

Binomial and Poisson Distributions

From Michael Barron Book

3.4.1 Bernoulli distribution

The simplest random variable (excluding non-random ones!) takes just two possible values. Call them 0 and 1.

DEFINITION 3.10

A random variable with two possible values, 0 and 1, is called a **Bernoulli variable**, its distribution is **Bernoulli distribution**, and any experiment with a *binary outcome* is called a **Bernoulli trial**.

3.4.2 Binomial distribution

Now consider a sequence of independent Bernoulli trials and count the number of successes in it. This may be the number of defective computers in a shipment, the number of updated files in a folder, the number of girls in a family, the number of e-mails with attachments, etc.

DEFINITION 3.11

A variable described as the number of successes in a sequence of independent Bernoulli trials has **Binomial distribution**. Its parameters are n , the number of trials, and p , the probability of success.

**Binomial
distribution**

$$\begin{array}{ll} n & = \text{number of trials} \\ p & = \text{probability of success} \\ P(x) & = \binom{n}{x} p^x q^{n-x} \\ \mathbf{E}(X) & = np \\ \text{Var}(X) & = npq \end{array}$$

Example 3.16. As part of a business strategy, randomly selected 20% of new internet service subscribers receive a special promotion from the provider. A group of 10 neighbors signs for the service. What is the probability that at least 4 of them get a special promotion?

Solution. We need to find the probability $\mathbf{P}\{X \geq 4\}$, where X is the number of people, out of 10, who receive a special promotion. This is the number of successes in 10 Bernoulli trials, therefore, X has Binomial distribution with parameters $n = 10$ and $p = 0.2$. From [Table A2](#),

$$\mathbf{P}\{X \geq 4\} = 1 - F(3) = 1 - 0.8791 = \underline{0.1209}.$$



Example 3.17. An exciting computer game is released. Sixty percent of players complete all the levels. Thirty percent of them will then buy an advanced version of the game. Among 15 users, what is the expected number of people who will buy the advanced version? What is the probability that at least two people will buy it?

Solution. Let X be the number of people (successes), among the mentioned 15 users (trials), who will buy the advanced version of the game. It has Binomial distribution with $n = 15$ trials and the probability of success

$$\begin{aligned} p &= P\{\text{buy advanced}\} \\ &= P\{\text{buy advanced} \mid \text{complete all levels}\} P\{\text{complete all levels}\} \\ &= (0.30)(0.60) = 0.18. \end{aligned}$$

Then we have

$$\mathbf{E}(X) = np = (15)(0.18) = \underline{2.7}$$

and

$$P\{X \geq 2\} = 1 - P(0) - P(1) = 1 - (1 - p)^n - np(1 - p)^{n-1} = \underline{0.7813}.$$

The last probability was computed directly by formula (3.9) because the probability of success, 0.18, is not in [Table A2](#). \diamond

3.4.5 Poisson distribution

The next distribution is related to a concept of *rare events*, or Poissonian events. Essentially it means that two such events are extremely unlikely to occur simultaneously or within a very short period of time. Arrivals of jobs, telephone calls, e-mail messages, traffic accidents, network blackouts, virus attacks, errors in software, floods, and earthquakes are examples of rare events. The rigorous definition of rare events is given in Section 6.3.2.

DEFINITION 3.14

The number of rare events occurring within a fixed period of time has **Poisson distribution**.

This distribution bears the name of a famous French mathematician *Siméon-Denis Poisson* (1781–1840).

**Poisson
distribution**

$$\begin{array}{ll}\lambda &= \text{frequency, average number of events} \\ P(x) &= e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \dots \\ \mathbf{E}(X) &= \lambda \\ \text{Var}(X) &= \lambda\end{array}$$

PROOF: This probability mass function satisfies $\sum_0^\infty P(x) = 1$ because the Taylor series for $f(\lambda) = e^\lambda$ at $\lambda = 0$ is

$$e^\lambda = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}, \quad (3.13)$$

and this series converges for all λ .

The expectation $\mathbf{E}(X)$ can be derived from (3.13) as

$$\mathbf{E}(X) = \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} = 0 + e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1} \cdot \lambda}{(x-1)!} = e^{-\lambda} \cdot e^{\lambda} \cdot \lambda = \lambda.$$

Similarly,

$$\mathbf{E}(X^2) - \mathbf{E}(X) = \sum_{x=0}^{\infty} (x^2 - x) P(x) = \sum_{x=2}^{\infty} x(x-1) e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2} \cdot \lambda^2}{(x-2)!} = \lambda^2,$$

and therefore,

$$\text{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}^2(X) = (\lambda^2 + \mathbf{E}(X)) - \mathbf{E}^2(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

□

A Poisson variable can take any nonnegative integer value because there may be no rare events within the chosen period, on one end, and the possible number of events is not limited, on the other end. Poisson distribution has one parameter, $\lambda > 0$, which is the average number of the considered rare events. Values of its cdf are given in [Table A3](#) on p. 430.

Example 3.22 (NEW ACCOUNTS). Customers of an internet service provider initiate new accounts at the average rate of 10 accounts per day.

- (a) What is the probability that more than 8 new accounts will be initiated today?
- (b) What is the probability that more than 16 accounts will be initiated within 2 days?

Solution. (a) New account initiations qualify as rare events because no two customers open accounts simultaneously. Then the number X of today's new accounts has Poisson distribution with parameter $\lambda = 10$. From [Table A3](#),

$$P\{X > 8\} = 1 - F_X(8) = 1 - 0.333 = \underline{0.667}.$$

(b) The number of accounts, Y , opened within 2 days does *not* equal $2X$. Rather, Y is another Poisson random variable whose parameter equals 20. Indeed, the parameter is the average number of rare events, which, over the period of two days, doubles the one-day average. Using [Table A3](#) with $\lambda = 20$,

$$P\{Y > 16\} = 1 - F_Y(16) = 1 - 0.221 = \underline{0.779}.$$

◇

3.4.6 Poisson approximation of Binomial distribution

Poisson distribution can be effectively used to approximate Binomial probabilities when the number of trials n is large, and the probability of success p is small. Such an approximation is adequate, say, for $n \geq 30$ and $p \leq 0.05$, and it becomes more accurate for larger n .

Example 3.23 (NEW ACCOUNTS, CONTINUED). Indeed, the situation in Example 3.22 can be viewed as a sequence of Bernoulli trials. Suppose there are $n = 400,000$ potential internet users in the area, and on any specific day, each of them opens a new account with probability $p = 0.000025$. We see that the number of new accounts is the number of successes, hence a Binomial model with expectation $\mathbf{E}(X) = np = 10$ is possible. However, a distribution with such extreme n and p is unlikely to be found in any table, and computing its pmf by hand is tedious. Instead, one can use Poisson distribution with the same expectation $\lambda = 10$. \diamond

**Poisson
approximation
to Binomial**

$$\text{Binomial}(n, p) \approx \text{Poisson}(\lambda)$$

where $n \geq 30$, $p \leq 0.05$, $np = \lambda$

(3.14)

Remark: Mathematically, it means closeness of Binomial and Poisson pmf,

$$\lim_{\substack{n \rightarrow \infty \\ p \rightarrow 0 \\ np \rightarrow \lambda}} \binom{n}{x} p^x (1-p)^{n-x} = e^{-\lambda} \frac{\lambda^x}{x!}$$

and this is what S. D. Poisson has shown.

When p is large ($p \geq 0.95$), the Poisson approximation is applicable too. The probability of a failure $q = 1 - p$ is small in this case. Then, we can approximate the number of failures, which is also Binomial, by a Poisson distribution.

Example 3.24. Ninety-seven percent of electronic messages are transmitted with no error. What is the probability that out of 200 messages, at least 195 will be transmitted correctly?

Solution. Let X be the number of correctly transmitted messages. It is the number of successes in 200 Bernoulli trials, thus X is Binomial with $n = 200$ and $p = 0.97$. Poisson approximation cannot be applied to X because p is too large. However, the number of failures Y is also Binomial, with parameters $n = 200$ and $q = 0.03$, and it is approximately Poisson with $\lambda = nq = 6$. From [Table A3](#),

$$P\{X \geq 195\} = P\{Y \leq 5\} = F_Y(5) \approx \underline{0.446}.$$

◇

Exercises

3.20. A quality control engineer tests the quality of produced computers. Suppose that 5% of computers have defects, and defects occur independently of each other.

- (a) Find the probability of exactly 3 defective computers in a shipment of twenty.

3.21. A lab network consisting of 20 computers was attacked by a computer virus. This virus enters each computer with probability 0.4, independently of other computers. Find the probability that it entered at least 10 computers.

3.23. Every day, a lecture may be canceled due to inclement weather with probability 0.05. Class cancelations on different days are independent.

- (a) There are 15 classes left this semester. Compute the probability that at least 4 of them get canceled.

3.26. After a computer virus entered the system, a computer manager checks the condition of all important files. She knows that each file has probability 0.2 to be damaged by the virus, independently of other files.

- (a) Compute the probability that at least 5 of the first 20 files are damaged.

3.28. Messages arrive at an electronic message center at random times, with an average of 9 messages per hour.

- (a) What is the probability of receiving *at least* five messages during the next hour?
- (b) What is the probability of receiving *exactly* five messages during the next hour?

Table A2, continued. Binomial distribution

n	x	P																			
		.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550	.600	.650	.700	.750	.800	.850	.900	.950	
11	0	.569	.314	.167	.086	.042	.020	.009	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	1	.898	.697	.492	.322	.197	.113	.061	.030	.014	.006	.002	.001	.000	.000	.000	.000	.000	.000	.000	
	2	.985	.910	.779	.617	.455	.313	.200	.119	.065	.033	.015	.006	.002	.001	.000	.000	.000	.000	.000	
	3	.998	.981	.931	.839	.713	.570	.426	.296	.191	.113	.061	.029	.012	.004	.001	.000	.000	.000	.000	
	4	1.0	.997	.984	.950	.885	.790	.668	.533	.397	.274	.174	.099	.050	.022	.008	.002	.000	.000	.000	
	5	1.0	1.0	.997	.988	.966	.922	.851	.753	.633	.500	.367	.247	.149	.078	.034	.012	.003	.000	.000	
	6	1.0	1.0	1.0	.998	.992	.978	.950	.901	.826	.726	.603	.467	.332	.210	.115	.050	.016	.003	.000	
	7	1.0	1.0	1.0	1.0	.999	.996	.988	.971	.939	.887	.809	.704	.574	.430	.287	.161	.069	.019	.002	
	8	1.0	1.0	1.0	1.0	1.0	.999	.998	.994	.985	.967	.935	.881	.800	.687	.545	.383	.221	.090	.015	
	9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.998	.994	.986	.970	.939	.887	.803	.678	.508	.303	.102	
10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.996	.991	.980	.958	.914	.833	.686	.431		
12	0	.540	.282	.142	.069	.032	.014	.006	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	1	.882	.659	.443	.275	.158	.085	.042	.020	.008	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	
	2	.980	.889	.736	.558	.391	.253	.151	.083	.042	.019	.008	.003	.001	.000	.000	.000	.000	.000	.000	
	3	.998	.974	.908	.795	.649	.493	.347	.225	.134	.073	.036	.015	.006	.002	.000	.000	.000	.000	.000	
	4	1.0	.996	.976	.927	.842	.724	.583	.438	.304	.194	.112	.057	.026	.009	.003	.001	.000	.000	.000	
	5	1.0	.999	.995	.981	.946	.882	.787	.665	.527	.387	.261	.158	.085	.039	.014	.004	.001	.000	.000	
	6	1.0	1.0	.999	.996	.986	.961	.915	.842	.739	.613	.473	.335	.213	.118	.054	.019	.005	.001	.000	
	7	1.0	1.0	1.0	.999	.997	.991	.974	.943	.888	.806	.696	.562	.417	.276	.158	.073	.024	.004	.000	
	8	1.0	1.0	1.0	1.0	1.0	.998	.994	.985	.964	.927	.866	.775	.653	.507	.351	.205	.092	.026	.002	
	9	1.0	1.0	1.0	1.0	1.0	1.0	.999	.997	.992	.981	.958	.917	.849	.747	.609	.442	.264	.111	.020	
10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.997	.992	.980	.958	.915	.842	.725	.557	.341	.118		
11	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.998	.994	.986	.968	.931	.858	.718	.460		
13	0	.513	.254	.121	.055	.024	.010	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	1	.865	.621	.398	.234	.127	.064	.030	.013	.005	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	
	2	.975	.866	.692	.502	.333	.202	.113	.058	.027	.011	.004	.001	.000	.000	.000	.000	.000	.000	.000	
	3	.997	.966	.882	.747	.584	.421	.278	.169	.093	.046	.020	.008	.003	.001	.000	.000	.000	.000	.000	
	4	1.0	.994	.966	.901	.794	.654	.501	.353	.228	.133	.070	.032	.013	.004	.001	.000	.000	.000	.000	
	5	1.0	.999	.992	.970	.920	.835	.716	.574	.427	.291	.179	.098	.046	.018	.006	.001	.000	.000	.000	
	6	1.0	1.0	.999	.993	.976	.938	.871	.771	.644	.500	.356	.229	.129	.062	.024	.007	.001	.000	.000	
	7	1.0	1.0	1.0	.999	.994	.982	.954	.902	.821	.709	.573	.426	.284	.165	.080	.030	.008	.001	.000	
	8	1.0	1.0	1.0	1.0	.999	.996	.987	.968	.930	.867	.772	.647	.499	.346	.206	.099	.034	.006	.000	
	9	1.0	1.0	1.0	1.0	1.0	.999	.997	.992	.980	.954	.907	.831	.722	.579	.416	.253	.118	.034	.003	
10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.996	.989	.973	.942	.887	.798	.667	.498	.308	.134	.025		
11	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.998	.995	.987	.970	.936	.873	.766	.602	.379	.135		
12	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.996	.991	.980	.958	.914	.833	.686	.431		
14	0	.488	.229	.103	.044	.018	.007	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	1	.847	.585	.357	.198	.101	.047	.021	.008	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	2	.970	.842	.648	.448	.281	.161	.084	.040	.017	.006	.002	.001	.000	.000	.000	.000	.000	.000	.000	
	3	.996	.956	.853	.698	.521	.355	.220	.124	.063	.029	.011	.004	.001	.000	.000	.000	.000	.000	.000	
	4	1.0	.991	.953	.870	.742	.584	.423	.279	.167	.090	.043	.018	.006	.002	.000	.000	.000	.000	.000	
	5	1.0	.999	.988	.956	.888	.781	.641	.486	.337	.212	.119	.058	.024	.008	.002	.000	.000	.000	.000	
	6	1.0	1.0	.998	.988	.962	.907	.816	.692	.546	.395	.259	.150	.075	.031	.010	.002	.000	.000	.000	
	7	1.0	1.0	1.0	.998	.990	.969	.925	.850	.741	.605	.454	.308	.184	.093	.038	.012	.002	.000	.000	
	8	1.0	1.0	1.0	1.0	.998	.992	.976	.942	.881	.788	.663	.514	.359	.219	.112	.044	.012	.001	.000	
	9	1.0	1.0	1.0	1.0	1.0	.998	.994	.982	.957	.910	.833	.721	.577	.416	.258	.130	.047	.009	.000	
10	1.0	1.0	1.0	1.0	1.0	1.0	.999	.996	.989	.971	.937	.876	.780	.645	.479	.302	.147	.044	.004		
11	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.998	.994	.983	.960	.916	.839	.719	.552	.352	.158	.030		
12	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.997	.992	.979	.953	.899	.802	.643	.415	.153		
13	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.998	.994	.986	.993	.982	.956	.897	.771	.512	
15	0	.463	.206	.087	.035	.013	.005	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	1	.829	.549	.319	.167	.080	.035	.014	.005	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	2	.964	.816	.604	.398	.236	.127	.062	.027	.011	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	
	3	.995	.944	.823	.648	.461	.297	.173	.091	.042	.018	.006	.002	.000	.000	.000	.000	.000	.000	.000	
	4	.999	.987	.938	.836	.686	.515	.352	.217	.120	.059	.025	.009	.003	.001	.000	.000	.000	.000	.000	
	5	1.0	.998	.983	.939	.852	.722	.564	.403	.261	.151	.077	.034	.012	.004	.001	.000	.000	.000	.000	
	6	1.0	1.0	.996	.982	.943	.869	.755	.610	.452	.304	.182	.095	.042	.015	.004	.001	.000	.000	.000	
	7	1.0	1.0	.999	.996	.983	.950	.887	.787	.654	.500	.346	.213	.113	.050	.017	.004	.001	.000	.000	
	8	1.0	1.0	1.0	.999	.996	.985	.958	.905	.818	.696	.548	.390	.245	.131	.057	.018	.004	.000	.000	
	9	1.0	1.0	1.0	1.0	.999	.996	.988	.966	.923	.849	.739	.597	.436	.278	.148	.061	.017	.002	.000	
10	1.0	1.0	1.0	1.0	1.0	.999	.997	.991	.975	.941	.880	.783	.648	.485	.314	.164	.062	.013	.001		
11	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.982	.958	.909	.827	.703	.539	.352	.177	.056	.005		
12	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.996	.989	.973	.938	.873	.764	.602	.396	.184	.036		
13	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.995	.986	.965	.920	.833	.681	.451	.171		
14	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.995	.986	.965	.920	.833	.681	.451	.171	

Table A2, continued. Binomial distribution

n	x	P																			
		.050	.100	.150	.200	.250	.300	.350	.400	.450	.500	.550	.600	.650	.700	.750	.800	.850	.900	.950	
16	1	.811	.515	.284	.141	.063	.026	.010	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	2	.957	.789	.561	.352	.197	.099	.045	.018	.007	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	
	3	.993	.932	.790	.598	.405	.246	.134	.065	.028	.011	.003	.001	.000	.000	.000	.000	.000	.000	.000	
	4	.999	.983	.921	.798	.630	.450	.289	.167	.085	.038	.015	.005	.001	.000	.000	.000	.000	.000	.000	
	5	1.0	.997	.976	.918	.810	.660	.490	.329	.198	.105	.049	.019	.006	.001	.000	.000	.000	.000	.000	
	6	1.0	.999	.994	.973	.920	.825	.688	.527	.366	.227	.124	.058	.023	.007	.002	.000	.000	.000	.000	
	7	1.0	1.0	.999	.993	.973	.926	.841	.716	.563	.402	.256	.142	.067	.026	.007	.001	.000	.000	.000	
	8	1.0	1.0	1.0	.999	.993	.974	.933	.858	.744	.598	.437	.284	.159	.074	.027	.007	.001	.000	.000	
	9	1.0	1.0	1.0	1.0	1.0	.998	.993	.977	.942	.876	.773	.634	.473	.312	.175	.080	.027	.006	.001	
	10	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.981	.951	.895	.802	.671	.510	.340	.190	.082	.024	.003	
	11	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.995	.985	.962	.915	.833	.711	.550	.370	.202	.079	.017	
	12	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.997	.989	.972	.935	.866	.754	.595	.402	.210	.068	
	13	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.998	.993	.982	.955	.901	.803	.648	.439	.211	
	14	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.997	.990	.974	.937	.859	.716	.485	.189	
	15	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.997	.990	.972	.926	.815	
18	1	.774	.450	.224	.099	.039	.014	.005	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	2	.942	.734	.480	.271	.135	.060	.024	.008	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	3	.989	.902	.720	.501	.306	.165	.078	.033	.012	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	
	4	.998	.972	.879	.716	.519	.333	.189	.094	.041	.015	.005	.001	.000	.000	.000	.000	.000	.000	.000	
	5	1.0	.994	.958	.867	.717	.534	.355	.209	.108	.048	.018	.006	.001	.000	.000	.000	.000	.000	.000	
	6	1.0	.999	.988	.949	.861	.722	.549	.374	.226	.119	.054	.020	.006	.001	.000	.000	.000	.000	.000	
	7	1.0	1.0	.997	.984	.943	.859	.728	.563	.391	.240	.128	.058	.021	.006	.001	.000	.000	.000	.000	
	8	1.0	1.0	.999	.996	.981	.940	.861	.737	.578	.407	.253	.135	.060	.021	.005	.001	.000	.000	.000	
	9	1.0	1.0	1.0	.999	.995	.979	.940	.865	.747	.593	.422	.263	.139	.060	.019	.004	.001	.000	.000	
	10	1.0	1.0	1.0	1.0	.999	.994	.979	.942	.872	.760	.609	.437	.272	.141	.057	.016	.003	.000	.000	
	11	1.0	1.0	1.0	1.0	1.0	.999	.994	.980	.946	.881	.774	.626	.451	.278	.139	.051	.012	.001	.000	
	12	1.0	1.0	1.0	1.0	1.0	1.0	.999	.994	.982	.952	.892	.791	.645	.466	.283	.133	.042	.006	.000	
	13	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.995	.985	.959	.906	.811	.667	.481	.284	.121	.028	.002	
	14	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.996	.988	.967	.922	.835	.694	.499	.280	.098	.011	
	15	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.997	.992	.976	.940	.865	.729	.520	.266	.058	
16	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.995	.986	.961	.901	.776	.550	.226		
20	1	.736	.392	.176	.069	.024	.008	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	2	.925	.677	.405	.206	.091	.035	.012	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	3	.984	.867	.648	.411	.225	.107	.044	.016	.005	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	4	.997	.957	.830	.630	.415	.238	.118	.051	.019	.006	.002	.000	.000	.000	.000	.000	.000	.000	.000	
	5	1.0	.989	.933	.804	.617	.416	.245	.126	.055	.021	.006	.002	.000	.000	.000	.000	.000	.000	.000	
	6	1.0	.998	.978	.913	.786	.608	.417	.250	.130	.058	.021	.006	.002	.000	.000	.000	.000	.000	.000	
	7	1.0	1.0	.994	.968	.898	.772	.601	.416	.252	.132	.058	.021	.006	.001	.000	.000	.000	.000	.000	
	8	1.0	1.0	.999	.990	.959	.887	.762	.596	.414	.252	.131	.057	.020	.005	.001	.000	.000	.000	.000	
	9	1.0	1.0	1.0	.997	.986	.952	.878	.755	.591	.412	.249	.128	.053	.017	.004	.001	.000	.000	.000	
	10	1.0	1.0	1.0	.999	.996	.983	.947	.872	.751	.588	.409	.245	.122	.048	.014	.003	.000	.000	.000	
	11	1.0	1.0	1.0	1.0	.999	.995	.980	.943	.869	.748	.586	.404	.238	.113	.041	.010	.001	.000	.000	
	12	1.0	1.0	1.0	1.0	1.0	.999	.994	.979	.942	.868	.748	.584	.399	.228	.102	.032	.006	.000	.000	
	13	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.979	.942	.870	.750	.583	.392	.214	.087	.022	.002	.000	
	14	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.979	.945	.874	.755	.584	.383	.196	.067	.011	.000	
	15	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.994	.981	.949	.882	.762	.585	.370	.170	.043	.003	
16	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.995	.984	.956	.893	.775	.589	.352	.133	.016		
25	2	.873	.537	.254	.098	.032	.009	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	3	.966	.764	.471	.234	.096	.033	.010	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	4	.993	.902	.682	.421	.214	.090	.032	.009	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	5	.999	.967	.838	.617	.378	.193	.083	.029	.009	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000	
	6	1.0	.991	.930	.780	.561	.341	.173	.074	.026	.007	.002	.000	.000	.000	.000	.000	.000	.000	.000	
	7	1.0	.998	.975	.891	.727	.512	.306	.154	.064	.022	.006	.001	.000	.000	.000	.000	.000	.000	.000	
	8	1.0	1.0	.992	.953	.851	.677	.467	.274	.134	.054	.017	.004	.001	.000	.000	.000	.000	.000	.000	
	9	1.0	1.0	.998	.983	.929	.811	.630	.425	.242	.115	.044	.013	.003	.000	.000	.000	.000	.000	.000	
	10	1.0	1.0	1.0	.994	.970	.902	.771	.586	.384	.212	.096	.034	.009	.002	.000	.000	.000	.000	.000	
	11	1.0	1.0	1.0	.998	.989	.956	.875	.732	.543	.345	.183	.078	.025	.006	.001	.000	.000	.000	.000	
	12	1.0	1.0	1.0	1.0	.997	.983	.940	.846	.694	.500	.306	.154	.060	.017	.003	.000	.000	.000	.000	
	13	1.0	1.0	1.0	1.0	.999	.994	.975	.922	.817	.655	.457	.268	.125	.044	.011	.002	.000	.000	.000	
	14	1.0	1.0	1.0	1.0	1.0	.998	.991	.966	.904	.788	.616	.414	.229	.098	.030	.006	.000	.000	.000	
	15	1.0	1.0	1.0	1.0	1.0	1.0	.997	.987	.956	.885	.758	.575	.370	.189	.071	.017	.002	.000	.000	
	16	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.996	.983	.946	.866	.726	.533	.283	.149	.047	.008	.000	
17	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.999	.994	.978	.936	.846	.694	.488	.273	.109	.025	.002		
18	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.993	.974	.926	.827	.659	.439	.220	.070	.009		
19	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.991	.971	.917	.807	.622	.383	.162	.033		
20	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.998	.991	.968	.910	.786	.579	.318	.098		

Table A3. Poisson distribution

$$F(x) = P\{X \leq x\} = \sum_{k=0}^x \frac{e^{-\lambda} \lambda^k}{k!}$$

x	λ														
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
0	.905	.819	.741	.670	.607	.549	.497	.449	.407	.368	.333	.301	.273	.247	.223
1	.995	.982	.963	.938	.910	.878	.844	.809	.772	.736	.699	.663	.627	.592	.558
2	1.00	.999	.996	.992	.986	.977	.966	.953	.937	.920	.900	.879	.857	.833	.809
3	1.00	1.00	1.00	.999	.998	.997	.994	.991	.987	.981	.974	.966	.957	.946	.934
4	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.998	.996	.995	.992	.989	.986	.981
5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.998	.998	.997	.996
6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999
7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

x	λ														
	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	.202	.183	.165	.150	.135	.122	.111	.100	.091	.082	.074	.067	.061	.055	.050
1	.525	.493	.463	.434	.406	.380	.355	.331	.308	.287	.267	.249	.231	.215	.199
2	.783	.757	.731	.704	.677	.650	.623	.596	.570	.544	.518	.494	.469	.446	.423
3	.921	.907	.891	.875	.857	.839	.819	.799	.779	.758	.736	.714	.692	.670	.647
4	.976	.970	.964	.956	.947	.938	.928	.916	.904	.891	.877	.863	.848	.832	.815
5	.994	.992	.990	.987	.983	.980	.975	.970	.964	.958	.951	.943	.935	.926	.916
6	.999	.998	.997	.997	.995	.994	.993	.991	.988	.986	.983	.979	.976	.971	.966
7	1.00	1.00	.999	.999	.999	.999	.998	.997	.997	.996	.995	.993	.992	.990	.988
8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.999	.999	.998	.998	.997	.996
9	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.999	.999
10	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

x	λ														
	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	10.5
0	.030	.018	.011	.007	.004	.002	.002	.001	.001	.000	.000	.000	.000	.000	.000
1	.136	.092	.061	.040	.027	.017	.011	.007	.005	.003	.002	.001	.001	.000	.000
2	.321	.238	.174	.125	.088	.062	.043	.030	.020	.014	.009	.006	.004	.003	.002
3	.537	.433	.342	.265	.202	.151	.112	.082	.059	.042	.030	.021	.015	.010	.007
4	.725	.629	.532	.440	.358	.285	.224	.173	.132	.100	.074	.055	.040	.029	.021
5	.858	.785	.703	.616	.529	.446	.369	.301	.241	.191	.150	.116	.089	.067	.050
6	.935	.889	.831	.762	.686	.606	.527	.450	.378	.313	.256	.207	.165	.130	.102
7	.973	.949	.913	.867	.809	.744	.673	.599	.525	.453	.386	.324	.269	.220	.179
8	.990	.979	.960	.932	.894	.847	.792	.729	.662	.593	.523	.456	.392	.333	.279
9	.997	.992	.983	.968	.946	.916	.877	.830	.776	.717	.653	.587	.522	.458	.397
10	.999	.997	.993	.986	.975	.957	.933	.901	.862	.816	.763	.706	.645	.583	.521
11	1.00	.999	.998	.995	.989	.980	.966	.947	.921	.888	.849	.803	.752	.697	.639
12	1.00	1.00	.999	.998	.996	.991	.984	.973	.957	.936	.909	.876	.836	.792	.742
13	1.00	1.00	1.00	.999	.998	.996	.993	.987	.978	.966	.949	.926	.898	.864	.825
14	1.00	1.00	1.00	1.00	.999	.999	.997	.994	.990	.983	.973	.959	.940	.917	.888
15	1.00	1.00	1.00	1.00	1.00	.999	.999	.998	.995	.992	.986	.978	.967	.951	.932
16	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998	.996	.993	.989	.982	.973	.960
17	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998	.997	.995	.991	.986	.978
18	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998	.996	.993	.988	.988
19	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.998	.997	.994
20	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998	.998	.997

Table A3, continued. Poisson distribution

x	λ														
	11	12	13	14	15	16	17	18	19	20	22	24	26	28	30
0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
1	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
2	.001	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
3	.005	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
4	.015	.008	.004	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
5	.038	.020	.011	.006	.003	.001	.001	.000	.000	.000	.000	.000	.000	.000	.000
6	.079	.046	.026	.014	.008	.004	.002	.001	.001	.000	.000	.000	.000	.000	.000
7	.143	.090	.054	.032	.018	.010	.005	.003	.002	.001	.000	.000	.000	.000	.000
8	.232	.155	.100	.062	.037	.022	.013	.007	.004	.002	.001	.000	.000	.000	.000
9	.341	.242	.166	.109	.070	.043	.026	.015	.009	.005	.002	.000	.000	.000	.000
10	.460	.347	.252	.176	.118	.077	.049	.030	.018	.011	.004	.001	.000	.000	.000
11	.579	.462	.353	.260	.185	.127	.085	.055	.035	.021	.008	.003	.001	.000	.000
12	.689	.576	.463	.358	.268	.193	.135	.092	.061	.039	.015	.005	.002	.001	.000
13	.781	.682	.573	.464	.363	.275	.201	.143	.098	.066	.028	.011	.004	.001	.000
14	.854	.772	.675	.570	.466	.368	.281	.208	.150	.105	.048	.020	.008	.003	.001
15	.907	.844	.764	.669	.568	.467	.371	.287	.215	.157	.077	.034	.014	.005	.002
16	.944	.899	.835	.756	.664	.566	.468	.375	.292	.221	.117	.056	.025	.010	.004
17	.968	.937	.890	.827	.749	.659	.564	.469	.378	.297	.169	.087	.041	.018	.007
18	.982	.963	.930	.883	.819	.742	.655	.562	.469	.381	.232	.128	.065	.030	.013
19	.991	.979	.957	.923	.875	.812	.736	.651	.561	.470	.306	.180	.097	.048	.022
20	.995	.988	.975	.952	.917	.868	.805	.731	.647	.559	.387	.243	.139	.073	.035
21	.998	.994	.986	.971	.947	.911	.861	.799	.725	.644	.472	.314	.190	.106	.054
22	.999	.997	.992	.983	.967	.942	.905	.855	.793	.721	.556	.392	.252	.148	.081
23	1.00	.999	.996	.991	.981	.963	.937	.899	.849	.787	.637	.473	.321	.200	.115
24	1.00	.999	.998	.995	.989	.978	.959	.932	.893	.843	.712	.554	.396	.260	.157
25	1.00	1.00	.999	.997	.994	.987	.975	.955	.927	.888	.777	.632	.474	.327	.208
26	1.00	1.00	1.00	.999	.997	.993	.985	.972	.951	.922	.832	.704	.552	.400	.267
27	1.00	1.00	1.00	.999	.998	.996	.991	.983	.969	.948	.877	.768	.627	.475	.333
28	1.00	1.00	1.00	1.00	.999	.998	.995	.990	.980	.966	.913	.823	.697	.550	.403
29	1.00	1.00	1.00	1.00	1.00	.999	.997	.994	.988	.978	.940	.868	.759	.623	.476
30	1.00	1.00	1.00	1.00	1.00	.999	.999	.997	.993	.987	.959	.904	.813	.690	.548
31	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998	.996	.992	.973	.932	.859	.752	.619
32	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998	.995	.983	.953	.896	.805	.685
33	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.997	.989	.969	.925	.850	.744
34	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.999	.994	.979	.947	.888	.797
35	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.996	.987	.964	.918	.843
36	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.998	.992	.976	.941	.880
37	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.995	.984	.959	.911
38	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.997	.990	.972	.935
39	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.998	.994	.981	.954
40	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.996	.988	.968
41	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998	.992	.978
42	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.995	.985
43	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.997	.990
44	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.998	.994
45	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.996
46	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999	.998
47	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999
48	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999
49	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.999
50	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00