MID-2

9 Relations

6 Counting.

8 Advanced Counting Techniques.

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4 Number Theory and Cryptography.

Relations

CHAPTER 9

Chapter Summary

Relations and Their Properties

Representing Relations

Equivalence Relations

Partial Orderings

Relations and Their Properties

SECTION 9.1

Section Summary

Relations and Functions

Properties of Relations

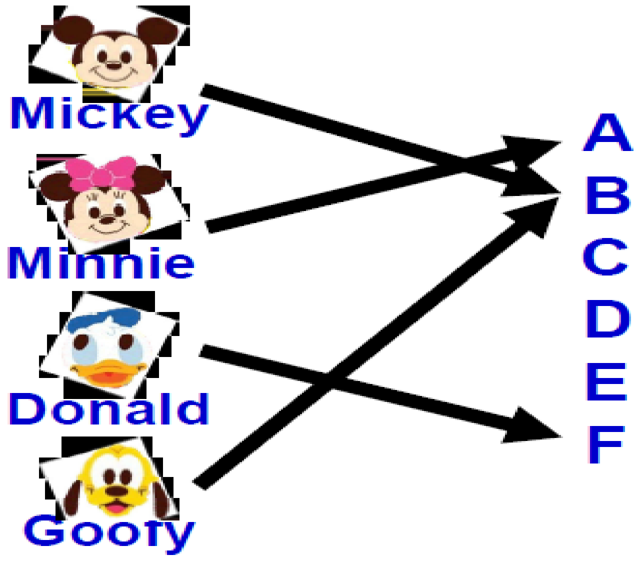
- Reflexive Relations
- Symmetric Relations
- Antisymmetric Relations
- Transitive Relations
- Irreflexive Relations
- Asymmetric Relations

Combining Relations

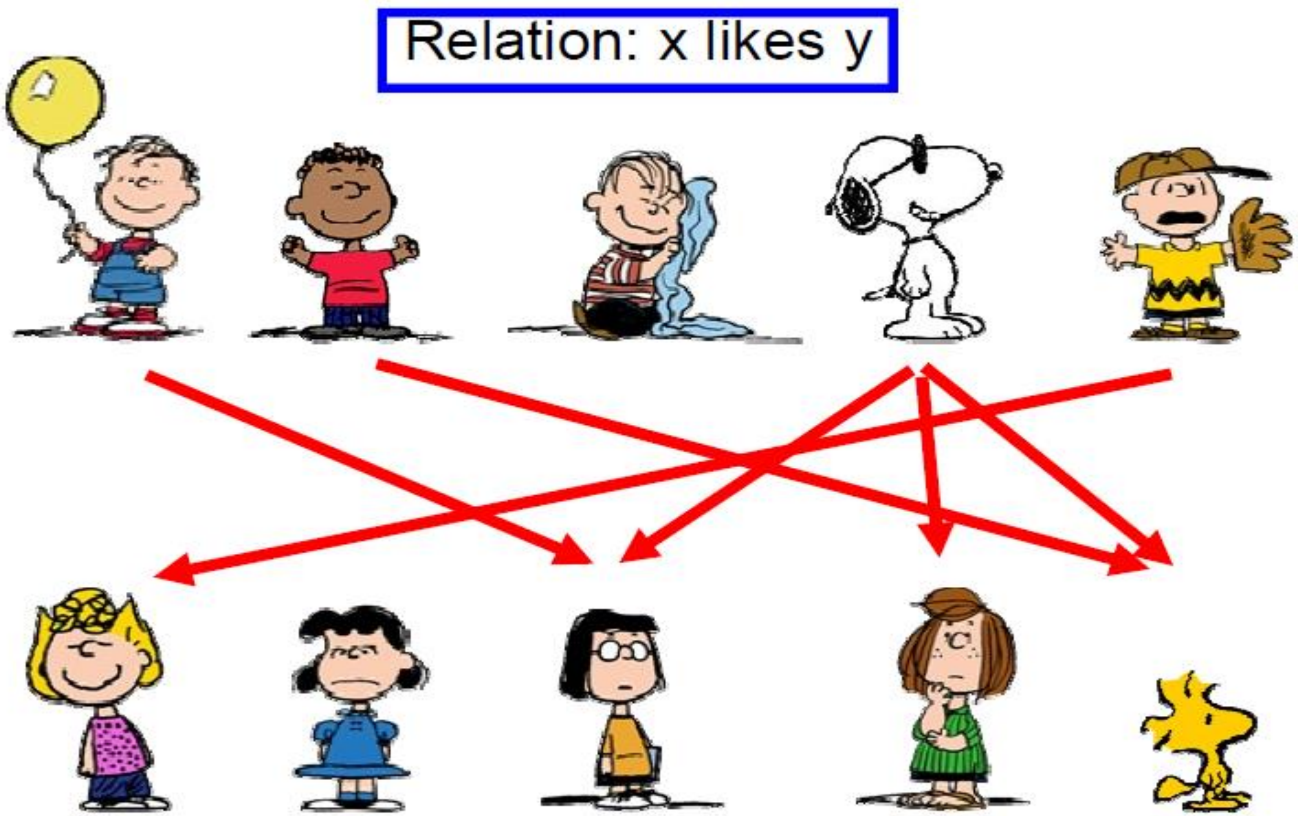
Recall, Function is…

Let **A** and **B** be nonempty sets Function ***f*** from **A** to **B** is an assignment of exactly one element of **B** to each element of **A.**

By **defining** using a **relation**, a **function** from A to B contains **unique** ordered pair (a, b) for **every** element a ∈ A.



What is Relation?



Binary Relations

Definition: A *binary relation R* from a set *A* to a set *B* is a subset $R \subseteq A \times B$.

Recall, for example:

A = {a1, a2} and B = {b1, b2, b3}

A x B = { (a1, b1), (a1, b2), (a1, b3),

(a2, b1), (a2, b2), (a2, b3)}

Ordered pairs, which

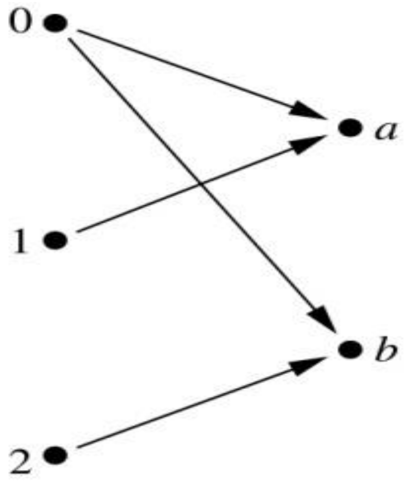
- First element comes from A
- Second element comes from B
- **aRb**: (a, b) ∈ R
- **a$\not$Rb** b) ∉ R

Moreover, when (a, b) belongs to R, a is said to be related to b by R.

Binary Relations

Example:

- Let A = {0,1,2} and B = {a,b}
- {(0, a), (0, b), (1,a) , (2, b)} is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Binary Relations

EXAMPLE:

Let A = {eggs, milk, corn} and B = {cows, goats, hens}

Define a relation R from A to B by $(a, b) \in R$ *iff* a is produced by b.

Then R = {(eggs, hens), (milk, cows), (milk, goats)}

Thus, with respect to this relation eggs R hens , milk R cows, etc.

Binary Relations

EXAMPLE #1:

- S = {Peter, Paul, Mary}
- C = {C++, DisMath}
- Given
  - Peter takes C++ — Peter R C++   Peter ~~R~~ DisMath
  - Paul takes DisMath — Paul ~~R~~ C++   Paul R DisMath
  - Mary takes none of them — Mary ~~R~~ C++   Mary ~~R~~ DisMath
- R = {(Peter, C++), (Paul, DisMath)}

Domain and Range of a Relation

DOMAIN OF A RELATION:

The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R denoted by Dom(R).

Symbolically, Dom (R) = $\{a \in A \mid (a, b) \in R\}$

RANGE OF A RELATION:

The range of a relation R from A to B is the set of all second elements of the ordered pairs which belong to R denoted Ran(R).

Symbolically, Ran(R) = $\{b \in B \mid (a, b) \in R\}$

Domain and Range of a Relation

EXERCISE:

Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$,

Define a binary relation R from A to B as follows:

$R = \{(a, b) \in A \times B \mid a < b\}$ Then

- a. Find the ordered pairs in R.
- b. Find the Domain and Range of R.
- c. Is 1R3, 2R2?

SOLUTION:

Given $A = \{1, 2\}$, $B = \{1, 2, 3\}$,

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

a. $R = \{(a, b) \in A \times B \mid a < b\}$

$R = \{(1,2), (1,3), (2,3)\}$

Domain and Range of a Relation

Given A = {1, 2}, B = {1, 2, 3},

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

b. Find the Domain and Range of R.

Solution:

Dom(R) = {1,2} and Ran(R) = {2, 3}

c. Is 1R3, 2R2?

Solution:

c. Since (1, 3)∈R so 1R3.

Since (2, 2)∈R so 2R2.

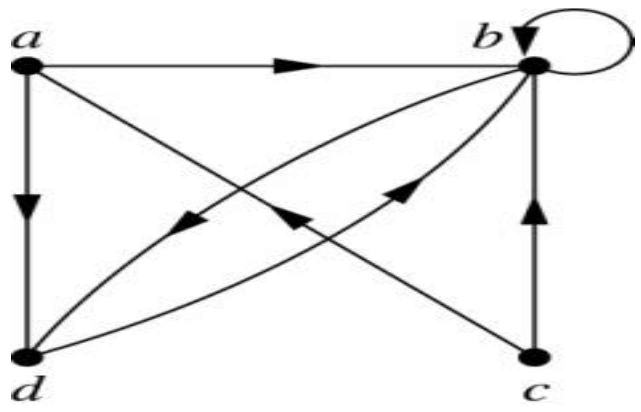
Representing Relations

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

- An edge of the form (a,a) is called a *loop*.

Example: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is shown here.



Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix.

Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

- The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.

The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

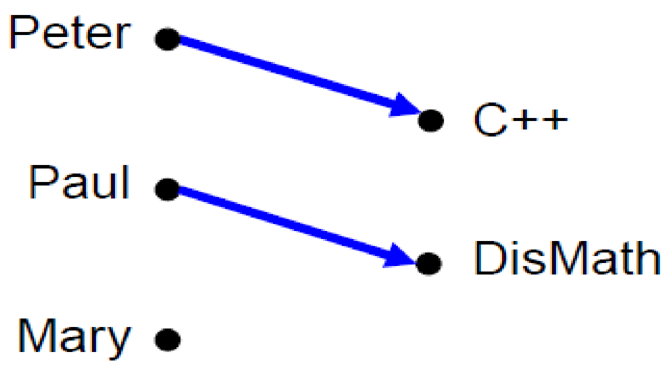
Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Binary Relations

EXAMPLE #1: (cont.)

- Peter R C++, Peter $\not R$ DisMath
  Paul $\not R$ C++, Paul R DisMath
  Mary $\not R$ C++, Mary $\not R$ DisMath



Directed Graph

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Matrix

Binary Relation on a Set

Definition: A binary relation *R on a set A* is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a,b) \mid a \text{ divides } b\}$ are

  $\{(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), \text{and } (4, 4)\}$.

REMARK:

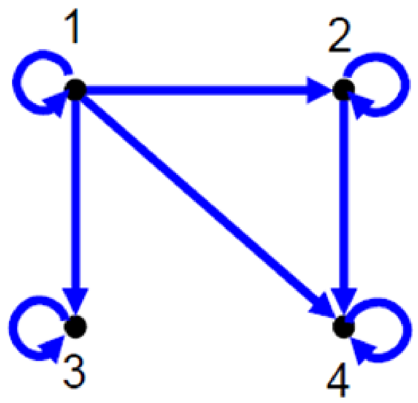
For any set A

1. A × A is known as the universal relation.
2. ∅ is known as the empty relation.

Binary Relation on a Set

- Let A be the set $\{1, 2, 3, 4\}$, which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

- $R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$



$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Relations and Their Properties

Binary Relation on a Set (*cont.*)

Question: How many different relations are there on a set A with n elements?

Solution:

Suppose A has n elements

Recall, a relation on a set A is a subset of A x A.

A x A has n^2elements.

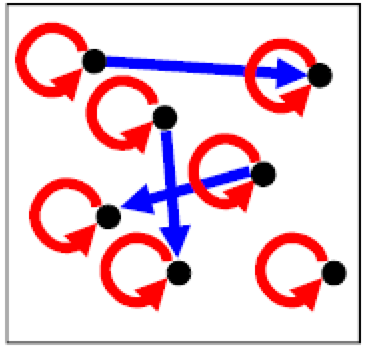
If a set has m element, its has 2^m subsets.

Therefore, the answer is 2^{n^2}.

Reflexive Relations

Definition: *R* is *reflexive* iff (*a*,*a*) ∈ *R* for every element *a* ∈ A. Written symbolically, R is reflexive if and only if

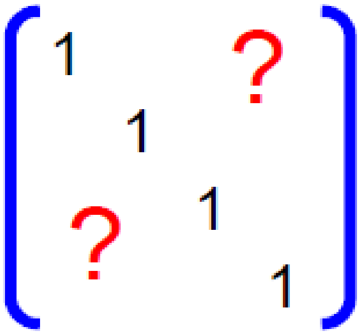
$$\forall a[a \in U \longrightarrow (a,a) \in R]$$



Reflexive

$\forall a\ ((a, a) \in R)$

Every node has a self-loop



Reflexive

$\forall a\ ((a, a) \in R)$

All 1's on diagonal

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

Reflexive Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R1, R2, R3, and R4 are Reflexive?

R1 = {(1, 1), (3, 3), (2, 2), (4, 4)}

R2 = {(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)}

R3 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R4 = {(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)}

Solution:

R1 is reflexive, since (a, a) ∈R1 for all a ∈A.

R2 is not reflexive, because (4, 4) ∉R2.

R3 is reflexive, since (a, a) ∈R3 for all a ∈A.

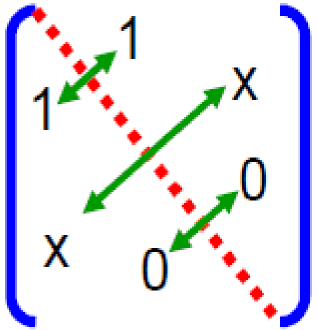
R4 is not reflexive, because (1, 1) ∉R4, (3, 3) ∉R4.

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall a \forall b\, [(a,b) \in R \longrightarrow (b,a) \in R]$$

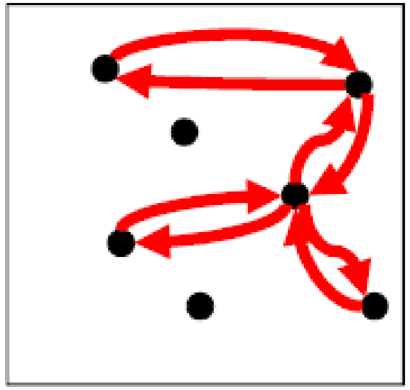
More precisely, M is a symmetric matrix i.e. M = M_t



Symmetric

$\forall a\ \forall b\ (\ ((a,\ b) \in R) \rightarrow ((b,\ a) \in R)\ )$

All identical across diagonal



Symmetric

$\forall a\ \forall b\ (\ ((a,\ b) \in R) \rightarrow ((b,\ a) \in R)\ )$

Every link is bidirectional

Accordingly, R is symmetric if the elements in the ith row are the same as the elements in the ith column of the matrix M representing R. More precisely, M is a symmetric matrix i.e. M = M^t

Symmetric Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R1, R2, R3, and R4 are Symmetric?

R1 = {(1, 1), (1, 3), (2, 4), (3, 1), (4,2)}

R2 = {(1, 1), (2, 2), (3, 3), (4, 4)}

R3 = {(2, 2), (2, 3), (3, 4)}

R4 = {(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)}

Solution:

R1 is Symmetric, since (a, b) and (b, a) $\in$R1 for all (a, b) $\in$A.

R2 is also symmetric. We say it is vacuously true.

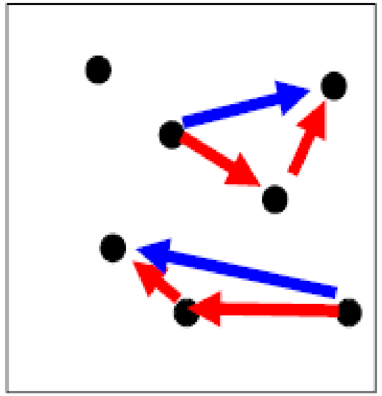
R3 is not symmetric, because (2,3) $\in$ R3 but (3,2) $\notin$ R3.

R4 is not symmetric because (4,3) $\in$ R4 but (3,4) $\notin$ R4.

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall a \forall b \forall c[(a,b) \in R \land (b,c) \in R \longrightarrow (a,c) \in R]$$



Transitive

$\forall a \forall b \forall c\ (((a,b) \in R \land (b,c) \in R) \rightarrow ((a,c) \in R))$

Every two adjacent forms a triangle
(Not easy to observe in Graph)

?

Transitive

$\forall a \forall b \forall c\ (((a,b) \in R \land (b,c) \in R) \rightarrow ((a,c) \in R))$

Not easy to observe in Matrix

For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the second to the third, there is an arrow going directly from the first to the third.

Transitive Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R1, R2 and R3 are Transitive?

R1 = {(1, 1), (1, 2), (1, 3), (2, 3)}

R2 = {(1, 2), (1, 4), (2, 3), (3, 4)}

R3 = {(2, 1), (2, 4), (2, 3), (3,4)}

Solution:

R1 is transitive because (1, 1), (1, 2) are in R, then to be transitive relation(1,2) must be there and it belongs to R.

R2 is not transitive since (1,2) and (2,3) $\in$ R2 but (1,3) $\notin$ R2.

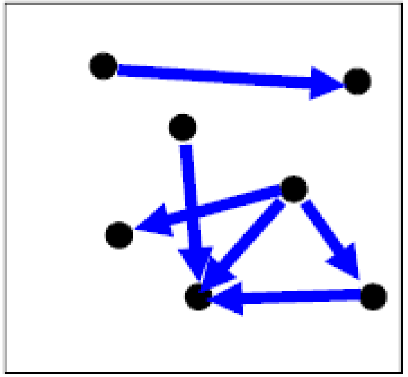
R3 is transitive.(check by definition)

Irreflexive Relations

Definition: R is irreflexive iff for all a∈A,(a,a) ∉R. That is, R is irreflexive if no element in A is related to itself by R.

Written symbolically, *R* is irreflexive if and only if

$$\forall a\,[(a \in A) \longrightarrow (a, a) \notin R]$$



Irreflexive

$\forall a\,(\,(a \in A) \to ((a, a) \notin R)\,)$

No node links to itself

$$\begin{bmatrix} 0 & & ? & \\ & 0 & & \\ ? & & 0 & \\ & & & 0 \end{bmatrix}$$

Irreflexive

$\forall a\,(\,(a \in A) \to ((a, a) \notin R)\,)$

All 0's on diagonal

R is not irreflexive iff there is an element a∈A such that (a,a) ∈R.

Irreflexive Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R1, R2 and R3 are Irreflexive?

R1 = {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)}

R2 = {(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)}

R3= {(1,2), (2,3), (3,3), (3,4)}

Solution:

R1 is irreflexive since no element of A is related to itself in R1. i.e. (1,1)$\notin$ R1, (2,2) $\notin$ R1, (3,3) $\notin$ R1,(4,4) $\notin$ R1.

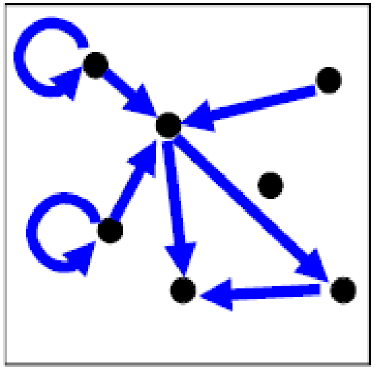
R2 is not irreflexive, since all elements of A are related to themselves in R2.

R3 is not irreflexive since (3,3) $\in$R3. Note that R3 is not reflexive.

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if $\forall a \forall b\,[(a,b) \in R \land (b,a) \in R \longrightarrow a = b]$

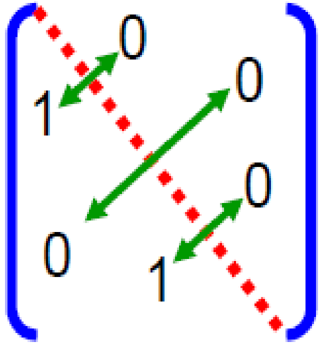
Note: (a,a) may be an element in R.



Antisymmetric

$\forall a\ \forall b\ (\ ((a,\ b) \in R \land (b,\ a) \in R) \rightarrow (a = b)\ )$

No link is bidirectional



Antisymmetric

$\forall a\ \forall b\ (\ ((a,\ b) \in R \land (b,\ a) \in R) \rightarrow (a = b)\ )$

All 1's are across from 0's

Let R be an anti-symmetric relation on a set A = {a1, a2, …, an}. Then if (ai, aj) ∈R for i ≠ j then (ai, aj) ∉R. Thus in the matrix representation of R there is a 1 in the ith row and jth column iff the jth row and ith column contains 0 vice versa.

Antisymmetric Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R1, R2, R3, and R4 are Antisymmetric?

R1 = {(1,1),(2,2),(3,3)}

R2 = {(1,2),(2,2), (2,3), (3,4), (4,1)}

R3={(1,3),(2,2), (2,4), (3,1), (4,2)}

R4={(1,3),(2,4), (3,1), (4,3)}

Solution:

R1 is anti-symmetric and symmetric.

R2 is anti-symmetric but not symmetric because $(1,2) \in R2$ but $(2,1) \notin R2$.

R3 is not anti-symmetric since $(1,3)$ & $(3,1) \in R3$ but $1 \neq 3$. Note that R3 is symmetric.

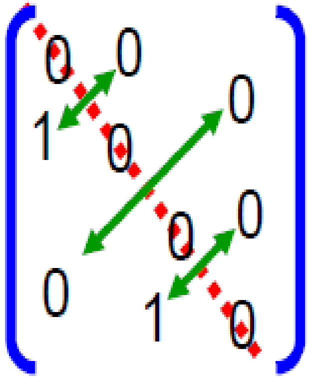
R4 is neither anti-symmetric because $(1,3)$ & $(3,1) \in R4$ but $1 \neq 3$ nor symmetric because $(2,4) \in R4$ but $(4,2) \notin R4$.

Asymmetric Relations

Definition: R is Asymmetric iff for all $(a,b) \in R$ than $(b,a) \notin R$. Written symbolically, R is Asymmetric if and only if

$$\forall a \forall b[((a,b) \in R)) \longrightarrow ((b,a) \notin R)]$$

Note:(a,a) cannot be an element in R.

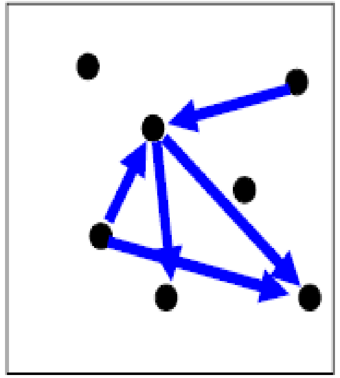


Asymmetric

$\forall a \forall b\ (((a, b) \in R) \rightarrow ((b, a) \notin R))$

All 1's are across from 0's (Antisymmetric)
All 0's on diagonal (Irreflexive)

Asymmetry = Antisymmetry + Irreflexivity



Asymmetric

$\forall a \forall b\ (((a, b) \in R) \rightarrow ((b, a) \notin R))$

No link is bidirectional (Antisymmetric)
No node links to itself (Irreflexive)

Asymmetric Relations

EXAMPLE: Let A = {1, 2, 3, 4} and determine whether relations R1, R2 and R3 are Asymmetric?

R1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}

R2 = {(1,2),(2,3),(3,4)}

R3 = {(2,3),(3,3),(3,4)}

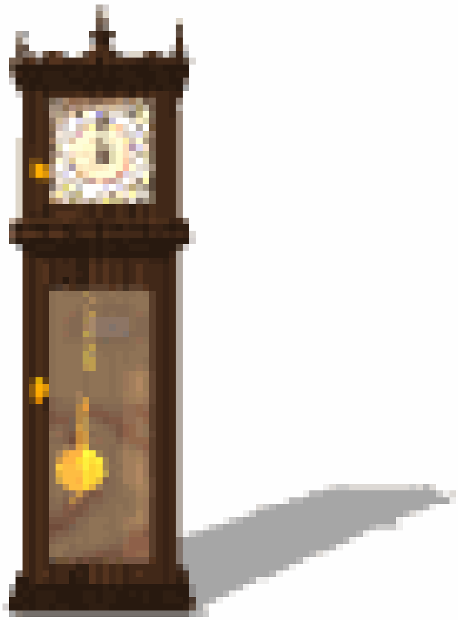
Solution:

R1 is not Asymmetric since R1 is neither Antisymmetric nor Irreflexive.

R2 is Asymmetric since R2 is both Antisymmetric and Irreflexive.

R3 is not Asymmetric since it is Antisymmetric but not irreflexive.

Activity Time



Consider the following relations on {1, 2, 3, 4}:

$R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R6 = \{(3, 4)\}$.

Determine which of these relation are Reflexive, Symmetric, Transitive, Antisymmetric, Irreflexive and Asymmetric.

Combining Relations

As R is a subsets of A x B, the set operations can be applied

Union (U)

Intersection (∩)

Difference (-)

Symmetric Complement (⊕)

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$ and $R_1 \oplus R_2$.

Combining Relations

Given, $A = \{1,2,3\}$, $B = \{1,2,3,4\}$

$R_1 = \{(1,1),(2,2),(3,3)\}$,

$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2),(3,3)\}$

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

$R_1 \oplus R_2 = \{(1,2),(1,3),(1,4),(2,2),(3,3)\}$

Composition of Relations

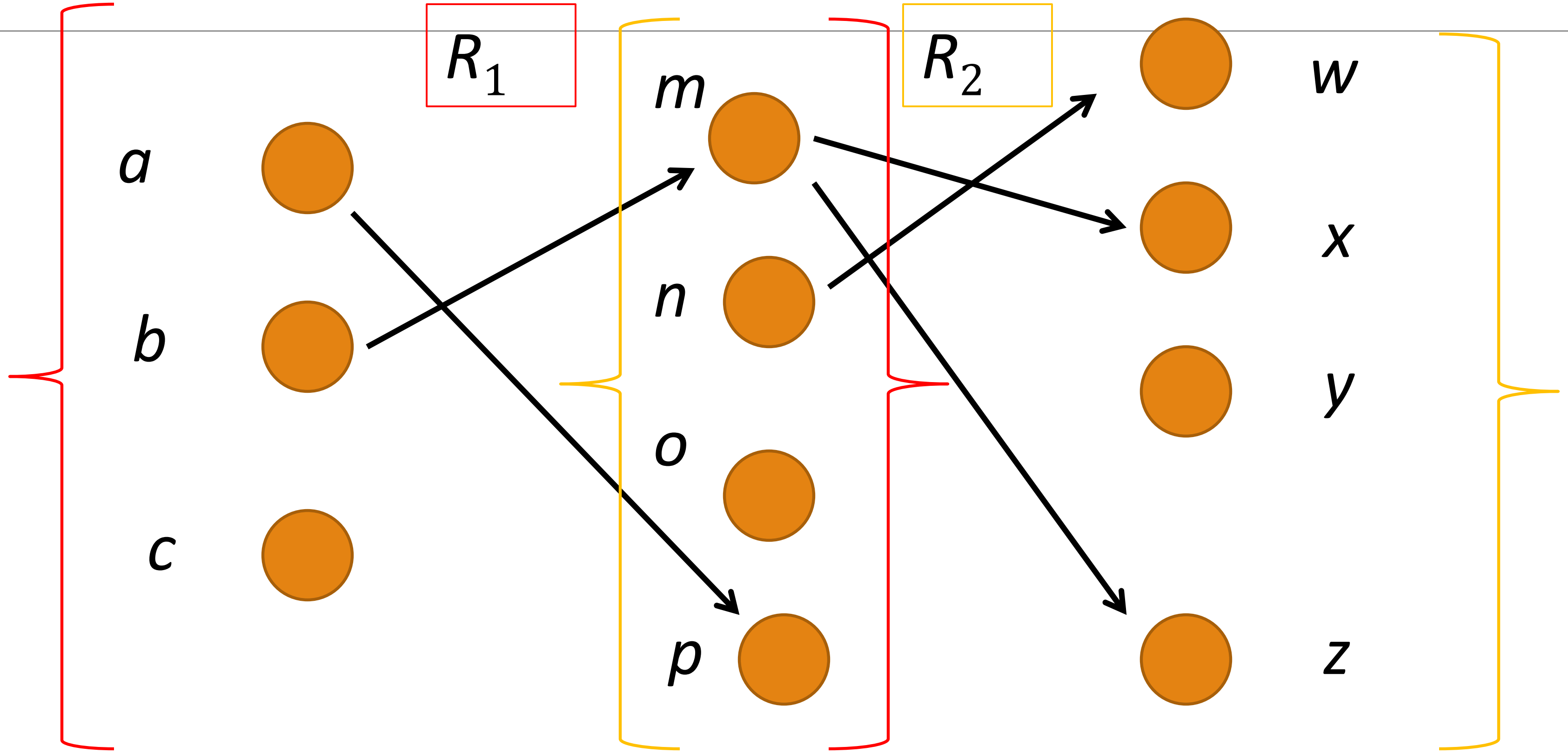
Definition: Suppose

- R_1 is a relation from a set *A* to a set *B*.
- R_2 is a relation from *B* to a set *C*.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from *A* to *C* where

- if (*x*,*y*) is a member of R_1 and (*y*,*z*) is a member of R_2, then (*x*,*z*) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$R_1 \circ R_2 = \{(b,D),(b,B)\}$

Composition of Relations

What is the composite of the relations R and S, where

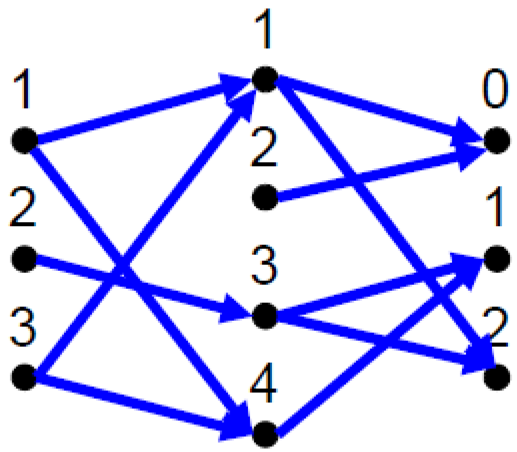
R is the relation from **{1,2,3}** to **{1,2,3,4}** with

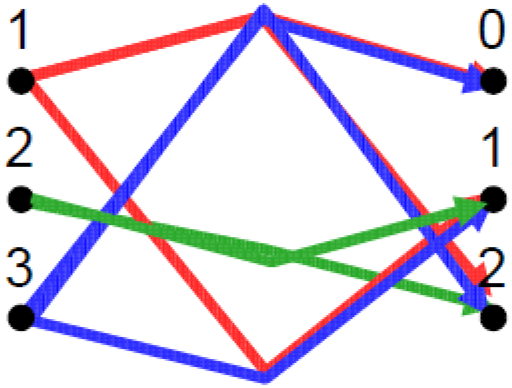
R = {(1,1),(1,4),(2,3),(3,1),(3,4)}

S is the relation from **{1,2,3,4}** to **{0,1,2}** with

S = {(1,0),(1,2),(2,0),(3,1),(3,2),(4,1)}?

S o R = {(1,0),(1,2),(1,1), (2,2),(2,1), (3,0),(3,2),(3,1)}





INVERSE OF A RELATION

Let R be a relation from A to B. The inverse relation R^{-1} from B to A is defined as:

$$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$$

More simply, the inverse relation R^{-1} of R is obtained by interchanging the elements of all the ordered pairs in R.

Example

X = {a, b, c} and Y={1, 2}

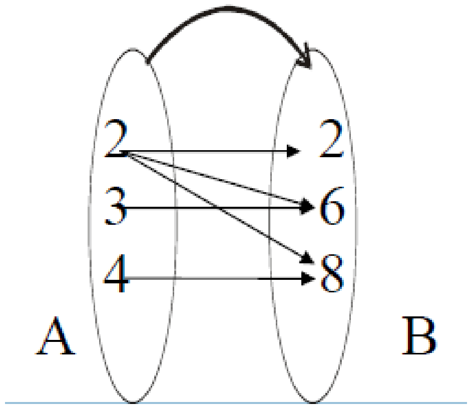
R = {(a, 1), (b, 2), (c, 1)}

R^{-1} = {(1, a), (2, b), (1, c)}

INVERSE OF A RELATION

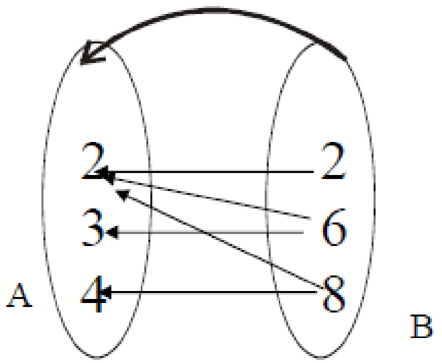
The relation

R = {(2,2), (2,6), (2,8), (3,6), (4,8)} is represented by the arrow diagram.



Then inverse of the above relation can be obtained simply changing the directions of the arrows and hence the diagram is

R^{-1}



Equivalence Relations

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent.* The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example:

Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$, and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that *a* divides *b* and *b* divides *c*. Then there are positive integers *k* and *l* such that *b* = *ak* and *c* = *bl*. Hence, *c* = *a*(*kl*), so *a* divides *c*. Therefore, the relation is transitive.

Partial Orderings

Partial Orderings

Definition 1: A relation *R* on a set S is called a *partial ordering,* or *partial order,* if it is reflexive, antisymmetric, and transitive.

A set together with a partial ordering *R* is called a *partially ordered set*, or *poset*, and is denoted by (*S*, *R*). Members of *S* are called *elements* of the poset.

Partial Orderings (*continued*)

Example 1: Show that the “greater than or equal” relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation (|) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

$(\boldsymbol{Z}^+, |)$ is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Problem-1

Home Activity

A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Problem-2

There are 2504 computer science students at a school. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these three programming languages?

a) Draw venn diagram

b) Use Principal of inclusion-Exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

EXAMPLE 5 Consider these relations on the set of integers:

Activity Time

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

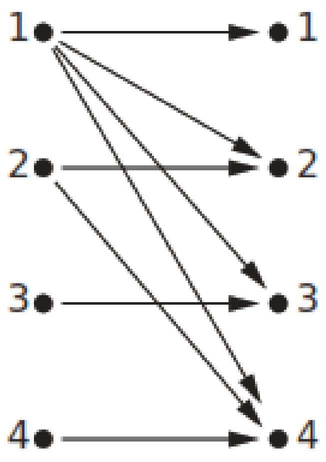
Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution: The pair (1, 1) is in R_1, R_3, R_4, and R_6; (1, 2) is in R_1 and R_6; (2, 1) is in R_2, R_5, and R_6; (1, −1) is in R_2, R_3, and R_6; and finally, (2, 2) is in R_1, R_3, and R_4. ◀

EXAMPLE 4 Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

graphically and in tabular form i



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

EXAMPLE 7 Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Which of the relations from Example 7 are symmetric and which are antisymmetric?

Which of the relations in Example 7 are transitive?

• Definition

Let R be a relation defined on a set A. R is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

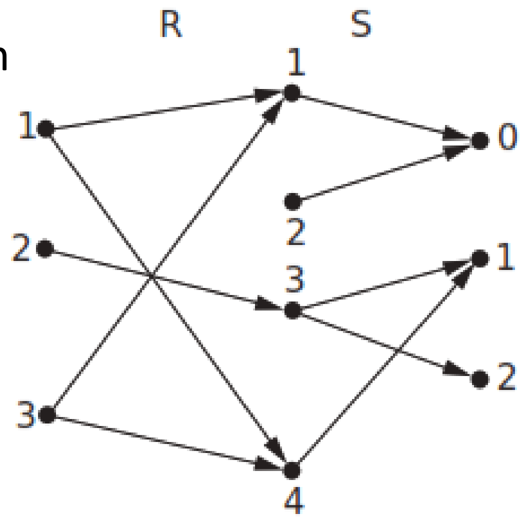
1. R is reflexive $\Leftrightarrow$ for all x in A, $(x, x) \in R$.
2. R is symmetric $\Leftrightarrow$ for all x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive $\Leftrightarrow$ for all x, y and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

EXAMPLE 20

What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution:

Draw arrow diagram



$1 \to 1 \to 0$ $(1, 0)$

$1 \to 4 \to 1$ $(1, 1)$

$2 \to 3 \to 1$ $(2, 1)$

$2 \to 3 \to 2$ $(2, 2)$

$3 \to 1 \to 0$ $(3, 0)$

$3 \to 4 \to 1$ $(3, 1)$

Constructing $S \circ R$.

$$\boxed{S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.}$$

EXAMPLE 3 Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

EXAMPLE 4 Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \quad \text{and} \quad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE 22 Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.

Solution:

$R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

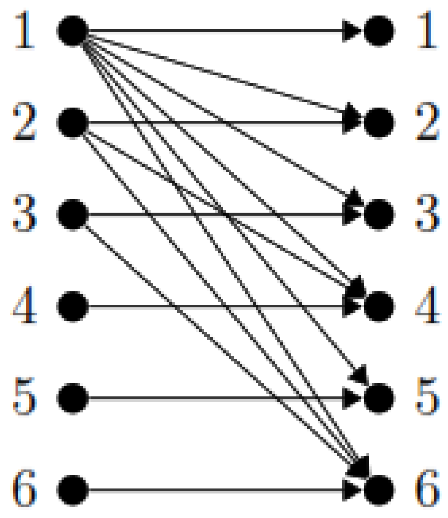
Exercises

2. a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.

b) Display this relation graphically, as was done in Example 4.

c) Display this relation in tabular form, as was done in Example 4.

Solution: **a)** $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,2)$, $(2,4)$, $(2,6)$, $(3,3)$, $(3,6)$, $(4,4)$, $(5,5)$, $(6,6)$



R	1	2	3	4	5	6
1	×	×	×	×	×	×
2		×		×		×
3			×			×
4				×		
5					×	
6						×

44. List the 16 different relations on the set $\{0, 1\}$.

46. Which of the 16 relations on $\{0, 1\}$, which you listed in Exercise 44, are

a) reflexive?
b) irreflexive?
c) symmetric?
d) antisymmetric?
e) asymmetric?
f) transitive?

Solution 44.

These are just the 16 different subsets of $\{(0,0),(0,1),(1,0),(1,1)\}$.

1. $\emptyset$
2. $\{(0,0)\}$
3. $\{(0,1)\}$
4. $\{(1,0)\}$
5. $\{(1,1)\}$
6. $\{(0,0),(0,1)\}$
7. $\{(0,0),(1,0)\}$
8. $\{(0,0),(1,1)\}$
9. $\{(0,1),(1,0)\}$
10. $\{(0,1),(1,1)\}$
11. $\{(1,0),(1,1)\}$
12. $\{(0,0),(0,1),(1,0)\}$
13. $\{(0,0),(0,1),(1,1)\}$
14. $\{(0,0),(1,0),(1,1)\}$
15. $\{(0,1),(1,0),(1,1)\}$
16. $\{(0,0),(0,1),(1,0),(1,1)\}$

14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$. **c)** $R_2 \circ R_1$. **d)** $R_1 \circ R_1$.

Solution:

$$R_1 \cup R_2 \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \qquad R_1 \cap R_2. \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

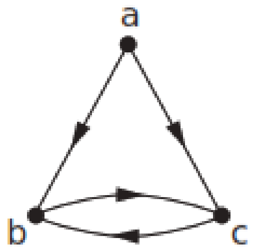
$$\text{Boolean product } \boldsymbol{M}_{R_1} \odot \boldsymbol{M}_{R_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\boldsymbol{M}_{R_1} \odot \boldsymbol{M}_{R_1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}. R_1 \circ R_1.$$

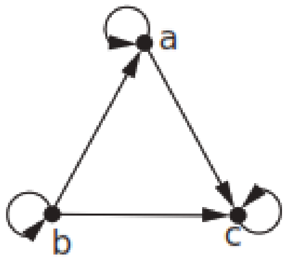
$$\boxed{\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \quad \text{and} \quad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.}$$

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs.

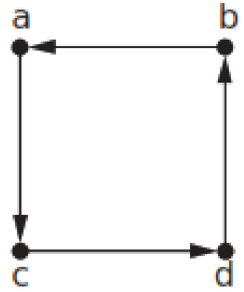
23.



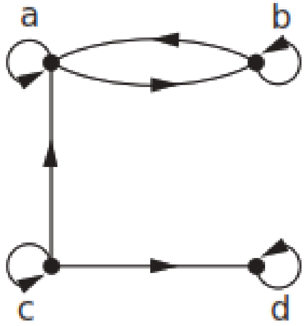
24.



25.



26.



Solution:

24) $\{(a,a),(a,c),(b,a),(b,b),(b,c),(c,c)\}$.

26) $\{(a,a),(a,b),(b,a),(b,b),(c,a),(c,c),(c,d),(d,d)\}$.

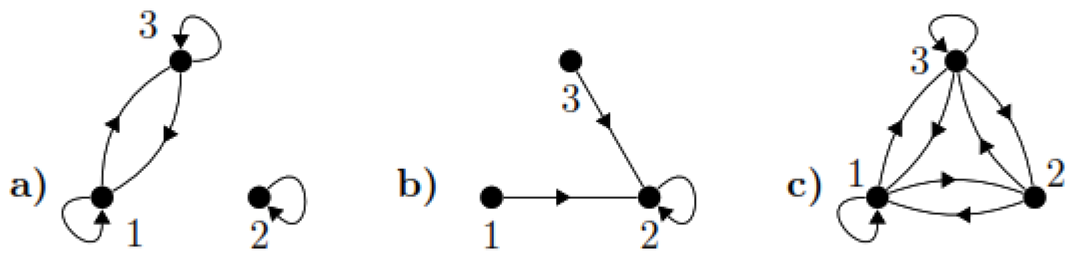
3. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

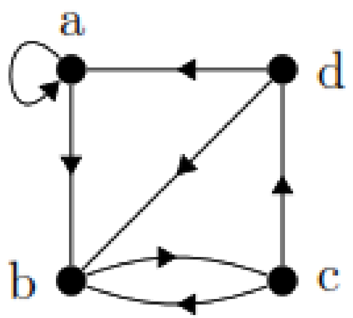
Draw the directed graph representing each of the relations from Exercise 3.

Solution:



22. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

Solution:



The Basics of Counting

SECTION 6.1

COMBINATORICS

Combinatorics is the mathematics of counting and arranging objects. Counting of objects with certain properties (enumeration) is required to solve many different types of problem.

Applications, include topics as diverse as codes, circuit design and algorithm complexity [and gambling]

Counting

Enumeration, the counting of objects with certain properties, is an important part of combinatorics.

We must count objects to solve many different types of problems. For example, counting is used to:

1. Determine number of ordered or unordered arrangement of objects.
2. Generate all the arrangements of a specified kind which is important in computer simulations.
3. Compute probabilities of events.
4. Analyze the chance of winning games, lotteries etc.
5. Determine the complexity of algorithms.

Section Summary

The Sum Rule

The Product Rule

The Subtraction Rule

The Division Rule

Examples, Examples, and Examples

Tree Diagrams

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways to do the second task, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

The Sum Rule in terms of sets.

The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when $A_i \cap A_j = \emptyset$ for all i, j.

The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

Basic Counting Principles: The Sum Rule

Example:

Suppose there are 7 different optional courses in Computer Science and 3 different optional courses in Mathematics. How many ways student can choose a course.

Solution: By the sum rule it follows that there are $7 + 3 = 10$ choices for a student who wants to take one optional course.

Basic Counting Principles: The Sum Rule

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

Basic Counting Principles: The Sum Rule

Example: A student can choose a computer project from one of the three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from?

Solution: The student can choose a project from the first list in 23 ways, from the second list in 15 ways, and from the third list in 19 ways. Hence, there are

$23 + 15 + 19 = 57$ projects to choose from.

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Product Rule in Terms of Sets

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, ..., and an element in A_m.

By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

The Product Rule

Example: How many ways a student can choose one optional course each from computer science and mathematics courses if there are 7 different optional courses in Computer Science and 3 different optional courses in Mathematics.

Solution:

A student who wants to take one optional course of each subject, there are $7 \times 3 = 21$ choices.

The Product Rule

Example: The chairs of an auditorium are to be labeled with two characters, a letter followed by a digit. What is the largest number of chairs that can be labeled differently?

Solution:

The procedure of labeling a chair consists of two events, namely,

Assigning one of the 26 letters: A, B, C, …, Z and

Assigning one of the 10 digits: 0, 1, 2, …, 9

By product rule, there are $26 \times 10 = 260$ different ways that a chair can be labeled by both a letter and a digit.

The Product Rule

Example: Find the number n of ways that an organization consisting of 15 members can elect a president, treasurer, and secretary. (assuming no person is elected to more than one position)

Solution:

The president can be elected in 15 different ways; following this, the treasurer can be elected in 14 different ways; and following this, the secretary can be elected in 13 different ways. Thus, by product rule, there are

$$n = 15 \times 14 \times 13 = 2730$$

different ways in which the organization can elect the officers.

The Product Rule

Example: There are four bus lines between A and B; and three bus lines between B and C.

Find the number of ways a person can travel:

a) By bus from A to C by way of B;

b) Round trip by bus from A to C by way of B;

c) Round trip by bus from A to C by way of B, if

the person does not want to use a bus line more than once.

The Product Rule

a) By bus from A to C by way of B;

Solution:

$$A \xrightarrow{4} B \xrightarrow{3} C.$$

There are 4 ways to go from A to B and 3 ways to go from B to C; hence there are $4 \times 3 = 12$ ways to go from A to C by way of B.

The Product Rule

b) Round trip by bus from A to C by way of B;

Solution:

The person will travel from A to B to C to B to A for the round trip. i.e. (A →B →C →B →A)

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{3} B \xrightarrow{4} A$$

The person can travel 4 ways from A to B and 3 way from B to C and back.

Thus there are 4 ×3 ×3 × 4 = 144 ways to travel the round trip.

The Product Rule

c) Round trip by bus from A to C by way of B, if the person does not want to use a bus line more than once.

Solution:

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

The person can travel 4 ways from A to B and 3 ways from B to C, but only 2 ways from C to B and 3 ways from B to A, since bus line cannot be used more than once. Hence there are

$4 \times 3 \times 2 \times 3 = 72$ ways

to travel the round trip without using a bus line more than once.

The Product Rule

Example: A bit string is a sequence of 0's and 1's.How many bit string are there of length 4?

Solution:

Each bit (binary digit) is either 0 or 1.

Hence, there are 2 ways to choose each bit. Since we have to choose four bits therefore,

$$2 \times 2 \times 2 \times 2 = 2^4 = 16$$

the product rule shows, there are a total of different bit strings of length four.

The Product Rule

Example: How many bit strings of length 8:

(i) begin with a 1?

(ii) begin and end with a 1?

Solution:

(i) If the first bit (left most bit) is a 1, then it can be filled in only one way. Each of the remaining seven positions in the bit string can be filled in 2 ways (i.e., either by 0 or 1). Hence, there are

different bit $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$

The Product Rule

(ii) begin and end with a 1?

Solution:

If the first and last bit in an 8 bit string is a 1, then only the intermediate six bits can be filled in 2 ways, i.e. by a 0 or 1. Hence there are

$$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 = 2^6 = 64$$

different bit strings of length 8 that begin and end with a 1.

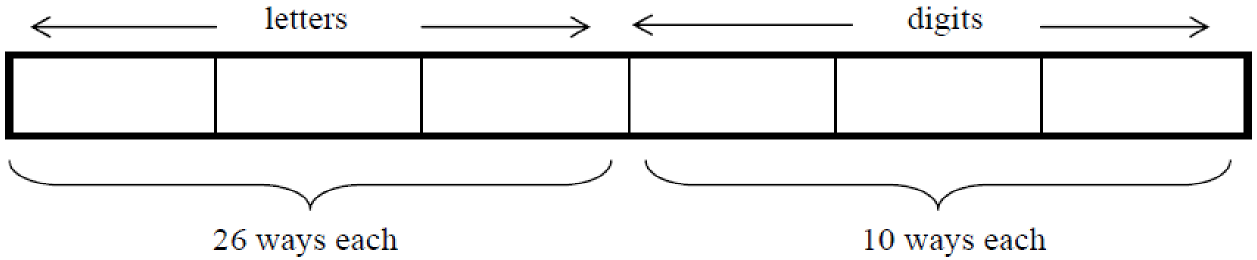
The Product Rule

Example: Suppose that an automobile license plate has three letters followed by three digits.

(a) How many different license plates are possible?

Solution:

Each of the three letters can be written in 26 different ways, and each of the three digits can be written in 10 different ways.



Hence, by the product rule, there is a total of

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$$

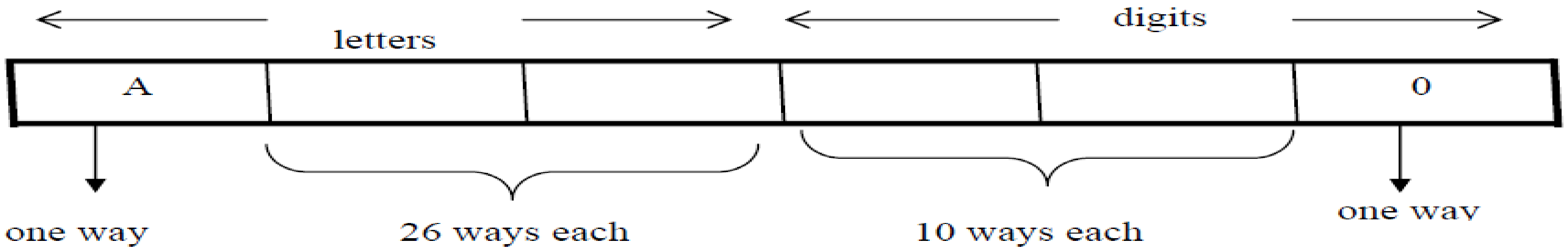
different License plates possible.

The Product Rule

(b) How many license plates could begin with A and end on 0?

Solution:

The first and last place can be filled in one way only, while each of second and third place can be filled in 26 ways and each of fourth and fifth place can be filled in 10 ways.



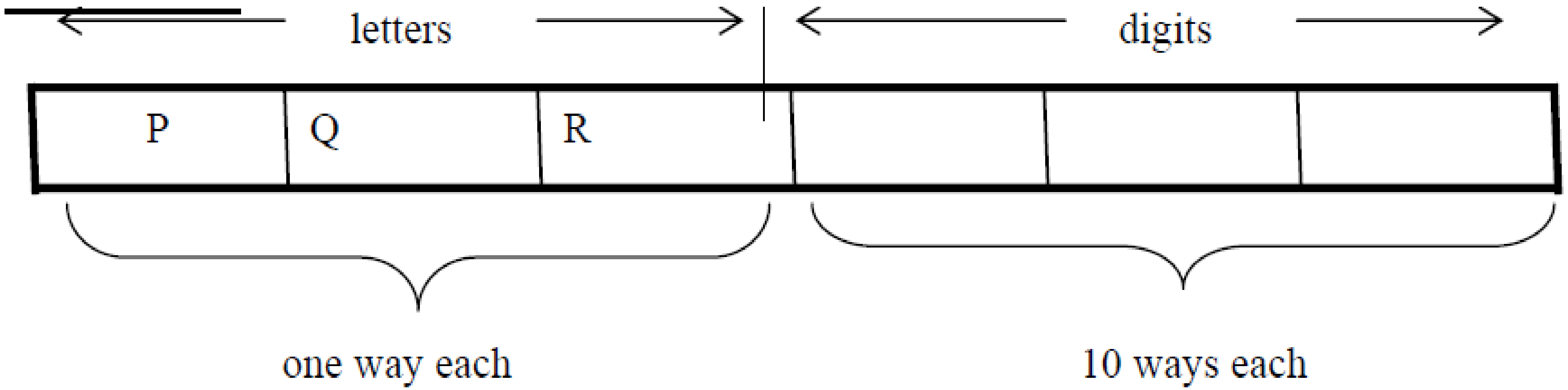
Number of license plates that begin with A and end in 0 are

$$1 \times 26 \times 26 \times 10 \times 10 \times 1 = 67600$$

The Product Rule

(c) How many license plates begin with PQR.

Solution:



Number of license plates that begin with PQR are

$$1 \times 1 \times 1 \times 10 \times 10 \times 10 = 1000 \text{ ways.}$$

The Product Rule

(d) How many license plates are possible in which all the letters and digits are distinct?

Solution:

The first letter place can be filled in 26 ways. Since, the second letter place should contain a different letter than the first, so it can be filled in 25 ways. Similarly, the third letter place can be filled in 24 ways. And the digits can be respectively filled in 10, 9, and 8 ways.

Hence;

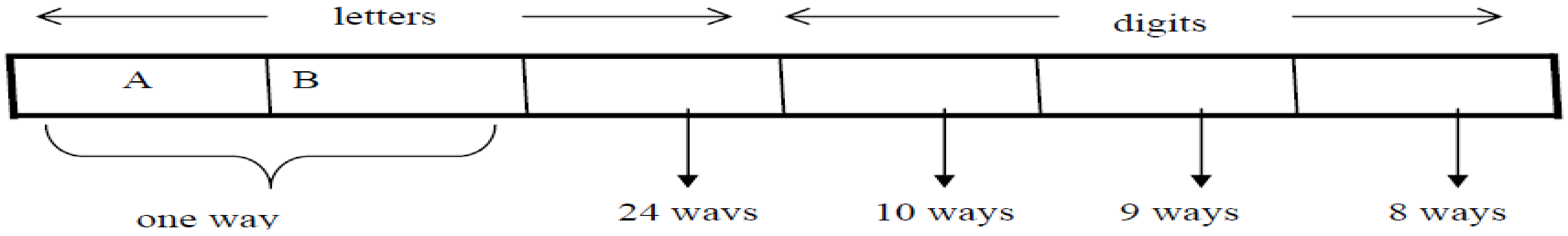
number of license plates in which all the letters and digits are distinct are

$$26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$$

The Product Rule

(e) How many license plates could begin with AB and have all three letters and digits distinct.

Solution:



The first two letters places are fixed (to be filled with A and B), so there is only one way to fill them. The third letter place should contain a letter different from A & B, so there

are 24 ways to fill it.

The three digit positions can be filled in 10 and 8 ways to have distinct digits. Hence, desired number of license plates are

$$1 \times 1 \times 24 \times 10 \times 9 \times 8 = 17280$$

Telephone Numbering Plan

Example: The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let *X* denote a digit from 0 through 9.
- Let *N* denote a digit from 2 through 9.
- Let *Y* denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was *NYX-NNX-XXXX*.
- In the new plan, the format is *NXX-NXX-XXXX*.

How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.

- There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format *NYX.*
- There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format *NXX.*
- There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format *NNX.*
- There are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format *XXXX.*

Number of old plan telephone numbers: $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$.

Number of new plan telephone numbers: $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$.

NUMBER OF ITERATIONS OF A NESTED LOOP

Example: Determine how many times the inner loop will be iterated when the following algorithm is implemented and run

For i: = 1 to 4

For j : = 1 to 3

[Statement in body of inner loop. None contain branching statements that lead out of the inner loop.]

next j

next i

Solution:

The outer loop is iterated four times, and during each iteration of the outer loop, there are three iterations of the inner loop.

Hence, by product rules the total number of iterations of inner loop is $4 \cdot 3 = 12$

Example: Determine how many times the inner loop will be iterated when the following algorithm is implemented and run.

for i = 5 to 50

for j: = 10 to 20

[Statement in body of inner loop. None contain branching statements that lead out of the inner loop.]

next j

next i

Solution:

The outer loop is iterated 50 - 5 + 1 = 46 times and during each iteration of the outer loop there are 20 - 10 + 1 = 11 iterations of the inner loop. Hence by product rule, the total number of iterations of the inner loop is $46 \times 11 = 506$.

Example: Determine how many times the inner loop will be iterated when the following algorithm is implemented and run.

for i: = 1 to 4

for j: = 1 to i

[Statements in body of inner loop. None contain branching statements that lead outside the loop.]

next j

next i

Solution:

The outer loop is iterated 4 times, but during each iteration of the outer loop, the inner loop iterates different number of times.

For first iteration of outer loop, inner loop iterates 1 times.

For second iteration of outer loop, inner loop iterates 2 times.

For third iteration of outer loop, inner loop iterates 3 times.

For fourth iteration of outer loop, inner loop iterates 4 times.

Hence, total number of iterations of inner loop = 1 + 2 + 3 + 4 = 10.

Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution:

First consider variable names one character in length. Since such names consist of a single letter, there are 26 variable names of length 1.

Next, consider variable names two characters in length. Since the first character is a letter, there are 26 ways to choose it. The second character is a digit, there are 10 ways to choose it. Hence, to construct variable name of two characters in length, there are $26 \times 10 = 260$ ways.

Finally, by sum rule, there are $26 + 260 = 286$ possible variable names in the programming language.

Combining the Sum and Product Rule

Example: A computer access code word consists of from one to three letters of English alphabets with repetitions allowed. How many different code words are possible.

Solution:

Number of code words of length 1 = 26^1

Number of code words of length 2 = 26^2

Number of code words of length 3 = 26^3

Hence, the total number of code words =

$$26^1 + 26^2 + 26^3 = 18,278$$

Counting Passwords

Combining the sum and product rule allows us to solve more complex problems.

Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.

  Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 36^6 ,and the number of strings with no digits is 26^6

Counting Passwords(Continued)

To find each of P6, P7, and P8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

P_6 = $36^6 - 26^6$ = 2,176,782,336 − 308,915,776 =1,867,866,560.

P_7 = $36^7 - 26^7$ = 78,364,164,096 − 8,031,810,176 = 70,332,353,920.

P_8 = $36^8 - 26^8$ = 2,821,109,907,456 − 208,827,064,576 =2,612,282,842,880.

Consequently, P = $P_6 + P_7 + P_8$ = 2,684,483,063,360.

Internet Addresses

Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

Class A Addresses: used for the largest networks, a 0,followed by a 7-bit netid and a 24-bit hostid.

Class B Addresses: used for the medium-sized networks, a 10,followed by a 14-bit netid and a 16-bit hostid.

Class C Addresses: used for the smallest networks, a 110,followed by a 21-bit netid and a 8-bit hostid.

- Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network.
- Hostids consisting of all 0s and all 1s are not available in any network.

Counting Internet Addresses

Example: How many different IPv4 addresses are available for computers on the internet?

Solution: Use both the sum and the product rule. Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.

- To find, x_A: $2^7 - 1 = 127$ netids. $2^{24} - 2 = 16{,}777{,}214$ hostids.

  $x_A = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178.$
- To find, x_B: $2^{14} = 16{,}384$ netids. $2^{16} - 2 = 16{,}534$ hostids.

  $x_B = 16{,}384 \cdot 16{,}534 = 1{,}073{,}709{,}056.$
- To find, x_C: $2^{21} = 2{,}097{,}152$ netids. $2^8 - 2 = 254$ hostids.

  $x_C = 2{,}097{,}152 \cdot 254 = 532{,}676{,}608.$
- Hence, the total number of available IPv4 addresses is

$$x = x_A + x_B + x_C$$
$$= 2{,}130{,}706{,}178 + 1{,}073{,}709{,}056 + 532{,}676{,}608$$
$$= 3{,}737{,}091{,}842.$$

Not Enough Today !!
The newer IPv6 protocol solves the problem of too few addresses.

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

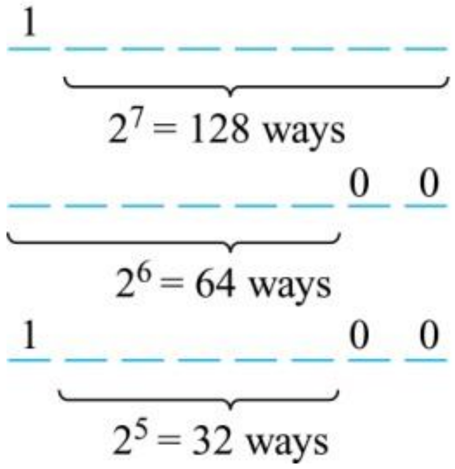
Counting Bit Strings

Example: How many bit strings of length eight start either with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.



Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?

Solution: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions.

Counting One-to-One Functions: How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2, etc. The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

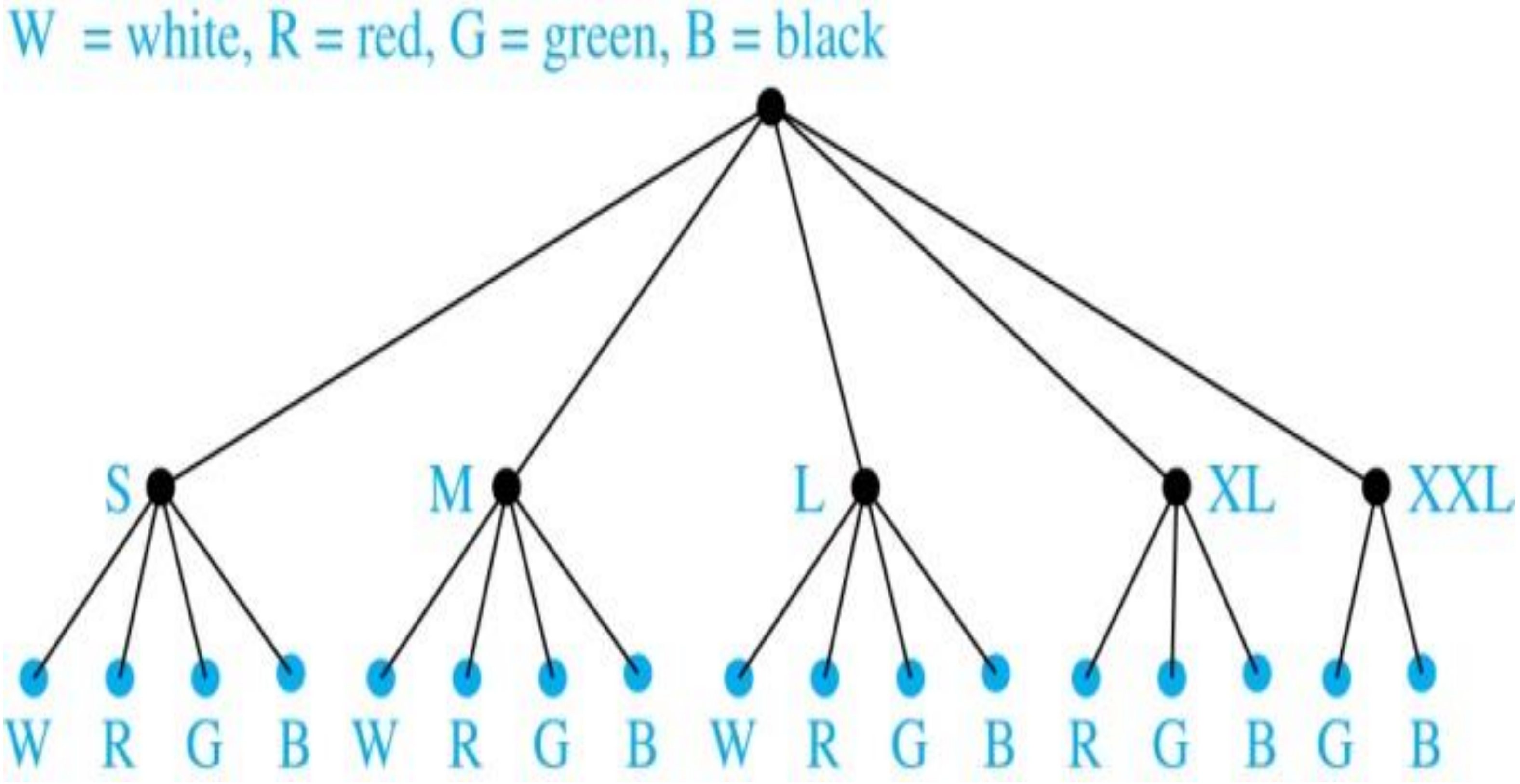
Tree Diagrams

Tree Diagrams: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

Example: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?

Tree Diagrams

Solution: Draw the tree diagram.



The store must stock 17 T-shirts.

Example:

Consider the following problem:

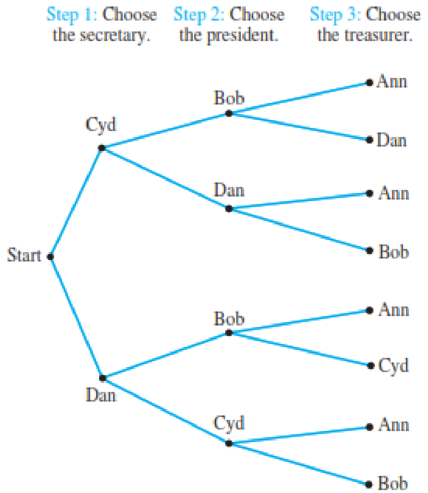
> Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

It is natural to try to solve this problem using the multiplication rule. A person might answer as follows:

> There are three choices for president (all except Ann), three choices for treasurer (all except the one chosen as president), and two choices for secretary (Cyd or Dan). Therefore, by the multiplication rule, there are $3 \cdot 3 \cdot 2 = 18$ choices in all.

Unfortunately, this analysis is incorrect. The number of ways to choose the secretary varies depending on who is chosen for president and treasurer. For instance, if Bob is chosen for president and Ann for treasurer, then there are two choices for secretary: Cyd and Dan. But if Bob is chosen for president and Cyd for treasurer, then there is just one choice for secretary: Dan. The clearest way to see all the possible choices is to construct the possibility tree, as is shown in Figure 9.2.3.

2nd way



First way

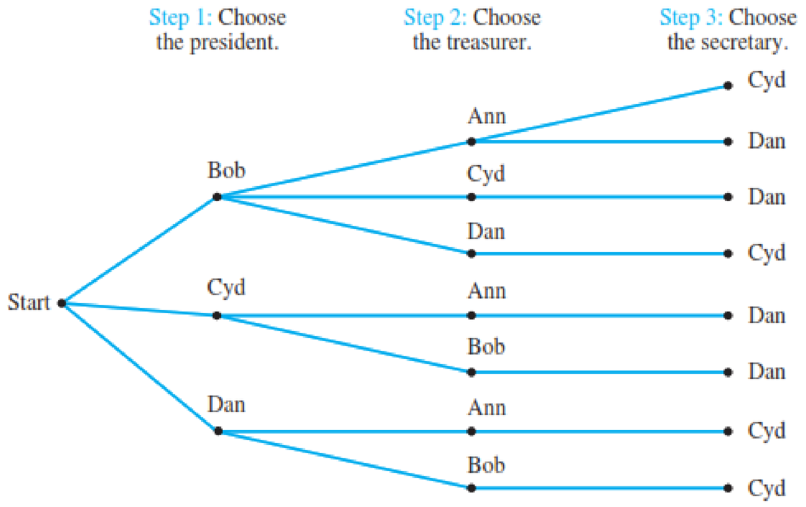


Figure 9.2.3

From the tree it is easy to see that there are only eight ways to choose a president, treasurer, and secretary so as to satisfy the given conditions.

The Pigeonhole Principle

SECTION 6.2

Section Summary

The Pigeonhole Principle

The Generalized Pigeonhole Principle

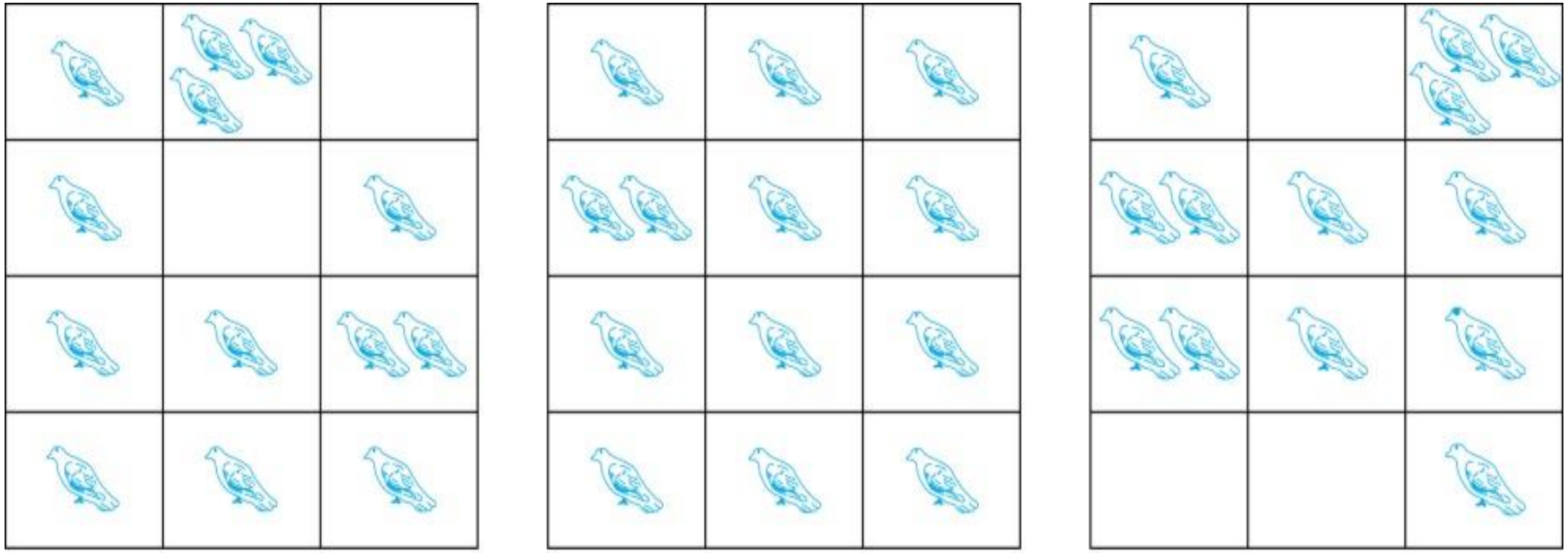
The Pigeonhole Principle

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects.

The Pigeonhole Principle

If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a) (b) (c)

The Pigeonhole Principle

Corollary 1: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f .
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $[N/k]$ objects.

Proof: We use a proof by contraposition. Suppose that none of the boxes contains more than $[N/k] - 1$ objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $[N/k] < [N/k] + 1$ has been used. This is a contradiction because there are a total of n objects.

Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays. $\lceil 367/366 \rceil = 2$

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example: In any set of 27 English , must be at least two that begin with the same letter, since there are 26 letters in the English alphabet. $\lceil 27/26 \rceil = 2$

The Generalized Pigeonhole Principle

Example: What is the minimum number of students required in a Discrete Mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F.

Solution:

The minimum number of students needed to guarantee that at least six students receive the same grade is the smallest integer N such that $\lceil N/K \rceil = \lceil N/5 \rceil = 6$. The smallest such integer is

$$N = K(\lceil N/K \rceil - 1) + 1 = 5(6-1)+1 = 5 \cdot 5 + 1 = 26.$$

Thus 26 is the minimum number of students needed to be sure that at least 6 students will receive the same grades.

Permutations and Combinations

SECTION 6.3

Section Summary

Permutations

Combinations

Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement $3,1,2$ is a permutation of S.
- The ordered arrangement $3,2$ is a 2-permutation of S.

The number of r-permutations of a set with n elements is denoted by $P(n,r)$.

The 2-permutations of $S = \{1,2,3\}$ are $1,2$; $1,3$; $2,1$; $2,3$; $3,1$; and $3,2$. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element.

Note that $P(n,0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Permutations of the Letters in a Word

a. How many ways can the letters in the word *COMPUTER* be arranged in a row?

b. How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

Permutations of Selected Letters of a Word

a. How many different ways can three of the letters of the word *BYTES* be chosen and written in a row?

b. How many different ways can this be done if the first letter must be *B*?

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Solution: The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are

$$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$$

possible ways to award the medals.

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$P(7,7) = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

P(6,6)= $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Counting Elements of a General Union

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

b. How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

Counting the Number of Elements in an Intersection

A professor in a discrete mathematics class passes out a form asking students to check all the mathematics and computer science courses they have recently taken. The finding is that out of a total of 50 students in the class,

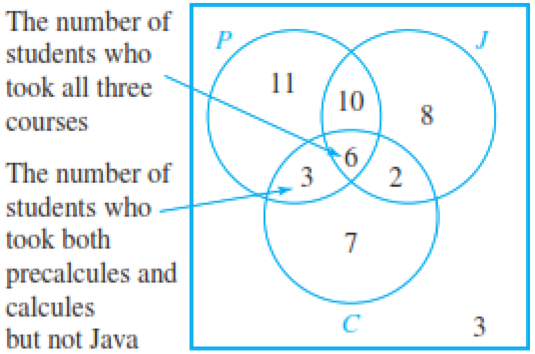
30 took precalculus;
18 took calculus;
26 took Java;
9 took both precalculus and calculus;
16 took both precalculus and Java;
8 took both calculus and Java;
47 took at least one of the three courses.

a. How many students did not take any of the three courses?

b. How many students took all three courses?

c. How many students took precalculus and calculus but not Java? How many students took precalculus but neither calculus nor Java?

Solution



Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$.

The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section*. 6.4)

Teams with Members of Two Types

Suppose the group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

c. How many five-person teams contain at most one man?

Combinations

Example:

Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

$C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Calculating the Number of Teams

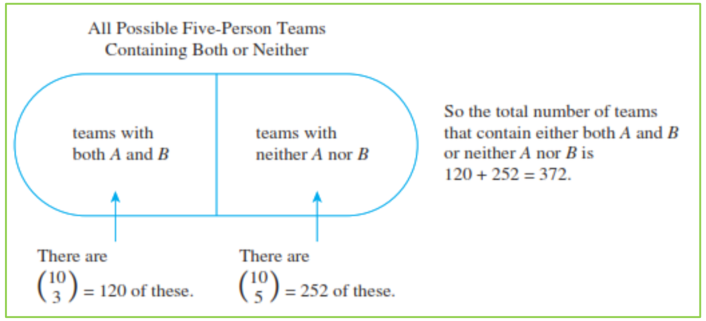
Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Teams That Contain Both or Neither

Suppose two members of the group of twelve insist on working as a pair—any team must contain either both or neither. How many five-person teams can be formed?

Solution Call the two members of the group that insist on working as a pair A and B. Then any team formed must contain both A and B or neither A nor B. The set of all possible

$$\begin{bmatrix}\text{number of teams containing}\\ \text{both } A \text{ and } B \text{ or}\\ \text{neither } A \text{ nor } B\end{bmatrix} = \begin{bmatrix}\text{number of teams}\\ \text{containing}\\ \text{both } A \text{ and } B\end{bmatrix} + \begin{bmatrix}\text{number of teams}\\ \text{containing}\\ \text{neither } A \text{ nor } B\end{bmatrix}$$
$$= 120 + 252 = 372.$$



Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n, r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!} .$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

Corollary 2: Let *n* and *r* be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}\ .$$

Hence, $C(n, r) = C(n, n - r)$.

◀

This result can be proved without using algebraic manipulation. →

Combinatorial Proofs

Definition 1: A *combinatorial proof* of an identity is a proof that uses one of the following methods.

- A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
- A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

- *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements.

- *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements.

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775\ .$$

THEOREM 4 The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations of a Set with Repeated Elements

Consider various ways of ordering the letters in the word *MISSISSIPPI*:

IIMSSPISSIP, *ISSSPMIIPIS*, *PIMISSSSIIP*, and so on.

How many distinguishable orderings are there?

Solution

$$\begin{aligned}
\begin{bmatrix}\text{number of ways to}\\ \text{position all the letters}\end{bmatrix} &= \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1} \\
&= \frac{11!}{4!\cancel{7!}}\cdot\frac{\cancel{7!}}{4!\cancel{3!}}\cdot\frac{\cancel{3!}}{2!\cancel{1!}}\cdot\frac{\cancel{1!}}{1!\cancel{0!}} \\
&= \frac{11!}{4!\cdot 4!\cdot 2!\cdot 1!} = 34{,}650.
\end{aligned}$$

Binomial Coefficients and Identities

SECTION 6.4

Section Summary

The Binomial Theorem

Pascal’s Identity and Triangle

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums.

Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$.

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.

To illustrate this idea, we first look at the process of expanding $(x + y)^3$.

$(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.

Terms of the form x^3, x^2y, xy^2, y^3 arise. The question is what are the coefficients?

- To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
- To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There $\binom{3}{2}$ are ways to do this and so the coefficient of x^2y is 3.
- To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There $\binom{3}{1}$ are ways to do this and so the coefficient of xy^2 is 3.
- To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.

We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Using the Binomial Theorem

Example:

What is the expansion of $(x + y)^4$?

Solution: From the binomial theorem it follows that

$$
\begin{aligned}
(x+y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j \\
&= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 \\
&= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.
\end{aligned}
$$

Using the Binomial Theorem

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^{j}.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With *n* ≥ 0,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}.$$

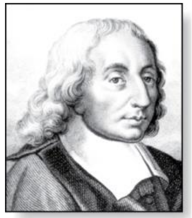
Proof (*combinatorial*): Consider the subsets of a set with *n* elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with *n* elements. Therefore the total is

$$\sum_{k=0}^{n} \binom{n}{k}.$$

Since, we know that a set with *n* elements has 2^n subsets, we conclude:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Blaise Pascal
(1623-1662)



Pascal's Identity

Pascal's Identity: If *n* and *k* are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof (*combinatorial*): Let *T* be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of *T* containing *k* elements. Each of these subsets either:

- contains *a* with $k - 1$ other elements, or
- contains *k* elements of *S* and not *a*.

There are

- $\binom{n}{k}$ subsets of *k* elements that contain *a*, since there are subsets of $k - 1$ elements of *S*,
- $\binom{n}{k}$ subsets of $\binom{n}{k}$ *k* elements of *T* that do not contain *a*, because there are $\binom{n}{k-1}$ subsets of k elements of S.

Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

◀

See Exercise 19 *for an algebraic proof.*

Pascal's Triangle

The *n*th row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$
$$\cdots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1\quad 1$$
$$1\quad 2\quad 1$$
$$1\quad 3\quad 3\quad 1$$
$$1\quad 4\quad 6\quad 4\quad 1$$
$$1\quad 5\quad 10\quad 10\quad 5\quad 1$$
$$1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1$$
$$1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1$$
$$1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1$$
$$\cdots$$

(b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Exercises

1. Find the expansion of $(x + y)^4$

a) using combinatorial reasoning, as in Example 1.

b) using the binomial theorem.

2. Find the expansion of $(x + y)^5$

a) using combinatorial reasoning, as in Example 1.

b) using the binomial theorem.

3. Find the expansion of $(x + y)^6$.

4. Find the coefficient of x^5y^8 in $(x + y)^{13}$.

5. How many terms are there in the expansion of $(x + y)^{100}$ after like terms are collected?

6. What is the coefficient of x^7 in $(1 + x)^{11}$?

7. What is the coefficient of x^9 in $(2 - x)^{19}$?

8. What is the coefficient of x^8y^9 in the expansion of $(3x + 2y)^{17}$?

9. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?

10. Use the binomial theorem to expand $(3x - y^2)^4$ into a sum of terms of the form cx^ay^b, where c is a real number and a and b are nonnegative integers.

11. Use the binomial theorem to expand $(3x^4 - 2y^3)^5$ into a sum of terms of the form cx^ay^b, where c is a real number and a and b are nonnegative integers.

12. Use the binomial theorem to find the coefficient of x^ay^b in the expansion of $(5x^2 + 2y^3)^6$, where

a) $a = 6, b = 9$.

b) $a = 2, b = 15$.

c) $a = 3, b = 12$.

d) $a = 12, b = 0$.

e) $a = 8, b = 9$.