Limit and Continuity

Exercise-13.2

Limit

13.2.1 DEFINITION Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \tag{3}$$

General Limit:

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

▶ Example 2

$$\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] = \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9$$

$$= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3 \left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9$$

$$= 5(1)^3(4)^2 - 9 = 71 \blacktriangleleft$$

EXAMPLE 2 If $f(x, y) = \dfrac{xy}{x^2 + y^2}$, does $\lim_{(x, y)\to(0, 0)} f(x, y)$ exist?

Indeterminate form (0/0)

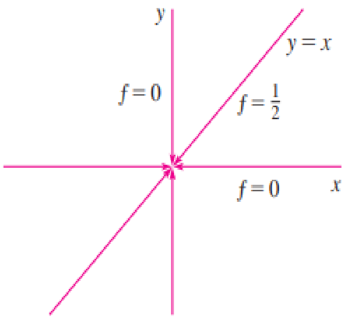

FIGURE 5

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach (0, 0) along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $\quad f(x, y) \to \frac{1}{2} \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } y = x$

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist.

13. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

Indeterminate form (0/0)

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^2-y^2)}{(x^2+y^2)}$$

$$\lim_{(x,y)\to(0,0)} x^2-y^2 = 0$$

Thus, $\displaystyle \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^2-y^2)}{(x^2+y^2)} = 0$

LIMITS AT DISCONTINUITIES

Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x, y)\to(0, 0)$ along any smooth curve (Figure 13.2.9). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0\cdot\infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.

▶ **Example 7** Find $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution. Let (r, θ) be polar coordinates of the point (x, y) with $r \geq 0$. Then we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2+y^2$$

Moreover, since $r \geq 0$ we have $r = \sqrt{x^2+y^2}$, so that $r\to 0^+$ if and only if $(x, y)\to(0, 0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2\ln r^2 \\
&= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} && \boxed{\text{This converts the limit to an indeterminate form of type } \infty/\infty.} \\
&= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} && \boxed{\text{L'Hôpital's rule}} \\
&= \lim_{r\to 0^+} (-r^2) = 0 \;◀
\end{aligned}$$

Continuity

13.2.3 DEFINITION A function $f(x, y)$ is said to be ***continuous at*** $(\boldsymbol{x_0}, \boldsymbol{y_0})$ if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In general,

If *f* is continuous at every point in the xy-plane (domain), then we say that *f* is continuous everywhere.

13.2.4 THEOREM

(*a*) *If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).*

(*b*) *If $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x, y) = g(h(x, y))$ is continuous at (x_0, y_0).*

(*c*) *If $f(x, y)$ is continuous at (x_0, y_0), and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0.*

▶ **Example 4** Use Theorem 13.2.4 to show that the functions $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution:

The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number.
Therefore, by part (a) of theorem
$$f(x,y) = 3x^2y^5 = g(x)h(y)$$
is continuous everywhere.

Since $h(x,y) = 3x^2y^5$ is continuous everywhere and $g(u) = \sin u$ is continuous at every real number 'u', therefore by part (b) of theorem the composition
$$f(x,y) = \sin(3x^2y^5)$$
is continuous everywhere.

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

▶ Example 5 Evaluate $\lim_{(x,y)\to(-1,2)} \dfrac{xy}{x^2+y^2}$.

Solution. Since $f(x, y) = xy/(x^2 + y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+(2)^2} = -\frac{2}{5} \blacktriangleleft$$

▶ Example 6 Since the function

$$f(x, y) = \frac{x^3y^2}{1-xy}$$

is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, $f(x, y)$ is continuous everywhere except on the hyperbola $xy = 1$. ◀

EXAMPLE 6 Where is the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

SOLUTION The function f is discontinuous at $(0, 0)$ because it is not defined there. Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. ■

EXAMPLE 7 Let

$$g(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Here g is defined at $(0, 0)$ but g is still discontinuous there because $\lim_{(x, y)\to(0, 0)} g(x, y)$ does not exist (see Example 1). ■

EXAMPLE 1 Show that $\lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x-axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Since *f* has two different limits along two different lines, the given limit does not exist.