Inner Product Spaces

Weighted Euclidean Inner Product:

Although the Euclidean inner product is the most important inner product on R^n, there are various applications in which it is desirable to modify it by *weighting* each term differently. More precisely, if

$$w_1, w_2, \ldots, w_n$$

are *positive* real numbers, called ***weights***, and if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then it can be shown that the formula

$$\boxed{\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n} \tag{2}$$

defines an inner product on R^n that we call the ***weighted Euclidean inner product with weights*** $\boldsymbol{w_1, w_2, \ldots, w_n}$.

EXAMPLE 1 | Weighted Euclidean Inner Product

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2 \tag{3}$$

satisfies the four inner product axioms.

Solution

Axiom 1: Interchanging $\mathbf{u}$ and $\mathbf{v}$ in Formula (3) does not change the sum on the right side, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

Axiom 2: If $\mathbf{w} = (w_1, w_2)$, then

$$\begin{aligned}
\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\
&= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2) \\
&= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2) \\
&= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

Axiom 3: $\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2$

$$\begin{aligned}
&= k(3u_1 v_1 + 2u_2 v_2) \\
&= k\langle \mathbf{u}, \mathbf{v} \rangle
\end{aligned}$$

Axiom 4: Observe that $\langle \mathbf{v}, \mathbf{v} \rangle = 3(v_1 v_1) + 2(v_2 v_2) = 3v_1^2 + 2v_2^2 \geq 0$ with equality if and only if $v_1 = v_2 = 0$, that is, if and only if $\mathbf{v} = \mathbf{0}$.

Question:

1. Let R^2 have the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$$

and let $\mathbf{u} = (1, 1)$, $\mathbf{v} = (3, 2)$, $\mathbf{w} = (0, -1)$, and $k = 3$. Compute the stated quantities.

a. $\langle \mathbf{u}, \mathbf{v} \rangle$ b. $\langle k\mathbf{v}, \mathbf{w} \rangle$ c. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle$

d. $\|\mathbf{v}\|$ e. $d(\mathbf{u}, \mathbf{v})$ f. $\|\mathbf{u} - k\mathbf{v}\|$

Solution:

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = 2(1)(3) + 3(1)(2) = 12$

(b) $\langle k\mathbf{v}, \mathbf{w} \rangle = 2((3)(3))(0) + 3((3)(2))(-1) = -18$

(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 2(1+3)(0) + 3(1+2)(-1) = -9$

(d) $\| \mathbf{v} \| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = [2(3)(3) + 3(2)(2)]^{1/2} = \sqrt{30}$

(e) $d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \langle (-2,-1), (-2,-1) \rangle^{1/2} = [2(-2)(-2) + 3(-1)(-1)]^{1/2} = \sqrt{11}$

(f) $\| \mathbf{u} - k\mathbf{v} \| = \langle (-8,-5), (-8,-5) \rangle^{1/2} = [2(-8)(-8) + 3(-5)(-5)]^{1/2} = \sqrt{203}$

Question:

In Exercises 3–4, compute the quantities in parts (a)–(f) of Exercise 1 using the inner product on R^2 generated by A.

3. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Solution:

(a) $$\langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} = 34$$

(b) $$\langle k\mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 24 \\ 15 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -39$$

(c) $$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 11 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -18$$

(d) $$\| \mathbf{v} \| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} 8 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} \right)^{1/2} = \sqrt{89}$$

(e) $$d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right) \right]^{1/2} = \left(\begin{bmatrix} -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -3 \end{bmatrix} \right)^{1/2} = \sqrt{34}$$

Question:

In Exercises 9–10, compute the standard inner product on M_{22} of the given matrices.

9. $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

Solution:

If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\langle \mathbf{u}, \mathbf{v} \rangle = \operatorname{tr}\left(U^T V\right) = \operatorname{tr}\left(\begin{bmatrix} 1 & 13 \\ 10 & 2 \end{bmatrix} \right) = 3$.

Question:

*In Exercises **11–12**, find the standard inner product on P_2 of the given polynomials.*

11. $\mathbf{p} = -2 + x + 3x^2$, $\mathbf{q} = 4 - 7x^2$

Solution:

$$\langle \mathbf{p}, \mathbf{q} \rangle = (-2)(4) + (1)(0) + (3)(-7) = -29$$

Question:

*In Exercises **15–16**, a sequence of sample points is given. Use the evaluation inner product on P_3 at those sample points to find $\langle \mathbf{p}, \mathbf{q} \rangle$ for the polynomials*

$$\mathbf{p} = x + x^3 \quad \textit{and} \quad \mathbf{q} = 1 + x^2$$

15. $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$

Solution:

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(-2)q(-2) + p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

$$= (-10)(5) + (-2)(2) + (0)(1) + (2)(2) = -50$$

Question:

*In Exercises **5–6**, find a matrix that generates the stated weighted inner product on R^2.*

5. $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$ **6.** $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}u_1v_1 + 5u_2v_2$

Solution:

5. $$\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

6. $$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

Angle Between Vectors

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

EXAMPLE 1 | Cosine of the Angle Between Vectors in M_{22}

Let M_{22} have the standard inner product. Find the cosine of the angle between the vectors

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

Solution We showed in Example 6 of the previous section that

$$\langle \mathbf{u}, \mathbf{v}\rangle = 16, \quad \|\mathbf{u}\| = \sqrt{30}, \quad \|\mathbf{v}\| = \sqrt{14}$$

from which it follows that

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{16}{\sqrt{30}\sqrt{14}} \approx 0.78$$

Question:

Find the cosine of the angle between the vectors with respect to the Euclidean inner product.

b. $\mathbf{u} = (-1, 5, 2)$, $\mathbf{v} = (2, 4, -9)$

Solution:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{(-1)(2)+(5)(4)+(2)(-9)}{\sqrt{(-1)^2+5^2+2^2}\sqrt{2^2+4^2+(-9)^2}} = 0$$

Question:

Find the cosine of the angle between the vectors with respect to the Euclidean inner product.

3. $\mathbf{p} = -1 + 5x + 2x^2, \quad \mathbf{q} = 2 + 4x - 9x^2$

Solution:

$$\cos\theta = \frac{\langle \mathbf{p},\mathbf{q}\rangle}{\|\mathbf{p}\|\,\|\mathbf{q}\|} = \frac{(-1)(2)+(5)(4)+(2)(-9)}{\sqrt{(-1)^2+5^2+2^2}\sqrt{2^2+4^2+(-9)^2}} = 0$$

Question:

Find the cosine of the angle between the vectors with respect to the Euclidean inner product.

5. $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

Solution:

$$\cos\theta = \frac{\langle U,V\rangle}{\|U\|\,\|V\|} = \frac{\operatorname{tr}\left(U^T V\right)}{\sqrt{\operatorname{tr}\left(U^T U\right)}\sqrt{\operatorname{tr}\left(V^T V\right)}} = \frac{(2)(3)+(6)(2)+(1)(1)+(-3)(0)}{\sqrt{2^2+6^2+1^2+(-3)^2}\sqrt{3^2+2^2+1^2+0^2}} = \frac{19}{\sqrt{50}\sqrt{14}} = \frac{19}{10\sqrt{7}}$$