

## 8.2 Solving Linear Recurrence Relations

### Definition 1

A linear homogeneous recurrence relation of degree  $k$  with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where  $c_1, c_2, \dots, c_k$  are real numbers, and  $c_k \neq 0$ .

The recurrence relation in the definition is **linear** because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of  $n$ . The recurrence relation is **homogeneous** because no terms occur that are not multiples of the  $a_j$ s. The coefficients of the terms of the sequence are all **constants**, rather than functions that depend on  $n$ . The **degree** is  $k$  because  $a_n$  is expressed in terms of the previous  $k$  terms of the sequence.

### THEOREM 1

Let  $c_1$  and  $c_2$  be real numbers. Suppose that  $r^2 - c_1 r - c_2 = 0$  has two distinct roots  $r_1$  and  $r_2$ . Then the sequence  $\{a_n\}$  is a solution of the recurrence relation  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$  if and only if  $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$  for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants.

### EXAMPLE 3 What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with  $a_0 = 2$  and  $a_1 = 7$ ?

### THEOREM 2

Let  $c_1$  and  $c_2$  be real numbers with  $c_2 \neq 0$ . Suppose that  $r^2 - c_1 r - c_2 = 0$  has only one root  $r_0$ . A sequence  $\{a_n\}$  is a solution of the recurrence relation  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$  if and only if  $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ , for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants.

### EXAMPLE 5 What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions  $a_0 = 1$  and  $a_1 = 6$ ?

**THEOREM 3**

Let  $c_1, c_2, \dots, c_k$  be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \dots - c_k = 0$$

has  $k$  distinct roots  $r_1, r_2, \dots, r_k$ . Then a sequence  $\{a_n\}$  is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$$

for  $n = 0, 1, 2, \dots$ , where  $\alpha_1, \alpha_2, \dots, \alpha_k$  are constants.

**EXAMPLE 6** Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions  $a_0 = 2$ ,  $a_1 = 5$ , and  $a_2 = 15$ .

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

- a)  $a_n = 3a_{n-2}$                       b)  $a_n = 3$   
 c)  $a_n = a_{n-1}^2$                       d)  $a_n = a_{n-1} + 2a_{n-3}$   
 e)  $a_n = a_{n-1}/n$   
 f)  $a_n = a_{n-1} + a_{n-2} + n + 3$   
 g)  $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

**Exercises**

4. Solve these recurrence relations together with the initial conditions given.

- a)  $a_n = a_{n-1} + 6a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 3$ ,  $a_1 = 6$   
 b)  $a_n = 7a_{n-1} - 10a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 2$ ,  $a_1 = 1$   
 c)  $a_n = 6a_{n-1} - 8a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 4$ ,  $a_1 = 10$   
 d)  $a_n = 2a_{n-1} - a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 4$ ,  $a_1 = 1$   
 e)  $a_n = a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 5$ ,  $a_1 = -1$   
 f)  $a_n = -6a_{n-1} - 9a_{n-2}$  for  $n \geq 2$ ,  $a_0 = 3$ ,  $a_1 = -3$   
 g)  $a_{n+2} = -4a_{n+1} + 5a_n$  for  $n \geq 0$ ,  $a_0 = 2$ ,  $a_1 = 8$

12. Find the solution to  $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$  for  $n = 3, 4, 5, \dots$ , with  $a_0 = 3$ ,  $a_1 = 6$ , and  $a_2 = 0$ .  
 13. Find the solution to  $a_n = 7a_{n-2} + 6a_{n-3}$  with  $a_0 = 9$ ,  $a_1 = 10$ , and  $a_2 = 32$ .  
 14. Find the solution to  $a_n = 5a_{n-2} - 4a_{n-4}$  with  $a_0 = 3$ ,  $a_1 = 2$ ,  $a_2 = 6$ , and  $a_3 = 8$ .