QR-Decomposition

In recent years a numerical algorithm based on the Gram–Schmidt process, and known as ***QR-decomposition***, has assumed growing importance as the mathematical foundation for a wide variety of numerical algorithms, including those for computing eigenvalues of large matrices. The technical aspects of such algorithms are discussed in books that specialize in the numerical aspects of linear algebra. However, we will discuss some of the underlying ideas here. We begin by posing the following problem.

> **Problem** If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram–Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ and that Q has orthonormal column vectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$. Thus, A and Q can be written in partitioned form as

$$A = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n] \quad \text{and} \quad Q = [\mathbf{q}_1 \mid \mathbf{q}_2 \mid \cdots \mid \mathbf{q}_n]$$

It follows from Theorem 6.3.2(b) that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are expressible in terms of the vectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ as

$$\begin{aligned}
\mathbf{u}_1 &= \langle \mathbf{u}_1, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_1, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_1, \mathbf{q}_n \rangle \mathbf{q}_n \\
\mathbf{u}_2 &= \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_2, \mathbf{q}_n \rangle \mathbf{q}_n \\
&\ \ \vdots \\
\mathbf{u}_n &= \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_n, \mathbf{q}_n \rangle \mathbf{q}_n
\end{aligned}$$

Recalling from Section 1.3 (Example 9) that the jth column vector of a matrix product is a linear combination of the column vectors of the first factor with coefficients coming from the jth column of the second factor, it follows that these relationships can be expressed in matrix form as

$$[\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n] = [\mathbf{q}_1 \mid \mathbf{q}_2 \mid \cdots \mid \mathbf{q}_n] \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{q}_2 \rangle & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{q}_n \rangle & \langle \mathbf{u}_2, \mathbf{q}_n \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix}$$

or more briefly as

$$A = QR \tag{15}$$

where R is the second factor in the product. However, it is a property of the Gram–Schmidt process that for $j \geq 2$, the vector $\mathbf{q}_j$ is orthogonal to $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{j-1}$. Thus, all entries below the main diagonal of R are zero, and R has the form

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix} \tag{16}$$

We leave it for you to show that R is invertible by showing that its diagonal entries are nonzero. Thus, Equation (15) is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R. We call Equation (15) a ***QR-decomposition of A***. In summary, we have the following theorem.

Theorem 6.3.7

***QR*-Decomposition**

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

EXAMPLE 10 | *QR*-Decomposition of a 3×3 Matrix

Find a QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution The column vectors of A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Applying the Gram–Schmidt process with normalization to these column vectors yields the orthonormal vectors (see Example 8)

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, it follows from Formula (16) that R is

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

from which it follows that a QR-decomposition of A is

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

Question:

44. Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 6 & 1 & -5 \\ 2 & 1 & 1 \\ -2 & -2 & 5 \\ 6 & 8 & -7 \end{bmatrix}$$

Solution:

The reduced row echelon form of the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the three column vectors of the original matrix, $\mathbf{u}_1 = (6, 2, -2, 6)$, $\mathbf{u}_2 = (1, 1, -2, 8)$, and $\mathbf{u}_3 = (-5, 1, 5, -7)$ form a basis for the column space. Applying the Gram-Schmidt process yields an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

$$\mathbf{v}_1 = \mathbf{u}_1 = (6,2,-2,6)$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \tfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1,1,-2,8) - \tfrac{6+2+4+48}{36+4+4+36}(6,2,-2,6) = (1,1,-2,8) - \tfrac{3}{4}(6,2,-2,6)$$

$$= \left(-\tfrac{7}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{7}{2}\right)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \tfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \tfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (-5,1,5,-7) - \tfrac{-30+2-10-42}{36+4+4+36}(6,2,-2,6) - \tfrac{\frac{35}{2}-\frac{1}{2}-\frac{5}{2}-\frac{49}{2}}{\frac{49}{4}+\frac{1}{4}+\frac{1}{4}+\frac{49}{4}}\left(-\tfrac{7}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{7}{2}\right)$$

$$= (-5,1,5,-7) + (6,2,-2,6) + \tfrac{2}{5}\left(-\tfrac{7}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{7}{2}\right) = \left(-\tfrac{2}{5}, \tfrac{14}{5}, \tfrac{14}{5}, \tfrac{2}{5}\right)$$

Question:

In Exercises 45–48, we obtained the column vectors of Q by applying the Gram–Schmidt process to the column vectors of A. Find a QR-decomposition of the matrix A.

48. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$

Solution:

Let $\mathbf{u}_1=(1,1,0)$, $\mathbf{u}_2=(2,1,3)$, $\mathbf{u}_3=(1,1,1)$, $\mathbf{q}_1=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$, $\mathbf{q}_2=\left(\frac{\sqrt{2}}{2\sqrt{19}},\ -\frac{\sqrt{2}}{2\sqrt{19}},\ \frac{3\sqrt{2}}{\sqrt{19}}\right)$, and

$\mathbf{q}_3=\left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right)$. A QR-decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R=\begin{bmatrix}\langle\mathbf{u}_1,\ \mathbf{q}_1\rangle & \langle\mathbf{u}_2,\ \mathbf{q}_1\rangle & \langle\mathbf{u}_3,\ \mathbf{q}_1\rangle\\ 0 & \langle\mathbf{u}_2,\ \mathbf{q}_2\rangle & \langle\mathbf{u}_3,\ \mathbf{q}_2\rangle\\ 0 & 0 & \langle\mathbf{u}_3,\ \mathbf{q}_3\rangle\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+0 & \frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}}+0 & \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+0\\ 0 & \frac{2\sqrt{2}}{2\sqrt{19}}-\frac{\sqrt{2}}{2\sqrt{19}}+\frac{9\sqrt{2}}{\sqrt{19}} & \frac{\sqrt{2}}{2\sqrt{19}}-\frac{\sqrt{2}}{2\sqrt{19}}+\frac{3\sqrt{2}}{\sqrt{19}}\\ 0 & 0 & -\frac{3}{\sqrt{19}}+\frac{3}{\sqrt{19}}+\frac{1}{\sqrt{19}}\end{bmatrix}$$

$$=\begin{bmatrix}\sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2}\\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}}\\ 0 & 0 & \frac{1}{\sqrt{19}}\end{bmatrix}.$$

Question:

49. Find a QR-decomposition of the matrix

$$A=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 1\\ 1 & 0 & 1\\ -1 & 1 & 1\end{bmatrix}$$

Solution:

In partitioned form, $A=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 1\\ 1 & 0 & 1\\ -1 & 1 & 1\end{bmatrix}=\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix}$. By inspection, $\mathbf{u}_3=\mathbf{u}_1+2\mathbf{u}_2$, so the column vectors of A are not linearly independent and A does not have a QR-decomposition.