Chp 02

Minor and Cofactors

Q.1) Find cofactor & minor

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$

Solution:

Minor M

$\Rightarrow M_{11} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix}$

$\Rightarrow M_{11} = 7(4) - (-1)(1)$

$\Rightarrow M_{11} = 28 + 1$

$\therefore M_{11} = 29$

Cofactor C

$C_{ij} = (-1)^{i+j} M_{ij}$

$$\text{Cofactor} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$C_{11} = (-1)^{1+1}(29)$

$C_{11} = (-1)^2(29)$

$C_{11} = 29$

$M_{12} = \begin{vmatrix} 6 & -1 \\ -3 & 4 \end{vmatrix}$

$M_{12} = 6(4) - (-3)(-1)$

$24 - 3$

$M_{12} = 21$

Cofactor

$C_{12} = (-1)^{1+2}(21)$

$-1(21)$

$\therefore \boxed{C_{12} = -21}$

$M_{13} = \begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix}$

$M_{13} = 6(1) - (7)(-3)$

$6 + 21$

$M_{13} = 27$

$M_{13} = (-1)^{1+3}(-27)$

$(1)(-27)$

$M_{13} = +27$

$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix}$

$(-2)(4) - (3)(1)$

$-8 - 3$

$M_{21} = -11$

$C_{21} = (-1)^{2+1}(-11)$

$(-1)^{3}(-11)$

$C_{21} = 11$

$M_{22} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix}$

$1(4) - 3(-3)$

$4 + 9$

$M_{22} = 13$

$C_{22} = (-1)^{2+2}(13)$

$C_{22} = 13$

$M_{23} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix}$

$1(1) - (-2)(-3)$

$1 - 6$

$M_{23} = -5$

$C_{23} = (-1)^{2+3}(-5)$

$C_{23} = (-1)(-5)$

$C_{23} = 5$

$M_{31} = \begin{vmatrix} -2 & 3 \\ 7 & -1 \end{vmatrix}$

$M_{31} = -2(-1) - 3(7)$

$M_{31} = 2 - 21$

$M_{31} = -19$

$C_{31} = (-1)^{3+1}(-19)$

$C_{31} = (1) - 19$

$C_{31} = -19$

$M_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 6 & -1 \end{vmatrix}$

$(-1)^{5} \quad (1)(-1) - 3(6)$

$(-1) \quad -1 - 18$

-19

$M_{32} = 19$

$C_{32} = 19$

$$M_{33} = \begin{vmatrix} 1 & -2 \\ 6 & 7 \end{vmatrix}$$

$1(7) - (-2)(6)$

$7 + 12$

19

$(-1)^{3+3}(19)$

$C_{33} = 19$

$$\begin{bmatrix} 29 & -21 & 27 \\ 11 & 13 & 5 \\ -19 & 19 & 19 \end{bmatrix}$$

Q.2) Find the inverse

$$\begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj of }(A)}{|A|}$$

$|A| = (-5)(-2) - 7(-7)$

$|A| = 10 + 49$

$|A| = 59$

$|A| > 0$ hence it is non-singular matrix

$$\text{Adj of }(A) = \begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix}}{59}$$

$$A^{-1} = \begin{bmatrix} \frac{-2}{59} & \frac{-7}{59} \\ \frac{7}{59} & \frac{-5}{59} \end{bmatrix}$$

Q.) Find the determinant of (A)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 5 & 6 \\ -8 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 5 \\ 7 & -8 \end{vmatrix}$$

$= 1\left(5(9) - 6(-8)\right) - 2\left(-4(9) - 6(7)\right) + 3$

$-4 \times -8 - 5 \times 7$

$= (45 + 48) - 2(-36 - 42) + 3(32 - 35)$

$= 93 - 2(-78) + 3(-3)$

$= 93 + 156 - 9$

$= 240$

$156 + 93 = 249$

ARROW METHOD

Find det of A

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{vmatrix}$$

$= -[7 \times 5 \times 3 + -8 \times 6 \times 1 + 9 \times -4 \times 2]$
$+ [1 \times 5 \times 9 + 2 \times 6 \times 7 + 3 \times -4 \times$

$= -105 + 48 + 72 + 45 + 84 + 96$

$= +240$

CRAMER'S RULE

For example, a general 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

$a_{23} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{|A_1|}{|A|}$$

$$x_2 = \frac{|A_2|}{|A|}$$

$$x_3 = \frac{|A_3|}{|A|}$$

$$A_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

$$\Rightarrow A_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{bmatrix}$$

Q.1) Solve the system of equations

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 9x_2 + 3x_3 = -1$$
$$x_1 + 4x_3 = 9$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}$$

$$x_1 = \frac{|A_1|}{|A|}$$

First finding determinant

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

$$1\begin{vmatrix} 9 & 3 \\ 0 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 9 \\ 1 & 0 \end{vmatrix}$$

$$(36 - 0) - 2(8 - 3) + 3(0 - 9)$$
$$36 - 2(5) + 3(-9)$$
$$36 - 10 - 27$$
$$= 26 - 27$$
$$= -1$$

$$|A_1| = \begin{vmatrix} 5 & 2 & 3 \\ -1 & 9 & 3 \\ 9 & 0 & 4 \end{vmatrix}$$

$$|A_1| = 5\begin{vmatrix} 9 & 3 \\ 0 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 9 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 9 \\ 9 & 0 \end{vmatrix}$$

$$= 5(36 - 0) - 2(-4 - 27) + 3(0 - 81)$$
$$= 5(36) - 2(-31) + 3(-81)$$
$$= 180 + 62 - 243$$

$$\begin{array}{r} 243 \\ -180 \\ \hline 063 \end{array}$$

$$= 62 - 63$$
$$= -1$$

$$|A_2| = \begin{vmatrix} 1 & 5 & 3 \\ 2 & -1 & 3 \\ 1 & 9 & 4 \end{vmatrix}$$

$$|A_2| = 1\begin{vmatrix} -1 & 3 \\ 9 & 4 \end{vmatrix} - 5\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix}$$

$$= (-4 - 27) - 5(8 - 3) + 3(18 + 1)$$
$$= -31 - 5(5) + 3(19)$$
$$= -31 - 25 + 57$$
$$= -56 + 57$$
$$= 1$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 9 & -1 \\ 1 & 0 & 9 \end{vmatrix}$$

$$|A_3| = 1\begin{vmatrix} 9 & -1 \\ 0 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & 9 \end{vmatrix} + 5\begin{vmatrix} 2 & 9 \\ 1 & 0 \end{vmatrix}$$

$= (81-0) - 2(18+1) + 5(0-9)$

$= 81 - 2(19) + 5(-9)$

$= 81 - 38 - 45$

$= 81 - 83$

$= -2$

$$\begin{array}{r} 38 \\ +45 \\ \hline 83 \end{array}$$

$x_1 = \frac{|A_1|}{|A|} = \frac{-1}{-1} = 1$

$x_2 = \frac{|A_2|}{|A|} = \frac{1}{-1} = -1$

$x_3 = \frac{|A_3|}{|A|} = \frac{-2}{-1} = 2$

CHAPTER 04 "REAL & VECTOR SPACES"

LINEAR COMBINATION:-

If w is a vector space V, then w is said to be a l.c of the vector $V_1, V_2, \ldots V_r$ in V, if w can be expanded in the form

$$w = K_1 V_1 + K_2 V_2 + \cdots K_r V_r$$

R in scalar

Q.1) Consider the vectors

$V = (1, 2, -1)$ & $v = (6, 4, 2)$ in $\mathbb{R}^2$

Show that $w = (9, 2, 9)$ is a linear Combination.

(b) v & v and then $w' = (4, -1, 8)$ is not linear combination of u & v.

A) SOLUTION :-

Let $u = (1, 2, -1)$, $v = (6, 4, 2)$
and $w = (9, 2, 7)$

By linear combination

$$w = k_1 u + k_2 v$$

$$(9, 2, 7) = k_1 (1, 2, -1) + k_2 (6, 4, 2)$$

$$k_1 + 6k_2 = 9$$
$$2k_1 + 4k_2 = 2$$
$$-1k_1 + 2k_3 = 7$$

The augmented matrix for the system of equation is

$$= \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix}$$

$-2R_1 + R_2 \longrightarrow R_2$ & $+1R_1 + R_3 \longrightarrow R_3$

$$= \begin{bmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{bmatrix}$$

Multiplying R_2 by $-1/8$

$$= \begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{bmatrix}.$$

$-8R_2 + R_3 \longrightarrow R_3$

$$= \begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k_1 + 6k_2 = 9$$
$$k_2 = 2$$

$k_2 = 2$

$k_1 = 9 - 6(2)$

$9 - 12$

$\therefore k_1 = -3$

$\therefore w = -3u + 2v$

so, w is a linear combination of u & v

b) SOLUTION

Let $u = (1, 2, -1)$, $v = (6, 4, 2)$.
and $w = (4, -1, 8)$

By Linear Combination

$w = k_1 u + k_2 v$

$4, -1, 8 = k(1, 2, -1) + k_2 (6, 4, 2)$

$k_1 + 6k_2 = 4$
$2k_1 + 4k_2 = -1$
$-k_1 + 2k_2 = 8$

The augmented matrix for the system of equations is

$$= \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix} \quad \begin{matrix} -2R + R_2 \to R_2 \\ 9R_1' + R_3 \to R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{bmatrix}$$

$R_2 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus the system is inconsistent if
$0 + 0 = 4$ It is not a linear combination

SPAN

If (det) of a matrix $\neq 0$, then it is span, otherwise not span

If system is consistent, then it is span, otherwise, it is not span

$|A| \neq 0$

Determine whether the vector $v = (1, 1, 2)$ $v_2 = (1, 0, 1)$ $v_3 = (2, 1, 3)$ is a span vector space in R^3

Let
$v = (1, 1, 2)$ $v_2 = (1, 0, 1)$
& $v_3 = (2, 1, 3)$

By linear Combination

$$w = K_1 v_1 + K_2 v_2 + K_3 v_3$$

$$(w_1, w_2, w_3) = K(1, 1, 2) + K_2(1, 0, 1) + K_3(2, 1, 3)$$

$$K_1 + K_2 + 2K_3 = w_1$$
$$K_1 + 0K_2 + 1K_3 = w_2$$
$$2K_1 + 1K_2 + 3K_3 = w_3$$

It is a square matrix

$$= \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$1 \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 1(0 - 1) - 1(3 - 2) + 2((1) - 0)$$
$$= 1(-1) - 1(+1) + 2$$
$$= -1 - 1 + 2$$
$$= 0$$

Thus, $|A| = 0$, therefore it is not a span.

Which of the following are linear combination of $u(0, -2, 2)$ & $v = (1, 3, -1)$?

a) $(2, 2, 2)$

b) $(0, 4, 5)$

c) $(0, 0, 0)$

A) Let

$U = (0, -2, 2)$, $v = (1, 3, -1)$

By linear combination

$w = K_1 U + K_2 V$

$(2, 2, 2) = K_1(0, -2, 2) + K_2(1, 3, -1)$

$0K_1 + K_2 = 2$

$-2K_1 + 3K_2 = 2$

$2K_1 - K_2 = 2$

$$= \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

By Echelon form

Interchange R_1 with R_2

$$= \begin{bmatrix} -2 & 3 & 2 \\ 0 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

$\frac{1}{2}R_1$

$$= \begin{bmatrix} 1 & -3/2 & -1 \\ 0 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

$-2R_1 + R_3 \longrightarrow R_3$

$$= \begin{bmatrix} 1 & -3/2 & -1 \\ 0 & 1 & 2 \\ 0 & +2 & 4 \end{bmatrix}$$

$-2R_1 + R_3 \longrightarrow R_3$

$$= \begin{bmatrix} 1 & -3/2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$K_1 - \frac{3}{2}K_2 = -1$

$K_2 = 2$

$K_1 = -1 + \frac{3}{2}(2)$

$K_1 = 2$

$w = 2U + 2V$

Thus, the system is consistent

$(0, 4, 5) = K_1(0, -2, 2) + K_2(1, 3, -1)$

$0K_1 + K_2 = 0$
$-2K_1 + 3K_2 = 4$
$2K_1 - 1K_2 = 5$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{bmatrix}$$

Interchange R_1 with R_2

$$= \begin{bmatrix} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 2 & -1 & 5 \end{bmatrix}$$

$-\frac{1}{2} R_1$

$$= \begin{bmatrix} 1 & -3/2 & -2 \\ 0 & 1 & 0 \\ 2 & -1 & 5 \end{bmatrix}$$

$-2R_1 + R_3 \rightarrow R_3$

$$= \begin{bmatrix} 1 & -3/2 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 9 \end{bmatrix}$$

$-2R_2 + R_3 \rightarrow R_3$

$$= \begin{bmatrix} 1 & -3/2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Thus, the system is inconsistent, so, it is not a span

$(0, 0, 0) = K_1(0, -2, 2) + K_2(1, 3, -1)$

$0K_1 + 1K_2 = 0$
$-2K_1 + 3K_2 = 0$
$2K_1 + (-K_2) = 0$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

Interchange R_1 with R_2

$$= \begin{bmatrix} -2 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

$-\frac{1}{2} R_1$

$$= \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

$-R_1 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Thus

$-2R_2 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$K_1 - 3/2 K_2 = 0$

$K_2 = 0$

$K_2 = 0$

$K_1 = -3/2(0)$

$K_1 = 0$

$w = 0 + 0$

Thus, it is a linear combination of u & v

Q.2)

which of the linear combination

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

a.) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$

b.) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

c.) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

By linear combination

$K_1 A + K_2 B + K_3 C = w$

$$K_1 \begin{bmatrix} 4 & 0 \\ -2 & 2 \end{bmatrix} + K_2 \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} + K_3 \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

$$4k_1 + k_2 + 0k_3 = 6$$
$$0k_1 - k_2 + 2k_3 = -8$$
$$-2k_1 + 2k_2 + k_3 = -1$$
$$-2k_1 + 3k_2 + 4k_3 = -8$$

The augmented matrices for the system of equation is:

$$= \begin{bmatrix} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{bmatrix}$$

$+\frac{1}{4} R_1$

$$= \begin{bmatrix} 1 & 1/4 & 0 & 6/4 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{bmatrix}$$

$+2R_1 + R_3 \rightarrow R_3$ & $+2R_1 + R_4 \rightarrow R_4$

$$= \begin{bmatrix} 1 & 1/4 & 0 & 6/4 \\ 0 & -1 & 2 & -8 \\ 0 & 5/2 & 1 & 2 \\ 0 & 7/2 & 4 & -5 \end{bmatrix}$$

$+(-R_2)$

$$= \begin{bmatrix} 1 & 1/4 & 0 & 6/4 \\ 0 & +1 & -2 & +8 \\ 0 & 5/2 & 1 & 2 \\ 0 & 7/2 & 4 & -5 \end{bmatrix}$$

$-\frac{5}{2} R_2 + R_3 \rightarrow R_3$ & $-\frac{7}{2} R_2 + R_4 \rightarrow R_4$

$$= \begin{bmatrix} 1 & 1/4 & 0 & 6/4 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 6 & -18 \\ 0 & 0 & 11 & -33 \end{bmatrix}$$

$-\cancel{8}^{4} \times \frac{7}{2}$

$-28-5$

$$= \begin{bmatrix} 1 & 1/4 & 0 & 6/4 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 6 & -18 \\ 0 & 0 & 11 & -33 \end{bmatrix}$$

$+\frac{1}{6} R_3$

$$= \begin{bmatrix} 1 & 1/4 & 0 & 6/4 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 11 & -33 \end{bmatrix}$$

$-11 R_3 + R_4$

$$= \begin{bmatrix} 1 & 1/4 & 0 & 6/4 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_1 + \tfrac{1}{4}K_2 = 6/4$$
$$K_2 - 2K_3 = 8$$
$$K_3 = -3$$

$$K_3 = -3$$
$$K_2 = 8 + 2(-3)$$
$$8 - 6$$
$$K_2 = 2$$

$$K_1 = 6/4 - \tfrac{1}{4}(2)$$
$$K_1 = \frac{6}{4} - \frac{1}{2}$$
$$K_1 = \frac{6-2}{4}$$
$$\frac{4}{4}$$
$$K_1 = 1$$

$\therefore$ ① $\Rightarrow 1A + 2B - 3C = W$

Q-3)

Which of the following polynomial is in Linear combination

$$P_1 = 2 + x + x^2$$
$$P_2 = 1 - x^2$$
$$P_3 = 1 + 2x$$

A) $1 + x$

B) $1 + x^2$

C) $1 + x + x^2$

By Linear combination

$$K_1 P_2 + K_2 P_2 + K_3 P_3 = P$$

$$K_1(2 + x + x^2) + K_1(1 - x^2) + K_2(1 + 2x) = 1 + x$$

$$2K_1 + K_2 + K_3 = 1$$
$$K_1 + 0K_2 + 2K_3 = 1$$
$$K_1 - K_2 + 0K_3 = 0$$

$$= \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$+1/2 R_1$

$$= \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$-1R_1 + R_2 \to R_2$ & $-R_1 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 0 & -1/2 & -1/2 & \\ 0 & +3/2 & & \end{bmatrix}$$

Ex 4.3

Q.1) Let $T_A : R^2 \to R^2$ be mull by A. Determine whether the vector $v_1 = (1,2)$ is in the span of $[T_A(e_1), T_A(e_2)]$

a.) $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

b.) $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T_A(e_1) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T_A(e_2) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

By Linear Combination

$K_1[T_A(e_1)] + K_2[T_A(e_2)] = U$

$K_1(1, 0) + K_2(2, 1) = (1, 2)$

$K_1 + 2K_2 = 1$
$0K_1 + K_2 = 2$

Augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

This is echelon form

$K_2 = 2$
$K_1 = -3$

$\therefore \; ① \Rightarrow -3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = (1, 2)$

Hence U is the system of $[T_A(e_1), T_A(e_2)]$

It is a span

If the system has infinitely many solution then it is not span

Q.2) Let w be the sol space to the system $Ax = 0$. Determine whether the set $[u, v]$ span w.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

a.) $U = (1, 0, -1, 0)$, $V = (0, 1, 0, -1)$

Augmented matrix

By $Ax = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Reduced echelon

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

Let $x = s$
$y = t$

$x + z = 0$
$y + w = 0$

$z = -s$
$w = -t$

$(x, y, z, w) = (s, t, -s, -t)$

$= s(1, 0, -1, 0) + t(0, 1,$

$= [u, v]$ spam

Q-3) $T_A : R^2 \rightarrow R^3$ be mul by A and let $u_1 = (1, 2)$ & $u_2 = (-1, 1)$ Determine whether the set $[T_A(u_1), T_A(u_2)]$ span.

$T_A(u_1) =$

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$T_A(u_1) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$T_A(u_2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T_A(u_1) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$T_A(u_2) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

By linear combination

K_1

By L.C

$K_1[T_A(U_1) + K_2[T_A(U_2) = (1, 2)$

$$|A| = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Q-4) Let

$v_1 = (1, 6, 4)$

$v_2 = (2, 4, -1)$

$v_3 = (-1, 2, 5)$

$w_1 = (1, -2, -5)$

$w_2 = (0, 8, 9)$

Show that span $[v_1, v_2, v_3] = [w_1, w_2]$

i) By Linear combination

$K_1 v_1 + K_2 v_2 + K_3 v_3 = w_1$

$K_1, K_2, K_3 = ?$

$K_1(1, 6, 4) + K_2(2, 4, -1) + K_3(-1, 2, 5) = (1, -2, -5)$

$1K_1 + 2K_2 - K_3 = 1$

$6K_1 + 4K_2 + 2K_3 = -2$

$4K_1 - K_2 + 5K_3 = -5$

$$= \begin{bmatrix} 1 & 2 & -1 & 1 \\ 6 & 4 & 2 & -2 \\ 4 & -1 & 5 & -5 \end{bmatrix}$$

$-6R_1+R_2 \to R_2$ & $-4R_1+R_3 \to R_3$

$$=\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -8 & 8 & -8 \\ 0 & -9 & 9 & -9 \end{bmatrix}$$

$-\times 1/8\, R_2$

$$=\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -9 & 9 & -9 \end{bmatrix}$$

$+9R_2+R_3 \to R_3$

$$=\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1/17\, R_3$

$$=\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$+R_3+R_2 \to R_2$ & $+R_3+R_1 \to R_1$

$$=\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$-2R_2+R_1 \to R_1$

$$=\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$k_3 = 0$
$k_2 = 1$
$k_1 = -1$

$k_1 + 2k_2 - k_3 = 1$
$k_2 - k_3 = 1$

$k_3 = t$
$k_2 = t+1$
$k_1 = 1+t-2(t+1)$
$\therefore k_1 = 1+t-2t-2$
$k_1 = -1-t$
$(-1-t,\ t+1,\ t)$

ii) $k_1 v_1 + k_2 v_2 + k_3 v_3 = w_2$

$k_1, k_2, k_3 = ?$

$k_1(1,6,4) + k_2(2,4,-1) + k_3(-1,2,$
$= (0,8,9)$

$k_1 + 2k_2 - k_3 = 0$
$6k_1 + 4k_2 + 2k_3 = 8$
$4k_1 - k_2 + 5k_3 = 9$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 6 & 4 & 2 & 8 \\ 4 & -1 & 5 & 9 \end{bmatrix}$$

$-6R_1 + R_2 \to R_2$ & $-4R_1 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -8 & 8 & 8 \\ 0 & -9 & 9 & 9 \end{bmatrix}$$

$-1/8 R_2$

$$= \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -9 & 9 & 9 \end{bmatrix}$$

$+1/9 R_2 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= k_1 + 2k_2 - k_3 = 0$
$k_2 - k_3 = -1$

$k_3 = t$

$k_2 = -1 + t$
$k_1 = 0 + t - 2(-1+t)$
$t + 2 - 2t$

$k_1 = 2 - t$

$= (2-t, -1+t, t)$

$(2-t, -1+t, t)$ $(-1-t, t+1, t)$

$(-1, 1, 0)$ $(2, -1, 0)$

Linearly Independent

$K_1 v_1 + K_2 v_2 + K_3 v_3 + \dots K_n v_n = 0$

$K_1 = K_2 = K_3 \dots K_n = 0$

Q.1) Determine whether the vector

$v = (1, -2, 3)$

$v_2 = (5, 6, -1)$

$v_3 = (3, 2, 1)$

are linearly independent in $\mathbb{R}^3$

By linear combination

$K_1 v_1 + K_2 v_2 + K_3 v_3 = 0$

$K_1(1, -2, 3) + K_2(5, 6, -1) + K_3(3, 2, 1)$

$K_1 + 5K_2 + 3K_3 = 0$

$-2K_1 + 6K_2 + 2K_3 = 0$

$3K_1 - K_2 + K_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right]$$

$+2R_1 + R_2 \to R_2$ & $-3R_1 + R_3 \to R_3$

$$= \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{array}\right]$$

$+\frac{1}{16} R_2$

$$= \left[\begin{array}{cccc} 1 & 5 & 3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & -16 & -8 & 0 \end{array}\right]$$

$+16R_2 + R_3 \to R_3$

$$= \left[\begin{array}{cccc} 1 & 5 & 3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$K_1 + 5K_2 + 3K_3 = 0$

$K_2 + \frac{1}{2}K_3 = 0$

$K_3 = t$

$K_2 = -\frac{1}{2}t$

$K_1 = -3t - 5(-\frac{1}{2}t)$

$K_1 = -3t + \frac{5}{2}t = -\frac{t}{2}$

They are linearly dependent

In a coplanar, three

Q.) In each part, determine whether three vector lies in plane $\mathbb{R}^3$

$v_1 = (2, -2, 0)$
$v_2 = (6, 1, 4)$
$v_3 = (2, 0, -4)$

By Linear Combination

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$k_1(2, -2, 0) + k_2(6, 1, 4) + k_3(2, 0, -4) = 0$

$2k_1 + 6k_2 + 2k_3 = 0$
$-2k_1 + 1k_2 + 0k_3 = 0$
$0k_1 + 4k_2 - 4k_3 = 0$

$$\begin{bmatrix} 2 & 6 & 2 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 4 & -4 & 0 \end{bmatrix}$$

$+\frac{1}{2} R_1$

$$= \begin{bmatrix} 1 & 3 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 4 & -4 & 0 \end{bmatrix}$$

$+2R_1 + R_2 \to R_2$

$$= \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 7 & 2 & 0 \\ 0 & 4 & -4 & 0 \end{bmatrix}$$

$+\frac{1}{7} R_2$

$$= \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2/7 & 0 \\ 0 & 4 & -4 & 0 \end{bmatrix}$$

$-4R_2 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2/7 & 0 \\ 0 & 0 & -36/7 & 0 \end{bmatrix}$$

$\frac{2}{7} \times -4$
$\frac{-8}{7}$
$-4 - \frac{8}{7}$
$\frac{-28-8}{7}$
$-\frac{36}{7}$

$-7/36 \; R_3$

$$= \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2/7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$k_1 + 3k_2 + k_3 = 0$
$k_2 + 2/7 \, k_3 = 0$
$k_3 = 0$

$k_2 = 0,\ k_3 = 0$

$k_1 = 0$

Since $k_1 = 0, k_2 = 0, k_3 = 0$, so they are linearly independent.

Q) In each part, whether the three vector lie on the same line in $\mathbb{R}^3$

$v_1 = (-1, 2, 3)$
$v_2 = (2, -4, -6)$
$v_3 = (-3, 6, 0)$

By linear combination

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$k_1(-1, 2, 3) + k_2(2, -4, -6) + k_3(-3, 6, 0) = 0$

$-1k_1 + 2k_3 - 3k_3 = 0$
$2k_1 - 4k_2 + 6k_3 = 0$
$3k_1 - 6k_2 + 0k_3 = 0$

$$\begin{bmatrix} -1 & 2 & -3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 0 & 0 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$ & $-3R_1 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 0 \end{bmatrix}$$

$-1/9\, R_3$

$$= \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ Interchange R_2 with R_3

$k_3 = 1$

$k_1 - 2k_2 + 3k_3 = 0$

$k_2 = t$
$k_1 = -3 + 2t$

$(-3 + 2t, t, 1)$

Since $k_1, k_2, k_3 \neq 0$, so they are linearly dependent

Ex 4.5

Basis
If $S = \{v_1, v_2, \dots v_n\}$ is set of vectors in a finite-dimensional space V, then S is called a basis for V, If

① S span V
② S is linearly independent

Q.) Show that $v_1 = (1, 2, 1)$
$v_2 = (2, 9, 0)$
$v_3 = (3, 3, 4)$
form a basis of $\mathbb{R}^3$?

By Linear combination
$K_1 v_1 + k_2 v_2 + K_3 v_3 = \omega$

By linearly independent

$K_1 v_1 + K_2 v_2 + k_3 w_3 = 0$

$\therefore$ ② $\Rightarrow K_1(1, 2, 1) + K_2(2, 9, 0) + K_3(3, 3, 4) = \omega$

$\Rightarrow$

$K_1 + 2K_2 + 3K_3 = \omega_1$
$2K_1 + 9K_2 + 3K_3 = \omega_2$
$1K_1 + 0K_2 + 4K_3 = \omega_3$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$ & $-R_1 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -3 \\ 0 & -2 & 1 \end{bmatrix}$$

$\frac{1}{5} R_2$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3/5 \\ 0 & -2 & 1 \end{bmatrix}$$

$\frac{-3 \times 2}{5}$

$\frac{1-6}{5}$

$+2R_2 + R_3 \to R_3$

$\frac{5-6}{5}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3/5 \\ 0 & 0 & -1/5 \end{bmatrix}$$

$\frac{-1}{5}$

$-5R_3$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1 \end{bmatrix}$$

~~$x_1 + 2x$~~

$k_1 + 2k_2 + 3k_3 = 0$

$k_2 - 3/5\, k_3 = 0$

$k_3 = 1$

$k_3 = 1$

$k_2 = 3/5$

$k_1 = -3 - 2(3/5)$

$k_1 = -1 - 6/5$

$k_1 = -5 - \frac{6}{5}$

$k_1 = -21/5$

$-3 - \frac{6}{5}$

$\frac{-15-6}{5}$

$\frac{-21}{5}$

& system is consistent

So, k_1, k_2, k_3 exist, so it a span and it is a linearly independent

, Therefore, it is a basis

Q.2) Show that the following polynomial form a basis for P_2

$x^2+1,\ x^2-1,\ 2x-1$

By the linear combination

$k_1 p_1 + k_2 p_2 + k_3 p_3 = w$

By the linearly independent

$k_1 p_1 + k_2 p_2 + k_3 p_3 = 0$

$k_1(x^2+1) + k_2(x^2-1) + k_3(2x-1) = 0$

~~$k\ x^2+1$~~

$k_1 + \cancel{0}k_2 + 0k_3 = 0$

$0k_1 + 0k_2 + 2k_3 = 0$

$1k_1 - 1k_2 - k_3 = 0$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

Cofactor expansion along ~~for~~ second row

$$= -2 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$= -2(1(-1) - 1)$

$= -2(-1-1)$

$= +4$

Hence, P_1, P_2, P_3 are basis of polynomial

Whrouskain (W)

$$w = \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ \vdots \\ f_1^n & f_2^n & f_3^n & & f_n^n \end{vmatrix}$$

$w \neq 0$; linearly independent

otherwise linearly dependent

Q.1) Use the wrouskian to show that following set of vectors are linearly independent.

a.) $1, x, e^x$

$f_1 = 1, f_2 = x, f_3 = e^x$

By Row determinant

$$w = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$$

$e^x \neq 0$

This is linearly independent

Ex 4.5

CO-ORDINATE

$V = c_1 V_1 + c_2 V_2 + \dots c_m V_n$

$(V)_S = (c_1, c_2, \dots c_n)$

co-ordinate of v relative to the basis S

Q. 1) Show that

$V_1 = (1, 2, 1)$
$V_2 = (2, 9, 0)$
$V_3 = (3, 3, 4)$

form a basis for R^3

Find the co-ordinate vector

$V = (5, -1, 9)$

relative to the basis
$S = \{V_1, V_2, V_3\}$

By linear combination

$c_1 V_1 + c_2 V_2 + c_3 V_3 = w \longrightarrow (1)$

By linear independent

$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$

$= \begin{array}{l} c_1 + 2c_2 + 3c_3 = 0 \\ 2c_1 + 9c_2 + 3c_3 = 0 \\ c_1 + 0c_2 + 4c_3 = 0 \end{array}$

$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$

= Co factor expansion along third row

$= 1 \begin{vmatrix} 2 & 3 \\ 9 & 3 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix}$

$= 1(6 - \cancel{18}\,27) + 4(9 - 4)$

$= 1(-\cancel{12}\,21) + 4(5)$

$= 1(-\cancel{12}\,21) + 20$

$= \cancel{8} \quad -1$

$\begin{array}{r} 27 \\ -6 \\ \hline 21 \end{array}$

Hence Proved v_1, v_2, v_3 form a basis in $\mathbb{R}^3$

Since determinant is $\neq 0$, therefore exist (c_1, c_2, c_3)

$= 1$

$$\therefore (1) \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{bmatrix}$$

$-2R_1 + R_2 \rightarrow R_2$, $-1R_1 + R_3 \rightarrow R_3$

$$= \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 5 & -3 & -11 \\ 0 & -2 & 1 & 4 \end{bmatrix}$$

$\frac{1}{5}R_2$

$$= \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & -3/5 & -11/5 \\ 0 & -2 & 1 & 4 \end{bmatrix}$$

$+2R_2 + R_3 \rightarrow R_3$

$$= \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & -3/5 & -11/5 \\ 0 & 0 & -1/5 & -2/5 \end{bmatrix}$$

$\times -5$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & -3/5 & -11/5 \\ 0 & 0 & 1 & +2 \end{bmatrix}$$

$\therefore \boxed{c_3 = 2}$

$c_2 - 3/5\, c_3 = -11/5$

$c_2 = -11/5 + 3/5(2)$

$c_2 = \frac{-11}{5} + \frac{6}{5}$

$c_2 = \frac{-5}{5}$

$\therefore \boxed{c_2 = -1}$

$c_1 + 2c_2 + 3c_3 = 5$

$c_1 = 5 - 3(2) - 2(-1)$

$c_1 = 5 - 6 + 2$

$-1 + 2$

$\therefore \boxed{c_1 = 1}$

Hence

$(V)_S = (c_1, c_2, c_3) = (1, -1, 2)$

Q.17) $P_1 = 1+x+x^2$ $P_2 = x+x^2$ $P_3 = x^2$

$p = 7-x+2x^2$

By linear combination
$c_1P_1 + c_2P_2 + c_3P_3 = p \rightarrow (1)$

By linear independent

$c_1P_1 + c_2P_2 + c_3P_3 = 0$

$c_1 + 0c_2 + 0c_3 = 0$
$1c_1 + 1c_2 + 0c_3 = 0$
$1c_1 + 1c_2 + 1c_3 = 0$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Since it is a lower triangular matrix its determinant will be the product of diagonal elements

$= 1 \times 1 \times 1$
$= 1$

Hence Proved P_1, P_2, P_3 form a basis in $\mathbb{R}^3$

$$\therefore (1) \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 7 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$-R_1 + R_2 \rightarrow R_2$ $-R_1 + R_3 \rightarrow R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -8 \\ 0 & 1 & 1 & -5 \end{bmatrix}$$

$-R_2 + R_3 \rightarrow R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$c_3 = 3$
$c_2 = -8$
$c_1 = 7$

Exercise 4.6

Dimension

Q.) Find a basis for the solution space of the following homogeneous linear system and find the dimension of the space?

$$x_1 + x_2 - x_3 = 0$$
$$-2x_1 - x_2 + 2x_3 = 0$$
$$-x_1 + x_3 = 0$$

The augmented matrix for the system of equations is

$$= \begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$+2R_1 + R_2 \to R_2$ & $R_1 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$-R_2 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_2 - x_3 = 0$$
$$x_2 = 0$$

$x_3 = t$

$x_1 = t - 0$ Parametric equation

$x_1 = t$

$x_1 = t$

$x_2 = 0$

$x_3 = t$

For $x_1, x_2, x_3 = (t, 0, t)$

$= t(1, 0, 1)$

$=$

$V_1 = (1, 0, 1)$ is a basis one dimension

Q.) Find the standard basis of vector in R^3 that can be added to the set $\{v_1, v_2\}$ to produce a basis for R^3

a.) $v_1 = (-1, 2, 3)$ $v_2 = (1, -2, -2)$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{bmatrix}$$

Co-factor expansion along third col

$\Rightarrow |A| = 1 \begin{vmatrix} 2 & -2 \\ 3 & -2 \end{vmatrix}$

$\Rightarrow |A| = 1 \quad 2(-2) - (-2)(3)$

$\Rightarrow |A| = -4 + 6$

$\therefore |A| = 2$

$|A| \neq 0$, so, it is a span

It is also a linear independent

Hence it is a basis

Q.3) The vector $v_1 = (1, -2, 3)$
$v_2 = (0, 5, -3)$

Enlarge $[v_1, v_2]$ to a basis for R^3.

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$|A| = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & 0 \\ 3 & -3 & 1 \end{bmatrix}$$

Cofactor expansion along third column

$|A| = 1 \begin{vmatrix} -2 & 5 \\ 3 & -3 \end{vmatrix}$ $\quad |A| = \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix}$

$\cancel{|A| = 1(-2(-3) - 5(3))}$ $\quad |A| = (5)$

$\cancel{|A| = 1(6 - 15)}$ $\quad |A| = 5$

$\cancel{|A| = 1(-9)}$

$\cancel{|A| = -9}$

Q.4) Find a basis for the subspace of R^3 that is spanned by the vectors

$v_1 = (1, 0, 0)$
$v_2 = (1, 0, 1)$
$v_3 = (2, 0, 1)$
$v_4 = (0, 0, -1)$

$$= \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Interchange R_2 with R_3

$$= \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1 - R_2 + R_1 \rightarrow R_1$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_3 \cancel{+} x_4 = 0$
$x_2 + x_3 - x_4 = 0$

$x_4 = t$
$x_3 = \cancel{t}\, s$
$\therefore x_2 = t - s$
$x_1 = -t - s$

$x_1, x_2, x_3, x_4 = (-s-t, t-s, s, t)$

(I) $t = 0, s = 1$
$(-1-0, 0-1, 1, 0)$
$= (-1, -1, 1, 0)$

(II) $t = 1, s = 0 = (-0-1, 1-0, 0, 1)$
$= (-1, 1, 0, 1)$

Q.5) Let $T_A : R^3 \to R^3$ be mul by A and find the dimension of the subspace of R^3 consisting of all vectors for which $T_A(x) = 0$ $\quad Ax = 0$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The augmented matrix for the system of equation is :

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$-1R_1 + R_2 \to R_2$ & $-R_1 + R_3 \to$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$-R_2$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$R_2 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-R_2 + R_1 \to R_1$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$x_1 + x_3 = 0$

$x_2 - x_3 = 0$

$x_3 = t$

$x_2 = +t$

$x_1 = -t$

$(x_1, x_2, x_3) = (-t, t, t)$

$= t(-1, 1, 1)$

$v_1(-1, 1, 1)$ is a bases one-dimension

11th October 2024

Exercise 4.8

Bases for the row and Column Spaces of a matrix

Q.1) Find row and column space?

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row

$R_1 = [1 \ -2 \ 5 \ 0 \ 3]$

$R_2 = [0 \ 1 \ 3 \ 0 \ 0]$

$R_3 = [0 \ 0 \ 0 \ 1 \ 0]$

Column

$C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $C_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $C_3 =$

Q.2) Find a basis for Row and column

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

$-2R_1 + R_2 \longrightarrow R_2$, $-2R_1 + R_3 \longrightarrow R_3$

$+R_1 + R_4 \longrightarrow R_4$

$$= \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$-R_2 + R_3 \longrightarrow R_3$

$$= \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & +1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row

$R_1 = [1 \ -3 \ 4 \ -2 \ 5 \ 4]$

$R_2 = [0 \ 0 \ 1 \ +3 \ -2 \ -6]$

$R_3 = [0 \ 0 \ 0 \ 0 \ 1 \ 5]$

Column =

$$C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Q. 2 Express the product as a linear combination of the column vectors of A

a.)

$$= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution

Linear combination

$$= 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

b.)

$$\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$$

Linear combination

$$= -2 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

Q.3) Determine whether b is in the column space of A and if so express b as a linear combination of the column vectors of A.

a.) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \; b = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$$Ax = b$$

The augmented matrix for the system is:

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

$-1R_1 + R_2 \to R_2$ & $-2R_1 + R_3 \to R_3$

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

$-R_2$

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

$+R_2 + R_3 \longrightarrow R_3$

$$= \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The system is inconsistent as linear combination is not possible

b.) $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

$AX = B$

Finding

The augmented matrix for the system of equations

$$= \begin{bmatrix} 1 & -1 & 1 & : 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$-9R_1 + R_2 \longrightarrow R_2$ & $-1R_1 + R_3 \longrightarrow R_3$

$$= \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -44 \\ 0 & 2 & 0 & -6 \end{bmatrix}$$

$\frac{1}{12} R_2$

$$= \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & -8/12 & -44/12 \\ 0 & 2 & 0 & -6 \end{bmatrix}$$

$-2R_2 + R_3$

$$= \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & -8/12 & -44/12 \\ 0 & 0 & 4/3 & 4/3 \end{bmatrix}$$

$\frac{-44\times2}{12}$

$\frac{22}{3}$

$\frac{-8\times2}{12}$

$\frac{16}{12} = \frac{4}{3}$

$\frac{22}{3} - 6$

$\frac{22-18}{3}$

$$= \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & -2/3 & -11/3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 5$$

$$x_2 - 2/3\,x_3 = -11/3$$

$$\boxed{x_3 = 1}$$

$$x_2 = -11/3 + 2/3$$

$$x_2 = \frac{-\cancel{9}\,3}{\cancel{3}}$$

$$\therefore \boxed{x_2 = -3}$$

$$x_1 = 5 - 1 + (-3)$$

$$x_1 = 5 - 1 - 3$$

$$5 - 4$$

$$\therefore \boxed{x_1 = 1}$$

Linear Combination

$$= 1\begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Exercise 4.8

NULL SPACES

Q.1) Find the basis of null & row space of A.

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

The augmented matrix for the system of equations is:

$$Ax = 0$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

$$-5R_1 + R_2 \rightarrow R_2 \text{ \& } -7R_1 + R_3 \rightarrow R_3$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$$

$$-R_2 + R_3$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

$+R_2 + R_1 \rightarrow R_1$

$$= \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 16x_3 = 0$

$x_2 - 19x_3 = 0$ Checking basis

$x_3 = t$ $\quad = (16t, 19t, t)$

$x_2 = 19t$ $\quad = t(16, 19, 1)$

$x_1 = 16t$ $\quad$ = Therefore, it is one dimension

Row

$[1 \quad 0 \quad -16]$

$[0 \quad 1 \quad -19]$

$$V_1 = \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

Column

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

It is also called null spaces

Q-2) Construct a matrix where null spaces consists of all linear combination of the vectors

$$V_1 = \begin{bmatrix} +1 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 4 \end{bmatrix}$$

$Ax = 0$

$[V_1 \mid V_2 \mid Ax = 0]$

$$= \begin{bmatrix} 1 & -1 & 3 & 2 & 0 \\ 2 & 0 & -2 & 4 & 0 \end{bmatrix}$$

$-2R_1 + R_2 \rightarrow R_2$

$$= \begin{bmatrix} 1 & -1 & 3 & 2 & 0 \\ 0 & 2 & -8 & 0 & 0 \end{bmatrix}$$

$\frac{1}{2}R_2$

$$= \begin{bmatrix} 1 & -1 & 3 & 2 & 0 \\ 0 & 1 & -4 & 0 & 0 \end{bmatrix}$$

$+R_2 + R_1 \rightarrow R_1$

$$= \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -4 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -4 & 0 & 0 \end{bmatrix}$$

$x_1 - x_3 + 2x_4 = 0$

$x_2 - 4x_3 = 0$

$x_3 = t$

$x_2 = +4t$

$x_4 = s$

$x_1 = t - 2s$

$(x_1, x_2, x_3, x_4) = \cancel{s}(t-2s, 4t, t, s)$

$= t(1,4,1,0) + s(-2,0,0,1)$

$$= t \begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

= Two dimension

Consider the linear system equation

$$\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

a) Find a general solution of homogeneous system?

b) Confirm that $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ is a solution of the homogeneous system

c.) use result (a) & (b), find the result general solution of non-homogeneous system

d.) Check your result in (c) by solving the non-homogeneous system

A.)

$$= \begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}$$

$\cancel{\&}$ $\frac{1}{3}R_1$

$$= \begin{bmatrix} 1 & 2/3 & -1/3 \\ \cancel{2} & 4/3 & -2/3 \\ & & \end{bmatrix}$$

$\frac{1}{3}R_1$

$$= \begin{bmatrix} 1 & 2/3 & -1/3 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}$$

$-6R_1 + R_2 \rightarrow R_2$

$$= \begin{bmatrix} 1 & 2/3 & -1/3 \\ 0 & 0 & 0 \\ -3 & -2 & 1 \end{bmatrix}$$

$\frac{2\times}{\cancel{3}}$ $= -4$ $-\frac{1}{3}$

Interchange R_2 with R_3

$$= \begin{bmatrix} 1 & 2/3 & -1/3 \\ \cancel{0}-3 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$+3R_1 + R_2 \rightarrow$

$$= \begin{bmatrix} 1 & 2/3 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = 0$$

$x_3 = t$

$x_2 = s$

$x_1 = -\frac{2}{3}s + \frac{1}{3}t$

B)

$$\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times1 + -1\times1 \\ 6\times1 + -2\times1 \\ -3\times1 + 1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$