CHAPTER 4: GENERAL VECTOR SPACES

4.1 Real Vector Spaces

1. **(a)** $\mathbf{u}+\mathbf{v}=(-1+3,2+4)=(2,6)$; $k\mathbf{u}=(0,3\cdot 2)=(0,6)$

(b) For any $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$ is an ordered pair of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2)$ in V and for any scalar k, $k\mathbf{u}=(0,ku_2)$. is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^2.

(d) Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(0,k(u_2+v_2))=(0,ku_2)+(0,kv_2)$
$=k(u_1,u_2)+k(v_1,v_2)$ for all real k, u_1, u_2, v_1, and v_2;

Axiom 8: $(k+m)(u_1,u_2)=(0,(k+m)u_2)=(0,ku_2+mu_2)=(0,ku_2)+(0,mu_2)$
$=k(u_1,u_2)+m(u_1,u_2)$ for all real k, m, u_1, and u_2;

Axiom 9: $k(m(u_1,u_2))=k(0,mu_2)=(0,kmu_2)=(km)(u_1,u_2)$ for all real k, m, u_1, and u_2;

(e) Axiom 10 fails to hold: $1(u_1,u_2)=(0,u_2)$ does not generally equal (u_1,u_2).
Consequently, V is not a vector space.

2. **(a)** $\mathbf{u}+\mathbf{v}=(0+1+1,4-3+1)=(2,2)$; $k\mathbf{u}=(2\cdot 0,2\cdot 4)=(0,8)$

(b) $(0,0)+(u_1,u_2)=(0+u_1+1,0+u_2+1)=(u_1+1,\ u_2+1)\neq(u_1,u_2)$ therefore $(0,0)$ is not the zero vector $\mathbf{0}$ required by Axiom 4

(c) For all real numbers u_1 and u_2, we have
$(-1,-1)+(u_1,u_2)=(-1+u_1+1,\ -1+u_2+1)=(u_1,\ u_2)$ and
$(u_1,u_2)+(-1,-1)=(u_1-1+1,\ u_2-1+1)=(u_1,\ u_2)$ therefore Axiom 4 holds for
$\mathbf{0}=(-1,-1)$

d) For any pair of real numbers $\mathbf{u}=(u_1,u_2)$, letting $-\mathbf{u}=(-2-u_1,-2-u_2)$ yields
$\mathbf{u}+(-\mathbf{u})=(u_1+(-2-u_1)+1,u_2+(-2-u_2)+1)=(-1,-1)=\mathbf{0}$;
Since $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$ holds as well, Axiom 5 holds.

(e) Axiom 7 fails to hold:
$k(\mathbf{u}+\mathbf{v})=k(u_1+v_1+1,u_2+v_2+1)=(ku_1+kv_1+k,ku_2+kv_2+k)$

$k\mathbf{u}+k\mathbf{v}=(ku_1,ku_2)+(kv_1,kv_2)=(ku_1+kv_1+1,ku_2+kv_2+1)$
therefore in general $k(\mathbf{u}+\mathbf{v})\neq k\mathbf{u}+k\mathbf{v}$

Axiom 8 fails to hold:
$(k+m)\mathbf{u}=((k+m)u_1,(k+m)u_2)=(ku_1+mu_1,ku_2+mu_2)$
$k\mathbf{u}+m\mathbf{u}=(ku_1,ku_2)+(mu_1,mu_2)=(ku_1+mu_1+1,ku_2+mu_2+1)$
therefore in general $(k+m)\mathbf{u}\neq k\mathbf{u}+m\mathbf{u}$

3. Let V denote the set of all real numbers.

Axiom 1: $x+y$ is in V for all real x and y;

Axiom 2: $x+y=y+x$ for all real x and y;

Axiom 3: $x+(y+z)=(x+y)+z$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=0$, we have $0+x=x+0=x$ for all real x;

Axiom 5: for each $\mathbf{u}=x$, let $-\mathbf{u}=-x$; then $x+(-x)=(-x)+x=0$

Axiom 6: kx is in V for all real k and x;

Axiom 7: $k(x+y)=kx+ky$ for all real k, x, and y;

Axiom 8: $(k+m)x=kx+mx$ for all real k, m, and x;

Axiom 9: $k(mx)=(km)x$ for all real k, m, and x;

Axiom 10: $1x=x$ for all real x.

This is a vector space – all axioms hold.

4. Let V denote the set of all pairs of real numbers of the form $(x,0)$.

Axiom 1: $(x,0)+(y,0)=(x+y,0)$ is in V for all real x and y;

Axiom 2: $(x,0)+(y,0)=(x+y,0)=(y+x,0)=(y,0)+(x,0)$ for all real x and y;

Axiom 3: $(x,0)+((y,0)+(z,0))=(x,0)+(y+z,0)=(x+y+z,0)=(x+y,0)+(z,0)$
$=((x,0)+(y,0))+(z,0)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=(0,0)$, we have $(0,0)+(x,0)=(x,0)$ and $(x,0)+(0,0)=(x,0)$ for all real x;

Axiom 5: for each $\mathbf{u}=(x,0)$, let $-\mathbf{u}=(-x,0)$;
then $(x,0)+(-x,0)=(0,0)$ and $(-x,0)+(x,0)=(0,0)$;

Axiom 6: $k(x,0)=(kx,0)$ is in V for all real k and x;

Axiom 7: $k\left(\left(x,0\right)+\left(y,0\right)\right)=k\left(x+y,0\right)=\left(kx+ky,0\right)=k\left(x,0\right)+k\left(y,0\right)$
for all real k, x, and y;

Axiom 8: $\left(k+m\right)\left(x,0\right)=\left(\left(k+m\right)x,0\right)=\left(kx+mx,0\right)=k\left(x,0\right)+m\left(x,0\right)$
for all real k, m, and x;

Axiom 9: $k\left(m\left(x,0\right)\right)=k\left(mx,0\right)=\left(kmx,0\right)=\left(km\right)\left(x,0\right)$ for all real k, m, and x;

Axiom 10: $1\left(x,0\right)=\left(x,0\right)$ for all real x.

This is a vector space – all axioms hold.

5. Axiom 5 fails whenever $x\neq 0$ since it is then impossible to find $\left(x',y'\right)$ satisfying $x'\geq 0$ for which $\left(x,y\right)+\left(x',y'\right)=\left(0,0\right)$. (The zero vector from axiom 4 must be $\mathbf{0}=\left(0,0\right)$.)

Axiom 6 fails whenever $k<0$ and $x\neq 0$.

This is not a vector space.

6. Let V denote the set of all n-tuples of real numbers of the form $\left(x,x,\ldots,x\right)$.

Axiom 1: $\left(x,x,\ldots,x\right)+\left(y,y,\ldots,y\right)=\left(x+y,x+y,\ldots,x+y\right)$ is in V for all real x and y;

Axiom 2: $\left(x,x,\ldots,x\right)+\left(y,y,\ldots,y\right)=\left(x+y,x+y,\ldots,x+y\right)=\left(y+x,y+x,\ldots,y+x\right)$
$=\left(y,y,\ldots,y\right)+\left(x,x,\ldots,x\right)$ for all real x and y;

Axiom 3: $\left(x,x,\ldots,x\right)+\left(\left(y,y,\ldots,y\right)+\left(z,z,\ldots,z\right)\right)=\left(x,x,\ldots,x\right)+\left(y+z,y+z,\ldots,y+z\right)$
$=\left(x+y+z,x+y+z,\ldots,x+y+z\right)=\left(x+y,x+y,\ldots,x+y\right)+\left(z,z,\ldots,z\right)$
$=\left(\left(x,x,\ldots,x\right)+\left(y,y,\ldots,y\right)\right)+\left(z,z,\ldots,z\right)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=\left(0,0,\ldots,0\right)$, we have $\left(0,0,\ldots,0\right)+\left(x,x,\ldots,x\right)=\left(x,x,\ldots,x\right)$ and
$\left(x,x,\ldots,x\right)+\left(0,0,\ldots,0\right)=\left(x,x,\ldots,x\right)$ for all real x;

Axiom 5: for each $\mathbf{u}=\left(x,x,\ldots,x\right)$, let $-\mathbf{u}=\left(-x,-x,\ldots,-x\right)$;
then $\left(x,x,\ldots,x\right)+\left(-x,-x,\ldots,-x\right)=\left(0,0,\ldots,0\right)$ and
$\left(-x,-x,\ldots,-x\right)+\left(x,x,\ldots,x\right)=\left(0,0,\ldots,0\right)$;

Axiom 6: $k\left(x,x,\ldots,x\right)=\left(kx,kx,\ldots,kx\right)$ is in V for all real k and x;

Axiom 7: $k\left(\left(x,x,\ldots,x\right)+\left(y,y,\ldots,y\right)\right)=k\left(x+y,x+y,\ldots,x+y\right)=\left(kx+ky,kx+ky,\ldots,kx+ky\right)$
$=k\left(x,x,\ldots,x\right)+k\left(y,y,\ldots,y\right)$ for all real k, x, and y;

Axiom 8: $(k+m)(x,x,\ldots,x)=((k+m)x,(k+m)x,\ldots,(k+m)x)$
$=(kx+mx,kx+mx,\ldots,kx+mx)=k(x,x,\ldots,x)+m(x,x,\ldots,x)$
for all real k, m, and x;

Axiom 9: $k(m(x,x,\ldots,x))=k(mx,mx,\ldots,mx)=(kmx,kmx,\ldots,kmx)=(km)(x,x,\ldots,x)$
for all real k, m, and x;

Axiom 10: $1(x,x,\ldots,x)=(x,x,\ldots,x)$ for all real x.

This is a vector space – all axioms hold.

7. Axiom 8 fails to hold:

$$(k+m)\mathbf{u}=\left((k+m)^2 x,(k+m)^2 y,(k+m)^2 z\right)$$

$$k\mathbf{u}+m\mathbf{u}=\left(k^2x,k^2y,k^2z\right)+\left(m^2x,m^2y,m^2z\right)=\left(\left(k^2+m^2\right)x,\left(k^2+m^2\right)y,\left(k^2+m^2\right)z\right)$$

therefore in general $(k+m)\mathbf{u}\neq k\mathbf{u}+m\mathbf{u}$.

This is not a vector space.

8. Axiom 1 fails since a sum of two 2×2 invertible matrices may or may not be invertible, e.g. both $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ are invertible, but $\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is not invertible.

Axiom 6 fails whenever $k=0$.

9. Let V be the set of all 2×2 matrices of the form $\begin{bmatrix}a&0\\0&b\end{bmatrix}$ (i.e., all diagonal 2×2 matrices)

Axiom 1: the sum of two diagonal 2×2 matrices is also a diagonal 2×2 matrix.

Axiom 2: follows from part (a) of Theorem 1.4.1.

Axiom 3: follows from part (b) of Theorem 1.4.1.

Axiom 4: taking $0=\begin{bmatrix}0&0\\0&0\end{bmatrix}$; follows from part (a) of Theorem 1.4.2.

Axiom 5: let the negative of $\begin{bmatrix}a&0\\0&b\end{bmatrix}$ be $\begin{bmatrix}-a&0\\0&-b\end{bmatrix}$; follows from part (c) of Theorem 1.4.2 and Axiom 2.

Axiom 6: the scalar multiple of a diagonal 2×2 matrix is also a diagonal 2×2 matrix.

Axiom 7: follows from part (h) of Theorem 1.4.1.

Axiom 8: follows from part (j) of Theorem 1.4.1.

Axiom 9: follows from part (l) of Theorem 1.4.1.

Axiom 10: $1\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for all real a and b.

This is a vector space – all axioms hold.

10. Let V be the set of all real-valued functions f defined for all real numbers and such that $f(1)=0$.

Axiom 1: If f and g are in V then $f+g$ is a function defined for all real numbers and $(f+g)(1)=f(1)+g(1)=0$ therefore V is closed under the operation of addition defined by Formula (2).

Axiom 6: If k is a scalar and f is in V then kf is a function defined for all real numbers and $(kf)(1)=k(f(1))=0$ therefore V is closed under the operation of scalar multiplication defined by Formula (3).

Verification of the eight remaining axioms proceeds analogously to Example 6.

This is a vector space – all axioms hold.

11. Let V denote the set of all pairs of real numbers of the form $(1,x)$.

Axiom 1: $(1,y)+(1,y')=(1,y+y')$ is in V for all real y and y';

Axiom 2: $(1,y)+(1,y')=(1,y+y')=(1,y'+y)=(1,y')+(1,y)$ for all real y and y';

Axiom 3: $(1,y)+((1,y')+(1,y''))=(1,y)+(1,y'+y'')=(1,y+y'+y'')=(1,y+y')+(1,y'')$
$=((1,y)+(1,y'))+(1,y'')$ for all real y, y', and y'';

Axiom 4: taking $\mathbf{0}=(1,0)$, we have $(1,0)+(1,y)=(1,y)$ and $(1,y)+(1,0)=(1,y)$ for all real y;

Axiom 5: for each $\mathbf{u}=(1,y)$, let $-\mathbf{u}=(1,-y)$;
then $(1,y)+(1,-y)=(1,0)$ and $(1,-y)+(1,y)=(1,0)$;

Axiom 6: $k(1,y)=(1,ky)$ is in V for all real k and y;

Axiom 7: $k((1,y)+(1,y'))=k(1,y+y')=(1,ky+ky')=(1,ky)+(1,ky')=k(1,y)+k(1,y')$ for all real k, y, and y';

Axiom 8: $(k+m)(1,y)=(1,(k+m)y)=(1,ky+my)=(1,ky)+(1,my)=k(1,y)+m(1,y)$ for all real k, m, and y;

Axiom 9: $k(m(1,y))=k(1,my)=(1,kmy)=(km)(1,y)$ for all real k, m, and y;

Axiom 10: $1(1,y)=(1,y)$ for all real y.

This is a vector space – all axioms hold.

12. Let V be the set of polynomials of the form $a+bx$.

Axiom 1: $(a_0+b_0x)+(a_1+b_1x)=(a_0+a_1)+(b_0+b_1)x$ is in V for all real a_0, a_1, b_0, and b_1;

Axiom 2: $(a_0+b_0x)+(a_1+b_1x)=(a_0+a_1)+(b_0+b_1)x=(a_1+a_0)+(b_1+b_0)x$
$=(a_1+b_1x)+(a_0+b_0x)$ for all real a_0, a_1, b_0, and b_1;

Axiom 3: $(a_0+b_0x)+((a_1+b_1x)+(a_2+b_2x))=(a_0+a_1+a_2)+(b_0+b_1+b_2)x$
$((a_0+b_0x)+(a_1+b_1x))+(a_2+b_2x)$ for all real a_0, a_1, a_2, b_0, b_1, and b_2;

Axiom 4: taking $\mathbf{0}=0+0x$, we have $(0+0x)+(a+bx)=a+bx$ and
$(a+bx)+(0+0x)=a+bx$ for all real a and b;

Axiom 5: for each $\mathbf{u}=a+bx$, let $-\mathbf{u}=-a-bx$;
then $(a+bx)+(-a-bx)=0+0x=(-a-bx)+(a+bx)$ for all real a and b;

Axiom 6: $k(a+bx)=ka+(kb)x$ is in V for all real a, b, and k;

Axiom 7: $k((a_0+b_0x)+(a_1+b_1x))=k((a_0+a_1)+(b_0+b_1)x)=k(a_0+b_0x)+k(a_1+b_1x)$ for all real a_0, a_1, b_0, b_1, and k;

Axiom 8: $(k+m)(a+bx)=(k+m)a+(k+m)bx=k(a+bx)+m(a+bx)$
for all real a, b, k, and m;

Axiom 9: $k(m(a+bx))=k(ma+mbx)=kma+kmbx=(km)(a+bx)$
for all real a, b, k, and m;

Axiom 10: $1(a+bx)=a+bx$ for all real a and b.

This is a vector space – all axioms hold.

13. Axiom 3: follows from part (b) of Theorem 1.4.1 since

$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\left(\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}+\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right)+\begin{bmatrix}w_{11} & w_{12}\\ w_{21} & w_{22}\end{bmatrix}=(\mathbf{u}+\mathbf{v})+\mathbf{w}$$

Axiom 7: follows from part (h) of Theorem 1.4.1 since

$$k(\mathbf{u}+\mathbf{v})=k\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}\right)=k\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+k\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=k\mathbf{u}+k\mathbf{v}$$