Sample Space, Tree Diagram and Set Theory

Sample Space:

The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol S.

Each outcome in a sample space is called an **element** or a **member** of the sample space, or simply a **sample point**. If the sample space has a finite number of elements, we may *list* the members separated by commas and enclosed in braces. Thus, the sample space S, of possible outcomes when a coin is flipped, may be written

$$S = \{H, T\},$$

where H and T correspond to heads and tails, respectively.

Example 2.1: Consider the experiment of tossing a die. If we are interested in the number that shows on the top face, the sample space is

$$S_1 = \{1, 2, 3, 4, 5, 6\}.$$

If we are interested only in whether the number is even or odd, the sample space is simply

$$S_2 = \{\text{even}, \text{odd}\}.$$

Tree Diagram:

Example 2.2: An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once. To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.1. The various paths along the branches of the tree give the distinct sample points. Starting with the top left branch and moving to the right along the first path, we get the sample point *HH*, indicating the possibility that heads occurs on two successive flips of the coin. Likewise, the sample point $T3$ indicates the possibility that the coin will show a tail followed by a 3 on the toss of the die. By proceeding along all paths, we see that the sample space is

$$S = \{HH,\ HT,\ T1,\ T2,\ T3,\ T4,\ T5,\ T6\}.$$

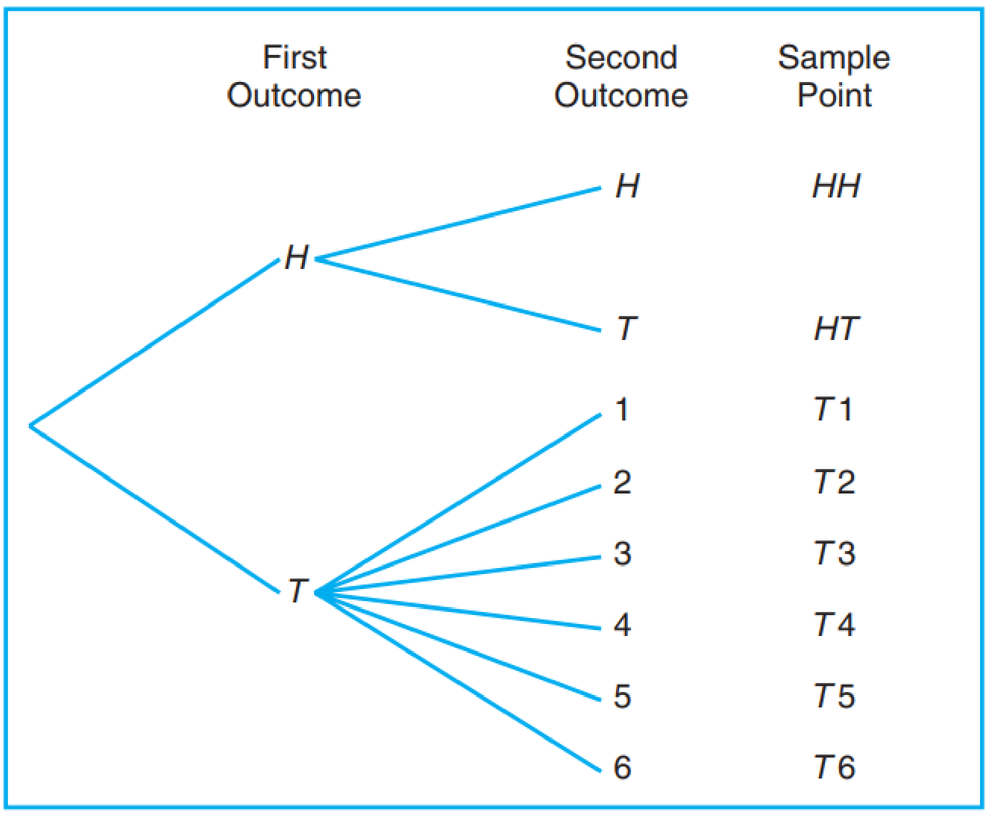

Figure 2.1: Tree diagram for Example 2.2.

Example 2.3: Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, D, or nondefective, N. To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.2. Now, the various paths along the branches of the tree give the distinct sample points. Starting with the first path, we get the sample point DDD, indicating the possibility that all three items inspected are defective. As we proceed along the other paths, we see that the sample space is

$$S = \{DDD,\ DDN,\ DND,\ DNN,\ NDD,\ NDN,\ NND,\ NNN\}.$$

Sample spaces with a large or infinite number of sample points are best described by a **statement** or **rule method**. For example, if the possible outcomes of an experiment are the set of cities in the world with a population over 1 million, our sample space is written

$$S = \{x \mid x \text{ is a city with a population over 1 million}\},$$

which reads "S is the set of all x such that x is a city with a population over 1 million." The vertical bar is read "such that." Similarly, if S is the set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the origin, we write the **rule**

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}.$$

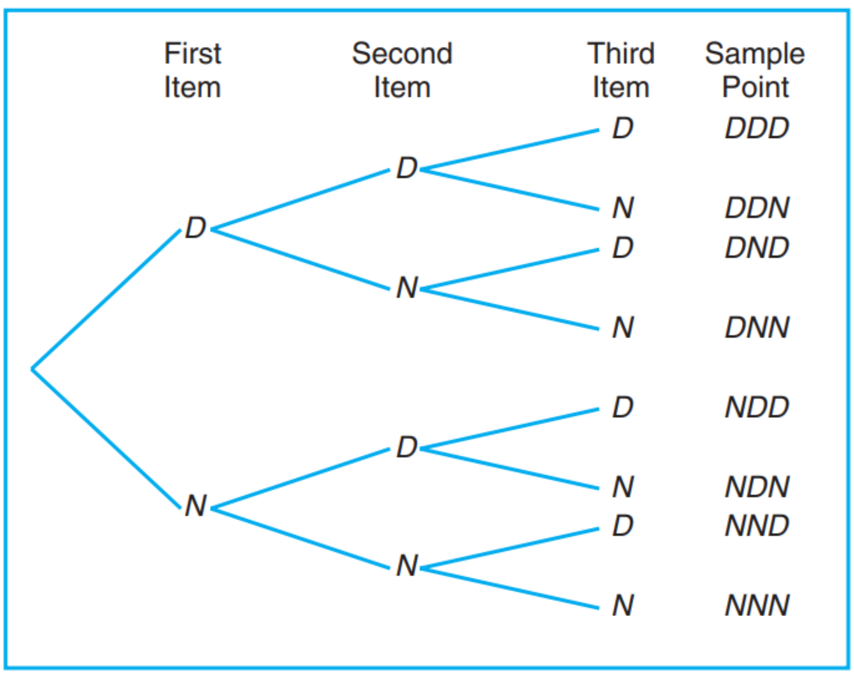

Figure 2.2: Tree diagram for Example 2.3.

Event:

An **event** is a subset of a sample space.

Example 2.4: Given the sample space $S = \{t \mid t \geq 0\}$, where t is the life in years of a certain electronic component, then the event A that the component fails before the end of the fifth year is the subset $A = \{t \mid 0 \leq t < 5\}$.

It is conceivable that an event may be a subset that includes the entire sample space S or a subset of S called the **null set** and denoted by the symbol ϕ, which contains no elements at all. For instance, if we let A be the event of detecting a microscopic organism by the naked eye in a biological experiment, then $A = \phi$. Also, if

$$B = \{x \mid x \text{ is an even factor of } 7\},$$

then B must be the null set, since the only possible factors of 7 are the odd numbers 1 and 7.

Complement of Event:

The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A'.

Example 2.5: Let R be the event that a red card is selected from an ordinary deck of 52 playing cards, and let S be the entire deck. Then R' is the event that the card selected from the deck is not a red card but a black card.

Example 2.6: Consider the sample space

$$S = \{\text{book, cell phone, mp3, paper, stationery, laptop}\}.$$

Let $A = \{\text{book, stationery, laptop, paper}\}$. Then the complement of A is $A' = \{\text{cell phone, mp3}\}$.

Intersection of two events:

The **intersection** of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

Example 2.7: Let E be the event that a person selected at random in a classroom is majoring in engineering, and let F be the event that the person is female. Then $E \cap F$ is the event of all female engineering students in the classroom.

Example 2.8: Let $V = \{a, e, i, o, u\}$ and $C = \{l, r, s, t\}$; then it follows that $V \cap C = \phi$. That is, V and C have no elements in common and, therefore, cannot both simultaneously occur.

Mutually Exclusive or Disjoint Events:

Two events A and B are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is, if A and B have no elements in common.

Example 2.9: A cable television company offers programs on eight different channels, three of which are affiliated with ABC, two with NBC, and one with CBS. The other two are an educational channel and the ESPN sports channel. Suppose that a person subscribing to this service turns on a television set without first selecting the channel. Let A be the event that the program belongs to the NBC network and B the event that it belongs to the CBS network. Since a television program cannot belong to more than one network, the events A and B have no programs in common. Therefore, the intersection $A \cap B$ contains no programs, and consequently the events A and B are mutually exclusive.

Often one is interested in the occurrence of at least one of two events associated with an experiment. Thus, in the die-tossing experiment, if

$$A = \{2, 4, 6\} \text{ and } B = \{4, 5, 6\},$$

we might be interested in either A or B occurring or both A and B occurring. Such an event, called the **union** of A and B, will occur if the outcome is an element of the subset $\{2, 4, 5, 6\}$.

Union of the two events:

The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Example 2.10: Let $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$; then $A \cup B = \{a, b, c, d, e\}$.

Example 2.11: Let P be the event that an employee selected at random from an oil drilling company smokes cigarettes. Let Q be the event that the employee selected drinks alcoholic beverages. Then the event $P \cup Q$ is the set of all employees who either drink or smoke or do both.

Example 2.12: If $M = \{x \mid 3 < x < 9\}$ and $N = \{y \mid 5 < y < 12\}$, then

$$M \cup N = \{z \mid 3 < z < 12\}.$$

The relationship between events and the corresponding sample space can be illustrated graphically by means of **Venn diagrams**. In a Venn diagram we let the sample space be a rectangle and represent events by circles drawn inside the rectangle. Thus, in Figure 2.3, we see that

$$\begin{aligned} A \cap B &= \text{regions 1 and 2,} \\ B \cap C &= \text{regions 1 and 3,} \end{aligned}$$

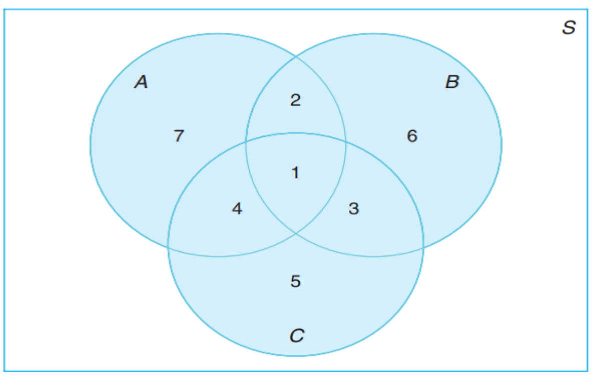

Figure 2.3: Events represented by various regions.

$$\begin{aligned} A \cup C &= \text{regions 1, 2, 3, 4, 5, and 7,} \\ B' \cap A &= \text{regions 4 and 7,} \\ A \cap B \cap C &= \text{region 1,} \\ (A \cup B) \cap C' &= \text{regions 2, 6, and 7,} \end{aligned}$$

and so forth.

Exercises

2.1 List the elements of each of the following sample spaces:

(a) the set of integers between 1 and 50 divisible by 8;

(b) the set $S = \{x \mid x^2 + 4x - 5 = 0\}$;

(c) the set of outcomes when a coin is tossed until a tail or three heads appear;

(d) the set $S = \{x \mid x \text{ is a continent}\}$;

(e) the set $S = \{x \mid 2x - 4 \geq 0 \text{ and } x < 1\}$.

2.2 Use the rule method to describe the sample space S consisting of all points in the first quadrant inside a circle of radius 3 with center at the origin.

2.3 Which of the following events are equal?

(a) $A = \{1, 3\}$;

(b) $B = \{x \mid x \text{ is a number on a die}\}$;

(c) $C = \{x \mid x^2 - 4x + 3 = 0\}$;

(d) $D = \{x \mid x \text{ is the number of heads when six coins are tossed}\}$.

2.4 An experiment involves tossing a pair of dice, one green and one red, and recording the numbers that come up. If x equals the outcome on the green die and y the outcome on the red die, describe the sample space S

(a) by listing the elements (x, y);

(b) by using the rule method.

2.5 An experiment consists of tossing a die and then flipping a coin once if the number on the die is even. If the number on the die is odd, the coin is flipped twice. Using the notation $4H$, for example, to denote the outcome that the die comes up 4 and then the coin comes up heads, and $3HT$ to denote the outcome that the die comes up 3 followed by a head and then a tail on the coin, construct a tree diagram to show the 18 elements of the sample space S.

2.6 Two jurors are selected from 4 alternates to serve at a murder trial. Using the notation A_1A_3, for example, to denote the simple event that alternates 1 and 3 are selected, list the 6 elements of the sample space S.

2.7 Four students are selected at random from a chemistry class and classified as male or female. List the elements of the sample space S_1, using the letter M for male and F for female. Define a second sample space S_2 where the elements represent the number of females selected.

2.8 For the sample space of Exercise 2.4,

(a) list the elements corresponding to the event A that the sum is greater than 8;

(b) list the elements corresponding to the event B that a 2 occurs on either die;

(c) list the elements corresponding to the event C that a number greater than 4 comes up on the green die;

(d) list the elements corresponding to the event $A \cap C$;

(e) list the elements corresponding to the event $A \cap B$;

(f) list the elements corresponding to the event $B \cap C$;

(g) construct a Venn diagram to illustrate the intersections and unions of the events A, B, and C.

2.9 For the sample space of Exercise 2.5,

(a) list the elements corresponding to the event A that a number less than 3 occurs on the die;

(b) list the elements corresponding to the event B that two tails occur;

(c) list the elements corresponding to the event A';

(d) list the elements corresponding to the event $A' \cap B$;

(e) list the elements corresponding to the event $A \cup B$.

2.10 An engineering firm is hired to determine if certain waterways in Virginia are safe for fishing. Samples are taken from three rivers.

(a) List the elements of a sample space S, using the letters F for safe to fish and N for not safe to fish.

(b) List the elements of S corresponding to event E that at least two of the rivers are safe for fishing.

(c) Define an event that has as its elements the points

$$\{FFF, NFF, FFN, NFN\}.$$

2.11 The resumés of two male applicants for a college teaching position in chemistry are placed in the same file as the resumés of two female applicants. Two positions become available, and the first, at the rank of assistant professor, is filled by selecting one of the four applicants at random. The second position, at the rank of instructor, is then filled by selecting at random one of the remaining three applicants. Using the notation M_2F_1, for example, to denote the simple event that the first position is filled by the second male applicant and the second position is then filled by the first female applicant,

(a) list the elements of a sample space S;

(b) list the elements of S corresponding to event A that the position of assistant professor is filled by a male applicant;

(c) list the elements of S corresponding to event B that exactly one of the two positions is filled by a male applicant;

(d) list the elements of S corresponding to event C that neither position is filled by a male applicant;

(e) list the elements of S corresponding to the event $A \cap B$;

(f) list the elements of S corresponding to the event $A \cup C$;

(g) construct a Venn diagram to illustrate the intersections and unions of the events A, B, and C.

2.12 Exercise and diet are being studied as possible substitutes for medication to lower blood pressure. Three groups of subjects will be used to study the effect of exercise. Group 1 is sedentary, while group 2 walks and group 3 swims for 1 hour a day. Half of each of the three exercise groups will be on a salt-free diet. An additional group of subjects will not exercise or restrict their salt, but will take the standard medication. Use Z for sedentary, W for walker, S for swimmer, Y for salt, N for no salt, M for medication, and F for medication free.

(a) Show all of the elements of the sample space S.

(b) Given that A is the set of nonmedicated subjects and B is the set of walkers, list the elements of $A \cup B$.

(c) List the elements of $A \cap B$.

2.13 Construct a Venn diagram to illustrate the possible intersections and unions for the following events relative to the sample space consisting of all automobiles made in the United States.

F : Four door, S : Sun roof, P : Power steering.

2.14 If $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{2, 3, 4, 5\}$, and $D = \{1, 6, 7\}$, list the elements of the sets corresponding to the following events:

(a) $A \cup C$;

(b) $A \cap B$;

(c) C';

(d) $(C' \cap D) \cup B$;

(e) $(S \cap C)'$;

(f) $A \cap C \cap D'$.

2.15 Consider the sample space $S = \{$copper, sodium, nitrogen, potassium, uranium, oxygen, zinc$\}$ and the events

$$A = \{\text{copper, sodium, zinc}\},$$
$$B = \{\text{sodium, nitrogen, potassium}\},$$
$$C = \{\text{oxygen}\}.$$

List the elements of the sets corresponding to the following events:

(a) A';

(b) $A \cup C$;

(c) $(A \cap B') \cup C'$;

(d) $B' \cap C'$;

(e) $A \cap B \cap C$;

(f) $(A' \cup B') \cap (A' \cap C)$.

2.16 If $S = \{x \mid 0 < x < 12\}$, $M = \{x \mid 1 < x < 9\}$, and $N = \{x \mid 0 < x < 5\}$, find

(a) $M \cup N$;

(b) $M \cap N$;

(c) $M' \cap N'$.

2.17 Let A, B, and C be events relative to the sample space S. Using Venn diagrams, shade the areas representing the following events:

(a) $(A \cap B)'$;

(b) $(A \cup B)'$;

(c) $(A \cap C) \cup B$.