Row Space, Column Space, and Null Space

Matrix Spaces

Definition 1

For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ &\vdots \qquad\qquad \vdots \\ \mathbf{r}_m &= [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{aligned}$$

in R^n formed from the rows of A are called the ***row vectors*** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from the columns of A are called the ***column vectors*** of A.

EXAMPLE 1 | Row and Column Vectors of a 2×3 Matrix

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of A are

$$\mathbf{r}_1 = [2 \quad 1 \quad 0] \quad \text{and} \quad \mathbf{r}_2 = [3 \quad -1 \quad 4]$$

and the column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Row Space, Column Space

Definition 2

If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is denoted by row(A) and is called the ***row space*** of A, and the subspace of R^m spanned by the column vectors of A is denoted by col(A) and is called the ***column space*** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is denoted by null(A) and is called the ***null space*** of A.

Theorem 4.8.1

A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

EXAMPLE 2 | A Vector $\mathbf{b}$ in the Column Space of A

Let $A\mathbf{x} = \mathbf{b}$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\mathbf{b}$ is in the column space of A by expressing it as a linear combination of the column vectors of A.

Solution Solving the system by Gaussian elimination yields (verify)

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

It follows from this and Formula (2) that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

The Relationship Between $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

In this subsection we will explore the relationship between the solutions of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ and the solutions (if any) of the nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ with the same coefficient matrix. These are called ***corresponding linear systems.*** By way of example, we will consider the following linear systems that we first discussed in Examples 5 and 6 of Section 1.2 and then again in Example 3 of Section 4.6.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

In Section 1.2 we found the general solutions of these systems to be

homogeneous $\longrightarrow x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$

nonhomogeneous $\longrightarrow x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$

which we can express in column-vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{bmatrix}$$

By splitting the entries on the right apart and collecting terms with like parameters we can rewrite these general solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{3}$$

Homogeneous Case

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} \tag{4}$$

Nonhomogeneous Case

In Example 3 of Section 4.6 we observed that the three vectors on the right side of (3) are linearly independent and therefore form a basis for the solution space of the homogeneous system. Thus, as illustrated in (5), the general solution $\mathbf{x}$ of the nonhomogeneous system can be divided into two parts, a basis $\mathbf{x}_h$ for the null space of the homogeneous system and a term $\mathbf{x}_0$ that is a solution of the nonhomogeneous system (in this case, the solution resulting from setting the parameters to zero).

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}}_{\mathbf{x}_0} + \underbrace{r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}_h} \tag{5}$$

This example illustrates the following general theorem.

Theorem 4.8.2

If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{6}$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

The vector $\mathbf{x}_0$ in Formula (6) is called a ***particular solution of*** $A\mathbf{x} = \mathbf{b}$, and the remaining part of the formula is called the ***general solution of*** $A\mathbf{x} = \mathbf{0}$. With this terminology Theorem 4.8.2 can be rephrased as:

> *The general solution of a consistent linear system can be expressed as the sum of a particular solution of that system and the general solution of the corresponding homogeneous system.*

Geometrically, the solution set of $A\mathbf{x} = \mathbf{b}$ can be viewed as the translation by $\mathbf{x}_0$ of the solution space of $A\mathbf{x} = \mathbf{0}$ (**Figure 4.8.1**).

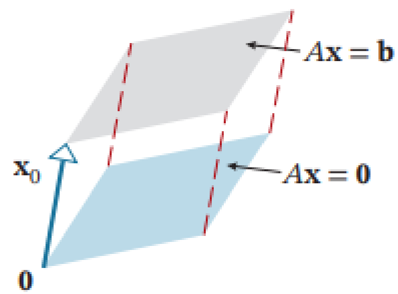

FIGURE 4.8.1 The solution space of $A\mathbf{x} = \mathbf{b}$ is a translation of the solution space of $A\mathbf{x} = \mathbf{0}$.

Bases for Row Spaces, Column Spaces, and Null Spaces

Theorem 4.8.3

(*a*) *Row equivalent matrices have the same row space.*

(*b*) *Row equivalent matrices have the same null space.*

Theorem 4.8.4

If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

EXAMPLE 3 | Bases for the Row and Column Spaces of a Matrix in Row Echelon Form

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution Since the matrix R is in row echelon form, it follows from Theorem 4.8.4 that the vectors

$$\begin{aligned} \mathbf{r}_1 &= [1 \quad -2 \quad 5 \quad 0 \quad 3] \\ \mathbf{r}_2 &= [0 \quad 1 \quad 3 \quad 0 \quad 0] \\ \mathbf{r}_3 &= [0 \quad 0 \quad 0 \quad 1 \quad 0] \end{aligned}$$

form a basis for the row space of R, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

EXAMPLE 4 | Basis for a Row Space by Row Reduction

Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of A by finding a basis for the row space of any row echelon form of A. Reducing A to row echelon form, we obtain (verify)

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 4.8.4, the nonzero row vectors of R form a basis for the row space of R and hence form a basis for the row space of A. These basis vectors are

$$\begin{aligned} \mathbf{r}_1 &= [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4] \\ \mathbf{r}_2 &= [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6] \\ \mathbf{r}_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5] \end{aligned}$$

Theorem 4.8.5

If A and B are row equivalent matrices, then:

(*a*) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

(*b*) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

EXAMPLE 5 | Basis from the Columns of A

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

that consists of column vectors of A.

Solution We observed in Example 4 that the matrix

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a row echelon form of A. Keeping in mind that A and R can have different column spaces, we cannot find a basis for the column space of A directly from the column vectors of R. However, it follows from Theorem 4.8.5(*b*) that if we can find a set of column vectors of R that forms a basis for the column space of R, then the *corresponding* column vectors of A will form a basis for the column space of A.

Since the first, third, and fifth columns of R contain the leading 1's of the row vectors, the vectors

$$\mathbf{c}_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R. Thus, the corresponding column vectors of A, which are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

EXAMPLE 6 | Basis from the Rows of A

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

consisting entirely of row vectors from A.

Solution We will transpose A, thereby converting the row space of A into the column space of A^T; then we will use the method of Example 5 to find a basis for the column space of A^T; and then we will transpose again to convert column vectors back to row vectors.

Transposing A yields

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

and then reducing this matrix to row echelon form we obtain

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second, and fourth columns contain the leading 1's, so the corresponding column vectors in A^T form a basis for the column space of A^T; these are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

Transposing again and adjusting the notation appropriately yields the basis vectors

$$\mathbf{r}_1 = [1 \quad -2 \quad 0 \quad 0 \quad 3], \qquad \mathbf{r}_2 = [2 \quad -5 \quad -3 \quad -2 \quad 6],$$
$$\mathbf{r}_4 = [2 \quad 6 \quad 18 \quad 8 \quad 6]$$

for the row space of A.

EXAMPLE 8 | Basis and Linear Combinations

(*a*) Find a subset of the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6),$$
$$\mathbf{v}_3 = (0, 1, 3, 0), \quad \mathbf{v}_4 = (2, -1, 4, -7), \quad \mathbf{v}_5 = (5, -8, 1, 2)$$

that forms a basis for the subspace of R^4 spanned by these vectors.

(*b*) Express each vector not in the basis as a linear combination of the basis vectors.

Solution (*a*) We begin by constructing a matrix that has $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ as its column vectors:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{array} \end{array} \tag{9}$$

The first part of our problem can be solved by finding a basis for the column space of this matrix. Reducing the matrix to *reduced* row echelon form and denoting the column vectors of the resulting matrix by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$, and $\mathbf{w}_5$ yields

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 & \mathbf{w}_5 \end{array} \end{array} \tag{10}$$

The leading 1's occur in columns 1, 2, and 4, so by Theorem 4.8.4,

$$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$$

is a basis for the column space of (6), and consequently,

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$$

is a basis for the column space of (9).

Solution (*b*) We will start by expressing $\mathbf{w}_3$ and $\mathbf{w}_5$ as linear combinations of the basis vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_4$. The simplest way of doing this is to express $\mathbf{w}_3$ and $\mathbf{w}_5$ in terms of basis vectors with numerically smaller subscripts. Accordingly, we will express $\mathbf{w}_3$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$, and we will express $\mathbf{w}_5$ as a linear combination of the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_4$. By inspection of (10), these linear combinations are

$$\begin{aligned} \mathbf{w}_3 &= 2\mathbf{w}_1 - \mathbf{w}_2 \\ \mathbf{w}_5 &= \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_4 \end{aligned}$$

We call these the ***dependency equations***. The corresponding relationships in (9) are

$$\begin{aligned} \mathbf{v}_3 &= 2\mathbf{v}_1 - \mathbf{v}_2 \\ \mathbf{v}_5 &= \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4 \end{aligned}$$