# Basic Linear Algebra for Quantum Computing

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**Topics:**

1. Vector spaces
2. Hilbert spaces
3. Dimension, basis, tensor products, orthonormality

# Vector spaces are sets...

$V = \{ |v_1\rangle, |v_2\rangle, \cdots \}$

$\mathbb{C} = \{ a, b, c, \cdots \}$

$a = 2 + 3i$

**Axioms:**

1. Closure w.r.t. add

$|v_1\rangle + |v_2\rangle = |v_3\rangle \in V$

2. Commutative

$|v_1\rangle + |v_2\rangle = |v_2\rangle + |v_1\rangle$

3. $\exists \ |N\rangle \in V$

$|N\rangle + |v_1\rangle = |v_1\rangle$

$|v\rangle + (-|v\rangle) = |N\rangle$

4. Associtative

$|v_1\rangle + (|v_2\rangle + |v_3\rangle) = (|v_1\rangle + |v_2\rangle) + |v_3\rangle$

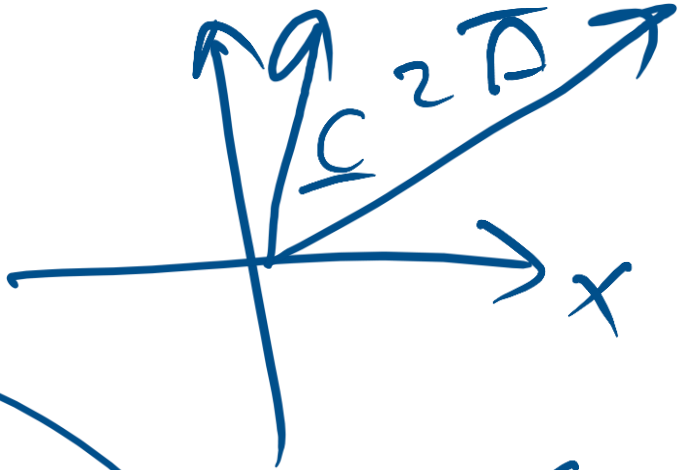
5. $a|v\rangle \in V$

6. Distributive

$a(|v_1\rangle + |v_2\rangle) = a|v_1\rangle + a|v_2\rangle$

7. $a(b|v\rangle) = ab|v\rangle$

- $I|v\rangle = |v\rangle$
- $0|v\rangle = 0$



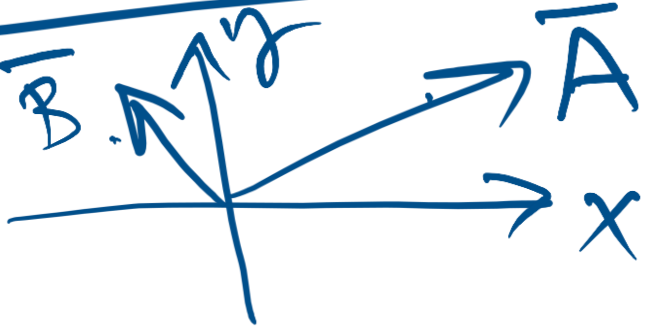

3D space
- a vector space

Summarized:

$$\boxed{a|v_1\rangle + b|v_2\rangle \in V}$$

Closure          vector space!

Example: 2D Euclidean space




$V = \{\bar{A}\}$

$S = \{\mathbb{R}\}$

Hilbert spaces are vector spaces with... an additional requirement: — It has inner product defined

example: $\bar{A} \cdot \bar{B} = A_x B_x + A_y B_y$

$$V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix}_{2\times 1} \right\} \quad a, b \in \mathbb{C}$$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\langle\psi| = (|\psi\rangle)^* = |\psi\rangle^\dagger =$$

$$\langle\psi|\phi\rangle = \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = a^* c + b^* d$$

Hilbert spaces: $\rightarrow$ inner products $\rightarrow$ norm of vector

$$\text{norm} = \sqrt{\bar{A}\cdot\bar{A}} = \sqrt{A_x^2 + A_y^2}$$

$$\text{norm} = \sqrt{\langle\phi|\phi\rangle} = \sqrt{\begin{bmatrix} c^* & d^* \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}} = \sqrt{|c|^2 + |d|^2}$$

$$= \sqrt{c^* c + d^* d}$$

$c = 2+3i$

$c^* = 2-3i$

$$\boxed{\langle\psi|\phi\rangle \neq \langle\phi|\psi\rangle}$$

$\langle\psi|\phi\rangle \neq$ real

$$\boxed{\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*}$$

$c c^* = |c|^2$

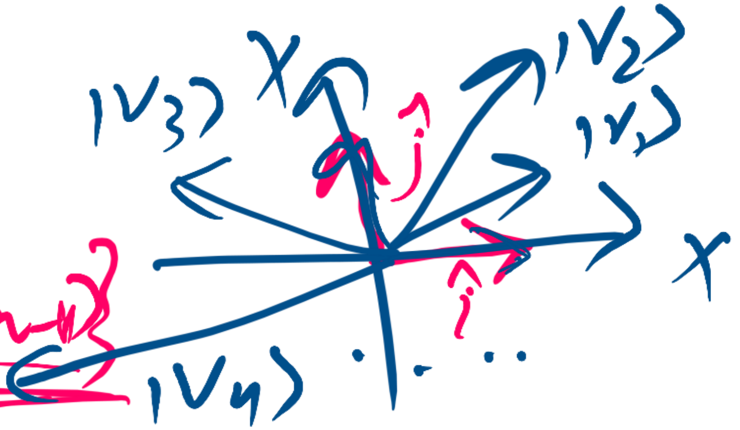
$= 4+9 = 13$

$\langle\psi|\psi\rangle =$ real

# Basis is the linearly independent set...

$\underline{V} = \{ |V_1\rangle, |V_2\rangle, \ldots \} \rightarrow \text{Basis} = \{ |w_0\rangle, |w_1\rangle, \ldots |w_{N-1}\rangle \}$

$N$ members




$|V\rangle = a\hat{i} + b\hat{j}$

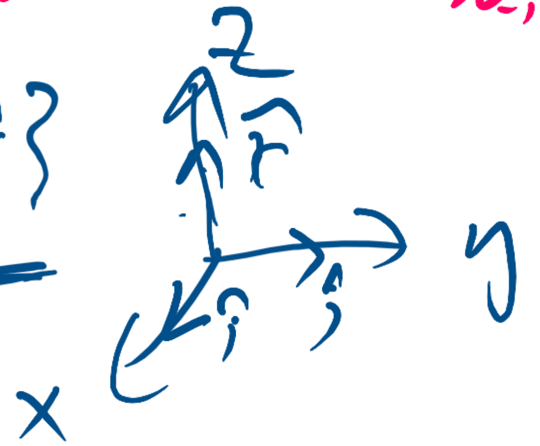
$\boxed{\text{Basis} = \{ \hat{i}, \hat{j} \}}$

$|V_i\rangle = \sum_{j=0}^{N-1} c_j |w_j\rangle$

If: $|w_i\rangle \neq \sum_{j=0, j \neq i}^{N-1} c_j |w_j\rangle = c_0|w_0\rangle + c_1|w_1\rangle \ldots + c_{N-1}|w_{N-1}\rangle$

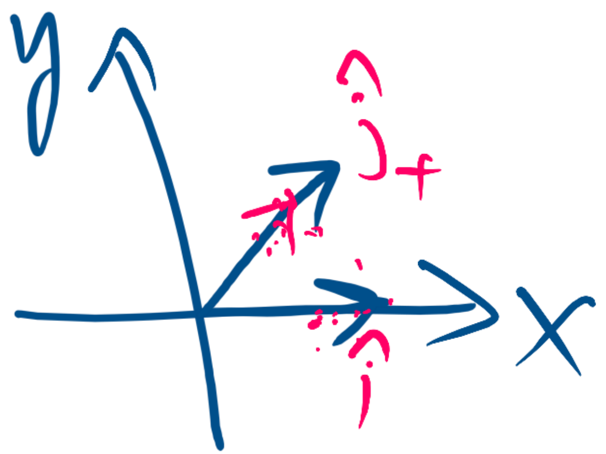
Linearly independent set

$\{ \hat{i}, \hat{j}, \hat{k} \}$



Basis does not have to have orthogonal vectors:

$\{\hat{i}, \hat{j}_f\}$ $\quad \hat{i} \neq c\,\hat{j}_f$




Orthogonal: $\langle\psi|\phi\rangle = 0$

$|\psi\rangle, |\phi\rangle$ are orthogonal!

$\{|w_0\rangle, |w_1\rangle, |w_2\rangle\}$

A basis with orthogonal vectors is preferred but not necessary!

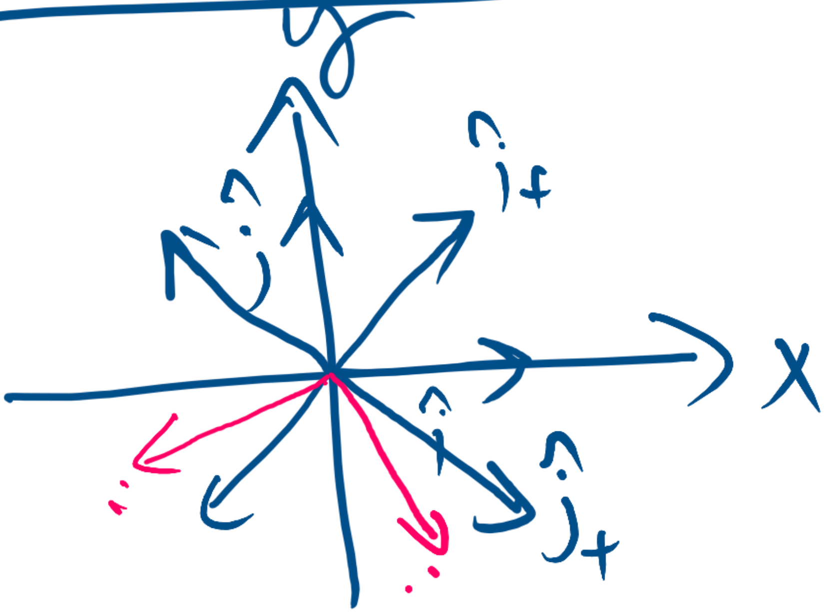
Normal vectors: $\sqrt{\langle\psi|\psi\rangle} = 1$

## Orthonormal basis:

→ All vectors in it are orthogonal

and

→ All vectors have norm 1.




## Dimension:

number of elements in a basis

$$V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \right\}$$

$$\text{Dim} = 2$$

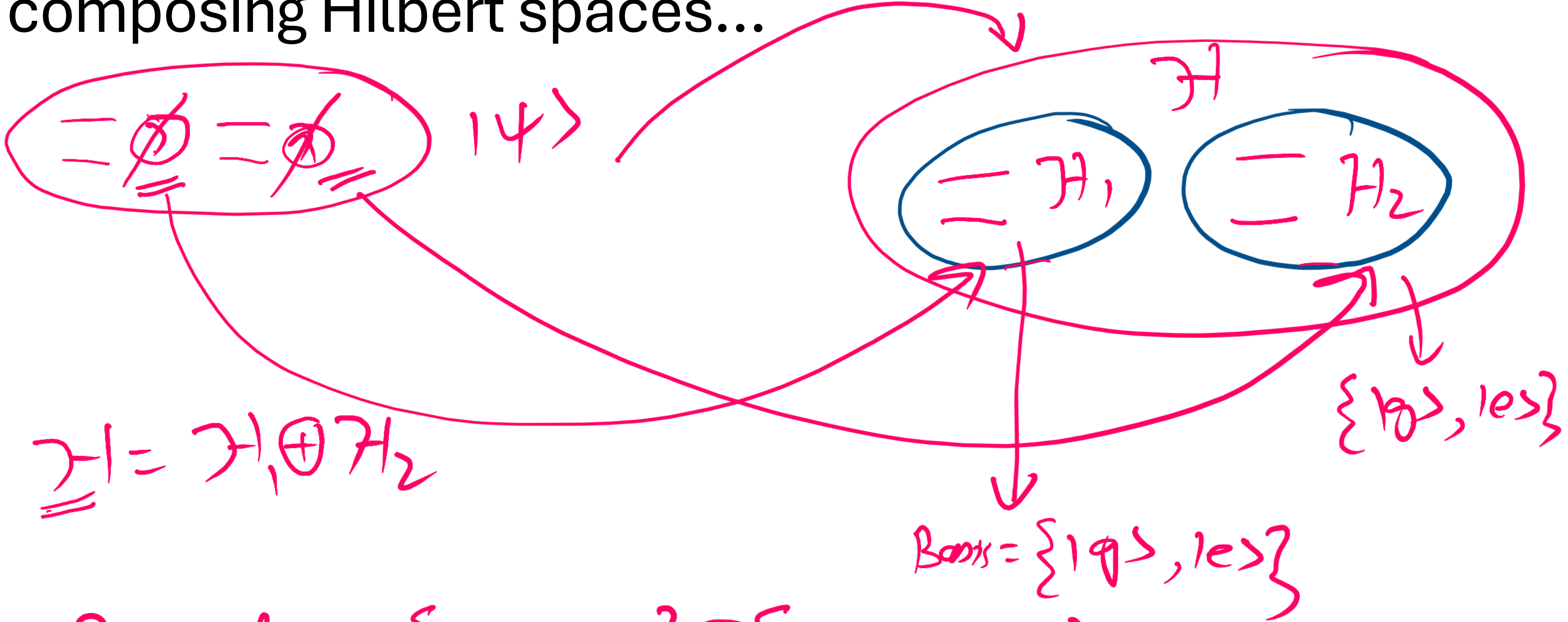
$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

## 2D C.S.

$$\text{Dim} = 2$$

# Basis of big Hilbert space is tensor product of bases of composing Hilbert spaces...




$$\text{Basis of } \mathcal{H} = \{|g\rangle, |e\rangle\} \otimes \{|g\rangle, |e\rangle\}$$

$$= \{|g\rangle \otimes |g\rangle, |g\rangle \otimes |e\rangle, |e\rangle \otimes |g\rangle, |e\rangle \otimes |e\rangle\}$$

# Examples

$|g\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \ |e\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

$\{|g\rangle, |e\rangle\}$ $\quad$ $\{|g\rangle, |e\rangle\}$

$|\psi_1\rangle = a|g\rangle + b|e\rangle = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$

$= \begin{bmatrix}a\\b\end{bmatrix}$

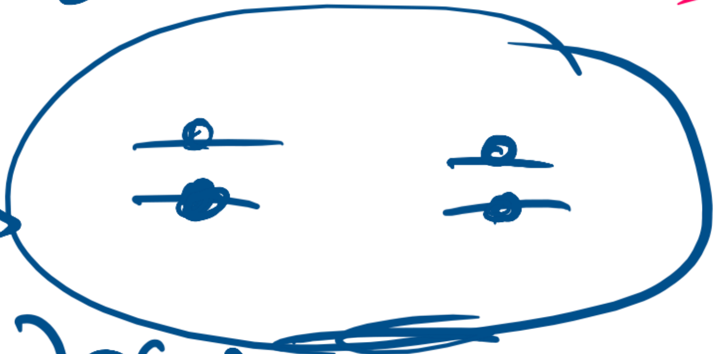
$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

Dim = 2 $\qquad$ Dim = 2

$|\psi_2\rangle = c|g\rangle + d|e\rangle$

$= \begin{bmatrix}c\\d\end{bmatrix}$

Basis = $\{ |g\rangle \otimes |g\rangle, \ |g\rangle \otimes |e\rangle, \ |e\rangle \otimes |g\rangle,$
$|e\rangle \otimes |e\rangle$



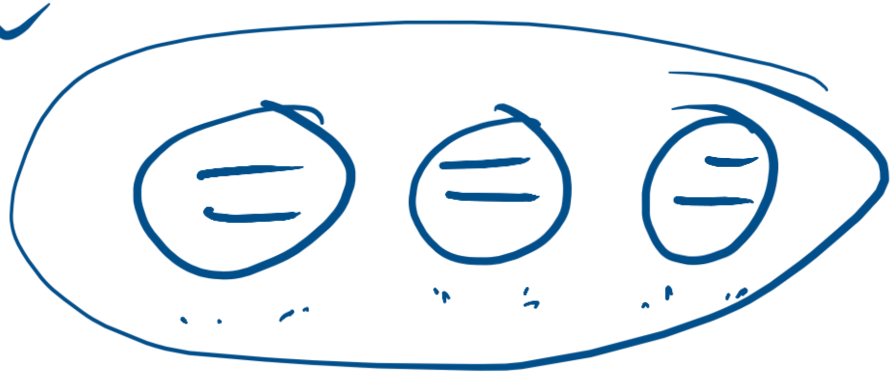
$|g\rangle \otimes |g\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad |g\rangle|e\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

$$|e\rangle|g\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |e\rangle|e\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$= a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \cdots$$

$$\text{Basis} = \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \underset{|g\rangle|e\rangle}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{|e\rangle|g\rangle}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{|e\rangle|e\rangle}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\}}$$

$$\text{Dim} = 4 = 2 \times 2 = 2^2$$



$$2^3 = 8$$

# Summary

- Vector spaces are sets with addition and scalar multiplication
- Hilbert spaces have additional structure of inner products
- Basis is the linearly independent set to represent vectors
- Composite Hilbert spaces have bases that can be constructed as tensor products of bases of individual Hilbert spaces
- Orthogonal vectors have inner product of zero
- Normal vectors have zero norm