

13.3 PARTIAL DERIVATIVES

In this section we will develop the mathematical tools for studying rates of change that involve two or more independent variables.

The limits in (1) and (2) show the relationship between partial derivatives and derivatives of functions of one variable. In practice, our usual method for computing partial derivatives is to hold one variable fixed and then differentiate the resulting function using the derivative rules for functions of one variable.

13.3.1 DEFINITION If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f , then the **partial derivative of f with respect to x** at (x_0, y_0) [also called the **partial derivative of z with respect to x** at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)] \Big|_{x=x_0} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \quad (1)$$

Similarly, the **partial derivative of f with respect to y** at (x_0, y_0) [also called the **partial derivative of z with respect to y** at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)] \Big|_{y=y_0} = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \quad (2)$$

► **Example 1** Find $f_x(1, 3)$ and $f_y(1, 3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

Solution. Since

$$f_x(x, 3) = \frac{d}{dx}[f(x, 3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4$$

we have $f_x(1, 3) = 54 + 4 = 58$. Also, since

$$f_y(1, y) = \frac{d}{dy}[f(1, y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2$$

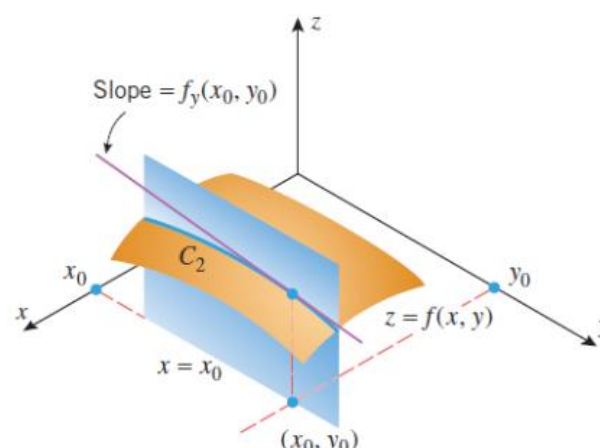
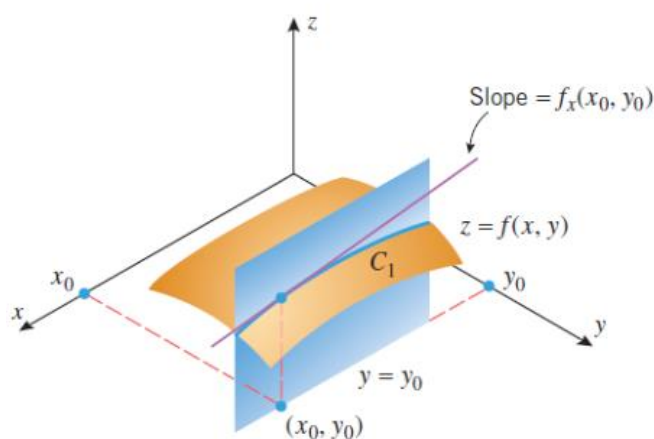
we have $f_y(1, 3) = 4(3) + 2 = 14$. ◀

THE PARTIAL DERIVATIVE FUNCTIONS

Formulas (1) and (2) define the partial derivatives of a function at a specific point (x_0, y_0) . However, often it will be desirable to omit the subscripts and think of the partial derivatives as functions of the variables x and y . These functions are

$$f_x(x, y) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \quad f_y(x, y) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

The following example gives an alternative way of performing the computations in Example 1.



PARTIAL DERIVATIVE NOTATION

If $z = f(x, y)$, then the partial derivatives f_x and f_y are also denoted by the symbols

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x, y)$ at a point (x_0, y_0) are

$$\left. \frac{\partial f}{\partial x} \right|_{x=x_0, y=y_0}, \quad \left. \frac{\partial z}{\partial x} \right|_{(x_0, y_0)}, \quad \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

► **Example 3** Find $\partial z / \partial x$ and $\partial z / \partial y$ if $z = x^4 \sin(xy^3)$.

Solution.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial x}(x^4) \\ &= x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3 = x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3) \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial y}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial y}(x^4) \\ &= x^4 \cos(xy^3) \cdot 3xy^2 + \sin(xy^3) \cdot 0 = 3x^5 y^2 \cos(xy^3) \quad \blacktriangleleft \end{aligned}$$

ESTIMATING PARTIAL DERIVATIVES FROM TABULAR DATA

For functions that are presented in tabular form, we can estimate partial derivatives by using adjacent entries within the table.

Table 13.3.1

TEMPERATURE T ($^{\circ}\text{F}$)

	20	25	30	35
5	13	19	25	31
10	9	15	21	27
15	6	13	19	25
20	4	11	17	24

► Example 6 Use the values of the wind chill index function $W(T, v)$ displayed in Table 13.3.1 to estimate the partial derivative of W with respect to v at $(T, v) = (25, 10)$. Compare this estimate with the value of the partial derivative obtained in Example 4.

Solution. Since

$$\frac{\partial W}{\partial v}(25, 10) = \lim_{\Delta v \rightarrow 0} \frac{W(25, 10 + \Delta v) - W(25, 10)}{\Delta v} = \lim_{\Delta v \rightarrow 0} \frac{W(25, 10 + \Delta v) - 15}{\Delta v}$$

we can approximate the partial derivative by

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 + \Delta v) - 15}{\Delta v}$$

With $\Delta v = 5$ this approximation is

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 + 5) - 15}{5} = \frac{W(25, 15) - 15}{5} = \frac{13 - 15}{5} = -\frac{2}{5} \frac{^{\circ}\text{F}}{\text{mi/h}}$$

and with $\Delta v = -5$ this approximation is

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 - 5) - 15}{-5} = \frac{W(25, 5) - 15}{-5} = \frac{19 - 15}{-5} = -\frac{4}{5} \frac{^{\circ}\text{F}}{\text{mi/h}}$$

We will take the average, $-\frac{3}{5} = -0.6^{\circ}\text{F}/(\text{mi/h})$, of these two approximations as our estimate of $(\partial W/\partial v)(25, 10)$. This is close to the value

$$\frac{\partial W}{\partial v}(25, 10) = (-4.01)10^{-0.84} \approx -0.58 \frac{^{\circ}\text{F}}{\text{mi/h}}$$

found in Example 4. ◀

IMPLICIT PARTIAL DIFFERENTIATION

► Example 7 Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y -direction at the points $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$ (Figure 13.3.2).

Solution. The point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ lies on the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, and the point $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$ lies on the lower hemisphere $z = -\sqrt{1 - x^2 - y^2}$. We could find the slopes by differentiating each expression for z separately with respect to y and then evaluating the derivatives at $x = \frac{2}{3}$ and $y = \frac{1}{3}$. However, it is more efficient to differentiate the given equation

$$x^2 + y^2 + z^2 = 1$$

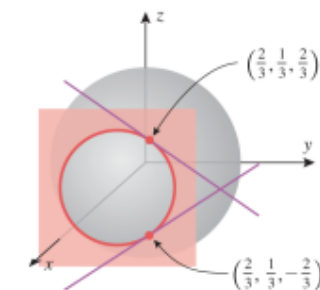
implicitly with respect to y , since this will give us both slopes with one differentiation. To perform the implicit differentiation, we view z as a function of x and y and differentiate both sides with respect to y , taking x to be fixed. The computations are as follows:

$$\frac{\partial}{\partial y}[x^2 + y^2 + z^2] = \frac{\partial}{\partial y}[1]$$

$$0 + 2y + 2z \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

Substituting the y - and z -coordinates of the points $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$ in this expression, we find that the slope at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ is $-\frac{1}{2}$ and the slope at $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$ is $\frac{1}{2}$. ◀



▲ Figure 13.3.2

Check the results in Example 7 by differentiating the functions

$$z = \sqrt{1 - x^2 - y^2}$$

and

$$z = -\sqrt{1 - x^2 - y^2}$$

directly.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = f_{xx}$$

Differentiate twice
with respect to x .

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = f_{yy}$$

Differentiate twice
with respect to y .

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = f_{xy}$$

Differentiate first with
respect to x and then
with respect to y .

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{yx}$$

Differentiate first with
respect to y and then
with respect to x .

Higher-Order Partial Derivatives

13.3.2 THEOREM Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.

► **Example 12** Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Solution. We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

so that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (2xy^3 + 4x^3y) = 6xy^2 + 4x^3 \quad \blacktriangleleft$$

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial x^2} \right) = f_{xxx} \quad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y} \left(\frac{\partial^3 f}{\partial y^3} \right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial y \partial x} \right) = f_{xyy} \quad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y} \left(\frac{\partial^3 f}{\partial y \partial x^2} \right) = f_{xxyy}$$

► **Example 13** Let $f(x, y) = y^2e^x + y$. Find f_{xyy} .

Solution.

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^2}{\partial y^2} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2}{\partial y^2} (y^2e^x) = \frac{\partial}{\partial y} (2ye^x) = 2e^x \quad \blacktriangleleft$$