Orthogonal and Orthonormal Sets

Definition 1

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

EXAMPLE 1 | An Orthogonal Set in R^3

Let

$$\mathbf{v}_1 = (0,1,0), \quad \mathbf{v}_2 = (1,0,1), \quad \mathbf{v}_3 = (1,0,-1)$$

and assume that R^3 has the Euclidean inner product. It follows that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set since $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$.

EXAMPLE 2 | Constructing an Orthonormal Set

The Euclidean norms of the vectors in Example 1 are

$$\|\mathbf{v}_1\| = 1, \quad \|\mathbf{v}_2\| = \sqrt{2}, \quad \|\mathbf{v}_3\| = \sqrt{2}$$

Consequently, normalizing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ yields

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (0,1,0), \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

We leave it for you to verify that the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthonormal by showing that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0 \quad \text{and} \quad \|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$$

Theorem 6.3.1

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

EXAMPLE 3 | An Orthonormal Basis for P_n

Recall from Example 7 of Section 6.1 that the standard inner product of the polynomials

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n \quad \text{and} \quad \mathbf{q} = b_0 + b_1x + \cdots + b_nx^n$$

is

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$$

and the norm of $\mathbf{p}$ relative to this inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$$

Using these formulas you should be able to show that the standard basis

$$S = \{1, x, x^2, \ldots, x^n\}$$

is orthonormal with respect to this inner product (verify).

EXAMPLE 4 | An Orthonormal Basis

In Example 2 we showed that the vectors

$$\mathbf{u}_1 = (0, 1, 0), \quad \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \text{and} \quad \mathbf{u}_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

form an orthonormal set with respect to the Euclidean inner product on R^3. By Theorem 6.3.1, these vectors form a linearly independent set, and since R^3 is three-dimensional, it follows from Theorem 4.6.4 that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for R^3.

Coordinates Relative to Orthonormal Bases

Theorem 6.3.2

(*a*) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \tag{3}$$

(*b*) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n \tag{4}$$

Using the terminology and notation from Definition 2 of Section 4.5, it follows from Theorem 6.3.2 that the coordinate vector of a vector $\mathbf{u}$ in V relative to an orthogonal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

$$\boxed{(\mathbf{u})_S = \left(\frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}, \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}, \ldots, \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\right)} \tag{6}$$

and relative to an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

$$\boxed{(\mathbf{u})_S = \left(\langle \mathbf{u}, \mathbf{v}_1\rangle, \langle \mathbf{u}, \mathbf{v}_2\rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n\rangle\right)} \tag{7}$$

EXAMPLE 5 | A Coordinate Vector Relative to an Orthonormal Basis

Let

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = \left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right), \quad \mathbf{v}_3 = \left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right)$$

It is easy to check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for R^3 with the Euclidean inner product. Express the vector $\mathbf{u} = (1, 1, 1)$ as a linear combination of the vectors in S, and find the coordinate vector $(\mathbf{u})_S$.

Solution We leave it for you to verify that

$$\langle \mathbf{u}, \mathbf{v}_1\rangle = 1, \quad \langle \mathbf{u}, \mathbf{v}_2\rangle = -\tfrac{1}{5}, \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v}_3\rangle = \tfrac{7}{5}$$

Therefore, by Theorem 6.3.2 we have

$$\mathbf{u} = \mathbf{v}_1 - \tfrac{1}{5}\mathbf{v}_2 + \tfrac{7}{5}\mathbf{v}_3$$

that is,

$$(1, 1, 1) = (0, 1, 0) - \tfrac{1}{5}\left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right) + \tfrac{7}{5}\left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right)$$

Thus, the coordinate vector of $\mathbf{u}$ relative to S is

$$(\mathbf{u})_S = \left(\langle \mathbf{u}, \mathbf{v}_1\rangle, \langle \mathbf{u}, \mathbf{v}_2\rangle, \langle \mathbf{u}, \mathbf{v}_3\rangle\right) = \left(1, -\tfrac{1}{5}, \tfrac{7}{5}\right)$$

EXAMPLE 6 | An Orthonormal Basis from an Orthogonal Basis

(*a*) Show that the vectors

$$\mathbf{w}_1 = (0, 2, 0), \quad \mathbf{w}_2 = (3, 0, 3), \quad \mathbf{w}_3 = (-4, 0, 4)$$

form an orthogonal basis for R^3 with the Euclidean inner product, and use that basis to find an orthonormal basis by normalizing each vector.

(*b*) Express the vector $\mathbf{u} = (1, 2, 4)$ as a linear combination of the orthonormal basis vectors obtained in part (*a*).

Solution (*a*) The given vectors form an orthogonal set since

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0, \quad \langle \mathbf{w}_1, \mathbf{w}_3 \rangle = 0, \quad \langle \mathbf{w}_2, \mathbf{w}_3 \rangle = 0$$

It follows from Theorem 6.3.1 that these vectors are linearly independent and hence form a basis for R^3 by Theorem 4.6.4. We leave it for you to calculate the norms of $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_3$ and then obtain the orthonormal basis

$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = (0, 1, 0), \quad \mathbf{v}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{v}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Solution (*b*) It follows from Formula (4) that

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3$$

We leave it for you to confirm that

$$\langle \mathbf{u}, \mathbf{v}_1 \rangle = (1, 2, 4) \cdot (0, 1, 0) = 2$$

$$\langle \mathbf{u}, \mathbf{v}_2 \rangle = (1, 2, 4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$$

$$\langle \mathbf{u}, \mathbf{v}_3 \rangle = (1, 2, 4) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$$

and hence that

$$(1, 2, 4) = 2(0, 1, 0) + \frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + \frac{3}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Question:

10. Verify that the vectors

$$\mathbf{v}_1 = (1,-1,2,-1), \quad \mathbf{v}_2 = (-2,2,3,2),$$
$$\mathbf{v}_3 = (1,2,0,-1), \quad \mathbf{v}_4 = (1,0,0,1)$$

form an orthogonal basis for R^4 with respect to the Euclidean inner product, and then use Theorem 6.3.2(a) to express the vector $\mathbf{u} = (1,1,1,1)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$.

Solution:

$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -2-2+6-2=0$ $\quad$ $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 1-2+0+1=0$ $\quad$ $\langle \mathbf{v}_1, \mathbf{v}_4 \rangle = 1+0+0-1=0$

$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = -2+4+0-2=0$ $\quad$ $\langle \mathbf{v}_2, \mathbf{v}_4 \rangle = -2+0+0+2=0$ $\quad$ $\langle \mathbf{v}_3, \mathbf{v}_4 \rangle = 1+0+0-1=0$

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim(R^4)=4$, this set forms a basis for R^4 by Theorem 4.6.4. We use Theorem 6.3.2(a):

$$\begin{aligned}
\mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 + \frac{\langle \mathbf{u}, \mathbf{v}_4 \rangle}{\|\mathbf{v}_4\|^2}\mathbf{v}_4 \\
&= \frac{1-1+2-1}{1+1+4+1}\mathbf{v}_1 + \frac{-2+2+3+2}{4+4+9+4}\mathbf{v}_2 + \frac{1+2+0-1}{1+4+0+1}\mathbf{v}_3 + \frac{1+0+0+1}{1+0+0+1}\mathbf{v}_4 \\
&= \frac{1}{7}\mathbf{v}_1 + \frac{5}{21}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3 + 1\mathbf{v}_4.
\end{aligned}$$