

## Determinants

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if  $ad - bc \neq 0$  and that the expression  $ad - bc$  is called the **determinant** of the matrix  $A$ . Recall also that this determinant is denoted by writing

$$\det(A) = ad - bc \quad \text{or} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (1)$$

and that the inverse of  $A$  can be expressed in terms of the determinant as

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (2)$$

### Definition 1

If  $A$  is a square matrix, then the **minor of entry  $a_{ij}$**  is denoted by  $M_{ij}$  and is defined to be the determinant of the submatrix that remains after the  $i$ th row and  $j$ th column are deleted from  $A$ . The number  $(-1)^{i+j}M_{ij}$  is denoted by  $C_{ij}$  and is called the **cofactor of entry  $a_{ij}$** .

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry  $a_{11}$  is

$$M_{11} = \begin{vmatrix} 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

The cofactor of  $a_{11}$  is

$$C_{11} = (-1)^{1+1}M_{11} = M_{11} = 16$$

Similarly, the minor of entry  $a_{32}$  is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of  $a_{32}$  is

$$C_{32} = (-1)^{3+2}M_{32} = -M_{32} = -26$$

### Definition 2

If  $A$  is an  $n \times n$  matrix, then the number obtained by multiplying the entries in any row or column of  $A$  by the corresponding cofactors and adding the resulting products is called the **determinant of  $A$** , and the sums themselves are called **cofactor expansions of  $A$** . That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad (7)$$

**[cofactor expansion along the  $j$ th column]**

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad (8)$$

**[cofactor expansion along the  $i$ th row]**

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

**Solution**

$$\begin{aligned} \det(A) &= \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0 \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} \\ &= 3(-4) - (1)(-11) + 0 = -1 \end{aligned}$$

**Solution**

$$\begin{aligned} \det(A) &= \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} \\ &= 3(-4) - (-2)(-2) + 5(3) = -1 \end{aligned}$$

### Theorem 2.1.2

If  $A$  is an  $n \times n$  triangular matrix (upper triangular, lower triangular, or diagonal), then  $\det(A)$  is the product of the entries on the main diagonal of the matrix; that is,  $\det(A) = a_{11}a_{22} \cdots a_{nn}$ .

### EXAMPLE 7 | A Technique for Evaluating $2 \times 2$ and $3 \times 3$ Determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{vmatrix} = [45 + 84 + 96] - [105 - 48 - 72] = 240$$

## Evaluating Determinants by Row Reduction Elementary Row Operations

### Theorem 2.2.3

Let  $A$  be an  $n \times n$  matrix.

- (a) If  $B$  is the matrix that results when a single row or single column of  $A$  is multiplied by a scalar  $k$ , then  $\det(B) = k \det(A)$ .
- (b) If  $B$  is the matrix that results when two rows or two columns of  $A$  are interchanged, then  $\det(B) = -\det(A)$ .
- (c) If  $B$  is the matrix that results when a multiple of one row of  $A$  is added to another or when a multiple of one column is added to another, then  $\det(B) = \det(A)$ .

TABLE 1

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k \det(A)$	In the matrix $B$ the first row of $A$ was multiplied by $k$ .
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	In the matrix $B$ the first and second rows of $A$ were interchanged.
$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	In the matrix $B$ a multiple of the second row of $A$ was added to the first row.

### Theorem 2.2.1

Let  $A$  be a square matrix. If  $A$  has a row of zeros or a column of zeros, then  $\det(A) = 0$ .

### Theorem 2.2.2

Let  $A$  be a square matrix. Then  $\det(A) = \det(A^T)$ .

### Theorem 2.2.5

If  $A$  is a square matrix with two proportional rows or two proportional columns, then  $\det(A) = 0$ .

## EXAMPLE 3 | Using Row Reduction to Evaluate a Determinant

Evaluate  $\det(A)$  where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

**Solution** We will reduce  $A$  to row echelon form (which is upper triangular) and then apply Theorem 2.1.2.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} && \leftarrow \text{The first and second rows of } A \text{ were interchanged.} \\ &= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} && \leftarrow \text{A common factor of 3 from the first row was taken through the determinant sign.} \end{aligned}$$

$$\begin{aligned}
&= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} && \leftarrow -2 \text{ times the first row was added to the third row.} \\
&= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} && \leftarrow -10 \text{ times the second row was added to the third row.} \\
&= (-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix} && \leftarrow \text{A common factor of } -55 \text{ from the last row was taken through the determinant sign.} \\
&= (-3)(-55)(1) = 165
\end{aligned}$$

#### EXAMPLE 4 | Using Column Operations to Evaluate a Determinant

Compute the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

**Solution** This determinant could be computed as above by using elementary row operations to reduce  $A$  to row echelon form, but we can put  $A$  in lower triangular form in one step by adding  $-3$  times the first column to the fourth to obtain

$$\det(A) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{bmatrix} = (1)(7)(3)(-26) = -546$$

#### EXAMPLE 5 | Row Operations and Cofactor Expansion

Evaluate  $\det(A)$  where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

**Solution** By adding suitable multiples of the second row to the remaining rows, we obtain

$$\begin{aligned}
 \det(A) &= \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \\
 &= - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \quad \leftarrow \text{Cofactor expansion along the first column} \\
 &= - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} \quad \leftarrow \text{We added the first row to the third row.} \\
 &= -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} \quad \leftarrow \text{Cofactor expansion along the first column} \\
 &= -18
 \end{aligned}$$

## Basic Properties of Determinants

Suppose that  $A$  and  $B$  are  $n \times n$  matrices and  $k$  is any scalar. We begin by considering possible relationships among  $\det(A)$ ,  $\det(B)$ , and

$$\det(kA), \quad \det(A+B), \quad \text{and} \quad \det(AB)$$

Since a common factor of any row of a matrix can be moved through the determinant sign, and since each of the  $n$  rows in  $kA$  has a common factor of  $k$ , it follows that

$$\boxed{\det(kA) = k^n \det(A)} \quad (1)$$

For example,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

### Theorem 2.3.1

Let  $A$ ,  $B$ , and  $C$  be  $n \times n$  matrices that differ only in a single row, say the  $r$ th, and assume that the  $r$ th row of  $C$  can be obtained by adding corresponding entries in the  $r$ th rows of  $A$  and  $B$ . Then

$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns.

### EXAMPLE 1 | $\det(A + B) \neq \det(A) + \det(B)$

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

We have  $\det(A) = 1$ ,  $\det(B) = 8$ , and  $\det(A + B) = 23$ ; thus

$$\det(A + B) \neq \det(A) + \det(B)$$

### EXAMPLE 2 | Sums of Determinants

We leave it to you to confirm the following equality by evaluating the determinants.

$$\det \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

## Determinant of a Matrix Product

### Lemma 2.3.2

If  $B$  is an  $n \times n$  matrix and  $E$  is an  $n \times n$  elementary matrix, then

$$\det(EB) = \det(E) \det(B)$$

## Determinant Test for Invertibility

### Theorem 2.3.3

A square matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .

### Theorem 2.3.4

If  $A$  and  $B$  are square matrices of the same size, then

$$\det(AB) = \det(A) \det(B)$$

### Theorem 2.3.5

If  $A$  is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

## Adjoint of a Matrix

### Definition 1

If  $A$  is any  $n \times n$  matrix and  $C_{ij}$  is the cofactor of  $a_{ij}$ , then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the **matrix of cofactors from  $A$** . The transpose of this matrix is called the **adjoint of  $A$**  and is denoted by  $\text{adj}(A)$ .

### EXAMPLE 6 | Adjoint of a $3 \times 3$ Matrix

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

As noted in Example 5, the cofactors of  $A$  are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of  $A$  is

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$



### Theorem 2.3.6

#### Inverse of a Matrix Using Its Adjoint

If  $A$  is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \quad (6)$$

### EXAMPLE 7 | Using the Adjoint to Find an Inverse Matrix

Use Formula (6) to find the inverse of the matrix  $A$  in Example 6.

**Solution** We showed in Example 5 that  $\det(A) = 64$ . Thus,

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

### Theorem 2.3.7

#### Cramer's Rule

If  $A\mathbf{x} = \mathbf{b}$  is a system of  $n$  linear equations in  $n$  unknowns such that  $\det(A) \neq 0$ , then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \dots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where  $A_j$  is the matrix obtained by replacing the entries in the  $j$ th column of  $A$  by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

### EXAMPLE 8 | Using Cramer's Rule to Solve a Linear System

Use Cramer's rule to solve

$$\begin{aligned}x_1 + \quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8\end{aligned}$$

**Solution**

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$