# Change of Basis

## Question:

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for $R^2$, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{u}'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

**a.** Find the transition matrix from $B'$ to $B$.

**b.** Find the transition matrix from $B$ to $B'$.

**c.** Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use (11) to compute $[\mathbf{w}]_{B'}$.

**d.** Check your work by computing $[\mathbf{w}]_{B'}$ directly.

## Solution:

**(a)** In this part, $B'$ is the start basis and $B$ is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{13}{10} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{5} & 0 \end{array}\right]$$

The transition matrix is $P_{B' \to B} = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}$.

**(b)** In this part, $B$ is the start basis and $B'$ is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & -2 & -\frac{13}{2} \end{array}\right]$$

The transition matrix is $P_{B\to B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}$.

**(c)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 2c_1 & + & 4c_2 & = & 3 \\ 2c_1 & - & c_2 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{17}{10} \\ 0 & 1 & \frac{8}{5} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{17}{10}$, $c_2 = \frac{8}{5}$, therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix} \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$.

**(d)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}'_1$ and $\mathbf{u}'_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} c_1 & - & c_2 & = & 3 \\ 3c_1 & - & c_2 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -7 \end{bmatrix}$. The solution of the linear system is $c_1 = -4$, $c_2 = -7$, therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$. This matches the result obtained in part (c).