Applications of Linear Systems

Network Analysis:

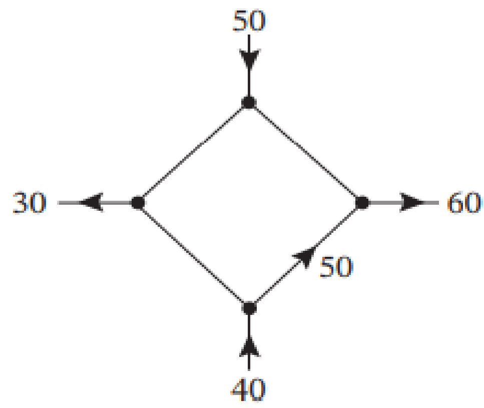
The concept of a *network* appears in a variety of applications. Loosely stated, a ***network*** is a set of ***branches*** through which something "flows." For example, the branches might be electrical wires through which electricity flows, pipes through which water or oil flows, traffic lanes through which vehicular traffic flows, or economic linkages through which money flows, to name a few possibilities.

In most networks, the branches meet at points, called ***nodes*** or ***junctions***, where the flow divides. For example, in an electrical network, nodes occur where three or more wires join, in a traffic network they occur at street intersections, and in a financial network they occur at banking centers where incoming money is distributed to individuals or other institutions.

In the study of networks, there is generally some numerical measure of the rate at which the medium flows through a branch. For example, the flow rate of electricity is often measured in amperes, the flow rate of water or oil in gallons per minute, the flow rate of traffic in vehicles per hour, and the flow rate of European currency in millions of Euros per day. We will restrict our attention to networks in which there is ***flow conservation*** at each node, by which we mean that *the rate of flow into any node is equal to the rate of flow out of that node.* This ensures that the flow medium does not build up at the nodes and block the free movement of the medium through the network.

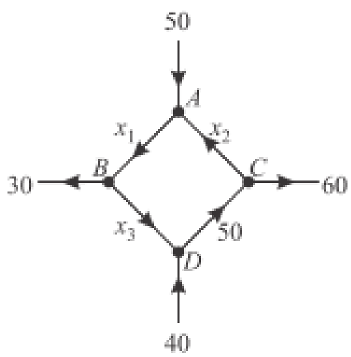
Question:

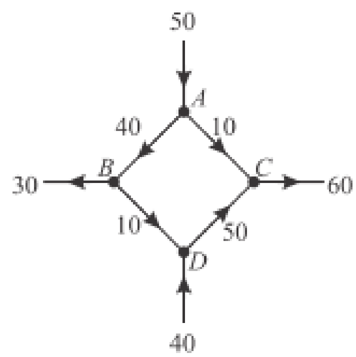
1. The accompanying figure shows a network in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.



Solution:

There are four nodes, which we denote by A, B, C, and D (see the figure on the left).
We determine the unknown flow rates x_1, x_2, and x_3 assuming the counterclockwise direction (if any of these quantities are found to be negative then the flow direction along the corresponding branch will be reversed).





Network node	Flow In		Flow Out
A	$x_2 + 50$	=	x_1
B	x_1	=	$x_3 + 30$
C	50	=	$x_2 + 60$
D	$x_3 + 40$	=	50

This system can be rearranged as follows

$$\begin{array}{rcrcrcr} -x_1 & + & x_2 & & & = & -50 \\ x_1 & & & - & x_3 & = & 30 \\ & - & x_2 & & & = & 10 \\ & & & & x_3 & = & 10 \end{array}$$

By inspection, this system has a unique solution $x_1 = 40$, $x_2 = -10$, $x_3 = 10$. This yields the flow rates and directions shown in the figure on the right.

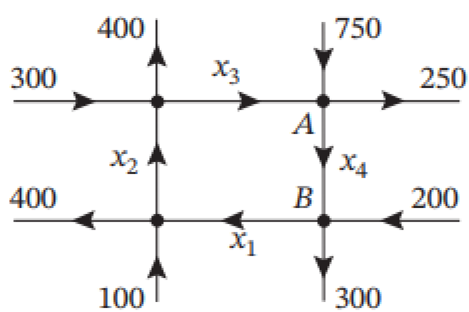
Question:

3. The accompanying figure shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour.

a. Set up a linear system whose solution provides the unknown flow rates.

b. Solve the system for the unknown flow rates.

c. If the flow along the road from A to B must be reduced for construction, what is the minimum flow that is required to keep traffic flowing on all roads?



Solution:

(a) There are four nodes – each of them corresponds to an equation.

Network node	Flow In		Flow Out
top left	$x_2 + 300$	$=$	$x_3 + 400$
top right (A)	$x_3 + 750$	$=$	$x_4 + 250$
bottom left	$x_1 + 100$	$=$	$x_2 + 400$
bottom right (B)	$x_4 + 200$	$=$	$x_1 + 300$

This system can be rearranged as follows

$$\begin{array}{rcrcrcrcr} & & x_2 & - & x_3 & & & = & 100 \\ & & & & x_3 & - & x_4 & = & -500 \\ x_1 & - & x_2 & & & & & = & 300 \\ -x_1 & & & & & + & x_4 & = & 100 \end{array}$$

(b) The augmented matrix of the linear system obtained in part (a) $\begin{bmatrix} 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 1 & -1 & 0 & 0 & 300 \\ -1 & 0 & 0 & 1 & 100 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & -1 & -400 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we assign x_4 the arbitrary value s, the general solution is given by the formulas

$x_1 = -100 + s$, $x_2 = -400 + s$, $x_3 = -500 + s$, $x_4 = s$

(c) In order for all x_i values to remain positive, we must have $s > 500$. Therefore, to keep the traffic flowing on all roads, the flow from A to B must exceed 500 vehicles per hour.