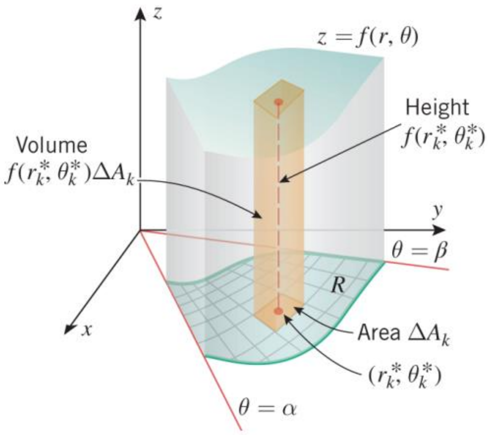
14.3 DOUBLE INTEGRALS IN POLAR COORDINATES

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates. Such integrals are important for two reasons: first, they arise naturally in many applications, and second, many double integrals in rectangular coordinates can be evaluated more easily if they are converted to polar coordinates.

DOUBLE INTEGRALS IN POLAR COORDINATES

Next we will consider the polar version of Problem 14.1.1.

14.3.2 THE VOLUME PROBLEM IN POLAR COORDINATES Given a function $f(r, \theta)$ that is continuous and nonnegative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$ (Figure 14.3.4).

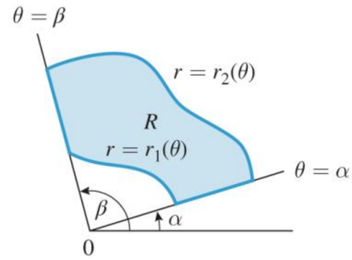


which is called the ***polar double integral*** of $f(r, \theta)$ over R. If $f(r, \theta)$ is continuous and nonnegative on R, then the volume formula (1) can be expressed as

$$V = \iint_R f(r, \theta)\, dA \qquad (4)$$

14.3.3 THEOREM *If R is a simple polar region whose boundaries are the rays $\theta=\alpha$ and $\theta=\beta$ and the curves $r=r_1(\theta)$ and $r=r_2(\theta)$ shown in Figure 14.3.8, and if $f(r,\theta)$ is continuous on R, then*

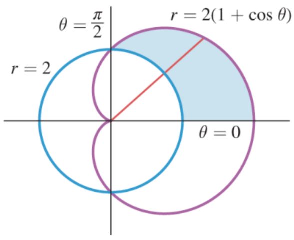
$$\iint_R f(r,\theta)\,dA=\int_\alpha^\beta\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)r\,dr\,d\theta \tag{7}$$



► Example 1 Evaluate

$$\iint_R \sin\theta\,dA$$

where R is the region in the first quadrant that is outside the circle $r=2$ and inside the cardioid $r=2(1+\cos\theta)$.

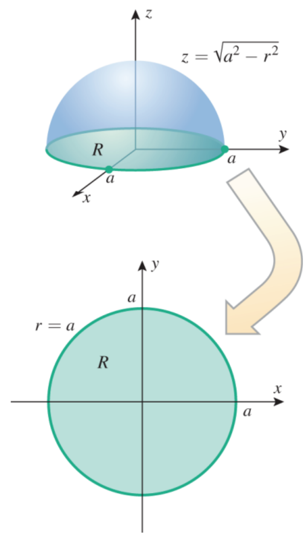


▲ **Figure 14.3.10**

Solution. The region R is sketched in Figure 14.3.10. Following the two steps outlined above we obtain

$$\begin{aligned}\iint_R \sin\theta\,dA&=\int_0^{\pi/2}\int_2^{2(1+\cos\theta)}(\sin\theta)r\,dr\,d\theta\\&=\int_0^{\pi/2}\left[\frac{1}{2}r^2\sin\theta\right]_{r=2}^{2(1+\cos\theta)}d\theta\\&=2\int_0^{\pi/2}[(1+\cos\theta)^2\sin\theta-\sin\theta]\,d\theta\\&=2\left[-\frac{1}{3}(1+\cos\theta)^3+\cos\theta\right]_0^{\pi/2}\\&=2\left[-\frac{1}{3}-\left(-\frac{5}{3}\right)\right]=\frac{8}{3}\ ◄\end{aligned}$$

► Example 2 The sphere of radius a centered at the origin is expressed in rectangular coordinates as $x^2+y^2+z^2=a^2$, and hence its equation in cylindrical coordinates is $r^2+z^2=a^2$. Use this equation and a polar double integral to find the volume of the sphere.



▲ **Figure 14.3.11**

Solution. In cylindrical coordinates the upper hemisphere is given by the equation

$$z=\sqrt{a^2-r^2}$$

so the volume enclosed by the entire sphere is

$$V=2\iint_R\sqrt{a^2-r^2}\,dA$$

where R is the circular region shown in Figure 14.3.11. Thus,

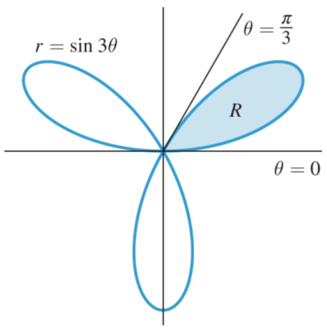
$$\begin{aligned}V=2\iint_R\sqrt{a^2-r^2}\,dA&=\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}(2r)\,dr\,d\theta\\&=\int_0^{2\pi}\left[-\frac{2}{3}(a^2-r^2)^{3/2}\right]_{r=0}^{a}d\theta=\int_0^{2\pi}\frac{2}{3}a^3\,d\theta\\&=\left[\frac{2}{3}a^3\theta\right]_0^{2\pi}=\frac{4}{3}\pi a^3\ ◄\end{aligned}$$

FINDING AREAS USING POLAR DOUBLE INTEGRALS

Recall from Formula (7) of Section 14.2 that the area of a region R in the xy-plane can be expressed as

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA \tag{8}$$

The argument used to derive this result can also be used to show that the formula applies to polar double integrals over regions in polar coordinates.



▲ **Figure 14.3.12**

▶ **Example 3** Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.

Solution. The rose is sketched in Figure 14.3.12. We will use Formula (8) to calculate the area of the petal R in the first quadrant and multiply by 3.

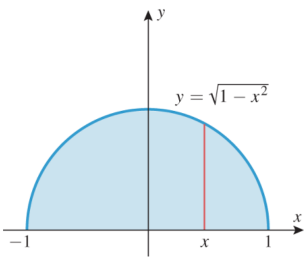
$$A = 3\iint_R dA = 3\int_0^{\pi/3}\int_0^{\sin 3\theta} r\, dr\, d\theta$$
$$= \frac{3}{2}\int_0^{\pi/3} \sin^2 3\theta\, d\theta = \frac{3}{4}\int_0^{\pi/3} (1-\cos 6\theta)\, d\theta$$
$$= \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{1}{4}\pi \;◀$$

CONVERTING DOUBLE INTEGRALS FROM RECTANGULAR TO POLAR COORDINATES

Sometimes a double integral that is difficult to evaluate in rectangular coordinates can be evaluated more easily in polar coordinates by making the substitution $x = r\cos\theta$, $y = r\sin\theta$ and expressing the region of integration in polar form; that is, we rewrite the double integral in rectangular coordinates as

$$\iint_R f(x,y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, dA = \iint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta \tag{9}$$

▶ **Example 4** Use polar coordinates to evaluate $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$.



▲ **Figure 14.3.13**

Solution. In this problem we are starting with an iterated integral in rectangular coordinates rather than a double integral, so before we can make the conversion to polar coordinates we will have to identify the region of integration. Observe that for fixed x the y-integration runs from $y = 0$ to $y = \sqrt{1-x^2}$, which tells us that the lower boundary of the region is the x-axis and the upper boundary is a semicircle of radius 1 centered at the origin. From the x-integration we see that x varies from -1 to 1, so we conclude that the region of integration is as shown in Figure 14.3.13. In polar coordinates, this is the region swept out as r varies between 0 and 1 and θ varies between 0 and π. Thus,

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx = \iint_R (x^2+y^2)^{3/2}\, dA$$
$$= \int_0^{\pi}\int_0^1 (r^3) r\, dr\, d\theta = \int_0^{\pi} \frac{1}{5}\, d\theta = \frac{\pi}{5} \;◀$$

REMARK The reason the conversion to polar coordinates worked so nicely in Example 4 is that the substitution $x = r\cos\theta$, $y = r\sin\theta$ collapsed the sum x^2+y^2 into the single term r^2, thereby simplifying the integrand. Whenever you see an expression involving x^2+y^2 in the integrand, you should consider the possibility of converting to polar coordinates.

1–6 Evaluate the iterated integral.

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\,dr\,d\theta$

2. $\displaystyle\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$

3. $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta} r^2\,dr\,d\theta$

4. $\displaystyle\int_0^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta$

5. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\,dr\,d\theta$

6. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} r^3\,dr\,d\theta$

7–10 Use a double integral in polar coordinates to find the area of the region described.

7. The region enclosed by the cardioid $r = 1 - \cos\theta$.
8. The region enclosed by the rose $r = \sin 2\theta$.
9. The region in the first quadrant bounded by $r = 1$ and $r = \sin 2\theta$, with $\pi/4 \le \theta \le \pi/2$.
10. The region inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$.

23–26 Use polar coordinates to evaluate the double integral.

23. $\displaystyle\iint_R \sin(x^2+y^2)\,dA$, where R is the region enclosed by the circle $x^2+y^2=9$.

24. $\displaystyle\iint_R \sqrt{9-x^2-y^2}\,dA$, where R is the region in the first quadrant within the circle $x^2+y^2=9$.

25. $\displaystyle\iint_R \frac{1}{1+x^2+y^2}\,dA$, where R is the sector in the first quadrant bounded by $y=0$, $y=x$, and $x^2+y^2=4$.

26. $\displaystyle\iint_R 2y\,dA$, where R is the region in the first quadrant bounded above by the circle $(x-1)^2+y^2=1$ and below by the line $y=x$.

27–34 Evaluate the iterated integral by converting to polar coordinates.

27. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$

28. $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\, dx\, dy$

29. $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$

30. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2)\, dx\, dy$

31. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\, dx}{(1 + x^2 + y^2)^{3/2}} \quad (a > 0)$

32. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \sqrt{x^2 + y^2}\, dx\, dy$

33. $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1 + x^2 + y^2}}\, dx\, dy$

34. $\displaystyle\int_{-4}^0 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 3x\, dy\, dx$