1. **(a)** In this part, B' is the start basis and B is the end basis:

$$\left[\text{end basis} \mid \text{start basis}\right]=\left[\begin{array}{cc|cc}2 & 4 & 1 & -1\\ 2 & -1 & 3 & -1\end{array}\right]$$

The reduced row echelon form of this matrix is

$$\left[I \mid \text{transition from start to end}\right]=\left[\begin{array}{cc|cc}1 & 0 & \frac{13}{10} & -\frac{1}{2}\\ 0 & 1 & -\frac{2}{5} & 0\end{array}\right]$$

The transition matrix is $P_{B'\to B}=\begin{bmatrix}\frac{13}{10} & -\frac{1}{2}\\ -\frac{2}{5} & 0\end{bmatrix}$.

(b) In this part, B is the start basis and B' is the end basis:

$$\left[\text{end basis} \mid \text{start basis}\right]=\left[\begin{array}{cc|cc}1 & -1 & 2 & 4\\ 3 & -1 & 2 & -1\end{array}\right]$$

The reduced row echelon form of this matrix is

$$\left[I \mid \text{transition from start to end}\right]=\left[\begin{array}{cc|cc}1 & 0 & 0 & -\frac{5}{2}\\ 0 & 1 & -2 & -\frac{13}{2}\end{array}\right]$$

The transition matrix is $P_{B\to B'}=\begin{bmatrix}0 & -\frac{5}{2}\\ -2 & -\frac{13}{2}\end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix}3\\-5\end{bmatrix}=c_1\begin{bmatrix}2\\2\end{bmatrix}+c_2\begin{bmatrix}4\\-1\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned}2c_1 + 4c_2 &= 3\\ 2c_1 - c_2 &= -5\end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & -\frac{17}{10}\\ 0 & 1 & \frac{8}{5}\end{bmatrix}$. The solution of the linear system is $c_1=-\frac{17}{10}$, $c_2=\frac{8}{5}$, therefore the coordinate vector is $\left[\mathbf{w}\right]_B=\begin{bmatrix}-\frac{17}{10}\\ \frac{8}{5}\end{bmatrix}$.

Using Formula (12), $\left[\mathbf{w}\right]_{B'}=P_{B\to B'}\left[\mathbf{w}\right]_B=\begin{bmatrix}0 & -\frac{5}{2}\\ -2 & -\frac{13}{2}\end{bmatrix}\begin{bmatrix}-\frac{17}{10}\\ \frac{8}{5}\end{bmatrix}=\begin{bmatrix}-4\\-7\end{bmatrix}$.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$ and $\mathbf{u}_2'$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} c_1 - c_2 &= 3 \\ 3c_1 - c_2 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -7 \end{bmatrix}$. The solution of the linear system is $c_1 = -4$, $c_2 = -7$, therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$. This matches the result obtained in part (c).

2. **(a)** In this part, B' is the start basis and B is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & 4 \end{array}\right] = [I \mid \text{transition from start to end}]$$

No row operations were necessary to obtain the transition matrix $P_{B' \to B} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$.

(b) In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right]$$

The transition matrix is $P_{B \to B'} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}$.

(c) Clearly, $[\mathbf{w}]_B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Using Formula (12), $[\mathbf{w}]_{B'} = P_{B \to B'} [\mathbf{w}]_B = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$ and $\mathbf{u}_2'$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 2c_1 - 3c_2 &= 3 \\ c_1 + 4c_2 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{3}{11} \\ 0 & 1 & -\frac{13}{11} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{3}{11}$, $c_2 = -\frac{13}{11}$, therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

This matches the result obtained in part (c).

3. **(a)** In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

The transition matrix is $P_{B \to B'} = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2c_1 & + & 2c_2 & + & c_3 & = & -5 \\ c_1 & - & c_2 & + & 2c_3 & = & 8 \\ c_1 & + & c_2 & + & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{bmatrix}$. The solution of the linear system is $c_1 = 9$, $c_2 = -9$, $c_3 = -5$ therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B \to B'} [\mathbf{w}]_B = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$, $\mathbf{u}_2'$ and $\mathbf{u}_3'$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 3c_1 & + & c_2 & - & c_3 & = & -5 \\ c_1 & + & c_2 & & & = & 8 \\ -5c_1 & - & 3c_2 & + & 2c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & \frac{23}{2} \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

The solution of the linear system is $c_1 = -\frac{7}{2}$, $c_2 = \frac{23}{2}$, $c_3 = 6$ therefore the coordinate vector is

$[\mathbf{w}]_{B'} = \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$, which matches the result we obtained in part (b).

4. **(a)** In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{rrr|rrr} -6 & -2 & -2 & -3 & -3 & 1 \\ -6 & -6 & -3 & 0 & 2 & 6 \\ 0 & 4 & 7 & -3 & -1 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{2}{3} \end{array}\right]$$

The transition matrix is $P_{B\to B'} = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} -3c_1 & - & 3c_2 & + & c_3 & = & -5 \\ & & 2c_2 & + & 6c_3 & = & 8 \\ -3c_1 & - & c_2 & - & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = 1$, $c_2 = 1$, $c_3 = 1$ therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}'_1$, $\mathbf{u}'_2$ and $\mathbf{u}'_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1\begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} -6c_1 & - & 2c_2 & - & 2c_3 & = & -5 \\ -6c_1 & - & 6c_2 & - & 3c_3 & = & 8 \\ & & 4c_2 & + & 7c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{19}{12} \\ 0 & 1 & 0 & -\frac{43}{12} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix}$.

The solution of the linear system is $c_1 = \frac{19}{12}$, $c_2 = -\frac{43}{12}$, $c_3 = \frac{4}{3}$ therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{bmatrix}$, which matches the result we obtained in part (b).

5. **(a)** The set $\{\mathbf{f}_1, \mathbf{f}_2\}$ is linearly independent since neither vector is a scalar multiple of the other. Thus $\{\mathbf{f}_1, \mathbf{f}_2\}$ is a basis for V and $\dim(V) = 2$.

Likewise, the set $\{\mathbf{g}_1, \mathbf{g}_2\}$ of vectors in V is linearly independent since neither vector is a scalar multiple of the other. By Theorem 4.6.4, $\{\mathbf{g}_1, \mathbf{g}_2\}$ is a basis for V.

(b) Clearly, $[\mathbf{g}_1]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $[\mathbf{g}_2]_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ hence $P_{B'\to B} = \left[[\mathbf{g}_1]_B \mid [\mathbf{g}_2]_B\right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

(c) We find the two columns of the transitions matrix $P_{B\to B'} = \left[\left[\mathbf{f}_1\right]_{B'} \mid \left[\mathbf{f}_2\right]_{B'}\right]$

$$\mathbf{f}_1 = a_1\mathbf{g}_1 + a_2\mathbf{g}_2 \qquad\qquad \mathbf{f}_2 = b_1\mathbf{g}_1 + b_2\mathbf{g}_2$$

$$\sin x = a_1\left(2\sin x + \cos x\right) + a_2\left(3\cos x\right) \qquad\qquad \cos x = b_1\left(2\sin x + \cos x\right) + b_2\left(3\cos x\right)$$

equate the coefficients corresponding to the same function on both sides of each equation

$$\begin{array}{rcrcr} 2a_1 & & & = & 1 \\ a_1 & + & 3a_2 & = & 0 \end{array} \qquad\qquad \begin{array}{rcrcr} 2b_1 & & & = & 0 \\ b_1 & + & 3b_2 & = & 1 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{6} \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$$

We obtain the transition matrix $P_{B\to B'} = \left[\left[\mathbf{f}_1\right]_{B'} \mid \left[\mathbf{f}_2\right]_{B'}\right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$.

(An alternate way to solve this part is to use Theorem 4.7.1 to yield

$$P_{B\to B'} = P_{B'\to B}^{-1} = \left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\right)^{-1} = \frac{1}{(2)(3)-(0)(1)}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}.)$$

(d) Clearly, the coordinate vector is $\left[\mathbf{h}\right]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Using Formula (12), we obtain $\left[\mathbf{h}\right]_{B'} = P_{B\to B'}\left[\mathbf{h}\right]_B = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(e) By inspection, $2\sin x - 5\cos x = \left(2\sin x + \cos x\right) - 2\left(3\cos x\right)$, hence the coordinate vector is $\left[\mathbf{p}\right]_{B'} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, which matches the result obtained in part (d).

6. **(a)** We find the two columns of the transitions matrix $P_{B'\to B} = \left[\left[\mathbf{q}_1\right]_B \mid \left[\mathbf{q}_2\right]_B\right]$

$$\mathbf{q}_1 = a_1\mathbf{p}_1 + a_2\mathbf{p}_2 \qquad\qquad \mathbf{q}_2 = b_1\mathbf{p}_1 + b_2\mathbf{p}_2$$

$$2 = a_1\left(6 + 3x\right) + a_2\left(10 + 2x\right) \qquad\qquad 3 + 2x = b_1\left(6 + 3x\right) + b_2\left(10 + 2x\right)$$

equate the coefficients corresponding to like powers of x on both sides of each equation

$$\begin{array}{rcrcr} 6a_1 & + & 10a_2 & = & 2 \\ 3a_1 & + & 2a_2 & = & 0 \end{array} \qquad\qquad \begin{array}{rcrcr} 6b_1 & + & 10b_2 & = & 3 \\ 3b_1 & + & 2b_2 & = & 2 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\begin{bmatrix} 1 & 0 & -\frac{2}{9} \\ 0 & 1 & \frac{1}{3} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{1}{6} \end{bmatrix}$$

We obtain the transition matrix $P_{B'\to B} = \left[[\mathbf{q}_1]_B \mid [\mathbf{q}_2]_B \right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$.

(b) We find the two columns of the transitions matrix $P_{B\to B'} = \left[[\mathbf{p}_1]_{B'} \mid [\mathbf{p}_2]_{B'} \right]$

$$\mathbf{p}_1 = a_1\mathbf{q}_1 + a_2\mathbf{q}_2 \qquad \mathbf{p}_2 = b_1\mathbf{q}_1 + b_2\mathbf{q}_2$$

$$6+3x = a_1(2) + a_2(3+2x) \qquad 10+2x = b_1(2) + b_2(3+2x)$$

equate the coefficients corresponding to like powers of x on both sides of each equation

$$\begin{aligned} 2a_1 + 3a_2 &= 6 \\ 2a_2 &= 3 \end{aligned} \qquad \begin{aligned} 2b_1 + 3b_2 &= 10 \\ 2b_2 &= 2 \end{aligned}$$

reduced row echelon form of the augmented matrix of each system

$$\begin{bmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{3}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

We obtain the transition matrix $P_{B\to B'} = \left[[\mathbf{p}_1]_{B'} \mid [\mathbf{p}_2]_{B'} \right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$.

(c) Since $-4+x = (6+3x)-(10+2x)$, the coordinate vector is $[\mathbf{p}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Using Formula (12), we obtain $[\mathbf{p}]_{B'} = P_{B\to B'}[\mathbf{p}]_B = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{2} \end{bmatrix}$.

(d) We are looking for the coordinate vector $[\mathbf{p}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ with c_1 and c_2 satisfying the equality

$$-4+x = c_1(2) + c_2(3+2x)$$

for all real values x. Equating the coefficients associated with like powers of x on both sides yields the linear system

$$\begin{aligned} 2c_1 + 3c_2 &= -4 \\ 2c_2 &= 1 \end{aligned}$$

which can easily be solved by back-substitution: $c_2 = \frac{1}{2}$, $c_1 = \frac{-4-3\left(\frac{1}{2}\right)}{2} = -\frac{11}{4}$. We conclude that $[\mathbf{p}]_{B'} = \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{2} \end{bmatrix}$, which matches the result obtained in part (c).

7. **(a)** In this part, B_2 is the start basis and B_1 is the end basis:

$$\left[\text{end basis} \mid \text{start basis}\right] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right].$$

The reduced row echelon form of this matrix is

$$\left[I \mid \text{transition from start to end}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right].$$

The transition matrix is $P_{B_2 \to B_1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$.

(b) In this part, B_1 is the start basis and B_2 is the end basis:

$$\left[\text{end basis} \mid \text{start basis}\right] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array}\right].$$

The reduced row echelon form of this matrix is

$$\left[I \mid \text{transition from start to end}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right].$$

The transition matrix is $P_{B_1 \to B_2} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$.

(c) Since $\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it follows that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} c_1 & + & 2c_2 & = & 0 \\ 2c_1 & + & 3c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$. The solution of the linear system is $c_1 = 2$, $c_2 = -1$, therefore the coordinate vector is $\left[\mathbf{w}\right]_{B_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

From Formula (12), $\left[\mathbf{w}\right]_{B_2} = P_{B_1 \to B_2}\left[\mathbf{w}\right]_{B_1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(e) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ we obtain

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 1c_1 & + & 1c_2 & = & 2 \\ 3c_1 & + & 4c_2 & = & 5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$. The solution of the linear system is $c_1 = 3$, $c_2 = -1$, therefore the coordinate vector is $[\mathbf{w}]_{B_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_{B_1} = P_{B_2 \to B_1} [\mathbf{w}]_{B_2} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

8. (a) By Theorem 4.7.2, $P_{B \to S} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$.

(b) In this part, S is the start basis and B is the end basis: $[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{rr|rr} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$.

The reduced row echelon form of this matrix is

$[I \mid \text{transition from start to end}] = \left[\begin{array}{rr|rr} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right]$.

The transition matrix is $P_{S \to B} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}$.

(c) Since $\begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it follows that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another.

(d) Since $(5,-3) = (2,1) - (-3,4)$ the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_S = P_{B \to S} [\mathbf{w}]_B = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

(e) By inspection, $[\mathbf{w}]_S = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. From Formula (12), $[\mathbf{w}]_B = P_{S \to B} [\mathbf{w}]_S = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

9. **(a)** By Theorem 4.7.2, $P_{B\to S}=\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}$.

(b) In this part, S is the start basis and B is the end basis:

$$[\text{end basis}\mid\text{start basis}]=\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&5&3&0&1&0\\1&0&8&0&0&1\end{array}\right].$$

The reduced row echelon form of this matrix is

$$[I\mid\text{transition from start to end}]=\left[\begin{array}{ccc|ccc}1&0&0&-40&16&9\\0&1&0&13&-5&-3\\0&0&1&5&-2&-1\end{array}\right].$$

The transition matrix is $P_{S\to B}=\begin{bmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{bmatrix}$.

(c) Since $\begin{bmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{bmatrix}\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ and $\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}\begin{bmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ it follows that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$\begin{bmatrix}5\\-3\\1\end{bmatrix}=c_1\begin{bmatrix}1\\2\\1\end{bmatrix}+c_2\begin{bmatrix}2\\5\\0\end{bmatrix}+c_3\begin{bmatrix}3\\3\\8\end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr}c_1&+&2c_2&+&3c_3&=&5\\2c_1&+&5c_2&+&3c_3&=&-3\\c_1&&&+&8c_3&=&1\end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1&0&0&-239\\0&1&0&77\\0&0&1&30\end{bmatrix}$. The solution of the linear system is $c_1=-239$, $c_2=77$, $c_3=30$ therefore the coordinate vector is

$[\mathbf{w}]_B=\begin{bmatrix}-239\\77\\30\end{bmatrix}$. From Formula (12), $[\mathbf{w}]_S=P_{B\to S}[\mathbf{w}]_B=\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}\begin{bmatrix}-239\\77\\30\end{bmatrix}=\begin{bmatrix}5\\-3\\1\end{bmatrix}$.

(e) By inspection, $[\mathbf{w}]_S = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_B = P_{S\to B}[\mathbf{w}]_S = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -200 \\ 64 \\ 25 \end{bmatrix}$.

10. Reflecting $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ about the line $y = x$ results in $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Likewise for $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we obtain $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

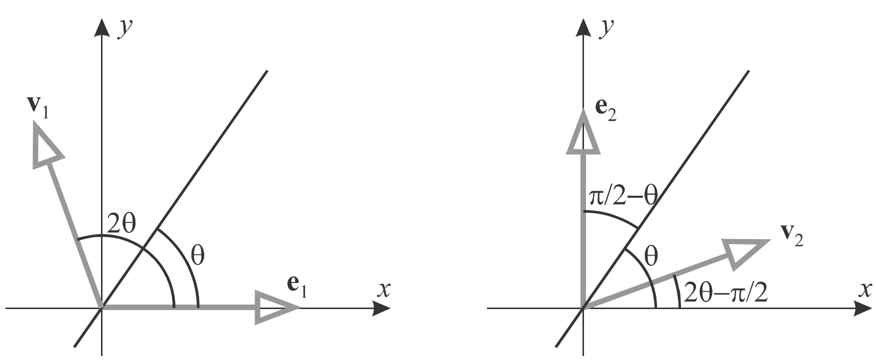
(a) From Theorem 4.7.5, $P_{B\to S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) Denoting $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, it follows from Theorem 4.7.5 that $P_{S\to B} = P^{-1}$. In our case, $PP = I$ therefore $P = P^{-1}$. Furthermore, since P is symmetric, we also have $P_{S\to B} = P^T$.

11. **(a)** Clearly, $\mathbf{v}_1 = (\cos(2\theta), \sin(2\theta))$. Referring to the figure on the right, we see that the angle between the positive x-axis and $\mathbf{v}_2$ is $\frac{\pi}{2} - 2\left(\frac{\pi}{2} - \theta\right) = 2\theta - \frac{\pi}{2}$. Hence,

$$\mathbf{v}_2 = \left(\cos\left(2\theta - \tfrac{\pi}{2}\right), \sin\left(2\theta - \tfrac{\pi}{2}\right)\right) = \left(\sin(2\theta), -\cos(2\theta)\right)$$

From Theorem 4.7.5, $P_{B\to S} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.



(b) Denoting $P = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$, it follows from Theorem 4.7.5 that $P_{S\to B} = P^{-1}$. In our case, $PP = I$ therefore $P = P^{-1}$. Furthermore, since P is symmetric, we also have $P_{S\to B} = P^T$.

12. Since for every vector $\mathbf{v}$ in R^2 we have $[\mathbf{v}]_{B_2} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}[\mathbf{v}]_{B_1}$ and $[\mathbf{v}]_{B_3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}[\mathbf{v}]_{B_2}$, it follows that $[\mathbf{v}]_{B_3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}[\mathbf{v}]_{B_1} = \begin{bmatrix} 31 & 11 \\ 7 & 2 \end{bmatrix}[\mathbf{v}]_{B_1}$ so that $P_{B_1 \to B_3} = \begin{bmatrix} 31 & 11 \\ 7 & 2 \end{bmatrix}$.

From Theorem 4.7.1, $P_{B_3 \to B_1}$ is the inverse of this matrix: $\begin{bmatrix} -\frac{2}{15} & \frac{11}{15} \\ \frac{7}{15} & -\frac{31}{15} \end{bmatrix}$.

13. Since for every vector $\mathbf{v}$ we have $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$ and $[\mathbf{v}]_C = Q[\mathbf{v}]_B$, it follows that $[\mathbf{v}]_C = QP[\mathbf{v}]_{B'}$ so that $P_{B' \to C} = QP$. From Theorem 4.7.1, $P_{C \to B'} = (QP)^{-1} = P^{-1}Q^{-1}$.

15. **(a)** By Theorem 4.7.2, P is the transition matrix from $B = \{(1,1,0),(1,0,2),(0,2,1)\}$ to S.

(b) By Theorem 4.7.1, $P^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$ is the transition matrix from B to S, hence by Theorem 4.7.2, $B = \left\{\left(\frac{4}{5}, \frac{1}{5}, -\frac{2}{5}\right), \left(\frac{1}{5}, -\frac{1}{5}, \frac{2}{5}\right), \left(-\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)\right\}$.

16. Let the given basis be denoted as $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,1,0)$, $\mathbf{v}_3 = (1,0,0)$ and denote the unknown basis as $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

We have $P_{B \to B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \left[[\mathbf{u}_1]_{B'} \,\middle|\, [\mathbf{u}_2]_{B'} \,\middle|\, [\mathbf{u}_3]_{B'}\right]$. Equating the respective columns yields

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{u}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 = (1,1,1)$$

$$[\mathbf{u}_2]_{B'} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \mathbf{u}_2 = 0\mathbf{v}_1 + 3\mathbf{v}_2 + 1\mathbf{v}_3 = (4,3,0)$$

$$[\mathbf{u}_3]_{B'} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{u}_3 = 0\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = (3,2,0)$$

Thus the given matrix is the transition matrix from the basis $\{(1,1,1),(4,3,0),(3,2,0)\}$.

17. From $T(1,0) = (2,5)$, $T(0,1) = (3,-1)$, and Theorem 4.7.2 we obtain $P_{B \to S} = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$.

18. From $T(1,0,0)=(1,2,0)$, $T(0,1,0)=(1,-1,1)$, $T(0,0,1)=(0,4,3)$, and Theorem 4.7.2 we obtain

$$P_{B\to S}=\begin{bmatrix}1&1&0\\2&-1&4\\0&1&3\end{bmatrix}.$$

19. By Formula (10), the transition matrix from the standard basis $S=\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ to B is $P_{S\to B}=\left[[\mathbf{e}_1]_B|\ldots|[\mathbf{e}_n]_B\right]=\left[\mathbf{e}_1|\ldots|\mathbf{e}_n\right]=I_n$ therefore B must be the standard basis.

True-False Exercises

(a) True. The matrix can be constructed according to Formula (10).

(b) True. This follows from Theorem 4.7.1.

(c) True.

(d) True.

(e) False. For instance, $B_1=\{(0,2),(3,0)\}$ is a basis for R^2 made up of scalar multiples of vectors in the standard basis $B_2=\{(1,0),(0,1)\}$. However, $P_{B_1\to B_2}=\begin{bmatrix}0&3\\2&0\end{bmatrix}$ (obtained by Theorem 4.7.2) is not a diagonal matrix.

(f) False. A must be invertible.

4.8 Row Space, Column Space, and Null Space

1. **(a)** $$\begin{bmatrix}2&3\\-1&4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=1\begin{bmatrix}2\\-1\end{bmatrix}+2\begin{bmatrix}3\\4\end{bmatrix}$$

(b) $$\begin{bmatrix}4&0&-1\\3&6&2\\0&-1&4\end{bmatrix}\begin{bmatrix}-2\\3\\5\end{bmatrix}=-2\begin{bmatrix}4\\3\\0\end{bmatrix}+3\begin{bmatrix}0\\6\\-1\end{bmatrix}+5\begin{bmatrix}-1\\2\\4\end{bmatrix}$$

2. **(a)** $$\begin{bmatrix}-3&6&2\\5&-4&0\\2&3&-1\\1&8&3\end{bmatrix}\begin{bmatrix}-1\\2\\5\end{bmatrix}=-1\begin{bmatrix}-3\\5\\2\\1\end{bmatrix}+2\begin{bmatrix}6\\-4\\3\\8\end{bmatrix}+5\begin{bmatrix}2\\0\\-1\\3\end{bmatrix}$$

(b) $$\begin{bmatrix}2&1&5\\6&3&-8\end{bmatrix}\begin{bmatrix}3\\0\\-5\end{bmatrix}=3\begin{bmatrix}2\\6\end{bmatrix}+0\begin{bmatrix}1\\3\end{bmatrix}-5\begin{bmatrix}5\\-8\end{bmatrix}$$