4.9 Rank, Nullity, and the Fundamental Matrix Spaces

In the last section we investigated relationships between a system of linear equations and the row space, column space, and null space of its coefficient matrix. In this section we will be concerned with the dimensions of those spaces. The results we obtain will provide a deeper insight into the relationship between a linear system and its coefficient matrix.

Row and Column Spaces Have Equal Dimensions

In Examples 6 and 7 of Section 4.8 we found that the row and column spaces of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

both have three basis vectors and hence are both three-dimensional. The fact that these spaces have the same dimension is not accidental, but rather a consequence of the following theorem.

Theorem 4.9.1

The row space and the column space of a matrix A have the same dimension.

Proof It follows from Theorems 4.8.4 and 4.8.6 (b) that elementary row operations do not change the dimension of the row space or of the column space of a matrix. Thus, if R is any row echelon form of A, it must be true that

$$\begin{aligned} \dim(\text{row space of } A) &= \dim(\text{row space of } R) \\ \dim(\text{column space of } A) &= \dim(\text{column space of } R) \end{aligned}$$

so it suffices to show that the row and column spaces of R have the same dimension. But the dimension of the row space of R is the number of nonzero rows, and by Theorem 4.8.5 the dimension of the column space of R is the number of leading 1's. Since these two numbers are the same, the row and column space have the same dimension. ■

Rank and Nullity

The dimensions of the row space, column space, and null space of a matrix are such important numbers that there is some notation and terminology associated with them.

The proof of Theorem 4.9.1 shows that the rank of A can be interpreted as the number of leading 1's in any row echelon form of A.

Definition 1

The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by $\text{rank}(A)$; the dimension of the null space of A is called the ***nullity*** of A and is denoted by $\text{nullity}(A)$.

EXAMPLE 1 | Rank and Nullity of a 4×6 Matrix

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

(verify). Since this matrix has two leading 1's, its row and column spaces are two-dimensional and $\text{rank}(A) = 2$. To find the nullity of A, we must find the dimension of the solution space of the linear system $A\mathbf{x} = \mathbf{0}$. This system can be solved by reducing its augmented matrix to reduced row echelon form. The resulting matrix will be identical to (1), except that it will have an additional last column of zeros, and hence the corresponding system of equations will be

$$\begin{aligned} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 &= 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 &= 0 \end{aligned}$$

Solving these equations for the leading variables yields

$$\begin{aligned} x_1 &= 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ x_2 &= 2x_3 + 12x_4 + 16x_5 - 5x_6 \end{aligned} \tag{2}$$

from which we obtain the general solution

$$\begin{aligned} x_1 &= 4r + 28s + 37t - 13u \\ x_2 &= 2r + 12s + 16t - 5u \\ x_3 &= r \\ x_4 &= s \\ x_5 &= t \\ x_6 &= u \end{aligned}$$

or in column vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

Because the four vectors on the right side of Formula (3) form a basis for the solution space it follows that $\text{nullity}(A) = 4$.

EXAMPLE 2 | Maximum Value for Rank

What is the maximum possible rank of an $m \times n$ matrix A that is not square?

Solution Since the row vectors of A lie in R^n and the column vectors in R^m, the row space of A is at most n-dimensional and the column space is at most m-dimensional. Since the rank of A is the common dimension of its row and column space, it follows that the rank is at most the smaller of m and n. We denote this by writing

$$\text{rank}(A) \le \min(m, n)$$

in which $\min(m, n)$ is the minimum of m and n.

The following theorem establishes a fundamental relationship between the rank and nullity of a matrix.

Theorem 4.9.2

Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n \tag{4}$$

Proof Since A has n columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has n unknowns (variables). These fall into two distinct categories: the leading variables and the free variables. Thus,

$$\begin{bmatrix} \text{number of leading} \\ \text{variables} \end{bmatrix} + \begin{bmatrix} \text{number of free} \\ \text{variables} \end{bmatrix} = n$$

But the number of leading variables is the same as the number of leading 1's in any row echelon form of A, which is the same as the dimension of the row space of A, which is the same as the rank of A. Also, the number of free variables in the general solution of $A\mathbf{x} = \mathbf{0}$ is the same as the number of parameters in that solution, which is the same as the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, which is the same as the nullity of A. This yields Formula (4). ■

EXAMPLE 3 | The Sum of Rank and Nullity

The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns, so

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 6$$

This is consistent with Example 1, where we showed that

$$\operatorname{rank}(A) = 2 \quad \text{and} \quad \operatorname{nullity}(A) = 4$$

The following theorem, which summarizes results already obtained, interprets rank and nullity in the context of a homogeneous linear system.

Theorem 4.9.3

If A is an $m \times n$ matrix, then

(*a*) $\operatorname{rank}(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.

(*b*) $\operatorname{nullity}(A)$ = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

EXAMPLE 4 | Rank, Nullity, and Linear Systems

(*a*) Find the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ if A is a 5×7 matrix of rank 3.

(*b*) Find the rank of a 5×7 matrix A for which $A\mathbf{x} = \mathbf{0}$ has a two-dimensional solution space.

Solution (*a*) From (4),

$$\text{nullity}(A) = n - \text{rank}(A) = 7 - 3 = 4$$

Thus, there are four parameters.

Solution (*b*) The matrix A has nullity 2, so

$$\text{rank}(A) = n - \text{nullity}(A) = 7 - 2 = 5$$

Recall from Section 4.8 that if $A\mathbf{x} = \mathbf{b}$ is a consistent linear system, then its general solution can be expressed as the sum of a particular solution of this system and the general solution of $A\mathbf{x} = \mathbf{0}$. We leave it as an exercise for you to use this fact and Theorem 4.9.3 to prove the following result.

Theorem 4.9.4

If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

The Fundamental Spaces of a Matrix

There are six important vector spaces associated with an $m \times n$ matrix A and its transpose A^T:

row space of A	row space of A^T
column space of A	column space of A^T
null space of A	null space of A^T

However, transposing a matrix converts row vectors into column vectors and conversely, so except for a difference in notation, the row space of A^T is the same as the column space of A, and the column space of A^T is the same as the row space of A. Thus, of the six spaces listed above, only the following four are distinct:

row space of A	column space of A
null space of A	null space of A^T

These are called the ***fundamental spaces*** of the matrix A. The row space and null space of A are subspaces of R^n, whereas the column space of A and the null space of A^T are subspaces of R^m. The null space of A^T is also called the ***left null space of A*** because transposing both sides of the equation $A^T\mathbf{x} = \mathbf{0}$ produces the equation $\mathbf{x}^T A = \mathbf{0}^T$ in which the unknown is on the left. The dimension of the left null space of A is called the ***left nullity of A***. We will now consider how the four fundamental spaces are related.

Let us focus for a moment on the matrix A^T. Since the row space and column space of a matrix have the same dimension, and since transposing a matrix converts its columns to rows and its rows to columns, the following result should not be surprising.

Theorem 4.9.5

If A is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

Proof

$$\text{rank}(A) = \dim(\text{row space of } A) = \dim(\text{column space of } A^T) = \text{rank}(A^T). \quad \blacksquare$$

This result has some important implications. For example, if A is an $m \times n$ matrix, then applying Formula (4) to the matrix A^T and using the fact that this matrix has m columns yields

$$\text{rank}(A^T) + \text{nullity}(A^T) = m$$

which, by virtue of Theorem 4.9.5, can be rewritten as

$$\boxed{\text{rank}(A) + \text{nullity}(A^T) = m} \tag{5}$$

This alternative form of Formula (4) makes it possible to express the dimensions of all four fundamental spaces in terms of the size and rank of A. Specifically, if $\text{rank}(A) = r$, then

$$\boxed{\begin{array}{ll} \dim[\text{row}(A)] = r & \dim[\text{col}(A)] = r \\ \dim[\text{null}(A)] = n - r & \dim[\text{null}(A^T)] = m - r \end{array}} \tag{6}$$

Bases for the Fundamental Spaces

An efficient way to obtain bases for the four fundamental spaces of an $m \times n$ matrix A is to adjoin the $m \times m$ identity matrix to A to obtain an augmented matrix $[A \,|\, I]$ and apply elementary row operations to this matrix to put A in reduced row echelon form R, thereby putting the augmented matrix in the form $[R \,|\, E]$. In the case where A is invertible the matrix E will be A^{-1}, but in general it will not. The rank r of A can then be obtained by counting the number of pivots (leading l's) in R, and the nullity of A^T can be obtained from the relationship

$$\text{nullity}(A^T) = m - r \tag{7}$$

that follows from Formula (5). Bases for three of the fundamental spaces can be obtained directly from $[R \,|\, E]$ as follows:

- A basis for $\text{row}(A)$ will be the r rows of R that contain the leading 1's (the pivot rows).
- A basis for $\text{col}(A)$ will be the r columns of A that contain the leading 1's of R (the pivot columns).
- A basis for $\text{null}(A^T)$ will be the bottom $m - r$ rows of E (see the proof at the end of this section)

EXAMPLE 5 | Bases for the Fundamental Spaces

In Example 1 we found a basis for the null space of the 4×6 matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

so in this example we will focus on finding bases for the remaining three fundamental spaces starting with the matrix

$$\underbrace{\left[\begin{array}{rrrrrr|rrrr} -1 & 2 & 0 & 4 & 5 & -3 & 1 & 0 & 0 & 0 \\ 3 & -7 & 2 & 0 & 1 & 4 & 0 & 1 & 0 & 0 \\ 2 & -5 & 2 & 4 & 6 & 1 & 0 & 0 & 1 & 0 \\ 4 & -9 & 2 & -4 & -4 & 7 & 0 & 0 & 0 & 1 \end{array}\right]}_{A \qquad\qquad\qquad\qquad I}$$

in which a 4×4 identity matrix has been adjoined to A. Using Gaussian elimination to reduce the left side to reduced row echelon form R yields (verify)

$$\underbrace{\left[\begin{array}{rrrrrr|rrrr} 1 & 0 & -4 & -28 & -37 & 13 & 0 & 0 & -\frac{9}{2} & \frac{5}{2} \\ 0 & 1 & -2 & -12 & -16 & 5 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]}_{R \qquad\qquad\qquad\qquad E}$$

From R we see that A has rank $r = 2$ (two nonzero rows), has nullity $n - r = 6 - 2 = 4$, and from (7) has left nullity $m - r = 2$. The two pivot rows of R (rows 1 and 2) form a basis for the row space of A, the two pivot columns of A (columns 1 and 2) form a basis for the column space of A, and the bottom two rows of E form a basis for the left null space of A. Expressing these bases in column form we have:

$$\text{row space basis: } \left\{ \begin{bmatrix} -1 \\ 0 \\ -4 \\ -28 \\ -37 \\ 13 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ -12 \\ -16 \\ 5 \end{bmatrix} \right\}, \quad \text{column space basis: } \left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ -5 \\ -9 \end{bmatrix} \right\}$$

$$\text{left null space basis: } \left\{ \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\}$$

A Geometric Link Between the Fundamental Spaces

The four formulas in (6) provide an *algebraic* relationship between the size of a matrix and the dimensions of its fundamental spaces. Our next objective is to find a *geometric* relationship between the fundamental spaces themselves. For this purpose recall from Theorem 3.4.3 that if A is an $m \times n$ matrix, then the null space of A consists of those vectors that are orthogonal to each of the row vectors of A. To develop that idea in more detail, we make the following definition.

Definition 2

If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^\perp$.

The following theorem lists three basic properties of orthogonal complements. We will omit the formal proof because a more general version of this theorem will be proved later in the text.

Theorem 4.9.6

If W is a subspace of R^n, then:

(a) $W^\perp$ is a subspace of R^n.

(b) The only vector common to W and $W^\perp$ is $\mathbf{0}$.

(c) The orthogonal complement of $W^\perp$ is W.

Part (b) of Theorem 4.9.6 can be expressed as

$$W \cap W^\perp = \{\mathbf{0}\}$$

and part (c) as

$$(W^\perp)^\perp = W$$

EXAMPLE 6 | Orthogonal Complements

In R^2 the orthogonal complement of a line W through the origin is the line through the origin that is perpendicular to W (**Figure 4.9.1a**); and in R^3 the orthogonal complement of a plane W through the origin is the line through the origin that is perpendicular to that plane (**Figure 4.9.1b**).

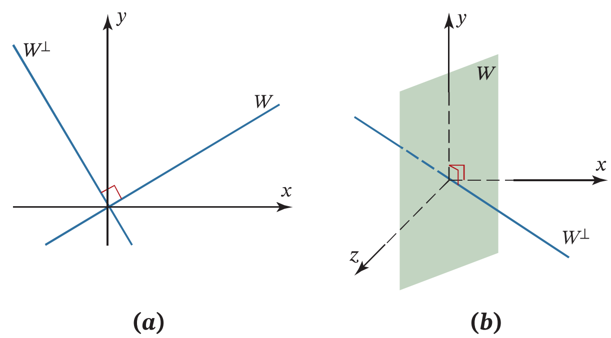

FIGURE 4.9.1

The next theorem will provide a geometric link between the fundamental spaces of a matrix. In the exercises we will ask you to prove that if a vector in R^n is orthogonal to each vector in a *basis* for a subspace of R^n, then it is orthogonal to *every* vector in that subspace. Thus, part (a) of the following theorem is essentially a restatement of Theorem 3.4.3 in the language of orthogonal complements; it is illustrated in Example 6 of Section 3.4. The proof of part (b), which is left as an exercise, follows from part (a). The essential idea of the theorem is illustrated in **Figure 4.9.2**.

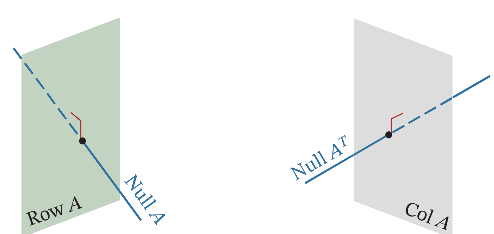

FIGURE 4.9.2

Theorem 4.9.7

If A is an $m \times n$ matrix, then:

(a) The null space of A and the row space of A are orthogonal complements in R^n.

(b) The null space of A^T and the column space of A are orthogonal complements in R^m.

Explain why $\{\mathbf{0}\}$ and R^n are orthogonal complements.

The results in Theorem 4.9.7 are often illustrated as in **Figure 4.9.3**, which conveys the orthogonality properties in the theorem as well as the dimensions of the fundamental spaces.

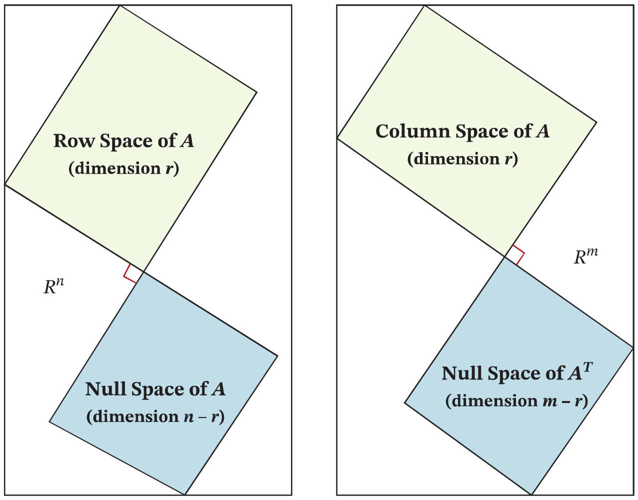

FIGURE 4.9.3

More on the Equivalence Theorem

In Theorem 2.3.8 we listed seven results that are equivalent to the invertibility of a square matrix A. We are now in a position to add ten more statements to that list to produce a single theorem that summarizes and links together all of the topics that we have covered thus far. We will prove some of the equivalences and leave others as exercises.

Theorem 4.9.8

Equivalent Statements

If A is an $n \times n$ matrix in which there are no duplicate rows and no duplicate columns, then the following statements are equivalent.

(*a*) A is invertible.

(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*c*) The reduced row echelon form of A is I_n.

(*d*) A is expressible as a product of elementary matrices.

(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) The column vectors of A are linearly independent.

(*i*) The row vectors of A are linearly independent.

(*j*) The column vectors of A span R^n.

(*k*) The row vectors of A span R^n.

(*l*) The column vectors of A form a basis for R^n.

(*m*) The row vectors of A form a basis for R^n.

(*n*) A has rank n.

(*o*) A has nullity 0.

(*p*) The orthogonal complement of the null space of A is R^n.

(*q*) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

Proof The following proofs show that (b) implies (h) through (q). In the exercises we will ask you to complete the proof by showing that (q) implies (b).

(b) $\Rightarrow$ ***(h)*** By Formula (10) of Section 1.3, $A\mathbf{x}$ is a linear combination of the column vectors of A. Since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, the column vectors of A must be linearly independent.

(h) $\Rightarrow$ ***(j)***, ***(h)*** $\Rightarrow$ ***(l)***, ***(h)*** $\Rightarrow$ ***(n)*** Since we now know that the n column vectors of A are linearly independent vectors in the n-dimensional vector space R^n, they must span R^n by Theorem 4.6.4 and hence form a basis for R^n. This also means that $\text{rank}(A) = n$.

(h) $\Rightarrow$ ***(i)***, ***(h)*** $\Rightarrow$ ***(k)***, ***(h)*** $\Rightarrow$ ***(m)*** Since we have shown that the column vectors form a basis for R^n, and since the row space and column space of A have the same dimension by Theorem 4.9.1, the n row vectors of A must also form a basis for R^n.

(n) $\Rightarrow$ ***(o)*** Since $\text{rank}(A) = n$, it follows from Theorem 4.9.2 that $\text{nullity}(A) = 0$.

(o) $\Rightarrow$ ***(p)*** $\text{nullity}(A) = 0$ means that the null space of A is $\{\mathbf{0}\}$, and since every vector in R^n is orthogonal to $\mathbf{0}$, it follows that the orthogonal complement of the null space of A is R^n.

(p) $\Rightarrow$ ***(q)*** It follows from Theorem 4.9.7 that orthogonal complement of the row space of A is the null space of A, which is $\{\mathbf{0}\}$. ■

Applications of Rank

The advent of the Internet has stimulated research on finding efficient methods for transmitting large amounts of digital data over communications lines with limited bandwidths. Digital data are commonly stored in matrix form, and many techniques for improving transmission speed use the rank of a matrix in some way. Rank plays a role because it measures the "redundancy" in a matrix in the sense that if A is an $m \times n$ matrix of rank k, then $n - k$ of the column vectors and $m - k$ of the row vectors can be expressed in terms of k linearly independent column or row vectors. The essential idea in many data compression schemes is to approximate the original data set by a data set with smaller rank that conveys nearly the same information, then eliminate redundant vectors in the approximating set to speed up the transmission time.

OPTIONAL: Overdetermined and Underdetermined Systems

In many applications the equations in a linear system correspond to physical constraints or conditions that must be satisfied. In general, the most desirable systems are those that have the same number of constraints as unknowns since such systems often have a unique solution. Unfortunately, it is not always possible to match the number of constraints and unknowns, so researchers are often faced with linear systems that have more constraints than unknowns, called ***overdetermined systems***, or with fewer constraints than unknowns, called ***underdetermined systems***. The following theorem will help us to analyze both overdetermined and underdetermined systems.

Theorem 4.9.9

Let A be an $m \times n$ matrix.

(*a*) (***Overdetermined Case***). If $m > n$, then the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent for at least one vector $\mathbf{b}$ in R^n.

(*b*) (***Underdetermined Case***). If $m < n$, then for each vector $\mathbf{b}$ in R^m the linear system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.

In engineering and physics, the occurrence of an overdetermined or underdetermined linear system often signals that one or more variables were omitted in formulating the problem or that extraneous variables were included. This often leads to some kind of complication.

Proof (a) Assume that $m > n$, in which case the column vectors of A cannot span R^m (fewer vectors than the dimension of R^m). Thus, there is at least one vector $\mathbf{b}$ in R^m that is not in the column space of A, and for any such $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ is inconsistent by Theorem 4.8.1.

Proof (b) Assume that $m < n$. For each vector $\mathbf{b}$ in R^n there are two possibilities: either the system $A\mathbf{x} = \mathbf{b}$ is consistent or it is inconsistent. If it is inconsistent, then the proof is complete. If it is consistent, then Theorem 4.9.4 implies that the general solution has $n - r$ parameters, where $r = \text{rank}(A)$. But we know from Example 2 that rank(A) is at most the smaller of m and n (which is m), so

$$n - r \geq n - m > 0$$

This means that the general solution has at least one parameter and hence there are infinitely many solutions. ■

EXAMPLE 7 | Overdetermined and Underdetermined Systems

(*a*) What can you say about the solutions of an overdetermined system $A\mathbf{x} = \mathbf{b}$ of 7 equations in 5 unknowns in which A has rank $r = 4$?

(*b*) What can you say about the solutions of an underdetermined system $A\mathbf{x} = \mathbf{b}$ of 5 equations in 7 unknowns in which A has rank $r = 4$?

Solution (*a*) The system is consistent for some vector $\mathbf{b}$ in R^7, and for any such $\mathbf{b}$ the number of parameters in the general solution is $n - r = 5 - 4 = 1$.

Solution (*b*) The system may be consistent or inconsistent, but if it is consistent for the vector $\mathbf{b}$ in R^5, then the general solution has $n - r = 7 - 4 = 3$ parameters.

EXAMPLE 8 | An Overdetermined System

The linear system

$$\begin{aligned} x_1 - 2x_2 &= b_1 \\ x_1 - x_2 &= b_2 \\ x_1 + x_2 &= b_3 \\ x_1 + 2x_2 &= b_4 \\ x_1 + 3x_2 &= b_5 \end{aligned}$$

is overdetermined, so it cannot be consistent for all possible values of b_1, b_2, b_3, b_4, and b_5. Conditions under which the system is consistent can be obtained by solving the linear system by Gauss–Jordan elimination. We leave it for you to show that the augmented matrix is row equivalent to

$$\begin{bmatrix} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 3b_2 + 2b_1 \\ 0 & 0 & b_4 - 4b_2 + 3b_1 \\ 0 & 0 & b_5 - 5b_2 + 4b_1 \end{bmatrix} \tag{8}$$

Thus, the system is consistent if and only if b_1, b_2, b_3, b_4, and b_5 satisfy the conditions

$$\begin{aligned} 2b_1 - 3b_2 + b_3 \qquad\qquad &= 0 \\ 3b_1 - 4b_2 \qquad + b_4 \qquad &= 0 \\ 4b_1 - 5b_2 \qquad\qquad + b_5 &= 0 \end{aligned}$$

Solving this homogeneous linear system yields

$$b_1 = 5r - 4s, \quad b_2 = 4r - 3s, \quad b_3 = 2r - s, \quad b_4 = r, \quad b_5 = s$$

where r and s are arbitrary.

Remark The coefficient matrix for the given linear system in the last example has $n = 2$ columns, and it has rank $r = 2$ because there are two nonzero rows in its reduced row echelon form. This implies that when the system is consistent its general solution will contain $n - r = 0$ parameters; that is, the solution will be unique. With a moment's thought, you should be able to see that this is so from (8).

OPTIONAL: Left Null Space Proof

Suppose that A is an $m \times n$ matrix of rank r and its reduced row echelon form is R. We will conclude this section by proving that if the augmented matrix $[A \mid I]$ is reduced to $[R \mid E]$ by Gauss-Jordan elimination, then the bottom $m - r$ rows of E form a basis for the left null space of A.

Proof The left null space of A is the solution space of the system $A^T\mathbf{x} = \mathbf{0}$, which, on transposing both sides, we can rewrite as

$$\mathbf{x}^T A = \mathbf{0}^T \tag{9}$$

Let $[R \mid E]$ denote the augmented matrix that results from $[A \mid I]$, when elementary row operations are applied to put the left side in reduced row echelon form R. The matrices A, R, and E are related by the equation

$$EA = R$$

where E is a product of elementary matrices. Since A has rank r and size $m \times n$, the matrix R has r nonzero rows and $m - r$ zero rows. By Formula (9) of Section 1.3 the ith row vector of R is the product

$$[i\text{th row vector of } E]\ A = i\text{th row vector of } R$$

But the last $m - r$ row vectors of R are zero, so the last $m - r$ row vectors of E are solutions of (9) and hence lie in the left null space of A. We leave it as an exercise to use Theorem 4.9.8 to show that these vectors form a basis for the left null space of A. ■

Exercise Set 4.9

In Exercises 1–2, find the rank and nullity of the matrix A by reducing it to row echelon form.

1. a. $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

2. a. $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$

In Exercises 3–6, the matrix R is the reduced row echelon form of the matrix A.

a. *By inspection of the matrix R, find the rank and nullity of A.*

b. *Confirm that the rank and nullity satisfy Formula (4).*

c. *Find the number of leading variables and the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ without solving the system.*

3. $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{bmatrix}$; $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & -6 \end{bmatrix}$; $R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 1 & 3 \\ -4 & 2 & 6 \end{bmatrix}$; $R = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}$; $R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. In each part, find the largest possible value for the rank of A and the smallest possible value for the nullity of A.

a. A is 4×4 **b.** A is 3×5 **c.** A is 5×3

8. If A is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value for its nullity?

9. In each part, use the information in the table to:

i. find the dimensions of the row space of A, column space of A, null space of A, and null space of A^T;

ii. determine whether the linear system $A\mathbf{x} = \mathbf{b}$ is consistent;

iii. find the number of parameters in the general solution of each system in (ii) that is consistent.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Rank$[A \mid \mathbf{b}]$	3	3	1	2	3	0	2

10. Verify that $\text{rank}(A) = \text{rank}(A^T)$.

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

In Exercises 11–14 find the dimensions and bases for the four fundamental spaces of the matrix.

11. $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$ **12.** $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & -1 & -4 \\ -1 & 0 & -4 \\ -2 & 3 & 4 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & -3 \\ 1 & -1 & 2 & 2 \end{bmatrix}$

In Exercises 15–18 confirm the orthogonality statements in the two parts of Theorem 4.9.7 for the given matrix.

15. The matrix in Exercise 11. **16.** The matrix in Exercise 12.

17. The matrix in Exercise 13. **18.** The matrix in Exercise 14.

In Exercises 19–20 use the method of Example 5 to find bases for the four fundamental spaces of the matrix.

19. $A = \begin{bmatrix} 0 & 2 & 8 & -7 \\ 2 & -2 & 4 & 0 \\ -3 & 4 & -2 & 5 \end{bmatrix}$ **20.** $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 2 & 8 & 0 & 1 & 2 \\ 0 & 4 & -6 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

21. a. Find an equation relating $\text{nullity}(A)$ and $\text{nullity}(A^T)$ for the matrix in Exercise 10.

b. Find an equation relating $\text{nullity}(A)$ and $\text{nullity}(A^T)$ for a general $m \times n$ matrix.

22. Let $T: R^2 \to R^3$ be the linear transformation defined by the formula

$$T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, x_1)$$

a. Find the rank of the standard matrix for T.

b. Find the nullity of the standard matrix for T.

23. Let $T: R^5 \to R^3$ be the linear transformation defined by the formula

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2, x_2 + x_3 + x_4, x_4 + x_5)$$

a. Find the rank of the standard matrix for T.

b. Find the nullity of the standard matrix for T.

24. Discuss how the rank of A varies with t.

a. $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$ **b.** $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

25. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1? Has rank 2? If so, find those values.

26. **a.** Give an example of a 3×3 matrix whose column space is a plane through the origin in 3-space.

b. What kind of geometric object is the null space of your matrix?

c. What kind of geometric object is the row space of your matrix?

27. Suppose that A is a 3×3 matrix whose null space is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

28. **a.** If A is a 3×5 matrix, then the rank of A is at most ________. Why?

b. If A is a 3×5 matrix, then the nullity of A is at most ________. Why?

c. If A is a 3×5 matrix, then the rank of A^T is at most ________. Why?

d. If A is a 3×5 matrix, then the nullity of A^T is at most ________. Why?

29. **a.** If A is a 3×5 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ________. Why?

b. If A is a 3×5 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ________. Why?

c. If A is a 5×3 matrix, then the number of leading 1's in the reduced row echelon form of A is at most ________. Why?

d. If A is a 5×3 matrix, then the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ is at most ________. Why?

30. Let A be a 7×6 matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Find the rank and nullity of A.

31. Let A be a 5×7 matrix with rank 4.

a. What is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

b. Is $A\mathbf{x} = \mathbf{b}$ consistent for all vectors $\mathbf{b}$ in R^5? Explain.

32. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Show that A has rank 2 if and only if one or more of the following determinants is nonzero.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

33. Use the result in Exercise 22 to show that the set of points (x, y, z) in R^3 for which the matrix

$$\begin{bmatrix} x & y & z \\ 1 & x & y \end{bmatrix}$$

has rank 1 is the curve with parametric equations $x = t, y = t^2$, $z = t^3$.

34. Find matrices A and B for which $\text{rank}(A) = \text{rank}(B)$, but $\text{rank}(A^2) \neq \text{rank}(B^2)$.

35. In Example 6 of Section 4.7 we showed that the row space and the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

are orthogonal complements in R^6, as guaranteed by part (a) of Theorem 4.9.7. Show that null space of A^T and the column space of A are orthogonal complements in R^4, as guaranteed by part (b) of Theorem 4.9.7. [*Suggestion:* Show that each column vector of A is orthogonal to each vector in a basis for the null space of A^T.]

36. Confirm the results stated in Theorem 4.9.7 for the matrix.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

37. In each part, state whether the system is overdetermined or underdetermined. If overdetermined, find all values of the b's for which it is inconsistent, and if underdetermined, find all values of the b's for which it is inconsistent and all values for which it has infinitely many solutions.

a. $\begin{bmatrix} 1 & -1 \\ -3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -3 & 4 \\ -2 & -6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -3 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

38. What conditions must be satisfied by b_1, b_2, b_3, b_4, and b_5 for the overdetermined linear system

$$\begin{aligned} x_1 - 3x_2 &= b_1 \\ x_1 - 2x_2 &= b_2 \\ x_1 + x_2 &= b_3 \\ x_1 - 4x_2 &= b_4 \\ x_1 + 5x_2 &= b_5 \end{aligned}$$

to be consistent?

Working with Proofs

39. Prove: If $k \neq 0$, then A and kA have the same rank.

40. Prove: If a matrix A is not square, then either the row vectors or the column vectors of A are linearly dependent.

41. Use Theorem 4.9.3 to prove Theorem 4.9.4.

42. Prove Theorem 4.9.7(b).

43. Prove: If a vector $\mathbf{v}$ in R^n is orthogonal to each vector in a basis for a subspace W of R^n, then $\mathbf{v}$ is orthogonal to every vector in W.

44. Prove: (q) implies (b) in Theorem 4.9.8.

True-False Exercises

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

a. Either the row vectors or the column vectors of a square matrix are linearly independent.

b. A matrix with linearly independent row vectors and linearly independent column vectors is square.

c. The nullity of a nonzero $m \times n$ matrix is at most m.

d. Adding one additional column to a matrix increases its rank by one.

e. The nullity of a square matrix with linearly dependent rows is at least one.

f. If A is square and $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, then the nullity of A is zero.

g. If a matrix A has more rows than columns, then the dimension of the row space is greater than the dimension of the column space.

h. If $\text{rank}(A^T) = \text{rank}(A)$, then A is square.

i. There is no 3×3 matrix whose row space and null space are both lines in 3-space.

j. If V is a subspace of R^n and W is a subspace of V, then $W^\perp$ is a subspace of $V^\perp$.

Working with Technology

T1. It can be proved that a nonzero matrix A has rank k if and only if some $k \times k$ submatrix has a nonzero determinant and all square submatrices of larger size have determinant zero. Use this fact to find the rank of

$$A = \begin{bmatrix} 3 & -1 & 3 & 2 & 5 \\ 5 & -3 & 2 & 3 & 4 \\ 1 & -3 & -5 & 0 & -7 \\ 7 & -5 & 1 & 4 & 1 \end{bmatrix}$$

Check your result by computing the rank of A in a different way.

T2. ***Sylvester's inequality*** states that if A and B are $n \times n$ matrices with rank r_A and r_B, respectively, then the rank r_{AB} of AB satisfies the inequality

$$r_A + r_B - n \leq r_{AB} \leq \min(r_A, r_B)$$

where $\min(r_A, r_B)$ denotes the smaller of r_A and r_B or their common value if the two ranks are the same. Use your technology utility to confirm this result for some matrices of your choice.

Chapter 4 Supplementary Exercises

1. Let V be the set of all ordered triples of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3), \quad k\mathbf{u} = (ku_1, 0, 0)$$

a. Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (3, -2, 4)$, $\mathbf{v} = (1, 5, -2)$, and $k = -1$.

b. In words, explain why V is closed under addition and scalar multiplication.

c. Since the addition operation on V is the standard addition operation on R^3, certain vector space axioms hold for V because they are known to hold for R^3. Which axioms in Definition 1 of Section 4.1 are they?

d. Show that Axioms 7, 8, and 9 hold.

e. Show that Axiom 10 fails for the given operations.

2. In each part, the solution space of the system is a subspace of R^3 and so must be a line through the origin, a plane through the origin, all of R^3, or the origin only. For each system, determine which is the case. If the subspace is a plane, find an equation for it, and if it is a line, find parametric equations.

a. $0x + 0y + 0z = 0$

b.
$$\begin{aligned} 2x - 3y + z &= 0 \\ 6x - 9y + 3z &= 0 \\ -4x + 6y - 2z &= 0 \end{aligned}$$

c.
$$\begin{aligned} x - 2y + 7z &= 0 \\ -4x + 8y + 5z &= 0 \\ 2x - 4y + 3z &= 0 \end{aligned}$$

d.
$$\begin{aligned} x + 4y + 8z &= 0 \\ 2x + 5y + 6z &= 0 \\ 3x + y - 4z &= 0 \end{aligned}$$

3. For what values of s is the solution space of

$$\begin{aligned} x_1 + x_2 + sx_3 &= 0 \\ x_1 + sx_2 + x_3 &= 0 \\ sx_1 + x_2 + x_3 &= 0 \end{aligned}$$

the origin only, a line through the origin, a plane through the origin, or all of R^3?

4. a. Express $(4a, a - b, a + 2b)$ as a linear combination of $(4, 1, 1)$ and $(0, -1, 2)$.

b. Express $(3a + b + 3c, -a + 4b - c, 2a + b + 2c)$ as a linear combination of $(3, -1, 2)$ and $(1, 4, 1)$.

c. Express $(2a - b + 4c, 3a - c, 4b + c)$ as a linear combination of three nonzero vectors.

5. Let W be the space spanned by $\mathbf{f} = \sin x$ and $\mathbf{g} = \cos x$.

a. Show that for any value of θ, $\mathbf{f}_1 = \sin(x + \theta)$ and $\mathbf{g}_1 = \cos(x + \theta)$ are vectors in W.

b. Show that $\mathbf{f}_1$ and $\mathbf{g}_1$ form a basis for W.

6. a. Express $\mathbf{v} = (1, 1)$ as a linear combination of $\mathbf{v}_1 = (1, -1)$, $\mathbf{v}_2 = (3, 0)$, and $\mathbf{v}_3 = (2, 1)$ in two different ways.

b. Explain why this does not violate Theorem 4.5.1.