Linear Combination

Definition 1

If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{1}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

Theorem 4.3.1

If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

(*a*) The set W of all possible linear combinations of the vectors in S is a subspace of V.

(*b*) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

EXAMPLE 4 | Linear Combinations

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is *not* a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution In order for $\mathbf{w}$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Solving this system using Gaussian elimination yields $k_1 = -3$, $k_2 = 2$, so

$$\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$$

Similarly, for $\mathbf{w}'$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

This system of equations is inconsistent (verify), so no such scalars k_1 and k_2 exist. Consequently, $\mathbf{w}'$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Question:

2. Express the following as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$.

a. $(-9, -7, -15)$ **b.** $(6, 11, 6)$ **c.** $(0, 0, 0)$

Solution:

(a) For $(-9,-7,-15)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4)+b(1,-1,3)+c(3,2,5)=(-9,-7,-15)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a=-2$, $b=1$, $c=-2$, therefore $(-9,-7,-15)=-2\mathbf{u}+1\mathbf{v}-2\mathbf{w}$.

(b) For $(6,11,6)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4)+b(1,-1,3)+c(3,2,5)=(6,11,6)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a=4$, $b=-5$, $c=1$, therefore $(6,11,6)=4\mathbf{u}-5\mathbf{v}+1\mathbf{w}$.

(c) For $(0,0,0)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4)+b(1,-1,3)+c(3,2,5)=(0,0,0)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 0 \\ 1a & - & 1b & + & 2c & = & 0 \\ 4a & + & 3b & + & 5c & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. There is only one solution to this system, $a=0$, $b=0$, $c=0$, therefore $(0,0,0)=0\mathbf{u}+0\mathbf{v}+0\mathbf{w}$.

Question:

3. Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

a. $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

Solution:

(a) For $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & 6 \\ 0a & - & 1b & + & 2c & = & -8 \\ -2a & + & 2b & + & 1c & = & -1 \\ -2a & + & 3b & + & 4c & = & -8 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The linear system is consistent, therefore $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of A, B, and C.

(b) The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a linear combination of A, B, and C since $0A + 0B + 0C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) For $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & -1 \\ 0a & - & 1b & + & 2c & = & 5 \\ -2a & + & 2b & + & 1c & = & 7 \\ -2a & + & 3b & + & 4c & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0 = 1$ which is contradictory. We conclude that $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$ is not a linear combination of A, B, and C.

Question:

6. In each part express the vector as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$.

a. $-9 - 7x - 15x^2$ **b.** $6 + 11x + 6x^2$

c. 0 **d.** $7 + 8x + 9x^2$

Solution:

(a) For $-9-7x-15x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a\left(2+x+4x^2\right)+b\left(1-x+3x^2\right)+c\left(3+2x+5x^2\right)=-9-7x-15x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$\left(2a+b+3c\right)+\left(a-b+2c\right)x+\left(4a+3b+5c\right)x^2=-9-7x-15x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a=-2$, $b=1$, $c=-2$, therefore

$$-9-7x-15x^2 = -2\mathbf{p}_1+1\mathbf{p}_2-2\mathbf{p}_3 .$$

(b) For $6+11x+6x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a\left(2+x+4x^2\right)+b\left(1-x+3x^2\right)+c\left(3+2x+5x^2\right)=6+11x+6x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a+b+3c)+(a-b+2c)x+(4a+3b+5c)x^2=6+11x+6x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a=4$, $b=-5$, $c=1$, therefore $6+11x+6x^2=4\mathbf{p}_1-5\mathbf{p}_2+1\mathbf{p}_3$.

(c) By inspection, $0 = 0\mathbf{p}_1+0\mathbf{p}_2+0\mathbf{p}_3$.

(d) For $7+8x+9x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a\left(2+x+4x^2\right)+b\left(1-x+3x^2\right)+c\left(3+2x+5x^2\right)=7+8x+9x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a+b+3c)+(a-b+2c)x+(4a+3b+5c)x^2=7+8x+9x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 7 \\ 1a & - & 1b & + & 2c & = & 8 \\ 4a & + & 3b & + & 5c & = & 9 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. There is only one solution to this system, $a=0$, $b=-2$, $c=3$, therefore $7+8x+9x^2 = 0\mathbf{p}_1-2\mathbf{p}_2+3\mathbf{p}_3$.