Orthogonal Matrices

Definition 1

A square matrix A is said to be ***orthogonal*** if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^TA = I \tag{1}$$

A matrix transformation $T_A: R^n \to R^n$ is said to be an ***orthogonal transformation*** or an ***orthogonal operator*** if A is an orthogonal matrix.

EXAMPLE 1 | A 3 × 3 Orthogonal Matrix

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^TA = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}\begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 2 | Rotation and Reflection Matrices Are Orthogonal

Recall from Table 5 of Section 1.8 that the standard matrix for the counterclockwise rotation about the origin of R^2 through an angle θ is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This matrix is orthogonal for all choices of θ since

$$A^TA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We leave it for you to verify that the reflection matrices in Tables 1 and 2 of Section 1.8 are all orthogonal.

Theorem 7.1.1

The following are equivalent for an $n \times n$ matrix A.

(*a*) A is orthogonal.

(*b*) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.

(*c*) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Theorem 7.1.2

(*a*) The transpose of an orthogonal matrix is orthogonal.

(*b*) The inverse of an orthogonal matrix is orthogonal.

(*c*) A product of orthogonal matrices is orthogonal.

(*d*) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

EXAMPLE 3 | $\det(A) = \pm 1$ for an Orthogonal Matrix A

The matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal since its row (and column) vectors form orthonormal sets in R^2 with the Euclidean inner product. We leave it for you to verify that $\det(A) = 1$ and that interchanging the rows produces an orthogonal matrix whose determinant is -1.

Question:

In each part of Exercises 1–4, determine whether the matrix is orthogonal, and if so find it inverse.

4. a. $$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Solution:

$AA^T = A^T A = I$ therefore A is an orthogonal matrix; $A^{\;1} = A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

Question:

In each part of Exercises 1–4, determine whether the matrix is orthogonal, and if so find it inverse.

b. $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution:

$\| \mathbf{r}_2 \| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{7}{12}} \neq 1$. The matrix is not orthogonal.

Question:

In Exercises 5–6, show that the matrix is orthogonal three ways: first by calculating ATA, then by using part (b) of Theorem 7.1.1, and then by using part (c) of Theorem 7.1.1.

5. $$A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

Solution:

$$A^T A = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} = I;$$

row vectors of A, $\mathbf{r}_1 = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \end{bmatrix}$, $\mathbf{r}_3 = \begin{bmatrix} \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$, form an orthonormal set since $\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$ and $\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = \|\mathbf{r}_3\| = 1$;

column vectors of A, $\mathbf{c}_1 = \begin{bmatrix} \frac{4}{5} \\ -\frac{9}{25} \\ \frac{12}{25} \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 0 \\ \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix} -\frac{3}{5} \\ -\frac{12}{25} \\ \frac{16}{25} \end{bmatrix}$, form an orthonormal set since

$\mathbf{c}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1 \cdot \mathbf{c}_3 = \mathbf{c}_2 \cdot \mathbf{c}_3 = 0$ and $\|\mathbf{c}_1\| = \|\mathbf{c}_2\| = \|\mathbf{c}_3\| = 1$.