CHAPTER 2: DETERMINANTS

2.1 Determinants by Cofactor Expansion

1.

$$M_{11} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 29 \qquad C_{11} = (-1)^{1+1} M_{11} = M_{11} = 29$$

$$M_{12} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -3 & 4 \end{vmatrix} = 21 \qquad C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -21$$

$$M_{13} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix} = 27 \qquad C_{13} = (-1)^{1+3} M_{13} = M_{13} = 27$$

$$M_{21} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -11 \qquad C_{21} = (-1)^{2+1} M_{21} = -M_{21} = 11$$

$$M_{22} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 13 \qquad C_{22} = (-1)^{2+2} M_{22} = M_{22} = 13$$

$$M_{23} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} = -5 \qquad C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 5$$

$$M_{31} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 7 & -1 \end{vmatrix} = -19 \qquad C_{31} = (-1)^{3+1} M_{31} = M_{31} = -19$$

$$M_{32} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 6 & -1 \end{vmatrix} = -19 \qquad C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 19$$

$$M_{33} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 6 & 7 \end{vmatrix} = 19 \qquad C_{33} = (-1)^{3+3} M_{33} = M_{33} = 19$$

3. **(a)** $M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 0\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 0 - 0 + 3(0) = 0$

$C_{13} = (-1)^{1+3} M_{13} = M_{13} = 0$

(b) $M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 6\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 4(-12) + 1(-48) + 6(0) = -96$

$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 96$

(c) $M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix} = -4\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 6 \\ 4 & 2 \end{vmatrix} - 14\begin{vmatrix} 4 & 1 \\ 4 & 3 \end{vmatrix}$ ← cofactor expansion along the second row

$= -4(-16) + 0 - 14(8) = -48$

$C_{22} = (-1)^{2+2} M_{22} = M_{22} = -48$

(d) $M_{21} = \begin{vmatrix} -1 & 1 & 6 \\ 1 & 0 & 14 \\ 1 & 3 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 6 \\ 1 & 2 \end{vmatrix} - 14\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix}$ ← cofactor expansion along the second row

$= -1(-16) + 0 - 14(-4) = 72$

$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -72$

5. $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = (3)(4) - (5)(-2) = 12 + 10 = 22 \neq 0$. Inverse: $\frac{1}{22}\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{11} & \frac{-5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{bmatrix}$

7. $\begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = (-5)(-2) - (7)(-7) = 10 + 49 = 59 \neq 0$. Inverse: $\frac{1}{59}\begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} \frac{-2}{59} & \frac{-7}{59} \\ \frac{7}{59} & \frac{-5}{59} \end{bmatrix}$

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = \begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = (a-3)(a-2) - 5(-3) = a^2 - 5a + 6 + 15 = a^2 - 5a + 21$

11. $$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}\begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix} = [-20-7+72]-[20+84+6] = -65$$

13. $$\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}\begin{matrix} 3 & 0 \\ 2 & -1 \\ 1 & 9 \end{matrix} = [12+0+0]-[0+135+0] = -123$$

15. $$\det(A) = \begin{vmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{vmatrix} = (\lambda-2)(\lambda+4)-(1)(-5) = \lambda^2+2\lambda-3 = (\lambda+3)(\lambda-1)$$

The determinant is zero if $\lambda = -3$ or $\lambda = 1$.

17. $$\det(A) = \begin{vmatrix} \lambda-1 & 0 \\ 2 & \lambda+1 \end{vmatrix} = (\lambda-1)(\lambda+1)$$

The determinant is zero if $\lambda = 1$ or $\lambda = -1$.

19. **(a)** $$3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0 + 0 = 3(-41) = -123$$

(b) $$3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} = 3(-41)-2(0)+1(0) = -123$$

(c) $$-2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} = -2(0)-1(-12)-5(27) = -123$$

(d) $$-0 + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} = -1(-12)-9(15) = -123$$

(e) $$1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = 1(0)-9(15)-4(-3) = -123$$

(f) $$0 - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -5(27)-4(-3) = -123$$

21. Calculate the determinant by a cofactor expansion along the second column:

$$-0 + 5\begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} - 0 = 5(-8) = -40$$

23. Calculate the determinant by a cofactor expansion along the first column:

$$1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} + 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} = 1(0)-1(0)+1(0) = 0$$

25. Calculate the determinant by a cofactor expansion along the third column:

$$\det(A) = 0 - 0 + (-3)\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix}$$

Calculate the determinants in the third and fourth terms by a cofactor expansion along the first row:

$$\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 2 & 10 \end{vmatrix} = 3(24) - 3(8) + 5(16) = 128$$

$$\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = 3(2) - 3(8) + 5(-6) = -48$$

Therefore $\det(A) = 0 - 0 - 3(128) - 3(-48) = -240$.

27. By Theorem 2.1.2, determinant of a diagonal matrix is the product of the entries on the main diagonal: $\det(A) = (1)(-1)(1) = -1$.

29. By Theorem 2.1.2, determinant of a lower triangular matrix is the product of the entries on the main diagonal: $\det(A) = (0)(2)(3)(8) = 0$.

31. By Theorem 2.1.2, determinant of an upper triangular matrix is the product of the entries on the main diagonal: $\det(A) = (1)(1)(2)(3) = 6$.

33. **(a)** $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = (\sin\theta)(\sin\theta) - (\cos\theta)(-\cos\theta) = \sin^2\theta + \cos^2\theta = 1$

(b) Calculate the determinant by a cofactor expansion along the third column:

$0 - 0 + 1\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = 0 - 0 + (1)(1) = 1$ (we used the result of part (a))

35. The minor M_{11} in both determinants is $\begin{vmatrix} 1 & f \\ 0 & 1 \end{vmatrix} = 1$. Expanding both determinants along the first row yields $d_1 + \lambda = d_2$.

43. Calculate the determinant by a cofactor expansion along the first column:

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} x_2 & x_2^2 \\ x_3 & x_3^2 \end{vmatrix} + \begin{vmatrix} x_1 & x_1^2 \\ x_3 & x_3^2 \end{vmatrix} + \begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix}$$

$$= (x_2 x_3^2 - x_3 x_2^2) - (x_1 x_3^2 - x_3 x_1^2) + (x_1 x_2^2 - x_2 x_1^2) = \left[x_3^2(x_2 - x_1) - x_3\left(x_2^2 - x_1^2\right)\right] + x_1 x_2^2 - x_2 x_1^2.$$

Factor out $(x_2 - x_1)$ to get $(x_2 - x_1)\left[x_3^2 - x_2 x_3 - x_1 x_3 + x_1 x_2\right] = (x_2 - x_1)\left[x_3^2 - (x_2 + x_1)x_3 + x_1 x_2\right]$.

Since $x_3^2 - (x_2 + x_1)x_3 + x_1 x_2 = (x_3 - x_1)(x_3 - x_2)$, the determinant is $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

True-False Exercises

(a) False. The determinant is $ad-bc$.

(b) False. E.g., $\det(I_2)=\det(I_3)=1$.

(c) True. If $i+j$ is even then $(-1)^{i+j}=1$ therefore $C_{ij}=(-1)^{i+j}M_{ij}=M_{ij}$.

(d) True. Let $A=\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

Then $C_{12}=(-1)^{1+2}\begin{vmatrix} b & e \\ c & f \end{vmatrix}=-(bf-ec)$ and $C_{21}=(-1)^{2+1}\begin{vmatrix} b & c \\ e & f \end{vmatrix}=-(bf-ce)$ therefore $C_{12}=C_{21}$. In the same way, one can show $C_{13}=C_{31}$ and $C_{23}=C_{32}$.

(e) True. This follows from Theorem 2.1.1.

(f) True. In formulas (7) and (8), each cofactor C_{ij} is zero.

(g) False. The determinant of a lower triangular matrix is the *product* of the entries along the main diagonal.

(h) False. E.g. $\det(2I_2)=4\neq 2=2\det(I_2)$.

(i) False. E.g., $\det(I_2+I_2)=4\neq 2=\det(I_2)+\det(I_2)$.

(j) True. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right)=\begin{vmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{vmatrix}=(a^2+bc)(bc+d^2)-(ab+bd)(ac+cd)$

$=a^2bc+a^2d^2+b^2c^2+bcd^2-a^2bc-abcd-abcd-bcd^2=a^2d^2+b^2c^2-2abcd$.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}^2=(ad-bc)^2=a^2d^2-2adbc+b^2c^2$ therefore $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right)=\left(\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)^2$.

2.2 Evaluating Determinants by Row Reduction

1. $\det(A)=\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix}=(-2)(4)-(3)(1)=-11$; $\det(A^T)=\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix}=(-2)(4)-(1)(3)=-11$

3. $\det(A)=\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{vmatrix}=[24-20-9]-[30-24-6]=-5$;

$\det(A^T)=\begin{vmatrix} 2 & 1 & 5 \\ -1 & 2 & -3 \\ 3 & 4 & 6 \end{vmatrix}=[24-9-20]-[30-24-6]=-5$ (we used the arrow technique)

5. The third row of I_4 was multiplied by -5. By Theorem 2.2.4, the determinant equals -5.

7. The second and the third rows of I_4 were interchanged. By Theorem 2.2.4, the determinant equals -1.

9.

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix}$$

← A common factor of 3 from the first row was taken through the determinant sign.

$$= 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix}$$

← 2 times the first row was added to the second row.

$$= 3(-1)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 4 \end{vmatrix}$$

← The second and third rows were interchanged.

$$= (3)(-1)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{vmatrix}$$

← -3 times the second row was added to the third row.

$$= (3)(-1)(-11)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

← A common factor of -11 from the last row was taken through the determinant sign.

$$= (3)(-1)(-11)(1) = 33$$

Another way to evaluate the determinant would be to use cofactor expansion along the first column after the second step above:

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = 3\left[1\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} - 0 + 0\right] = 3\left[(1)(11)\right] = 33.$$

11.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

← The first and second rows were interchanged.

$$= (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

← -2 times the first row was added to the second row.

$$=(-1)\begin{vmatrix}1&0&1&1\\0&1&1&-1\\0&0&-1&2\\0&1&2&3\end{vmatrix}$$ ← -2 times the second row was added to the third row.

$$=(-1)\begin{vmatrix}1&0&1&1\\0&1&1&-1\\0&0&-1&2\\0&0&1&4\end{vmatrix}$$ ← -1 times the second row was added to the fourth row.

$$=(-1)(-1)\begin{vmatrix}1&0&1&1\\0&1&1&-1\\0&0&1&-2\\0&0&1&4\end{vmatrix}$$ ← A common factor of -1 from the third row was taken through the determinant sign.

$$=(-1)(-1)\begin{vmatrix}1&0&1&1\\0&1&1&-1\\0&0&1&-2\\0&0&0&6\end{vmatrix}$$ ← -1 times the third row was added to the fourth row.

$$=(-1)(-1)(6)\begin{vmatrix}1&0&1&1\\0&1&1&-1\\0&0&1&-2\\0&0&0&1\end{vmatrix}$$ ← A common factor of 6 from the third row was taken through the determinant sign.

$$=(-1)(-1)(6)(1)=6$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the fourth step above:

$$\begin{vmatrix}2&1&3&1\\1&0&1&1\\0&2&1&0\\0&1&2&3\end{vmatrix}=(-1)\begin{vmatrix}1&0&1&1\\0&1&1&-1\\0&0&-1&2\\0&0&1&4\end{vmatrix}=(-1)(1)\begin{vmatrix}1&1&-1\\0&-1&2\\0&1&4\end{vmatrix}=(-1)(1)(1)\begin{vmatrix}-1&2\\1&4\end{vmatrix}$$

$=(-1)(1)(1)(-6)=6$.

13. $$\begin{vmatrix}1&3&1&5&3\\-2&-7&0&-4&2\\0&0&1&0&1\\0&0&2&1&1\\0&0&0&1&1\end{vmatrix}$$

$$=\begin{vmatrix}1 & 3 & 1 & 5 & 3\\0 & -1 & 2 & 6 & 8\\0 & 0 & 1 & 0 & 1\\0 & 0 & 2 & 1 & 1\\0 & 0 & 0 & 1 & 1\end{vmatrix}$$

← 2 times the first row was added to the second row.

$$=(-1)\begin{vmatrix}1 & 3 & 1 & 5 & 3\\0 & 1 & -2 & -6 & -8\\0 & 0 & 1 & 0 & 1\\0 & 0 & 2 & 1 & 1\\0 & 0 & 0 & 1 & 1\end{vmatrix}$$

← A common factor of -1 from the second row
was taken through the determinant sign.

$$=(-1)\begin{vmatrix}1 & 3 & 1 & 5 & 3\\0 & 1 & -2 & -6 & -8\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & -1\\0 & 0 & 0 & 1 & 1\end{vmatrix}$$

← -2 times the third row was added to the fourth row.

$$=(-1)\begin{vmatrix}1 & 3 & 1 & 5 & 3\\0 & 1 & -2 & -6 & -8\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & -1\\0 & 0 & 0 & 0 & 2\end{vmatrix}$$

← -1 times the fourth row was added to the fifth row.

$$=(-1)(2)\begin{vmatrix}1 & 3 & 1 & 5 & 3\\0 & 1 & -2 & -6 & -8\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & -1\\0 & 0 & 0 & 0 & 1\end{vmatrix}$$

← A common factor of 2 from the fifth row was taken through the determinant sign.

$$=(-1)(2)(1)=-2$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the third step above:

$$\begin{vmatrix}1 & 3 & 1 & 5 & 3\\-2 & -7 & 0 & -4 & 2\\0 & 0 & 1 & 0 & 1\\0 & 0 & 2 & 1 & 1\\0 & 0 & 0 & 1 & 1\end{vmatrix}=(-1)\begin{vmatrix}1 & 3 & 1 & 5 & 3\\0 & 1 & -2 & -6 & -8\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & -1\\0 & 0 & 0 & 1 & 1\end{vmatrix}=(-1)(1)\begin{vmatrix}1 & -2 & -6 & -8\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\\0 & 0 & 1 & 1\end{vmatrix}$$

$$=(-1)(1)(1)\begin{vmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 1 & 1\end{vmatrix}=(-1)(1)(1)(1)\begin{vmatrix}1 & -1\\1 & 1\end{vmatrix}=(-1)(1)(1)(1)(2)=-2 \ .$$

15.

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = (-1)\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$$

The first and third rows were interchanged.

$$= (-1)(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

The second and third rows were interchanged.

$$= (-1)(-1)(-6) = -6$$

17.

$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

$$= 3(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix}$$

A common factor of -1 from the second row was taken through the determinant sign.

$$= 3(-1)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

A common factor of 4 from the third row was taken through the determinant sign.

$$= 3(-1)(4)(-6) = 72$$

19.

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

-1 times the third row was added to the first row.

$$= -6$$

21.

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

A common factor of -3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

4 times the second row was added to the last row.

$$=(-3)(-6)=18$$

23.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$-a$ times the first row was added to the second row.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$-a^2$ times the first row was added to the third row.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{vmatrix}$$

$-(b+a)$ times the second row was added to the third row.

$$=(1)(b-a)(c-a)(c+a-b-a)$$

$$=(b-a)(c-a)(c-b)$$

25.

$$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & b_1+c_1 \\ a_2 & b_2 & b_2+c_2 \\ a_3 & b_3 & b_3+c_3 \end{vmatrix}$$

-1 times the first column was added to the third column.

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

-1 times the second column was added to the third column.

27.

$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+b_1 & -2b_1 & c_1 \\ a_2+b_2 & -2b_2 & c_2 \\ a_3+b_3 & -2b_3 & c_3 \end{vmatrix}$$

-1 times the first column was added to the second column.

$$=-2 \begin{vmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{vmatrix}$$

A common factor of -2 from the second column was taken through the determinant sign.

$$= -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \longleftarrow \text{-1 times the second column was added to the first column.}$$

29. The second column vector is a scalar multiple of the fourth. By Theorem 2.2.5, the determinant is 0.

31. $\det(M) = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ -1 & 3 & 2 \end{vmatrix}\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 8 & -4 \end{vmatrix} = \left(0-0+2\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}\right)\left(0-0+(-4)\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}\right) = (2)(-12) = -24$

33. In order to reverse the order of rows in 2×2 and 3×3 matrix, the first and the last rows can be interchanged, so $\det(B) = -\det(A)$.

For 4×4 and 5×5 matrices, two such interchanges are needed: the first and last rows can be swapped, then the second and the penultimate one can follow.

Thus, $\det(B) = (-1)(-1)\det(A) = \det(A)$ in this case.

Generally, to rows in an $n\times n$ matrix can be reversed by

- interchanging row 1 with row n,
- interchanging row 2 with row $n-1$,
- $\vdots$
- interchanging row $\lfloor n/2 \rfloor$ with row $n - \lfloor n/2 \rfloor$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x (also known as the "floor" of x).

We conclude that $\det(B) = (-1)^{\lfloor n/2 \rfloor}\det(A)$.

True-False Exercises

(a) True. $\det(B) = (-1)(-1)\det(A) = \det(A)$.

(b) True. $\det(B) = (4)(\frac{3}{4})\det(A) = 3\det(A)$.

(c) False. $\det(B) = \det(A)$.

(d) False. $\det(B) = n(n-1)\cdots 3\cdot 2\cdot 1\cdot \det(A) = (n!)\det(A)$.

(e) True. This follows from Theorem 2.2.5.

(f) True. Let B be obtained from A by adding the second row to the fourth row, so $\det(A) = \det(B)$. Since the fourth row and the sixth row of B are identical, by Theorem 2.2.5 $\det(B) = 0$.

2.3 Properties of Determinants; Cramer's Rule

1. $\det(2A) = \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} = (-2)(8) - (4)(6) = -40$

$(2)^2 \det(A) = 4\begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = 4\left((-1)(4) - (2)(3)\right) = (4)(-10) = -40$

3. We are using the arrow technique to evaluate both determinants.

$\det(-2A) = \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -10 \end{vmatrix} = (-160 + 8 - 288) - (-48 - 64 + 120) = -448$

$(-2)^3 \det(A) = -8\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} = (-8)\left((20 - 1 + 36) - (6 + 8 - 15)\right) = (-8)(56) = -448$

5. We are using the arrow technique to evaluate the determinants in this problem.

$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix} = (18 - 170 + 0) - (80 + 0 - 62) = -170$;

$\det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix} = (-22 - 120 + 510) - (660 - 20 - 102) = -170$;

$\det(A+B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix} = (45 + 0 + 0) - (75 + 0 + 0) = -30$;

$\det(A) = (16 + 0 + 0) - (0 + 0 + 6) = 10$;

$\det(B) = (1 - 10 + 0) - (15 + 0 - 7) = -17$;

$\det(A+B) \neq \det(A) + \det(B)$

7. $\det(A) = (-6 + 0 - 20) - (-10 + 0 - 15) = -1 \neq 0$ therefore A is invertible by Theorem 2.3.3

9. $\det(A) = (2)(1)(2) = 4 \neq 0$ therefore A is invertible by Theorem 2.3.3

11. $\det(A) = (24 - 24 - 16) - (24 - 16 - 24) = 0$ therefore A is not invertible by Theorem 2.3.3

13. $\det(A) = (2)(1)(6) = 12 \neq 0$ therefore A is invertible by Theorem 2.3.3

15. $\det(A)=(k-3)(k-2)-(-2)(-2)=k^2-5k+2=\left(k-\frac{5-\sqrt{17}}{2}\right)\left(k-\frac{5+\sqrt{17}}{2}\right)$. By Theorem 2.3.3, A is invertible if $k\neq\frac{5-\sqrt{17}}{2}$ and $k\neq\frac{5+\sqrt{17}}{2}$.

17. $\det(A)=(2+12k+36)-(4k+18+12)=8+8k=8(1+k)$.
By Theorem 2.3.3, A is invertible if $k\neq-1$.

19. $\det(A)=(-6+0-20)-(-10+0-15)=-1\neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}-1&0\\4&3\end{vmatrix}=-3\quad C_{12}=-\begin{vmatrix}-1&0\\2&3\end{vmatrix}=3\quad C_{13}=\begin{vmatrix}-1&-1\\2&4\end{vmatrix}=-2$$
$$C_{21}=-\begin{vmatrix}5&5\\4&3\end{vmatrix}=5\quad C_{22}=\begin{vmatrix}2&5\\2&3\end{vmatrix}=-4\quad C_{23}=-\begin{vmatrix}2&5\\2&4\end{vmatrix}=2$$
$$C_{31}=\begin{vmatrix}5&5\\-1&0\end{vmatrix}=5\quad C_{32}=-\begin{vmatrix}2&5\\-1&0\end{vmatrix}=-5\quad C_{33}=\begin{vmatrix}2&5\\-1&-1\end{vmatrix}=3$$

The matrix of cofactors is $\begin{bmatrix}-3&3&-2\\5&-4&2\\5&-5&3\end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix}-3&5&5\\3&-4&-5\\-2&2&3\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{-1}\begin{bmatrix}-3&5&5\\3&-4&-5\\-2&2&3\end{bmatrix}=\begin{bmatrix}3&-5&-5\\-3&4&5\\2&-2&-3\end{bmatrix}$.

21. $\det(A)=(2)(1)(2)=4\neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}1&-3\\0&2\end{vmatrix}=2\quad C_{12}=-\begin{vmatrix}0&-3\\0&2\end{vmatrix}=0\quad C_{13}=\begin{vmatrix}0&1\\0&0\end{vmatrix}=0$$
$$C_{21}=-\begin{vmatrix}-3&5\\0&2\end{vmatrix}=6\quad C_{22}=\begin{vmatrix}2&5\\0&2\end{vmatrix}=4\quad C_{23}=-\begin{vmatrix}2&-3\\0&0\end{vmatrix}=0$$
$$C_{31}=\begin{vmatrix}-3&5\\1&-3\end{vmatrix}=4\quad C_{32}=-\begin{vmatrix}2&5\\0&-3\end{vmatrix}=6\quad C_{33}=\begin{vmatrix}2&-3\\0&1\end{vmatrix}=2$$

The matrix of cofactors is $\begin{bmatrix}2&0&0\\6&4&0\\4&6&2\end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix}2&6&4\\0&4&6\\0&0&2\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{4}\begin{bmatrix}2&6&4\\0&4&6\\0&0&2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{3}{2}&1\\0&1&\frac{3}{2}\\0&0&\frac{1}{2}\end{bmatrix}$.

23.

$$\begin{vmatrix}1&3&1&1\\2&5&2&2\\1&3&8&9\\1&3&2&2\end{vmatrix}=\begin{vmatrix}1&3&1&1\\0&-1&0&0\\0&0&7&8\\0&0&1&1\end{vmatrix}$$

-2 times the first row was added to the second row; -1 times the first row was added to the third and fourth rows.

$$=-\begin{vmatrix}1&3&1&1\\0&-1&0&0\\0&0&1&1\\0&0&7&8\end{vmatrix}$$

The third row and the fourth row were interchanged.

$$=-\begin{vmatrix}1&3&1&1\\0&-1&0&0\\0&0&1&1\\0&0&0&1\end{vmatrix}$$

-7 times the third row was added to the fourth row

$$=-(1)(-1)(1)(1)=1$$

The determinant of A is nonzero therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}5&2&2\\3&8&9\\3&2&2\end{vmatrix}=(80+54+12)-(48+90+12)=-4$$

$$C_{12}=-\begin{vmatrix}2&2&2\\1&8&9\\1&2&2\end{vmatrix}=-\left[(32+18+4)-(16+36+4)\right]=2$$

$$C_{13}=\begin{vmatrix}2&5&2\\1&3&9\\1&3&2\end{vmatrix}=(12+45+6)-(6+54+10)=-7$$

$$C_{14}=-\begin{vmatrix}2&5&2\\1&3&8\\1&3&2\end{vmatrix}=-\left[(12+40+6)-(6+48+10)\right]=6$$

$$C_{21}=-\begin{vmatrix}3&1&1\\3&8&9\\3&2&2\end{vmatrix}=-\left[(48+27+6)-(24+54+6)\right]=3$$

$$C_{22}=\begin{vmatrix}1&1&1\\1&8&9\\1&2&2\end{vmatrix}=(16+9+2)-(8+18+2)=-1$$

$$C_{23}=-\begin{vmatrix}1&3&1\\1&3&9\\1&3&2\end{vmatrix}=-\left[(6+27+3)-(3+27+6)\right]=0$$

$$C_{24} = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = (6+24+3)-(3+24+6) = 0$$

$$C_{31} = \begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 2 & 2 \end{vmatrix} = (12+6+10)-(6+12+10) = 0$$

$$C_{32} = -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{vmatrix} = -\left[(4+2+4)-(2+4+4)\right] = 0$$

$$C_{33} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = (10+6+6)-(5+6+12) = -1$$

$$C_{34} = -\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -\left[(10+6+6)-(5+6+12)\right] = 1$$

$$C_{41} = -\begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 8 & 9 \end{vmatrix} = -\left[(54+6+40)-(6+48+45)\right] = -1$$

$$C_{42} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 8 & 9 \end{vmatrix} = (18+2+16)-(2+16+18) = 0$$

$$C_{43} = -\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 9 \end{vmatrix} = -\left[(45+6+6)-(5+6+54)\right] = 8$$

$$C_{44} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 8 \end{vmatrix} = (40+6+6)-(5+6+48) = -7$$

The matrix of cofactors is $\begin{bmatrix} -4 & 2 & -7 & 6 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 8 & -7 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{1}\begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

25. $\det(A)=\begin{vmatrix}4&5&0\\11&1&2\\1&5&2\end{vmatrix}=(8+10+0)-(0+40+110)=-132$,

$\det(A_1)=\begin{vmatrix}2&5&0\\3&1&2\\1&5&2\end{vmatrix}=(4+10+0)-(0+20+30)=-36$,

$\det(A_2)=\begin{vmatrix}4&2&0\\11&3&2\\1&1&2\end{vmatrix}=(24+4+0)-(0+8+44)=-24$,

$\det(A_3)=\begin{vmatrix}4&5&2\\11&1&3\\1&5&1\end{vmatrix}=(4+15+110)-(2+60+55)=12$;

$x=\frac{\det(A_1)}{\det(A)}=\frac{-36}{-132}=\frac{3}{11}$, $\quad y=\frac{\det(A_2)}{\det(A)}=\frac{-24}{-132}=\frac{2}{11}$, $\quad z=\frac{\det(A_3)}{\det(A)}=\frac{12}{-132}=-\frac{1}{11}$.

27. $\det(A)=\begin{vmatrix}1&-3&1\\2&-1&0\\4&0&-3\end{vmatrix}=(3+0+0)-(-4+0+18)=-11$,

$\det(A_1)=\begin{vmatrix}4&-3&1\\-2&-1&0\\0&0&-3\end{vmatrix}=-3\begin{vmatrix}4&-3\\-2&-1\end{vmatrix}=(-3)(-4-6)=30$,

$\det(A_2)=\begin{vmatrix}1&4&1\\2&-2&0\\4&0&-3\end{vmatrix}=(6+0+0)-(-8+0-24)=38$,

$\det(A_3)=\begin{vmatrix}1&-3&4\\2&-1&-2\\4&0&0\end{vmatrix}=4\begin{vmatrix}-3&4\\-1&-2\end{vmatrix}=(4)(6+4)=40$;

$x_1=\frac{\det(A_1)}{\det(A)}=\frac{30}{-11}=-\frac{30}{11}$, $\quad x_2=\frac{\det(A_2)}{\det(A)}=\frac{38}{-11}=-\frac{38}{11}$, $\quad x_3=\frac{\det(A_3)}{\det(A)}=\frac{40}{-11}=-\frac{40}{11}$.

29. $\det(A)=0$ therefore Cramer's rule does not apply.

31. $\det(A)=\begin{vmatrix}4&1&1&1\\3&7&-1&1\\7&3&-5&8\\1&1&1&2\end{vmatrix}=-424$; $\det(A_2)=\begin{vmatrix}4&6&1&1\\3&1&-1&1\\7&-3&-5&8\\1&3&1&2\end{vmatrix}=0$; $\quad y=\frac{\det(A_2)}{\det(A)}=\frac{0}{-424}=0$

33. **(a)** $\det(3A)=3^3\det(A)=(27)(-7)=-189$ (using Formula (1))

(b) $\det(A^{-1})=\frac{1}{\det(A)}=\frac{1}{-7}=-\frac{1}{7}$ (using Theorem 2.3.5)

(c) $\det\left(2A^{-1}\right)=2^3\det\left(A^{-1}\right)=\frac{8}{\det(A)}=\frac{8}{-7}=-\frac{8}{7}$ (using Formula (1) and Theorem 2.3.5)

(d) $\det\left(\left(2A\right)^{-1}\right)=\frac{1}{\det(2A)}=\frac{1}{2^3\det(A)}=\frac{1}{(8)(-7)}=-\frac{1}{56}$ (using Theorem 2.3.5 and Formula (1))

(e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}=-\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}=-\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}=-(-7)=7$ (in the first step we interchanged the last two columns applying Theorem 2.2.3(b); in the second step we transposed the matrix applying Theorem 2.2.2)

35. **(a)** $\det(3A)=3^3\det(A)=(27)(7)=189$ (using Formula (1))

(b) $\det\left(A^{-1}\right)=\frac{1}{\det(A)}=\frac{1}{7}$ (using Theorem 2.3.5)

(c) $\det\left(2A^{-1}\right)=2^3\det\left(A^{-1}\right)=\frac{8}{\det(A)}=\frac{8}{7}$ (using Formula (1) and Theorem 2.3.5)

(d) $\det\left(\left(2A\right)^{-1}\right)=\frac{1}{\det(2A)}=\frac{1}{2^3\det(A)}=\frac{1}{(8)(7)}=\frac{1}{56}$ (using Theorem 2.3.5 and Formula (1))

True-False Exercises

(a) False. By Formula (1), $\det(2A)=2^3\det(A)=8\det(A)$.

(b) False. E.g. $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have $\det(A)=\det(B)=0$ but $\det(A+B)=1\neq 2\det(A)$.

(c) True. By Theorems 2.3.4 and 2.3.5,
$\det\left(A^{-1}BA\right)=\det\left(A^{-1}\right)\det(B)\det(A)=\frac{1}{\det(A)}\det(B)\det(A)=\det(B)$.

(d) False. A square matrix A is invertible if and only if $\det(A)\neq 0$.

(e) True. This follows from Definition 1.

(f) True. This is Formula (8).

(g) True. If $\det(A)\neq 0$ then by Theorem 2.3.8 $A\mathbf{x}=\mathbf{0}$ must have only the trivial solution, which contradicts our assumption. Consequently, $\det(A)=0$.

(h) True. If the reduced row echelon form of A is I_n then by Theorem 2.3.8 $A\mathbf{x}=\mathbf{b}$ is consistent for every $\mathbf{b}$, which contradicts our assumption. Consequently, the reduced row echelon form of A cannot be I_n.

(i) True. Since the reduced row echelon form of E is I then by Theorem 2.3.8 $E\mathbf{x}=\mathbf{0}$ must have only the trivial solution.

(j) True. If A is invertible, so is A^{-1}. By Theorem 2.3.8, each system has only the trivial solution.

(k) True. From Theorem 2.3.6, $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)$ therefore $\operatorname{adj}(A)=\det(A)A^{-1}$. Consequently,

$\left(\frac{1}{\det(A)}A\right)\operatorname{adj}(A)=\left(\frac{1}{\det(A)}A\right)\left(\det(A)\,A^{-1}\right)=\frac{\det(A)}{\det(A)}\left(AA^{-1}\right)=I_n$ so $\left(\operatorname{adj}(A)\right)^{-1}=\frac{1}{\det(A)}A$.

(l) False. If the kth row of A contains only zeros then all cofactors C_{jk} where $j\neq i$ are zero (since each of them involves a determinant of a matrix with a zero row). This means the matrix of cofactors contains at least one zero row, therefore $\operatorname{adj}(A)$ has a *column* of zeros.

Chapter 2 Supplementary Exercises

1. **(a)** Cofactor expansion along the first row: $\begin{vmatrix}-4 & 2\\ 3 & 3\end{vmatrix}=(-4)(3)-(2)(3)=-12-6=-18$

(b)

$$\begin{vmatrix}-4 & 2\\ 3 & 3\end{vmatrix}=-\begin{vmatrix}3 & 3\\ -4 & 2\end{vmatrix}$$ ← The first and second rows were interchanged.

$$=-(3)\begin{vmatrix}1 & 1\\ -4 & 2\end{vmatrix}$$ ← A common factor of 3 from the first row was taken through the determinant sign.

$$=-(3)\begin{vmatrix}1 & 1\\ 0 & 6\end{vmatrix}$$ ← 4 times the first row was added to the second row

$$=-(3)(1)(6)=-18$$ ← Use Theorem 2.1.2.

3. **(a)** Cofactor expansion along the second row:

$$\begin{vmatrix}-1 & 5 & 2\\ 0 & 2 & -1\\ -3 & 1 & 1\end{vmatrix}=-0+2\begin{vmatrix}-1 & 2\\ -3 & 1\end{vmatrix}-(-1)\begin{vmatrix}-1 & 5\\ -3 & 1\end{vmatrix}=0+2\left[(-1)(1)-(2)(-3)\right]-(-1)\left[(-1)(1)-(5)(-3)\right]$$

$$=0+(2)(5)-(-1)(14)=0+10+14=24$$

(b)

$$\begin{vmatrix}-1 & 5 & 2\\ 0 & 2 & -1\\ -3 & 1 & 1\end{vmatrix}=(-1)\begin{vmatrix}1 & -5 & -2\\ 0 & 2 & -1\\ -3 & 1 & 1\end{vmatrix}$$ ← A common factor of -1 from the first row was taken through the determinant sign.

$$=(-1)\begin{vmatrix}1 & -5 & -2\\ 0 & 2 & -1\\ 0 & -14 & -5\end{vmatrix}$$ ← 3 times the first row was added to the third row.

$$=(-1)\begin{vmatrix}1 & -5 & -2\\ 0 & 2 & -1\\ 0 & 0 & -12\end{vmatrix}$$ ← 7 times the second row was added to the third.

$$=(-1)(1)(2)(-12)=24 \longleftarrow \text{Use Theorem 2.1.2.}$$

5. **(a)** Cofactor expansion along the first row:

$$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = (3)\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}$$

$$=(3)\left[(1)(2)-(1)(4)\right]-0+(-1)\left[(1)(4)-(1)(0)\right]$$

$$=(3)(-2)-0+(-1)(4)=-6+0-4=-10$$

(b)

$$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 4 & 2 \end{vmatrix} \longleftarrow \text{The first and second rows were interchanged.}$$

$$=(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 4 & 2 \end{vmatrix} \longleftarrow -3 \text{ times the first row was added to the second.}$$

$$=(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 1 & -2 \end{vmatrix} \longleftarrow \text{The second row was added to the third row}$$

$$=(-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -3 & -4 \end{vmatrix} \longleftarrow \text{The second and third rows were interchanged.}$$

$$=(-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -10 \end{vmatrix} \longleftarrow 3 \text{ times the second row was added to the third.}$$

$$=(-1)(-1)(1)(1)(-10)=-10 \longleftarrow \text{Use Theorem 2.1.2.}$$

7. **(a)** We perform cofactor expansions along the first row in the 4x4 determinant. In each of the 3x3 determinants, we expand along the second row:

$$\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} - 6\begin{vmatrix} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} + 0 - 1\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix}$$

$$=3\left(-0+(-1)\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix}-1\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}\right)-6\left(-1\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix}+(-1)\begin{vmatrix} -2 & 4 \\ -9 & 2 \end{vmatrix}-1\begin{vmatrix} -2 & 1 \\ -9 & -2 \end{vmatrix}\right)$$

$$
\begin{aligned}
&+0-1\left(-1\begin{vmatrix}3&1\\2&-2\end{vmatrix}+0-(-1)\begin{vmatrix}-2&3\\-9&2\end{vmatrix}\right)\\
&=3\big(0-1(-2)-1(-8)\big)-6\big(-1(10)-1(32)-1(13)\big)+0-1\big(-1(-8)+0+1(23)\big)\\
&=3(10)-6(-55)+0-1(31)\\
&=329
\end{aligned}
$$

(b)

$$\begin{vmatrix}3&6&0&1\\-2&3&1&4\\1&0&-1&1\\-9&2&-2&2\end{vmatrix}=(-1)\begin{vmatrix}1&0&-1&1\\-2&3&1&4\\3&6&0&1\\-9&2&-2&2\end{vmatrix}$$

The first and third rows were interchanged.

$$=(-1)\begin{vmatrix}1&0&-1&1\\0&3&-1&6\\0&6&3&-2\\0&2&-11&11\end{vmatrix}$$

2 times the first row was added to the second, -3 times the first row was added to the third and 9 times the first row was added to the fourth.

$$=(-1)\begin{vmatrix}1&0&-1&1\\0&3&-1&6\\0&0&5&-14\\0&0&-\frac{31}{3}&7\end{vmatrix}$$

-2 times the second row was added to the third and $-\frac{2}{3}$ times the second row was added to the fourth.

$$=(-1)\begin{vmatrix}1&0&-1&1\\0&3&-1&6\\0&0&5&-14\\0&0&0&-\frac{329}{15}\end{vmatrix}$$

$\frac{31}{15}$ times the third row was added to the fourth.

$$=(-1)(1)(3)(5)\left(-\tfrac{329}{15}\right)=329$$

Use Theorem 2.1.2.

9. $$\begin{vmatrix}-1&5&2\\0&2&-1\\-3&1&1\end{vmatrix}=\begin{vmatrix}-1&5&2\\0&2&-1\\-3&1&1\end{vmatrix}\begin{matrix}-1&5\\0&2\\-3&1\end{matrix}=[-2+15+0]-[-12+1+0]=24$$

$$\begin{vmatrix}-1&-2&-3\\-4&-5&-6\\-7&-8&-9\end{vmatrix}=\begin{vmatrix}-1&-2&-3\\-4&-5&-6\\-7&-8&-9\end{vmatrix}\begin{matrix}-1&-2\\-4&-5\\-7&-8\end{matrix}=[-45-84-96]-[-105-48-72]=0$$

$$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = \left|\begin{array}{ccc|cc} 3 & 0 & -1 & 3 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & 2 & 0 & 4 \end{array}\right. = [6+0-4]-[0+12+0] = -10$$

$$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = \left|\begin{array}{ccc|cc} -5 & 1 & 4 & -5 & 1 \\ 3 & 0 & 2 & 3 & 0 \\ 1 & -2 & 2 & 1 & -2 \end{array}\right. = [0+2-24]-[0+20+6] = -48$$

11. In Exercise 1: $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} = -18 \neq 0$ therefore the matrix is invertible.

In Exercise 2: $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = -44 \neq 0$ therefore the matrix is invertible.

In Exercise 3: $\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = 24 \neq 0$ therefore the matrix is invertible.

In Exercise 4: $\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = 0$ therefore the matrix is not invertible.

13. $\begin{vmatrix} 5 & b-3 \\ b-2 & -3 \end{vmatrix} = (5)(-3)-(b-3)(b-2) = -15-b^2+2b+3b-6 = -b^2+5b-21$

15.

$$\begin{vmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix}$$

← The first row and the fifth row were interchanged.

$$=(-1)(-1)\begin{vmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix}$$

The second row and the fourth row were interchanged.

$$=(-1)(-1)(5)(2)(-1)(-4)(-3)=-120$$

17. It was shown in the solution of Exercise 1 that $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix}=-18$. The determinant is nonzero, therefore by Theorem 2.3.3, the matrix $A=\begin{bmatrix} -4 & 2 \\ 3 & 3 \end{bmatrix}$ is invertible.

The cofactors are:

$$\begin{array}{ll} C_{11}=3 & C_{12}=-3 \\ C_{21}=-2 & C_{22}=-4 \end{array}$$

The matrix of cofactors is $\begin{bmatrix} 3 & -3 \\ -2 & -4 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix} 3 & -2 \\ -3 & -4 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{-18}\begin{bmatrix} 3 & -2 \\ -3 & -4 \end{bmatrix}=\begin{bmatrix} -\frac{1}{6} & \frac{1}{9} \\ \frac{1}{6} & \frac{2}{9} \end{bmatrix}$.

19. It was shown in the solution of Exercise 3 that $\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix}=24$. The determinant is nonzero, therefore by Theorem 2.3.3, $A=\begin{bmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$\begin{array}{lll} C_{11}=\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}=3 & C_{12}=-\begin{vmatrix} 0 & -1 \\ -3 & 1 \end{vmatrix}=3 & C_{13}=\begin{vmatrix} 0 & 2 \\ -3 & 1 \end{vmatrix}=6 \\ C_{21}=-\begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix}=-3 & C_{22}=\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix}=5 & C_{23}=-\begin{vmatrix} -1 & 5 \\ -3 & 1 \end{vmatrix}=-14 \\ C_{31}=\begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix}=-9 & C_{32}=-\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}=-1 & C_{33}=\begin{vmatrix} -1 & 5 \\ 0 & 2 \end{vmatrix}=-2 \end{array}$$

The matrix of cofactors is $\begin{bmatrix} 3 & 3 & 6 \\ -3 & 5 & -14 \\ -9 & -1 & -2 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix} 3 & -3 & -9 \\ 3 & 5 & -1 \\ 6 & -14 & -2 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{24}\begin{bmatrix}3&-3&-9\\3&5&-1\\6&-14&-2\end{bmatrix}=\begin{bmatrix}\frac{1}{8}&-\frac{1}{8}&-\frac{3}{8}\\\frac{1}{8}&\frac{5}{24}&-\frac{1}{24}\\\frac{1}{4}&-\frac{7}{12}&-\frac{1}{12}\end{bmatrix}$.

21. It was shown in the solution of Exercise 5 that $\begin{vmatrix}3&0&-1\\1&1&1\\0&4&2\end{vmatrix}=-10$. The determinant is nonzero, therefore by Theorem 2.3.3, $A=\begin{bmatrix}3&0&-1\\1&1&1\\0&4&2\end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}1&1\\4&2\end{vmatrix}=-2 \qquad C_{12}=-\begin{vmatrix}1&1\\0&2\end{vmatrix}=-2 \qquad C_{13}=\begin{vmatrix}1&1\\0&4\end{vmatrix}=4$$

$$C_{21}=-\begin{vmatrix}0&-1\\4&2\end{vmatrix}=-4 \qquad C_{22}=\begin{vmatrix}3&-1\\0&2\end{vmatrix}=6 \qquad C_{23}=-\begin{vmatrix}3&0\\0&4\end{vmatrix}=-12$$

$$C_{31}=\begin{vmatrix}0&-1\\1&1\end{vmatrix}=1 \qquad C_{32}=-\begin{vmatrix}3&-1\\1&1\end{vmatrix}=-4 \qquad C_{33}=\begin{vmatrix}3&0\\1&1\end{vmatrix}=3$$

The matrix of cofactors is $\begin{bmatrix}-2&-2&4\\-4&6&-12\\1&-4&3\end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix}-2&-4&1\\-2&6&-4\\4&-12&3\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{-10}\begin{bmatrix}-2&-4&1\\-2&6&-4\\4&-12&3\end{bmatrix}=\begin{bmatrix}\frac{1}{5}&\frac{2}{5}&-\frac{1}{10}\\\frac{1}{5}&-\frac{3}{5}&\frac{2}{5}\\-\frac{2}{5}&\frac{6}{5}&-\frac{3}{10}\end{bmatrix}$.

23. It was shown in the solution of Exercise 7 that $\begin{vmatrix}3&6&0&1\\-2&3&1&4\\1&0&-1&1\\-9&2&-2&2\end{vmatrix}=329$. The determinant of A is nonzero therefore by Theorem 2.3.3, $A=\begin{bmatrix}3&6&0&1\\-2&3&1&4\\1&0&-1&1\\-9&2&-2&2\end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}3&1&4\\0&-1&1\\2&-2&2\end{vmatrix}=(-6+2+0)-(-8-6+0)=10$$

$$C_{12} = -\begin{vmatrix} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} = -\left[(4-9-8)-(36+4+2)\right] = 55$$

$$C_{13} = \begin{vmatrix} -2 & 3 & 4 \\ 1 & 0 & 1 \\ -9 & 2 & 2 \end{vmatrix} = (0-27+8)-(0-4+6) = -21$$

$$C_{14} = -\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix} = -\left[(0+27+2)-(0+4-6)\right] = -31$$

$$C_{21} = -\begin{vmatrix} 6 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} = -\left[(-12+0+0)-(-2-12+0)\right] = -2$$

$$C_{22} = \begin{vmatrix} 3 & 0 & 1 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} = (-6+0-2)-(9-6+0) = -11$$

$$C_{23} = -\begin{vmatrix} 3 & 6 & 1 \\ 1 & 0 & 1 \\ -9 & 2 & 2 \end{vmatrix} = -\left[(0-54+2)-(0+6+12)\right] = 70$$

$$C_{24} = \begin{vmatrix} 3 & 6 & 0 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix} = (0+54+0)-(0-6-12) = 72$$

$$C_{31} = \begin{vmatrix} 6 & 0 & 1 \\ 3 & 1 & 4 \\ 2 & -2 & 2 \end{vmatrix} = (12+0-6)-(2-48+0) = 52$$

$$C_{32} = -\begin{vmatrix} 3 & 0 & 1 \\ -2 & 1 & 4 \\ -9 & -2 & 2 \end{vmatrix} = -\left[(6+0+4)-(-9-24+0)\right] = -43$$

$$C_{33} = \begin{vmatrix} 3 & 6 & 1 \\ -2 & 3 & 4 \\ -9 & 2 & 2 \end{vmatrix} = (18-216-4)-(-27+24-24) = -175$$

$$C_{34} = -\begin{vmatrix} 3 & 6 & 0 \\ -2 & 3 & 1 \\ -9 & 2 & -2 \end{vmatrix} = -\left[(-18-54+0)-(0+6+24)\right] = 102$$

$$C_{41} = -\begin{vmatrix} 6 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & -1 & 1 \end{vmatrix} = -\left[(6+0-3)-(0-24+0)\right] = -27$$

$$C_{42}=\begin{vmatrix}3&0&1\\-2&1&4\\1&-1&1\end{vmatrix}=(3+0+2)-(1-12+0)=16$$

$$C_{43}=-\begin{vmatrix}3&6&1\\-2&3&4\\1&0&1\end{vmatrix}=-\big[(9+24+0)-(3+0-12)\big]=-42$$

$$C_{44}=\begin{vmatrix}3&6&0\\-2&3&1\\1&0&-1\end{vmatrix}=(-9+6+0)-(0+0+12)=-15$$

The matrix of cofactors is $\begin{bmatrix}10&55&-21&-31\\-2&-11&70&72\\52&-43&-175&102\\-27&16&-42&-15\end{bmatrix}$ and $\operatorname{adj}(A)=\begin{bmatrix}10&-2&52&-27\\55&-11&-43&16\\-21&70&-175&-42\\-31&72&102&-15\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{329}\begin{bmatrix}10&-2&52&-27\\55&-11&-43&16\\-21&70&-175&-42\\-31&72&102&-15\end{bmatrix}=\begin{bmatrix}\frac{10}{329}&-\frac{2}{329}&\frac{52}{329}&-\frac{27}{329}\\\frac{55}{329}&-\frac{11}{329}&-\frac{43}{329}&\frac{16}{329}\\-\frac{3}{47}&\frac{10}{47}&-\frac{25}{47}&-\frac{6}{47}\\-\frac{31}{329}&\frac{72}{329}&\frac{102}{329}&-\frac{15}{329}\end{bmatrix}$.

25. $A=\begin{bmatrix}\frac{3}{5}&-\frac{4}{5}\\\frac{4}{5}&\frac{3}{5}\end{bmatrix}$, $\det(A)=\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)-\left(-\frac{4}{5}\right)\left(\frac{4}{5}\right)=\frac{9}{25}+\frac{16}{25}=1$; $A_1=\begin{bmatrix}x&-\frac{4}{5}\\y&\frac{3}{5}\end{bmatrix}$, $A_2=\begin{bmatrix}\frac{3}{5}&x\\\frac{4}{5}&y\end{bmatrix}$;

$x'=\frac{\det(A_1)}{\det(A)}=\frac{3}{5}x+\frac{4}{5}y$, $y'=\frac{\det(A_2)}{\det(A)}=\frac{3}{5}y-\frac{4}{5}x$

27. The coefficient matrix of the given system is $A=\begin{bmatrix}1&1&\alpha\\1&1&\beta\\\alpha&\beta&1\end{bmatrix}$. Coefficient expansion along the first row yields

$$\det(A)=1\begin{vmatrix}1&\beta\\\beta&1\end{vmatrix}-1\begin{vmatrix}1&\beta\\\alpha&1\end{vmatrix}+\alpha\begin{vmatrix}1&1\\\alpha&\beta\end{vmatrix}$$

$$=1-\beta^2-(1-\alpha\beta)+\alpha(\beta-\alpha)=-\alpha^2+2\alpha\beta-\beta^2=-(\alpha-\beta)^2$$

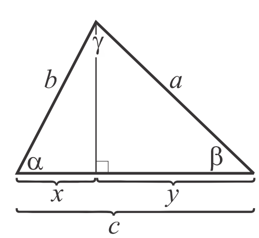
By Theorem 2.3.8, the given system has a nontrivial solution if and only if $\det(A)=0$, i.e., $\alpha=\beta$.

29. **(a)** We will justify the third equality, $a\cos\beta+b\cos\alpha=c$ by considering three cases:

CASE I: $\alpha\le\frac{\pi}{2}$ and $\beta\le\frac{\pi}{2}$

Referring to the figure on the right side, we have $x=b\cos\alpha$ and $y=a\cos\beta$.

Since $x+y=c$ we obtain, $a\cos\beta+b\cos\alpha=c$.

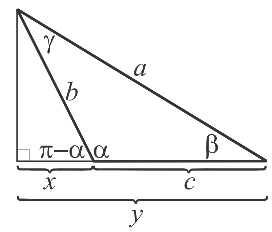


CASE II: $\alpha > \frac{\pi}{2}$ and $\beta < \frac{\pi}{2}$

Referring to the picture on the right side, we can write
$x = b\cos(\pi - \alpha) = -b\cos\alpha$ and $y = a\cos\beta$

This time we can write $c = y - x = a\cos\beta - (-b\cos\alpha)$

therefore once again $a\cos\beta + b\cos\alpha = c$.



CASE III: $\beta > \frac{\pi}{2}$ and $\alpha < \frac{\pi}{2}$ (similarly to case II, $c = b\cos\alpha - a\cos(\pi - \beta) = b\cos\alpha + a\cos\beta$)

The first two equations can be justified in the same manner.

Denoting $X = \cos\alpha$, $Y = \cos\beta$, and $Z = \cos\gamma$ we can rewrite the linear system as

$$\begin{array}{ccccccc} & & cY & + & bZ & = & a \\ cX & & & + & aZ & = & b \\ bX & + & aY & & & = & c \end{array}$$

We have $\det(A) = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = [0 + abc + abc] - [0 + 0 + 0] = 2abc$ and

$\det(A_1) = \begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix} = [0 + ac^2 + ab^2] - [0 + a^3 + 0] = a(b^2 + c^2 - a^2)$ therefore by

Cramer's rule $\cos\alpha = X = \frac{\det(A_1)}{\det(A)} = \frac{a(b^2+c^2-a^2)}{2abc} = \frac{b^2+c^2-a^2}{2bc}$.

(b) Using the results obtained in part (a) along with

$\det(A_2) = \begin{vmatrix} 0 & a & b \\ c & b & a \\ b & c & 0 \end{vmatrix} = [0 + a^2b + bc^2] - [b^3 + 0 + 0] = b(a^2 + c^2 - b^2)$ and

$\det(A_3) = \begin{vmatrix} 0 & c & a \\ c & 0 & b \\ b & a & c \end{vmatrix} = [0 + b^2c + a^2c] - [0 + 0 + c^3] = c(a^2 + b^2 - c^2)$ therefore by

Cramer's rule

$\cos\beta = Y = \frac{\det(A_2)}{\det(A)} = \frac{a^2+c^2-b^2}{2ac}$ and $\cos\gamma = Z = \frac{\det(A_3)}{\det(A)} = \frac{a^2+b^2-c^2}{2ab}$.

31. From Theorem 2.3.6, $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ therefore $\operatorname{adj}(A) = \det(A)A^{-1}$. Consequently,

$$\left(\frac{1}{\det(A)}A\right)\operatorname{adj}(A) = \left(\frac{1}{\det(A)}A\right)\left(\det(A)\,A^{-1}\right) = \frac{\det(A)}{\det(A)}\left(AA^{-1}\right) = I_n \text{ so } \left(\operatorname{adj}(A)\right)^{-1} = \frac{1}{\det(A)}A .$$

Using Theorem 2.3.5, we can also write $\operatorname{adj}(A^{-1}) = \det(A^{-1})(A^{-1})^{-1} = \frac{1}{\det(A)}A.$

33. The equality $A\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ means that the homogeneous system $A\mathbf{x}=\mathbf{0}$ has a nontrivial solution $\mathbf{x}=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$.

Consequently, it follows from Theorem 2.3.8 that $\det(A)=0$.

37. In the special case that $n=3$, the augmented matrix for the system (13) of Section 1.10 is $\begin{bmatrix}1 & x_1 & x_1^2 & y_1\\ 1 & x_2 & x_2^2 & y_2\\ 1 & x_3 & x_3^2 & y_3\end{bmatrix}$.

We apply Cramer's Rule to the coefficient matrix $A=\begin{bmatrix}1 & x_1 & x_1^2\\ 1 & x_2 & x_2^2\\ 1 & x_3 & x_3^2\end{bmatrix}$.

$A_1=\begin{bmatrix}y_1 & x_1 & x_1^2\\ y_2 & x_2 & x_2^2\\ y_3 & x_3 & x_3^2\end{bmatrix}, A_2=\begin{bmatrix}1 & y_1 & x_1^2\\ 1 & y_2 & x_2^2\\ 1 & y_3 & x_3^2\end{bmatrix}$, and $A_3=\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}$ so the coefficients of the desired interpolating polynomial $y=a_0+a_1x+a_2x^2$ are: $a_0=\frac{\det(A_1)}{\det(A)}$, $a_1=\frac{\det(A_2)}{\det(A)}$, and $a_2=\frac{\det(A_3)}{\det(A)}$. From the result of Exercise 43 of Section 2.1, $\det(A)=(x_2-x_1)(x_3-x_1)(x_3-x_2)$.

$$\det\left(\begin{bmatrix}y_1 & x_1 & x_1^2\\ y_2 & x_2 & x_2^2\\ y_3 & x_3 & x_3^2\end{bmatrix}\right)=y_3x_1x_2(x_2-x_1)-y_2x_1x_3(x_3-x_1)+y_1x_2x_3(x_3-x_2),$$

$$\det\left(\begin{bmatrix}1 & y_1 & x_1^2\\ 1 & y_2 & x_2^2\\ 1 & y_3 & x_3^2\end{bmatrix}\right)=-y_3(x_2^2-x_1^2)+y_2(x_3^2-x_1^2)-y_1(x_3^2-x_2^2),$$

and $\det\left(\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}\right)=y_3(x_2-x_1)-y_2(x_3-x_1)+y_1(x_3-x_2)$.

Therefore,

$$a_0=\frac{y_3x_1x_2(x_2-x_1)-y_2x_1x_3(x_3-x_1)+y_1x_2x_3(x_3-x_2)}{(x_2-x_1)(x_3-x_1)(x_3-x_2)}=\frac{y_3x_1x_2}{(x_3-x_1)(x_3-x_2)}-\frac{y_2x_1x_3}{(x_2-x_1)(x_3-x_2)}+\frac{y_1x_2x_3}{(x_3-x_1)(x_2-x_1)},$$

$$a_1=\frac{-y_3(x_2^2-x_1^2)+y_2(x_3^2-x_1^2)-y_1(x_3^2-x_2^2)}{(x_2-x_1)(x_3-x_1)(x_3-x_2)}=\frac{y_2(x_3+x_1)}{(x_3-x_2)(x_2-x_1)}-\frac{y_3(x_2+x_1)}{(x_3-x_1)(x_3-x_2)}-\frac{y_1(x_3+x_2)}{(x_3-x_1)(x_2-x_1)}$$

and $$a_2=\frac{y_3(x_2-x_1)-y_2(x_3-x_1)+y_1(x_3-x_2)}{(x_2-x_1)(x_3-x_1)(x_3-x_2)}=\frac{y_3}{(x_3-x_1)(x_3-x_2)}-\frac{y_2}{(x_2-x_1)(x_3-x_2)}+\frac{y_1}{(x_3-x_1)(x_2-x_1)}.$$