CHAPTER **6**

Inner Product Spaces

CHAPTER CONTENTS

Introduction

In Chapter 3 we defined the dot product of vectors in R^n, and we used that concept to define notions of length, angle, distance, and orthogonality. In this chapter we will generalize those ideas so they are applicable in any vector space, not just R^n. We will also discuss various applications of these ideas.

6.1 Inner Products

In this section we will use the most important properties of the dot product on R^n as axioms, which, if satisfied by the vectors in a vector space V, will enable us to extend the notions of length, distance, angle, and perpendicularity to general vector spaces.

General Inner Products

Most, but not all, of the concepts we will develop in this section apply to both real and complex vector spaces. We will limit the text discussion to real vector spaces and leave the comparable ideas for complex vector spaces for the exercises. Thus, it should be understood that all vector spaces in this section are real, even if not stated explicitly.

Definition 1

An ***inner product*** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetry axiom]**
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity axiom]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ **[Homogeneity axiom]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity axiom]**

A real vector space with an inner product is called a ***real inner product space***.

Because the axioms for a real inner product space are based on properties of the dot product, these inner product space axioms will be satisfied automatically if we define the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n to be

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n} \tag{1}$$

This inner product is commonly called the ***Euclidean inner product*** (or the ***standard inner product***) on R^n to distinguish it from other possible inner products that might be defined on R^n. We call R^n with the Euclidean inner product ***Euclidean n-space***.

Inner products can be used to define notions of norm and distance in a general inner product space just as we did with dot products in R^n. Recall from Formulas (11) and (19) of Section 3.2 that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in Euclidean n-space, then norm and distance can be expressed in terms of the dot product as

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \quad \text{and} \quad d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}$$

Motivated by these formulas, we make the following definition.

Definition 2

If V is a real inner product space, then the ***norm*** (or ***length***) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the ***distance*** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

The following theorem, whose proof is left for the exercises, shows that norms and distances in real inner product spaces have many of the properties that you might expect.

Theorem 6.1.1

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, and if k is a scalar, then:

(*a*) $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$.
(*b*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.
(*c*) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.
(*d*) $d(\mathbf{u}, \mathbf{v}) \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{v}$.

Although the Euclidean inner product is the most important inner product on R^n, there are various applications in which it is desirable to modify it by *weighting* each term differently. More precisely, if

$$w_1, w_2, \dots, w_n$$

are *positive* real numbers, called ***weights***, and if $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in R^n, then it can be shown that the formula

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n} \tag{2}$$

defines an inner product on R^n that we call the ***weighted Euclidean inner product with weights*** $\boldsymbol{w_1, w_2, \dots, w_n}$.

EXAMPLE 1 | Weighted Euclidean Inner Product

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2 \tag{3}$$

satisfies the four inner product axioms.

Solution

Axiom 1: Interchanging $\mathbf{u}$ and $\mathbf{v}$ in Formula (3) does not change the sum on the right side, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

Axiom 2: If $\mathbf{w} = (w_1, w_2)$, then

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\ &= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2) \\ &= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2) \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

Axiom 3: $\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2$

$$\begin{aligned} &= k(3u_1 v_1 + 2u_2 v_2) \\ &= k\langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

Axiom 4: Observe that $\langle \mathbf{v}, \mathbf{v} \rangle = 3(v_1 v_1) + 2(v_2 v_2) = 3v_1^2 + 2v_2^2 \geq 0$ with equality if and only if $v_1 = v_2 = 0$, that is, if and only if $\mathbf{v} = \mathbf{0}$.

Note that the standard Euclidean inner product in Formula (1) is the special case of the weighted Euclidean inner product in which all the weights are 1.

In Example 1, we are using subscripted w's to denote the components of the vector $\mathbf{w}$, not the weights. The weights are the numbers 3 and 2 in Formula (3).

An Application of Weighted Euclidean Inner Products

To illustrate one way in which a weighted Euclidean inner product can arise, suppose that some physical experiment has n possible numerical outcomes

$$x_1, x_2, \dots, x_n$$

and that a series of m repetitions of the experiment yields these values with various frequencies. Specifically, suppose that x_1 occurs f_1 times, x_2 occurs f_2 times, and so forth. Since there is a total of m repetitions of the experiment, it follows that

$$f_1 + f_2 + \cdots + f_n = m$$

Thus, the ***arithmetic average*** of the observed numerical values (denoted by $\bar{x}$) is

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + \cdots + f_n} = \frac{1}{m}(f_1 x_1 + f_2 x_2 + \cdots + f_n x_n) \tag{4}$$

If we let

$$\begin{aligned} \mathbf{f} &= (f_1, f_2, \dots, f_n) \\ \mathbf{x} &= (x_1, x_2, \dots, x_n) \\ w_1 = w_2 &= \cdots = w_n = 1/m \end{aligned}$$

then (4) can be expressed as the weighted Euclidean inner product

$$\bar{x} = \langle \mathbf{f}, \mathbf{x} \rangle = w_1 f_1 x_1 + w_2 f_2 x_2 + \cdots + w_n f_n x_n$$

EXAMPLE 2 | Calculating with a Weighted Euclidean Inner Product

It is important to keep in mind that norm and distance depend on the inner product being used. If the inner product is changed, then the norms and distances between vectors also change. For example, for the vectors $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1)$ in R^2 with the Euclidean inner product we have

$$\|\mathbf{u}\| = \sqrt{1^2 + 0^2} = 1$$

and

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(1, -1)\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

but if we change to the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

we have

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = [3(1)(1) + 2(0)(0)]^{1/2} = \sqrt{3}$$

and

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| &= \langle (1, -1), (1, -1) \rangle^{1/2} \\ &= [3(1)(1) + 2(-1)(-1)]^{1/2} = \sqrt{5} \end{aligned}$$

Unit Circles and Spheres in Inner Product Spaces

Definition 3

If V is an inner product space, then the set of points in V that satisfy

$$\|\mathbf{u}\| = 1$$

is called the ***unit sphere*** in V (or the ***unit circle*** in the case where $V = R^2$).

EXAMPLE 3 | Unusual Unit Circles in R^2

(*a*) Sketch the unit circle in an xy-coordinate system in R^2 using the Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_2v_2$.

(*b*) Sketch the unit circle in an xy-coordinate system in R^2 using the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9}u_1v_1 + \frac{1}{4}u_2v_2$.

Solution (*a*) If $\mathbf{u} = (x, y)$, then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{x^2 + y^2}$, so the equation of the unit circle is $\sqrt{x^2 + y^2} = 1$, or on squaring both sides,

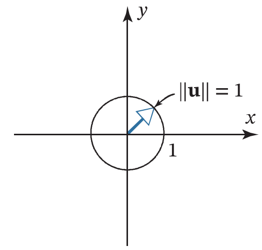
$$x^2 + y^2 = 1$$

As expected, the graph of this equation is a circle of radius 1 centered at the origin (**Figure 6.1.1*a***).

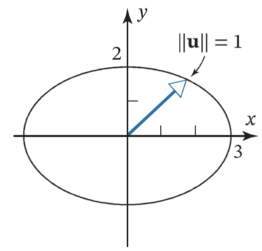
Solution (*b*) If $\mathbf{u} = (x, y)$, then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\frac{1}{9}x^2 + \frac{1}{4}y^2}$, so the equation of the unit circle is $\sqrt{\frac{1}{9}x^2 + \frac{1}{4}y^2} = 1$, or on squaring both sides,

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

The graph of this equation is the ellipse shown in **Figure 6.1.1*b***. Though this may seem odd when viewed geometrically, it makes sense algebraically since all points on the ellipse are 1 unit away from the origin relative to the given weighted Euclidean inner product. In short, weighting has the effect of distorting the space that we are used to seeing through "unweighted Euclidean eyes."



(*a*) The unit circle using the standard Euclidean inner product.



(*b*) The unit circle using a weighted Euclidean inner product.

FIGURE 6.1.1

Inner Products Generated by Matrices

The Euclidean inner product and the weighted Euclidean inner products are special cases of a general class of inner products on R^n called ***matrix inner products***. To define this class of inner products, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n that are expressed in *column form*, and let A be an *invertible* $n \times n$ matrix. It can be shown (Exercise 47) that if $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on R^n, then the formula

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}} \tag{5}$$

also defines an inner product; it is called the ***inner product on R^n generated by A***.

Recall from Table 1 of Section 3.2 that if $\mathbf{u}$ and $\mathbf{v}$ are in column form, then $\mathbf{u} \cdot \mathbf{v}$ can be written as $\mathbf{v}^T\mathbf{u}$ from which it follows that (5) can be expressed as

$$\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{v})^T A\mathbf{u}$$

or equivalently as

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A^T A\mathbf{u}} \tag{6}$$

EXAMPLE 4 | Matrices Generating Weighted Euclidean Inner Products

The standard Euclidean and weighted Euclidean inner products are special cases of matrix inner products. The standard Euclidean inner product on R^n is generated by the $n \times n$ identity matrix, since setting $A = I$ in Formula (5) yields

$$\langle \mathbf{u}, \mathbf{v} \rangle = I\mathbf{u} \cdot I\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$$

and the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n \tag{7}$$

is generated by the matrix

$$A = \begin{bmatrix} \sqrt{w_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{w_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{w_n} \end{bmatrix}$$

This can be seen by observing that A^TA is the $n \times n$ diagonal matrix whose diagonal entries are the weights $w_1, w_2, \ldots, w_n$.

EXAMPLE 5 | Example 1 Revisited

The weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ discussed in Example 1 is the inner product on R^2 generated by

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

Every diagonal matrix with positive diagonal entries generates a weighted inner product. Why?

Other Examples of Inner Products

So far, we have considered only examples of inner products on R^n. We will now consider examples of inner products on some of the other kinds of vector spaces that we discussed earlier.

EXAMPLE 6 | The Standard Inner Product on M_{nn}

If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space M_{nn}, then the formula

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V)} \tag{8}$$

defines an inner product on M_{nn} called the ***standard inner product*** on that space (see Definition 8 of Section 1.3 for a definition of trace). This can be proved by confirming that the four inner product space axioms are satisfied, but we can illustrate the idea by computing (8) for the 2×2 matrices

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

This yields

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

which is just the dot product of the corresponding entries in the two matrices. And it follows from this that

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}\langle U^T U \rangle} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$

For example, if

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}(U^T V) = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

and

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\operatorname{tr}(U^T U)} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$
$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{\operatorname{tr}(V^T V)} = \sqrt{(-1)^2 + 0^2 + 3^2 + 2^2} = \sqrt{14}$$

EXAMPLE 7 | The Standard Inner Product on P_n

If

$$\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad \mathbf{q} = b_0 + b_1 x + \cdots + b_n x^n$$

are polynomials in P_n, then the following formula defines an inner product on P_n (verify) that we will call the ***standard inner product*** on this space:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n \tag{9}$$

The norm of a polynomial $\mathbf{p}$ relative to this inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$$

EXAMPLE 8 | The Evaluation Inner Product on P_n

If

$$\mathbf{p} = p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad \mathbf{q} = q(x) = b_0 + b_1 x + \cdots + b_n x^n$$

are polynomials in P_n, and if $x_0, x_1, \ldots, x_n$ are distinct real numbers (called ***sample points***), then the formula

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + \cdots + p(x_n)q(x_n) \tag{10}$$

defines an inner product on P_n called the ***evaluation inner product*** at $x_0, x_1, \ldots, x_n$. Algebraically, this can be viewed as the dot product in R^n of the n-tuples

$$(p(x_0), p(x_1), \ldots, p(x_n)) \quad \text{and} \quad (q(x_0), q(x_1), \ldots, q(x_n))$$

and hence the first three inner product axioms follow from properties of the dot product. The fourth inner product axiom follows from the fact that

$$\langle \mathbf{p}, \mathbf{p} \rangle = [p(x_0)]^2 + [p(x_1)]^2 + \cdots + [p(x_n)]^2 \geq 0$$

with equality holding if and only if

$$p(x_0) = p(x_1) = \cdots = p(x_n) = 0$$

But a nonzero polynomial of degree n or less can have at most n distinct roots, so it must be that $\mathbf{p} = \mathbf{0}$, which proves that the fourth inner product axiom holds.

The norm of a polynomial $\mathbf{p}$ relative to the evaluation inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{[p(x_0)]^2 + [p(x_1)]^2 + \cdots + [p(x_n)]^2} \tag{11}$$

EXAMPLE 9 | Working with the Evaluation Inner Product

Let P_2 have the evaluation inner product at the points

$$x_0 = -2, \quad x_1 = 0, \quad \text{and} \quad x_2 = 2$$

Compute $\langle \mathbf{p}, \mathbf{q} \rangle$ and $\|\mathbf{p}\|$ for the polynomials $\mathbf{p} = p(x) = x^2$ and $\mathbf{q} = q(x) = 1 + x$.

Solution It follows from (10) and (11) that

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2) = (4)(-1) + (0)(1) + (4)(3) = 8$$

$$\begin{aligned} \|\mathbf{p}\| &= \sqrt{[p(x_0)]^2 + [p(x_1)]^2 + [p(x_2)]^2} = \sqrt{[p(-2)]^2 + [p(0)]^2 + [p(2)]^2} \\ &= \sqrt{4^2 + 0^2 + 4^2} = \sqrt{32} = 4\sqrt{2} \end{aligned}$$

EXAMPLE 10 | An Integral Inner Product on $C[a, b]$

CALCULUS REQUIRED

Let $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ be two functions in $C[a, b]$ and define

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\, dx \tag{12}$$

We will show that this formula defines an inner product on $C[a, b]$ by verifying the four inner product axioms for functions $\mathbf{f} = f(x)$, $\mathbf{g} = g(x)$, and $\mathbf{h} = h(x)$ in $C[a, b]$.

Axiom 1: $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle \mathbf{g}, \mathbf{f} \rangle$

Axiom 2:
$$\begin{aligned}\langle \mathbf{f} + \mathbf{g}, \mathbf{h} \rangle &= \int_a^b (f(x) + g(x))h(x)\,dx \\ &= \int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx \\ &= \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle\end{aligned}$$

Axiom 3: $\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x)\,dx = k\int_a^b f(x)g(x)\,dx = k\langle \mathbf{f}, \mathbf{g} \rangle$

Axiom 4: If $\mathbf{f} = f(x)$ is any function in $C[a, b]$, then

$$\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b f^2(x)\,dx \geq 0 \tag{13}$$

since $f^2(x) \geq 0$ for all x in the interval $[a, b]$. Moreover, because f is continuous on $[a, b]$, the equality in Formula (13) holds if and only if the function f is identically zero on $[a, b]$, that is, if and only if $\mathbf{f} = \mathbf{0}$; and this proves that Axiom 4 holds.

CALCULUS REQUIRED

EXAMPLE 11 | Norm of a Vector in $C[a, b]$

If $C[a, b]$ has the inner product that was defined in Example 10, then the norm of a function $\mathbf{f} = f(x)$ relative to this inner product is

$$\|\mathbf{f}\| = \langle \mathbf{f}, \mathbf{f} \rangle^{1/2} = \sqrt{\int_a^b f^2(x)\,dx} \tag{14}$$

and the unit sphere in this space consists of all functions $\mathbf{f}$ in $C[a, b]$ that satisfy the equation

$$\int_a^b f^2(x)\,dx = 1$$

Remark Note that the vector space P_n is a subspace of $C[a, b]$ because polynomials are continuous functions. Thus, Formula (12) defines an inner product on P_n that is different from both the standard inner product and the evaluation inner product.

Warning Recall from calculus that the arc length of a curve $y = f(x)$ over an interval $[a, b]$ is given by the formula

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \tag{15}$$

Do not confuse this concept of arc length with $\|\mathbf{f}\|$, which is the length (norm) of $\mathbf{f}$ when $\mathbf{f}$ is viewed as a vector in $C[a, b]$. Formulas (14) and (15) have different meanings.

Algebraic Properties of Inner Products

The following theorem lists some of the algebraic properties of inner products that follow from the inner product axioms. This result is a generalization of Theorem 3.2.3, which applied only to the dot product on R^n.

Theorem 6.1.2

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a real inner product space V, and if k is a scalar, then:

(a) $\langle \mathbf{0}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$

(b) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$

(c) $\langle \mathbf{u}, \mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{w}\rangle$

(d) $\langle \mathbf{u}-\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{w}\rangle$

(e) $k\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, k\mathbf{v}\rangle$

Proof We will prove part (b) and leave the proofs of the remaining parts for the reader.

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle &= \langle \mathbf{v}+\mathbf{w}, \mathbf{u}\rangle && \text{[By symmetry]} \\ &= \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{w}, \mathbf{u}\rangle && \text{[By additivity]} \\ &= \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle && \text{[By symmetry]} \end{aligned}$$

The following example illustrates how Theorem 6.1.2 and the defining properties of inner products can be used to perform algebraic computations with inner products. As you read through the example, you will find it instructive to justify the steps.

EXAMPLE 12 | Calculating with Inner Products

$$\begin{aligned} \langle \mathbf{u}-2\mathbf{v}, 3\mathbf{u}+4\mathbf{v}\rangle &= \langle \mathbf{u}, 3\mathbf{u}+4\mathbf{v}\rangle - \langle 2\mathbf{v}, 3\mathbf{u}+4\mathbf{v}\rangle \\ &= \langle \mathbf{u}, 3\mathbf{u}\rangle + \langle \mathbf{u}, 4\mathbf{v}\rangle - \langle 2\mathbf{v}, 3\mathbf{u}\rangle - \langle 2\mathbf{v}, 4\mathbf{v}\rangle \\ &= 3\langle \mathbf{u}, \mathbf{u}\rangle + 4\langle \mathbf{u}, \mathbf{v}\rangle - 6\langle \mathbf{v}, \mathbf{u}\rangle - 8\langle \mathbf{v}, \mathbf{v}\rangle \\ &= 3\|\mathbf{u}\|^2 + 4\langle \mathbf{u}, \mathbf{v}\rangle - 6\langle \mathbf{u}, \mathbf{v}\rangle - 8\|\mathbf{v}\|^2 \\ &= 3\|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle - 8\|\mathbf{v}\|^2 \end{aligned}$$

Exercise Set 6.1

1. Let R^2 have the weighted Euclidean inner product
$$\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 + 3u_2v_2$$
and let $\mathbf{u} = (1, 1)$, $\mathbf{v} = (3, 2)$, $\mathbf{w} = (0, -1)$, and $k = 3$. Compute the stated quantities.

   a. $\langle \mathbf{u}, \mathbf{v}\rangle$ **b.** $\langle k\mathbf{v}, \mathbf{w}\rangle$ **c.** $\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle$

   d. $\|\mathbf{v}\|$ **e.** $d(\mathbf{u}, \mathbf{v})$ **f.** $\|\mathbf{u}-k\mathbf{v}\|$

2. Follow the directions of Exercise 1 using the weighted Euclidean inner product
$$\langle \mathbf{u}, \mathbf{v}\rangle = \tfrac{1}{2}u_1v_1 + 5u_2v_2$$

In Exercises 3–4, compute the quantities in parts (a)–(f) of Exercise 1 using the inner product on R^2 generated by A.

3. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ 4. $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

In Exercises 5–6, find a matrix that generates the stated weighted inner product on R^2.

5. $\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 + 3u_2v_2$ 6. $\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{2}u_1v_1 + 5u_2v_2$

In Exercises 7–8, use the inner product on R^2 generated by the matrix A to find $\langle \mathbf{u}, \mathbf{v}\rangle$ for the vectors $\mathbf{u} = (0, -3)$ and $\mathbf{v} = (6, 2)$.

7. $A = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$ 8. $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

In Exercises 9–10, compute the standard inner product on M_{22} of the given matrices.

9. $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

10. $U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix},\ V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

In Exercises 11–12, find the standard inner product on P_2 of the given polynomials.

11. $\mathbf{p} = -2 + x + 3x^2,\ \mathbf{q} = 4 - 7x^2$

12. $\mathbf{p} = -5 + 2x + x^2,\ \mathbf{q} = 3 + 2x - 4x^2$

In Exercises 13–14, a weighted Euclidean inner product on R^2 is given for the vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Find a matrix that generates it.

13. $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$

14. $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 6u_2v_2$

In Exercises 15–16, a sequence of sample points is given. Use the evaluation inner product on P_3 at those sample points to find $\langle \mathbf{p}, \mathbf{q} \rangle$ for the polynomials

$$\mathbf{p} = x + x^3 \quad \text{and} \quad \mathbf{q} = 1 + x^2$$

15. $x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1$

16. $x_0 = -1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 2$

In Exercises 17–18, find $\|\mathbf{u}\|$ and $d(\mathbf{u}, \mathbf{v})$ relative to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$ on R^2.

17. $\mathbf{u} = (-3, 2)$ and $\mathbf{v} = (1, 7)$

18. $\mathbf{u} = (-1, 2)$ and $\mathbf{v} = (2, 5)$

In Exercises 19–20, find $\|\mathbf{p}\|$ and $d(\mathbf{p}, \mathbf{q})$ relative to the standard inner product on P_2.

19. $\mathbf{p} = -2 + x + 3x^2,\ \ \mathbf{q} = 4 - 7x^2$

20. $\mathbf{p} = -5 + 2x + x^2,\ \ \mathbf{q} = 3 + 2x - 4x^2$

In Exercises 21–22, find $\|U\|$ and $d(U, V)$ relative to the standard inner product on M_{22}.

21. $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix},\ V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

22. $U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix},\ V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

In Exercises 23–24, let

$$\mathbf{p} = x + x^3 \quad \text{and} \quad \mathbf{q} = 1 + x^2$$

Find $\|\mathbf{p}\|$ and $d(\mathbf{p}, \mathbf{q})$ relative to the evaluation inner product on P_3 at the stated sample points.

23. $x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1$

24. $x_0 = -1,\ x_1 = 0,\ x_2 = 1,\ x_3 = 2$

In Exercises 25–26, find $\|\mathbf{u}\|$ and $d(\mathbf{u}, \mathbf{v})$ for the vectors $\mathbf{u} = (-1, 2)$ and $\mathbf{v} = (2, 5)$ relative to the inner product on R^2 generated by the matrix A.

25. $A = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

*In Exercises 27–28, suppose that **u**, **v**, and **w** are vectors in an inner product space such that*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2, \quad \langle \mathbf{v}, \mathbf{w} \rangle = -6, \quad \langle \mathbf{u}, \mathbf{w} \rangle = -3$$
$$\|\mathbf{u}\| = 1, \quad \|\mathbf{v}\| = 2, \quad \|\mathbf{w}\| = 7$$

Evaluate the given expression.

27. a. $\langle 2\mathbf{v} - \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle$ **b.** $\|\mathbf{u} + \mathbf{v}\|$

28. a. $\langle \mathbf{u} - \mathbf{v} - 2\mathbf{w}, 4\mathbf{u} + \mathbf{v} \rangle$ **b.** $\|2\mathbf{w} - \mathbf{v}\|$

In Exercises 29–30, sketch the unit circle in R^2 using the given inner product.

29. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}u_1v_1 + \frac{1}{16}u_2v_2$

30. $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_2v_2$

In Exercises 31–32, find a weighted Euclidean inner product on R^2 for which the "unit circle" is the ellipse shown in the accompanying figure.

31.

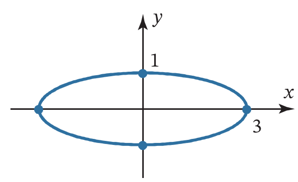


FIGURE Ex-31

32.

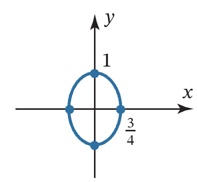


FIGURE Ex-32

In Exercises 33–34, let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Show that the expression does not define an inner product on R^3, and list all inner product axioms that fail to hold.

33. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2$

34. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - u_2v_2 + u_3v_3$

*In Exercises 35–36, suppose that **u** and **v** are vectors in an inner product space. Rewrite the given expression in terms of $\langle \mathbf{u}, \mathbf{v} \rangle$, $\|\mathbf{u}\|^2$, and $\|\mathbf{v}\|^2$.*

35. $\langle 2\mathbf{v} - 4\mathbf{u}, \mathbf{u} - 3\mathbf{v} \rangle$

36. $\langle 5\mathbf{u} + 6\mathbf{v}, 4\mathbf{v} - 3\mathbf{u} \rangle$

37. **(*Calculus required*)** Let the vector space P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Find the following for $\mathbf{p} = 1$ and $\mathbf{q} = x^2$.

a. $\langle \mathbf{p}, \mathbf{q} \rangle$ **b.** $d(\mathbf{p}, \mathbf{q})$

c. $\|\mathbf{p}\|$ **d.** $\|\mathbf{q}\|$

38. **(*Calculus required*)** Let the vector space P_3 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Find the following for $\mathbf{p} = 2x^3$ and $\mathbf{q} = 1 - x^3$.

a. $\langle \mathbf{p}, \mathbf{q} \rangle$ **b.** $d(\mathbf{p}, \mathbf{q})$

c. $\|\mathbf{p}\|$ **d.** $\|\mathbf{q}\|$

(*Calculus required*) *In Exercises 39–40, use the inner product*

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{0}^{1} f(x)g(x)dx$$

on $C[0, 1]$ to compute $\langle \mathbf{f}, \mathbf{g} \rangle$.

39. $\mathbf{f} = \cos 2\pi x,\ \ \mathbf{g} = \sin 2\pi x$

40. $\mathbf{f} = x,\ \ \mathbf{g} = e^x$