CHAPTER 1: SYSTEMS OF LINEAR EQUATIONS AND MATRICES

1.1 Introduction to Systems of Linear Equations

1. **(a)** This is a linear equation in x_1, x_2, and x_3.

(b) This is not a linear equation in x_1, x_2, and x_3 because of the term x_1x_3.

(c) We can rewrite this equation in the form $x_1 + 7x_2 - 3x_3 = 0$ therefore it is a linear equation in x_1, x_2, and x_3.

(d) This is not a linear equation in x_1, x_2, and x_3 because of the term x_1^{-2}.

(e) This is not a linear equation in x_1, x_2, and x_3 because of the term $x_1^{3/5}$.

(f) This is a linear equation in x_1, x_2, and x_3.

3. **(a)**
$$\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

(b)
$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array}$$

(c)
$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & a_{14}x_4 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & a_{24}x_4 & = & b_2 \end{array}$$

5. **(a)**
$$\begin{array}{ccccc} 2x_1 & & & = & 0 \\ 3x_1 & - & 4x_2 & = & 0 \\ & & x_2 & = & 1 \end{array}$$

(b)
$$\begin{array}{ccccccc} 3x_1 & & & - & 2x_3 & = & 5 \\ 7x_1 & + & x_2 & + & 4x_3 & = & -3 \\ & - & 2x_2 & + & x_3 & = & 7 \end{array}$$

7. **(a)**
$$\begin{bmatrix} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{bmatrix}$$

9. The values in (a), (d), and (e) satisfy all three equations – these 3-tuples are solutions of the system. The 3-tuples in (b) and (c) are not solutions of the system.

11. **(a)** We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the system

$$\begin{array}{rcrcr} 3x & - & 2y & = & 4 \\ & & 0 & = & 1 \end{array}$$

The second equation is contradictory, so the original system has no solutions. The lines represented by the equations in that system have no points of intersection (the lines are parallel and distinct).

(b) We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the system

$$\begin{array}{rcrcr} 2x & - & 4y & = & 1 \\ & & 0 & = & 0 \end{array}$$

The second equation does not impose any restriction on x and y therefore we can omit it. The lines represented by the original system have infinitely many points of intersection. Solving the first equation for x we obtain $x = \frac{1}{2} + 2y$. This allows us to represent the solution using parametric equations

$$x = \frac{1}{2} + 2t, \qquad y = t$$

where the parameter t is an arbitrary real number.

(c) We can eliminate x from the second equation by adding -1 times the first equation to the second. This yields the system

$$\begin{array}{rcrcr} x & - & 2y & = & 0 \\ & - & 2y & = & 8 \end{array}$$

From the second equation we obtain $y = -4$. Substituting -4 for y into the first equation results in $x = -8$. Therefore, the original system has the unique solution

$$x = -8, \qquad y = -4$$

The represented by the equations in that system have one point of intersection: $(-8, -4)$.

13. **(a)** Solving the equation for x we obtain $x = \frac{3}{7} + \frac{5}{7}y$ therefore the solution set of the original equation can be described by the parametric equations

$$x = \frac{3}{7} + \frac{5}{7}t, \qquad y = t$$

where the parameter t is an arbitrary real number.

(b) Solving the equation for x_1 we obtain $x_1 = \frac{7}{3} + \frac{5}{3}x_2 - \frac{4}{3}x_3$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = \frac{7}{3} + \frac{5}{3}r - \frac{4}{3}s, \qquad x_2 = r, \qquad x_3 = s$$

where the parameters r and s are arbitrary real numbers.

(c) Solving the equation for x_1 we obtain $x_1 = -\frac{1}{8} + \frac{1}{4}x_2 - \frac{5}{8}x_3 + \frac{3}{4}x_4$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = -\frac{1}{8} + \frac{1}{4}r - \frac{5}{8}s + \frac{3}{4}t, \qquad x_2 = r, \qquad x_3 = s, \qquad x_4 = t$$

where the parameters r, s, and t are arbitrary real numbers.

(d) Solving the equation for v we obtain $v = \frac{8}{3}w - \frac{2}{3}x + \frac{1}{3}y - \frac{4}{3}z$ therefore the solution set of the original equation can be described by the parametric equations

$$v = \frac{8}{3}t_1 - \frac{2}{3}t_2 + \frac{1}{3}t_3 - \frac{4}{3}t_4, \qquad w = t_1, \qquad x = t_2, \qquad y = t_3, \qquad z = t_4$$

where the parameters t_1, t_2, t_3, and t_4 are arbitrary real numbers.

15. **(a)** We can eliminate x from the second equation by adding -3 times the first equation to the second. This yields the system

$$\begin{array}{rcrcr} 2x & - & 3y & = & 1 \\ & & 0 & = & 0 \end{array}$$

The second equation does not impose any restriction on x and y therefore we can omit it. Solving the first equation for x we obtain $x = \frac{1}{2} + \frac{3}{2}y$. This allows us to represent the solution using parametric equations

$$x = \frac{1}{2} + \frac{3}{2}t, \qquad y = t$$

where the parameter t is an arbitrary real number.

(b) We can see that the second and the third equation are multiples of the first: adding -3 times the first equation to the second, then adding the first equation to the third yields the system

$$\begin{array}{c} x_1 + 3x_2 - x_3 = -4 \\ 0 = 0 \\ 0 = 0 \end{array}$$

The last two equations do not impose any restriction on the unknowns therefore we can omit them. Solving the first equation for x_1 we obtain $x_1 = -4 - 3x_2 + x_3$. This allows us to represent the solution using parametric equations

$$x_1 = -4 - 3r + s, \qquad x_2 = r, \qquad x_3 = s$$

where the parameters r and s are arbitrary real numbers.

17. **(a)** Add 2 times the second row to the first to obtain $\begin{bmatrix} 1 & -7 & 8 & 8 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$.

(b) Add the third row to the first to obtain $\begin{bmatrix} 1 & 3 & -8 & 3 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

(another solution: interchange the first row and the third row to obtain $\begin{bmatrix} 1 & 4 & -3 & 3 \\ 2 & -9 & 3 & 2 \\ 0 & -1 & -5 & 0 \end{bmatrix}$).

19. **(a)** Add -4 times the first row to the second to obtain $\begin{bmatrix} 1 & k & -4 \\ 0 & 8-4k & 18 \end{bmatrix}$ which corresponds to the system

$$\begin{aligned} x + \quad ky &= -4 \\ (8-4k)y &= 18 \end{aligned}$$

If $k = 2$ then the second equation becomes $0 = 18$, which is contradictory thus the system becomes inconsistent.

If $k \neq 2$ then we can solve the second equation for y and proceed to substitute this value into the first equation and solve for x.

Consequently, for all values of $k \neq 2$ the given augmented matrix corresponds to a consistent linear system.

(b) Add -4 times the first row to the second to obtain $\begin{bmatrix} 1 & k & -1 \\ 0 & 8-4k & 0 \end{bmatrix}$ which corresponds to the system

$$\begin{aligned} x + \quad ky &= -1 \\ (8-4k)y &= 0 \end{aligned}$$

If $k = 2$ then the second equation becomes $0 = 0$, which does not impose any restriction on x and y therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If $k \neq 2$ then the second equation yields $y = 0$ and the first equation becomes $x = -1$.

Consequently, for all values of k the given augmented matrix corresponds to a consistent linear system.

21. Substituting the coordinates of the first point into the equation of the curve we obtain

$$y_1 = ax_1^2 + bx_1 + c$$

Repeating this for the other two points and rearranging the three equations yields

$$x_1^2 a + x_1 b + c = y_1$$

$$x_2^2 a + x_2 b + c = y_2$$

$$x_3^2 a + x_3 b + c = y_3$$

This is a linear system in the unknowns a, b, and c. Its augmented matrix is $\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$.

23. Solving the first equation for x_1 we obtain $x_1 = c - kx_2$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = c - kt, \qquad x_2 = t$$

where the parameter t is an arbitrary real number.

Substituting these into the second equation yields

$$c - kt + lt = d$$

which can be rewritten as

$$c - kt = d - lt$$

This equation must hold true for all real values t, which requires that the coefficients associated with the same power of t on both sides must be equal. Consequently, $c = d$ and $k = l$.

25. $$\begin{array}{rcrcrcr} 2x & + & 3y & + & z & = & 7 \\ 2x & + & y & + & 3z & = & 9 \\ 4x & + & 2y & + & 5z & = & 16 \end{array}$$

27. $$\begin{array}{rcrcrcr} x & + & y & + & z & = & 12 \\ 2x & + & y & + & 2z & = & 5 \\ -x & & & + & z & = & 1 \end{array}$$

True-False Exercises

(a) True. $(0,0,\ldots,0)$ is a solution.

(b) False. Only multiplication by a **non**zero constant is a valid elementary row operation.

(c) True. If $k = 6$ then the system has infinitely many solutions; otherwise the system is inconsistent.

(d) True. According to the definition, $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is a linear equation if the a's are not all zero. Let us assume $a_j \neq 0$. The values of all *x*'s except for x_j can be set to be arbitrary parameters, and the equation can be used to express x_j in terms of those parameters.

(e) False. E.g. if the equations are all homogeneous then the system must be consistent. (See True-False Exercise (a) above.)

(f) False. If $c \neq 0$ then the new system has the same solution set as the original one.

(g) True. Adding -1 times one row to another amounts to the same thing as subtracting one row from another.

(h) False. The second row corresponds to the equation $0 = -1$, which is contradictory.

1.2 Gaussian Elimination

1. **(a)** This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(b) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(c) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(d) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(e) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(f) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(g) This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (-7) above it. The matrix is in row echelon form but not reduced row echelon form.

3. **(a)** The linear system

$$\begin{array}{rcrcrcl} x & - & 3y & + & 4z & = & 7 \\ & & y & + & 2z & = & 2 \\ & & & & z & = & 5 \end{array} \quad \text{can be rewritten as} \quad \begin{array}{rcl} x & = & 7+3y-4z \\ y & = & 2-2z \\ z & = & 5 \end{array}$$

and solved by back-substitution:

$$z=5$$

$$y=2-2(5)=-8$$

$$x=7+3(-8)-4(5)=-37$$

therefore the original linear system has a unique solution: $x=-37$, $y=-8$, $z=5$.

(b) The linear system

$$\begin{array}{rcrcrcrcl} w & & & + & 8y & - & 5z & = & 6 \\ & & x & + & 4y & - & 9z & = & 3 \\ & & & & y & + & z & = & 2 \end{array} \quad \text{can be rewritten as} \quad \begin{array}{rcl} w & = & 6-8y+5z \\ x & = & 3-4y+9z \\ y & = & 2-z \end{array}$$

Let $z=t$. Then

$$y=2-t$$

$$x=3-4(2-t)+9t=-5+13t$$

$$w=6-8(2-t)+5t=-10+13t$$

therefore the original linear system has infinitely many solutions:

$$w=-10+13t,\ x=-5+13t,\ y=2-t,\ z=t$$

where t is an arbitrary value.

(c) The linear system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 7x_2 & - & 2x_3 & & & - & 8x_5 & = & -3 \\ & & & & x_3 & + & x_4 & + & 6x_5 & = & 5 \\ & & & & & & x_4 & + & 3x_5 & = & 9 \\ & & & & & & & & 0 & = & 0 \end{array}$$

can be rewritten: $x_1=-3-7x_2+2x_3+8x_5$, $x_3=5-x_4-6x_5$, $x_4=9-3x_5$.

Let $x_2=s$ and $x_5=t$. Then

$$x_4 = 9 - 3t$$

$$x_3 = 5 - (9 - 3t) - 6t = -4 - 3t$$

$$x_1 = -3 - 7s + 2(-4 - 3t) + 8t = -11 - 7s + 2t$$

therefore the original linear system has infinitely many solutions:

$$x_1 = -11 - 7s + 2t, \quad x_2 = s, \quad x_3 = -4 - 3t, \quad x_4 = 9 - 3t, \quad x_5 = t$$

where s and t are arbitrary values.

(d) The system is inconsistent since the third row of the augmented matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

5. $\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix}$ ← The first row was added to the second row.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$ ← -3 times the first row was added to the third row.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$ ← The second row was multiplied by -1.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$ ← 10 times the second row was added to the third row.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← The third row was multiplied by $-\frac{1}{52}$.

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ & & x_2 & - & 5x_3 & = & -9 \\ & & & & x_3 & = & 2 \end{array} \qquad \text{and can be rewritten as} \qquad \begin{array}{rcl} x_1 & = & 8 - x_2 - 2x_3 \\ x_2 & = & -9 + 5x_3 \\ x_3 & = & 2 \end{array}$$

Back-substitution yields

$$x_3 = 2$$

$$x_2 = -9 + 5(2) = 1$$

$$x_1 = 8 - 1 - 2(2) = 3$$

The linear system has a unique solution: $x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

7.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{The first row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow -3 \text{ times the first row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow -3 \text{ times the second row was added to the fourth row.}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{array}{rcrcrcrcr} x & - & y & + & 2z & - & w & = & -1 \\ & & y & - & 2z & & & = & 0 \\ & & & & & & 0 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

Solve the equations for the leading variables

$$\begin{aligned} x &= -1 + y - 2z + w \\ y &= 2z \end{aligned}$$

then substitute the second equation into the first

$$\begin{aligned} x &= -1 + 2z - 2z + w = -1 + w \\ y &= 2z \end{aligned}$$

If we assign z and w the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x = -1 + t, \qquad y = 2s, \qquad z = s, \qquad w = t$$

9. $\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix}$ ← The first row was added to the second row.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$ ← -3 times the first row was added to the third row.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$ ← The second row was multiplied by -1.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$ ← 10 times the second row was added to the third row.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← The third row was multiplied by $-\frac{1}{52}$.

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← 5 times the third row was added to the second row.

$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← -2 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the first row.}$$

The linear system has a unique solution: $x_1 = 3$, $x_2 = 1$, $x_3 = 2$.

11.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow -3 \text{ times the first row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow -3 \text{ times the second row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{the second row was added to the first row.}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x \qquad\qquad\quad - w &= -1 \\ y - 2z \qquad &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Solve the equations for the leading variables

$$x = -1 + w$$

$$y = 2z$$

If we assign z and w the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x = -1 + t, \quad y = 2s, \quad z = s, \quad w = t\ .$$

13. Since the number of unknowns (4) exceeds the number of equations (3), it follows from Theorem 1.2.2 that this system has infinitely many solutions. Those include the trivial solution and infinitely many nontrivial solutions.

15. We present two different solutions.
Solution I uses Gauss-Jordan elimination

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{2}{3}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \begin{array}{l}\text{The third row was added to the second row} \\ \text{and } -\tfrac{3}{2} \text{ times the third row was added to the first row}\end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow -\tfrac{1}{2} \text{ times the second row was added to the first row.}$$

Unique solution: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

Solution II. This time, we shall choose the order of the elementary row operations differently in order to avoid introducing fractions into the computation. (Since every matrix has a unique reduced row echelon form, the exact sequence of elementary row operations being used does not matter – see part 1 of the discussion “Some Facts About Echelon Forms” in Section 1.2)

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \begin{array}{l}\text{The first and second rows were interchanged} \\ \text{(to avoid introducing fractions into the first row).}\end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } -\tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{The third row was added to the second row.}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow -2 \text{ times the second row was added to the first row.}$$

Unique solution: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

17.

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{bmatrix} \longleftarrow -5 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } -\tfrac{3}{8}.$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix} \longleftarrow -\tfrac{1}{3} \text{ times the second row was added to the first row.}$$

If we assign x_3 and x_4 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x_1 = -\frac{1}{4}s, \quad x_2 = -\frac{1}{4}s - t, \quad x_3 = s, \quad x_4 = t.$$

(Note that fractions in the solution could be avoided if we assigned $x_3 = 4s$ instead, which along with $x_4 = t$ would yield $x_1 = -s$, $x_2 = -s - t$, $x_3 = 4s$, $x_4 = t$.)

19.

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \longleftarrow \text{The first and second rows were interchanged.}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \longleftarrow$$ -2 times the first row was added to the third row and 2 times the first row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \longleftarrow$$ The second row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \longleftarrow$$ -3 times the second row was added to the third and -1 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow$$ 10 times the third row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow$$ -2 times the third row was added to the second and 3 times the third row was added to the first row.

If we assign y an arbitrary value t the general solution is given by the formulas

$$w = t, \quad x = -t, \quad y = t, \quad z = 0.$$

21.

$$\begin{bmatrix} 2 & -1 & 3 & 4 & 9 \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 2 & -1 & 3 & 4 & 9 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix} \longleftarrow$$ The first and second rows were interchanged (to avoid introducing fractions into the first row).

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & -1 & 7 & -10 & -13 \\ 0 & -3 & 7 & -16 & -25 \\ 0 & 1 & 8 & -10 & -12 \end{bmatrix} \longleftarrow$$ -2 times the first row was added to the second row, -3 times the first row was added to the third row, and -2 times the first row was added to the fourth.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & -3 & 7 & -16 & -25 \\ 0 & 1 & 8 & -10 & -12 \end{bmatrix}$$ ← The second row was multiplied by -1.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & -14 & 14 & 14 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix}$$ ← 3 times the second row was added to the third row and -1 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix}$$ ← The third row was multiplied by $-\frac{1}{14}$.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -5 & -10 \end{bmatrix}$$ ← -15 times the third row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ ← The fourth row was multiplied by $-\frac{1}{5}$.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & -7 & 0 & -7 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ ← The fourth row was added to the third row, -10 times the fourth row was added to the second, and -7 times the fourth row was added to the first.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ ← 7 times the third row was added to the second row, and 2 times the third row was added to the first row.

Unique solution: $I_1 = -1$, $I_2 = 0$, $I_3 = 1$, $I_4 = 2$.

23. **(a)** The system is consistent; it has a unique solution (back-substitution can be used to solve for all three unknowns).

(b) The system is consistent; it has infinitely many solutions (the third unknown can be assigned an arbitrary value t, then back-substitution can be used to solve for the first two unknowns).

(c) The system is inconsistent since the third equation $0 = 1$ is contradictory.

(d) There is insufficient information to decide whether the system is consistent as illustrated by these examples:

- For $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$ the system is consistent with infinitely many solutions.

- For $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ the system is inconsistent (the matrix can be reduced to $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$).

25. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$ ⟵ -3 times the first row was added to the second row and -4 times the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$ ⟵ -1 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$ ⟵ The second row was multiplied by $-\frac{1}{7}$.

The system has no solutions when $a=-4$ (since the third row of our last matrix would then correspond to a contradictory equation $0=-8$).

The system has infinitely many solutions when $a=4$ (since the third row of our last matrix would then correspond to the equation $0=0$).

For all remaining values of a (i.e., $a\neq -4$ and $a\neq 4$) the system has exactly one solution.

27. $\begin{bmatrix} 1 & 3 & -1 & a \\ 1 & 1 & 2 & b \\ 0 & 2 & -3 & c \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 3 & -1 & a \\ 0 & -2 & 3 & -a+b \\ 0 & 2 & -3 & c \end{bmatrix}$ ⟵ -1 times the first row was added to the second row.

$\begin{bmatrix} 1 & 3 & -1 & a \\ 0 & -2 & 3 & -a+b \\ 0 & 0 & 0 & -a+b+c \end{bmatrix}$ ⟵ The second row was added to the third row.

$$\begin{bmatrix} 1 & 3 & -1 & a \\ 0 & 1 & -\frac{3}{2} & \frac{a}{2}-\frac{b}{2} \\ 0 & 0 & 0 & -a+b+c \end{bmatrix} \longleftarrow \text{The second row was multiplied by } -\tfrac{1}{2}.$$

If $-a+b+c=0$ then the linear system is consistent. Otherwise (if $-a+b+c\neq 0$) it is inconsistent.

29.

$$\begin{bmatrix} 2 & 1 & a \\ 3 & 6 & b \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 3 & 6 & b \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 0 & \frac{9}{2} & -\frac{3}{2}a+b \end{bmatrix} \longleftarrow -3 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 0 & 1 & -\frac{1}{3}a+\frac{2}{9}b \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{2}{9}.$$

$$\begin{bmatrix} 1 & 0 & \frac{2}{2}a-\frac{1}{9}b \\ 0 & 1 & -\frac{1}{3}a+\frac{2}{9}b \end{bmatrix} \longleftarrow -\tfrac{1}{2} \text{ times the second row was added to the first row.}$$

The system has exactly one solution: $x=\frac{2}{3}a-\frac{1}{9}b$ and $y=-\frac{1}{3}a+\frac{2}{9}b$.

31. Adding -2 times the first row to the second yields a matrix in row echelon form $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

Adding -3 times its second row to the first results in $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is also in row echelon form.

33. We begin by substituting $x=\sin\alpha$, $y=\cos\beta$, and $z=\tan\gamma$ so that the system becomes

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ 2x & + & 5y & + & 3z & = & 0 \\ -x & - & 5y & + & 5z & = & 0 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{bmatrix} \longleftarrow \begin{array}{l} -2 \text{ times the first row was added to the second row} \\ \text{and the first row was added to the third row.} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \longleftarrow 3 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } -1.$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

3 times the third row was added to the second row and -3 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

-2 times the second row was added to the first row.

This system has exactly one solution $x = 0, \ y = 0, \ z = 0.$

On the interval $0 \le \alpha \le 2\pi$, the equation $\sin\alpha = 0$ has three solutions: $\alpha = 0$, $\alpha = \pi$, and $\alpha = 2\pi$.

On the interval $0 \le \beta \le 2\pi$, the equation $\cos\beta = 0$ has two solutions: $\beta = \frac{\pi}{2}$ and $\beta = \frac{3\pi}{2}$.

On the interval $0 \le \gamma \le 2\pi$, the equation $\tan\gamma = 0$ has three solutions: $\gamma = 0$, $\gamma = \pi$, and $\gamma = 2\pi$.

Overall, $3 \cdot 2 \cdot 3 = 18$ solutions (α, β, γ) can be obtained by combining the values of α, β, and γ listed above: $\left(0, \frac{\pi}{2}, 0\right), \left(\pi, \frac{\pi}{2}, 0\right)$, etc.

35. We begin by substituting $X = x^2$, $Y = y^2$, and $Z = z^2$ so that the system becomes

$$\begin{aligned} X + Y + Z &= 6 \\ X - Y + 2Z &= 2 \\ 2X + Y - Z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 3 \end{bmatrix} \longleftarrow$$

The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 0 & -1 & -3 & -9 \end{bmatrix} \longleftarrow$$

–1 times the first row was added to the second row and –2 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -9 \\ 0 & -2 & 1 & -4 \end{bmatrix} \longleftarrow$$

The second and third rows were interchanged (to avoid introducing fractions into the second row).

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & -2 & 1 & -4 \end{bmatrix} \longleftarrow$$

The second row was multiplied by –1.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 7 & 14 \end{bmatrix} \leftarrow \text{2 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow \text{The third row was multiplied by } \tfrac{1}{7}.$$

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow \begin{matrix}\text{–3 times the third row was added to the second row} \\ \text{and –1 times the third row was added to the first row.}\end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow \text{–1 times the second row was added to the first row.}$$

We obtain

$$\begin{aligned} X &= 1 &&\Rightarrow && x = \pm 1 \\ Y &= 3 &&\Rightarrow && y = \pm\sqrt{3} \\ Z &= 2 &&\Rightarrow && z = \pm\sqrt{2} \end{aligned}$$

37. Each point on the curve yields an equation, therefore we have a system of four equations

equation corresponding to $(1,7)$: $\quad a + b + c + d = 7$
equation corresponding to $(3,-11)$: $\quad 27a + 9b + 3c + d = -11$
equation corresponding to $(4,-14)$: $\quad 64a + 16b + 4c + d = -14$
equation corresponding to $(0,10)$: $\quad d = 10$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 27 & 9 & 3 & 1 & -11 \\ 64 & 16 & 4 & 1 & -14 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \leftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & -18 & -24 & -26 & -200 \\ 0 & -48 & -60 & -63 & -462 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \leftarrow \begin{matrix}-27 \text{ times the first row was added to the second row} \\ \text{and } -64 \text{ times the first row was added to the third.}\end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & -48 & -60 & -63 & -462 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \leftarrow \text{The second row was multiplied by } -\tfrac{1}{18}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & 0 & 4 & \frac{19}{3} & \frac{214}{3} \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \longleftarrow \text{48 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & 0 & 1 & \frac{19}{12} & \frac{107}{6} \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{4}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & -3 \\ 0 & 1 & \frac{4}{3} & 0 & -\frac{10}{3} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \longleftarrow$$

$-\frac{19}{12}$ times the fourth row was added to the third row, $-\frac{13}{9}$ times the fourth row was added to the second row, and -1 times the fourth row was added to the first.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \longleftarrow$$

$-\frac{4}{3}$ times the third row was added to the second row and -1 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \longleftarrow \text{-1 times the second row was added to the first row.}$$

The linear system has a unique solution: $a=1$, $b=-6$, $c=2$, $d=10$. These are the coefficient values required for the curve $y=ax^3+bx^2+cx+d$ to pass through the four given points.

39. Since the homogeneous system has only the trivial solution, its augmented matrix must be possible to reduce via a sequence of elementary row operations to the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Applying the **same** sequence of elementary row operations to the augmented matrix of the nonhomogeneous system yields the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \end{bmatrix}$ where r, s, and t are some real numbers. Therefore, the nonhomogeneous system has one solution.

41. **(a)** There are eight possible reduced row echelon forms:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where r and s can be any real numbers.

(b) There are sixteen possible reduced row echelon forms:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&r\\0&1&0&s\\0&0&1&t\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&r&0\\0&1&s&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&r&t\\0&1&s&u\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&r&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&r&0&s\\0&0&1&t\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\begin{bmatrix}1&r&s&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&r&s&t\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&r\\0&0&1&s\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&r&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&r&s\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1&r\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \text{ and } \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

where r, s, t, and u can be any real numbers.

43. (a) We consider two possible cases: (i) $a=0$, and (ii) $a \neq 0$.

(i) If $a=0$ then the assumption $ad-bc \neq 0$ implies that $b \neq 0$ and $c \neq 0$. Gauss-Jordan elimination yields

$\begin{bmatrix}0&b\\c&d\end{bmatrix}$ ⟵ We assumed $a=0$

$\begin{bmatrix}c&d\\0&b\end{bmatrix}$ ⟵ The rows were interchanged.

$\begin{bmatrix}1&\frac{d}{c}\\0&1\end{bmatrix}$ ⟵ The first row was multiplied by $\frac{1}{c}$ and the second row was multiplied by $\frac{1}{b}$. (Note that $b, c \neq 0$.)

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$ ⟵ $-\frac{d}{c}$ times the second row was added to the first row.

(ii) If $a \neq 0$ then we perform Gauss-Jordan elimination as follows:

$\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$\begin{bmatrix}1&\frac{b}{a}\\c&d\end{bmatrix}$ ⟵ The first row was multiplied by $\frac{1}{a}$.

$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$ ⟵ $-c$ times the first row was added to the second row.

$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$ ⟵ The second row was multiplied by $\frac{a}{ad-bc}$.
(Note that both a and $ad-bc$ are nonzero.)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ⟵ $-\frac{b}{a}$ times the second row was added to the first row.

In both cases ($a=0$ as well as $a \neq 0$) we established that the reduced row echelon form of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ provided that $ad-bc \neq 0$.

(b) Applying the **same** elementary row operation steps as in part (a) the augmented matrix $\begin{bmatrix} a & b & k \\ c & d & l \end{bmatrix}$ will be transformed to a matrix in reduced row echelon form $\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \end{bmatrix}$ where p and q are some real numbers. We conclude that the given linear system has exactly one solution: $x=p$, $y=q$.

True-False Exercises

(a) True. A matrix in reduced row echelon form has all properties required for the row echelon form.

(b) False. For instance, interchanging the rows of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yields a matrix that is not in row echelon form.

(c) False. See Exercise 31.

(d) True. In a reduced row echelon form, the number of nonzero rows equals to the number of leading 1's. The result follows from Theorem 1.2.1.

(e) True. This is implied by the third property of a row echelon form (see Section 1.2).

(f) False. Nonzero entries are permitted above the leading 1's in a row echelon form.

(g) True. In a reduced row echelon form, the number of nonzero rows equals to the number of leading 1's. From Theorem 1.2.1 we conclude that the system has $n-n=0$ free variables, i.e. it has only the trivial solution.

(h) False. The row of zeros imposes no restriction on the unknowns and can be omitted. Whether the system has infinitely many, one, or no solution(s) depends *solely* on the nonzero rows of the reduced row echelon form.

(i) False. For example, the following system is clearly inconsistent:

$$\begin{aligned} x+y+z&=1 \\ x+y+z&=2 \end{aligned}$$

1.3 Matrices and Matrix Operations

1. **(a)** Undefined (the number of columns in B does not match the number of rows in A)

(b) Defined; 4×4 matrix

(c) Defined; 4×2 matrix

(d) Defined; 5×2 matrix

(e) Defined; 4×5 matrix

(f) Defined; 5×5 matrix

3. **(a)** $\begin{bmatrix} 1+6 & 5+1 & 2+3 \\ -1+(-1) & 0+1 & 1+2 \\ 3+4 & 2+1 & 4+3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1-6 & 5-1 & 2-3 \\ -1-(-1) & 0-1 & 1-2 \\ 3-4 & 2-1 & 4-3 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 5\cdot 3 & 5\cdot 0 \\ 5\cdot(-1) & 5\cdot 2 \\ 5\cdot 1 & 5\cdot 1 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -7\cdot 1 & -7\cdot 4 & -7\cdot 2 \\ -7\cdot 3 & -7\cdot 1 & -7\cdot 5 \end{bmatrix} = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$

(e) Undefined (a 2×3 matrix C cannot be subtracted from a 2×2 matrix $2B$)

(f) $\begin{bmatrix} 4\cdot 6 & 4\cdot 1 & 4\cdot 3 \\ 4\cdot(-1) & 4\cdot 1 & 4\cdot 2 \\ 4\cdot 4 & 4\cdot 1 & 4\cdot 3 \end{bmatrix} - \begin{bmatrix} 2\cdot 1 & 2\cdot 5 & 2\cdot 2 \\ 2\cdot(-1) & 2\cdot 0 & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 2 & 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 24-2 & 4-10 & 12-4 \\ -4-(-2) & 4-0 & 8-2 \\ 16-6 & 4-4 & 12-8 \end{bmatrix}$

$= \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$

(g) $-3\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 2\cdot 6 & 2\cdot 1 & 2\cdot 3 \\ 2\cdot(-1) & 2\cdot 1 & 2\cdot 2 \\ 2\cdot 4 & 2\cdot 1 & 2\cdot 3 \end{bmatrix}\right) = -3\begin{bmatrix} 1+12 & 5+2 & 2+6 \\ -1+(-2) & 0+2 & 1+4 \\ 3+8 & 2+2 & 4+6 \end{bmatrix}$

$= \begin{bmatrix} -3\cdot 13 & -3\cdot 7 & -3\cdot 8 \\ -3\cdot(-3) & -3\cdot 2 & -3\cdot 5 \\ -3\cdot 11 & -3\cdot 4 & -3\cdot 10 \end{bmatrix} = \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$

(h) $\begin{bmatrix} 3-3 & 0-0 \\ -1-(-1) & 2-2 \\ 1-1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(i) $1+0+4=5$

(j) $\operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 3\cdot 6 & 3\cdot 1 & 3\cdot 3 \\ 3\cdot(-1) & 3\cdot 1 & 3\cdot 2 \\ 3\cdot 4 & 3\cdot 1 & 3\cdot 3 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 1-18 & 5-3 & 2-9 \\ -1-(-3) & 0-3 & 1-6 \\ 3-12 & 2-3 & 4-9 \end{bmatrix}\right)$

$= \operatorname{tr}\left(\begin{bmatrix} -17 & 2 & -7 \\ 2 & -3 & -5 \\ -9 & -1 & -5 \end{bmatrix}\right) = -17-3-5 = -25$

(k) $4\operatorname{tr}\left(\begin{bmatrix} 7\cdot 4 & 7\cdot(-1) \\ 7\cdot 0 & 7\cdot 2 \end{bmatrix}\right) = 4\operatorname{tr}\left(\begin{bmatrix} 28 & -7 \\ 0 & 14 \end{bmatrix}\right) = 4(28+14) = 4\cdot 42 = 168$

(l) Undefined (trace is only defined for square matrices)

5. **(a)** $\begin{bmatrix} (3\cdot 4)+(0\cdot 0) & -(3\cdot 1)+(0\cdot 2) \\ -(1\cdot 4)+(2\cdot 0) & (1\cdot 1)+(2\cdot 2) \\ (1\cdot 4)+(1\cdot 0) & -(1\cdot 1)+(1\cdot 2) \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$

(b) Undefined (the number of columns of B does not match the number of rows in A)

(c) $\begin{bmatrix} 3\cdot 6 & 3\cdot 1 & 3\cdot 3 \\ 3\cdot(-1) & 3\cdot 1 & 3\cdot 2 \\ 3\cdot 4 & 3\cdot 1 & 3\cdot 3 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} (18\cdot 1)-(3\cdot 1)+(9\cdot 3) & (18\cdot 5)+(3\cdot 0)+(9\cdot 2) & (18\cdot 2)+(3\cdot 1)+(9\cdot 4) \\ -(3\cdot 1)-(3\cdot 1)+(6\cdot 3) & -(3\cdot 5)+(3\cdot 0)+(6\cdot 2) & -(3\cdot 2)+(3\cdot 1)+(6\cdot 4) \\ (12\cdot 1)-(3\cdot 1)+(9\cdot 3) & (12\cdot 5)+(3\cdot 0)+(9\cdot 2) & (12\cdot 2)+(3\cdot 1)+(9\cdot 4) \end{bmatrix}$

$= \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$

(d) $\begin{bmatrix} (3\cdot 4)+(0\cdot 0) & -(3\cdot 1)+(0\cdot 2) \\ -(1\cdot 4)+(2\cdot 0) & (1\cdot 1)+(2\cdot 2) \\ (1\cdot 4)+(1\cdot 0) & -(1\cdot 1)+(1\cdot 2) \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

$= \begin{bmatrix} (12\cdot 1)-(3\cdot 3) & (12\cdot 4)-(3\cdot 1) & (12\cdot 2)-(3\cdot 5) \\ -(4\cdot 1)+(5\cdot 3) & -(4\cdot 4)+(5\cdot 1) & -(4\cdot 2)+(5\cdot 5) \\ (4\cdot 1)+(1\cdot 3) & (4\cdot 4)+(1\cdot 1) & (4\cdot 2)+(1\cdot 5) \end{bmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (4\cdot 1)-(1\cdot 3) & (4\cdot 4)-(1\cdot 1) & (4\cdot 2)-(1\cdot 5) \\ (0\cdot 1)+(2\cdot 3) & (0\cdot 4)+(2\cdot 1) & (0\cdot 2)+(2\cdot 5) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$

$= \begin{bmatrix} (3\cdot 1)+(0\cdot 6) & (3\cdot 15)+(0\cdot 2) & (3\cdot 3)+(0\cdot 10) \\ -(1\cdot 1)+(2\cdot 6) & -(1\cdot 15)+(2\cdot 2) & -(1\cdot 3)+(2\cdot 10) \\ (1\cdot 1)+(1\cdot 6) & (1\cdot 15)+(1\cdot 2) & (1\cdot 3)+(1\cdot 10) \end{bmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(f) $$\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix}\begin{bmatrix}1&3\\4&1\\2&5\end{bmatrix}=\begin{bmatrix}(1\cdot1)+(4\cdot4)+(2\cdot2)&(1\cdot3)+(4\cdot1)+(2\cdot5)\\(3\cdot1)+(1\cdot4)+(5\cdot2)&(3\cdot3)+(1\cdot1)+(5\cdot5)\end{bmatrix}=\begin{bmatrix}21&17\\17&35\end{bmatrix}$$

(g) $$\left(\begin{bmatrix}(1\cdot3)-(5\cdot1)+(2\cdot1)&(1\cdot0)+(5\cdot2)+(2\cdot1)\\-(1\cdot3)-(0\cdot1)+(1\cdot1)&-(1\cdot0)+(0\cdot2)+(1\cdot1)\\(3\cdot3)-(2\cdot1)+(4\cdot1)&(3\cdot0)+(2\cdot2)+(4\cdot1)\end{bmatrix}\right)^T=\begin{bmatrix}0&-2&11\\12&1&8\end{bmatrix}$$

(h) $$\left(\begin{bmatrix}1&3\\4&1\\2&5\end{bmatrix}\begin{bmatrix}4&-1\\0&2\end{bmatrix}\right)\begin{bmatrix}3&-1&1\\0&2&1\end{bmatrix}=\begin{bmatrix}(1\cdot4)+(3\cdot0)&-(1\cdot1)+(3\cdot2)\\(4\cdot4)+(1\cdot0)&-(4\cdot1)+(1\cdot2)\\(2\cdot4)+(5\cdot0)&-(2\cdot1)+(5\cdot2)\end{bmatrix}\begin{bmatrix}3&-1&1\\0&2&1\end{bmatrix}$$

$$=\begin{bmatrix}4&5\\16&-2\\8&8\end{bmatrix}\begin{bmatrix}3&-1&1\\0&2&1\end{bmatrix}=\begin{bmatrix}(4\cdot3)+(5\cdot0)&-(4\cdot1)+(5\cdot2)&(4\cdot1)+(5\cdot1)\\(16\cdot3)-(2\cdot0)&-(16\cdot1)-(2\cdot2)&(16\cdot1)-(2\cdot1)\\(8\cdot3)+(8\cdot0)&-(8\cdot1)+(8\cdot2)&(8\cdot1)+(8\cdot1)\end{bmatrix}$$

$$=\begin{bmatrix}12&6&9\\48&-20&14\\24&8&16\end{bmatrix}$$

(i) $$\operatorname{tr}\left(\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix}\begin{bmatrix}1&-1&3\\5&0&2\\2&1&4\end{bmatrix}\right)$$

$$=\operatorname{tr}\left(\begin{bmatrix}(1\cdot1)+(5\cdot5)+(2\cdot2)&-(1\cdot1)+(5\cdot0)+(2\cdot1)&(1\cdot3)+(5\cdot2)+(2\cdot4)\\-(1\cdot1)+(0\cdot5)+(1\cdot2)&(1\cdot1)+(0\cdot0)+(1\cdot1)&-(1\cdot3)+(0\cdot2)+(1\cdot4)\\(3\cdot1)+(2\cdot5)+(4\cdot2)&-(3\cdot1)+(2\cdot0)+(4\cdot1)&(3\cdot3)+(2\cdot2)+(4\cdot4)\end{bmatrix}\right)$$

$$=\operatorname{tr}\left(\begin{bmatrix}30&1&21\\1&2&1\\21&1&29\end{bmatrix}\right)=30+2+29=61$$

(j) $$\operatorname{tr}\left(4\begin{bmatrix}6&-1&4\\1&1&1\\3&2&3\end{bmatrix}-\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix}\right)=\operatorname{tr}\left(\begin{bmatrix}4\cdot6-1&4\cdot(-1)-5&4\cdot4-2\\4\cdot1-(-1)&4\cdot1-0&4\cdot1-1\\4\cdot3-3&4\cdot2-2&4\cdot3-4\end{bmatrix}\right)$$

$$=\operatorname{tr}\left(\begin{bmatrix}23&-9&14\\5&4&3\\9&6&8\end{bmatrix}\right)=23+4+8=35$$

(k) $$\operatorname{tr}\left(\begin{bmatrix}1&3\\4&1\\2&5\end{bmatrix}\begin{bmatrix}3&-1&1\\0&2&1\end{bmatrix}+2\begin{bmatrix}6&-1&4\\1&1&1\\3&2&3\end{bmatrix}\right)$$

$$\mathrm{tr}\left(\begin{bmatrix}(1\cdot3)+(3\cdot0) & -(1\cdot1)+(3\cdot2) & (1\cdot1)+(3\cdot1)\\(4\cdot3)+(1\cdot0) & -(4\cdot1)+(1\cdot2) & (4\cdot1)+(1\cdot1)\\(2\cdot3)+(5\cdot0) & -(2\cdot1)+(5\cdot2) & (2\cdot1)+(5\cdot1)\end{bmatrix}+\begin{bmatrix}2\cdot6 & 2\cdot(-1) & 2\cdot4\\2\cdot1 & 2\cdot1 & 2\cdot1\\2\cdot3 & 2\cdot2 & 2\cdot3\end{bmatrix}\right)$$

$$\mathrm{tr}\left(\begin{bmatrix}3 & 5 & 4\\12 & -2 & 5\\6 & 8 & 7\end{bmatrix}+\begin{bmatrix}12 & -2 & 8\\2 & 2 & 2\\6 & 4 & 6\end{bmatrix}\right)=\mathrm{tr}\left(\begin{bmatrix}15 & 3 & 12\\14 & 0 & 7\\12 & 12 & 13\end{bmatrix}\right)=15+0+13=28$$

(l) $$\mathrm{tr}\left(\left(\begin{bmatrix}6 & 1 & 3\\-1 & 1 & 2\\4 & 1 & 3\end{bmatrix}\begin{bmatrix}1 & 3\\4 & 1\\2 & 5\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}3 & 0\\-1 & 2\\1 & 1\end{bmatrix}\right)$$

$$=\mathrm{tr}\left(\left(\begin{bmatrix}(6\cdot1)+(1\cdot4)+(3\cdot2) & (6\cdot3)+(1\cdot1)+(3\cdot5)\\-(1\cdot1)+(1\cdot4)+(2\cdot2) & -(1\cdot3)+(1\cdot1)+(2\cdot5)\\(4\cdot1)+(1\cdot4)+(3\cdot2) & (4\cdot3)+(1\cdot1)+(3\cdot5)\end{bmatrix}\right)^{T}\begin{bmatrix}3 & 0\\-1 & 2\\1 & 1\end{bmatrix}\right)$$

$$=\mathrm{tr}\left(\left(\begin{bmatrix}16 & 34\\7 & 8\\14 & 28\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}3 & 0\\-1 & 2\\1 & 1\end{bmatrix}\right)=\mathrm{tr}\left(\begin{bmatrix}16 & 7 & 14\\34 & 8 & 28\end{bmatrix}\begin{bmatrix}3 & 0\\-1 & 2\\1 & 1\end{bmatrix}\right)$$

$$=\mathrm{tr}\left(\begin{bmatrix}(16\cdot3)-(7\cdot1)+(14\cdot1) & (16\cdot0)+(7\cdot2)+(14\cdot1)\\(34\cdot3)-(8\cdot1)+(28\cdot1) & (34\cdot0)+(8\cdot2)+(28\cdot1)\end{bmatrix}\right)$$

$$=\mathrm{tr}\left(\begin{bmatrix}55 & 28\\122 & 44\end{bmatrix}\right)=55+44=99$$

7. **(a)** first row of $AB=$ [first row of A] $B=\begin{bmatrix}3 & -2 & 7\end{bmatrix}\begin{bmatrix}6 & -2 & 4\\0 & 1 & 3\\7 & 7 & 5\end{bmatrix}$

$$=\begin{bmatrix}(3\cdot6)-(2\cdot0)+(7\cdot7) & -(3\cdot2)-(2\cdot1)+(7\cdot7) & (3\cdot4)-(2\cdot3)+(7\cdot5)\end{bmatrix}$$
$$=\begin{bmatrix}67 & 41 & 41\end{bmatrix}$$

(b) third row of $AB=$ [third row of A] $B=\begin{bmatrix}0 & 4 & 9\end{bmatrix}\begin{bmatrix}6 & -2 & 4\\0 & 1 & 3\\7 & 7 & 5\end{bmatrix}$

$$=\begin{bmatrix}(0\cdot6)+(4\cdot0)+(9\cdot7) & -(0\cdot2)+(4\cdot1)+(9\cdot7) & (0\cdot4)+(4\cdot3)+(9\cdot5)\end{bmatrix}$$
$$=\begin{bmatrix}63 & 67 & 57\end{bmatrix}$$

(c) second column of $AB=A$ [second column of B]

$$=\begin{bmatrix}3 & -2 & 7\\6 & 5 & 4\\0 & 4 & 9\end{bmatrix}\begin{bmatrix}-2\\1\\7\end{bmatrix}=\begin{bmatrix}-(3\cdot2)-(2\cdot1)+(7\cdot7)\\-(6\cdot2)+(5\cdot1)+(4\cdot7)\\-(0\cdot2)+(4\cdot1)+(9\cdot7)\end{bmatrix}=\begin{bmatrix}41\\21\\67\end{bmatrix}$$

(d) first column of $BA = B$ [first column of A]

$$=\begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} (6\cdot 3)-(2\cdot 6)+(4\cdot 0) \\ (0\cdot 3)+(1\cdot 6)+(3\cdot 0) \\ (7\cdot 3)+(7\cdot 6)+(5\cdot 0) \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix}$$

(e) third row of $AA =$ [third row of A] $A = \begin{bmatrix} 0 & 4 & 9 \end{bmatrix}\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$

$$=\begin{bmatrix} (0\cdot 3)+(4\cdot 6)+(9\cdot 0) & -(0\cdot 2)+(4\cdot 5)+(9\cdot 4) & (0\cdot 7)+(4\cdot 4)+(9\cdot 9) \end{bmatrix}$$
$$=\begin{bmatrix} 24 & 56 & 97 \end{bmatrix}$$

(f) third column of $AA = A$ [third column of A]

$$=\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} (3\cdot 7)-(2\cdot 4)+(7\cdot 9) \\ (6\cdot 7)+(5\cdot 4)+(4\cdot 9) \\ (0\cdot 7)+(4\cdot 4)+(9\cdot 9) \end{bmatrix} = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$$

9. **(a)** first column of $AA = 3\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 48 \\ 24 \end{bmatrix}$

second column of $AA = -2\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 5\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 4\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 12 \\ 29 \\ 56 \end{bmatrix}$

third column of $AA = 7\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 9\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$

(b) first column of $BB = 6\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 64 \\ 21 \\ 77 \end{bmatrix}$

second column of $BB = -2\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 28 \end{bmatrix}$

third column of $BB = 4\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 38 \\ 18 \\ 74 \end{bmatrix}$

11. **(a)** $A = \begin{bmatrix} 2 & -3 & 5 \\ 9 & -1 & 1 \\ 1 & 5 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$; the matrix equation: $\begin{bmatrix} 2 & -3 & 5 \\ 9 & -1 & 1 \\ 1 & 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$

(b) $A=\begin{bmatrix}4&0&-3&1\\5&1&0&-8\\2&-5&9&-1\\0&3&-1&7\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}1\\3\\0\\2\end{bmatrix}$; the matrix equation: $\begin{bmatrix}4&0&-3&1\\5&1&0&-8\\2&-5&9&-1\\0&3&-1&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\3\\0\\2\end{bmatrix}$

13. **(a)**
$$\begin{aligned}5x_1 + 6x_2 - 7x_3 &= 2\\ -x_1 - 2x_2 + 3x_3 &= 0\\ 4x_2 - x_3 &= 3\end{aligned}$$
(b)
$$\begin{aligned}x + y + z &= 2\\ 2x + 3y \quad &= 2\\ 5x - 3y - 6z &= -9\end{aligned}$$

15. $$\begin{bmatrix}k&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&2\\0&2&-3\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix}=\begin{bmatrix}k&1&1\end{bmatrix}\begin{bmatrix}k+1\\k+2\\-1\end{bmatrix}=k^2+k+k+2-1=k^2+2k+1=(k+1)^2$$

The only value of k that satisfies the equation is $k=-1$.

17. $$\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}0&1&2\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}\begin{bmatrix}-2&3&1\end{bmatrix}=\begin{bmatrix}0&4&8\\0&2&4\end{bmatrix}+\begin{bmatrix}6&-9&-3\\2&-3&-1\end{bmatrix}=\begin{bmatrix}6&-5&5\\2&-1&3\end{bmatrix}$$

19. $$\begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}1&2\end{bmatrix}+\begin{bmatrix}2\\5\end{bmatrix}\begin{bmatrix}3&4\end{bmatrix}+\begin{bmatrix}3\\6\end{bmatrix}\begin{bmatrix}5&6\end{bmatrix}=\begin{bmatrix}1&2\\4&8\end{bmatrix}+\begin{bmatrix}6&8\\15&20\end{bmatrix}+\begin{bmatrix}15&18\\30&36\end{bmatrix}=\begin{bmatrix}22&28\\49&64\end{bmatrix}$$

21. $$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}-3r-4s-2t\\r\\-2s\\s\\t\\\frac{1}{3}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\\frac{1}{3}\end{bmatrix}+\begin{bmatrix}-3r\\r\\0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-4s\\0\\-2s\\s\\0\\0\end{bmatrix}+\begin{bmatrix}-2t\\0\\0\\0\\t\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\\frac{1}{3}\end{bmatrix}+r\begin{bmatrix}-3\\1\\0\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}-4\\0\\-2\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-2\\0\\0\\0\\1\\0\end{bmatrix}$$

23. The given matrix equation is equivalent to the linear system

$$a=4$$
$$3=d-2c$$
$$-1=d+2c$$
$$a+b=-2$$

After subtracting first equation from the fourth, adding the second to the third, and back-substituting, we obtain the solution: $a=4$, $b=-6$, $c=-1$, and $d=1$.

25. **(a)** If the i th row vector of A is $\begin{bmatrix}0&\cdots&0\end{bmatrix}$ then it follows from Formula (9) in Section 1.3 that

i th row vector of AB = $\begin{bmatrix}0&\cdots&0\end{bmatrix}B$ = $\begin{bmatrix}0&\cdots&0\end{bmatrix}$

(b) If the j th column vector of B is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that the j th column vector of $AB = A\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

27. Setting the left hand side $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}$ equal to $\begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$ yields

$$a_{11}x + a_{12}y + a_{13}z = x + y$$

$$a_{21}x + a_{22}y + a_{23}z = x - y$$

$$a_{31}x + a_{32}y + a_{33}z = 0$$

Assuming the entries of A are real numbers that do not depend on x, y, and z, this requires that the coefficients corresponding to the same variable on both sides of each equation must match.

Therefore, the only matrix satisfying the given condition is $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

29. **(a)** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

(b) Four square roots can be found: $\begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} -\sqrt{5} & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} \sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$, and $\begin{bmatrix} -\sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$.

33. The given matrix product represents $\begin{bmatrix} \text{the total cost of items purchased in January} \\ \text{the total cost of items purchased in February} \\ \text{the total cost of items purchased in March} \\ \text{the total cost of items purchased in April} \end{bmatrix}$.

True-False Exercises

(a) True. The main diagonal is only defined for square matrices.

(b) False. An $m \times n$ matrix has m row vectors and n column vectors.

(c) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not equal $BA = B$.

(d) False. The i th row vector of AB can be computed by multiplying the i th row vector of A by B.

(e) True. Using Formula (14), $((A^T)^T)_{ij} = (A^T)_{ji} = (A)_{ij}$.

(f) False. E.g., if $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$ then the trace of $AB=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is 0, which does not equal $\operatorname{tr}(A)\operatorname{tr}(B)=1$.

(g) False. E.g., if $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $B=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ then $(AB)^T=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ does not equal $A^TB^T=\begin{bmatrix}0&1\\0&0\end{bmatrix}$.

(h) True. The main diagonal entries in a square matrix A are the same as those in A^T.

(i) True. Since A^T is a 4×6 matrix, it follows from B^TA^T being a 2×6 matrix that B^T must be a 2×4 matrix. Consequently, B is a 4×2 matrix.

(j) True.

$$\operatorname{tr}\left(c\begin{bmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{n1}&\cdots&a_{nn}\end{bmatrix}\right)=\operatorname{tr}\left(\begin{bmatrix}ca_{11}&\cdots&ca_{1n}\\\vdots&\ddots&\vdots\\ca_{n1}&\cdots&ca_{nn}\end{bmatrix}\right)$$

$$=ca_{11}+\cdots+ca_{nn}=c\left(a_{11}+\cdots+a_{nn}\right)=c\operatorname{tr}\left(\begin{bmatrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{n1}&\cdots&a_{nn}\end{bmatrix}\right)$$

(k) True. The equality of the matrices $A-C$ and $B-C$ implies that $a_{ij}-c_{ij}=b_{ij}-c_{ij}$ for all i and j. Adding c_{ij} to both sides yields $a_{ij}=b_{ij}$ for all i and j. Consequently, the matrices A and B are equal.

(l) False. E.g., if $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $B=C=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ then $AC=BC=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ even though $A\neq B$.

(m) True. If A is a $p\times q$ matrix and B is an $r\times s$ matrix then AB being defined requires $q=r$ and BA being defined requires $s=p$. For the $p\times p$ matrix AB to be possible to add to the $q\times q$ matrix BA, we must have $p=q$.

(n) True. If the jth column vector of B is $\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that

the jth column vector of $AB = A\begin{bmatrix}0\\\vdots\\0\end{bmatrix}=\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$.

(o) False. E.g., if $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\1&0\end{bmatrix}$ then $BA=A$ does not have a column of zeros even though B does.

1.4 Inverses; Algebraic Properties of Matrices

1. **(a)** $A+(B+C)=(A+B)+C=\begin{bmatrix}7 & 2\\ 0 & -2\end{bmatrix}$ **(b)** $A(BC)=(AB)C=\begin{bmatrix}-34 & -21\\ 52 & 28\end{bmatrix}$

(c) $A(B+C)=AB+AC=\begin{bmatrix}14 & 15\\ 0 & -18\end{bmatrix}$ **(d)** $(a+b)C=aC+bC=\begin{bmatrix}-12 & -3\\ 9 & 6\end{bmatrix}$

3. **(a)** $\left(A^T\right)^T=A=\begin{bmatrix}3 & -1\\ 2 & 4\end{bmatrix}$ **(b)** $(AB)^T=B^TA^T=\begin{bmatrix}-1 & 4\\ 10 & -12\end{bmatrix}$

5. The determinant of A, $\det(A)=(2)(4)-(-3)(4)=20$, is nonzero. Therefore A is invertible and its inverse is $A^{-1}=\frac{1}{20}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & \frac{3}{20}\\ -\frac{1}{5} & \frac{1}{10}\end{bmatrix}$.

7. The determinant of C, $\det(C)=(2)(3)-(0)(0)=6$, is nonzero. Therefore C is invertible and its inverse is $C^{-1}=\frac{1}{6}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}$.

9. The determinant of $A=\begin{bmatrix}\frac{1}{2}\left(e^x+e^{-x}\right) & \frac{1}{2}\left(e^x-e^{-x}\right)\\ \frac{1}{2}\left(e^x-e^{-x}\right) & \frac{1}{2}\left(e^x+e^{-x}\right)\end{bmatrix}$,

$\det(A)=\frac{1}{4}\left(e^x+e^{-x}\right)^2-\frac{1}{4}\left(e^x-e^{-x}\right)^2=\frac{1}{4}\left(e^{2x}+2+e^{-2x}\right)-\frac{1}{4}\left(e^{2x}-2+e^{-2x}\right)=\frac{1}{4}(2+2)=1$ is nonzero. Therefore A is invertible and its inverse is $A^{-1}=\begin{bmatrix}\frac{1}{2}\left(e^x+e^{-x}\right) & -\frac{1}{2}\left(e^x-e^{-x}\right)\\ -\frac{1}{2}\left(e^x-e^{-x}\right) & \frac{1}{2}\left(e^x+e^{-x}\right)\end{bmatrix}$.

11. $A^T=\begin{bmatrix}2 & 4\\ -3 & 4\end{bmatrix}$; $\left(A^T\right)^{-1}=\frac{1}{(2)(4)-(4)(-3)}\begin{bmatrix}4 & -4\\ 3 & 2\end{bmatrix}=\frac{1}{20}\begin{bmatrix}4 & -4\\ 3 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & -\frac{1}{5}\\ \frac{3}{20} & \frac{1}{10}\end{bmatrix}$

$A^{-1}=\frac{1}{(2)(4)-(-3)(4)}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}=\frac{1}{20}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & \frac{3}{20}\\ -\frac{1}{5} & \frac{1}{10}\end{bmatrix}$; $\left(A^{-1}\right)^T=\begin{bmatrix}\frac{1}{5} & -\frac{1}{5}\\ \frac{3}{20} & \frac{1}{10}\end{bmatrix}$

13. $ABC=\begin{bmatrix}-18 & -12\\ 64 & 36\end{bmatrix}$; $(ABC)^{-1}=\frac{1}{(-18)(36)-(-12)(64)}\begin{bmatrix}36 & 12\\ -64 & -18\end{bmatrix}=\frac{1}{120}\begin{bmatrix}36 & 12\\ -64 & -18\end{bmatrix}=\begin{bmatrix}\frac{3}{10} & \frac{1}{10}\\ -\frac{8}{15} & -\frac{3}{20}\end{bmatrix}$

$C^{-1}B^{-1}A^{-1}=\left(\frac{1}{6}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\right)\begin{bmatrix}2 & -1\\ -5 & 3\end{bmatrix}\left(\frac{1}{20}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}2 & -1\\ -5 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{5} & \frac{3}{20}\\ -\frac{1}{5} & \frac{1}{10}\end{bmatrix}=\begin{bmatrix}\frac{3}{10} & \frac{1}{10}\\ -\frac{8}{15} & -\frac{3}{20}\end{bmatrix}$

15. From part (a) of Theorem 1.4.7 it follows that the inverse of $(7A)^{-1}$ is $7A$.

Thus $7A=\frac{1}{(-3)(-2)-(7)(1)}\begin{bmatrix}-2 & -7\\ -1 & -3\end{bmatrix}=\frac{1}{-1}\begin{bmatrix}-2 & -7\\ -1 & -3\end{bmatrix}=\begin{bmatrix}2 & 7\\ 1 & 3\end{bmatrix}$. Consequently, $A=\frac{1}{7}\begin{bmatrix}2 & 7\\ 1 & 3\end{bmatrix}=\begin{bmatrix}\frac{2}{7} & 1\\ \frac{1}{7} & \frac{3}{7}\end{bmatrix}$.

17. From part (a) of Theorem 1.4.7 it follows that the inverse of $(I+2A)^{-1}$ is $I+2A$.

Thus $I+2A=\frac{1}{(-1)(5)-(2)(4)}\begin{bmatrix}5 & -2\\ -4 & -1\end{bmatrix}=\frac{1}{-13}\begin{bmatrix}5 & -2\\ -4 & -1\end{bmatrix}=\begin{bmatrix}-\frac{5}{13} & \frac{2}{13}\\ \frac{4}{13} & \frac{1}{13}\end{bmatrix}$.

Consequently, $A=\frac{1}{2}\left(\begin{bmatrix}-\frac{5}{13} & \frac{2}{13}\\ \frac{4}{13} & \frac{1}{13}\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}-\frac{9}{13} & \frac{1}{13}\\ \frac{2}{13} & -\frac{6}{13}\end{bmatrix}$

19. **(a)** $A^3=AAA=\begin{bmatrix}41 & 15\\ 30 & 11\end{bmatrix}$

(b) $\left(A^3\right)^{-1}=\frac{1}{(41)(11)-(15)(30)}\begin{bmatrix}11 & -15\\ -30 & 41\end{bmatrix}=\begin{bmatrix}11 & -15\\ -30 & 41\end{bmatrix}$

(c) $A^2-2A+I=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}-2\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}11 & 4\\ 8 & 3\end{bmatrix}-\begin{bmatrix}6 & 2\\ 4 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}6 & 2\\ 4 & 2\end{bmatrix}$

21. **(a)** $A-2I=\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$ **(b)** $2A^2-A+I=\begin{bmatrix}20 & 7\\ 14 & 6\end{bmatrix}$ **(c)** $A^3-2A+I=\begin{bmatrix}36 & 13\\ 26 & 10\end{bmatrix}$

23. $AB=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & a\\ 0 & c\end{bmatrix}$; $BA=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}c & d\\ 0 & 0\end{bmatrix}$.

The matrices A and B commute if $\begin{bmatrix}0 & a\\ 0 & c\end{bmatrix}=\begin{bmatrix}c & d\\ 0 & 0\end{bmatrix}$, i.e.

$$0=c$$
$$a=d$$

$$0=0$$
$$c=0$$

Therefore, $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ commute if $c=0$ and $a=d$.

If we assign b and d the arbitrary values s and t, respectively, the general solution is given by the formulas

$$a=t,\quad b=s,\quad c=0,\quad d=t$$

25. $x_1=\frac{(5)(-1)-(-2)(3)}{(3)(5)-(-2)(4)}=\frac{1}{23}$, $x_2=\frac{(3)(3)-(4)(-1)}{(3)(5)-(-2)(4)}=\frac{13}{23}$

27. $x_1=\frac{(-3)(0)-(1)(-2)}{(6)(-3)-(1)(4)}=\frac{2}{-22}=-\frac{1}{11}$, $x_2=\frac{(6)(-2)-(4)(0)}{(6)(-3)-(1)(4)}=\frac{-12}{-22}=\frac{6}{11}$

29. $p(A)=A^2-9I=\begin{bmatrix}2 & 4\\ 8 & -6\end{bmatrix}$,

$p_1(A)=A+3I=\begin{bmatrix}6 & 1\\ 2 & 4\end{bmatrix}$, $p_2(A)=A-3I=\begin{bmatrix}0 & 1\\ 2 & -2\end{bmatrix}$, $p_1(A)p_2(A)=\begin{bmatrix}2 & 4\\ 8 & -6\end{bmatrix}$

31. (a) If $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ then $(A+B)(A-B)=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}1&-1\\0&0\end{bmatrix}=\begin{bmatrix}1&-1\\0&0\end{bmatrix}$ does not equal $A^2-B^2=\begin{bmatrix}1&0\\0&0\end{bmatrix}-\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}$.

(b) Using the properties in Theorem 1.4.1 we can write

$$(A+B)(A-B)=A(A-B)+B(A-B)=A^2-AB+BA-B^2$$

(c) If the matrices A and B commute (i.e., $AB=BA$) then $(A+B)(A-B)=A^2-B^2$.

33. (a) We can rewrite the equation

$$A^2+2A+I=O$$

$$A^2+2A=-I$$

$$-A^2-2A=I$$

$$A(-A-2I)=I$$

which shows that A is invertible and $A^{-1}=-A-2I$.

(b) Let $p(x)=c_nx^n+\cdots+c_2x^2+c_1x+c_0$ with $c_0\neq 0$. The equation $p(A)=O$ can be rewritten as

$$c_nA^n+\cdots+c_2A^2+c_1A+c_0I=O$$

$$c_nA^n+\cdots+c_2A^2+c_1A=-c_0I$$

$$-\frac{c_n}{c_0}A^n-\cdots-\frac{c_2}{c_0}A^2-\frac{c_1}{c_0}A=I$$

$$A\left(-\frac{c_n}{c_0}A^{n-1}-\cdots-\frac{c_2}{c_0}A-\frac{c_1}{c_0}I\right)=I$$

which shows that A is invertible and $A^{-1}=-\frac{c_n}{c_0}A^{n-1}-\cdots-\frac{c_2}{c_0}A-\frac{c_1}{c_0}I$.

35. If the ith row vector of A is $\begin{bmatrix}0&\cdots&0\end{bmatrix}$ then it follows from Formula (9) in Section 1.3 that ith row vector of $AB=\begin{bmatrix}0&\cdots&0\end{bmatrix}B=\begin{bmatrix}0&\cdots&0\end{bmatrix}$.

Consequently no matrix B can be found to make the product $AB=I$ thus A does not have an inverse.

If the jth column vector of A is $\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that

the jth column vector of $BA=B\begin{bmatrix}0\\\vdots\\0\end{bmatrix}=\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$.

Consequently no matrix B can be found to make the product $BA=I$ thus A does not have an inverse.

37. Letting $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$, the matrix equation $AX = I$ becomes

$$\begin{bmatrix} x_{11}+x_{31} & x_{12}+x_{32} & x_{13}+x_{33} \\ x_{11}+x_{21} & x_{12}+x_{22} & x_{13}+x_{23} \\ x_{21}+x_{31} & x_{22}+x_{32} & x_{23}+x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Setting the first columns on both sides equal yields the system

$$\begin{aligned} x_{11}+x_{31} &= 1 \\ x_{11}+x_{21} &= 0 \\ x_{21}+x_{31} &= 0 \end{aligned}$$

Subtracting the second and third equations from the first leads to $-2x_{21} = 1$. Therefore $x_{21} = -\frac{1}{2}$ and (after substituting this into the remaining equations) $x_{11} = x_{31} = \frac{1}{2}$.

The second and the third columns can be treated in a similar manner to result in
$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. We conclude that A invertible and its inverse is $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

39. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$

$= (B^{-1}A^{-1})(AC^{-1})((C^{-1})^{-1}(D^{-1})^{-1})D^{-1}$ ← Theorem 1.4.6

$= (B^{-1}A^{-1})(AC^{-1})(CD)D^{-1}$ ← Theorem 1.4.7(a)

$= B^{-1}(A^{-1}A)(C^{-1}C)(DD^{-1})$ ← Theorem 1.4.1(c)

$= B^{-1}III$ ← Formula (1) in Section 1.4

$= B^{-1}$ ← Property $AI = IA = A$ in Section 1.4

41. If $R = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix}$ and $C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ then $CR = \begin{bmatrix} c_1r_1 & \cdots & c_1r_n \\ \vdots & \ddots & \vdots \\ c_nr_1 & \cdots & c_nr_n \end{bmatrix}$ and

$RC = \left[r_1c_1 + \cdots + r_nc_n\right] = \left[\operatorname{tr}(CR)\right]$.

43. **(a)** Assuming A is invertible, we can multiply (on the left) each side of the equation by A^{-1}:

$AB = AC$

$A^{-1}(AB) = A^{-1}(AC)$ ← Multiply (on the left) each side by A^{-1}

$(A^{-1}A)B = (A^{-1}A)C$ ← Theorem 1.4.1(c)

$IB = IC$ ← Formula (1) in Section 1.4

$B = C$ ← Property $AI = IA = A$ in Section 1.4

(b) If A is not an invertible matrix then $AB = AC$ does not generally imply $B = C$ as evidenced by Example 3.

45. (a) $A(A^{-1} + B^{-1})B(A+B)^{-1}$

$= (AA^{-1}B + AB^{-1}B)(A+B)^{-1}$ ← Theorem 1.4.1(d) and (e)

$= (IB + AI)(A+B)^{-1}$ ← Formula (1) in Section 1.4

$= (B + A)(A+B)^{-1}$ ← Property $AI = IA = A$ in Section 1.4

$= (A + B)(A+B)^{-1}$ ← Theorem 1.4.1(a)

$= I$ ← Formula (1) in Section 1.4

(b) We can multiply each side of the equality from part (a) on the left by A^{-1}, then on the right by A to obtain

$$(A^{-1} + B^{-1})B(A+B)^{-1}A = I$$

which shows that if A, B, and $A+B$ are invertible then so is $A^{-1} + B^{-1}$.
Furthermore, $\left(A^{-1} + B^{-1}\right)^{-1} = B\left(A+B\right)^{-1}A$.

47. Applying Theorem 1.4.1(d) and (g), property $AI = IA = A$, and the assumption $A^k = O$ we can write

$$\begin{aligned}
&\left(I - A\right)\left(I + A + A^2 + \cdots + A^{k-2} + A^{k-1}\right) \\
&= I - A + A - A^2 + A^2 - A^3 + \cdots + A^{k-2} - A^{k-1} + A^{k-1} - A^k \\
&= I - A^k \\
&= I - O \\
&= I
\end{aligned}$$

True-False Exercises

(a) False. A and B are inverses of one another if and only if $AB = BA = I$.

(b) False. $\left(A+B\right)^2 = \left(A+B\right)\left(A+B\right) = A^2 + AB + BA + B^2$ does not generally equal $A^2 + 2AB + B^2$ since AB may not equal BA.

(c) False. $\left(A-B\right)\left(A+B\right) = A^2 + AB - BA - B^2$ does not generally equal $A^2 - B^2$ since AB may not equal BA.

(d) False. $\left(AB\right)^{-1} = B^{-1}A^{-1}$ does not generally equal $A^{-1}B^{-1}$.

(e) False. $\left(AB\right)^T = B^T A^T$ does not generally equal $A^T B^T$.

(f) True. This follows from Theorem 1.4.5.

(g) True. This follows from Theorem 1.4.8.

(h) True. This follows from Theorem 1.4.9. (The inverse of A^T is the transpose of A^{-1}.)

(i) False. $p(I)=(a_0+a_1+a_2+\cdots+a_m)I$.

(j) True.
If the i th row vector of A is $\begin{bmatrix}0 & \cdots & 0\end{bmatrix}$ then it follows from Formula (9) in Section 1.3 that
i th row vector of $AB = \begin{bmatrix}0 & \cdots & 0\end{bmatrix}B = \begin{bmatrix}0 & \cdots & 0\end{bmatrix}$.

Consequently no matrix B can be found to make the product $AB=I$ thus A does not have an inverse.

If the j th column vector of A is $\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that

the j th column vector of $BA = B\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$.

Consequently no matrix B can be found to make the product $BA=I$ thus A does not have an inverse.

(k) False. E.g. I and $-I$ are both invertible but $I+(-I)=O$ is not.

1.5 Elementary Matrices and a Method for Finding A^{-1}

1. **(a)** Elementary matrix (corresponds to adding -5 times the first row to the second row)

(b) Not an elementary matrix

(c) Not an elementary matrix

(d) Not an elementary matrix

3. **(a)** Add 3 times the second row to the first row: $\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$

(b) Multiply the first row by $-\frac{1}{7}$: $\begin{bmatrix}-\frac{1}{7} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

(c) Add 5 times the first row to the third row: $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 5 & 0 & 1\end{bmatrix}$

(d) Interchange the first and third rows: $\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

5. **(a)** Interchange the first and second rows: $EA = \begin{bmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{bmatrix}$

(b) Add -3 times the second row to the third row: $EA = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{bmatrix}$

(c) Add 4 times the third row to the first row: $EA = \begin{bmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

7. **(a)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (B was obtained from A by interchanging the first row and the third row)

(b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (A was obtained from B by interchanging the first row and the third row)

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (C was obtained from A by adding -2 times the first row to the third row)

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ (A was obtained from C by adding 2 times the first row to the third row)

9. **(a)** (Method I: using Theorem 1.4.5)

The determinant of A, $\det(A) = (1)(7) - (4)(2) = -1$, is nonzero. Therefore A is invertible and its inverse is

$A^{-1} = \frac{1}{-1}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$.

(Method II: using the inversion algorithm)

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \longleftarrow \text{The second row was multiplied by } -1.$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right] \longleftarrow -4 \text{ times the second row was added to the first row.}$$

The inverse is $\begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$.

(b) (Method I: using Theorem 1.4.5)
The determinant of A, $\det(A) = (2)(8) - (-4)(-4) = 0$. Therefore A is not invertible. (Method II: using the inversion algorithm)

$\left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ -4 & 8 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$ ← 2 times the first row was added to the second row.

A row of zeros was obtained on the left side, therefore A is not invertible.

11. (a) $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$ ← -2 times the first row was added to the second row and -1 times the first row was added to the third row.

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$ ← 2 times the second row was added to the third row.

$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ ← The third row was multiplied by -1.

$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ ← 3 times the third row was added to the second row and -3 times the third row was added to the first row.

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$ ← -2 times the second row was added to the first row.

The inverse is $\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$.

(b)

$$\left[\begin{array}{rrr|rrr} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The first row was multiplied by } -1.$$

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right] \longleftarrow \begin{array}{l} -2 \text{ times the first row was added to the second row and} \\ 4 \text{ times the first row was added to the third row.} \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right] \longleftarrow \text{The second row was added to the third row.}$$

A row of zeros was obtained on the left side, therefore the matrix is not invertible.

13.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \longleftarrow -1 \text{ times the first row was added to the third row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \longleftarrow \text{The third row was multiplied by } -\tfrac{1}{2}.$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \longleftarrow \begin{array}{l} -1 \text{ times the third row was added to the second and} \\ -1 \text{ times the third row was added to the first row} \end{array}$$

The inverse is $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

15.

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \longleftarrow \begin{array}{l} -1 \text{ times the first row was added to the second and} \\ -1 \text{ times the first row was added to the third row} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longleftarrow -6 \text{ times the third row was added to the first row}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longleftarrow -6 \text{ times the second row was added to the first row}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

The inverse is $\begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

17.

$$\left[\begin{array}{cccc|cccc} 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The first and second rows were interchanged.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow -2 \text{ times the first row was added to the second.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right] \longleftarrow \text{The second and fourth rows were interchanged.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right] \longleftarrow \text{The second row was multiplied by } -1.$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right] \longleftarrow 8 \text{ times the second row was added to the fourth.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right] \longleftarrow \text{The third row was multiplied by } \tfrac{1}{2}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 40 & 1 & -2 & -4 & -8 \end{array}\right] \longleftarrow -8 \text{ times the third row was added to the fourth row.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right] \longleftarrow \text{The fourth row was multiplied by } \tfrac{1}{40}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right] \longleftarrow -5 \text{ times the fourth row was added to the second row.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & -6 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right] \longleftarrow -4 \text{ times the third row was added to the second row and } -12 \text{ times the third row was added to the first row.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$ ← -2 times the second row was added to the first row.

The inverse is $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{bmatrix}$.

19. **(a)** $\left[\begin{array}{cccc|cccc} k_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_4} \end{array}\right]$$ ← The first row was multiplied by $1/k_1$, the second row was multiplied by $1/k_2$, the third row was multiplied by $1/k_3$, and the fourth row was multiplied by $1/k_4$.

The inverse is $\begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix}$.

(b) $\left[\begin{array}{cccc|cccc} k & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & \frac{1}{k} & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{k} & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$ ← First row and third row were both multiplied by $1/k$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$

$-\frac{1}{k}$ times the fourth row was added to the third row and $-\frac{1}{k}$ times the second row was added to the first row.

The inverse is $\begin{bmatrix} \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

21. It follows from parts (a) and (d) of Theorem 1.5.3 that a square matrix is invertible if and only if its reduced row echelon form is identity.

$$\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & c \\ 1 & c & c \\ c & c & c \end{bmatrix} \longleftarrow$$ The first and third rows were interchanged.

$$\begin{bmatrix} 1 & 1 & c \\ 0 & -1+c & 0 \\ 0 & 0 & c-c^2 \end{bmatrix} \longleftarrow$$ -1 times the first row was added to the second row and $-c$ times the first row was added to the third row.

If $c-c^2=c(1-c)=0$ or $-1+c=0$, i.e. if $c=0$ or $c=1$ the last matrix contains at least one row of zeros, therefore it cannot be reduced to I by elementary row operations.

Otherwise (if $c\neq 0$ and $c\neq 1$), multiplying the second row by $\frac{1}{-1+c}$ and multiplying the third row by $\frac{1}{c-c^2}$ would result in a row echelon form with 1's on the main diagonal. Subsequent elementary row operations would then lead to the identity matrix.

We conclude that for any value of c other than 0 and 1 the matrix is invertible.

23. We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrix:

$$A=\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix} \longleftarrow$$ 2 times the second row was added to the first. $E_1=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 \\ 0 & -8 \end{bmatrix} \longleftarrow$$ -2 times the first row was added to the second. $E_2=\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ ⟵ The second row was multiplied by $-\frac{1}{8}$. $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ⟵ -5 times the second row was added to the first. $E_4 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

Since $E_4E_3E_2E_1A = I$, then

$A = \left(E_4E_3E_2E_1\right)^{-1} I = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ and

$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

25. We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrix:

$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$ ⟵ The second row was multiplied by $\frac{1}{4}$. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ⟵ $-\frac{3}{4}$ times the third row was added to the second. $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ⟵ 2 times the third row was added to the first row. $E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $E_3E_2E_1A = I$, we have $A = \left(E_3E_2E_1\right)^{-1} I = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$A^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

27. Let us perform a sequence of elementary row operations to produce B from A. As we do so, we keep track of each corresponding elementary matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \longleftarrow \text{-1 times the first row was added to the second row.} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \longleftarrow \text{-1 times the second row was added to the first row.} \quad E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix} \longleftarrow \text{-1 times the first row was added to the third row.} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Since $E_3E_2E_1A = B$, the equality $CA = B$ is satisfied by the matrix

$$C = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

29. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$ cannot result from interchanging two rows of I_3 (since that would create a nonzero entry above the main diagonal).

A can result from multiplying the third row of I_3 by a nonzero number c
(in this case, $a = b = 0,\ c \neq 0$).

The other possibilities are that A can be obtained by adding a times the first row to the third $(b = 0, c = 1)$ or by adding b times the second row to the third $(a = 0, c = 1)$.

In all three cases, at least one entry in the third row must be zero.

True-False Exercises

(a) False. An elementary matrix results from performing a *single* elementary row operation on an identity matrix; a product of two elementary matrices would correspond to a sequence of two such operations instead, which generally is not equivalent to a single elementary operation.

(b) True. This follows from Theorem 1.5.2.

(c) True. If A and B are row equivalent then there exist elementary matrices $E_1,\ldots,E_p$ such that $B = E_p \cdots E_1 A$. Likewise, if B and C are row equivalent then there exist elementary matrices $E_1^*,\ldots,E_q^*$ such that $C = E_q^* \cdots E_1^* B$. Combining the two equalities yields $C = E_q^* \cdots E_1^* E_p \cdots E_1 A$ therefore A and C are row equivalent.

(d) True. A homogeneous system $A\mathbf{x} = \mathbf{0}$ has either one solution (the trivial solution) or infinitely many solutions. If A is not invertible, then by Theorem 1.5.3 the system cannot have just one solution. Consequently, it must have infinitely many solutions.

(e) True. If the matrix A is not invertible then by Theorem 1.5.3 its reduced row echelon form is not I_n. However, the matrix resulting from interchanging two rows of A (an elementary row operation) must have the same reduced row echelon form as A does, so by Theorem 1.5.3 that matrix is not invertible either.

(f) True. Adding a multiple of the first row of a matrix to its second row is an elementary row operation. Denoting by E be the corresponding elementary matrix we can write $(EA)^{-1} = A^{-1}E^{-1}$ so the resulting matrix EA is invertible if A is.

(g) False. For instance, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

1.6 More on Linear Systems and Invertible Matrices

1. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

We begin by inverting the coefficient matrix A

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -5 & 1 \end{array}\right] \longleftarrow -5 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 1 & -5 & 1 \end{array}\right] \longleftarrow -1 \text{ times the second row was added to the first row.}$$

Since $A^{-1} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$:

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, i.e., $x_1 = 3$, $x_2 = -1$.

3. The given system can be written in matrix form as $A\mathbf{x}=\mathbf{b}$, where $A=\begin{bmatrix}1&3&1\\2&2&1\\2&3&1\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}4\\-1\\3\end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\2&2&1&0&1&0\\2&3&1&0&0&1\end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&-4&-1&-2&1&0\\0&-3&-1&-2&0&1\end{array}\right] \longleftarrow \begin{array}{l}-2 \text{ times the first row was added to the second and}\\ -2 \text{ times the first row was added to the third row.}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&-4&-1&-2&1&0\\0&1&0&0&-1&1\end{array}\right] \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&1&0&0&-1&1\\0&-4&-1&-2&1&0\end{array}\right] \longleftarrow \text{The second and third rows were interchanged.}$$

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&1&0&0&-1&1\\0&0&-1&-2&-3&4\end{array}\right] \longleftarrow 4 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&1&0&0&-1&1\\0&0&1&2&3&-4\end{array}\right] \longleftarrow \text{The third row was multiplied by } -1.$$

$$\left[\begin{array}{ccc|ccc}1&3&0&-1&-3&4\\0&1&0&0&-1&1\\0&0&1&2&3&-4\end{array}\right] \longleftarrow -1 \text{ times the third row was added to the first row.}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&-1&0&1\\0&1&0&0&-1&1\\0&0&1&2&3&-4\end{array}\right] \longleftarrow -3 \text{ times the second row was added to the first row.}$$

Since $A^{-1}=\begin{bmatrix}-1&0&1\\0&-1&1\\2&3&-4\end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x}=A^{-1}\mathbf{b}$:

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1&0&1\\0&-1&1\\2&3&-4\end{bmatrix}\begin{bmatrix}4\\-1\\3\end{bmatrix}=\begin{bmatrix}-1\\4\\-7\end{bmatrix}$, i.e., $x_1=-1, x_2=4,$ and $x_3=-7$.

5. The given system can be written in matrix form as $A\mathbf{x}=\mathbf{b}$, where $A=\begin{bmatrix}1&1&1\\1&1&-4\\-4&1&1\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}5\\10\\0\end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{rrr|rrr}1&1&1&1&0&0\\1&1&-4&0&1&0\\-4&1&1&0&0&1\end{array}\right]$$ ← The identity matrix was adjoined to the coefficient matrix.

$$\left[\begin{array}{rrr|rrr}1&1&1&1&0&0\\0&0&-5&-1&1&0\\0&5&5&4&0&1\end{array}\right]$$ ← -1 times the first row was added to the second row and 4 times the first row was added to the third row.

$$\left[\begin{array}{rrr|rrr}1&1&1&1&0&0\\0&5&5&4&0&1\\0&0&-5&-1&1&0\end{array}\right]$$ ← The second and third rows were interchanged.

$$\left[\begin{array}{rrr|rrr}1&1&1&1&0&0\\0&1&1&\frac{4}{5}&0&\frac{1}{5}\\0&0&1&\frac{1}{5}&-\frac{1}{5}&0\end{array}\right]$$ ← The second row was multiplied by $\frac{1}{5}$ and the third row was multiplied by $-\frac{1}{5}$.

$$\left[\begin{array}{rrr|rrr}1&1&0&\frac{4}{5}&\frac{1}{5}&0\\0&1&0&\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\0&0&1&\frac{1}{5}&-\frac{1}{5}&0\end{array}\right]$$ ← -1 times the third row was added to the second row and to the first row.

$$\left[\begin{array}{rrr|rrr}1&0&0&\frac{4}{5}&0&-\frac{1}{5}\\0&1&0&\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\0&0&1&\frac{1}{5}&-\frac{1}{5}&0\end{array}\right]$$ ← -1 times the second row was added to the first row.

Since $A^{-1}=\begin{bmatrix}\frac{1}{5}&0&-\frac{1}{5}\\\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\\frac{1}{5}&-\frac{1}{5}&0\end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x}=A^{-1}\mathbf{b}$:

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{5}&0&-\frac{1}{5}\\\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\\frac{1}{5}&-\frac{1}{5}&0\end{bmatrix}\begin{bmatrix}5\\10\\0\end{bmatrix}=\begin{bmatrix}1\\5\\-1\end{bmatrix}$$, i.e., $x=1, y=5,$ and $z=-1$.

7. The given system can be written in matrix form as $A\mathbf{x}=\mathbf{b}$, where $A=\begin{bmatrix}3&5\\1&2\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$. We begin by inverting the coefficient matrix A

$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the coefficient matrix.

$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array}\right]$ ← The first and second rows were interchanged.

$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right]$ ← -3 times the first row was added to the second row.

$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right]$ ← The second row was multiplied by -1.

$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right]$ ← -2 times the second row was added to the first row.

Since $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$:

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 \\ -b_1 + 3b_2 \end{bmatrix}$, i.e., $x_1 = 2b_1 - 5b_2$, $x_2 = -b_1 + 3b_2$.

9. $\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 3 & 2 & 4 & 5 \end{array}\right]$ ← We augmented the coefficient matrix with two columns of constants on the right hand sides of the systems (i) and (ii) – refer to Example 2.

$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 17 & 1 & 11 \end{array}\right]$ ← -3 times the first row was added to the second row.

$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right]$ ← The second row was multiplied by $\frac{1}{17}$.

$\left[\begin{array}{cc|c|c} 1 & 0 & \frac{22}{17} & \frac{21}{17} \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right]$ ← 5 times the second row was added to the first row.

We conclude that the solutions of the two systems are:

(i) $x_1 = \frac{22}{17}$, $x_2 = \frac{1}{17}$ **(ii)** $x_1 = \frac{21}{17}$, $x_2 = \frac{11}{17}$

11. $\left[\begin{array}{cc|c|c|c|c} 4 & -7 & 0 & -4 & -1 & -5 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right]$ ← We augmented the coefficient matrix with four columns of constants on the right hand sides of the systems (i), (ii), (iii), and (iv) – refer to Example 2.

$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 4 & -7 & 0 & -4 & -1 & -5 \end{array}\right]$ ← The first and second rows were interchanged.

$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & -15 & -4 & -28 & -13 & -9 \end{array}\right]$ ← -4 times the first row was added to the second row.

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & 1 & \frac{4}{15} & \frac{28}{15} & \frac{13}{15} & \frac{3}{5} \end{array}\right] \longleftarrow \text{The second row was multiplied by } -\tfrac{1}{15}.$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 0 & \frac{7}{15} & \frac{34}{15} & \frac{19}{15} & -\frac{1}{5} \\ 0 & 1 & \frac{4}{15} & \frac{28}{15} & \frac{13}{15} & \frac{3}{5} \end{array}\right] \longleftarrow -2 \text{ times the second row was added to the first row.}$$

We conclude that the solutions of the four systems are:

(i) $x_1 = \dfrac{7}{15},\ x_2 = \dfrac{4}{15}$ **(ii)** $x_1 = \dfrac{34}{15},\ x_2 = \dfrac{28}{15}$

(iii) $x_1 = \dfrac{19}{15},\ x_2 = \dfrac{13}{15}$ **(iv)** $x_1 = -\dfrac{1}{5},\ x_2 = \dfrac{3}{5}$

13.

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array}\right] \longleftarrow \text{The augmented matrix for the system.}$$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & 2b_1 + b_2 \end{array}\right] \longleftarrow 2 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{2}{7}b_1 + \frac{1}{7}b_2 \end{array}\right] \longleftarrow \text{The second row was multiplied by } \tfrac{1}{7}.$$

The system is consistent for all values of b_1 and b_2.

15.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right] \longleftarrow \text{The augmented matrix for the system.}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & -3 & 12 & 3b_1 + b_3 \end{array}\right] \longleftarrow \begin{array}{l} -4 \text{ times the first row was added to the second row} \\ \text{and } 3 \text{ times the first row was added to the third row.} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right] \longleftarrow \text{The second row was added to the third row.}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & -\frac{4}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right] \longleftarrow \text{The second row was multiplied by } \tfrac{1}{3}.$$

The system is consistent if and only if $-b_1 + b_2 + b_3 = 0$, i.e. $b_1 = b_2 + b_3$.

17. $\left[\begin{array}{rrrr|l} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right]$ ← The augmented matrix for the system.

$\left[\begin{array}{rrrr|r} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 0 & 1 & -11 & -5 & -4b_1+b_4 \end{array}\right]$ ← 2 times the first row was added to the second row, 3 times the first row was added to the third row, and -4 times the first row was added to the fourth row.

$\left[\begin{array}{rrrr|r} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 0 & 1 & -11 & -5 & -4b_1+b_4 \end{array}\right]$ ← The second row was multiplied by -1.

$\left[\begin{array}{rrrr|r} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$ ← The second row was added to the third row and -1 times the second row was added to the fourth row.

The system is consistent for all values of b_1, b_2, b_3, and b_4 that satisfy the equations $b_1-b_2+b_3=0$ and $-2b_1+b_2+b_4=0$.

These equations form a linear system in the variables b_1, b_2, b_3, and b_4 whose augmented matrix $\begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{bmatrix}$. Therefore the system is consistent if $b_1=b_3+b_4$ and $b_2=2b_3+b_4$.

19. $X=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$. Let us find $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1}$:

$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the matrix.

$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$ ← -2 times the first row was added to the second row.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{-2 times the third row was added to the second row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & -1 & 4 & -2 & 5 \end{array}\right] \longleftarrow \text{-2 times the second row was added to the third row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \text{The third row was multiplied by -1.}$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 5 & -2 & 5 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \text{-1 times the third row was added to the first row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \text{The second row was added to the first row.}$$

Using $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix}$ we obtain

$$X = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix}\begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$$

True-False Exercises

(a) True. By Theorem 1.6.1, if a system of linear equation has more than one solution then it must have infinitely many.

(b) True. If A is a square matrix such that $A\mathbf{x} = \mathbf{b}$ has a unique solution then the reduced row echelon form of A must be I. Consequently, $A\mathbf{x} = \mathbf{c}$ must have a unique solution as well.

(c) True. Since B is a square matrix then by Theorem 1.6.3(b) $AB = I_n$ implies $B = A^{-1}$.
Therefore, $BA = A^{-1}A = I_n$.

(d) True. Since A and B are row equivalent matrices, it must be possible to perform a sequence of elementary row operations on A resulting in B. Let E be the product of the corresponding elementary matrices, i.e., $EA = B$. Note that E must be an invertible matrix thus $A = E^{-1}B$.
Any solution of $A\mathbf{x} = \mathbf{0}$ is also a solution of $B\mathbf{x} = \mathbf{0}$ since $B\mathbf{x} = EA\mathbf{x} = E\mathbf{0} = \mathbf{0}$.
Likewise, any solution of $B\mathbf{x} = \mathbf{0}$ is also a solution of $A\mathbf{x} = \mathbf{0}$ since $A\mathbf{x} = E^{-1}B\mathbf{x} = E^{-1}\mathbf{0} = \mathbf{0}$.

(e) True. If $\left(S^{-1}AS\right)\mathbf{x}=\mathbf{b}$ then $SS^{-1}AS\mathbf{x}=A\left(S\mathbf{x}\right)=S\mathbf{b}$. Consequently, $\mathbf{y}=S\mathbf{x}$ is a solution of $A\mathbf{y}=S\mathbf{b}$.

(f) True. $A\mathbf{x}=4\mathbf{x}$ is equivalent to $A\mathbf{x}=4I_n\mathbf{x}$, which can be rewritten as $\left(A-4I_n\right)\mathbf{x}=\mathbf{0}$. By Theorem 1.6.4, this homogeneous system has a unique solution (the trivial solution) if and only if its coefficient matrix $A-4I_n$ is invertible.

(g) True. If AB were invertible, then by Theorem 1.6.5 both A and B would be invertible.

1.7 Diagonal, Triangular, and Symmetric Matrices

1. **(a)** The matrix is upper triangular. It is invertible (its diagonal entries are both nonzero).

(b) The matrix is lower triangular. It is not invertible (its diagonal entries are zero).

(c) This is a diagonal matrix, therefore it is also both upper and lower triangular. It is invertible (its diagonal entries are all nonzero).

(d) The matrix is upper triangular. It is not invertible (its diagonal entries include a zero).

3. $$\begin{bmatrix}3&0&0\\0&-1&0\\0&0&2\end{bmatrix}\begin{bmatrix}2&1\\-4&1\\2&5\end{bmatrix}=\begin{bmatrix}(3)(2)&(3)(1)\\(-1)(-4)&(-1)(1)\\(2)(2)&(2)(5)\end{bmatrix}=\begin{bmatrix}6&3\\4&-1\\4&10\end{bmatrix}$$

5. $$\begin{bmatrix}5&0&0\\0&2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}-3&2&0&4&-4\\1&-5&3&0&3\\-6&2&2&2&2\end{bmatrix}=\begin{bmatrix}(5)(-3)&(5)(2)&(5)(0)&(5)(4)&(5)(-4)\\(2)(1)&(2)(-5)&(2)(3)&(2)(0)&(2)(3)\\(-3)(-6)&(-3)(2)&(-3)(2)&(-3)(2)&(-3)(2)\end{bmatrix}$$

$$=\begin{bmatrix}-15&10&0&20&-20\\2&-10&6&0&6\\18&-6&-6&-6&-6\end{bmatrix}$$

7. $$A^2=\begin{bmatrix}1^2&0\\0&(-2)^2\end{bmatrix}=\begin{bmatrix}1&0\\0&4\end{bmatrix},\qquad A^{-2}=\begin{bmatrix}1^{-2}&0\\0&(-2)^{-2}\end{bmatrix}=\begin{bmatrix}1&0\\0&\frac{1}{4}\end{bmatrix},\quad A^{-k}=\begin{bmatrix}1^{-k}&0\\0&(-2)^{-k}\end{bmatrix}=\begin{bmatrix}1&0\\0&\frac{1}{(-2)^k}\end{bmatrix}$$

9. $$A^2=\begin{bmatrix}\left(\frac{1}{2}\right)^2&0&0\\0&\left(\frac{1}{3}\right)^2&0\\0&0&\left(\frac{1}{4}\right)^2\end{bmatrix}=\begin{bmatrix}\frac{1}{4}&0&0\\0&\frac{1}{9}&0\\0&0&\frac{1}{16}\end{bmatrix},\qquad A^{-2}=\begin{bmatrix}\left(\frac{1}{2}\right)^{-2}&0&0\\0&\left(\frac{1}{3}\right)^{-2}&0\\0&0&\left(\frac{1}{4}\right)^{-2}\end{bmatrix}=\begin{bmatrix}4&0&0\\0&9&0\\0&0&16\end{bmatrix},$$

$$A^{-k}=\begin{bmatrix}\left(\frac{1}{2}\right)^{-k}&0&0\\0&\left(\frac{1}{3}\right)^{-k}&0\\0&0&\left(\frac{1}{4}\right)^{-k}\end{bmatrix}=\begin{bmatrix}2^k&0&0\\0&3^k&0\\0&0&4^k\end{bmatrix}$$

11. $\begin{bmatrix} (1)(2)(0) & 0 & 0 \\ 0 & (0)(5)(2) & 0 \\ 0 & 0 & (3)(0)(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 1^{39} & 0 \\ 0 & (-1)^{39} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

15. **(a)** $\begin{bmatrix} au & av \\ bw & bx \\ cy & cz \end{bmatrix}$ **(b)** $\begin{bmatrix} ra & sb & tc \\ ua & vb & wc \\ xa & yb & zc \end{bmatrix}$

17. **(a)** $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 3 & 7 & 2 \\ 3 & 1 & -8 & -3 \\ 7 & -8 & 0 & 9 \\ 2 & -3 & 9 & 0 \end{bmatrix}$

19. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this upper triangular matrix has a 0 on its diagonal, it is not invertible.

21. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this lower triangular matrix has all four diagonal entries nonzero, it is invertible.

23. $AB = \begin{bmatrix} (3)(-1) & \times & \times \\ 0 & (1)(5) & \times \\ 0 & 0 & (-1)(6) \end{bmatrix}$. The diagonal entries of AB are: $-3, 5, -6$.

25. The matrix is symmetric if and only if $a+5=-3$. In order for A to be symmetric, we must have $a=-8$.

27. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Therefore, the given upper triangular matrix is invertible for any real number x such that $x \neq 1$, $x \neq -2$, and $x \neq 4$.

29. By Theorem 1.7.1, A^{-1} is also an upper triangular or lower triangular invertible matrix. Its diagonal entries must all be nonzero - they are reciprocals of the corresponding diagonal entries of the matrix A.

31. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

33. $AB = \begin{bmatrix} (-1)(2)+(2)(0)+(5)(0) & (-1)(-8)+(2)(2)+(5)(0) & (-1)(0)+(2)(1)+(5)(3) \\ (0)(2)+(1)(0)+(3)(0) & (0)(-8)+(1)(2)+(3)(0) & (0)(0)+(1)(1)+(3)(3) \\ (0)(2)+(0)(0)+(-4)(0) & (0)(-8)+(0)(2)+(-4)(0) & (0)(0)+(0)(1)+(-4)(3) \end{bmatrix}$

$= \begin{bmatrix} -2 & 12 & 17 \\ 0 & 2 & 10 \\ 0 & 0 & -12 \end{bmatrix}$. Since this is an upper triangular matrix, we have verified Theorem 1.7.1(b).

35. **(a)** $A^{-1} = \frac{1}{(2)(3)-(-1)(-1)} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$ is symmetric, therefore we verified Theorem 1.7.4.

(b) $\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ -2 & 1 & -7 & 0 & 1 & 0 \\ 3 & -7 & 4 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the matrix A.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -1 & 2 & 1 & 0 \\ 0 & -1 & -5 & -3 & 0 & 1 \end{array}\right]$ ← 2 times the first row was added to the second row and -3 times the first row was added to the third row.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -3 & 0 & 1 \\ 0 & -3 & -1 & 2 & 1 & 0 \end{array}\right]$ ← The second and third rows were interchanged.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -1 \\ 0 & -3 & -1 & 2 & 1 & 0 \end{array}\right]$ ← The second row was multiplied by -1.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -1 \\ 0 & 0 & 14 & 11 & 1 & -3 \end{array}\right]$ ← 3 times the second row was added to the third row.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -1 \\ 0 & 0 & 1 & \frac{11}{14} & \frac{1}{14} & -\frac{3}{14} \end{array}\right]$ ← The third row was multiplied by $\frac{1}{14}$.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & -\frac{19}{14} & -\frac{3}{14} & \frac{9}{14} \\ 0 & 1 & 0 & -\frac{13}{14} & -\frac{5}{14} & \frac{1}{14} \\ 0 & 0 & 1 & \frac{11}{14} & \frac{1}{14} & -\frac{3}{14} \end{array}\right]$ ← -5 times the third row was added to the second row and -3 times the third row was added to the first row.

$\left[\begin{array}{rrr|rrr} 1 & 0 & 3 & -\frac{45}{14} & -\frac{13}{14} & \frac{11}{14} \\ 0 & 1 & 0 & -\frac{13}{14} & -\frac{5}{14} & \frac{1}{14} \\ 0 & 0 & 1 & \frac{11}{14} & \frac{1}{14} & -\frac{3}{14} \end{array}\right]$ ← 2 times the second row was added to the first row.

Since $A^{-1} = \begin{bmatrix} -\frac{45}{14} & -\frac{13}{14} & \frac{11}{14} \\ -\frac{13}{14} & -\frac{5}{14} & \frac{1}{14} \\ \frac{11}{14} & \frac{1}{14} & -\frac{3}{14} \end{bmatrix}$ is symmetric, we have verified Theorem 1.7.4

37. **(a)** $a_{ji} = j^2 + i^2 = i^2 + j^2 = a_{ij}$ for all i and j therefore A is symmetric.

(b) $a_{ji} = j^2 - i^2$ does not generally equal $a_{ij} = i^2 - j^2$ for $i \neq j$ therefore A is not symmetric (unless $n = 1$).

(c) $a_{ji} = 2j + 2i = 2i + 2j = a_{ij}$ for all i and j therefore A is symmetric.

(d) $a_{ji} = 2j^2 + 2i^3$ does not generally equal $a_{ij} = 2i^2 + 2j^3$ for $i \neq j$ therefore A is not symmetric (unless $n = 1$).

39. For a general upper triangular 2×2 matrix $A=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ we have

$$A^3=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$$=\begin{bmatrix} a^2 & ab+bc \\ 0 & c^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}=\begin{bmatrix} a^3 & a^2b+(ab+bc)c \\ 0 & c^3 \end{bmatrix}=\begin{bmatrix} a^3 & (a^2+ac+c^2)b \\ 0 & c^3 \end{bmatrix}$$

Setting $A^3=\begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$ we obtain the equations $a^3=1$, $(a^2+ac+c^2)b=30$, $c^3=-8$.

The first and the third equations yield $a=1, c=-2$.

Substituting these into the second equation leads to $(1-2+4)b=30$, i.e., $b=10$.

We conclude that the only upper triangular matrix A such that $A^3=\begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$ is $A=\begin{bmatrix} 1 & 10 \\ 0 & -2 \end{bmatrix}$.

41. **(a)** $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 1 \\ -4 & -1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 0 & -8 \\ 0 & 0 & -4 \\ 8 & 4 & 0 \end{bmatrix}$

43. No. If $AB=BA$, $A^T=-A$, and $B^T=-B$ then $(AB)^T=B^TA^T=(-B)(-A)=BA=AB$ which does not generally equal $-AB$. (The product of skew-symmetric matrices that commute is symmetric.)

45. **(a)** $\left(A^{-1}\right)^T$

$=\left(A^T\right)^{-1}$ ⟵ Theorem 1.4.9(d)

$=(-A)^{-1}$ ⟵ The assumption: A is skew-symmetric

$=-A^{-1}$ ⟵ Theorem 1.4.7(c)

(b) $\left(A^T\right)^T$

$=A$ ⟵ Theorem 1.4.8(a)

$=-A^T$ ⟵ The assumption: A is skew-symmetric

$(A+B)^T$

$=A^T+B^T$ ⟵ Theorem 1.4.8(b)

$=-A-B$ ⟵ The assumption: A and B are skew-symmetric

$= -(A+B)$ ⟵ Theorem 1.4.1(h)

$(A-B)^T$

$= A^T - B^T$ ⟵ Theorem 1.4.8(c)

$= -A-(-B)$ ⟵ The assumption: A and B are skew-symmetric

$= -(A-B)$ ⟵ Theorem 1.4.1(i)

$(kA)^T$

$= kA^T$ ⟵ Theorem 1.4.8(d)

$= k(-A)$ ⟵ The assumption: A is skew-symmetric

$= -kA$ ⟵ Theorem 1.4.1(l)

47. $A^T = (A^TA)^T = A^T(A^T)^T = A^TA = A$ therefore A is symmetric; thus we have $A^2 = AA = A^TA = A$.

True-False Exercises

(a) True. Every diagonal matrix is symmetric: its transpose equals to the original matrix.

(b) False. The transpose of an upper triangular matrix is a *lower* triangular matrix.

(c) False. E.g., $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is not a diagonal matrix.

(d) True. Mirror images of entries across the main diagonal must be equal - see the margin note next to Example 4.

(e) True. All entries below the main diagonal must be zero.

(f) False. By Theorem 1.7.1(d), the inverse of an invertible lower triangular matrix is a lower triangular matrix.

(g) False. A diagonal matrix is invertible if and only if all or its diagonal entries are nonzero (positive or negative).

(h) True. The entries above the main diagonal are zero.

(i) True. If A is upper triangular then A^T is lower triangular. However, if A is also symmetric then it follows that $A^T = A$ must be both upper triangular and lower triangular. This requires A to be a diagonal matrix.

(j) False. For instance, neither $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ nor $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is symmetric even though $A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is.

(k) False. For instance, neither $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ nor $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is upper triangular even though $A+B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is.

(l) False. For instance, $A=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ is not symmetric even though $A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is.

(m) True. By Theorem 1.4.8(d), $(kA)^T = kA^T$. Since kA is symmetric, we also have $(kA)^T = kA$. For nonzero k the equality of the right hand sides $kA^T = kA$ implies $A^T = A$.

1.8 Matrix Transformations

1. **(a)** $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w} = A\mathbf{x}$ in R^3.
The domain of T_A is R^2; the codomain is R^3.

(b) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^3 into a vector $\mathbf{w} = A\mathbf{x}$ in R^2.
The domain of T_A is R^3; the codomain is R^2.

(c) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^3 into a vector $\mathbf{w} = A\mathbf{x}$ in R^3.
The domain of T_A is R^3; the codomain is R^3.

(d) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^6 into a vector $\mathbf{w} = A\mathbf{x}$ in $R^1 = R$.
The domain of T_A is R^6; the codomain is R.

3. **(a)** The transformation maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w}$ in R^2.
Its domain is R^2; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w}$ in R^3.
Its domain is R^2; the codomain is R^3.

5. **(a)** The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^2.
Its domain is R^3; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^3.
Its domain is R^2; the codomain is R^3.

7. **(a)** The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^2.
Its domain is R^2; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^2.
Its domain is R^3; the codomain is R^2.

9. The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^3. Its domain is R^2; the codomain is R^3.

11. **(a)** The given equations can be expressed in matrix form as $\begin{bmatrix}w_1\\w_2\end{bmatrix}=\begin{bmatrix}2&-3&1\\3&5&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix}2&-3&1\\3&5&-1\end{bmatrix}$

(b) The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$.

13. **(a)** $T(x_1, x_2) = \begin{bmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$

(b) $T(x_1, x_2, x_3, x_4) = \begin{bmatrix} 7x_1 + 2x_2 - x_3 + x_4 \\ x_2 + x_3 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$;

the standard matrix is $\begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

(c) $T(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $T(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$

15. The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ therefore the standard matrix for this operator is $\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$.

By directly substituting $(-1, 2, 4)$ for (x_1, x_2, x_3) into the given equation we obtain

$$w_1 = -(3)(1) + (5)(2) - (1)(4) = 3$$

$$w_2 = -(4)(1) - (1)(2) + (1)(4) = -2$$

$$w_3 = -(3)(1)+(2)(2)-(1)(4) = -3$$

By matrix multiplication, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -(3)(1)+(5)(2)-(1)(4) \\ -(4)(1)-(1)(2)+(1)(4) \\ -(3)(1)+(2)(2)-(1)(4) \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$.

17. **(a)** $T(x_1, x_2) = \begin{bmatrix} -x_1 + x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} (1)(1)+(1)(4) \\ -(0)(1)+(1)(4) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ matches $T(-1,4) = (1+4,4) = (5,4)$.

(b) $T(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ x_2 + x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} (2)(2)-(1)(1)-(1)(3) \\ (0)(2)+(1)(1)-(1)(3) \\ (0)(2)+(0)(1)-(0)(3) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

matches $T(2,1,-3) = (4-1-3, 1-3, 0) = (0,-2,0)$.

19. **(a)** $T_A(\mathrm{x}) = A\mathrm{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $T_A(\mathrm{x}) = A\mathrm{x} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$

21. **(a)** If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1+v_1, u_2+v_2) \\ &= (2(u_1+v_1)+(u_2+v_2), (u_1+v_1)-(u_2+v_2)) \\ &= (2u_1+u_2, u_1-u_2) + (2v_1+v_2, v_1-v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and $T(k\mathbf{u}) = T(ku_1, ku_2) = (2ku_1 + ku_2, ku_1 - ku_2) = k(2u_1 + u_2, u_1 - u_2) = kT(\mathbf{u})$.

(b) If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1+v_1, u_2+v_2, u_3+v_3) \\ &= (u_1+v_1, u_3+v_3, u_1+v_1+u_2+v_2) \end{aligned}$$

$$= (u_1, u_3, u_1 + u_2) + (v_1, v_3, v_1 + v_2)$$

$$= T(\mathbf{u}) + T(\mathbf{v})$$

and $T(k\mathbf{u}) = T(ku_1, ku_2, ku_3) = (ku_1, ku_3, ku_1 + ku_2) = k(u_1, u_3, u_1 + u_2) = kT(\mathbf{u})$.

23. **(a)** The homogeneity property fails to hold since $T(kx, ky) = \left((kx)^2, ky\right) = \left(k^2x^2, ky\right)$ does not generally equal $kT(x, y) = k\left(x^2, y\right) = \left(kx^2, ky\right)$. (It can be shown that the additivity property fails to hold as well.)

(b) The homogeneity property fails to hold since $T(kx, ky, kz) = (kx, ky, kxkz) = \left(kx, ky, k^2xz\right)$ does not generally equal $kT(x, y, z) = k(x, y, xz) = (kx, ky, kxz)$. (It can be shown that the additivity property fails to hold as well.)

25. The homogeneity property fails to hold since for $b \neq 0$, $f(kx) = m(kx) + b$ does not generally equal $kf(x) = k(mx + b) = kmx + kb$. (It can be shown that the additivity property fails to hold as well.)
On the other hand, both properties hold for $b = 0$: $f(x + y) = m(x + y) = mx + my = f(x) + f(y)$ and $f(kx) = m(kx) = k(mx) = kf(x)$.
Consequently, f is not a matrix transformation on R unless $b = 0$.

27. By Formula (13), the standard matrix for T is $A = [\,T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid T(\mathbf{e}_3)\,]$. Therefore

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} (1)(2)+(0)(1)+(4)(0) \\ (3)(2)+(0)(1)-(3)(0) \\ (0)(2)+(1)(1)-(1)(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}.$$

29. **(a)** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

31. **(a)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$ **(c)** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$

33. **(a)** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$

35. **(a)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

37. **(a)** $\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}}{2} + 2 \\ \frac{3}{2} - 2\sqrt{3} \end{bmatrix} \approx \begin{bmatrix} 4.60 \\ -1.96 \end{bmatrix}$

(b) $\begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - 2\sqrt{3} \\ -\frac{3\sqrt{3}}{2} - 2 \end{bmatrix} \approx \begin{bmatrix} -1.96 \\ -4.60 \end{bmatrix}$

(c) $\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \approx \begin{bmatrix} 4.95 \\ -0.71 \end{bmatrix}$

(d) $\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

39. By Formula (13), the standard matrix for T is $A = \left[T(\mathbf{e}_1) \quad | T(\mathbf{e}_2) \right]$. Therefore

$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $T(1,1) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$.

41. **(a)** $T_A(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, $T_A(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$, $T_A(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$.

(b) Since T_A is a matrix transformation,

$$T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = T_A(\mathbf{e}_1) + T_A(\mathbf{e}_2) + T_A(\mathbf{e}_3) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}.$$

(c) Since T_A is a matrix transformation, $T_A(7\boldsymbol{e}_3) = 7T_A(\mathbf{e}_3) = 7\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ -21 \end{bmatrix}$.

43. Reflection about the xy-plane: $T(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Reflection about the xz-plane: $T(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

Reflection about the yz-plane: $T(1,2,3) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

45. The standard matrix for T is $A = \left[T(\mathbf{e}_1) \quad | T(\mathbf{e}_2) \right]$. Observe that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Because

T_A is a transformation, $T_A(\mathbf{e}_1) = T_A\left(3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 3T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -11 \end{bmatrix}$.

Likewise, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so we obtain

$$T_A(\mathbf{e}_2) = T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) - 2T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}.$$

Therefore, the matrix for T_A is $A=\begin{bmatrix}5 & -4\\ -11 & 9\end{bmatrix}$.

47. The terminal point of the vector is first rotated about the origin through the angle θ, then it is translated by the vector $\mathbf{x}_0$. No, this is not a matrix transformation, for instance it fails the additivity property: $T(\mathbf{u}+\mathbf{v})=\mathbf{x}_0+R_\theta(\mathbf{u}+\mathbf{v})=\mathbf{x}_0+R_\theta\mathbf{u}+R_\theta\mathbf{v}\neq\mathbf{x}_0+R_\theta\mathbf{u}+\mathbf{x}_0+R_\theta\mathbf{v}=T(\mathbf{u})+T(\mathbf{v})$.

49. Since $\cos^2\theta-\sin^2\theta=\cos(2\theta)$ and $2\sin\theta\cos\theta=\sin(2\theta)$, we have $A=\begin{bmatrix}\cos(2\theta) & -\sin(2\theta)\\ \sin(2\theta) & \cos(2\theta)\end{bmatrix}$. The geometric effect of multiplying A by $\mathbf{x}$ is to rotate the vector through the angle 2θ.

True-False Exercises

(a) False. The domain of T_A is R^3.

(b) False. The codomain of T_A is R^m.

(c) True. Since the statement requires the given equality to hold for <u>*some*</u> vector $\mathbf{x}$ in R^n, we can let $\mathbf{x}=0$.

(d) False. (Refer to Theorem 1.8.3.)

(e) True. The columns of A are $T(\mathbf{e}_i)=0$.

(f) False. The given equality must hold for every matrix transformation since it follows from the homogeneity property.

(g) False. The homogeneity property fails to hold since $T(k\mathbf{x})=k\mathbf{x}+\mathbf{b}$ does not generally equal $kT(\mathbf{x})=k(\mathbf{x}+\mathbf{b})=k\mathbf{x}+k\mathbf{b}$.

1.9 Compositions of Matrix Transformations

1. **(a)** From Tables 1 and 3 in Section 1.8, $[T_1]=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $[T_2]=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$;

$[T_1\circ T_2]=[T_1][T_2]=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$; $[T_2\circ T_1]=[T_2][T_1]=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$.

For these transformations, $T_1\circ T_2\neq T_2\circ T_1$.

(b) From Table 1 in Section 1.8, $[T_1]=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $[T_2]=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$;

$[T_1\circ T_2]=[T_1][T_2]=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$; $[T_2\circ T_1]=[T_2][T_1]=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$.

For these transformations, $T_1\circ T_2\neq T_2\circ T_1$.

3. From Tables 2 and 4 in Section 1.8, $[T_1]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$ and $[T_2]=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$;

$[T_1\circ T_2]=[T_1][T_2]=\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}$; $[T_2\circ T_1]=[T_2][T_1]=\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}$.

For these transformations, $T_1\circ T_2=T_2\circ T_1$

5. $[T_B\circ T_A]=[T_B][T_A]=BA=\begin{bmatrix}-10&-7\\5&-10\end{bmatrix}$; $[T_A\circ T_B]=[T_A][T_B]=AB=\begin{bmatrix}-8&-3\\13&-12\end{bmatrix}$

7. **(a)** We are looking for the standard matrix of $T=T_2\circ T_1$ where T_1 is a rotation of $90°$ and T_2 is a reflection about the line $y=x$. From Tables 5 and 1 in Section 1.8,

$[T_1]=\begin{bmatrix}\cos 90°&-\sin 90°\\\sin 90°&\cos 90°\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, $[T_2]=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. Therefore, $[T]=[T_2][T_1]=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$.

(b) We are looking for the standard matrix of $T=T_2\circ T_1$ where T_1 is an orthogonal projection onto the y-axis and T_2 is a rotation of $45°$ about the origin. From Tables 3 and 5 in Section 1.8,

$[T_1]=\begin{bmatrix}0&0\\0&1\end{bmatrix}$, $[T_2]=\begin{bmatrix}\cos 45°&-\sin 45°\\\sin 45°&\cos 45°\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\end{bmatrix}$. Therefore, $[T]=[T_2][T_1]=\begin{bmatrix}0&\frac{-\sqrt{2}}{2}\\0&\frac{\sqrt{2}}{2}\end{bmatrix}$.

(c) We are looking for the standard matrix of $T=T_2\circ T_1$ where T_1 is a reflection about the x-axis and T_2 is a rotation of $60°$ about the origin. From Tables 1 and 5 in Section 1.8, $[T_1]=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ and

$[T_2]=\begin{bmatrix}\cos 60°&-\sin 60°\\\sin 60°&\cos 60°\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}$.

Therefore, $[T]=[T_2][T_1]=\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{bmatrix}$.

9. **(a)** We are looking for the standard matrix of $T=T_2\circ T_1$ where T_1 is a reflection about the yz-plane and T_2 is an orthogonal projection onto the xz-plane. From Tables 2 and 4 in Section 1.8,

$[T_1]=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ and $[T_2]=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$. Therefore, $[T]=[T_2][T_1]=\begin{bmatrix}-1&0&0\\0&0&0\\0&0&1\end{bmatrix}$.

(b) We are looking for the standard matrix of $T=T_2\circ T_1$ where T_1 is a reflection about the xy-plane and T_2 is an orthogonal projection onto the xy-plane. From Tables 2 and 4 in Section 1.8,

$[T_1]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$ and $[T_2]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$. Therefore, $[T]=[T_2][T_1]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

(c) We are looking for the standard matrix of $T=T_2\circ T_1$ where T_1 is an orthogonal projection on the xy-plane and T_2 is a reflection about the yz-plane. From Tables 4 and 2 in Section 1.8,

$[T_1]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$, $[T_2]=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. Therefore, $[T]=[T_2][T_1]=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

11. **(a)** In vector form, $T_1(x_1,x_2)=\begin{bmatrix}x_1+x_2\\x_1-x_2\end{bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ so that $[T_1]=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$.

Likewise, $T_2(x_1,x_2)=\begin{bmatrix}3x_1\\2x_1+4x_2\end{bmatrix}=\begin{bmatrix}3&0\\2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ so that $[T_2]=\begin{bmatrix}3&0\\2&4\end{bmatrix}$.

(b) $[T_2\circ T_1]=[T_2][T_1]=\begin{bmatrix}3&0\\2&4\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}=\begin{bmatrix}3&3\\6&-2\end{bmatrix}$

$[T_1\circ T_2]=[T_1][T_2]=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}3&0\\2&4\end{bmatrix}=\begin{bmatrix}5&4\\1&-4\end{bmatrix}$

(c) $T_1(T_2(x_1,x_2))=(5x_1+4x_2\ ,\ x_1-4x_2)$; $T_2(T_1(x_1,x_2))=(3x_1+3x_2,\ 6x_1-2x_2)$

13. **(a)** In vector form, $T_1(x_1,x_2)=\begin{bmatrix}x_1-x_2\\-x_1+2x_2\\3x_1\end{bmatrix}=\begin{bmatrix}1&-1\\-1&2\\3&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ so that $[T_1]=\begin{bmatrix}1&-1\\-1&2\\3&0\end{bmatrix}$.

Likewise, $T_2(x_1,x_2,x_3)=\begin{bmatrix}4x_2\\x_1+2x_2\end{bmatrix}=\begin{bmatrix}0&4&0\\1&2&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ so that $[T_2]=\begin{bmatrix}0&4&0\\1&2&0\end{bmatrix}$.

(b) $[T_2\circ T_1]=[T_2][T_1]=\begin{bmatrix}0&4&0\\1&2&0\end{bmatrix}\begin{bmatrix}1&-1\\-1&2\\3&0\end{bmatrix}=\begin{bmatrix}-4&8\\-1&3\end{bmatrix}$

$[T_1\circ T_2]=[T_1][T_2]=\begin{bmatrix}1&-1\\-1&2\\3&0\end{bmatrix}\begin{bmatrix}0&4&0\\1&2&0\end{bmatrix}=\begin{bmatrix}-1&2&0\\2&0&0\\0&12&0\end{bmatrix}$

(c) $T_1(T_2(x_1,x_2,x_3))=(-x_1+2x_2,2x_1,12x_2)$; $T_2(T_1(x_1,x_2))=(-4x_1+8x_2,-x_1+3x_2)$

15. **(a)** In vector form, $T_1(\mathrm{x},\mathrm{y})=\begin{bmatrix}\mathrm{y}\\\mathrm{x}\\x+y\\x-y\end{bmatrix}=\begin{bmatrix}0&1\\1&0\\1&1\\1&-1\end{bmatrix}\begin{bmatrix}x\\\mathrm{y}\end{bmatrix}$ so that $[T_1]=\begin{bmatrix}0&1\\1&0\\1&1\\1&-1\end{bmatrix}$.

Likewise, $T_2(\mathrm{x,y,z,w}) = \begin{bmatrix} \mathrm{x+w} \\ \mathrm{y+w} \\ z+w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \\ \mathrm{w} \end{bmatrix}$ so that $[T_2] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(b) $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 2 & 0 \end{bmatrix}$

(c) $T_1 \circ T_2$ is not defined because the outputs from T_2 are vectors in $\boldsymbol{R}^3$ but the inputs for T_1 are vectors in $\boldsymbol{R}^2$.

(d) $T_2(T_1(\mathrm{x,y})) = (x\ , 2x - y, 2x)$

17. **(a)** $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 8x_1 + 4x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$. Using Theorem 1.5.3(c), we attempt to find the inverse:

$$\left[\begin{array}{cc|cc} 8 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & -4 \\ 2 & 1 & 0 & 1 \end{array}\right] \longleftarrow \text{4 times the second row was subtracted from the first row.}$$

Since we obtained a row of zeros on the left side, the operator is not one-to-one.

(b) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 3x_2 + 2x_3 \\ 2x_1 + 4x_3 \\ x_1 + 3x_2 + 6x_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix}$. Using Theorem 1.5.3(c), we attempt to find the inverse:

$$\left[\begin{array}{ccc|ccc} -1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{ccc|ccc} -1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 6 & 8 & 2 & 1 & 0 \\ 0 & 6 & 8 & 1 & 0 & 1 \end{array}\right] \longleftarrow \text{2 times the first row was added to the second row and the first row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} -1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 6 & 8 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right] \longleftarrow \text{The second row was subtracted from the third row.}$$

Since we obtained a row of zeros on the left side, the operator is not one-to-one.

19. **(a)** $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$; since $\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3 \neq 0$, it follows from Theorem 1.4.5 that the operator is invertible;

the standard matrix of T^{-1} is $\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$; $T^{-1}(w_1, w_2) = \left(\frac{1}{3}w_1 - \frac{2}{3}w_2, \frac{1}{3}w_1 + \frac{1}{3}w_2\right)$

(b) $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 6x_2 \\ -2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$; since $\begin{vmatrix} 4 & -6 \\ -2 & 3 \end{vmatrix} = 0$, it follows from Theorem 1.4.5 that the operator is not invertible.

21. **(a)** From Table 1 in Section 1.8, the standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; since $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$, the matrix operator is invertible. The inverse is also a reflection about the x-axis.

(b) From Table 5 in Section 1.8, the standard matrix is $\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Since $\begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix} = 1 \neq 0,$ the matrix operator is invertible. The inverse is a rotation of $-60°$ (equivalent to $300°$) about the origin.

(c) From Table 3 in Section 1.8, the standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; since $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$, the matrix operator is not invertible.

23. **(a)** Since $\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$, it follows from Theorem 1.4.5 that the operator T_A is invertible;

$A^{-1} = -1\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$. Therefore, $T_A^{-1}(\mathbf{x}) = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

(b) Since $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$, it follows from Theorem 1.4.5 that the operator T_A is not invertible.

25. **(a)** In vector form, $T_A(\mathrm{x}, \mathrm{y}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$. The geometric effect of applying this transformation to $\mathbf{x}$ is to reflect $\mathbf{x}$ about $y = x$ and then to reflect the result about the origin.

(b) For instance, if $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (the standard matrix of the reflection about $y = x$) and

$C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (the standard matrix of the reflection about the origin) then $T_A = T_C \circ T_B$.

True-False Exercises

(a) False. For instance, Example 2 shows two matrix operators on R^2 whose composition is not commutative.

(b) True. This is stated as Theorem 1.9.1.

(c) True. This was established in Example 3.

(d) False. For instance, composition of any reflection operator with itself is the identity operator, which is not a reflection.

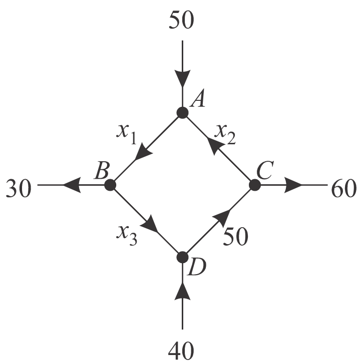
(e) True. The reflection of a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ about the line $y = x$ is $\begin{bmatrix} y \\ x \end{bmatrix}$ so a second reflection yields $\begin{bmatrix} x \\ y \end{bmatrix}$.

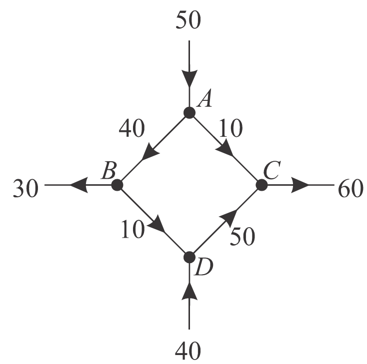
(f) False. This follows from Example 6.

(g) True. The reflection about the origin is given by the transformation $T(\mathbf{x}) = -\mathbf{x}$ so that T is its own inverse.

1.10 Applications of Linear Systems

1. There are four nodes, which we denote by A, B, C, and D (see the figure on the left).
We determine the unknown flow rates x_1, x_2, and x_3 assuming the counterclockwise direction (if any of these quantities are found to be negative then the flow direction along the corresponding branch will be reversed).





Network node	Flow In		Flow Out
A	$x_2 + 50$	$=$	x_1
B	x_1	$=$	$x_3 + 30$
C	50	$=$	$x_2 + 60$
D	$x_3 + 40$	$=$	50

This system can be rearranged as follows

$$\begin{array}{rcrcrcr} -x_1 & + & x_2 & & & = & -50 \\ x_1 & & & - & x_3 & = & 30 \\ & - & x_2 & & & = & 10 \\ & & & & x_3 & = & 10 \end{array}$$

By inspection, this system has a unique solution $x_1 = 40$, $x_2 = -10$, $x_3 = 10$. This yields the flow rates and directions shown in the figure on the right.

3. **(a)** There are four nodes – each of them corresponds to an equation.

Network node	Flow In		Flow Out
top left	$x_2 + 300$	=	$x_3 + 400$
top right (A)	$x_3 + 750$	=	$x_4 + 250$
bottom left	$x_1 + 100$	=	$x_2 + 400$
bottom right (B)	$x_4 + 200$	=	$x_1 + 300$

This system can be rearranged as follows

$$\begin{array}{rcrcrcrcr} & & x_2 & - & x_3 & & & = & 100 \\ & & & & x_3 & - & x_4 & = & -500 \\ x_1 & - & x_2 & & & & & = & 300 \\ -x_1 & & & & & + & x_4 & = & 100 \end{array}$$

(b) The augmented matrix of the linear system obtained in part (a) $\begin{bmatrix} 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 1 & -1 & 0 & 0 & 300 \\ -1 & 0 & 0 & 1 & 100 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & -1 & -400 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we assign x_4 the arbitrary value s, the general solution is given by the formulas

$x_1 = -100 + s$, $x_2 = -400 + s$, $x_3 = -500 + s$, $x_4 = s$

(c) In order for all x_i values to remain positive, we must have $s > 500$. Therefore, to keep the traffic flowing on all roads, the flow from *A* to *B* must exceed 500 vehicles per hour.

5. From Kirchhoff's current law at each node, we have $I_1 + I_2 - I_3 = 0$. Kirchhoff's voltage law yields

	Voltage Rises		Voltage Drops
Left Loop (clockwise)	$2I_1$	=	$2I_2 + 6$
Right Loop (clockwise)	$2I_2 + 4I_3$	=	8

(An equation corresponding to the outer loop is a combination of these two equations.)
The linear system can be rewritten as

$$\begin{array}{rcrcrcr} I_1 & + & I_2 & - & I_3 & = & 0 \\ 2I_1 & - & 2I_2 & & & = & 6 \\ & & 2I_2 & + & 4I_3 & = & 8 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{13}{5} \\ 0 & 1 & 0 & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{11}{5} \end{bmatrix}$.

The solution is $I_1 = 2.6\text{A}$, $I_2 = -0.4\text{A}$, and $I_3 = 2.2\text{A}$.

Since I_2 is negative, this current is opposite to the direction shown in the diagram.

7. From Kirchhoff's current law, we have

	Current In		Currrent Out
Top Left Node	I_1	=	$I_2 + I_4$
Top Right Node	I_4	=	$I_3 + I_5$
Bottom Left Node	$I_2 + I_6$	=	I_1
Bottom Right Node	$I_3 + I_5$	=	I_6

Kirchhoff's voltage law yields

	Voltage Rises		Voltage Drops
Left Loop (clockwise)	10	=	$20I_1 + 20I_2$
Middle Loop (clockwise)	$20I_2$	=	$20I_3$
Right Loop (clockwise)	$20I_3 + 10$	=	$20I_5$

(Equations corresponding to the other loops are combinations of these three equations.)

The linear system can be rewritten as

$$\begin{array}{rcrcrcrcrcrcrcr}
I_1 & - & I_2 & & & - & I_4 & & & & & = & 0 \\
 & & & - & I_3 & + & I_4 & - & I_5 & & & = & 0 \\
-I_1 & + & I_2 & & & & & & & + & I_6 & = & 0 \\
 & & & & I_3 & & & + & I_5 & - & I_6 & = & 0 \\
-20I_1 & - & 20I_2 & & & & & & & & & = & -10 \\
 & & 20I_2 & - & 20I_3 & & & & & & & = & 0 \\
 & & & & 20I_3 & & & - & 20I_5 & & & = & -10
\end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The solution is $I_1 = I_4 = I_5 = I_6 = 0.5\text{A}$, $I_2 = I_3 = 0\text{A}$.

9. We are looking for positive integers x_1, x_2, x_3, and x_4 such that

$$x_1\left(\mathrm{C_3H_8}\right) + x_2\left(\mathrm{O_2}\right) \to x_3\left(\mathrm{CO_2}\right) + x_4\left(\mathrm{H_2O}\right)$$

The number of atoms of carbon, hydrogen, and oxygen on both sides must equal:

	Left Side		Right Side
Carbon	$3x_1$	=	x_3
Hydrogen	$8x_1$	=	$2x_4$
Oxygen	$2x_2$	=	$2x_3 + x_4$

The linear system

$$\begin{array}{rcrcrcrcl} 3x_1 & & & - & x_3 & & & = & 0 \\ 8x_1 & & & & & - & 2x_4 & = & 0 \\ & & 2x_2 & - & 2x_3 & - & x_4 & = & 0 \end{array}$$

has the augmented matrix whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \end{bmatrix}$.

The general solution is $x_1 = \frac{1}{4}t$, $x_2 = \frac{5}{4}t$, $x_3 = \frac{3}{4}t$, $x_4 = t$ where t is arbitrary. The smallest positive integer values for the unknowns occur when $t = 4$, which yields the solution $x_1 = 1$, $x_2 = 5$, $x_3 = 3$, $x_4 = 4$. The balanced equation is

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$

11. We are looking for positive integers x_1, x_2, x_3, and x_4 such that

$$x_1\left(CH_3COF\right) + x_2\left(H_2O\right) \rightarrow x_3\left(CH_3COOH\right) + x_4\left(HF\right)$$

The number of atoms of carbon, hydrogen, oxygen, and fluorine on both sides must equal:

	Left Side		Right Side
Carbon	$2x_1$	=	$2x_3$
Hydrogen	$3x_1 + 2x_2$	=	$4x_3 + x_4$
Oxygen	$x_1 + x_2$	=	$2x_3$
Fluorine	x_1	=	x_4

The linear system

$$\begin{array}{rcrcrcrcl} 2x_1 & & & - & 2x_3 & & & = & 0 \\ 3x_1 & + & 2x_2 & - & 4x_3 & - & x_4 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & & & = & 0 \\ x_1 & & & & & - & x_4 & = & 0 \end{array}$$

has the augmented matrix whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x_1 = t$, $x_2 = t$, $x_3 = t$, $x_4 = t$ where t is arbitrary. The smallest positive integer values for the unknowns occur when $t = 1$, which yields the solution $x_1 = 1$, $x_2 = 1$ $x_3 = 1$, $x_4 = 1$. The balanced equation is

$$CH_3COF + H_2O \rightarrow CH_3COOH + HF$$

13. We are looking for a polynomial of the form $p(x) = a_0 + a_1x + a_2x^2$ such that $p(1) = 1$, $p(2) = 2$, and $p(3) = 5$. We obtain a linear system

$$\begin{array}{rcrcrcr} a_0 & + & a_1 & + & a_2 & = & 1 \\ a_0 & + & 2a_1 & + & 4a_2 & = & 2 \\ a_0 & + & 3a_1 & + & 9a_2 & = & 5 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

There is a unique solution $a_0 = 2$, $a_1 = -2$, $a_2 = 1$.

The quadratic polynomial is $p(x) = 2 - 2x + x^2$.

15. We are looking for a polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ such that $p(-1) = -1$, $p(0) = 1$, $p(1) = 3$ and $p(4) = -1$. We obtain a linear system

$$\begin{array}{rcrcrcrcr} a_0 & - & a_1 & + & a_2 & - & a_3 & = & -1 \\ a_0 & & & & & & & = & 1 \\ a_0 & + & a_1 & + & a_2 & + & a_3 & = & 3 \\ a_0 & + & 4a_1 & + & 16a_2 & + & 64a_3 & = & -1 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{13}{6} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix}$.

There is a unique solution $a_0 = 1$, $a_1 = \frac{13}{6}$, $a_2 = 0$, $a_3 = -\frac{1}{6}$.

The cubic polynomial is $p(x) = 1 + \frac{13}{6}x - \frac{1}{6}x^3$.

17. **(a)** We are looking for a polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2$ such that $p(0) = 1$ and $p(1) = 2$. We obtain a linear system

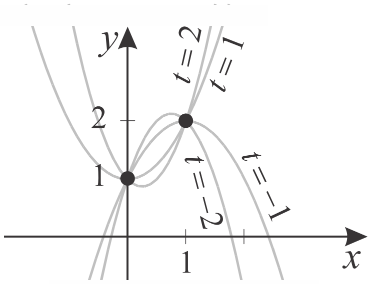
$$\begin{array}{rcrcrcr} a_0 & & & & & = & 1 \\ a_0 & + & a_1 & + & a_2 & = & 2 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

The general solution of the linear system is $a_0 = 1$, $a_1 = 1 - t$, $a_2 = t$ where t is arbitrary.

Consequently, the family of all second-degree polynomials that pass through $(0,1)$ and $(1,2)$ can be represented by $p(x) = 1 + (1-t)x + tx^2$ where t is an arbitrary real number.

(b)



True-False Exercises

(a) False. In general, networks may or may not satisfy the property of flow conservation at each node (although the ones discussed in this section do).

(b) False. When a current passes through a resistor, there is a drop in the electrical potential in a circuit.

(c) True.

(d) False. A chemical equation is said to be balanced if *for each type of atom in the reaction*, the same number of atoms appears on each side of the equation.

(e) False. By Theorem 1.10.1, this is true if the points have distinct x-coordinates.

1.11 Leontief Input-Output Models

1. **(a)** $C=\begin{bmatrix} 0.50 & 0.25 \\ 0.25 & 0.10 \end{bmatrix}$

(b) The Leontief matrix is $I-C=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.50 & 0.25 \\ 0.25 & 0.10 \end{bmatrix}=\begin{bmatrix} 0.50 & -0.25 \\ -0.25 & 0.90 \end{bmatrix}$;

the outside demand vector is $\mathrm{d}=\begin{bmatrix} 7,000 \\ 14,000 \end{bmatrix}$.

The Leontief equation $(I-C)\mathrm{x}=\mathrm{d}$ leads to the linear system with the augmented matrix

$\begin{bmatrix} 0.50 & -0.25 & 7,000 \\ -0.25 & 0.90 & 14,000 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & \frac{784,000}{31} \\ 0 & 1 & \frac{700,000}{31} \end{bmatrix}\approx\begin{bmatrix} 1 & 0 & 25,290.32 \\ 0 & 1 & 22,580.65 \end{bmatrix}$.

To meet the consumer demand, M must produce approximately \$25,290.32 worth of mechanical work and B must produce approximately \$22,580.65 worth of body work.

3. **(a)** $C=\begin{bmatrix} 0.10 & 0.60 & 0.40 \\ 0.30 & 0.20 & 0.30 \\ 0.40 & 0.10 & 0.20 \end{bmatrix}$

(b) The Leontief matrix is $I-C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.10 & 0.60 & 0.40 \\ 0.30 & 0.20 & 0.30 \\ 0.40 & 0.10 & 0.20 \end{bmatrix}=\begin{bmatrix} 0.90 & -0.60 & -0.40 \\ -0.30 & 0.80 & -0.30 \\ -0.40 & -0.10 & 0.80 \end{bmatrix}$;

the outside demand vector is $\mathrm{d}=\begin{bmatrix} 1930 \\ 3860 \\ 5790 \end{bmatrix}$.

The Leontief equation $(I-C)\text{x}=\text{d}$ leads to the linear system with the augmented matrix

$$\begin{bmatrix} 0.90 & -0.60 & -0.40 & 1930 \\ -0.30 & 0.80 & -0.30 & 3860 \\ -0.40 & -0.10 & 0.80 & 5790 \end{bmatrix}.$$

Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 31,500 \\ 0 & 1 & 0 & 26,500 \\ 0 & 0 & 1 & 26,300 \end{bmatrix}$.

The production vector that will meet the given demand is $\mathbf{x}=\begin{bmatrix} \$31,500 \\ \$26,500 \\ \$26,300 \end{bmatrix}$.

5. $I-C=\begin{bmatrix} 0.9 & -0.3 \\ -0.5 & 0.6 \end{bmatrix}$; $\quad (I-C)^{-1}=\frac{100}{39}\begin{bmatrix} 0.6 & 0.3 \\ 0.5 & 0.9 \end{bmatrix}=\begin{bmatrix} \frac{20}{13} & \frac{10}{13} \\ \frac{50}{39} & \frac{30}{13} \end{bmatrix}$

$$\text{x}=(I-C)^{-1}\text{d}=\begin{bmatrix} \frac{20}{13} & \frac{10}{13} \\ \frac{50}{39} & \frac{30}{13} \end{bmatrix}\begin{bmatrix} 50 \\ 60 \end{bmatrix}=\begin{bmatrix} \frac{1600}{13} \\ \frac{7900}{39} \end{bmatrix}\approx\begin{bmatrix} 123.08 \\ 202.56 \end{bmatrix}$$

7. **(a)** The Leontief matrix is $I-C=\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$.

The Leontief equation $(I-C)\mathbf{x}=\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ leads to the linear system with the augmented matrix $\begin{bmatrix} \frac{1}{2} & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ therefore a production vector can be found (namely, $\begin{bmatrix} 4 \\ t \end{bmatrix}$ for an arbitrary nonnegative t) to meet the demand.

On the other hand, the Leontief equation $(I-C)\mathbf{x}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ leads to the linear system with the augmented matrix $\begin{bmatrix} \frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; the system is inconsistent, therefore a production vector cannot be found to meet the demand.

(b) Mathematically, the linear system represented by $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ can be rewritten as $\begin{bmatrix} \frac{1}{2}x_1 \\ 0 \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

Clearly, if $d_2=0$ the system has infinitely many solutions: $x_1=2d_1$; $x_2=t$ where t is an arbitrary nonnegative number.
If $d_2\neq 0$ the system is inconsistent. (Note that the Leontief matrix is not invertible.)

An economic explanation of the result in part (a) is that $\mathbf{c}_2=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ therefore the second sector consumes all of its own output, making it impossible to meet any outside demand for its products.

9. From the assumption $c_{21}c_{12} < 1 - c_{11}$, it follows that the determinant of $\det(I - C) = \det\left(\begin{bmatrix} 1-c_{11} & -c_{12} \\ -c_{21} & 1 \end{bmatrix}\right) = 1 - c_{11} - c_{12}c_{21}$ is nonzero. Consequently, the Leontief matrix is invertible; its inverse is $(I - C)^{-1} = \frac{1}{1-c_{11}-c_{12}c_{21}}\begin{bmatrix} 1 & c_{12} \\ c_{21} & 1-c_{11} \end{bmatrix}$. Since the consumption matrix C has nonnegative entries and $1 - c_{11} > c_{21}c_{12} \geq 0$, we conclude that all entries of $(I - C)^{-1}$ are nonnegative as well. This economy is productive (see the discussion above Theorem 1.10.1) - the equation $\mathbf{x} - C\mathbf{x} = \mathbf{d}$ has a unique solution $\mathbf{x} = (I - C)^{-1}\mathbf{d}$ for every demand vector $\mathbf{d}$.

True-False Exercises

(a) False. Sectors that do *not* produce outputs are called open sectors.

(b) True.

(c) False. The i th row vector of a consumption matrix contains the monetary values required of the i th sector by the other sectors for each of them to produce one monetary unit of output.

(d) True. This follows from Theorem 1.11.1.

(e) True.

Chapter 1 Supplementary Exercises

1. The corresponding system of linear equations is

$$\begin{array}{rcrcrcrcr} 3x_1 & - & x_2 & & & + & 4x_4 & = & 1 \\ 2x_1 & & & + & 3x_3 & + & 3x_4 & = & -1 \end{array}$$

$\begin{bmatrix} 3 & -1 & 0 & 4 & 1 \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$ ⟵ The original augmented matrix.

$\begin{bmatrix} 1 & -1 & -3 & 1 & 2 \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$ ⟵ -1 times the second row was added to the first row.

$\begin{bmatrix} 1 & -1 & -3 & 1 & 2 \\ 0 & 2 & 9 & 1 & -5 \end{bmatrix}$ ⟵ -2 times the first row was added to the second row.

$\begin{bmatrix} 1 & -1 & -3 & 1 & 2 \\ 0 & 1 & \frac{9}{2} & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$ ⟵ The second row was multiplied by $\frac{1}{2}$.

This matrix is in row echelon form. It corresponds to the system of equations

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & - & 3x_3 & + & x_4 & = & 2 \\ & & x_2 & + & \frac{9}{2}x_3 & + & \frac{1}{2}x_4 & = & -\frac{5}{2} \end{array}$$

Solve the equations for the leading variables

$$x_1 = x_2 + 3x_3 - x_4 + 2$$

$$x_2 = -\frac{9}{2}x_3 - \frac{1}{2}x_4 - \frac{5}{2}$$

then substitute the second equation into the first

$$x_1 = -\frac{3}{2}x_3 - \frac{3}{2}x_4 - \frac{1}{2}$$

$$x_2 = -\frac{9}{2}x_3 - \frac{1}{2}x_4 - \frac{5}{2}$$

If we assign x_3 and x_4 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x_1 = -\frac{3}{2}s - \frac{3}{2}t - \frac{1}{2}, \quad x_2 = -\frac{9}{2}s - \frac{1}{2}t - \frac{5}{2}, \qquad x_3 = s, \qquad x_4 = t$$

3. The corresponding system of linear equations is

$$\begin{array}{rcrcrcr} 2x_1 & - & 4x_2 & + & x_3 & = & 6 \\ -4x_1 & & & + & 3x_3 & = & -1 \\ & & x_2 & - & x_3 & = & 3 \end{array}$$

$$\begin{bmatrix} 2 & -4 & 1 & 6 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix} \longleftarrow \text{The original augmented matrix.}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & -8 & 5 & 11 \\ 0 & 1 & -1 & 3 \end{bmatrix} \longleftarrow \text{4 times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & -8 & 5 & 11 \end{bmatrix} \longleftarrow \text{The second and third rows were interchanged.}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -3 & 35 \end{bmatrix} \longleftarrow \text{8 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -\frac{35}{3} \end{bmatrix} \longleftarrow \text{The third row was multiplied by } -\tfrac{1}{3}.$$

This matrix is in row echelon form. It corresponds to the system of equations

$$\begin{aligned} x_1 - 2x_2 + \frac{1}{2}x_3 &= 3 \\ x_2 - x_3 &= 3 \\ x_3 &= -\frac{35}{3} \end{aligned}$$

Solve the equations for the leading variables

$$x_1 = 2x_2 - \frac{1}{2}x_3 + 3$$

$$x_2 = x_3 + 3$$

$$x_3 = -\frac{35}{3}$$

then finish back-substituting to obtain the unique solution

$$x_1 = -\frac{17}{2}, \quad x_2 = -\frac{26}{3}, \quad x_3 = -\frac{35}{3}$$

5.

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & x \\ \frac{4}{5} & \frac{3}{5} & y \end{bmatrix} \longleftarrow \text{The augmented matrix corresponding to the system.}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ \frac{4}{5} & \frac{3}{5} & y \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{5}{3}.$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ 0 & \frac{5}{3} & -\frac{4}{3}x + y \end{bmatrix} \longleftarrow -\tfrac{4}{5} \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ 0 & 1 & -\frac{4}{5}x + \frac{3}{5}y \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{3}{5}.$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{5}x + \frac{4}{5}y \\ 0 & 1 & -\frac{4}{5}x + \frac{3}{5}y \end{bmatrix} \longleftarrow \tfrac{4}{3} \text{ times the second row was added to the first row.}$$

The system has exactly one solution: $x' = \frac{3}{5}x + \frac{4}{5}y$ and $y' = -\frac{4}{5}x + \frac{3}{5}y$.

7.

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 1 & 5 & 10 & 44 \end{bmatrix} \longleftarrow \text{The original augmented matrix.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 4 & 9 & 35 \end{bmatrix} \longleftarrow$$ -1 times the first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & \frac{9}{4} & \frac{35}{4} \end{bmatrix} \longleftarrow$$ The second row was multiplied by $\frac{1}{4}$.

$$\begin{bmatrix} 1 & 0 & -\frac{5}{4} & \frac{1}{4} \\ 0 & 1 & \frac{9}{4} & \frac{35}{4} \end{bmatrix} \longleftarrow$$ -1 times the second row was added to the first row.

If we assign z an arbitrary value t, the general solution is given by the formulas

$$x = \frac{1}{4} + \frac{5}{4}t, \qquad y = \frac{35}{4} - \frac{9}{4}t, \qquad z = t$$

The positivity of the three variables requires that $\frac{1}{4} + \frac{5}{4}t > 0$, $\frac{35}{4} - \frac{9}{4}t > 0$, and $t > 0$. The first inequality can be rewritten as $t > -\frac{1}{4}$, while the second inequality is equivalent to $t < \frac{35}{9}$. All three unknowns are positive whenever $0 < t < \frac{35}{9}$. There are three integer values of $t = z$ in this interval: 1, 2, and 3. Of those, only $z = t = 3$ yields integer values for the remaining variables: $x = 4$, $y = 2$.

9.

$$\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{bmatrix} \longleftarrow$$ -1 times the first row was added to the second row.

$$\begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{bmatrix} \longleftarrow$$ -1 times the second row was added to the third row.

(a) the system has a unique solution if $a \neq 0$ and $b \neq 2$ (multiplying the rows by $\frac{1}{a}$, $\frac{1}{a}$, and $\frac{1}{b-2}$, respectively, yields a row echelon form of the augmented matrix $\begin{bmatrix} 1 & 0 & \frac{b}{a} & \frac{2}{a} \\ 0 & 1 & \frac{4-b}{a} & \frac{2}{a} \\ 0 & 0 & 1 & 1 \end{bmatrix}$).

(b) the system has a one-parameter solution if $a \neq 0$ and $b = 2$ (multiplying the first two rows by $\frac{1}{a}$ yields a reduced row echelon form of the augmented matrix $\begin{bmatrix} 1 & 0 & \frac{2}{a} & \frac{2}{a} \\ 0 & 1 & \frac{2}{a} & \frac{2}{a} \\ 0 & 0 & 0 & 0 \end{bmatrix}$).

(c) the system has a two-parameter solution if $a=0$ and $b=2$

(the reduced row echelon form of the augmented matrix is $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$).

(d) the system has no solution if $a=0$ and $b\neq 2$

(the reduced row echelon form of the augmented matrix is $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$).

11. For the product AKB to be defined, K must be a 2×2 matrix. Letting $K=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we can write

$$ABC=\begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}=\begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2a & b & -b \\ 2c & d & -d \end{bmatrix}=\begin{bmatrix} 2a+8c & b+4d & -b-4d \\ -4a+6c & -2b+3d & 2b-3d \\ 2a-4c & b-2d & -b+2d \end{bmatrix}.$$

The matrix equation $AKB=C$ can be rewritten as a system of nine linear equations

$$\begin{array}{rrrrrrrrr}
2a & & & + & 8c & & & = & 8 \\
 & & b & & & + & 4d & = & 6 \\
 & - & b & & & - & 4d & = & -6 \\
-4a & & & + & 6c & & & = & 6 \\
 & - & 2b & & & + & 3d & = & -1 \\
 & & 2b & & & - & 3d & = & 1 \\
2a & & & - & 4c & & & = & -4 \\
 & & b & & & - & 2d & = & 0 \\
 & - & b & & & + & 2d & = & 0
\end{array}$$

which has a unique solution $a=0$, $b=2$, $c=1$, $d=1$. (An easy way to solve this system is to first split it into two smaller systems. The system $2a+8c=8$, $-4a+6c=6$, $2a-4c=-4$ involves a and c only, whereas the remaining six equations involve just b and d.) We conclude that $K=\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

13. **(a)** X must be a 2×3 matrix. Letting $X=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ we can write

$$X\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}=\begin{bmatrix} -a+b+3c & b+c & a-c \\ -d+e+3f & e+f & d-f \end{bmatrix}$$

therefore the given matrix equation can be rewritten as a system of linear equations:

$$\begin{array}{rcrcrcrcrcrcr}
-a & + & b & + & 3c & & & & & & & = & 1 \\
 & & b & + & c & & & & & & & = & 2 \\
a & & & - & c & & & & & & & = & 0 \\
 & & & & & - & d & + & e & + & 3f & = & -3 \\
 & & & & & & & & e & + & f & = & 1 \\
 & & & & & & d & & & - & f & = & 5
\end{array}$$

The augmented matrix of this system has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ so the system has a unique solution

$a=-1,\ b=3,\ c=-1,\ d=6,\ e=0,\ f=1$ and $X=\begin{bmatrix} -1 & 3 & -1 \\ 6 & 0 & 1 \end{bmatrix}$.

(An alternative to dealing with this large system is to split it into two smaller systems instead: the first three equations involve a, b, and c only, whereas the remaining three equations involve just d, e, and f. Since the coefficient matrix for both systems is the same, we can follow the procedure of Example 2 in Section 1.6; the reduced row echelon form of the matrix $\left[\begin{array}{rrr|r|r} -1 & 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 0 & 5 \end{array}\right]$ is $\left[\begin{array}{rrr|r|r} 1 & 0 & 0 & -1 & 6 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$.)

Yet another way of solving this problem would be to determine the inverse

$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ using the method introduced in Section 1.5, then multiply both sides of the given matrix equation on the right by this inverse to determine X:

$$X=\begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 3 & -1 \\ 6 & 0 & 1 \end{bmatrix}$$

(b) X must be a 2×2 matrix. Letting $X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we can write

$$X\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}=\begin{bmatrix} a+3b & -a & 2a+b \\ c+3d & -c & 2c+d \end{bmatrix}$$

therefore the given matrix equation can be rewritten as a system of linear equations:

$$\begin{array}{rcrcrcrcrcr} a & + & 3b & & & & & = & -5 \\ -a & & & & & & & = & -1 \\ 2a & + & b & & & & & = & 0 \\ & & & & c & + & 3d & = & 6 \\ & & & - & c & & & = & -3 \\ & & & & 2c & + & d & = & 7 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ so the system has a unique solution $a=1$, $b=-2$, $c=3$, $d=1$. We conclude that $X=\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$.

(An alternative to dealing with this large system is to split it into two smaller systems instead: the first three equations involve a and b only, whereas the remaining three equations involve just c and d. Since the coefficient matrix for both systems is the same, we can follow the procedure of Example 2 in Section 1.6; the reduced row echelon form of the matrix $\left[\begin{array}{rr|r|r} 1 & 3 & -5 & 6 \\ -1 & 0 & -1 & -3 \\ 2 & 1 & 0 & 7 \end{array}\right]$ is $\left[\begin{array}{rr|r|r} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.)

(c) X must be a 2×2 matrix. Letting $X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we can write

$$\begin{aligned} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}X - X\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3a+c & 3b+d \\ -a+2c & -b+2d \end{bmatrix} - \begin{bmatrix} a+2b & 4a \\ c+2d & 4c \end{bmatrix} \\ &= \begin{bmatrix} 2a-2b+c & -4a+3b+d \\ -a+c-2d & -b-4c+2d \end{bmatrix} \end{aligned}$$

therefore the given matrix equation can be rewritten as a system of linear equations:

$$\begin{array}{rcrcrcrcr} 2a & - & 2b & + & c & & & = & 2 \\ -4a & + & 3b & & & + & d & = & -2 \\ -a & & & + & c & - & 2d & = & 5 \\ & - & b & - & 4c & + & 2d & = & 4 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{113}{37} \\ 0 & 1 & 0 & 0 & -\frac{160}{37} \\ 0 & 0 & 1 & 0 & -\frac{20}{37} \\ 0 & 0 & 0 & 1 & -\frac{46}{37} \end{bmatrix}$

so the system has a unique solution $a=-\frac{113}{37}$, $b=-\frac{160}{37}$, $c=-\frac{20}{37}$, $d=-\frac{46}{37}$.

We conclude that $X=\begin{bmatrix} -\frac{113}{37} & -\frac{160}{37} \\ -\frac{20}{37} & -\frac{46}{37} \end{bmatrix}$.

15. We are looking for a polynomial of the form

$$p(x)=ax^2+bx+c$$

such that $p(1)=2$, $p(-1)=6$, and $p(2)=3$. We obtain a linear system

$$\begin{array}{rcrcrcr} a & + & b & + & c & = & 2 \\ a & - & b & + & c & = & 6 \\ 4a & + & 2b & + & c & = & 3 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

There is a unique solution $a=1$, $b=-2$, $c=3$.

17. When multiplying the matrix J_n by itself, each entry in the product equals n. Therefore, $J_nJ_n=nJ_n$.

$(I-J_n)\left(I-\frac{1}{n-1}J_n\right)$

$=I^2-I\frac{1}{n-1}J_n-J_nI+J_n\frac{1}{n-1}J_n$ ⟵ Theorem 1.4.1(f) and (g)

$=I-\frac{1}{n-1}J_n-J_n+J_n\frac{1}{n-1}J_n$ ⟵ Property $AI=IA=A$ in Section 1.4

$=I-\frac{1}{n-1}J_n-J_n+\frac{1}{n-1}J_nJ_n$ ⟵ Theorem 1.4.1(m)

$=I-\frac{1}{n-1}J_n-J_n+\frac{n}{n-1}J_n$ ⟵ $J_nJ_n=nJ_n$

$=I+\left(\frac{-1}{n-1}-1+\frac{n}{n-1}\right)J_n$ ⟵ Theorem 1.4.1(j) and (k)

$=I+\left(\frac{-1}{n-1}-\frac{n-1}{n-1}+\frac{n}{n-1}\right)J_n$

$=I$