Quantum Gates
in
Outer-Product Forms

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Topics:
1. Single qubit gates
2. Two qubit gates
3. Product gates
4. Projection Operators

Outer product form is an alternative to matrix form

- inner product $\langle 0 | 1 \rangle$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

- tensor product $|0\rangle \otimes |1\rangle$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

- $|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

X-gate:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{X = |0\rangle\langle 1| + |1\rangle\langle 0|}$$

$$X|0\rangle = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|0\rangle$$

$$= |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle$$

$$= |0\rangle \times 0 + |1\rangle \times 1$$

$$\boxed{X|0\rangle = |1\rangle}$$

$$X|1\rangle = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|1\rangle = |0\rangle$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\boxed{Z = |0\rangle\langle 0| - |1\rangle\langle 1|}$$

$$Z|0\rangle = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)|0\rangle$$

$$= |0\rangle 1 - |1\rangle 0$$

$$\boxed{Z|0\rangle = |0\rangle}$$

$$Z|1\rangle = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)|1\rangle$$

$$\boxed{Z|1\rangle = -|1\rangle}$$

Y-gate

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \leftarrow |1\rangle$$

$$Y|0\rangle = i|1\rangle$$

$$Y|1\rangle = -i|0\rangle$$

$$X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

Hadamard gate:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right]$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$

Two qubit outer-products

$$|00\rangle\langle 00| = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$|00\rangle\langle 01| = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{1n} \\ u_{21} & |00\rangle\langle 01| & & \cdot \\ \vdots & & & \ddots \end{bmatrix}$$

$|01\rangle\langle 00|$

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$|00\rangle\langle 01| = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$|00\rangle\langle 11| = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Two qubit gates

$|10\rangle = |1\rangle \otimes |0\rangle$

$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

input	output
$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 01\rangle$
$\lvert 10\rangle$	$\lvert 11\rangle$
$\lvert 11\rangle$	$\lvert 10\rangle$

$|11\rangle$

$|10\rangle$

$$\boxed{CX = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|}$$

$CX|00\rangle = \left(|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|\right)|00\rangle$

$\boxed{CX|00\rangle = |00\rangle}$ $\boxed{CX|11\rangle = |10\rangle}$

$\boxed{CX|01\rangle = |01\rangle}$

$\boxed{CX|10\rangle = |11\rangle}$

$CX = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$

$$CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

① Product operators

$\rightsquigarrow \cancel{\otimes}\ X$

$\rightsquigarrow \cancel{\otimes}\ Y$

True two-qubit operators

② $CX \neq A \otimes B$

$= \sum_i A_i \otimes B_i$

$$O = X \otimes Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{array}{c c} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\ & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \end{array}$$

$= i\,|11\rangle\langle 00| - i\,|10\rangle\langle 01| + i\,|01\rangle\langle 10|$

$- i\,|00\rangle\langle 11|$

$$O = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) \otimes \left(-i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|\right)$$

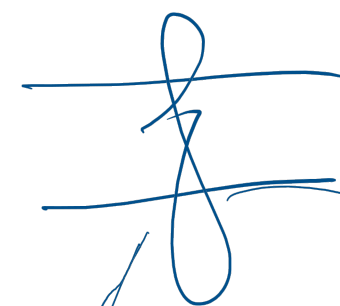
Projection Operators

Q. measurements are projective operators

$P_0 = |0\rangle\langle 0|$, $P_0|\psi\rangle = a|0\rangle$

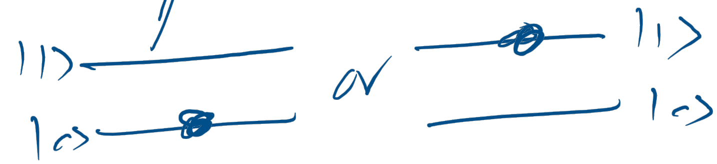
$P_1 = |1\rangle\langle 1|$, $P_1|\psi\rangle = b|1\rangle$

$P_{00} = |00\rangle\langle 00|$, $P_{+-} = |+-\rangle\langle +-|$



$|\psi\rangle = a|0\rangle + b|1\rangle$ P_1

P_0 measure



Summary

- Outer-product form is intuitive
- Handy when few non-zero terms in matrices
- More representative of quantum operations
- Quantum computers DO NOT implement matrices!
- Projection operators represents measurement process