Linear Independence

Definition 1

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a ***linearly independent set*** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be ***linearly dependent***. If S has only one vector, we will agree that it is linearly independent if and only if that vector is nonzero.

Theorem 4.4.1

A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

EXAMPLE 1 | Linear Independence of the Standard Unit Vectors in R^n

The most basic linearly independent set in R^n is the set of standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

To illustrate this in R^3, consider the standard unit vectors

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

To prove linear independence we must show that the only coefficients satisfying the vector equation

$$k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, k_3 = 0$. But this becomes evident by writing this equation in its component form

$$(k_1, k_2, k_3) = (0, 0, 0)$$

You should have no trouble adapting this argument to establish the linear independence of the standard unit vectors in R^n.

EXAMPLE 2 | Linear Independence in R^3

Determine whether the vectors

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1) \tag{2}$$

are linearly independent or linearly dependent in R^3.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0} \tag{3}$$

can be satisfied with coefficients that are not all zero. To see whether this is so, let us rewrite (3) in the component form

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned} \tag{4}$$

Thus, our problem reduces to determining whether this system has nontrivial solutions. There are various ways to do this; one possibility is to simply solve the system, which yields

$$k_1 = -\tfrac{1}{2}t, \quad k_2 = -\tfrac{1}{2}t, \quad k_3 = t$$

(we omit the details). This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent. A second method for establishing the linear dependence is to take advantage of the fact that the coefficient matrix

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

is square and compute its determinant. We leave it for you to show that $\det(A) = 0$ from which it follows that (4) has nontrivial solutions by parts (*b*) and (*g*) of Theorem 2.3.8.

Because we have established that the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ in (2) are linearly dependent, we know that at least one of them is a linear combination of the others. We leave it for you to confirm, for example, that

$$\mathbf{v}_3 = \tfrac{1}{2}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2$$

EXAMPLE 3 | Linear Independence in R^4

Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

in R^4 are linearly dependent or linearly independent.

Solution The linear independence or linear dependence of these vectors is determined by whether there exist nontrivial solutions of the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

or, equivalently, of

$$k_1(1, 2, 2, -1) + k_2(4, 9, 9, -4) + k_3(5, 8, 9, -5) = (0, 0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

We leave it for you to show that this system has only the trivial solution

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

from which you can conclude that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

EXAMPLE 4 | An Important Linearly Independent Set in P_n

Show that the polynomials

$$1, \quad x, \quad x^2, \ldots, \quad x^n$$

form a linearly independent set in P_n.

Solution For convenience, let us denote the polynomials as

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \ldots, \quad \mathbf{p}_n = x^n$$

We must show that the only coefficients satisfying the vector equation

$$a_0\mathbf{p}_0 + a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + \cdots + a_n\mathbf{p}_n = \mathbf{0} \tag{5}$$

are

$$a_0 = a_1 = a_2 = \cdots = a_n = 0$$

But (5) is equivalent to the statement that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0 \tag{6}$$

for all x in $(-\infty, \infty)$, so we must show that this is true if and only if each coefficient in (6) is zero. To see that this is so, recall from algebra that a nonzero polynomial of degree n has at most n distinct roots. That being the case, each coefficient in (6) must be zero, for otherwise the left side of the equation would be a nonzero polynomial with infinitely many roots. Thus, (5) has only the trivial solution.

EXAMPLE 5 | Linear Independence of Polynomials

Determine whether the polynomials

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in P_2.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0} \tag{7}$$

can be satisfied with coefficients that are not all zero. To see whether this is so, let us rewrite (7) in its polynomial form

$$k_1(1 - x) + k_2(5 + 3x - 2x^2) + k_3(1 + 3x - x^2) = 0 \tag{8}$$

or, equivalently, as

$$(k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

Since this equation must be satisfied by all x in $(-\infty, \infty)$, each coefficient must be zero (as explained in the previous example). Thus, the linear dependence or independence of the given polynomials hinges on whether the following linear system has a nontrivial solution:

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned} \tag{9}$$

We leave it for you to show that this linear system has nontrivial solutions either by solving it directly or by showing that the coefficient matrix has determinant zero. Thus, the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent.

Theorem 4.4.2

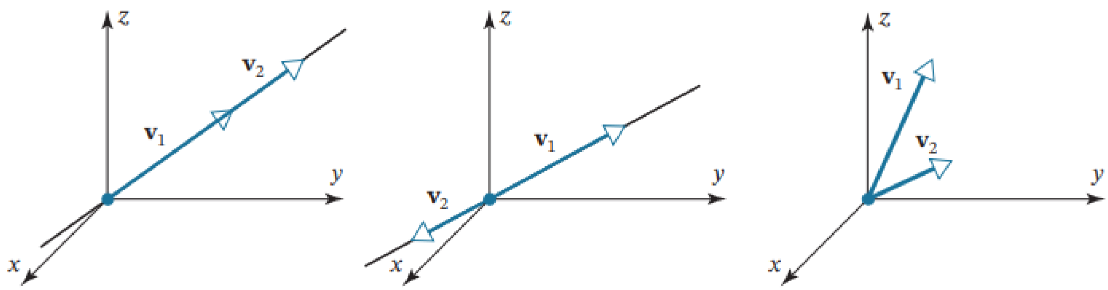
(*a*) A set with finitely many vectors that contains **0** is linearly dependent.

(*b*) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

A Geometric Interpretation of Linear Independence:

Linear independence has the following useful geometric interpretations in R^2 and R^3:

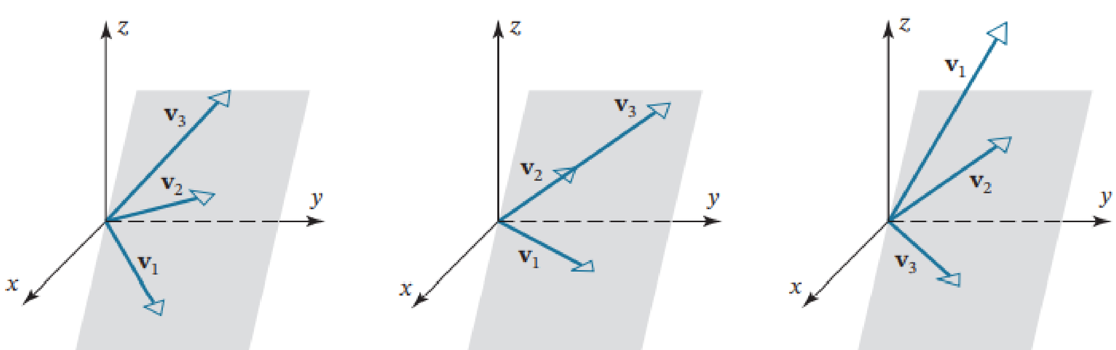
- Two vectors in R^2 or R^3 are linearly independent if and only if they do not lie on the same line when they have their initial points at the origin. Otherwise one would be a scalar multiple of the other (**Figure 4.4.3**).



(*a*) Linearly dependent **(*b*)** Linearly dependent **(*c*)** Linearly independent

FIGURE 4.4.3

- Three vectors in R^3 are linearly independent if and only if they do not lie in the same plane when they have their initial points at the origin. Otherwise at least one would be a linear combination of the other two (**Figure 4.4.4**).



(*a*) Linearly dependent **(*b*)** Linearly dependent **(*c*)** Linearly independent

FIGURE 4.4.4

Theorem 4.4.3

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

Question:

5. In each part, determine whether the matrices are linearly independent or dependent.

a. $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ in M_{23}

Solution:

(a) The matrix equation $a\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{ccccccc} 1a & + & 1b & + & 0c & = & 0 \\ 0a & + & 2b & + & 1c & = & 0 \\ 1a & + & 2b & + & 2c & = & 0 \\ 2a & + & 1b & + & 1c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given matrices are linearly independent.

(b) By inspection, the matrix equation $a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has only the trivial solution $a = b = c = 0$. We conclude that the given matrices are linearly independent.

Question:

6. Determine all values of k for which the following matrices are linearly independent in M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

Solution:

The matrix equation $a\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} + b\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix} + c\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1a & - & 1b & + & 2c & = & 0 \\ 0a & + & 0b & + & 0c & = & 0 \\ 1a & + & kb & + & 1c & = & 0 \\ ka & + & 1b & + & 3c & = & 0 \end{array}$$

Omitting the second equation (which imposes no restrictions on the unknowns), we obtain the coefficient matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix}$. Performing elementary row operations

- add -1 times the first row to the second row,
- add $-k$ times the first row to the third row, and
- add -1 times the second row to the third row

yields $B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1+k & -1 \\ 0 & 0 & 4-2k \end{bmatrix}$. We have $\det(A) = \det(B) = (1+k)(4-2k)$ therefore by Theorem 2.3.8, the system has only the trivial solution, whenever $(1+k)(4-2k) \neq 0$.

Consequently, the given matrices are linearly independent for all k values except -1 and 2.