Orthogonal Diagonalization

Definition 1

If A and B are square matrices, then we say that B is ***orthogonally similar*** to A if there is an orthogonal matrix P such that $B = P^TAP$.

Note that if B is orthogonally similar to A, then it is also true that A is orthogonally similar to B since we can express A as $A = PBP^T = Q^TBQ$, where $Q = P^T$. This being the case we will say that A and B are ***orthogonally similar matrices*** if either is orthogonally similar to the other.

If A is orthogonally similar to some diagonal matrix, say

$$P^TAP = D$$

then we say A is ***orthogonally diagonalizable*** and P ***orthogonally diagonalizes*** A.

Our first goal in this section is to determine what conditions a matrix must satisfy to be orthogonally diagonalizable. As an initial step, observe that there is no hope of orthogonally diagonalizing a matrix that is not symmetric. To see why this is so, suppose that

$$P^TAP = D \tag{1}$$

where P is an orthogonal matrix and D is a diagonal matrix. Multiplying the left side of (1) by P, the right side by P^T, and then using the fact that $PP^T = P^TP = I$, we can rewrite this equation as

$$A = PDP^T \tag{2}$$

Now transposing both sides of this equation and using the fact that a diagonal matrix is the same as its transpose we obtain

$$A^T = (PDP^T)^T = (P^T)^TD^TP^T = PDP^T = A$$

so A must be symmetric if it is orthogonally diagonalizable.

Conditions for Orthogonal Diagonalizability

Theorem 7.2.1

If A is an $n \times n$ matrix with real entries, then the following are equivalent.

(*a*) A is orthogonally diagonalizable.

(*b*) A has an orthonormal set of n eigenvectors.

(*c*) A is symmetric.

Properties of Symmetric Matrices

Theorem 7.2.2

If A is a symmetric matrix with real entries, then:

(*a*) The eigenvalues of A are all real numbers.

(*b*) Eigenvectors from different eigenspaces are orthogonal.

Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix

Step 1. Find a basis for each eigenspace of A.

Step 2. Apply the Gram–Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.

Step 3. Form the matrix P whose columns are the vectors constructed in Step 2. This matrix will orthogonally diagonalize A, and the eigenvalues on the diagonal of $D = P^TAP$ will be in the same order as their corresponding eigenvectors in P.

EXAMPLE 1 | Orthogonally Diagonalizing a Symmetric Matrix

Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Solution We leave it for you to verify that the characteristic equation of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{bmatrix} = (\lambda - 2)^2(\lambda - 8) = 0$$

Thus, the distinct eigenvalues of A are $\lambda = 2$ and $\lambda = 8$. By the method used in Example 7 of Section 5.1, it can be shown that

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

form a basis for the eigenspace corresponding to $\lambda = 2$. Applying the Gram–Schmidt process to $\{\mathbf{u}_1, \mathbf{u}_2\}$ yields the following orthonormal eigenvectors (verify):

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \tag{6}$$

The eigenspace corresponding to $\lambda = 8$ has

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as a basis. Applying the Gram–Schmidt process to $\{\mathbf{u}_3\}$ (i.e., normalizing $\mathbf{u}_3$) yields

$$\mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Finally, using $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as column vectors, we obtain

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

which orthogonally diagonalizes A. As a check, we leave it for you to confirm that

$$P^TAP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Question:

In Exercises 7–14, find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

11. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Solution:

$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda-3)^2$ therefore A has eigenvalues 3 and 0.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = \lambda_2 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - t$, $x_2 = s$, $x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace: $\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ orthogonally diagonalizes A resulting in

$$P^{-1}AP = P^{T}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$