in [Billingsley, 2013]) and (iii) follows from the Stone-Weierstrass theorem (c.f. [Rudin, 1991]), we will focus on the proof of (i).

In order to do this, we remind the readers that haven't been in touch with probability theory that a measure is called *probability measure* if it is countably additive, positive and normed, i.e. gives measure 1 to the whole space. Thus the family of probability measures naturally is a subset of the space $ba(\mathcal{A})$ and is given by

$$\mathcal{P}(\mathcal{A}) = ca(\mathcal{A}) \cap \Big\{\mu \mid \mu(\chi_A) \geq 0 \text{ for all } A \in \mathcal{A}\Big\} \cap \big\{\mu \mid \mu(\chi_X) = 1\big\}. \tag{3}$$

Because of $\mathcal{P}(\mathcal{A}) \subseteq ba(\mathcal{A}) \cong L_b^0(\mathcal{A})^*$ it is natural to consider the topology on the probability measures that arises from the evaluations

$$f \mapsto \mu(f) \quad \text{for } f \in L_b^0(\mathcal{A}).$$

Since we have $\mathcal{C}_b(X) \subseteq L_b^0(\mathcal{A})$ this topology is finer than the weak topology and thus it suffices to prove compactness in this finer topology.

Just like in the case of the weak topology we note that this topology is nothing but the restriction of the weak-$*$ topology in $ba(\mathcal{A})$ onto $\mathcal{P}(\mathcal{A})$. Since probability measures satisfy $|\mu(f)| \leq \|f\|_\infty$, we see that $\mathcal{P}(\mathcal{A}) \subseteq ba(\mathcal{A})$ is bounded. In fact the choice $f = \chi_X$ shows that the probability measures lie on the sphere. By the famous Banach-Alaoglu theorem the closed unit ball $B \subseteq ba(\mathcal{A})$ is compact with respect to the weak-$*$ topology. Since we have $\mathcal{P}(\mathcal{A}) \subseteq B$, we only have to show that $\mathcal{P}(\mathcal{A})$ is closed in the weak-$*$ topology to obtain that the probability measures are compact with respect to the weak-$*$ topology in $ba(\mathcal{A})$. To show this, we will use the representation (3) and we note that

$$\Big\{\mu \mid \mu(\chi_A) \geq 0 \text{ for all } A \in \mathcal{A}\Big\} = \bigcap_{A\in\mathcal{A}} \{\mu \mid \mu(\chi_A) \geq 0\}$$

and also

$$\{\mu \mid \mu(\chi_X) = 1\}$$

is closed in the weak-$*$ topology since they are preimages (or intersections of those) of closed sets. Hence it remains to show that $ca(\mathcal{A}) \subseteq ba(\mathcal{A})$ is weak-$*$ closed and thus we also seek a representation in terms of preimages of closed sets. Recall that μ is countably additive if and only if for all disjoint unions $A = \bigcup_{n\in\mathbb{N}} A_n$ we have

$$\mu(A) = \lim_{N\to\infty} \mu(A^N) \tag{4}$$

where $A^N := \bigcup_{n=1}^N A_n$. If we fix such a disjoint union, then the measures that satisfy (4) are exactly given by

$$\bigcap_{\varepsilon>0} \liminf_{N\to\infty} \left\{ \left|\mu(A) - \mu(A^N)\right| \leq \varepsilon \right\}. \tag{5}$$

We note that each of the sets in the intersection is closed in the weak-$*$ topology, since

$$\left\{ \left|\mu(A) - \mu(A^N)\right| \leq \varepsilon \right\} = \left\{ \mu(\chi_{A\setminus A^N}) \in [-\varepsilon, \varepsilon] \right\}.$$

We now note that $ca(\mathcal{A})$ is nothing but the intersection of (5) over all disjoint unions and therefore closed in the weak-$*$ topology. Hence we have proved the following results, where we note that we didn't use the metric structure in the proofs of the first two statements.

11 Theorem (Weak-$*$ closedness of $ca(\mathcal{A})$). *The subspace of countably additive measures $ca(\mathcal{A}) \subseteq ba(\mathcal{A}) \cong L_b^0(X)^*$ is closed in the weak-$*$ topology.*

12 Corollary. *The set of probability measures $\mathcal{P}(\mathcal{A}) \subseteq ba(\mathcal{A}) \cong L_b^0(\mathcal{A})^*$ is bounded and closed in the weak-$*$ topology and hence compact with respect to the the weak-$*$ topology.*

13 Corollary (Weak compactness of probability measures). *Let X be a topological space and let $\mathcal{P}(\mathcal{A})$ be the set of Borel probability measures. Then $\mathcal{P}(\mathcal{A})$ is compact with respect to the weak topology.*

14 Corollary (Metrisability of the weak topology). *Let X be a metric space and let $\mathcal{P}(\mathcal{A})$ be the set of Borel probability measures. Then the weak topology is metrisable if and only if X is compact.*

15 Remark. The assumption of the previous Corollary can be weakened. For example it is enough to assume that X is a topological space in where two points can be separated by a continuous bounded function, i.e. if for $x, y \in X$ there is $f \in \mathcal{C}_b(X)$ such that $f(x) \neq f(y)$. However in this case metrisability of the weak topology only implies sequential compactness of the space X.

Conclusion

We have seen that the Hahn-Jordan decomposition also holds for only finitely additive signed measures. To the authors knowledge there is no reference so far for this in the literature. Further we developed a duality between a function space and the Banach space of signed measures and proved that countable additivity is equivalent to the dominated or monotone convergence theorem. This duality provides an extremely short and self contained introduction to signed measures that avoids the mostly tedious calculation that are usually involved in the proof of the norm property of the total variation and the completeness of the space of signed measures.

In the last section it was shown that the set of probability measures is compact with respect to the weak topology. Further the motivating question was answered, namely it was the goal to investigate whether the Prokhorov metric metrises the weak topology like frequently stated or only the weak convergence (c.f. [Billingsley, 2013]).In fact the weak topology of probability measures over a complete and separable metric space is metrisable if and only the space is compact and thus only the weak convergence is metrised. However this is no contradiction but shows that the weak topology has no countable neighborhood basis and is therefore not uniquely specified by its notion of convergence.

cite

References

[Billingsley, 2013] Billingsley, P. (2013). *Convergence of probability measures*. John Wiley & Sons.

[Klenke, 2013] Klenke, A. (2013). *Probability theory: a comprehensive course*. Springer Science & Business Media.

[Rudin, 1991] Rudin, W. (1991). Functional analysis. 1991. *Internat. Ser. Pure Appl. Math*.