Grover Search problem

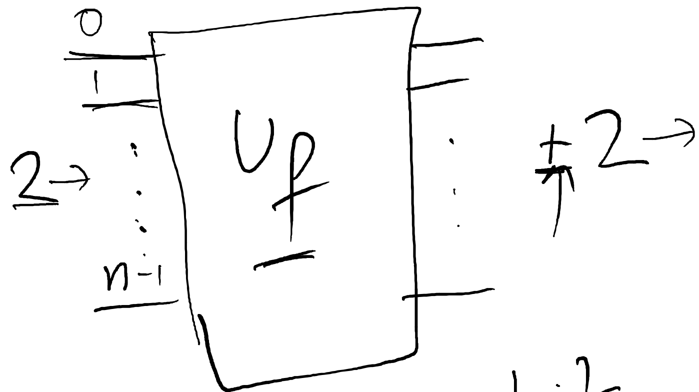
— Unstructured database

item 1, item 2, . . .

0, 1, 2, 3, . . . N-1

total entries = N

—



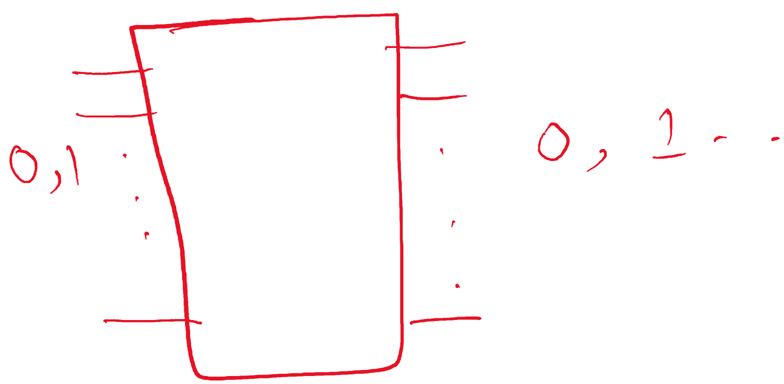
$$U_f|x\rangle = \begin{cases} -|x\rangle, & x=w \\ |x\rangle, & x\neq w \end{cases}$$

n bits of input

$$\boxed{N=2^n}$$

Classical Solution:

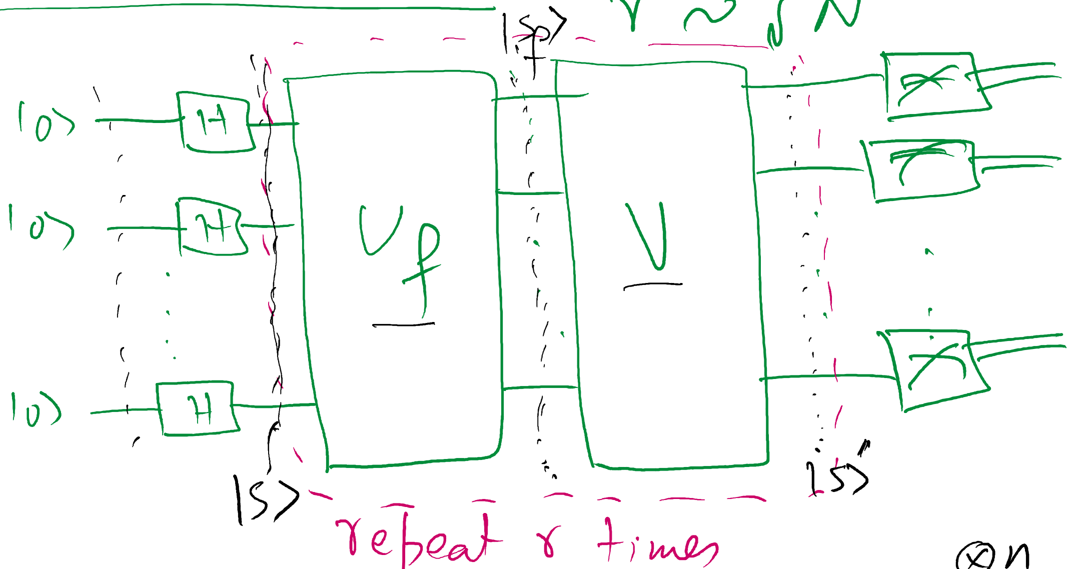
$\boxed{\text{Worst case} = N}$



Grover Theorem:

$r \sim \sqrt{N}$

Prob $w \sim 1$



Proof: ⓪ initial state: $|\psi_0\rangle = |0\rangle^{\otimes n} = |0000\cdots 0\rangle$

$= |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$

① After Hadamard transform:

$|s\rangle = H|0\rangle \otimes H|0\rangle \otimes H|0\rangle \otimes \cdots \otimes H|0\rangle = H^{\otimes n}|0\rangle^{\otimes n}$

$= |+\rangle \otimes |+\rangle \otimes |+\rangle \cdots \otimes |+\rangle$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$|s\rangle = \left(\frac{1}{\sqrt{2}}\right)^{n} \left[(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \cdots \otimes (|0\rangle + |1\rangle) \right]$$

$$= \frac{1}{\sqrt{2^n}} \left[|000\cdots 0\rangle + |00001\rangle + |00010\rangle + \cdots + |11111\rangle \right]$$

$$\boxed{|s\rangle = \frac{1}{\sqrt{N}} \left[|0\rangle + |1\rangle + |2\rangle + |3\rangle + \cdots + |N-1\rangle \right]}$$

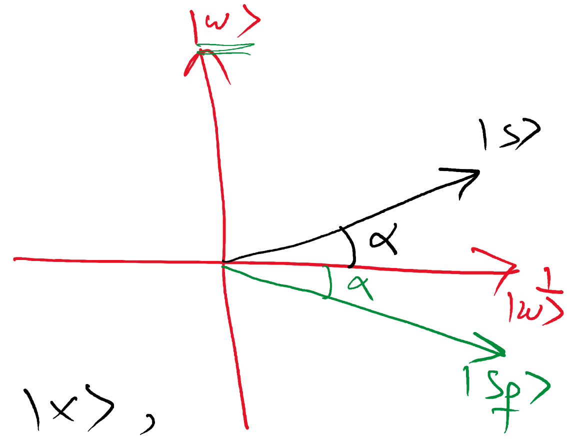
$$= \frac{1}{\sqrt{N}} \left[|0\rangle + |1\rangle + |2\rangle + |3\rangle + \cdots + |w\rangle + \cdots + |N-1\rangle \right]$$

$$= \underline{\frac{1}{\sqrt{N}} \left[|0\rangle + |1\rangle + |2\rangle \cdots + 0 + \cdots + |N-1\rangle \right]} + \frac{1}{\sqrt{N}} |w\rangle$$

$$= \sqrt{\frac{N-1}{N}} \overbrace{\frac{1}{\sqrt{N-1}} \left[|0\rangle + |1\rangle + |2\rangle + \cdots + 0 + \cdots + |N-1\rangle \right]} + \frac{1}{\sqrt{N}} |w\rangle$$

$$= \sqrt{\frac{N-1}{N}} \; |w^{\perp}\rangle + \frac{1}{\sqrt{N}} |w\rangle$$

$$|s\rangle = \sqrt{\frac{N-1}{N}}\,|w^\perp\rangle + \frac{1}{\sqrt{N}}\,|w\rangle$$



$$\boxed{|s\rangle = \cos\alpha\,|w^\perp\rangle + \sin\alpha\,|w\rangle}$$

$$\boxed{U_f|w\rangle = -|w\rangle}\,,\quad U_f|x\rangle = |x\rangle,\ x \neq w$$

$$\boxed{U_f|w^\perp\rangle = |w^\perp\rangle}$$

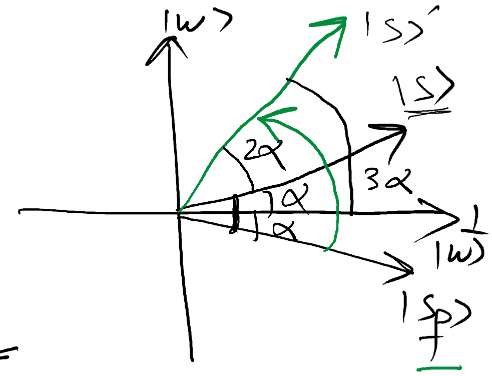
$$U_f\left[|0\rangle + |1\rangle + |2\rangle + \dots + 0 + \dots + |N-1\rangle\right]$$
$$= \left[|0\rangle + |1\rangle + |2\rangle + \dots + 0 + \dots + |N-1\rangle\right]$$

$$U_f = |w^\perp\rangle\langle w^\perp| - |w\rangle\langle w|$$
$$= 2|w^\perp\rangle\langle w^\perp| - |w^\perp\rangle\langle w^\perp| - |w\rangle\langle w| = 2|w^\perp\rangle\langle w^\perp| - I$$

$$I = |w\rangle\langle w| + |w^\perp\rangle\langle w^\perp| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{|s_f\rangle = U_f|s\rangle = \cos\alpha\,|w^\perp\rangle - \sin\alpha\,|w\rangle}$$

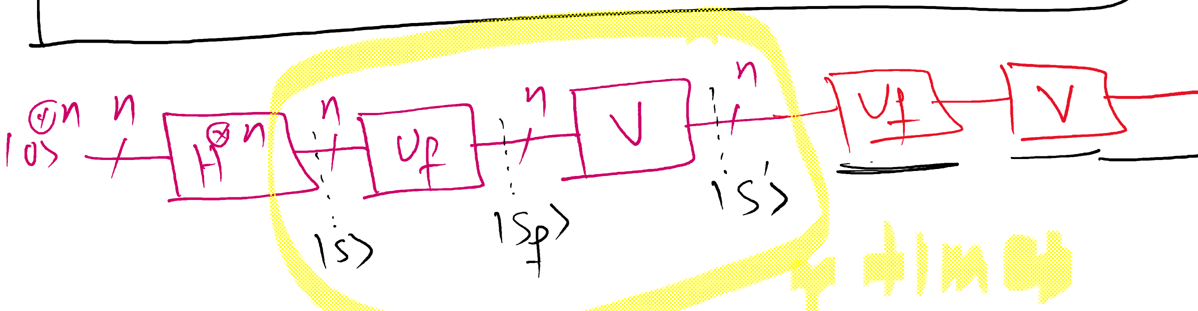
$$|s'\rangle = V|s_p\rangle$$

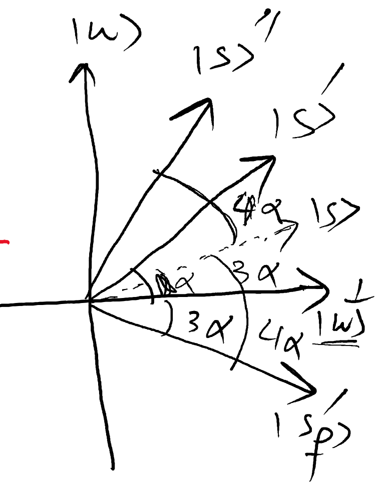


$$= V\left[\cos\alpha\,|w^{\perp}\rangle - \sin\alpha\,|w\rangle\right]$$

Let's take $\boxed{V = 2|s\rangle\langle s| - I}$

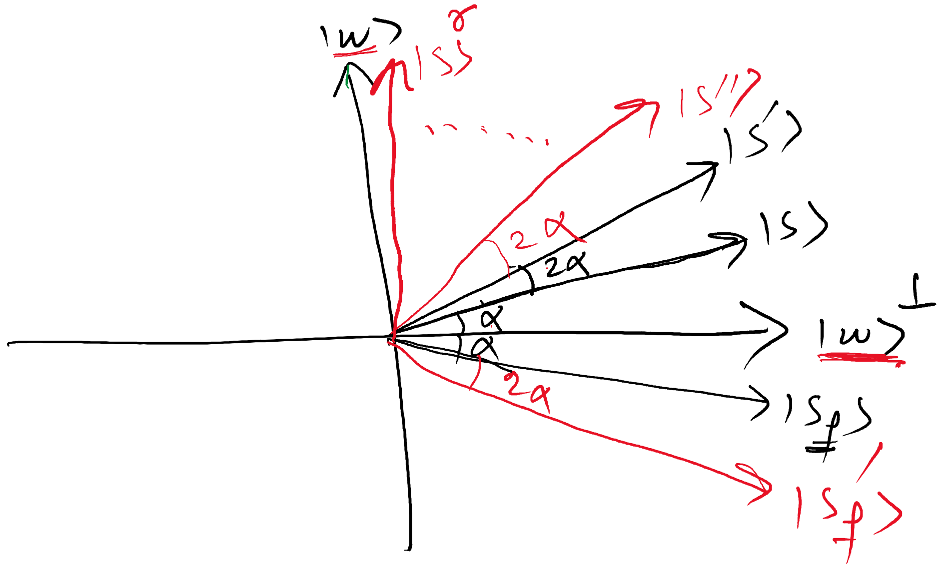
$$\boxed{|s'\rangle = \cos(3\alpha)\,|w^{\perp}\rangle + \sin(3\alpha)\,|w\rangle}$$





$$|s''\rangle = \cos(\alpha+4\alpha)\,|w^{\perp}\rangle + \sin(\alpha+4\alpha)\,|w\rangle$$

$$|s^{(r)}\rangle = \cos(\alpha+2r\alpha)\,|w^{\perp}\rangle + \sin(\alpha+2r\alpha)\,|w\rangle$$



$$\alpha + 2\gamma\alpha \approx \frac{\pi}{2}$$

$$2\gamma\alpha = \frac{\pi}{2} - \alpha$$

$$\boxed{\gamma = \frac{\pi}{4\alpha} - \frac{1}{2}}$$

$$\gamma = \frac{\pi}{4\sin^{-1}\left(\frac{1}{\sqrt{N}}\right)} - \frac{1}{2}$$

$$|s\rangle^{\gamma} = \cos(\underline{\alpha + 2\gamma\alpha})\,|w\rangle^{\perp} + \sin(\alpha + 2\gamma\alpha)\,|w\rangle$$

$$\boxed{|s\rangle^{\gamma} \approx |w\rangle}$$

$$\boxed{\text{Complexity} = \sqrt{N}}$$

when N is large

$$\alpha = \sin^{-1}\left(\frac{1}{\sqrt{N}}\right) \approx \frac{1}{\sqrt{N}}$$

$$\gamma \approx \frac{\pi}{4 \cdot \frac{1}{\sqrt{N}}} = \sqrt{N}\,\frac{\pi}{4}$$

Circuit for V:

$$V = 2|s\rangle\langle s| - I$$

where $|s\rangle = H^{\otimes n}|0\rangle^{\otimes n}$

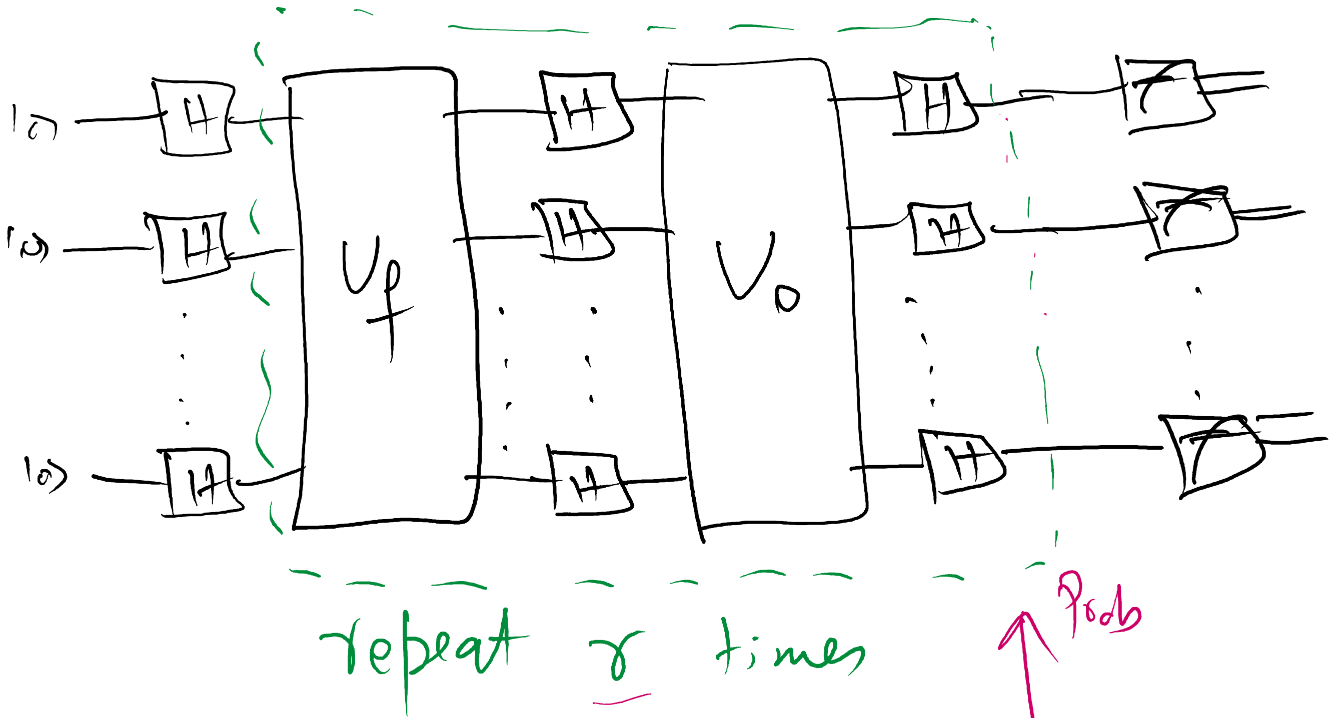
$= H|0\rangle \otimes H|0\rangle \otimes \cdots \otimes H|0\rangle$

$$V = 2H^{\otimes n}|0\rangle^{\otimes}\langle 0|^{\otimes n} H^{\otimes n} - I$$

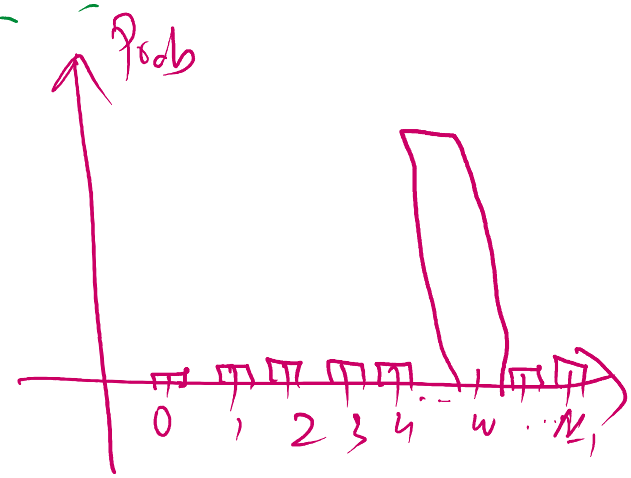
$$= 2H^{\otimes n}|0\rangle^{\otimes n}\langle 0|^{\otimes n} H^{\otimes n} - H^{\otimes n} H^{\otimes n}$$

$$= H^{\otimes n}\left[\underbrace{2|0\rangle^{\otimes n}\langle 0|^{\otimes n} - I}\right] H^{\otimes n}$$

$$= H^{\otimes n} U_0^{\otimes n} H^{\otimes n}$$



|0⟩
H
U_f
V_0
repeat r times



Prob
0
1
2
3
4
w
N-1