Single and Multi Qubit States

Muhammad Faryad

Associate Professor of Physics,

Lahore University of Management Sciences

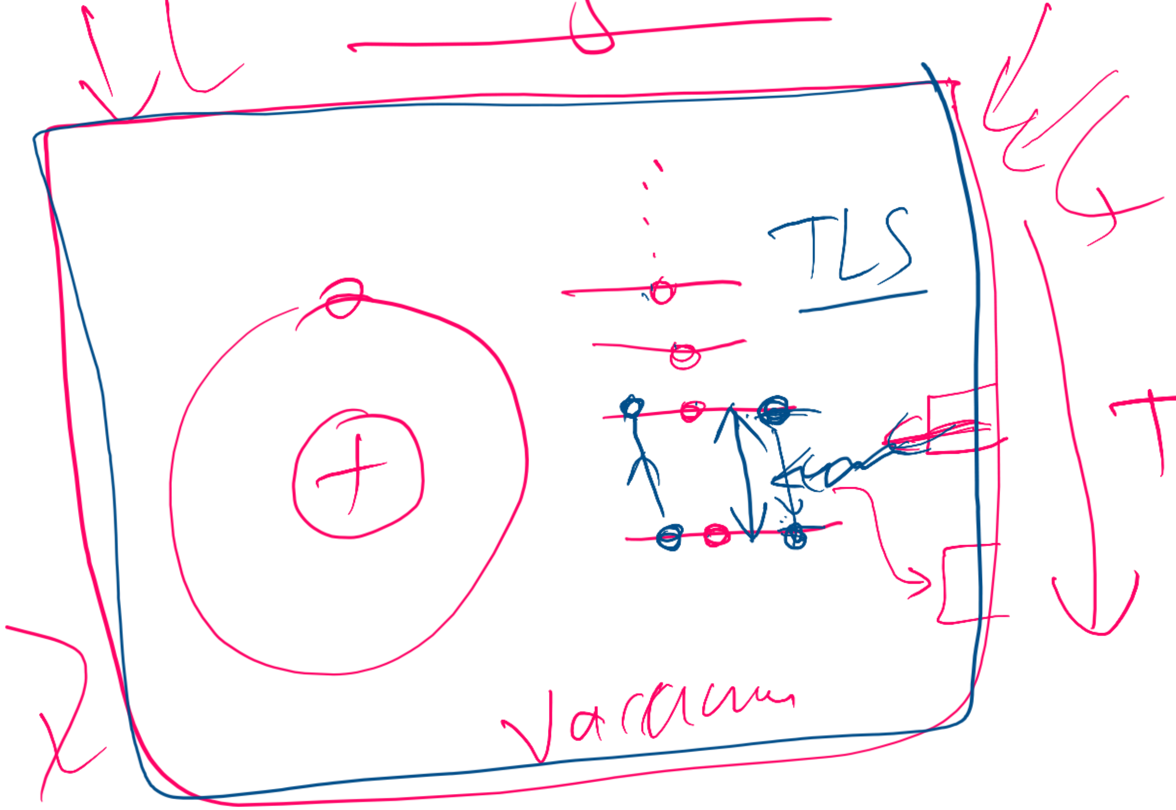
Topics:

1. Quantum computer basics
2. Single qubit states, Bloch sphere
3. Multi qubit states

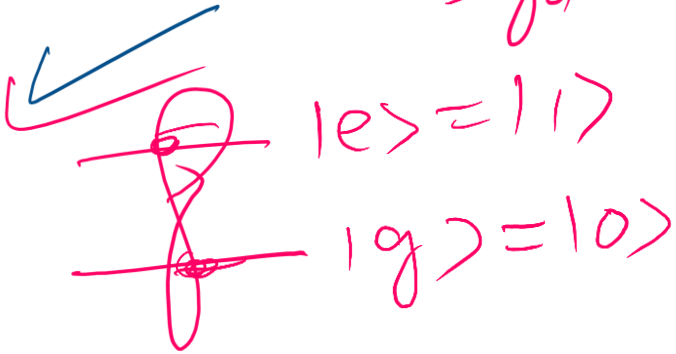
Quantum computer is a collection of qubits and ..

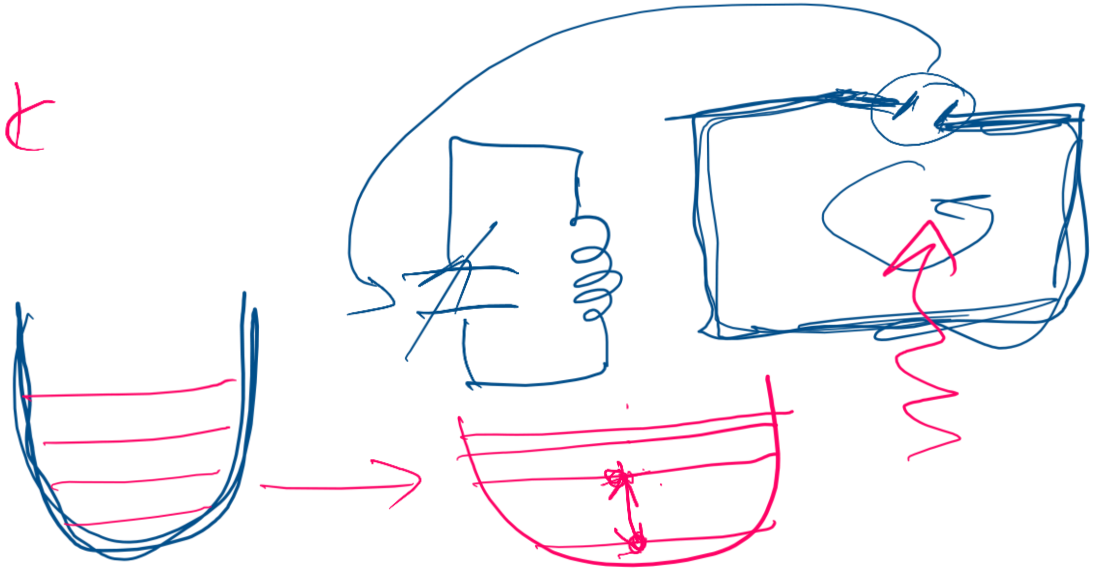
Two-level systems are idealizations

qubit ≡ Two-level system



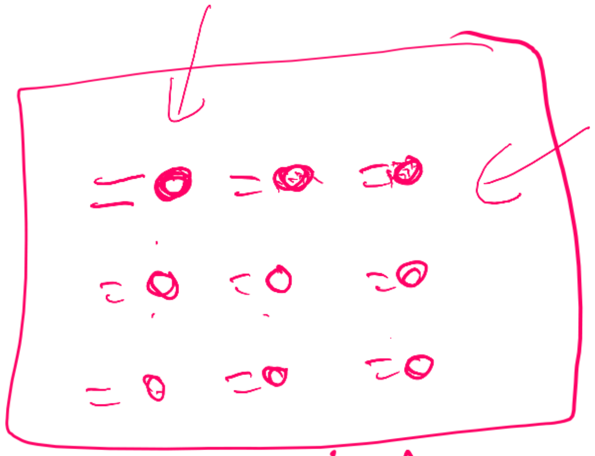
Two ok



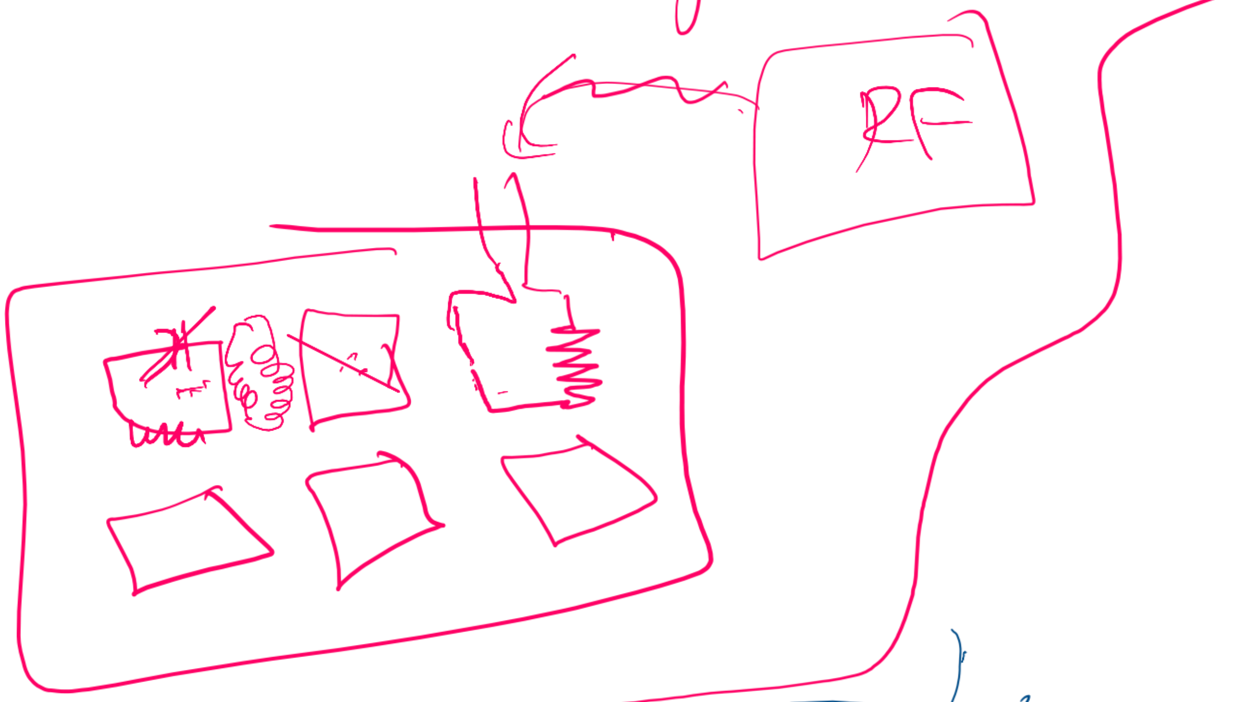


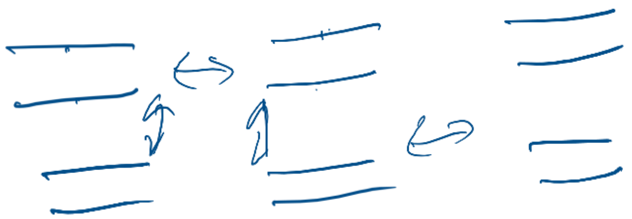
Q.C. is a collection of qubits with controlled interaction, s.t. qubits are isolated from the env.

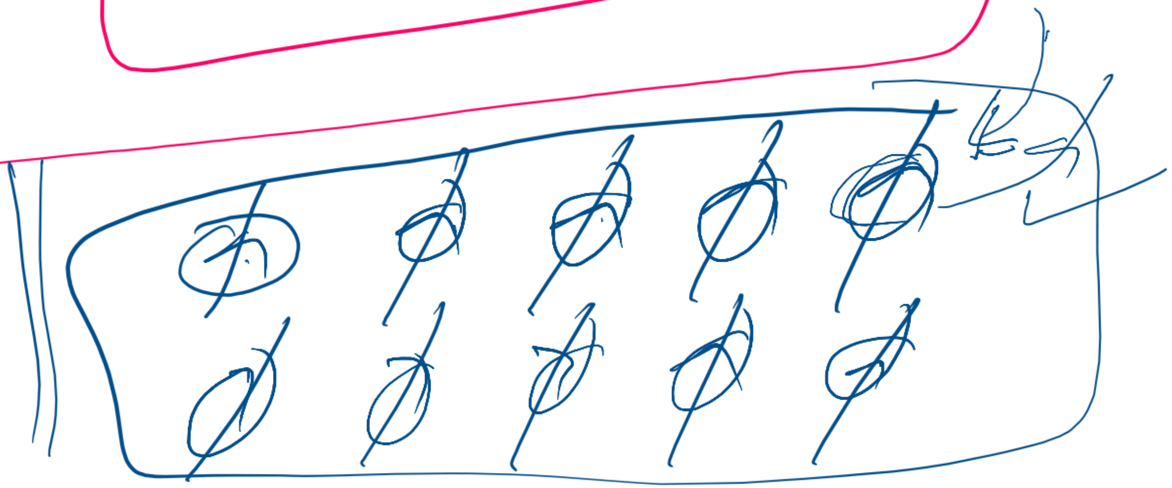
Trapped ion / atom



ions / atom







Qubit: State $|\psi\rangle = a|0\rangle + b|1\rangle$, $a, b \in \mathbb{C}$

$$\boxed{|a|^2 + |b|^2 = 1}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow |\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a|0\rangle + b|1\rangle$$

→ Basis: ① $\{|0\rangle, |1\rangle\}$, $\langle 0|1\rangle = (\langle 0|)(|1\rangle) = (|0\rangle)^{\dagger}(|1\rangle)$

$$\boxed{\langle 1|0\rangle = 0}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{T}\right)^{*} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\langle 1|1\rangle = 1$, $\langle 0|0\rangle = 1$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}^{*} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 + 0 = 0$$

② $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\langle -|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 - 1 \end{bmatrix} = 0$$

$$\langle +|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}^{\dagger}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}(1+1) = \frac{2}{2} = 1$$

$$\langle -|-\rangle = \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}(1+1) = 1$$

$\Rightarrow \underline{\{|+\rangle, |-\rangle\}}$ Hadamard basis!

$$|\psi\rangle = \underline{\alpha}\,\underline{|+\rangle} + \underline{\beta}\,\underline{|-\rangle}$$

$$|\psi\rangle = \underline{a}\,|0\rangle + \underline{b}\,|1\rangle$$

$=$ $\cancel{8}$ example: $\underline{|\psi\rangle} = \underline{\frac{1}{\sqrt{3}}}\underline{|0\rangle} + \underline{\sqrt{\frac{2}{3}}}\underline{|1\rangle}$

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right) + \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right)$$

$$= \frac{1}{\sqrt{6}}(\underline{|+\rangle} + \underline{|-\rangle}) + \frac{1}{\sqrt{3}}(\underline{|+\rangle} - \underline{|-\rangle}) = \underline{\left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right)}\underline{|+\rangle} + \underline{\left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\right)}\underline{|-\rangle}$$

still another basis:

$$|+i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

$$\langle +i|-i\rangle = 0$$

$$\langle +i|+i\rangle = 1$$

$$\langle -i|-i\rangle = 1$$

$$\{|+i\rangle, |-i\rangle\}$$

basis

General state:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$3+4i = \sqrt{9+16}\; e^{i\tan^{-1}\frac{4}{3}}$$

$$= r_a e^{i\theta_a}$$

$$a' + i a'' = r_a e^{i\theta_a}$$

$$|\psi\rangle = r_a e^{i\theta_a}|0\rangle + r_b e^{i\theta_b}|1\rangle$$

$$|a|^2 + |b|^2 = 1 \Rightarrow \boxed{r_a^2 + r_b^2 = 1}$$

$$a = r_a e^{i\theta_a} \Rightarrow a^* = r_a e^{-i\theta_a}$$

$$|a|^2 = a a^* = r_a^2$$

$$|\psi\rangle = r_a e^{i\theta_a}|0\rangle + r_b e^{i\theta_b}|1\rangle$$

$a, b \in \mathbb{C}$

$\boxed{|a|^2+|b|^2=1}$

$e^{i\theta} \rightarrow$ phase

$$= e^{i\theta_a}\left[r_a|0\rangle + r_b e^{i(\theta_b-\theta_a)}|1\rangle \right]$$

global phase

relative phase

$$= e^{i\theta_a}\left[r_a|0\rangle + r_b e^{i\phi}|1\rangle \right] = e^{i\theta_a}\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle\right)$$

$$\boxed{r_a^2 + r_b^2 = 1}$$

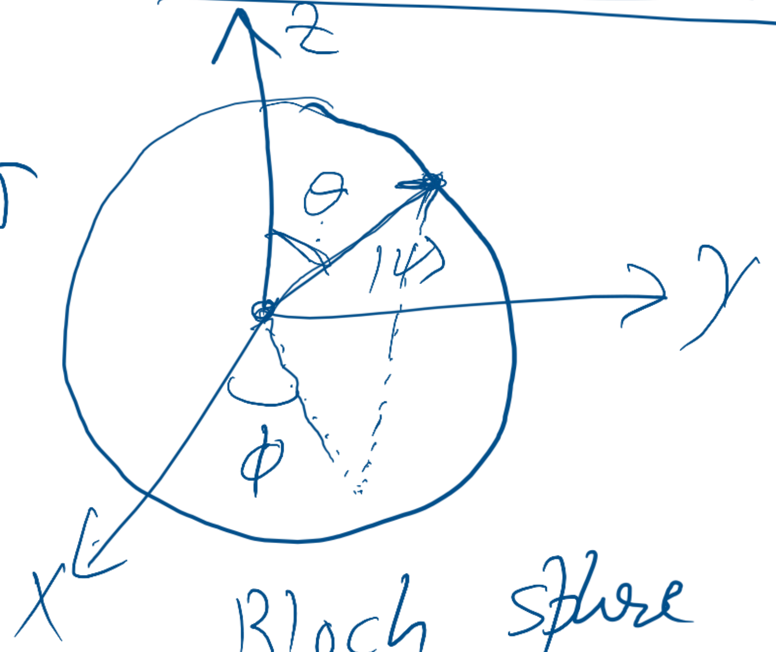
$$r_a = \sqrt{1 - r_b^2}$$

$$r_a = \cos\frac{\theta}{2} \neq \cos\theta$$

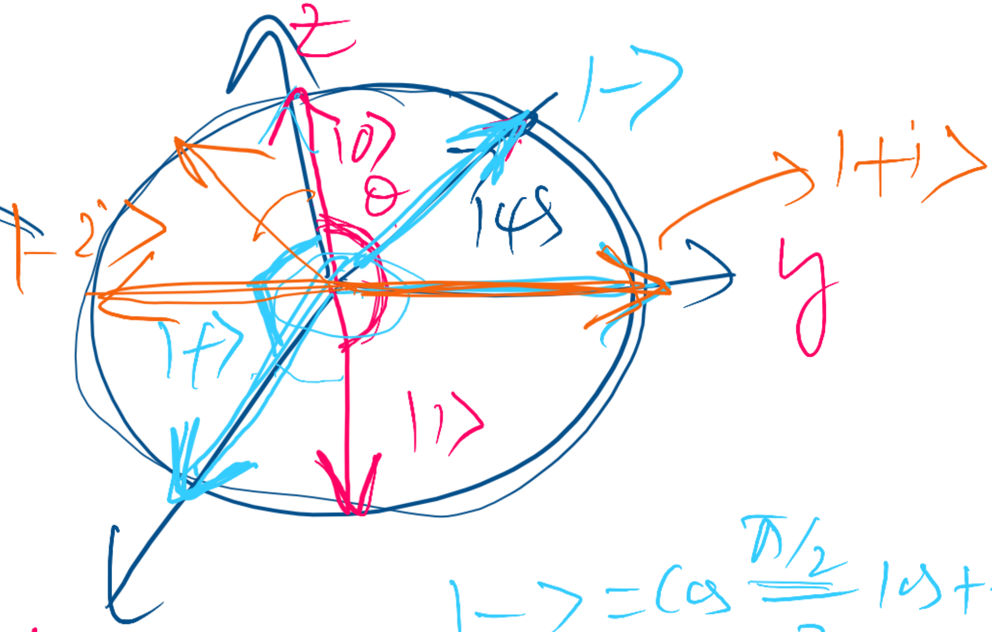
$$r_b = \sin\frac{\theta}{2}$$

$0 \le \phi \le 2\pi$

$0 \le \theta \le \pi$



Bloch sphere



$$|0\rangle = \left(\cos\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$|1\rangle = \cos\frac{\pi}{2}|0\rangle + e^{i\phi}\sin\frac{\pi}{2}|1\rangle \quad \times$$

θ

$$\theta = \pi, \ \phi$$

$$|+\rangle = \cos\frac{\pi/2}{2}|0\rangle + e^{i0}\sin\frac{\pi/2}{2}|1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\boxed{\begin{array}{c}\theta = \frac{\pi}{2} \\ \hline \phi = 0\end{array}}$$

$$|-\rangle = \cos\frac{\pi/2}{2}|0\rangle + e^{i\pi}\sin\frac{\pi}{4}|1\rangle$$

$$\theta = \pi/2$$

$$\phi = \pi$$

$$|+i\rangle \to \theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}$$

$$|-i\rangle \to \theta = \pi/2, \phi = 3\pi/2$$

Multi qubit states

observable state possibilities:

$|0\rangle|0\rangle$

$|0\rangle|1\rangle$

$|1\rangle|0\rangle$

$|1\rangle|1\rangle$

$|11\rangle = |1\rangle \otimes |1\rangle$

$|1\rangle|1\rangle = |11\rangle$

$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$|\psi\rangle = a\,|0\rangle|0\rangle + b\,|0\rangle|1\rangle + c\,|1\rangle|0\rangle + d\,|1\rangle|1\rangle$$

$$\boxed{|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1}$$

$$|0\rangle|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$|0\rangle|1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$|1\rangle|0\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad |1\rangle|1\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\langle 00|01\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}^\dagger \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0$$

$$|\psi\rangle = \begin{bmatrix}a\\b\\c\\d\end{bmatrix} = a\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$|\psi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

1- **Three and higher qubits:** $\{|0\rangle|0\rangle|0\rangle, |0\rangle|0\rangle|1\rangle, \dots\}$

8 basis states

2- **many choices for basis:**

$\{|0\rangle|+\rangle, |0\rangle|-\rangle, |1\rangle|+\rangle, |1\rangle|-\rangle\}$

$$(|0\rangle|+\rangle)^\dagger |0\rangle|-\rangle$$

$$= (\langle 0|\langle +|)(|0\rangle|-\rangle)$$

$$= \langle 0|0\rangle \langle +|-\rangle$$

$$= 1 \times 0$$

$$= 0$$

$$|0\rangle|+\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

$$|0\rangle|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$$

$$(|0\rangle|+\rangle)^\dagger(|0\rangle|-\rangle) = \frac{1}{2}(1 \;\; 1 \; 0 \; 0)\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$$

$$= \frac{1}{2}[1-1+0+0] = 0$$

Another basis: $\{|+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|+\rangle\}$

Summary

- Two-level systems are idealizations, and are also called qubits
- Quantum computers are a collection of qubits with the ability to manipulate them in a controlled way
- Single qubit states live in a Hilbert space of dimension 2 that can be mapped to Bloch sphere up to a global phase
- Multi qubit basis states can be constructed as tensor products of single qubit basis states