Weak compactness of probability measures

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Setting and definitions

Let $(X, \mathcal{A})$ be a measurable space and let $\mu \colon \mathcal{A} \to \mathbb{R}$. Then we say μ is

(i) *finitely additive* if we have

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Denote the space of bounded and finitely additive measures by $ba(\mathcal{A})$ and the space of bounded and countably additive measures by $ca(\mathcal{A})$.

Hahn-Jordan decomposition

Theorem (Hahn-Jordan decomposition)

For every signed measure $\mu \in ba(\mathcal{A})$ *there are two positive measures* $\mu_+, \mu_- \in ba(\mathcal{A})$ *such that* $\mu = \mu_+ - \mu_-$. *If* μ *is countably additive then* μ_+ *and* μ_- *can be chosen to be countably additive as well.*

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(*iii*) Choose now $\mu_- := \mu_+ - \mu$ to obtain the desired decomposition.

(*iv*) Use dominated convergence to show that μ_+ is countably additive if μ is.

□

Integration with respect to a signed measure

Theorem (Integral)

For every signed measure $\mu \in ba(\mathcal{A})$ there is a unique linear and continuous mapping

$$\int \mathrm{d}\mu \colon L_b^0(\mathcal{A}) \to R$$

called the integral wrt to μ such that we have

$$\int \sum_{k=1}^{N} \alpha_i \chi_{A_i} \mathrm{d}\mu := \sum_{k=1}^{N} \alpha_i \mu(A_i)$$

for all simple functions. Further we have

$$\left\| \int \mathrm{d}\mu \right\|_{L_b^0(\mathcal{A})'} \leq \|\mu\|_{BV} .$$

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Characterisation of measures as a dual space

Theorem (Measures as duals)

With the above notations the linear mapping

$$I\colon ba(\mathcal{A}) \to L_b^0(\mathcal{A})^*, \quad \mu \mapsto I\mu := \left(f \mapsto \int f \,\mathrm{d}\mu\right)$$

is an algebraic isomorphism and we have $\|I\mu\|_{L_b^0(\mathcal{A})^*} = \|\mu\|_{BV}$.
Further the inverse I^{-1} *is given by*

$$F \mapsto \big(A \mapsto F(\chi_A)\big).$$

For a measure $\mu \in ba(\mathcal{A})$ *the following three statements are equivalent:*

(i) μ *is countably additive.*

(ii) $I\mu$ *fulfills the monotone convergence theorem.*

(iii) $I\mu$ *fulfills the dominated convergence theorem.*

Consequences

Corollary

The quantity $\|\cdot\|_{BV}$ is a norm and $ba(\mathcal{A})$ is complete with respect to it. The space of countably additive signed measures is a Banach space with respect to the norm of total variation.

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Corollary

The quantity $\|\cdot\|_{BV}$ *is a norm and* $ba(\mathcal{A})$ *is complete with respect to it. The space of countably additive signed measures is a Banach space with respect to the norm of total variation.*

Proof.

(*i*) The first statement is an immediate consequence of the previous theorem.

(*ii*) For the second one, show that strong limits of countably additive signed measures satisfy the dominated convergence theorem.

□

Characterisation of $L^\infty(\mu)^*$ and further duals

Theorem (Dual of $L^\infty(\mu)^*$)

Let $\mu \in ba(\mathcal{A})$ and let $ba(\mu) \subseteq ba(\mathcal{A})$ be the subspace of measure that are absolutely continuous wrt μ. Then we have

$$ba(\mu) \cong L^\infty(\mu)^*$$

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Proof.

Use the surjection

$$\iota : L_b^0(\mathcal{A}) \to L^\infty(\mu), \quad f \mapsto [f]_\mu$$

and the property $\iota(B_1) = B_1$. □

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