(5) Axiom 3
(6) Axiom 5
(7) Axiom 4

True-False Exercises

(a) True. This is a part of Definition 1.

(b) False. Example 1 discusses a vector space containing only one vector.

(c) False. By part (d) of Theorem 4.1.1, if $k\mathbf{u} = \mathbf{0}$ then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

(d) False. Axiom 6 fails to hold if $k < 0$. (Also, Axiom 4 fails to hold.)

(e) True. This follows from part (c) of Theorem 4.1.1.

(f) False. This function must have a value of zero at *every* point in $(-\infty,\infty)$.

4.2 Subspaces

1. **(a)** Let W be the set of all vectors of the form $(a,0,0)$, i.e. all vectors in R^3 with last two components equal to zero.
This set contains at least one vector, e.g. $(0,0,0)$.
Adding two vectors in W results in another vector in W: $(a,0,0)+(b,0,0)=(a+b,0,0)$ since the result has zeros as the last two components.
Likewise, a scalar multiple of a vector in W is also in W: $k(a,0,0)=(ka,0,0)$ - the result also has zeros as the last two components.
According to Theorem 4.2.1, W is a subspace of R^3.

(b) Let W be the set of all vectors of the form $(a,1,1)$, i.e. all vectors in R^3 with last two components equal to one. The set W is not closed under the operation of vector addition since $(a,1,1)+(b,1,1)=(a+b,2,2)$ does not have ones as its last two components thus it is outside W.
According to Theorem 4.2.1, W is not a subspace of R^3.

(c) Let W be the set of all vectors of the form (a,b,c), where $b = a + c$.
This set contains at least one vector, e.g. $(0,0,0)$. (The condition $b = a + c$ is satisfied when $a = b = c = 0$.)
Adding two vectors in W results in another vector in W
$(a,a+c,c)+(a',a'+c',c')=(a+a',\ a+c+a'+c',\ c+c')$ since in this result, the second component is the sum of the first and the third: $a+c+a'+c'=(a+a')+(c+c')$.
Likewise, a scalar multiple of a vector in W is also in W: $k(a,a+c,c)=(ka,k(a+c),kc)$ since in this result, the second component is once again the sum of the first and the third:

$k(a+c) = ka + kc$.

According to Theorem 4.2.1, W is a subspace of R^3.

2. **(a)** Let W be the set of all vectors of the form (a,b,c), where $b = a + c + 1$. The set W is not closed under the operation of vector addition, since in the result of the following addition of two vectors from W

$$(a, a+c+1, c) + (a', a'+c'+1, c') = (a+a',\ a+c+a'+c'+2,\ c+c')$$

the second component does not equal to the sum of the first, the third, and 1:

$a+c+a'+c'+2 \neq (a+a') + (c+c') + 1$. Consequently, this result is not a vector in W.

According to Theorem 4.2.1, W is not a subspace of R^3.

(b) Let W be the set of all vectors of the form $(a,b,0)$, i.e. all vectors in R^3 with last component equal to zero.

This set contains at least one vector, e.g. $(0,0,0)$.

Adding two vectors in W results in another vector in W

$(a,b,0) + (a',b',0) = (a+a', b+b', 0)$ since the result has 0 as the last component.

Likewise, a scalar multiple of a vector in W is also in W: $k(a,b,0) = (ka,kb,0)$ - the result also has 0 as the last component.

According to Theorem 4.2.1, W is a subspace of R^3.

(c) Let W be the set of all vectors of the form (a,b,c), where $a+b=7$. The set W is not closed under the operation of vector addition, since in the result of the following addition of two vectors from W we obtain

$(a,b,c) + (a',b',c') = (a+a', b+b',\ c+c')$ where

$a+a'+b+b' = a+b+a'+b' = 7 + 7 = 14$. Consequently, this result is not a vector in W.

According to Theorem 4.2.1, W is not a subspace of R^3.

3. **(a)** Let W be the set of all $n \times n$ diagonal matrices.

This set contains at least one matrix, e.g. the zero $n \times n$ matrix.

Adding two matrices in W results in another $n \times n$ diagonal matrix, i.e. a matrix in W:

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

Likewise, a scalar multiple of a matrix in W is also in W:

$$k\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & 0 & \cdots & 0 \\ 0 & ka_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ka_{nn} \end{bmatrix}$$

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(b) Let W be the set of all $n \times n$ matrices such whose determinant is zero. We shall show that W is not closed under the operation of matrix addition. For instance, consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ - both have determinant equal 0, therefore both matrices are in W. However, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has nonzero determinant, thus it is outside W.

According to Theorem 4.2.1, W is not a subspace of M_{nn}.

(c) Let W be the set of all $n \times n$ matrices with zero trace.
This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.
Let us assume $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are both in W, i.e. $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$ and $\operatorname{tr}(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$.
Since $\operatorname{tr}(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn})$
$= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = 0 + 0 = 0$, it follows that $A + B$ is in W.
A scalar multiple of the same matrix A with a scalar k has $\operatorname{tr}(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = 0$ therefore kA is in W as well.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(d) Let W be the set of all symmetric $n \times n$ matrices (i.e., $n \times n$ matrices such that $A^T = A$).
This set contains at least one matrix, e.g., I_n is in W.

Let us assume A and B are both in W, i.e. $A^T = A$ and $B^T = B$. By Theorem 1.4.8(b), their sum satisfies $(A + B)^T = A^T + B^T = A + B$ therefore W is closed under addition.

From Theorem 1.4.8(d), a scalar multiple of a symmetric matrix is also symmetric: $(kA)^T = kA^T = kA$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

4. **(a)** Let W be the set of all $n \times n$ matrices such that $A^T = -A$.
This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.
Let us assume A and B are both in W, i.e. $A^T = -A$ and $B^T = -B$. By Theorem 1.4.8(b), their sum satisfies $(A + B)^T = A^T + B^T = -A - B = -(A + B)$ therefore W is closed under addition.

From Theorem 1.4.8(d), we have $(kA)^T = kA^T = k(-A) = -kA$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(b) Let W be the set of $n \times n$ matrices for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. It follows from Theorem 1.5.3 that the set W consists of all $n \times n$ matrices that are invertible. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{nn}.

(c) Let B be some fixed $n \times n$ matrix, and let W be the set of all $n \times n$ matrices A such that $AB = BA$. This set contains at least one matrix, e.g., I_n is in W.
Let us assume A and C are both in W, i.e. $AB = BA$ and $CB = BC$. By Theorem 1.4.1(d,e), their sum satisfies $(A+C)B = AB + CB = BA + BC = B(A+C)$ therefore W is closed under addition.
From Theorem 1.4.1(m), we have $(kA)B = k(AB) = k(BA) = B(kA)$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(d) Let W be the set of all invertible $n \times n$ matrices (i.e., $n \times n$ matrices such that A^{-1} exists). This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{nn}.

5. **(a)** Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
This set contains at least one polynomial, $0 + 0x + 0x^2 + 0x^3 = 0$.
Adding two polynomials in W results in another polynomial in W:
$$(0 + a_1x + a_2x^2 + a_3x^3) + (0 + b_1x + b_2x^2 + b_3x^3)$$
$$= 0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3.$$
Likewise, a scalar multiple of a polynomial in W is also in W:
$$k(0 + a_1x + a_2x^2 + a_3x^3) = 0 + (ka_1)x + (ka_2)x^2 + (ka_3)x^3.$$
According to Theorem 4.2.1, W is a subspace of P_3.

(b) Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$, i.e. all polynomials that can be expressed in the form $-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3$.
Adding two polynomials in W results in another polynomial in W
$$(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3) + (-b_1 - b_2 - b_3 + b_1x + b_2x^2 + b_3x^3)$$
$$= (-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$$
since we have $(-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0$.
Likewise, a scalar multiple of a polynomial in W is also in W
$$k(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3) = -ka_1 - ka_2 - ka_3 + ka_1x + ka_2x^2 + ka_3x^3$$

since it meets the condition $(-ka_1 - ka_2 - ka_3) + (ka_1) + (ka_2) + (ka_3) = 0$.
According to Theorem 4.2.1, W is a subspace of P_3.

6. **(a)** Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers. The set W is not closed under the operation of scalar multiplication, e.g., the scalar product of the polynomial x^3 in W by $k = \pi$ is πx^3, which is not in W.
According to Theorem 4.2.1, W is not a subspace of P_3.

(b) The set of all polynomials of degree ≤ 1 is a subset of P_3. It is also a vector space (called P_1) with same operations of addition and scalar multiplication as those defined in P_3. By Definition 1, we conclude that P_1 is a subspace of P_3.

7. **(a)** Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 0$.
This set contains at least one function, e.g., the constant function $f(x) = 0$.
Assume we have two functions f and g in W, i.e., $f(0) = g(0) = 0$. Their sum $f + g$ is also a function in $F(-\infty,\infty)$ and satisfies $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$ therefore W is closed under addition.
A scalar multiple of a function f in W, kf, is also a function in $F(-\infty,\infty)$ for which $(kf)(0) = k(f(0)) = 0$ making W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of $F(-\infty,\infty)$.

(b) Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 1$.
We will show that W is not closed under addition. For instance, let $f(x) = 1$ and $g(x) = \cos x$ be two functions in W. Their sum, $f + g$, is not in W since $(f+g)(0) = f(0) + g(0) = 1 + 1 = 2$.
We conclude that W is not a subspace of $F(-\infty,\infty)$.

8. **(a)** Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(-x) = f(x)$.
This set contains at least one function, e.g., the constant function $f(x) = 0$.
Assume we have two functions f and g in W, i.e., $f(-x) = f(x)$ and $g(-x) = g(x)$. Their sum $f + g$ is also a function in $F(-\infty,\infty)$ and satisfies $(f+g)(-x) = f(-x) + g(-x) =$ $f(x) + g(x) = (f+g)(x)$ therefore W is closed under addition.
A scalar multiple of a function f in W, kf, is also a function in $F(-\infty,\infty)$ for which $(kf)(-x) = k(f(-x)) = k(f(x)) = (kf)(x)$ making W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of $F(-\infty,\infty)$.

It is also closed under scalar multiplication because $k(x_1,y_1,z_1)=((a)(kt_1),(b)(kt_1),(c)(kt_1))$.
It follows from Theorem 4.2.1 that L is a subspace of R^3.

19. **(a)** The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ therefore the solution are $x=-\frac{1}{2}t$, $y=-\frac{3}{2}t$, $z=t$. These are parametric equations of a line through the origin.

(b) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ therefore the only solution is $x=y=z=0$ - the origin.

(c) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which corresponds to an equation of a plane through the origin $x-3y+z=0$.

(d) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the solutions are $x=-3t$, $y=-2t$, $z=t$. These are parametric equations of a line through the origin.

21. Let W denote the set of all continuous functions $f=f(x)$ on $[a,b]$ such that $\int_a^b f(x)\,dx=0$.
This set contains at least one function $f(x)\equiv 0$.
Let us assume $\mathbf{f}=f(x)$ and $\mathbf{g}=g(x)$ are functions in W. From calculus,
$\int_a^b f(x)+g(x)dx=\int_a^b f(x)dx+\int_a^b g(x)dx=0$ and $\int_a^b kf(x)dx=k\int_a^b f(x)dx=0$ therefore both $\mathbf{f}+\mathbf{g}$ and $k\mathbf{f}$ are in W for any scalar k. According to Theorem 4.2.1, W is a subspace of $C[a,b]$.

23. Since $T_A: R^3 \to R^m$, it follows from Theorem 4.2.5 that the kernel of T_A must be a subspace of R^3. Hence, according to Table 1 the kernel can be one of the following four geometric obects:

- the origin,
- a line through the origin,
- a plane through the origin,
- R^3.

25. Let W be the set of all function. of the form $x(t)=c_1\cos\omega t+c_2\sin\omega t$ - W is a subset of $C^\infty(-\infty,\infty)$.
This set contains at least one function $x(t)\equiv 0$.