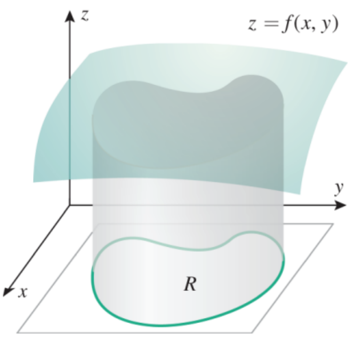
14.1 DOUBLE INTEGRALS

The notion of a definite integral can be extended to functions of two or more variables. In this section we will discuss the double integral, which is the extension to functions of two variables.

VOLUME

Recall that the definite integral of a function of one variable

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{1}$$



we use the "limit as $n\to+\infty$" to encapsulate the process by which we increase the number of subintervals of $[a, b]$ in such a way that the lengths of the subintervals approach zero.] Integrals of functions of two variables arise from the problem of finding volumes under surfaces.

14.1.1 THE VOLUME PROBLEM Given a function f of two variables that is continuous and nonnegative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R (Figure 14.1.1).

Later, we will place more restrictions on the region R, but for now we will just assume that

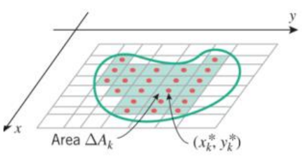
14.1.2 DEFINITION (*Volume Under a Surface*) If f is a function of two variables that is continuous and nonnegative on a region R in the xy-plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R is defined by

$$V = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{2}$$

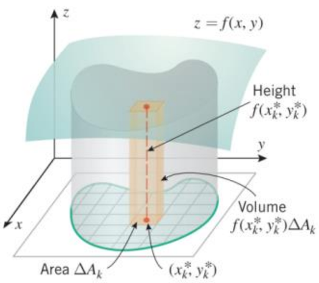
Here, $n\to+\infty$ indicates the process of increasing the number of subrectangles of the rectangle enclosing R in such a way that both the lengths and the widths of the subrectangles approach zero.

solid will be

$$V = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$



▲ **Figure 14.1.2**



▲ **Figure 14.1.3**

This suggests the following definition.

DEFINITION OF A DOUBLE INTEGRAL

As in Definition 14.1.2, the notation $n \to +\infty$ in (3) encapsulates a process in which the enclosing rectangle for R is repeatedly subdivided in such a way that both the lengths and the widths of the subrectangles approach zero. Thus, we have extended the notion conveyed by Formula (1) where the definite integral of a one-variable function is expressed as a limit of Riemann sums. By extension, the sums in (3) are also called ***Riemann sums***, and the limit of the Riemann sums is denoted by

$$\iint_R f(x, y)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k \tag{4}$$

which is called the ***double integral*** of $f(x, y)$ over R.

If f is continuous and nonnegative on the region R, then the volume formula in (2) can be expressed as

$$V = \iint_R f(x, y)\, dA \tag{5}$$

EVALUATING DOUBLE INTEGRALS

Except in the simplest cases, it is impractical to obtain the value of a double integral from the limit in (4). However, we will now show how to evaluate double integrals by calculating two successive single integrals. For the rest of this section we will limit our discussion to the case where R is a rectangle; in the next section we will consider double integrals over more complicated regions.

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, ***partial integration***. The symbols

$$\int_a^b f(x, y)\, dx \quad \text{and} \quad \int_c^d f(x, y)\, dy$$

denote ***partial definite integrals***; the first integral, called the ***partial definite integral with respect to x***, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the ***partial definite integral with respect to y***, is evaluated by holding x fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function of x.

▶ Example 1

$$\int_0^1 xy^2\, dx = y^2 \int_0^1 x\, dx = \frac{y^2x^2}{2}\bigg]_{x=0}^{1} = \frac{y^2}{2}$$

$$\int_0^1 xy^2\, dy = x \int_0^1 y^2\, dy = \frac{xy^3}{3}\bigg]_{y=0}^{1} = \frac{x}{3} \blacktriangleleft$$

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y; similarly, a partial definite integral with respect to y can be integrated with respect to x. This two-stage integration process is called ***iterated*** (or ***repeated***) ***integration***. We introduce the following notation:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy \tag{6}$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx \tag{7}$$

These integrals are called ***iterated integrals***.

14.1.3 THEOREM (*Fubini's Theorem*) *Let R be the rectangle defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d$$

If $f(x,y)$ is continuous on this rectangle, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Theorem 14.1.3 allows us to evaluate a double integral over a rectangle by converting it to an iterated integral. This can be done in two ways, both of which produce the value of the double integral.

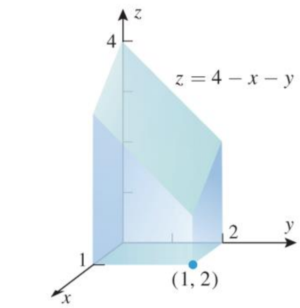


Figure 14.1.6

Example 3 Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0,1] \times [0,2]$ (Figure 14.1.6).

Solution. The volume is the double integral of $z = 4 - x - y$ over R. Using Theorem 14.1.3, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y)\,dx\,dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\,dy\,dx \tag{8}$$

Using the first of these, we obtain

$$V = \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$
$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy = \int_0^2 \left(\frac{7}{2} - y\right) dy$$
$$= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5$$

You can check this result by evaluating the second integral in (8). ◀

Theorem 14.1.3 guarantees that the double integral in Example 4 can also be evaluated by integrating first with respect to y and then with respect to x. Verify this.

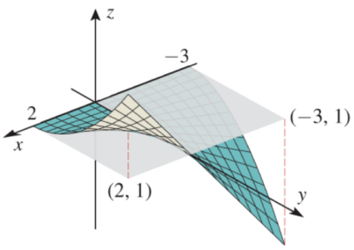
▶ Example 4 Evaluate the double integral

$$\iint_R y^2 x\, dA$$

over the rectangle $R = \{(x, y) : -3 \le x \le 2, 0 \le y \le 1\}$.

Solution. In view of Theorem 14.1.3, the value of the double integral can be obtained by evaluating one of two possible iterated double integrals. We choose to integrate first with respect to x and then with respect to y.

$$\iint_R y^2 x\, dA = \int_0^1 \int_{-3}^2 y^2 x\, dx\, dy = \int_0^1 \left[\frac{1}{2}y^2 x^2\right]_{x=-3}^{2} dy$$
$$= \int_0^1 \left(-\frac{5}{2}y^2\right) dy = -\frac{5}{6}y^3\Big]_0^1 = -\frac{5}{6} \quad ◀$$



$z = y^2 x$ on $[-3, 2] \times [0, 1]$

▲ **Figure 14.1.7**

The integral in Example 4 can be interpreted as the net signed volume between the rectangle $[-3, 2] \times [0, 1]$ and the surface $z = y^2 x$. That is, it is the volume below $z = y^2 x$ and above $[0, 2] \times [0, 1]$ minus the volume above $z = y^2 x$ and below $[-3, 0] \times [0, 1]$ (Figure 14.1.7).

EXERCISE SET 14.1 C CAS

1–12 Evaluate the iterated integrals.

1. $\int_0^1 \int_0^2 (x+3)\,dy\,dx$

2. $\int_1^3 \int_{-1}^1 (2x-4y)\,dy\,dx$

3. $\int_2^4 \int_0^1 x^2 y\,dx\,dy$

4. $\int_{-2}^0 \int_{-1}^2 (x^2+y^2)\,dx\,dy$

5. $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\,dy\,dx$

6. $\int_0^2 \int_0^1 y \sin x\,dy\,dx$

7. $\int_{-1}^0 \int_2^5 dx\,dy$

8. $\int_4^6 \int_{-3}^7 dy\,dx$

9. $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\,dy\,dx$

10. $\int_{\pi/2}^{\pi} \int_1^2 x \cos xy\,dy\,dx$

11. $\int_0^{\ln 2} \int_0^1 xye^{y^2 x}\,dy\,dx$

12. $\int_3^4 \int_1^2 \frac{1}{(x+y)^2}\,dy\,dx$

13–16 Evaluate the double integral over the rectangular region R.

13. $\iint_R 4xy^3\,dA;\; R = \{(x,y) : -1 \le x \le 1, -2 \le y \le 2\}$

14. $\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\,dA;$

$R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$

15. $\iint_R x\sqrt{1-x^2}\,dA;\; R = \{(x,y) : 0 \le x \le 1, 2 \le y \le 3\}$

16. $\iint_R (x \sin y - y \sin x)\,dA;$

$R = \{(x,y) : 0 \le x \le \pi/2, 0 \le y \le \pi/3\}$