# Linear Combination

**Definition 1**

If $\mathbf{w}$ is a vector in a vector space $V$, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in $V$ if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \qquad (1)$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

**Theorem 4.3.1**

If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space $V$, then:

(*a*) The set $W$ of all possible linear combinations of the vectors in $S$ is a subspace of $V$.

(*b*) The set $W$ in part (a) is the "smallest" subspace of $V$ that contains all of the vectors in $S$ in the sense that any other subspace that contains those vectors contains $W$.

## Spanning Set:

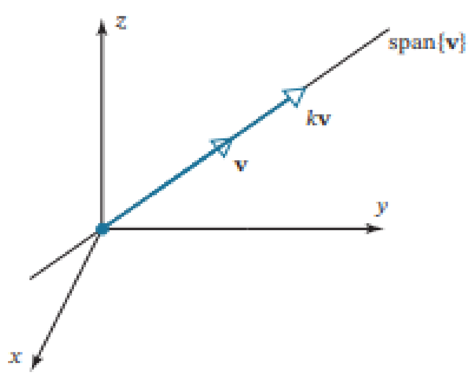
The subspace $W$ in Theorem 4.3.1 is called the subspace of $V$ ***spanned*** by $S$. The vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ in $S$ are said to ***span*** $W$, and we write

$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad W = \text{span}(S)$$

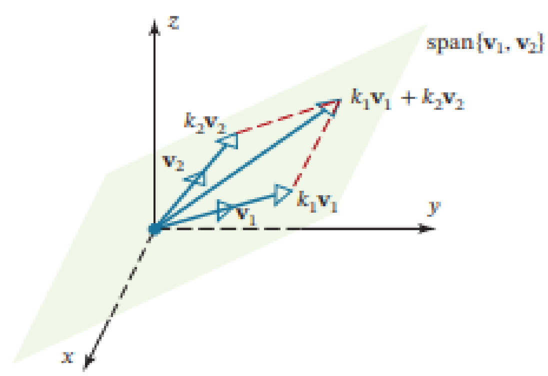
## EXAMPLE 2 | A Geometric View of Spanning in $R^2$ and $R^3$

(*a*) If $\mathbf{v}$ is a nonzero vector in $R^2$ or $R^3$ that has its initial point at the origin, then span{$\mathbf{v}$}, which is the set of all scalar multiples of $\mathbf{v}$, is the line through the origin determined by $\mathbf{v}$. You should be able to visualize this from Figure 4.3.1*a* by observing that the tip of the vector $k\mathbf{v}$ can be made to fall at any point on the line by choosing the value of $k$ to lengthen, shorten, or reverse the direction of $\mathbf{v}$ appropriately.

(*b*) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors in $R^3$ that have their initial points at the origin, then span$\{\mathbf{v}_1, \mathbf{v}_2\}$, which consists of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, is the plane through the origin determined by these two vectors. You should be able to visualize this from Figure 4.3.1*b* by observing that the tip of the vector $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$ can be made to fall at any point in the plane by adjusting the scalars $k_1$ and $k_2$ to lengthen, shorten, or reverse the directions of the vectors $k_1\mathbf{v}_1$ and $k_2\mathbf{v}_2$ appropriately.




(*a*) span{$\mathbf{v}$} is the line through the origin determined by $\mathbf{v}$




(*b*) span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$

**FIGURE 4.3.1**

## EXAMPLE 3 | A Spanning Set for $P_n$

The polynomials $1, x, x^2, \ldots, x^n$ span the vector space $P_n$ defined in Example 10 since each polynomial $\mathbf{p}$ in $P_n$ can be written as

$$\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. We can denote this by writing

$$P_n = \text{span}\{1, x, x^2, \ldots, x^n\}$$

The next two examples are concerned with two important types of problems:

- Given a nonempty set $S$ of vectors in $R^n$ and a vector $\mathbf{v}$ in $R^n$, determine whether $\mathbf{v}$ is a linear combination of the vectors in $S$.
- Given a nonempty set $S$ of vectors in $R^n$, determine whether the vectors span $R^n$.

## EXAMPLE 5 | Testing for Spanning

Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space $R^3$.

**Solution** We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in $R^3$ can be expressed as a linear combination

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Expressing this equation in terms of components gives

$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

or

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

or

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \phantom{{}+k_2} + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

Thus, our problem reduces to ascertaining whether this system is consistent for all values of $b_1$, $b_2$, and $b_3$. One way of doing this is to use parts ($e$) and ($g$) of Theorem 2.3.8, which state that the system is consistent if and only if its coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

has a nonzero determinant. But this is *not* the case here since $\det(A) = 0$ (verify), so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span $R^3$.

## A Procedure for Identifying Spanning Sets

**Step 1.** Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ be a given set of vectors in $V$, and let $\mathbf{x}$ be an arbitrary vector in $V$.

**Step 2.** Set up the augmented matrix for the linear system that results by equating corresponding components on the two sides of the vector equation

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r = \mathbf{x} \tag{2}$$

**Step 3.** Use the techniques developed in Chapters 1 and 2 to investigate the consistency or inconsistency of that system. If it is consistent for *all* choices of $\mathbf{x}$, the vectors in $S$ span $V$, and if it is inconsistent for *some* vector $\mathbf{x}$, they do not.

### EXAMPLE 6 | Testing for Spanning in $P_2$

Determine whether the set $S$ spans $P_2$.

(*a*) $S = \{1 + x + x^2, -1 - x, 2 + 2x + x^2\}$

(*b*) $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$

**Solution (*a*)** An arbitrary vector in $P_2$ is of the form $\mathbf{p} = a + bx + cx^2$, and so (2) becomes

$$k_1(1 + x + x^2) + k_2(-1 - x) + k_3(2 + 2x + x^2) = a + bx + cx^2$$

which we can rewrite as

$$(k_1 - k_2 + 2k_3) + (k_1 - k_2 + 2k_3)x + (k_1 + k_3)x^2 = a + bx + cx^2$$

Equating corresponding coefficients yields a linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -1 & 2 & a \\ 1 & -1 & 2 & b \\ 1 & 0 & 1 & c \end{bmatrix}$$

and whose coefficient matrix is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Because this matrix is square we can apply Theorem 2.3.8. Since the matrix $A$ has two identical rows it follows that $\det(A) = 0$, so parts (*e*) and (*g*) of that theorem imply that the system is inconsistent for *some* choice of $a$, $b$, and $c$; and this tells us that $S$ does *not* span $P_2$.

**Solution (*b*)** Using the same procedure as in part (*a*), the augmented matrix corresponding to (2) is

$$\begin{bmatrix} 0 & 0 & 1 & -1 & a \\ 1 & 1 & 1 & -1 & b \\ 1 & -1 & 0 & 0 & c \end{bmatrix} \tag{3}$$

Whereas Theorem 2.3.8 was applicable in part (*a*), it is not applicable here because the coefficient matrix is not square. However, reducing (3) to reduced row echelon form yields (verify)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & 0 & \frac{-a+b-c}{2} \\ 0 & 0 & 1 & -1 & a \end{bmatrix}$$

so (3) is consistent for every choice $a$, $b$, and $c$. Thus, the vectors in $S$ span $P_2$, which we can express by writing $\text{span}(S) = P_2$.

## EXAMPLE 7 | Testing for Spanning in $M_{22}$

In each part, determine whether the set $S$ spans $M_{22}$.

(*a*) $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

(*b*) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$

**Solution (*a*)** An arbitrary vector in $M_{22}$ is of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so Equation (2) becomes

$$k_1\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_3\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which we can rewrite as

$$\begin{bmatrix} k_1 + k_2 + k_3 + k_4 & 2k_1 + 2k_3 + k_4 \\ k_3 + k_4 & k_1 + k_2 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Equating corresponding entries produces a linear system whose augmented matrix is

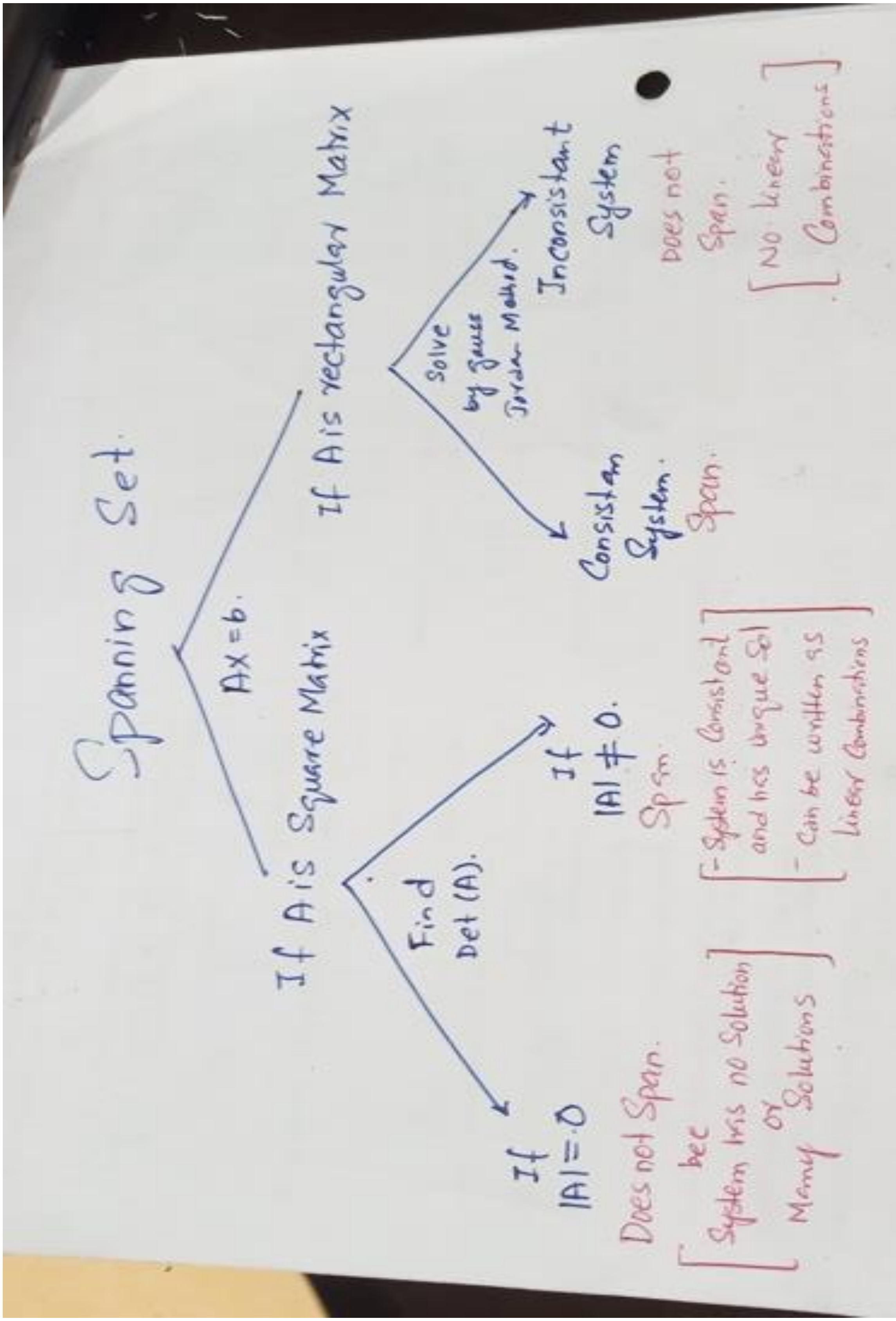
$$\begin{bmatrix} 1 & 1 & 1 & 1 & a \\ 2 & 0 & 2 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 1 & 1 & 0 & 1 & d \end{bmatrix} \text{ and whose coefficient matrix is } A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

As in part (*a*) of Example 6, the coefficient matrix is square, so we can apply parts (*e*) and (*g*) of Theorem 2.3.8. We leave it for you to verify that $\det(A) = -2 \neq 0$, so the system is consistent for *every* choice of $a$, $b$, $c$, and $d$, which implies that $\operatorname{span}(S) = M_{22}$.

**Solution (*b*)** Using the same procedure as in part (*a*), the augmented matrix for the linear system corresponding to Equation (2) is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 1 & b \\ 0 & 1 & 1 & -1 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix} \text{ and the coefficient matrix is } A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ which}$$

is square, so once again we can apply parts (*e*) and (*g*) of Theorem 2.3.8. Since the second and fourth rows of this matrix are identical, it follows that $\det(A) = 0$. Thus, the system is inconsistent for *some* choice of $a$, $b$, $c$, and $d$, which implies that $S$ does not span $M_{22}$.


Spanning Set.
Ax=b.
If A is Square Matrix
If A is rectangular Matrix
Find Det (A).
If |A|=0
Does not Span.
bec System has no Solution or Many Solutions
If |A|≠0.
Span.
- System is Consistent and has Unique Sol
- Can be written as linear Combinations
Solve by Gauss Jordan Method.
Consistan System.
Span.
Inconsistant System
Does not Span.
No linear Combinations

**Question:**

**9.** Determine whether the following polynomials span $P_2$.

$$\mathbf{p}_1 = 1 - x + 2x^2, \quad \mathbf{p}_2 = 3 + x,$$
$$\mathbf{p}_3 = 5 - x + 4x^2, \quad \mathbf{p}_4 = -2 - 2x + 2x^2$$

**Solution:**

The given polynomials span $P_2$ if an arbitrary polynomial in $P_2$, $\mathbf{p} = a_0 + a_1x + a_2x^2$ can be expressed as a linear combination

$$a_0 + a_1x + a_2x^2 = k_1\left(1 - x + 2x^2\right) + k_2\left(3 + x\right) + k_3\left(5 - x + 4x^2\right) + k_4\left(-2 - 2x + 2x^2\right)$$

Grouping the terms according to the powers of $x$ yields

$$a_0 + a_1x + a_2x^2 = \left(k_1 + 3k_2 + 5k_3 - 2k_4\right) + \left(-k_1 + k_2 - k_3 - 2k_4\right)x + \left(2k_1 + 4k_3 + 2k_4\right)x^2$$

Since this equality must hold for every real value $x$, the coefficients associated with the like powers of $x$ on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcrcl} 1k_1 & + & 3k_2 & + & 5k_3 & - & 2k_4 & = & a_0 \\ -1k_1 & + & 1k_2 & - & 1k_3 & - & 2k_4 & = & a_1 \\ 2k_1 & + & 0k_2 & + & 4k_3 & + & 2k_4 & = & a_2 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2 & 1 & \frac{1}{4}a_0 - \frac{3}{4}a_1 \\ 0 & 1 & 1 & -1 & \frac{1}{4}a_0 + \frac{1}{4}a_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \end{bmatrix}$ therefore the system has no solution if $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$.

Since polynomials $\mathbf{p} = a_0 + a_1x + a_2x^2$ for which $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$ cannot be expressed as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$, we conclude that the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ do not span $P_2$.

## Question:

**15.** Let $W$ be the solution space to the system $A\mathbf{x} = \mathbf{0}$. Determine whether the set $\{\mathbf{u}, \mathbf{v}\}$ spans $W$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

**a.** $\mathbf{u} = (1, 0, -1, 0)$, $\mathbf{v} = (0, 1, 0, -1)$

**b.** $\mathbf{u} = (1, 0, -1, 0)$, $\mathbf{v} = (1, 1, -1, -1)$

## Solution:

**(a)** The solution space $W$ to the homogenous system $A\mathbf{x} = \mathbf{0}$ where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ is obtained from the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution in vector form is $(x,y,z,w) = (s,t,-s,-t) = s(1,0,-1,0) + t(0,1,0,-1)$ therefore the solution space is spanned by the vectors $\mathbf{v}_1 = (1,0,-1,0)$ and $\mathbf{v}_2 = (0,1,0,-1)$. We conclude that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (0,1,0,-1)$ span the solution space $W$.

**(b)** From part (a) and Theorem 4.3.2 we need to show that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (1,1,-1,-1)$ are contained in the span of the vectors $\mathbf{v}_1 = (1,0,-1,0)$ and $\mathbf{v}_2 = (0,1,0,-1)$. Observe that $\mathbf{u} = \mathbf{v}_1$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. We conclude that the vectors $\mathbf{u} = (1,0,-1,0)$ and $\mathbf{v} = (1,1,-1,-1)$ span the solution space $W$.

## Question:

**18.** In each part, let $T_A : R^3 \to R^2$ be multiplication by $A$, and let $\mathbf{u}_1 = (0, 1, 1)$ and $\mathbf{u}_2 = (2, -1, 1)$ and $\mathbf{u}_3 = (1, 1, -2)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ spans $R^2$.

**a.** $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix}$

## Solution:

**(a)** The vectors $T_A(0,1,1)=(1,0)$, $T_A(2,-1,1)=(1,-2)$, and, $T_A(1,1,-2)=(2,3)$ span $R^2$ if an arbitrary vector

$\mathbf{b}=(b_1,b_2)$ can be expressed as a linear combination

$$(b_1,b_2)=k_1(1,0)+k_2(1,-2)+k_3(2,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 1k_1 & + & 1k_2 & + & 2k_3 & = & b_1 \\ 0k_1 & - & 2k_2 & + & 3k_3 & = & b_2 \end{array}$$

The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$, therefore the system is consistent for all right hand side vectors $\mathbf{b}$.

We conclude that $T_A(\mathbf{u}_1)$, $T_A(\mathbf{u}_2)$, and, $T_A(\mathbf{u}_3)$ span $R^2$.

**(b)** The vectors $T_A(0,1,1)=(1,4)$, $T_A(2,-1,1)=(-1,4)$, and, $T_A(1,1,-2)=(1,-4)$ span $R^2$ if an arbitrary vector

$\mathbf{b}=(b_1,b_2)$ can be expressed as a linear combination

$$(b_1,b_2)=k_1(1,4)+k_2(-1,4)+k_3(1,-4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 1k_1 & - & 1k_2 & + & 1k_3 & = & b_1 \\ 4k_1 & + & 4k_2 & - & 4k_3 & = & b_2 \end{array}$$

The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, therefore the system is consistent for all right hand side vectors $\mathbf{b}$. We conclude that $T_A(\mathbf{u}_1)$, $T_A(\mathbf{u}_2)$, and $T_A(\mathbf{u}_3)$ span $R^2$.