Angle and Orthogonality in Inner Product Spaces

Theorem 6.2.1

Cauchy–Schwarz Inequality

If **u** and **v** are vectors in a real inner product space V, then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\| \tag{3}$$

Angle Between Vectors

Our next goal is to define what is meant by the "angle" between vectors in a real inner product space. As a first step, we leave it as an exercise for you to use the Cauchy–Schwarz inequality to show that

$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1 \tag{6}$$

This being the case, there is a unique angle θ in radian measure for which

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \quad \text{and} \quad 0 \leq \theta \leq \pi \tag{7}$$

(**Figure 6.2.1**). This enables us to *define* the ***angle*** θ ***between* u *and* v** to be

$$\boxed{\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)} \tag{8}$$

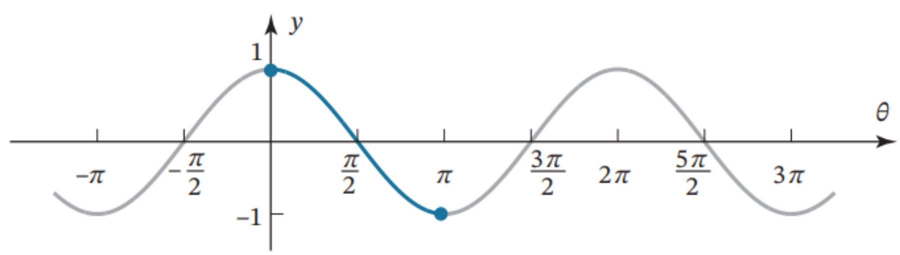

FIGURE 6.2.1

Theorem 6.2.2

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a real inner product space V, and if k is any scalar, then:

(a) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(b) $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ **[Triangle inequality for distances]**

Orthogonality

Definition 1

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V are called ***orthogonal*** if $\langle \mathbf{u},\mathbf{v}\rangle = 0$.

EXAMPLE 2 | Orthogonality Depends on the Inner Product

The vectors $\mathbf{u} = (1,1)$ and $\mathbf{v} = (1,-1)$ are orthogonal with respect to the Euclidean inner product on R^2 since

$$\mathbf{u}\cdot\mathbf{v} = (1)(1) + (1)(-1) = 0$$

However, they are not orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u},\mathbf{v}\rangle = 3u_1v_1 + 2u_2v_2$ since

$$\langle \mathbf{u},\mathbf{v}\rangle = 3(1)(1) + 2(1)(-1) = 1 \neq 0$$

EXAMPLE 3 | Orthogonal Vectors in M_{22}

If M_{22} has the inner product of Example 6 in the preceding section, then the matrices

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are orthogonal when viewed as vectors since

$$\langle U,V\rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0$$

Theorem 6.2.3

Generalized Theorem of Pythagoras

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in a real inner product space, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Definition 2

If W is a subspace of a real inner product space V, then the set of all vectors in V that are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$.

Question:

17. Do there exist scalars k and l such that the vectors

$$\mathbf{p}_1 = 2 + kx + 6x^2, \quad \mathbf{p}_2 = l + 5x + 3x^2, \quad \mathbf{p}_3 = 1 + 2x + 3x^2$$

are mutually orthogonal with respect to the standard inner product on P_2?

Solution:

Orthogonality of $\mathbf{p}_1$ and $\mathbf{p}_3$ implies $\langle \mathbf{p}_1, \mathbf{p}_3 \rangle = (2)(1)+(k)(2)+(6)(3) = 2k+20 = 0$ so $k=-10$. Likewise, orthogonality of $\mathbf{p}_2$ and $\mathbf{p}_3$ implies $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = (l)(1)+(5)(2)+(3)(3) = l+19$ so $l=-19$. Substituting the values of k and l obtained above yields the polynomials $\mathbf{p}_1 = 2-10x+6x^2$ and $\mathbf{p}_2 = -19+5x+3x^2$ which are not orthogonal since $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = (2)(-19)+(-10)(5)+(6)(3) = -70 \neq 0$. We conclude that no scalars k and l exist that make the three vectors mutually orthogonal.