4.7 Change of Basis

A basis that is suitable for one problem may not be suitable for another, so it is a common process in the study of vector spaces to change from one basis to another. Because a basis is the vector space generalization of a coordinate system, changing bases is akin to changing coordinate axes in R^2 and R^3. In this section we will study problems related to changing bases.

Coordinate Maps

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V, and if

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

is the coordinate vector of $\mathbf{v}$ relative to S, then, as illustrated in Figure 4.5.6, the mapping

$$\mathbf{v} \to (\mathbf{v})_S \tag{1}$$

creates a connection (a one-to-one correspondence) between vectors in the *general* vector space V and vectors in the *Euclidean* vector space R^n. We call (1) the ***coordinate map relative to S*** from V to R^n. In this section we will find it convenient to express coordinate vectors in the matrix form

$$[\mathbf{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{2}$$

where the square brackets emphasize the matrix notation (**Figure 4.7.1**).

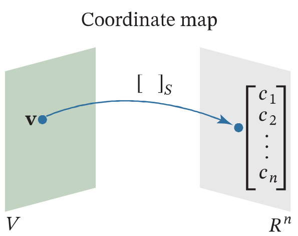

FIGURE 4.7.1

Change of Basis

There are many applications in which it is necessary to work with more than one coordinate system. In such cases it becomes important to know how the coordinates of a fixed vector relative to each coordinate system are related. This leads to the following problem.

The Change-of-Basis Problem

If $\mathbf{v}$ is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to a basis B', how are the coordinate vectors $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

Remark To solve this problem, it will be convenient to refer to the starting basis B as the "old basis" and the ending basis B' as the "new basis." Thus, our objective is to find a relationship between the old and new coordinates of a fixed vector $\mathbf{v}$ in V.

For simplicity, we will solve this problem for two-dimensional spaces. The solution for n-dimensional spaces is similar. Let

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} \quad \text{and} \quad B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$$

be the old and new bases, respectively. Suppose that the coordinate vectors for the old basis vectors relative to the new basis are

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad [\mathbf{u}_2]_{B'} = \begin{bmatrix} c \\ d \end{bmatrix} \tag{3}$$

That is,

$$\begin{aligned} \mathbf{u}_1 &= a\mathbf{u}'_1 + b\mathbf{u}'_2 \\ \mathbf{u}_2 &= c\mathbf{u}'_1 + d\mathbf{u}'_2 \end{aligned} \tag{4}$$

Now let $\mathbf{v}$ be any vector in V, and suppose that the old coordinate vector for $\mathbf{v}$ is

$$[\mathbf{v}]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{5}$$

so that

$$\mathbf{v} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{6}$$

In order to find the new coordinates of the vector $\mathbf{v}$ we must express $\mathbf{v}$ in terms of the new basis B'. To do this we will substitute (4) into (6), which yields

$$\mathbf{v} = k_1(a\mathbf{u}'_1 + b\mathbf{u}'_2) + k_2(c\mathbf{u}'_1 + d\mathbf{u}'_2)$$

or

$$\mathbf{v} = (k_1a + k_2c)\mathbf{u}'_1 + (k_1b + k_2d)\mathbf{u}'_2$$

Thus, the new coordinate vector for $\mathbf{v}$ is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix}$$

which, by using (5), we can rewrite as

$$[\mathbf{v}]_{B'} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}[\mathbf{v}]_B$$

This equation states that the new coordinate vector $[\mathbf{v}]_{B'}$ results when the old coordinate vector is multiplied on the left by the matrix

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

whose columns are the coordinate vectors of the old basis relative to the new basis [see (3)]. Thus, we are led to the following solution to the change-of-basis problem.

Solution to the Change-of-Basis Problem

If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$, then for each vector $\mathbf{v}$ in V, the new coordinate vector $[\mathbf{v}]_{B'}$ is related to the old coordinate vector $[\mathbf{v}]_B$ by the equation

$$[\mathbf{v}]_{B'} = P[\mathbf{v}]_B \tag{7}$$

where the columns of P are the coordinate vectors of the old basis vectors relative to the new basis; that is

$$P = \Big[[\mathbf{u}_1]_{B'} \,|\, [\mathbf{u}_2]_{B'} \,|\, \ldots \,|\, [\mathbf{u}_n]_{B'}\Big] \tag{8}$$

Transition Matrices

The matrix P in Equations (7) and (8) is called the ***transition matrix from B to B′*** and will be denoted in this text as

$$P_{B\to B'} = \Big[[\mathbf{u}_1]_{B'} \,|\, [\mathbf{u}_2]_{B'} \,|\, \ldots \,|\, [\mathbf{u}_n]_{B'}\Big] \tag{9}$$

to emphasize that it changes coordinates relative to B into coordinates relative to B'. Analogously, the ***transition matrix from B′ to B*** will be denoted by

$$P_{B'\to B} = \Big[[\mathbf{u}'_1]_B \,|\, [\mathbf{u}'_2]_B \,|\, \ldots \,|\, [\mathbf{u}'_n]_B\Big] \tag{10}$$

Remark In Formula (9) the old basis is B, and in Formula (10) the old basis is B'. Rather than memorizing these formulas, think about both in the following way.

The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.

EXAMPLE 1 | Finding Transition Matrices

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = (1, 0), \quad \mathbf{u}_2 = (0, 1), \quad \mathbf{u}'_1 = (1, 1), \quad \mathbf{u}'_2 = (2, 1)$$

(*a*) Find the transition matrix $P_{B \to B'}$ from B to B'.

(*b*) Find the transition matrix $P_{B' \to B}$ from B' to B.

Solution (*a*) Here the old basis vectors are $\mathbf{u}_1$ and $\mathbf{u}_2$ and the new basis vectors are $\mathbf{u}'_1$ and $\mathbf{u}'_2$. We want to find the coordinate matrices of the old basis vectors relative to the new basis vectors. To do this, observe that

$$\begin{aligned} \mathbf{u}_1 &= -\mathbf{u}'_1 + \mathbf{u}'_2 \\ \mathbf{u}_2 &= 2\mathbf{u}'_1 - \mathbf{u}'_2 \end{aligned}$$

from which it follows that

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{u}_2]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and hence that

$$P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Solution (*b*) Here the old basis vectors are $\mathbf{u}'_1$ and $\mathbf{u}'_2$ and the new basis vectors are $\mathbf{u}_1$ and $\mathbf{u}_2$. We want to find the coordinate matrices of the old basis vectors relative to the new basis vectors. To do this, observe that

$$\begin{aligned} \mathbf{u}'_1 &= \mathbf{u}_1 + \mathbf{u}_2 \\ \mathbf{u}'_2 &= 2\mathbf{u}_1 + \mathbf{u}_2 \end{aligned}$$

from which it follows that

$$[\mathbf{u}'_1]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{u}'_2]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and hence that

$$P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Transforming Coordinates

Suppose now that B and B' are bases for a finite-dimensional vector space V. Since multiplication by $P_{B \to B'}$ maps coordinate vectors relative to the basis B into coordinate vectors relative to a basis B', and $P_{B' \to B}$ maps coordinate vectors relative to B' into coordinate vectors relative to B, it follows that for every vector $\mathbf{v}$ in V we have

$$\boxed{[\mathbf{v}]_{B'} = P_{B \to B'}[\mathbf{v}]_B} \tag{11}$$

$$\boxed{[\mathbf{v}]_B = P_{B' \to B}[\mathbf{v}]_{B'}} \tag{12}$$

EXAMPLE 2 | Change of Coordinates

Let B and B' be the bases in Example 1. Use an appropriate formula to find $[\mathbf{v}]_{B'}$ given that

$$[\mathbf{v}]_B = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Solution To find $[\mathbf{v}]_{B'}$ we need to make the transition from B to B'. It follows from Formula (12) and part (a) of Example 1 that

$$[\mathbf{v}]_{B'} = P_{B\to B'}[\mathbf{v}]_B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -8 \end{bmatrix}$$

Invertibility of Transition Matrices

If B and B' are bases for a finite-dimensional vector space V, then

$$(P_{B'\to B})(P_{B\to B'}) = P_{B\to B}$$

because multiplication by the product $(P_{B'\to B})(P_{B\to B'})$ first maps the B-coordinates of a vector into its B'-coordinates, and then maps those B'-coordinates back into the original B-coordinates. Since the net effect of the two operations is to leave each coordinate vector unchanged, we are led to conclude that $P_{B\to B}$ must be the identity matrix, that is,

$$(P_{B'\to B})(P_{B\to B'}) = I \tag{13}$$

For example, for the transition matrices obtained in Example 1 we have

$$(P_{B'\to B})(P_{B\to B'}) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

It follows from (13) that $P_{B'\to B}$ is invertible and that its inverse is $P_{B\to B'}$. Thus, we have the following theorem.

Theorem 4.7.1

If P is the transition matrix from a basis B to a basis B' for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B' to B.

An Efficient Method for Computing Transition Matrices between Bases for R^n

Our next objective is to develop an efficient procedure for computing transition matrices between bases for R^n. As illustrated in Example 1, the first step in computing a transition matrix is to express each new basis vector as a linear combination of the old basis vectors. For R^n this involves solving n linear systems of n equations in n unknowns, each of which has the same coefficient matrix (why?). An efficient way to do this is by the method illustrated in Example 2 of Section 1.6, which is as follows:

A Procedure for Computing Transition Matrices

Step 1. Form the partitioned matrix **[new basis | old basis]** in which the basis vectors are in column form.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be **[I | transition matrix from old to new]** where I is an identity matrix.

Step 4. Extract the matrix on the right side of the matrix obtained in Step 3.

This procedure is captured in the diagram.

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}] \tag{14}$$

EXAMPLE 3 | Example 1 Revisited

In Example 1 we considered the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = (1, 0), \quad \mathbf{u}_2 = (0, 1), \quad \mathbf{u}'_1 = (1, 1), \quad \mathbf{u}'_2 = (2, 1)$$

(*a*) Use Formula (14) to find the transition matrix from B to B'.

(*b*) Use Formula (14) to find the transition matrix from B' to B.

Solution (*a*) Here B is the old basis and B' is the new basis, so

$$[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

By reducing this matrix, so the left side becomes the identity, we obtain (verify)

$$[I \mid \text{transition from old to new}] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

so the transition matrix is

$$P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

which agrees with the result in Example 1.

Solution (*b*) Here B' is the old basis and B is the new basis, so

$$[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Since the left side is already the identity matrix, no reduction is needed. We see by inspection that the transition matrix is

$$P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

which agrees with the result in Example 1.

Transition to the Standard Basis for R^n

Note that in part (*b*) of the last example the column vectors of the matrix that made the transition from the basis B to the standard basis turned out to be the vectors in B written in column form. This illustrates the following general result.

Theorem 4.7.2

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be any basis for R^n and let $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for R^n. If the vectors in these bases are written in column form, then

$$P_{B \to S} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n] \tag{15}$$

It follows from this theorem that if

$$A = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n]$$

is any *invertible* $n \times n$ matrix, then A can be viewed as the transition matrix from the basis $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ for R^n to the standard basis for R^n. Thus, for example, the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

which was shown to be invertible in Example 4 of Section 1.5, is the transition matrix from the basis

$$\mathbf{u}_1 = (1, 2, 1), \quad \mathbf{u}_2 = (2, 5, 0), \quad \mathbf{u}_3 = (3, 3, 8)$$

to the basis

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1)$$

Exercise Set 4.7

1. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{u}_1' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

a. Find the transition matrix from B' to B.

b. Find the transition matrix from B to B'.

c. Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use (11) to compute $[\mathbf{w}]_{B'}$.

d. Check your work by computing $[\mathbf{w}]_{B'}$ directly.

2. Repeat the directions of Exercise 1 with the same vector $\mathbf{w}$ but with

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2' = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

3. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3'\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_1' = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{u}_2' = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{u}_3' = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

a. Find the transition matrix from B to B'.

b. Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$$

and use (11) to compute $[\mathbf{w}]_{B'}$.

c. Check your work by computing $[\mathbf{w}]_{B'}$ directly.

4. Repeat the directions of Exercise 3 with the same vector $\mathbf{w}$, but with

$$\mathbf{u}_1 = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_1' = \begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2' = \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix}, \quad \mathbf{u}_3' = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

5. Let V be the space spanned by $\mathbf{f}_1 = \sin x$ and $\mathbf{f}_2 = \cos x$.

a. Show that $\mathbf{g}_1 = 2\sin x + \cos x$ and $\mathbf{g}_2 = 3\cos x$ form a basis for V.

b. Find the transition matrix from $B' = \{\mathbf{g}_1, \mathbf{g}_2\}$ to $B = \{\mathbf{f}_1, \mathbf{f}_2\}$.

c. Find the transition matrix from B to B'.

d. Compute the coordinate vector $[\mathbf{h}]_B$, where $\mathbf{h} = 2\sin x - 5\cos x$, and use (11) to obtain $[\mathbf{h}]_{B'}$.

e. Check your work by computing $[\mathbf{h}]_{B'}$ directly.

6. Consider the bases $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$ for P_1, where

$$\mathbf{p}_1 = 6 + 3x, \quad \mathbf{p}_2 = 10 + 2x, \quad \mathbf{q}_1 = 2, \quad \mathbf{q}_2 = 3 + 2x$$

a. Find the transition matrix from B' to B.

b. Find the transition matrix from B to B'.

c. Compute the coordinate vector $[\mathbf{p}]_B$, where $\mathbf{p} = -4 + x$, and use (11) to compute $[\mathbf{p}]_{B'}$.

d. Check your work by computing $[\mathbf{p}]_{B'}$ directly.

7. Let $B_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the bases for R^2 in which $\mathbf{u}_1 = (1, 2), \mathbf{u}_2 = (2, 3), \mathbf{v}_1 = (1, 3)$, and $\mathbf{v}_2 = (1, 4)$.

a. Use Formula (14) to find the transition matrix $P_{B_2 \to B_1}$.

b. Use Formula (14) to find the transition matrix $P_{B_1 \to B_2}$.

c. Confirm that the matrices $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another.

d. Let $\mathbf{w} = (0, 1)$. Find $[\mathbf{w}]_{B_1}$ and then use the matrix $P_{B_1 \to B_2}$ to compute $[\mathbf{w}]_{B_2}$ from $[\mathbf{w}]_{B_1}$.

e. Let $\mathbf{w} = (2, 5)$. Find $[\mathbf{w}]_{B_2}$ and then use the matrix $P_{B_2 \to B_1}$ to compute $[\mathbf{w}]_{B_1}$ from $[\mathbf{w}]_{B_2}$.