Markov Chains

A **Markov chain** is a mathematical system that experiences transitions from one state to another according to certain probabilistic rules.

In many dynamical systems the states of the variables are not known with certainty but can be expressed as probabilities; such dynamical systems are called ***stochastic processes*** (from the Greek word *stochastikos*, meaning "proceeding by guesswork"). A detailed study of stochastic processes requires a precise definition of the term *probability*, which is outside the scope of this course. However, the following interpretation will suffice for our present purposes:

> *Stated informally, the* **probability** *that an experiment or observation will have a certain outcome is the fraction of the time that the outcome would occur if the experiment could be repeated indefinitely under constant conditions—the greater the number of actual repetitions, the more accurately the probability describes the fraction of time that the outcome occurs.*

For example, when we say that the probability of tossing heads with a fair coin is $\frac{1}{2}$, we mean that if the coin were tossed many times under constant conditions, then we would expect about half of the outcomes to be heads. Probabilities are often expressed as decimals or percentages. Thus, the probability of tossing heads with a fair coin can also be expressed as 0.5 or 50%.

If an experiment or observation has n possible outcomes, then the probabilities of those outcomes must be nonnegative fractions whose sum is 1. The probabilities are nonnegative because each describes the fraction of occurrences of an outcome over the long term, and the sum is 1 because they account for all possible outcomes. For example, if a box containing 10 balls has one red ball, three green balls, and six yellow balls, and if a ball is drawn at random from the box, then the probabilities of the various outcomes are

$$
\begin{aligned}
p_1 &= \text{prob(red)} = 1/10 = 0.1 \\
p_2 &= \text{prob(green)} = 3/10 = 0.3 \\
p_3 &= \text{prob(yellow)} = 6/10 = 0.6
\end{aligned}
$$

Each probability is a nonnegative fraction and

$$p_1 + p_2 + p_3 = 0.1 + 0.3 + 0.6 = 1$$

In a stochastic process with n possible states, the state vector at each time t has the form

$$
\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}
\begin{array}{l} \textbf{Probability that the system is in state 1} \\ \textbf{Probability that the system is in state 2} \\ \vdots \\ \textbf{Probability that the system is in state } \boldsymbol{n} \end{array}
$$

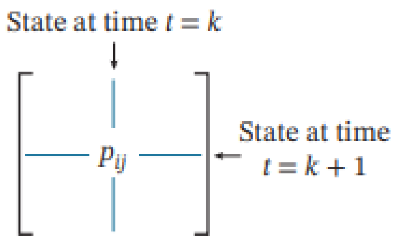
The entries in this vector must add up to 1 since they account for all n possibilities. In general, a vector with nonnegative entries that add up to 1 is called a ***probability vector***.

Definition 1

A ***Markov chain*** is a dynamical system whose state vectors at a succession of equally spaced times are probability vectors and for which the state vectors at successive times are related by an equation of the form

$$\mathbf{x}(k+1) = P\mathbf{x}(k)$$

in which $P = [p_{ij}]$ is a stochastic matrix and p_{ij} is the probability that the system will be in state i at time $t = k + 1$ if it is in state j at time $t = k$. The matrix P is called the ***transition matrix*** for the system.



The entry p_{ij} is the probability that the system is in state i at time $t = k + 1$ if it is in state j at time $t = k$.

FIGURE 5.5.2

EXAMPLE 4 | Wildlife Migration as a Markov Chain

Suppose that a tagged lion can migrate over three adjacent game reserves in search of food: Reserve 1, Reserve 2, and Reserve 3. Based on data about the food resources, researchers conclude that the monthly migration pattern of the lion can be modeled by a Markov chain with transition matrix

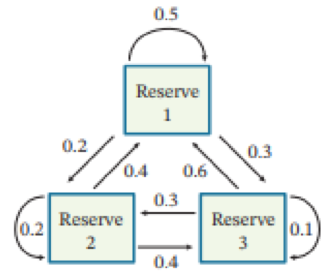


FIGURE 5.5.3

$$\begin{array}{c} \text{Reserve at time } t = k \\ \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ P = & \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.2 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \text{Reserve at time } t = k+1 \end{array} \end{array}$$

(see Figure 5.5.3). That is,

$p_{11} = 0.5 =$ probability that the lion will stay in Reserve 1 when it is in Reserve 1
$p_{12} = 0.4 =$ probability that the lion will move from Reserve 2 to Reserve 1
$p_{13} = 0.6 =$ probability that the lion will move from Reserve 3 to Reserve 1
$p_{21} = 0.2 =$ probability that the lion will move from Reserve 1 to Reserve 2
$p_{22} = 0.2 =$ probability that the lion will stay in Reserve 2 when it is in Reserve 2
$p_{23} = 0.3 =$ probability that the lion will move from Reserve 3 to Reserve 2
$p_{31} = 0.3 =$ probability that the lion will move from Reserve 1 to Reserve 3
$p_{32} = 0.4 =$ probability that the lion will move from Reserve 2 to Reserve 3
$p_{33} = 0.1 =$ probability that the lion will stay in Reserve 3 when it is in Reserve 3

Assuming that t is in months and the lion is released in Reserve 2 at time $t = 0$, track its probable locations over a six-month period, and find the reserve in which it is most likely to be at the end of that period.

Solution Let $x_1(k)$, $x_2(k)$, and $x_3(k)$ be the probabilities that the lion is in Reserve 1, 2, or 3, respectively, at time $t = k$, and let

$$\mathbf{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

be the state vector at that time. Since we know with certainty that the lion is in Reserve 2 at time $t = 0$, the initial state vector is

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We leave it for you to use a calculator or computer to show that the state vectors over a six-month period are

$$\mathbf{x}(1) = P\mathbf{x}(0) = \begin{bmatrix} 0.400 \\ 0.200 \\ 0.400 \end{bmatrix}, \quad \mathbf{x}(2) = P\mathbf{x}(1) = \begin{bmatrix} 0.520 \\ 0.240 \\ 0.240 \end{bmatrix}, \quad \mathbf{x}(3) = P\mathbf{x}(2) = \begin{bmatrix} 0.500 \\ 0.224 \\ 0.276 \end{bmatrix}$$

$$\mathbf{x}(4) = P\mathbf{x}(3) \approx \begin{bmatrix} 0.505 \\ 0.228 \\ 0.267 \end{bmatrix}, \quad \mathbf{x}(5) = P\mathbf{x}(4) \approx \begin{bmatrix} 0.504 \\ 0.227 \\ 0.269 \end{bmatrix}, \quad \mathbf{x}(6) = P\mathbf{x}(5) \approx \begin{bmatrix} 0.504 \\ 0.227 \\ 0.269 \end{bmatrix}$$

As in Example 2, the state vectors here seem to stabilize over time with a probability of approximately 0.504 that the lion is in Reserve 1, a probability of approximately 0.227 that it is in Reserve 2, and a probability of approximately 0.269 that it is in Reserve 3.

From $\mathbf{x}(6)$ we see that the lion is most likely to be in Reserve 1 at the end of six months.

EXAMPLE 5 | Finding a State Vector Directly

Use Formula (12) to find the state vector $\mathbf{x}(3)$ in Example 2.

Solution From (1) and (7), the initial state vector and transition matrix are

$$\mathbf{x}_0 = \mathbf{x}(0) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

We leave it for you to calculate P^3 and show that

$$\mathbf{x}(3) = \mathbf{x}_3 = P^3\mathbf{x}_0 = \begin{bmatrix} 0.562 & 0.219 \\ 0.438 & 0.781 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.3905 \\ 0.6095 \end{bmatrix}$$

which agrees with the result in (8).

Long-Term Behavior of a Markov Chain

Definition 2

A stochastic matrix P is said to be ***regular*** if P or some positive power of P has all positive entries, and a Markov chain whose transition matrix is regular is said to be a ***regular Markov chain***.

EXAMPLE 7 | Regular Stochastic Matrices

The transition matrices in Examples 2 and 4 are regular because their entries are positive. The matrix

$$P = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}$$

is regular because

$$P^2 = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$$

has positive entries. The matrix P in Example 6 is not regular because P and every positive power of P have some zero entries (verify).

EXAMPLE 9 | Example 4 Revisited

The transition matrix for the Markov chain in Example 4 is

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.2 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix}$$

Since the entries of P are positive, the Markov chain is regular and hence has a unique steady-state vector $\mathbf{q}$. To find $\mathbf{q}$ we will solve the system $(I - P)\mathbf{q} = \mathbf{0}$, which we can write (using fractions) as

$$\begin{bmatrix} \frac{1}{2} & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} & -\frac{3}{10} \\ -\frac{3}{10} & -\frac{2}{5} & \frac{9}{10} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

(We have converted to fractions to avoid roundoff error in this illustrative example.) We leave it for you to confirm that the reduced row echelon form of the coefficient matrix is

$$\begin{bmatrix} 1 & 0 & -\frac{15}{8} \\ 0 & 1 & -\frac{27}{32} \\ 0 & 0 & 0 \end{bmatrix}$$

and that the general solution of (15) is

$$q_1 = \tfrac{15}{8}s, \quad q_2 = \tfrac{27}{32}s, \quad q_3 = s \tag{16}$$

For $\mathbf{q}$ to be a probability vector we must have $q_1 + q_2 + q_3 = 1$, from which it follows that $s = \frac{32}{119}$ (verify). Substituting this value in (16) yields the steady-state vector

$$\mathbf{q} = \begin{bmatrix} \frac{60}{119} \\ \frac{27}{119} \\ \frac{32}{119} \end{bmatrix} \approx \begin{bmatrix} 0.5042 \\ 0.2269 \\ 0.2689 \end{bmatrix}$$

(verify), which is consistent with the results obtained in Example 4.

Question:

In a laboratory experiment, a mouse can choose one of two food types each day, type I or type II. Records show that if the mouse chooses type I on a given day, then there is a 75% chance that it will choose type I the next day, and if it chooses type II on one day, then there is a 50% chance that it will choose type II the next day.

a. Find a transition matrix for this phenomenon.

b. If the mouse chooses type I today, what is the probability that it will choose type I two days from now?
c. If the mouse chooses type II today, what is the probability that it will choose type II three days from now? d. If there is a 10% chance that the mouse will choose type I today, what is the probability that it will choose type I tomorrow?

Solution:

Ex 5.5.

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Type of Food Type I, Type II.

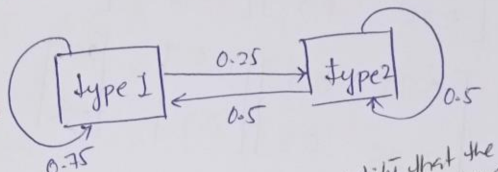
At time $t=k$ (Current state).

a) The transition Matrix

$$\begin{array}{cc} & \text{at time } t=k+1 \text{ Future state} \\ & \begin{array}{c|cc} & \text{typ I} & \text{type II} \\ \hline \text{type I} & 0.75 & 0.5 \\ \text{type II} & 0.25 & 0.5 \end{array} \end{array}$$

→ retention.

b) If a mouse chooses type 1 today, what is the probability that it will chose type I two days from now.



State Vector. $x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_0 / x(0)$

→ Probability that the system is in state 1.

Initial state Vector.

→ Prob. that the system is in state 2.

$X(k+1) = Px(k)$

$$x(1) = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

$$x(2) = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.6875 \\ 0.3125 \end{bmatrix}$$

↓ Entries in this Vector must add up to 1.

there is 0.687 %. 68.7%. Chance that mouse will choose type 1 after two days.

. or

$$x_k = P^k x_0$$

$$x(2) = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6875 \\ 0.3125 \end{bmatrix}$$

(c) If the mouse choose type II today, what is the probability that it will choose type II three days from now?

State vector $x(k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$X(k+1) = P X(k)$$

$$x(1) = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$x(2) = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 0.375 \end{bmatrix}$$

$$x(3) = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.625 \\ 0.375 \end{bmatrix} = \begin{bmatrix} 0.6562 \\ 0.34375 \end{bmatrix}$$

OR

$$X_k = P^k X_0$$

$$x(3) = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6562 \\ 0.34375 \end{bmatrix}$$

there is 0.6562 prob or 65.62% per. Change that mouse choose type II after three day.

d) If there is a 10% chance that the mouse will choose type 1 today, what is the probability that it will choose type 1 tomorrow?

State Vector $x(k) = \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}$

$$x(1) = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.525 \\ 0.475 \end{bmatrix}$$

there is 52.5% chace that mouse will choose type I tomorrow.

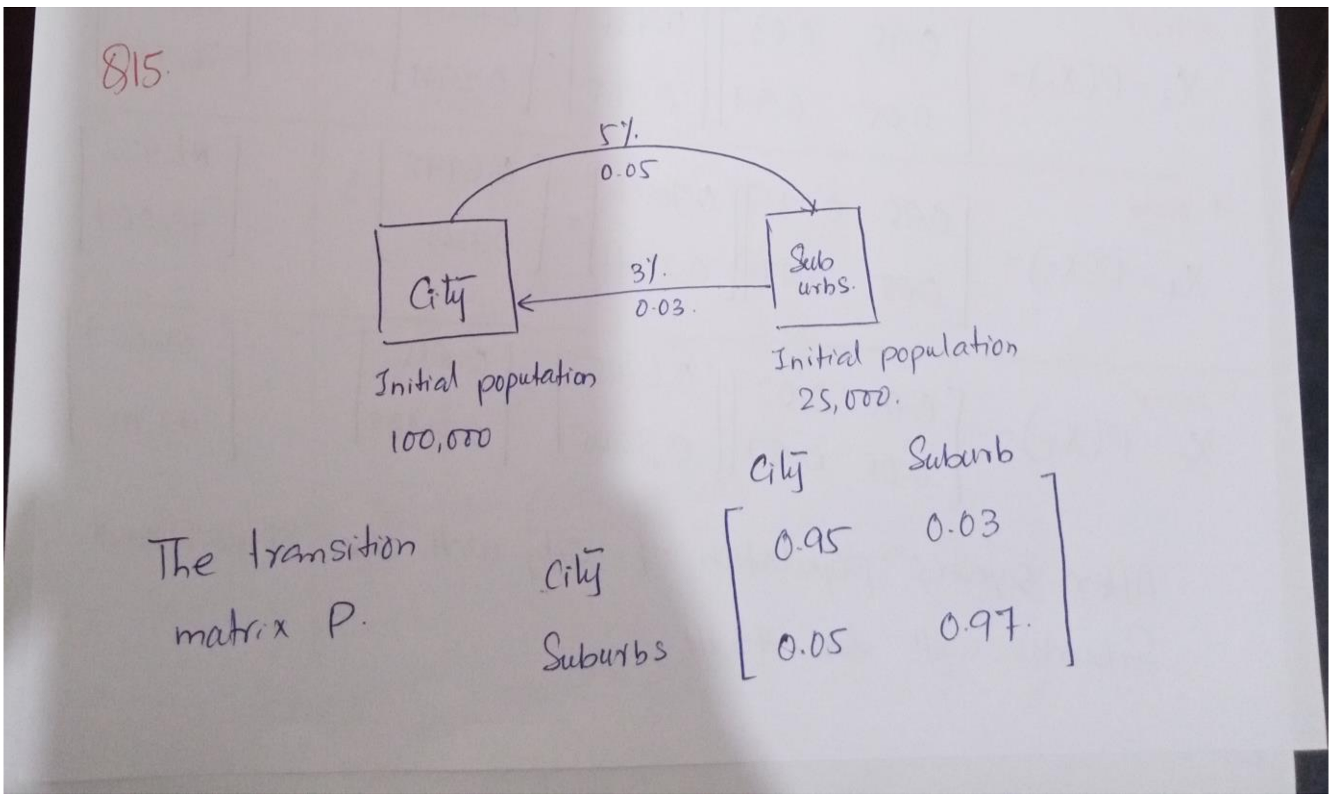
Question:

Suppose that at some initial point in time 100,000 people live in a certain city and 25,000 people live in its suburbs. The Regional Planning Commission determines that each year 5% of the city population moves to the suburbs and 3% of the sub□urban population moves to the city.

a. Assuming that the total population remains constant, make a table that shows the populations of the city and its suburbs over a five-year period (round to the nearest integer).

b. Over the long term, how will the population be distributed between the city and its suburbs?

Solution:



Assume that the total population remains constant, make a table that shows the population of the city and its suburbs over a five-year period.

State Vector.
Initial State

$$X_0 = \begin{bmatrix} \frac{100,000}{125,000} \\ \frac{25,000}{125,000} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

	$125,000 X_k$
After 1 Year $X_1 = P X_0 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix}$	$\begin{bmatrix} 95,750 \\ 29,250 \end{bmatrix}$
2 Year $X_2 = P(X_1) = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}\begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix} = \begin{bmatrix} 0.734 \\ 0.265 \end{bmatrix}$	$\begin{bmatrix} 91,840 \\ 33,160 \end{bmatrix}$
3 year $X_3 = P(X_2) = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}\begin{bmatrix} 0.734 \\ 0.265 \end{bmatrix} = \begin{bmatrix} 0.7059 \\ 0.2941 \end{bmatrix}$	$\begin{bmatrix} 88,243 \\ 36,757 \end{bmatrix}$
4 Year $X_4 = P(X_3) = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}\begin{bmatrix} 0.7059 \\ 0.2941 \end{bmatrix} = \begin{bmatrix} 0.6795 \\ 0.3205 \end{bmatrix}$	$\begin{bmatrix} 84,933 \\ 40,067 \end{bmatrix}$
5 Year $X_5 = P(X_4) = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}\begin{bmatrix} 0.6795 \\ 0.3205 \end{bmatrix} = \begin{bmatrix} 0.655 \\ 0.34489 \end{bmatrix}$	$\begin{bmatrix} 81,889 \\ 43,111 \end{bmatrix}$

After 5 years population of city will be 81,889 and Suburbs will be 43,111.

b) Over the long term, how will the population be distributed between the city and suburbs?

P is a regular stochastic matrix $\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$

To Find the Steady-State Vector

$(I-P)q=0.$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}\right)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.05 & -0.03 \\ -0.05 & 0.03 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

reduce the augmented matrix to reduced row echelon form.

$$\begin{bmatrix} 1 & -3/5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$q_1 - \frac{3}{5} q_2 = 0.$

$\Rightarrow q_1 = \frac{3}{5} t$

$q_2 = t$

As q_1 and q_2 are probabilities.

So, $q_1 + q_2 = 1$

$\frac{3}{5} t + t = 1 \Rightarrow t = \frac{5}{8}$

and $q_1 = \frac{3}{8}$

$q_2 = \frac{5}{8}.$

the steady state vector $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$.

Conclusion

The city population will approach $\frac{3}{8} \times 125,000 = 46,875$.

The Suburbs population will approach $\frac{5}{8} \times 125,000 = 78,125$.