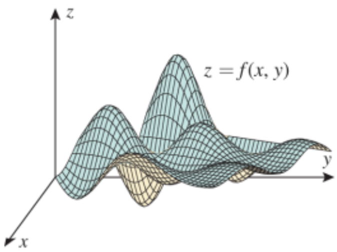
13.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Earlier in this text we learned how to find maximum and minimum values of a function of one variable. In this section we will develop similar techniques for functions of two variables.

EXTREMA

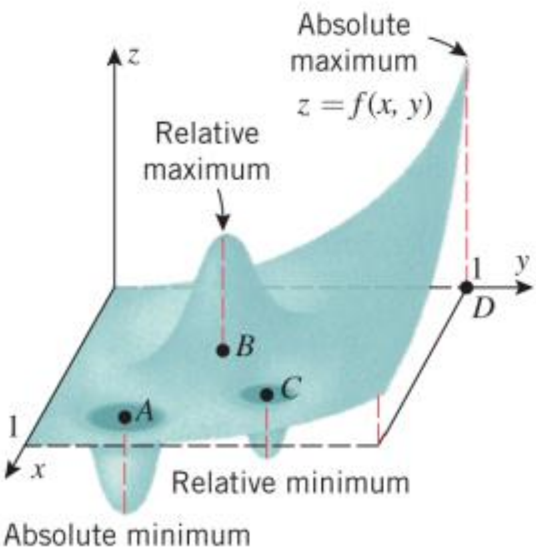


If we imagine the graph of a function f of two variables to be a mountain range (Figure 13.8.1), then the mountaintops, which are the high points in their immediate vicinity, are called *relative maxima* of f, and the valley bottoms, which are the low points in their immediate vicinity, are called *relative minima* of f.

Just as a geologist might be interested in finding the highest mountain and deepest valley in an entire mountain range, so a mathematician might be interested in finding the largest and smallest values of $f(x, y)$ over the *entire* domain of f. These are called the *absolute maximum* and *absolute minimum values* of f. The following definitions make these informal ideas precise.

13.8.1 DEFINITION A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

13.8.2 DEFINITION A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.



▲ **Figure 13.8.2**

THE EXTREME-VALUE THEOREM

For functions of one variable that are continuous on a closed interval, the Extreme-Value Theorem (Theorem 3.4.2) answered the existence question for absolute extrema. The following theorem, which we state without proof, is the corresponding result for functions of two variables.

> **13.8.3 THEOREM (Extreme-Value Theorem)** *If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.*

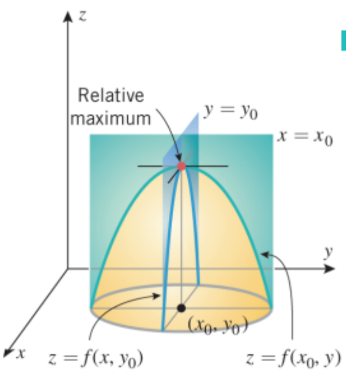


Figure 13.8.4

FINDING RELATIVE EXTREMA

Recall that if a function g of one variable has a relative extremum at a point x_0 where g is differentiable, then $g'(x_0) = 0$. To obtain the analog of this result for functions of two variables, suppose that $f(x, y)$ has a relative maximum at a point (x_0, y_0) and that the partial derivatives of f exist at (x_0, y_0). It seems plausible geometrically that the traces of the surface $z = f(x, y)$ on the planes $x = x_0$ and $y = y_0$ have horizontal tangent lines at (x_0, y_0) (Figure 13.8.4), so

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

The same conclusion holds if f has a relative minimum at (x_0, y_0), all of which suggests the following result, which we state without formal proof.

> **13.8.4 THEOREM** *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*
>
> $$f_x(x_0, y_0) = 0 \quad \textit{and} \quad f_y(x_0, y_0) = 0$$

Recall that the *critical points* of a function f of one variable are those values of x in the domain of f at which $f'(x) = 0$ or f is not differentiable. The following definition is the analog for functions of two variables.

> Explain why
>
> $$D_{\mathbf{u}} f(x_0, y_0) = 0$$
>
> for all $\mathbf{u}$ if (x_0, y_0) is a critical point of f and f is differentiable at (x_0, y_0).

> **13.8.5 DEFINITION** A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

It follows from this definition and Theorem 13.8.4 that relative extrema occur at critical points, just as for a function of one variable. However, recall that for a function of one variable a relative extremum need not occur at *every* critical point. For example, the function might have an inflection point with a horizontal tangent line at the critical point (see Figure 3.2.6). Similarly, a function of two variables need not have a relative extremum at every critical point. For example, consider the function

$$f(x, y) = y^2 - x^2$$

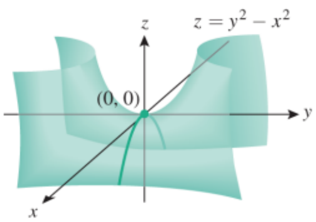
This function, whose graph is the hyperbolic paraboloid shown in Figure 13.8.5, has a critical point at $(0, 0)$, since

$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = 2y$$

from which it follows that

$$f_x(0, 0) = 0 \quad \text{and} \quad f_y(0, 0) = 0$$

However, the function f has neither a relative maximum nor a relative minimum at $(0, 0)$. For obvious reasons, the point $(0, 0)$ is called a *saddle point* of f. In general, we will say that a surface $z = f(x, y)$ has a ***saddle point*** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).



The function $f(x, y) = y^2 - x^2$ has neither a relative maximum nor a relative minimum at the critical point $(0, 0)$.

Figure 13.8.5

THE SECOND PARTIALS TEST

For functions of one variable the second derivative test (Theorem 3.2.4) was used to determine the behavior of a function at a critical point. The following theorem, which is usually proved in advanced calculus, is the analog of that theorem for functions of two variables.

> With the notation of Theorem 13.8.6, show that if $D > 0$, then $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ have the same sign. Thus, we can replace $f_{xx}(x_0, y_0)$ by $f_{yy}(x_0, y_0)$ in parts (*a*) and (*b*) of the theorem.

13.8.6 THEOREM (*The Second Partials Test*) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(*a*) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*

(*b*) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*

(*c*) *If $D < 0$, then f has a saddle point at (x_0, y_0).*

(*d*) *If $D = 0$, then no conclusion can be drawn.*

Example 3 Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$\begin{aligned} 6x - 2y &= 0 \\ -2x + 2y - 8 &= 0 \end{aligned}$$

Solving these for x and y yields $x = 2$, $y = 6$ (verify), so $(2, 6)$ is the only critical point. To apply Theorem 13.8.6 we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$

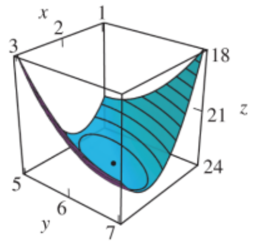
At the point $(2, 6)$ we have

$$D = f_{xx}(2, 6)f_{yy}(2, 6) - f_{xy}^2(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at $(2, 6)$ by part (a) of the second partials test. Figure 13.8.7 shows a graph of f in the vicinity of the relative minimum. ◄



$f(x, y) = 3x^2 - 2xy + y^2 - 8y$

▲ **Figure 13.8.7**

Example 4 Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

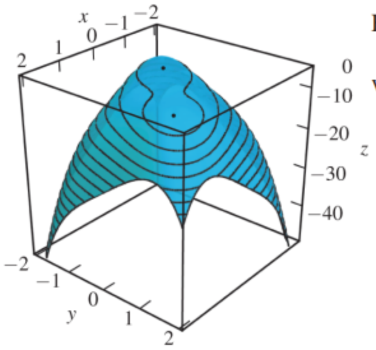
Solution. Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \tag{1}$$

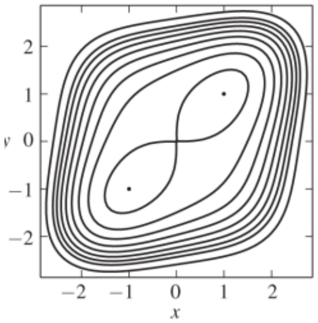
the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0, x = 1, x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0, y = 1, y = -1$. Thus, the critical points of f are $(0, 0), (1, 1)$, and $(-1, -1)$.



$f(x, y) = 4xy - x^4 - y^4$



From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128

At the points (1, 1) and (−1, −1), we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At (0, 0) there is a saddle point since $D < 0$. The surface and a contour plot are shown in Figure 13.8.8. ◀

The following theorem, which is the analog for functions of two variables of Theorem 3.4.3, will lead to an important method for finding absolute extrema.

13.8.7 THEOREM *If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.*

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

If $f(x, y)$ is continuous on a closed and bounded set R, then the Extreme-Value Theorem (Theorem 13.8.3) guarantees the existence of an absolute maximum and an absolute minimum of f on R. These absolute extrema can occur either on the boundary of R or in the interior of R, but if an absolute extremum occurs in the interior, then it occurs at a critical point by Theorem 13.8.7. Thus, we are led to the following procedure for finding absolute extrema:

How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R

Step 1. Find the critical points of f that lie in the interior of R.

Step 2. Find all boundary points at which the absolute extrema can occur.

Step 3. Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

▶ Example 5 Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7 \tag{3}$$

on the closed triangular region R with vertices $(0, 0)$, $(3, 0)$, and $(0, 5)$.

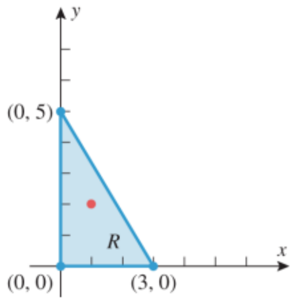


Figure 13.8.9

Solution. The region R is shown in Figure 13.8.9. We have

$$\frac{\partial f}{\partial x} = 3y - 6 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x - 3$$

so all critical points occur where

$$3y - 6 = 0 \quad \text{and} \quad 3x - 3 = 0$$

Solving these equations yields $x = 1$ and $y = 2$, so $(1, 2)$ is the only critical point. As shown in Figure 13.8.9, this critical point is in the interior of R.

Next we want to determine the locations of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we will treat separately:

The line segment between $(0, 0)$ *and* $(3, 0)$: On this line segment we have $y = 0$, so (3) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = -6x + 7, \quad 0 \le x \le 3$$

This function has no critical points because $u'(x) = -6$ is nonzero for all x. Thus the extreme values of $u(x)$ occur at the endpoints $x = 0$ and $x = 3$, which correspond to the points $(0, 0)$ and $(3, 0)$ of R.

The line segment between $(0, 0)$ *and* $(0, 5)$: On this line segment we have $x = 0$, so (3) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y + 7, \quad 0 \le y \le 5$$

This function has no critical points because $v'(y) = -3$ is nonzero for all y. Thus, the extreme values of $v(y)$ occur at the endpoints $y = 0$ and $y = 5$, which correspond to the points $(0, 0)$ and $(0, 5)$ of R.

The line segment between $(3, 0)$ *and* $(0, 5)$: In the xy-plane, an equation for this line segment is

$$y = -\tfrac{5}{3}x + 5, \quad 0 \le x \le 3 \tag{4}$$

so (3) simplifies to a function of the single variable x,

Since $w'(x) = -10x + 14$, the equation $w'(x) = 0$ yields $x = \frac{7}{5}$ as the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = \frac{7}{5}$ or at the endpoints $x = 0$ and $x = 3$. The endpoints correspond to the points $(0, 5)$ and $(3, 0)$ of R, and from (4) the critical point corresponds to $\left(\frac{7}{5}, \frac{8}{3}\right)$.

Finally, Table 13.8.1 lists the values of $f(x, y)$ at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is $f(0, 0) = 7$ and the absolute minimum value is $f(3, 0) = -11$. ◀

Table 13.8.1

(x, y)	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	7	−11	−8	$\frac{9}{5}$	1

EXERCISE SET 13.8 Graphing Utility C CAS

1–2 Locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus.

1. (a) $f(x, y) = (x - 2)^2 + (y + 1)^2$
(b) $f(x, y) = 1 - x^2 - y^2$
(c) $f(x, y) = x + 2y - 5$

2. (a) $f(x, y) = 1 - (x + 1)^2 - (y - 5)^2$
(b) $f(x, y) = e^{xy}$
(c) $f(x, y) = x^2 - y^2$

3–4 Complete the squares and locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus.

3. $f(x, y) = 13 - 6x + x^2 + 4y + y^2$

4. $f(x, y) = 1 - 2x - x^2 + 4y - 2y^2$

9–20 Locate all relative maxima, relative minima, and saddle points, if any.

9. $f(x, y) = y^2 + xy + 3y + 2x + 3$

10. $f(x, y) = x^2 + xy - 2y - 2x + 1$

11. $f(x, y) = x^2 + xy + y^2 - 3x$

12. $f(x, y) = xy - x^3 - y^2$

13. $f(x, y) = x^2 + y^2 + \dfrac{2}{xy}$

14. $f(x, y) = xe^y$

15. $f(x, y) = x^2 + y - e^y$

16. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$

17. $f(x, y) = e^x \sin y$

18. $f(x, y) = y \sin x$

19. $f(x, y) = e^{-(x^2+y^2+2x)}$

20. $f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{b^3}{y} \quad (a \neq 0, b \neq 0)$

31–36 Find the absolute extrema of the given function on the indicated closed and bounded set R.

31. $f(x, y) = xy - x - 3y$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(5, 0)$.

32. $f(x, y) = xy - 2x$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(4, 0)$.

33. $f(x, y) = x^2 - 3y^2 - 2x + 6y$; R is the region bounded by the square with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$, and $(2, 0)$.

34. $f(x, y) = xe^y - x^2 - e^y$; R is the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 1)$, and $(2, 0)$.

35. $f(x, y) = x^2 + 2y^2 - x$; R is the disk $x^2 + y^2 \leq 4$.

36. $f(x, y) = xy^2$; R is the region that satisfies the inequalities $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 1$.

37. Find three positive numbers whose sum is 48 and such that their product is as large as possible.

38. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.

39. Find all points on the portion of the plane $x + y + z = 5$ in the first octant at which $f(x, y, z) = xy^2z^2$ has a maximum value.

40. Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.