COMPANION NOTES AND SOLUTIONS

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Contents

1 Chapter 1: Matrices

1.1 Labs 0 and 1: Introduction and Matrix Basics

NOTES

1. Matrix addition is commutative, so addition in any order produces the same result.
$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1}$$

2. Scalar multiplication is applied on all elements of the given matrix as a regular multiplication. For example, for 3×3 matrix $\mathbf{A}$ and scalar s:
$$s\mathbf{A} = \begin{pmatrix} sA_{11} & sA_{12} & sA_{13} \\ sA_{21} & sA_{22} & sA_{23} \\ sA_{31} & sA_{32} & sA_{33} \end{pmatrix}. \tag{2}$$

3. A transposition is essentially swapping your rows and columns (do not, however, mistake it for a rotation), so 2×3 matrix $\mathbf{A}$ would produce 3×2 matrix $\mathbf{A}^{-1}$:
$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \\ A_{13} & A_{23} \end{pmatrix} \tag{3}$$

4. Matrix multiplication essentially computes the dot products of the rows of the first matrix with the columns of the second matrix, thus why the number of rows of the first matrix must equal the number of columns of the second. For example, for 2×3 matrix $\mathbf{A}$ and 3×2 matrix $\mathbf{B}$, the resulting matrix would be
$$\mathbf{A}\mathbf{B} = \begin{pmatrix} \mathbf{A}_{r=1} \cdot \mathbf{B}_{c=1} & \mathbf{A}_{r=1} \cdot \mathbf{B}_{c=2} \\ \mathbf{A}_{r=2} \cdot \mathbf{B}_{c=1} & \mathbf{A}_{r=2} \cdot \mathbf{B}_{c=2} \end{pmatrix} \text{ [1]}$$
$$= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \end{pmatrix} \tag{4}$$

[1] $M_{r=m}$ denotes "the vector comprised of the elements of row m in matrix $\mathbf{M}$" and $M_{c=n}$ denotes "the vector comprised of the elements of column n in matrix $\mathbf{M}$."

5. Matrix multiplication is not commutative, so multiplying matrices in different orders will produce different results.

$$\mathbf{AB} \neq \mathbf{BA} \tag{5}$$

6. Matrix addition and scalar multiplication are commutative with transposition, but multiplication is not.

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{6}$$
$$(s\mathbf{A})^T = s\mathbf{A}^T \tag{7}$$
$$(\mathbf{AB})^T \neq \mathbf{A}^T\mathbf{B}^T \tag{8}$$

7. The trace of a matrix is the sum of the elements on its main diagonal.

8. The trace is commutative with matrix addition (i.e., $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$), but non-commutative with matrix multiplication.

9. Matrix multiplication within trace in either order will produce the same result,
$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \tag{9}$$

1.2 Lab 2: A Matrix Representation of Linear Systems

EXERCISE 2(g)

Given
$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix},$$

we can produce $\mathbf{M} \rightarrow \mathbf{I}_3$ via:

1. Divide row 3 by 3.

2. Add multiples of -2 from row 2 to row 1.

3. Swap rows 2 and 3[2].

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[2]This was originally wrong and placed first. You need to swap last because matrix multiplication is right-associative, so the rightmost operation occurs first, not last. In other words, a "correct" way to approach this would be to do the entire process in reverse when multiplying.

This effect can be produced by the following elementary matrices.

The swap matrix can be created from emulation with I_3:

$$\mathbf{E}_{swap} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

similarly, the division matrix would be

$$\mathbf{E}_{div} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

and lastly, the additive matrix would be

$$\mathbf{E}_{add} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Altogether, the following multiplication should hold true:

$$\mathbf{E}_{div}\mathbf{E}_{add}\mathbf{E}_{swap}\mathbf{M} = \mathbf{I}_3$$

EXERCISE 2(h)

Given

$$x + 2y = 4$$
$$3z = 6$$
$$y = 8,$$

we can produce the following coefficient matrix $\mathbf{A}$ and constant vector $\mathbf{B}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix},$$

and given that $\mathbf{A} = \mathbf{M}$, from exercise 2(g) the solutions can be calculated by performing the RREF calculations on $\mathbf{B}$ to produce solution vector $\mathbf{S}$:

$$\mathbf{E}_{div}\mathbf{E}_{add}\mathbf{E}_{swap}\mathbf{B} = \mathbf{S},$$

yielding

$$\mathbf{S} = \begin{pmatrix} -12 \\ 8 \\ 2 \end{pmatrix},$$

showing that $x = -12, y = 8, z = 2$.

EXERCISE 2(i)

This isn't solvable because the best RREF form is

$$[\mathbf{M}|\mathbf{B}] = \begin{pmatrix} 1 & 0 & -6 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

or, in other words, equations (2) and (3) contradict each other.

EXERCISE 2(j)

The initial state of the system is

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 2 & 6 & 12 \end{pmatrix},$$

and the RREF of this system becomes

$$\begin{pmatrix} 1 & 0 & -6 & -8 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again, this system is not solvable as equations 2 and 3 are equivalent, resulting in this system not containing enough information to be adequately solved, but the reduced echelon form of the coefficient matrix is the same as in exercise 2(i).

1.3   Lab 3: Powers, Inverses, and Special Matrices

NOTES

1. A matrix multiplied by its transposition will produce a gram matrix which is symmetric (does not change on transposition).

$$\mathbf{M}^T\mathbf{M} = (\mathbf{M}^T\mathbf{M})^T \neq \mathbf{M}\mathbf{M}^T = (\mathbf{M}\mathbf{M}^T)^T \tag{10}$$

2. A matrix multiplied by its inverse, in any order, will produce an identity matrix.

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}_n \tag{11}$$

3. A matrix inverted twice produces itself.

$$(\mathbf{M}^{-1})^{-1} = \mathbf{M} \tag{12}$$

4. Matrix multiplication is not commutative with inversion, however, they are commutative in alternating order.

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \neq (\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1} \tag{13}$$

5. Matrix inversion is commutative with transposition.

$$(\mathbf{M}^T)^{-1} = (\mathbf{M}^{-1})^T \tag{14}$$

6. A square matrix $\mathbf{M}$ is

(a) Symmetric when $\mathbf{M} = \mathbf{M}^T$

(b) Diagonal when $\mathbf{M}_{ij} = 0$ if $i \neq j$

(c) Upper triangular when $\mathbf{M}_{ij} = 0$ if $i > j$ or lower triangular when $i < j$.

7. Assuming a matrix to be square, a matrix plus its transposition is symmetric.

$$\mathbf{M} + \mathbf{M}^T = (\mathbf{M} + \mathbf{M}^T)^T \tag{15}$$

THEOREM 1. The inverse of an elementary matrix is an elementary matrix.

Proof. There are three elementary operations to be considered: row swaps, multiplication/division by a scalar, and addition/subtraction of a row (by a scalar).

Let $\mathbf{E}_{swap}$ be the elementary matrix produced by a swap. By gaussian elimination, $\mathbf{E}_{swap}$ is its own inverse,

$$\mathbf{E}_{swap}^{-1} = \mathbf{E}_{swap},$$

as it can be undone (returned to the identity matrix $\mathbf{I}$) by repeating the swap that produced it.

Let $\mathbf{E}_{scale}$ be the elementary matrix produced by the multiplication of any scalar s. This multiplication by s can be undone by division by s, so $\mathbf{E}_{scale}^{-1}$ would be the elementary matrix produced by the divison of s.

Let $\mathbf{E}_{add}$ be the elementary matrix produced by the addition by any scalar multiple s from one row to another. As addition can be undone by subtraction, the addition by a scalar multiple s can be undone by the subtraction of the same values. $\square$

THEOREM 2. If $\mathbf{A}$ is invertible then the reduced row echelon form of $\mathbf{A}$ is $\mathbf{I}$.

Proof. If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible, then there exists some composite matrix $\mathbf{E}$ as the product of elementary operations that satisfies the following two equations:

$$\mathbf{EA} = \mathbf{I}, \qquad\qquad \mathbf{EI} = \mathbf{A}^{-1}.$$

These two equations correspond to the Gauss-Jordan elimination process, showing the transformation results on the left and right sides, and the left equation will reduce $\mathbf{A}$ into its reduced row echelon form. Therefore, the RREF form of $\mathbf{A}$ must be $\mathbf{I}$. $\qquad \square$

THEOREM 3. If the reduced row echelon form of $\mathbf{A}$ is $\mathbf{I}$, then $\mathbf{A}$ is invertible.

Proof. As shown in theorem 2, for $\mathbf{A} \in \mathbb{R}^{n \times n}$ to satisfy $\mathrm{RREF}(\mathbf{A}) = \mathbf{I}$, there must be a composite matrix $\mathbf{E} \in \mathbb{R}^{n \times n}$ such that $\mathbf{EA} = \mathbf{I}$, and therefore, this same matrix will simultaneously produce $\mathbf{EI} = \mathbf{A}^{-1}$ by Gauss-Jordan elimination, so $\mathbf{A}$ must be invertible. $\qquad \square$

THEOREM 4. $\mathbf{A}$ is a square invertible matrix if and only if $\mathbf{A}$ can be written as the product of elementary matrices.

Proof. Again, for $\mathbf{A} \in \mathbb{R}^{n \times n}$ to be invertible, it must satisfy $\mathbf{EA} = \mathbf{I}$ and $\mathbf{EI} = \mathbf{A}^{-1}$ for some product matrix $\mathbf{E}$ of elementary matrices in form

$$\mathbf{E} = \mathbf{E}_k \mathbf{E}_{k-1} \dots \mathbf{E}_2 \mathbf{E}_1 = \left(\prod_{n=1}^{k} \mathbf{E}_n \right)^{\sim} .^{3 4}$$

Therefore, $\mathbf{A} = \mathbf{E}^{-1}$, and as the inverse of elementary matrices is elementary, A inherently must be the product of elementary matrices if it is invertible. Furthermore, all composite matrices are inherently invertible because all elementary matrices are invertible, so altogehter, $\mathbf{A}$ is square and invertible if and only if its a composite product of elementary matrices. $\qquad \square$

THEOREM 5. If $\mathbf{A}$ is invertible then $\mathbf{A}^k$ is invertible for any natural number k.

Proof. If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible, then $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = I$, so the multiplication between them is commutative, meaning that

$$\mathbf{A}^k(\mathbf{A}^{-1})^k = (\mathbf{A}\mathbf{A}^{-1})^k = \mathbf{I}^k = \mathbf{I}.$$

$\qquad \square$

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[3]The tilde denotes that this product is in reverse order.
[4]From here on out, further matricies will be described as "composite."

THEOREM 6. If $\mathbf{A}$ is symmetric so is $\mathbf{A}^T$.

Proof. By the definition of symmetry, $\mathbf{A} = \mathbf{A}^T$, so $\mathbf{A}^T$ is symmetric. $\qquad\square$

THEOREM 7. If $\mathbf{A}$ is a symmetric invertible matrix then $\mathbf{A}^{-1}$ is symmetric.

Proof. Transposition is commutative with inversion, so $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$. Therefore, if $\mathbf{A} = \mathbf{A}^T$ then $\mathbf{A}^{-1} = (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$. $\qquad\square$

THEOREM 8. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the same size then $\mathbf{A} + \mathbf{B}$ is symmetric.

Proof. $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T$, so

$$\mathbf{A} + \mathbf{B} = \mathbf{A}^T + \mathbf{B}^T = \mathbf{A}^T + \mathbf{B} = \mathbf{A} + \mathbf{B}^T.$$

Matrix multiplication is inherently commutative with transposition, so $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T = \mathbf{A} + \mathbf{B}$. $\qquad\square$

THEOREM 9. If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the same size, then $\mathbf{A}\mathbf{B}$ is symmetric.

Proof. As $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T$, then $\mathbf{A}\mathbf{B} = \mathbf{A}^T\mathbf{B}^T = \mathbf{A}^T\mathbf{B} = \mathbf{A}\mathbf{B}^T$, and as matrix multiplication is commutative with transposition in a reverse order (i.e., $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$ and vice versa), because both $\mathbf{A}$ and $\mathbf{B}$ are symmetric, the matrix multiplication and transposition in this case are fully commutative, so $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} = \mathbf{A}^T\mathbf{B}^T = (\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T = (\mathbf{B}\mathbf{A})^T$. $\qquad\square$

THEOREM 10. If $\mathbf{A}$ is a square matrix then $\mathbf{A} + \mathbf{A}^T$ is symmetric.

Proof. Transposition is commutative with addition, thus making it distributive: $(\mathbf{A} + \mathbf{A}^T)^T = \mathbf{A}^T + (\mathbf{A}^T)^T = \mathbf{A}^T + \mathbf{A}$. $\qquad\square$

THEOREM 11. The sum of upper triangular matrices is upper triangular.

Proof. This is intuitively true based on how matrix addition works. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$. All $A_{ij} + B_{ij} = 0$ where $i > j$ as $A_{ij} = 0$ and $B_{ij} = 0$. $\qquad\square$

1.4   Lab 4: Graph Theory and Adjacency Matrices

NOTES

1. Graphs are comprised of nodes (points/places/etc) and edges (the relationships between the nodes, like paths/connections).

2. An edge and the node connected to it are called an indicent. Two nodes connected by an edge are considered adjacent.

3. Graphs can be represented as matrices in the form of an adjacency matrix, which are symmetric, square matrices such that indexing the matrix for the points in either order will produce the number of connections between the nodes at that depth, such as:

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

   For example, let $\mathbf{M}$ be a the adjacency matrix storing two points, a and b (where a and b each correspond to indices of the matrix respectively). The number of connections n long (passing through n edges) will be found at

$$\mathbf{M}^n(a, b) \qquad \text{and} \qquad \mathbf{M}^n(b, a). \tag{16}$$

1.5   Lab 5: Permutations and Determinants

NOTES

1. Given the number of options/variables, n, the number of permutations given n will be $n!$.

2. The determinant of a 2×2 matrix is

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{17}$$

3. If $\det \mathbf{M} = 0$ then $\mathbf{M}$ is not invertible.

4. For scalar k and $n \times n$ matrix $\mathbf{M}$, $\det(k\mathbf{M}) = k^n \det \mathbf{M}$

5. The determinant of the identity matrix, of any size, is always 1.

$$\det \mathbf{I}_n = 1 \text{ for } n \in \{1, 2, \dots\} \tag{18}$$

6. The determinant of a matrix is equal to the determinant of its transposition.

$$\det \mathbf{M} = \det \mathbf{M}^T \tag{19}$$

7. Matrix multiplication is fully commutative with determinants, so

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B} \tag{20}$$

8. Again, matrix addition is commutative, so

$$\det(\mathbf{A} + \mathbf{B}) = \det(\mathbf{B} + \mathbf{A}), \tag{21}$$

but matrix addition is not commutative with calculating determinants, so

$$\det(\mathbf{A} + \mathbf{B}) \neq \det \mathbf{A} \det \mathbf{B}. \tag{22}$$

9. The determinant of any triangular matrix (upper/lower triangular and diagonal matrices) are just the product of the main diagonal.

10. An alternative way to calculate the determinant is based on the trace of a matrix using the Cayley-Hamilton theorem[5] for a 2×2 matrix $\mathbf{M}_2$,

$$\det \mathbf{M}_2 = \frac{(\operatorname{tr} \mathbf{M}_2)^2 - \operatorname{tr} \mathbf{M}_2^2}{2}, \tag{23}$$

and for a 3×3 matrix $\mathbf{M}_3$,

$$\det \mathbf{M}_3 = \frac{(\operatorname{tr} \mathbf{M}_3)^3 - 3 \operatorname{tr} \mathbf{M}_3^2 \operatorname{tr} \mathbf{M}_3 + 2 \operatorname{tr} \mathbf{M}^3}{6} \tag{24}$$

EXERCISE 2(a-c)

Let $\mathbf{E}$ be an elementary matrix produced by one of the following operations.

1. If $\mathbf{E}$ is produced by scalar k in a row-scaling operation, $\det \mathbf{E} = k$.

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[5]Cayley–Hamilton theorem on Wikipedia.

2. If $\mathbf{E}$ is produced by a scalar addition, $\det \mathbf{E} = 1$ (unchanged).

3. If $\mathbf{E}$ is produced by a row swap, $\det \mathbf{E} = -1$.

THEOREM 12. If $\det \mathbf{A} \neq 0$ then $\mathbf{A}$ is invertible.

Proof. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. If $\det \mathbf{A} \neq 0$ then $\mathbf{A}$ can be reduced to $\mathbf{I}_n$ using only elementary operations, implying $\mathbf{A}$ satisfies $\mathbf{EA} = \mathbf{I}$ for some composite/product matrix of elementary matrices, $\mathbf{E}$. By definition, $\mathbf{E} = \mathbf{A}^{-1}$, making $\mathbf{A}$ invertible. $\square$

THEOREM 13. If $\mathbf{A}$ is invertible then $\det \mathbf{A} \neq 0$.

Proof. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an invertible matrix. $\mathbf{A}$ therefore satisfies $\mathbf{EA} = \mathbf{I}$ for some matrix $\mathbf{E}$ comprised of the product of elementary matrices. By definition, $\mathbf{A} = \mathbf{E}^{-1}$, and as the inverse of the product of elementary matrices will be the product of elementary matrices, $\mathbf{A}$ is also a product of elementary matrices. As the determinant of any elementary matrix is nonzero, and the determinant of a product is the product of determinants, $\det \mathbf{A} = \det \mathbf{E}^{-1} \neq 0$. $\square$

THEOREM 14. If $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices of the same size, then $\mathbf{A} + \mathbf{B}$ is invertible.

Proof. This is incorrect, because $\det \mathbf{A} + \det \mathbf{B} \neq \det(\mathbf{A} + \mathbf{B})$. For example, if $\mathbf{A} = -\mathbf{B}$. Let $\mathbf{A} = -\mathbf{I}_2$ and $\mathbf{B} = \mathbf{I}_2$. $\det \mathbf{A} = \det \mathbf{B} = 1$, but $\det(\mathbf{A} + \mathbf{B}) = 0$. $\square$

THEOREM 15. If $\mathbf{A}$ is a square matrix, then $\det \mathbf{A} = \det \mathbf{A}^T$.

Proof. Let $\mathbf{M} \in \mathbb{R}^{n \times n}$. $\det \mathbf{M}$ is defined via permutations of row and column indices of $\mathbf{M}$, so swapping the rows and columns will not affect $\det \mathbf{M}$. $\square$

THEOREM 16. If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size then $\mathbf{A}$ and $\mathbf{B}$ are invertible if and only if $\mathbf{AB}$ is invertible.

Proof. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. If $\mathbf{AB}$ is invertible, then $\det \mathbf{AB} \neq 0$, and as $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$, $\det \mathbf{A} \neq 0$ and $\det \mathbf{B} \neq 0$. Therefore, $\mathbf{A}$ and $\mathbf{B}$ are

invertible. Opposingly, if $\mathbf{A}$ and $\mathbf{B}$ are invertible, $\det \mathbf{A} \neq 0$ and $\det \mathbf{B} \neq 0$. Therefore, $\det \mathbf{AB} \neq 0$, so $\mathbf{AB}$ must be invertible. $\qquad\square$

1.6 Lab 6: 4×4 Determinants and Beyond

NOTES

1. The equation for calculating matrices higher than 2×2 follows a row or column-based expansion in form

$$\det \mathbf{M}\bigg|_{i=c} = \sum_{j=1}^{n} (-1)^{c+j} a_{cj} \det\left(\mathbf{M}_{cj}^{*} \in \mathbf{M}\right) \qquad (25)$$

for a row-based expansion at row c or

$$\det \mathbf{M}\bigg|_{j=c} = \sum_{i=1}^{n} (-1)^{i+c} a_{ic} \det(\mathbf{M}_{ic}^{*} \in \mathbf{M}) \qquad (26)$$

for a column-based expansion, given matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$, and $\mathbf{M}_{ij}^{*}$ denotes a submatrix anchored by (i, j), meaning that row i and column j are removed from $\mathbf{M}$ to produce $\mathbf{M}_{ij}^{*}$.

For example, for a 3×3 matrix $\mathbf{A}$,

$$\det \mathbf{A}\bigg|_{i=1} = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det\left(\mathbf{A}_{1j}^{*} \in \mathbf{A}\right)$$

$$= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

EXERCISE 1(a)

$$M_{41} = \det \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = 1$$

$$M_{42} = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = 1$$

$$M_{43} = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} - \det \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 2 - 1 = 1$$

$$M_{44} = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = 6 - 1 - (3 - 2) = 4$$

EXERCISE 2(b)

$$\det \mathbf{A}\Big|_{i=4} = \sum_{j=1}^{4} (-1)^{4+j} A_{4j} M_{4j}$$

$$= -A_{41} M_{41} + A_{42} M_{42} - A_{43} M_{43} + A_{44} M_{44}$$

$$= 4 M_{44} - M_{43} = 4(4) - 1 = 15$$

EXERCISE 2(c)

$$\det \mathbf{A}\Big|_{i=1} = \sum_{j=1}^{4} (-1)^{1+j} A_{1j} M_{1j}$$

$$= M_{11} - M_{12} = \det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix} - \det \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

$$= 2 \det \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} - \det \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} + \det \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$$

$$= 2(11) - 4 - (11) + 8 = 15.$$

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EXERCISE 2(d)

$$\det \mathbf{B} = \det \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix} = \det \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} - \det \begin{pmatrix} -1 & 1 \\ 0 & 4 \end{pmatrix}$$

$$= 11 - (-4) = 15$$

Because $\mathbf{P}$ is triangular,

$$\det \mathbf{P} = \prod_{i=1}^{n} P_{ii} = -15$$

EXERCISE 2(e)

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

EXERCISE 2(f)

As $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$, then

$$\det \mathbf{B} = \frac{\det \mathbf{P}}{\det \mathbf{E}_1 \det \mathbf{E}_2 \det \mathbf{E}_3} = \frac{15}{(1)(1)(-1)} = -15$$

2 Chapter 2: Invertibility

2.1 Lab 7: Singularity

NOTES

1. Given a linear system $\mathbf{A}x = \mathbf{B}$, if $\mathbf{A}$ is invertible, then the system has exactly one solution: $x = \mathbf{A}^{-1}\mathbf{B}$.

2. If the constants in a linear system $(\mathbf{Ax} = \mathbf{B})$ are all zero (i.e., $\mathbf{B} = \mathbf{0}$), we call the linear system a homogenous linear system, and these systems always have at least one solution: $\mathbf{x} = \mathbf{0}$.

EXERCISE 2(a)

$\mathbf{A}$ is invertible because $\det \mathbf{A} \neq 0$. As

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix},$$

then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} -20 \\ 27/2 \\ -7/6 \end{pmatrix}.$$

EXERCISE 2(b)

Likewise, for $\mathbf{Ax} = \mathbf{0}, \mathbf{x} = \mathbf{0}$, the trivial solution.

THEOREM 17. The inverse of a nonsingular upper triangular matrix is upper triangular.

Proof. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a nonsingular upper triangular matrix. Since $\mathbf{A}$ is invertible, there exists a matrix $\mathbf{A}^{-1}$ such that $\mathbf{AA}^{-1} = \mathbf{I}$.

Recall that the product of two upper triangular matrices is upper triangular. Moreover, the inverse of a nonsingular matrix is a product of elementary matrices. Among the elementary matrices, only those corresponding to row operations that preserve upper triangular form (scaling a row, adding a multiple of a row to another row above it, or swapping rows above the diagonal, which are disallowed in upper triangular matrices anyway) are involved in forming an upper triangular matrix.

Since $\mathbf{A}$ is upper triangular, its Gaussian elimination to the identity matrix involves only upper triangular-preserving row operations. Therefore, the inverse $\mathbf{A}^{-1}$, being the product of the inverses of those elementary matrices, also consists of upper triangular matrices.

Since a product of upper triangular matrices is upper triangular, it follows that $\mathbf{A}^{-1}$ is upper triangular. $\qquad\square$

THEOREM 18. The inverse of a nonsingular diagonal matrix is diagonal.

Proof. Let $\mathbf{A}$ be an invertible/nonsingular diagonal matrix. Therefore, $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{A}$ can be expressed as the product of elementary matrices, $\mathbf{E}$. As $\mathbf{I}$ is diagonal, the only elementary operations that can occur within $\mathbf{E}$ must maintain the diagonal property (with any operations that violate this property at any point being later undone). Similarly, $\mathbf{A}^{-1} = \mathbf{E}^{-1}$, and as the inverse of an elementary product is an elementary product of the inverse operations, the diagonal property must be maintained through $\mathbf{E}^{-1}$ as well. $\square$

THEOREM 19. The determinant of the inverse is equal to the multiplicative inverse of the determinant, or $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$.

Proof. We use the multiplicative property of the determinant: for any $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$,
$$\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}).$$

Suppose $\mathbf{A}$ is an invertible $n \times n$ matrix. Then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Taking the determinant of both sides, we get:

$$\det\left(\mathbf{A}^{-1}\mathbf{A}\right) = \det(\mathbf{I}) = 1.$$

Using the multiplicative property:

$$\det\left(\mathbf{A}^{-1}\right) \cdot \det(\mathbf{A}) = 1.$$

Solving for $\det(\mathbf{A}^{-1})$, we obtain:

$$\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})}.$$

$\square$

THEOREM 20. $\mathbf{A}$ is invertible if and only if $\mathbf{A}$ can be written as a product of elementary matrices.

Proof. For $\mathbf{A}$ to be invertible, there must exist some matrix $\mathbf{E}$ as the product of elementary matrices such that $\mathbf{EA} = \mathbf{I}$, with $\mathbf{E} = \mathbf{A}^{-1}$. Therefore, $\mathbf{A}^{-1}$ is a product of elementary matrices, and as the inverse of an elementary matrix

is elementary, the inverse of the product of elementary matrices is also the product of elementary matrices. Therefore, as $\mathbf{A} = \mathbf{E}^{-1}$, $\mathbf{A}$ can be expressed as the prodct of elementary matrices. $\square$

THEOREM 21. If $\mathbf{A}$ is an $n \times n$ invertible matrix, then the system $\mathbf{Ax} = \mathbf{B}$ has exactly one solution for all $n \times 1$ vectors $\mathbf{B}$.

Proof. By definition, for $\mathbf{Ax} = \mathbf{B}$ to have ≥ 1 solution, $\mathbf{A}$ must be either underrepresented ($n \times m$) or contradicting (det $\mathbf{A} = 0$). Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be invertible. Therefore, $\mathbf{A}$ cannot contradict (meaning that, when represented as linear equations, the equations either state the same information or directly state conflicting information) as det $\mathbf{A} \neq 0$. Additionally, as $\mathbf{A}$ is $n \times n$, $\mathbf{A}$ cannot be underrepresented: all values can be calculated. Therefore, $\mathbf{x}$ must have exactly one solution. $\square$

THEOREM 22. If $\mathbf{A}$ is an $n \times n$ matrix and the system $\mathbf{Ax} = \mathbf{B}$ is consistent (has at least one solution) for all $n \times 1$ vectors $\mathbf{B}$, then $\mathbf{A}$ is invertible.

Proof. As $\mathbf{Ax} = \mathbf{B}$ is consistent, it cannot contain a contradiction (two of the internal linear equations stating conflicting information), because if there was a contradiction, $\mathbf{Ax} = \mathbf{B}$ would yield no solutions and det $\mathbf{A} = 0$, making $\mathbf{A}$ singular. Therefore, $\mathbf{A}$ must be invertible to have ≥ 1 solution to $\mathbf{Ax} = \mathbf{B}$. $\square$

THEOREM 23. If $ad - bc \neq 0$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Proof. This can be verified by direct calculation. Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By

gaussian elimination:

$$(\mathbf{A}|\mathbf{I}) = \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{c}{a}R_1} \begin{pmatrix} a & b & 1 & 0 \\ 0 & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \leftarrow \frac{a}{ad-bc}R_2]{R_1 \leftarrow \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}.$$

Therefore,

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\square$

THEOREM 24. A is an $n \times n$ invertible matrix if and only if the system $\mathbf{Ax} = \mathbf{0}$ has only trivial solution.

Proof. Suppose $\mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible. Then, by Theorem 21, for any $\vec{b} \in \mathbb{R}^n$, the equation $\mathbf{Ax} = \vec{b}$ has a unique solution. In particular, taking $\vec{b} = \vec{0}$, the homogeneous equation $\mathbf{Ax} = \vec{0}$ must also have a unique solution. Since $\vec{x} = \vec{0}$ is always a solution to $\mathbf{Ax} = \vec{0}$, it must be the only solution — the solution is trivial.

Conversely, suppose the equation $\mathbf{Ax} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$. Then the null space of $\mathbf{A}$ is trivial, so the columns of $\mathbf{A}$ are linearly independent. For an $n \times n$ matrix, linear independence of columns implies that $\mathbf{A}$ has full rank n, and hence is invertible.

Therefore, $\mathbf{A}$ is invertible if and only if the homogeneous equation $\mathbf{Ax} = \vec{0}$ has only the trivial solution. $\square$

2.2 Lab 8: Modularity and Z_p

NOTES

1. If x and y are integers, they are said to be congruent modulo p (written $x \equiv y \pmod{p}$), meaning that $\frac{x-y}{p}$ is an integer.

2. $r = n \pmod{d}$ can be interpreted as the integer remainder of n/d where n, d and r are all integers.

3. Z_p is the set of all integers that can result from $\pmod{p}$ (not considering congruency), and for integer value p, this set is all integers $Z_p = \{0, 1, \ldots, p-2, p-1\}$.

EXERCISE 1(a)

Given $Z_5 = \{0, 1, 2, 3, 4\}$, the corresponding additive inverses would be $(0, 4, 3, 2, 1)$ (as each indexwise matchup would thereby result in a modular congruency with 0, such as $1 + 4 \equiv 0 \pmod{5}$).

EXERCISE 1(b)

The corresponding multiplicative inverses for $x \geq 0 \in Z_5$ would be (*undefined*, $1, 3, 2, 4$), as, for example, the integer remainder of $\frac{4 \times 4}{5}$ is 1.

EXERCISE 2(a)

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. $\mathbf{A} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod{2}$, thus $\det \mathbf{A} \pmod{2} \neq 0$, therefore $\mathbf{A}$ is invertible modulo 2. Similarly, $\mathbf{A} \pmod{5} = \mathbf{A}$ as all entries of $\mathbf{A} < 5$. Therefore, as $\det \mathbf{A} = \det \mathbf{A} \pmod{5} \neq 0$, $\mathbf{A}$ is invertible modulo 5.

EXERCISE 2(b)

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \pmod{2}$. $\mathbf{A} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod{2}$, so $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \pmod{2}$, therefore

$$\mathbf{x} \pmod{2} = \mathbf{A}^{-1}\mathbf{B} \pmod{2} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \pmod{2}.$$

Similarly, the calculation modulo 5 is just the standard calculation (as all values are < 5), so

$$\mathbf{x} \pmod{5} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} -10 \\ 17 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 2 \end{pmatrix} \pmod{5}.$$

2.3 Lab 9: Complex Entries and Eigenvalues

NOTES

1. Where a complex number is defined with a real and imaginary part in form $a + bi$, its complex conjugate will have the imaginary coefficient as the multiplicative inverse, $a - bi$.

2. The complex conjugate of a matrix with complex entries has all of the entries of the matrix as the complex conjugates of the original matrix's entires.

3. For an $n \times n$ matrix $\mathbf{A}$, the eigenvalues λ of $\mathbf{A}$ are values of λ which satisfy

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{27}$$

for each eigenvalue's corresponding eigenvector $\mathbf{x}$, and the characteristic equation,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{28}$$

   (a) The eigenvalues of a diagonal matrix are just the entries along the diagonal.

   (b) Singular matrices always have 0 as an eigenvalue, at least (they may have more).

   (c) The eigenvalues of a matrix $\mathbf{M}$ are the same as the eigenvalues of its transposition $\mathbf{M}^T$:

$$\operatorname{eig} \mathbf{M} = \operatorname{eig} \mathbf{M}^T \tag{29}$$

   (d) The eigenvalues of a complex-valued matrix $\mathbf{M}$ are the same as the eigenvalues of its complex transposition $\overline{\mathbf{M}}^T$:

$$\operatorname{eig} \mathbf{M} = \operatorname{eig} \overline{\mathbf{M}}^T \tag{30}$$

4. A matrix $\mathbf{M}$ is considered "Hermetian" if it is equal to its complex transpose: $\mathbf{M} = \overline{\mathbf{M}}^T$.

5. A matrix is considered "Unitary" if its complex transpose is its inverse: $\mathbf{M}^{-1} = \overline{\mathbf{M}}^T$.

THEOREM 25. Determinants and complex conjugates are commutative, or the determinant of the complex conjugate is equal to the complex conjugate of the determinant. I.E.,

$$\det \overline{\mathbf{A}} = \overline{\det \mathbf{A}}$$

.

THEOREM 26. If $\mathbf{A}$ is invertible, then inversion is commutative with the complex conjugate. I.E.,
$$\left(\overline{\mathbf{A}}\right)^{-1} = \overline{\mathbf{A}^{-1}}$$

THEOREM 27. If c is a complex number, $\overline{c\mathbf{A}} = \bar{c}\overline{\mathbf{A}}$.

THEOREM 28. The eigenvalues of a diagonal matrix are the entries on the diagonal.

THEOREM 29. All eigenvalues of Hermitian matrices are real numbers.

THEOREM 30. The complex conjugate of a Hermitian matrix is a Hermitian matrix.

THEOREM 31. $\mathbf{A}$ is Unitary if and only if $\mathbf{A}^{-1} = \overline{\mathbf{A}}^T$

THEOREM 32. λ is an eigenvalue of matrix $\mathbf{A}$ if and only if it is an eigenvalue of $\overline{\mathbf{A}}^T$.

2.4 Lab 10: Linear Combinations and Dependency

NOTES

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors (or perhaps, equivalently, a vector of vectors, making $\mathbf{S}$ a matrix) and there exists scalars within some vector $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ such that

$$\mathbf{w} = \sum_{i=1}^{n} k_i \mathbf{v}_i = \mathbf{Sk}, \tag{31}$$

   then we say that $\mathbf{w}$ is a linear combination of all $\mathbf{v} \in \mathbf{S}$.

2. For set $\mathbf{S} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $\mathbf{S}$ is linearly independent if

$$\mathbf{0} = \sum_{i=1}^{n} k_i \mathbf{v}_i = \mathbf{Sk} \tag{32}$$

   has only the trivial solution, $\mathbf{k} = \mathbf{0}$. Otherwise, it's linearly dependent, meaning that some nonzero value of $\mathbf{k}$ can result in $\mathbf{w} = \mathbf{0}$.

3. A set of vectors $\mathbf{S}$ is said to span a vector space $\mathbf{V}$ if every vector in $\mathbf{V}$ can be expressed as a linear combination of all of the vectors in $\mathbf{S}$.

   (a) A common example is dimensionality, like $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^{n \times n}, \ldots$.

   (b) The set $\mathbf{S}_1 = \{1\}$ spans $\mathbb{R}^1$ as all numbers can be written as a product with 1.

   (c) Similarly, $\mathbf{S}_2 = \{(1, 0), (0, 1)\}$ spans $\mathbb{R}^2$ because all vectors within two dimensions can be described as a linear combination of the two.

THEOREM 33. If $\mathbf{A}$ is invertible, then the rows of $\mathbf{A}$ are linearly independent.

THEOREM 34. If $\mathbf{A}$ is invertible, then the columns of $\mathbf{A}$ are linearly independent.

THEOREM 35. A set of vectors with only two vectors in it is linearly dependent if one is a scalar multiple of the other.

THEOREM 36. A set of vectors is linearly dependent if it contains the zero vector.

THEOREM 37. If $\mathbf{A}$ is an $n \times n$ invertible matrix, then the rows of $\mathbf{A}$ span $\mathbb{R}^n$.

THEOREM 38. If $\mathbf{A}$ is an $n \times n$ invertible matrix, then the columns of $\mathbf{A}$ span $\mathbb{R}^n$