14.2 DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

In this section we will show how to evaluate double integrals over regions other than rectangles.

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

Later in this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals of the following types:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx \tag{1}$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy \tag{2}$$

▶ **Example 1** Evaluate

$$\text{(a)}\ \int_0^1 \int_{-x}^{x^2} y^2 x\,dy\,dx \qquad \text{(b)}\ \int_0^{\pi/3} \int_0^{\cos y} x \sin y\,dx\,dy$$

Solution (a).

$$\int_0^1 \int_{-x}^{x^2} y^2 x\,dy\,dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x\,dy\right] dx = \int_0^1 \left.\frac{y^3 x}{3}\right]_{y=-x}^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3}\right] dx = \left.\left(\frac{x^8}{24} + \frac{x^5}{15}\right)\right]_0^1 = \frac{13}{120}$$

Solution (b).

$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y\,dx\,dy = \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y\,dx\right] dy = \int_0^{\pi/3} \left.\frac{x^2}{2} \sin y\right]_{x=0}^{\cos y} dy$$

$$= \int_0^{\pi/3} \left[\frac{1}{2}\cos^2 y \sin y\right] dy = -\frac{1}{6}\cos^3 y\Big]_0^{\pi/3} = \frac{7}{48} \blacktriangleleft$$

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Plane regions can be extremely complex, and the theory of double integrals over very general regions is a topic for advanced courses in mathematics. We will limit our study of double integrals to two basic types of regions, which we will call *type I* and *type II*; they are defined as follows.

$\overset{x}{\rightarrow}$

14.2.1 DEFINITION

(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$ (Figure 14.2.1*a*).

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$ (Figure 14.2.1*b*).

The following theorem will enable us to evaluate double integrals over type I and type II regions using iterated integrals.

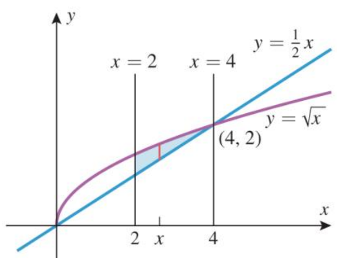
14.2.2 THEOREM

(*a*) *If R is a type I region on which f(x, y) is continuous, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{3}$$

(*b*) *If R is a type II region on which f(x, y) is continuous, then*

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{4}$$



▲ Figure 14.2.6

► Example 3 Evaluate

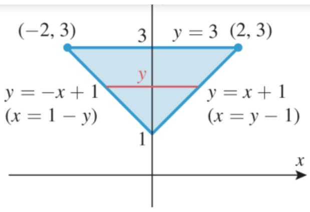
$$\iint_R xy\, dA$$

over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

Solution. We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure 14.2.6. This line meets the region R at the lower boundary $y = \frac{1}{2}x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x = 2$ and $x = 4$. Thus,

$$\iint_R xy\, dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6} \blacktriangleleft$$



▲ **Figure 14.2.8**

▶ **Example 4** Evaluate

$$\iint_R (2x - y^2)\, dA$$

over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

Solution. We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in Figure 14.2.8. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration. Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\begin{aligned}\iint_R (2x - y^2)\, dA &= \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy \\ &= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy \\ &= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3} \blacktriangleleft\end{aligned}$$

To integrate over a type II region, the left- and right-hand boundaries must be expressed in the form $x = h_1(y)$ and $x = h_2(y)$. This is why we rewrote the boundary equations

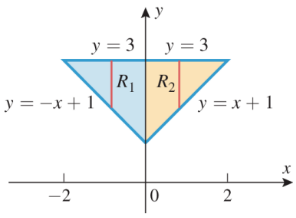
$$y = -x + 1 \quad \text{and} \quad y = x + 1$$

as

$$x = 1 - y \quad \text{and} \quad x = y - 1$$

in Example 4.

In Example 4 we could have treated R as a type I region, but with an added complication. Viewed as a type I region, the upper boundary of R is the line $y = 3$ (Figure 14.2.9) and the lower boundary consists of two parts, the line $y = -x + 1$ to the left of the y-axis and the line $y = x + 1$ to the right of the y-axis. To carry out the integration it is necessary to decompose the region R into two parts, R_1 and R_2, as shown in Figure 14.2.9, and write



▲ **Figure 14.2.9**

In Example 4 we could have treated R as a type I region, but with an added complication. Viewed as a type I region, the upper boundary of R is the line $y = 3$ (Figure 14.2.9) and the lower boundary consists of two parts, the line $y = -x + 1$ to the left of the y-axis and the line $y = x + 1$ to the right of the y-axis. To carry out the integration it is necessary to decompose the region R into two parts, R_1 and R_2, as shown in Figure 14.2.9, and write

$$\begin{aligned}\iint_R (2x - y^2)\, dA &= \iint_{R_1} (2x - y^2)\, dA + \iint_{R_2} (2x - y^2)\, dA \\ &= \int_{-2}^{0} \int_{-x+1}^{3} (2x - y^2)\, dy\, dx + \int_0^2 \int_{x+1}^{3} (2x - y^2)\, dy\, dx\end{aligned}$$

This will yield the same result that was obtained in Example 4. (Verify.)

▶ Example 5 Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution. The tetrahedron in question is bounded above by the plane

$$z = 4 - 4x - 2y \tag{5}$$

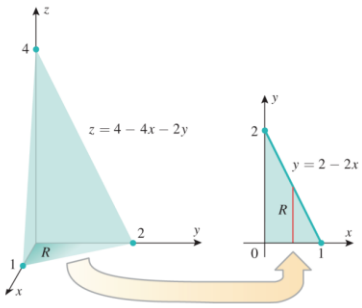
and below by the triangular region R shown in Figure 14.2.10. Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y)\, dA$$

The region R is bounded by the x-axis, the y-axis, and the line $y = 2 - 2x$ [set $z = 0$ in (5)], so that treating R as a type I region yields

$$V = \iint_R (4 - 4x - 2y)\, dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[4y - 4xy - y^2\right]_{y=0}^{2-2x} dx = \int_0^1 (4 - 8x + 4x^2)\, dx = \frac{4}{3} \quad ◀$$



▶ Figure 14.2.10

▶ Example 6 Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution. The solid shown in Figure 14.2.11 is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

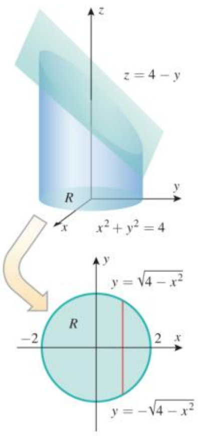
$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi$$

See Formula (3) of Section 7.4. ◀

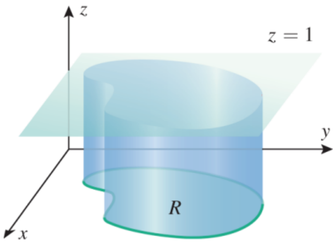


▲ Figure 14.2.11

AREA CALCULATED AS A DOUBLE INTEGRAL

Although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. To see why this is so, recall that a *right cylinder* is a solid that is generated when a plane region is translated along a line that is perpendicular to the region. In Formula (2) of Section 5.2 we stated that the volume V of a right cylinder with cross-sectional area A and height h is

$$V = A \cdot h \tag{6}$$



Cylinder with base R and height 1

Figure 14.2.13

Now suppose that we are interested in finding the area A of a region R in the xy-plane. If we translate the region R upward 1 unit, then the resulting solid will be a right cylinder that has cross-sectional area A, base R, and the plane $z = 1$ as its top (Figure 14.2.13). Thus, it follows from (6) that

$$\iint_R 1\, dA = (\text{area of } R) \cdot 1$$

which we can rewrite as

$$\boxed{\text{area of } R = \iint_R 1\, dA = \iint_R dA} \tag{7}$$

Example 8 Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution. The region R may be treated equally well as type I (Figure 14.2.14*a*) or type II (Figure 14.2.14*b*). Treating R as type I yields

$$\begin{aligned}\text{area of } R &= \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\, dx = \int_0^4 [y]_{y=x^2/2}^{2x}\, dx \\ &= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}\end{aligned}$$

Treating R as type II yields

$$\begin{aligned}\text{area of } R &= \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\, dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\, dy \\ &= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}\end{aligned}$$

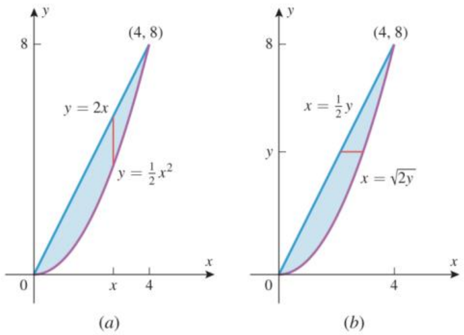


Figure 14.2.14

EXERCISE SET 14.2   Graphing Utility   C CAS

1–8 Evaluate the iterated integral.

1. $\displaystyle\int_0^1 \int_{x^2}^{x} xy^2\, dy\, dx$ **2.** $\displaystyle\int_1^{3/2} \int_y^{3-y} y\, dx\, dy$

3. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} y\, dx\, dy$ **4.** $\displaystyle\int_{1/4}^1 \int_{x^2}^{x} \sqrt{\frac{x}{y}}\, dy\, dx$

5. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^3} \sin\frac{y}{x}\, dy\, dx$ **6.** $\displaystyle\int_{-1}^1 \int_{-x^2}^{x^2} (x^2 - y)\, dy\, dx$

7. $\displaystyle\int_0^1 \int_0^{x} y\sqrt{x^2 - y^2}\, dy\, dx$ **8.** $\displaystyle\int_1^2 \int_0^{y^2} e^{x/y^2}\, dx\, dy$

FOCUS ON CONCEPTS

9. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

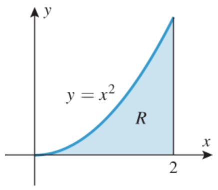
(a) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dy\, dx$

(b) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dx\, dy$

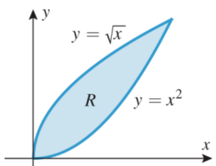
10. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dy\, dx$

(b) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dx\, dy$



▲ Figure Ex-9



▲ Figure Ex-10

11. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\displaystyle\iint_R f(x, y)\, dA = \int_1^2 \int_{\square}^{\square} f(x, y)\, dy\, dx$

$\displaystyle\qquad + \int_2^4 \int_{\square}^{\square} f(x, y)\, dy\, dx$

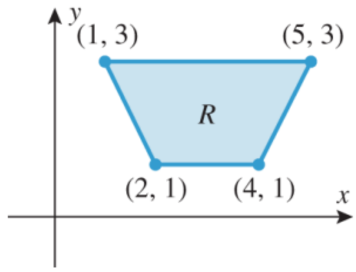
$\displaystyle\qquad + \int_4^5 \int_{\square}^{\square} f(x, y)\, dy\, dx$

(b) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dx\, dy$

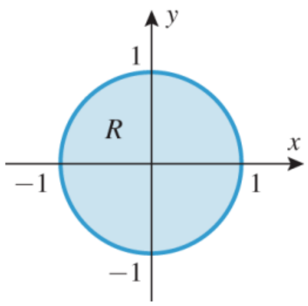
12. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\iint\limits_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dy\, dx$

(b) $\iint\limits_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dx\, dy$



▲ **Figure Ex-11**



▲ **Figure Ex-12**

15–18 Evaluate the double integral in two ways using iterated integrals: (a) viewing R as a type I region, and (b) viewing R as a type II region.

15. $\iint\limits_R x^2\, dA$; R is the region bounded by $y = 16/x$, $y = x$, and $x = 8$.

16. $\iint\limits_R xy^2\, dA$; R is the region enclosed by $y = 1$, $y = 2$, $x = 0$, and $y = x$.

17. $\iint\limits_R (3x - 2y)\, dA$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

18. $\iint\limits_R y\, dA$; R is the region in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

19–24 Evaluate the double integral.

19. $\iint_R x(1+y^2)^{-1/2}\,dA$; R is the region in the first quadrant enclosed by $y = x^2$, $y = 4$, and $x = 0$.

20. $\iint_R x\cos y\,dA$; R is the triangular region bounded by the lines $y = x$, $y = 0$, and $x = \pi$.

21. $\iint_R xy\,dA$; R is the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$.

22. $\iint_R x\,dA$; R is the region enclosed by $y = \sin^{-1} x$, $x = 1/\sqrt{2}$, and $y = 0$.

23. $\iint_R (x-1)\,dA$; R is the region in the first quadrant enclosed between $y = x$ and $y = x^3$.

24. $\iint_R x^2\,dA$; R is the region in the first quadrant enclosed by $xy = 1$, $y = x$, and $y = 2x$.

25. Evaluate $\iint_R \sin(y^3)\,dA$, where R is the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$. [*Hint:* Choose the order of integration carefully.]

47–52 Express the integral as an equivalent integral with the order of integration reversed.

47. $\int_0^2 \int_0^{\sqrt{x}} f(x, y)\,dy\,dx$

48. $\int_0^4 \int_{2y}^8 f(x, y)\,dx\,dy$

49. $\int_0^2 \int_1^{e^y} f(x, y)\,dx\,dy$

50. $\int_1^e \int_0^{\ln x} f(x, y)\,dy\,dx$

51. $\int_0^1 \int_{\sin^{-1} y}^{\pi/2} f(x, y)\,dx\,dy$

52. $\int_0^1 \int_{y^2}^{\sqrt{y}} f(x, y)\,dx\,dy$

53–56 Evaluate the integral by first reversing the order of integration.

53. $\int_0^1 \int_{4x}^4 e^{-y^2}\,dy\,dx$

54. $\int_0^2 \int_{y/2}^1 \cos(x^2)\,dx\,dy$

55. $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$

56. $\int_1^3 \int_0^{\ln x} x\,dy\,dx$