Introduction to Quantum Gates

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Topics:

1. Single qubit quantum gates
2. Pauli X, Y, Z, and Hadamard H gates
3. Two-qubit gates, CX

Quantum gates are controlled operations on qubits

Qubits → single particle → $\begin{bmatrix} a \\ b \end{bmatrix}$

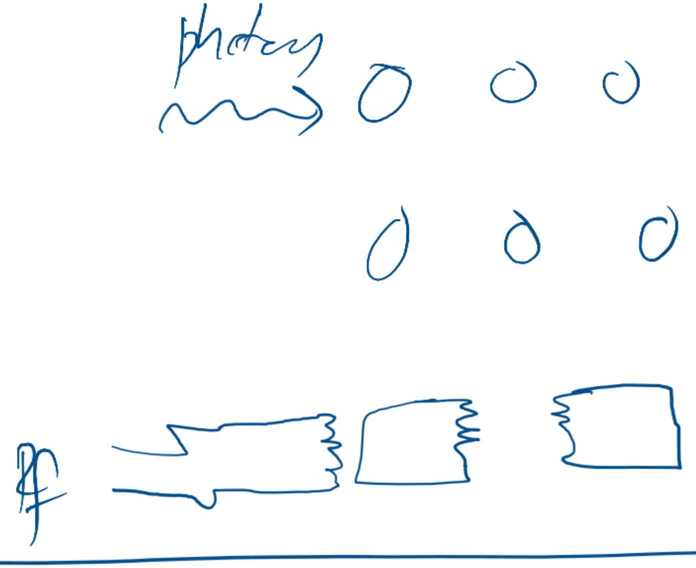
$\{|0\rangle, |1\rangle\}$ computational basis

$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix} = |1\rangle$

$$\begin{bmatrix}a\\b\end{bmatrix}_{2\times1} = \begin{bmatrix} \underline{U}_{2\times2}\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}_{2\times1}$$

$\boxed{U^\dagger = U^{-1}}$, $U^\dagger U = I$

$UU^\dagger = I$



$U^\dagger = (U^T)^*$

$U = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$, $U^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^*\end{bmatrix}$

$a = 2+3i$, $a^* = 2-3i$

Single qubit gates

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = a|0\rangle + b|1\rangle$$

$$|\psi'\rangle = U|\psi\rangle$$

$$\Rightarrow \langle\psi|\psi\rangle = (|\psi\rangle)^{\dagger}|\psi\rangle$$

$$= \begin{bmatrix} a \\ b \end{bmatrix}^{\dagger} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= |a|^2 + |b|^2 = 1$$

$$\langle\psi'|\psi'\rangle = (|\psi'\rangle)^{\dagger}|\psi'\rangle$$

$$= (U|\psi\rangle)^{\dagger}\, U|\psi\rangle$$

$$= (|\psi\rangle)^{\dagger}\, U^{\dagger}U\, |\psi\rangle$$

$$= \langle\psi|\, I\, |\psi\rangle = \langle\psi|\psi\rangle$$

$$\boxed{\langle\psi'|\psi'\rangle = 1}$$ ✓

Unitary operators preserve norm.

Examples:

1. Not gate | X gate

$|0\rangle \longrightarrow |1\rangle$

$|1\rangle \longrightarrow |0\rangle$

$$\boxed{X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$|1\rangle = X|0\rangle$

$|0\rangle = X|1\rangle$

$\boxed{\sigma_x}$ $\boxed{\sigma_x}$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

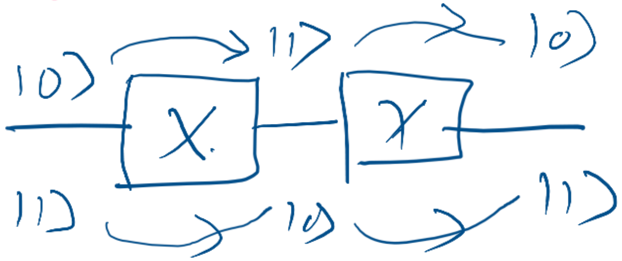
$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

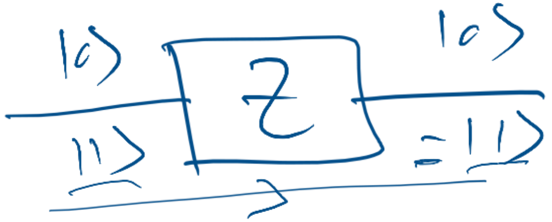
$$X^\dagger X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\boxed{X^2 = I}$$



Z gate:



, $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$, $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

$|0\rangle \rightarrow |0\rangle$

$|1\rangle \rightarrow -|1\rangle$

$$\boxed{Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}$$

$$Z|0\rangle = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \begin{bmatrix}1+0\\0-0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$= |0\rangle$$

$$Z|1\rangle = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{bmatrix}0\\-1\end{bmatrix} = -\begin{bmatrix}0\\1\end{bmatrix}$$

$$\boxed{Z|1\rangle = -|1\rangle}$$

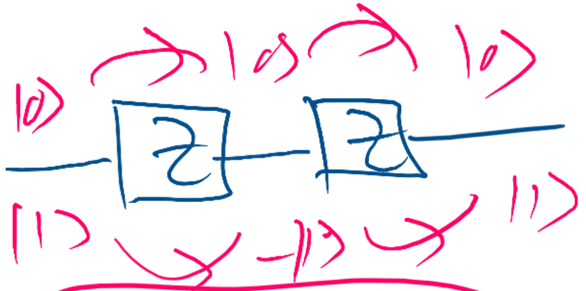
— Z gate = phase flip gate

$$Z^\dagger = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = Z$$

$$Z^\dagger Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$Z^2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

— $\boxed{H^\dagger = H}$ Hermitian operator



$$\boxed{Z^2 = I}$$

Pauli Y gate:

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = Y|0\rangle$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0 \\ i \end{pmatrix} = i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{Y|0\rangle = i|1\rangle}$$

$$i = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$Y|1\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} -i \\ 0 \end{bmatrix}$$

$$= -i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{Y|1\rangle = -i|0\rangle}$$

Phase-flip gate

$$Y^{\dagger} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y, \quad \boxed{Y^2 = I}$$

X	Not gate
Z	Phase ~~flip~~
Y	phase-flip gate
I	Identity gate

$\{I, X, Y, Z\}$ Pauli gate set

$$\boxed{U = aI + bX + cY + dZ}$$

Hadamard gate:

$$\boxed{H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

$$|0\rangle, |1\rangle \to \boxed{H} \to |+\rangle, |-\rangle \to \boxed{H} \to |0\rangle, |1\rangle$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

$$\boxed{H|0\rangle = |+\rangle} \qquad \boxed{H^2 = I}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\boxed{H|1\rangle = |-\rangle}$$

$$H|+\rangle = \frac{1}{\sqrt{2}} H\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(H|0\rangle + H|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|0\rangle + \cancel{|1\rangle}\right) + \frac{1}{\sqrt{2}}\left(|0\rangle - \cancel{|1\rangle}\right)\right)$$

$$= |0\rangle$$

Two qubit gates

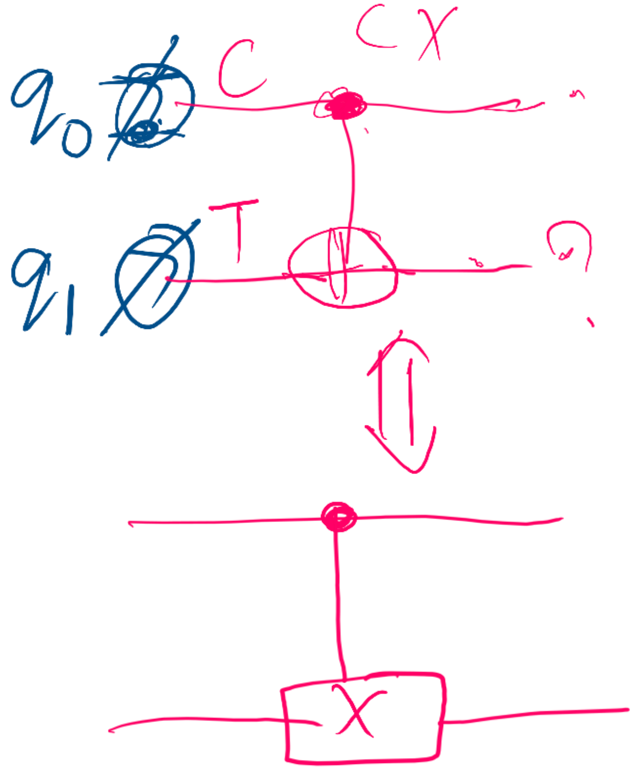
control-x gate

CX gate

$$|\psi\rangle = a|0\rangle|0\rangle + b|0\rangle|1\rangle + c|1\rangle|0\rangle + d|1\rangle|1\rangle$$

$$U|\psi\rangle = aU|0\rangle|0\rangle + bU|0\rangle|1\rangle + cU|1\rangle|0\rangle + dU|1\rangle|1\rangle$$

$$|\psi'\rangle = a|0\rangle|0\rangle + b|0\rangle|1\rangle + c|1\rangle|1\rangle + d|1\rangle|0\rangle$$



$q_0 q_1$	$q_0 q_1$
$\|\psi\rangle$	$\|\psi'\rangle$
$\|0\rangle\|0\rangle$	$\|0\rangle\|0\rangle$
$\|0\rangle\|1\rangle$	$\|0\rangle\|1\rangle$
$\|1\rangle\|0\rangle$	$\|1\rangle\|1\rangle$
$\|1\rangle\|1\rangle$	$\|1\rangle\|0\rangle$

CX

$$\boxed{CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$$

$$CX \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |0\rangle|0\rangle$$

$$\boxed{CX^{\dagger} CX = I_{4\times4}}$$

$$CX \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |1\rangle|0\rangle$$

Examples of CX

1) $CX\,|0\rangle|+\rangle$

$= CX\,|0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$= \frac{CX(|0\rangle|0\rangle + |0\rangle|1\rangle)}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}(CX|0\rangle|0\rangle + CX|0\rangle|1\rangle)$

$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle)$

$= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |0\rangle|+\rangle$

2) $CX\,|1\rangle|+\rangle$

$= \frac{1}{\sqrt{2}}(CX(|1\rangle|0\rangle + |1\rangle|1\rangle))$

$= \frac{1}{\sqrt{2}}(|1\rangle|1\rangle + |1\rangle|0\rangle)$

$= |1\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)$

$= |1\rangle|+\rangle$!!

3) $CX\,|1\rangle|-\rangle$

$= CX.\,|1\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}(|1\rangle|1\rangle - |1\rangle|0\rangle)$

$$CX|1\rangle|-\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}}\left(|1\rangle - |0\rangle\right)$$

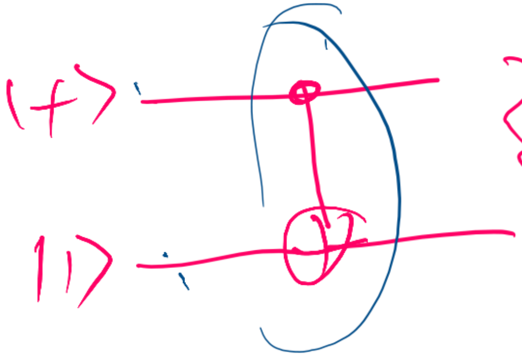
$$= -|1\rangle|-\rangle$$

④ $CX|\pm\rangle|1\rangle \longrightarrow$

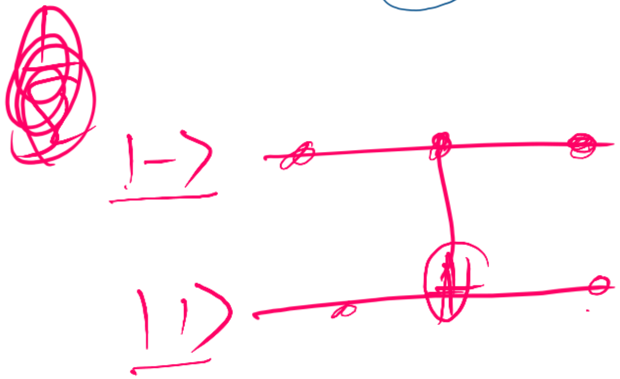
$$= CX\left[\frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left(CX|0\rangle|1\rangle \pm CX|1\rangle|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle \pm |1\rangle|0\rangle\right)$$



$\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right)$



$\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$

Entangled state

$\neq |\psi\rangle \otimes |\psi\rangle$

CX gate

= Entangling gate !

Summary

- Single qubit gates are 2×2 unitary matrices
- X gate flips the qubit state and Z gate flips the sign of |1> state
- Hadamard gate flips the basis
- Control X is the most important two-qubit gate
- Never think that CX only changes the state of one qubit!
- Two qubit gates are 4×4 unitary matrices