Topics in Vector Calculus

Exercise Set 15.1

1. (a) III, because the vector field is independent of y and the direction is that of the negative x-axis for negative x, and positive for positive.

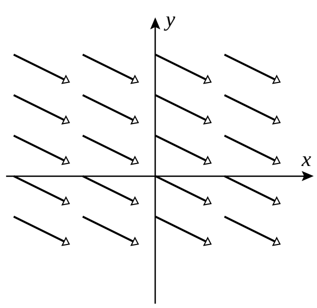
(b) IV, because the y-component is constant, and the x-component varies periodically with x.

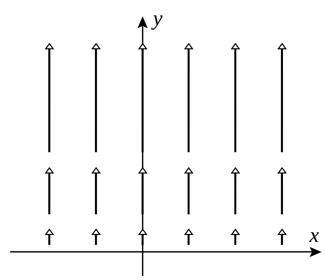
2. (a) I, since the vector field is constant.

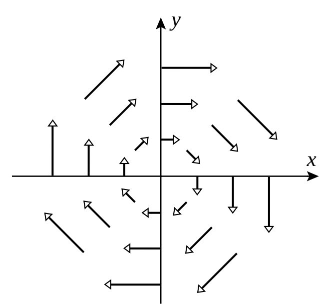
(b) II, since the vector field points away from the origin.

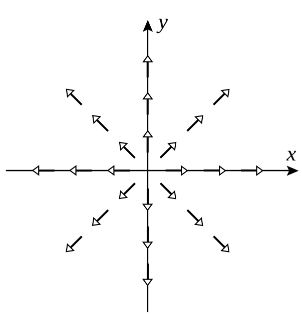
3. (a) True. **(b)** True. **(c)** True.

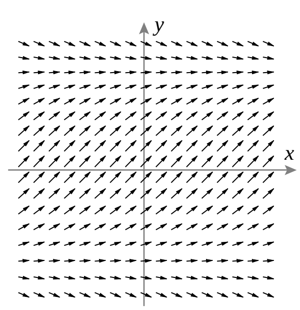
4. (a) False, the lengths are equal to 1. **(b)** False, the y-component is zero. **(c)** False, the x-component is zero.

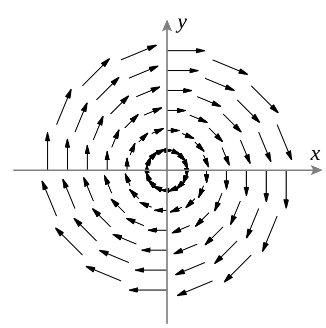
5. 

6. 

7. 

8. 

9. 

10. 

11. False, the $\mathbf{k}$-component is nonzero.

12. False, the power of $\|\mathbf{r}\|$ should be 3.

13. True (this example is the curl of $\mathbf{F}$).

14. False, ϕ is the divergence of $\mathbf{F}$.

15. **(a)** $\nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} = \dfrac{y}{1+x^2y^2}\mathbf{i} + \dfrac{x}{1+x^2y^2}\mathbf{j} = \mathbf{F}$, so $\mathbf{F}$ is conservative for all x, y.

(b) $\nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k} = 2x\mathbf{i} - 6y\mathbf{j} + 8z\mathbf{k} = \mathbf{F}$ so $\mathbf{F}$ is conservative for all x, y and z.

16. **(a)** $\nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} = (6xy - y^3)\mathbf{i} + (4y + 3x^2 - 3xy^2)\mathbf{j} = \mathbf{F}$, so $\mathbf{F}$ is conservative for all x, y.

(b) $\nabla\phi = \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k} = (\sin z + y\cos x)\mathbf{i} + (\sin x + z\cos y)\mathbf{j} + (x\cos z + \sin y)\mathbf{k} = \mathbf{F}$, so $\mathbf{F}$ is conservative for all x, y, and z.

17. div $\mathbf{F} = 2x + y$, curl $\mathbf{F} = z\mathbf{i}$.

18. div $\mathbf{F} = z^3 + 8y^3x^2 + 10zy$, curl $\mathbf{F} = 5z^2\mathbf{i} + 3xz^2\mathbf{j} + 4xy^4\mathbf{k}$.

19. div $\mathbf{F} = 0$, curl $\mathbf{F} = (40x^2z^4 - 12xy^3)\mathbf{i} + (14y^3z + 3y^4)\mathbf{j} - (16xz^5 + 21y^2z^2)\mathbf{k}$.

20. div $\mathbf{F} = ye^{xy} + \sin y + 2\sin z\cos z$, curl $\mathbf{F} = -xe^{xy}\mathbf{k}$.

21. div $\mathbf{F} = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$, curl $\mathbf{F} = \mathbf{0}$.

22. div $\mathbf{F} = \dfrac{1}{x} + xze^{xyz} + \dfrac{x}{x^2+z^2}$, curl $\mathbf{F} = -xye^{xyz}\mathbf{i} + \dfrac{z}{x^2+z^2}\mathbf{j} + yze^{xyz}\mathbf{k}$.

23. $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \nabla\cdot(-(z+4y^2)\mathbf{i} + (4xy+2xz)\mathbf{j} + (2xy - x)\mathbf{k}) = 4x$.

24. $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \nabla\cdot((x^2yz^2 - x^2y^2)\mathbf{i} - xy^2z^2\mathbf{j} + xy^2z\mathbf{k}) = -xy^2$.

25. $\nabla\cdot(\nabla\times\mathbf{F}) = \nabla\cdot(-\sin(x-y)\mathbf{k}) = 0$.

26. $\nabla\cdot(\nabla\times\mathbf{F}) = \nabla\cdot(-ze^{yz}\mathbf{i} + xe^{xz}\mathbf{j} + 3e^y\mathbf{k}) = 0$.

27. $\nabla\times(\nabla\times\mathbf{F}) = \nabla\times(xz\mathbf{i} - yz\mathbf{j} + y\mathbf{k}) = (1+y)\mathbf{i} + x\mathbf{j}$.

28. $\nabla\times(\nabla\times\mathbf{F}) = \nabla\times((x+3y)\mathbf{i} - y\mathbf{j} - 2xy\mathbf{k}) = -2x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$.

31. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$; div $(k\mathbf{F}) = k\dfrac{\partial f}{\partial x} + k\dfrac{\partial g}{\partial y} + k\dfrac{\partial h}{\partial z} = k$ div $\mathbf{F}$.

32. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$; curl $(k\mathbf{F}) = k\left(\dfrac{\partial h}{\partial y} - \dfrac{\partial g}{\partial z}\right)\mathbf{i} + k\left(\dfrac{\partial f}{\partial z} - \dfrac{\partial h}{\partial x}\right)\mathbf{j} + k\left(\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\right)\mathbf{k} = k$ curl $\mathbf{F}$.

33. Let $\mathbf{F} = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$ and $\mathbf{G} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, then div $(\mathbf{F}+\mathbf{G}) = \left(\dfrac{\partial f}{\partial x} + \dfrac{\partial P}{\partial x}\right) + \left(\dfrac{\partial g}{\partial y} + \dfrac{\partial Q}{\partial y}\right) + \left(\dfrac{\partial h}{\partial z} + \dfrac{\partial R}{\partial z}\right) = \left(\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}\right) + \left(\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}\right) =$ div $\mathbf{F} +$ div $\mathbf{G}$.

34. Let $\mathbf{F} = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$ and $\mathbf{G} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, then curl $(\mathbf{F}+\mathbf{G}) = \left[\dfrac{\partial}{\partial y}(h+R) - \dfrac{\partial}{\partial z}(g+Q)\right]\mathbf{i} + \left[\dfrac{\partial}{\partial z}(f+P) - \dfrac{\partial}{\partial x}(h+R)\right]\mathbf{j} + \left[\dfrac{\partial}{\partial x}(g+Q) - \dfrac{\partial}{\partial y}(f+P)\right]\mathbf{k}$; expand and rearrange terms to get curl $\mathbf{F}$ + curl $\mathbf{G}$.

35. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$; div $(\phi\mathbf{F}) = \left(\phi\dfrac{\partial f}{\partial x} + \dfrac{\partial\phi}{\partial x}f\right) + \left(\phi\dfrac{\partial g}{\partial y} + \dfrac{\partial\phi}{\partial y}g\right) + \left(\phi\dfrac{\partial h}{\partial z} + \dfrac{\partial\phi}{\partial z}h\right) = \phi\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}\right) + \left(\dfrac{\partial\phi}{\partial x}f + \dfrac{\partial\phi}{\partial y}g + \dfrac{\partial\phi}{\partial z}h\right) = \phi$ div $\mathbf{F} + \nabla\phi\cdot\mathbf{F}$.

36. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$; curl $(\phi\mathbf{F}) = \left[\frac{\partial}{\partial y}(\phi h) - \frac{\partial}{\partial z}(\phi g)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(\phi f) - \frac{\partial}{\partial x}(\phi h)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\phi g) - \frac{\partial}{\partial y}(\phi f)\right]\mathbf{k}$; use the product rule to expand each of the partial derivatives, rearrange to get ϕ curl $\mathbf{F} + \nabla\phi \times \mathbf{F}$.

37. Let $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$; div(curl $\mathbf{F}$) $= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) = \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 h}{\partial y \partial x} + \frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial y} = 0$, assuming equality of mixed second partial derivatives, which follows from the continuity assumptions.

38. curl $(\nabla\phi) = \left(\frac{\partial^2\phi}{\partial y \partial z} - \frac{\partial^2\phi}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2\phi}{\partial z \partial x} - \frac{\partial^2\phi}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2\phi}{\partial x \partial y} - \frac{\partial^2\phi}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}$, assuming equality of mixed second partial derivatives, which follows from the continuity assumptions.

39. $\nabla \cdot (k\mathbf{F}) = k\nabla \cdot \mathbf{F},\ \nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G},\ \nabla \cdot (\phi\mathbf{F}) = \phi\nabla \cdot \mathbf{F} + \nabla\phi \cdot \mathbf{F},\ \nabla \cdot (\nabla \times \mathbf{F}) = 0.$

40. $\nabla \times (k\mathbf{F}) = k\nabla \times \mathbf{F},\ \nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G},\ \nabla \times (\phi\mathbf{F}) = \phi\nabla \times \mathbf{F} + \nabla\phi \times \mathbf{F},\ \nabla \times (\nabla\phi) = \mathbf{0}.$

41. **(a)** curl $\mathbf{r} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$.

(b) $\nabla\|\mathbf{r}\| = \nabla\sqrt{x^2 + y^2 + z^2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$.

42. **(a)** div $\mathbf{r} = 1 + 1 + 1 = 3$.

(b) $\nabla\frac{1}{\|\mathbf{r}\|} = \nabla(x^2 + y^2 + z^2)^{-1/2} = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{\mathbf{r}}{\|\mathbf{r}\|^3}$.

43. **(a)** $\nabla f(r) = f'(r)\frac{\partial r}{\partial x}\mathbf{i} + f'(r)\frac{\partial r}{\partial y}\mathbf{j} + f'(r)\frac{\partial r}{\partial z}\mathbf{k} = f'(r)\nabla r = \frac{f'(r)}{r}\mathbf{r}$.

(b) $\text{div}[f(r)\mathbf{r}] = f(r)\text{div }\mathbf{r} + \nabla f(r) \cdot \mathbf{r} = 3f(r) + \frac{f'(r)}{r}\mathbf{r} \cdot \mathbf{r} = 3f(r) + rf'(r)$.

44. **(a)** $\text{curl}[f(r)\mathbf{r}] = f(r)\text{curl }\mathbf{r} + \nabla f(r) \times \mathbf{r} = f(r)\mathbf{0} + \frac{f'(r)}{r}\mathbf{r} \times \mathbf{r} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

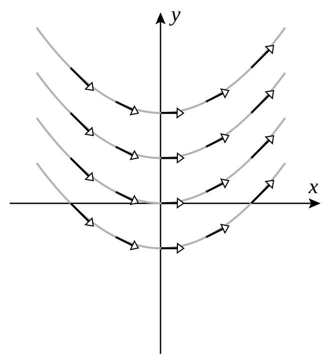
(b) $\nabla^2 f(r) = \text{div}[\nabla f(r)] = \text{div}\left[\frac{f'(r)}{r}\mathbf{r}\right] = \frac{f'(r)}{r}\text{div }\mathbf{r} + \nabla\frac{f'(r)}{r} \cdot \mathbf{r} = 3\frac{f'(r)}{r} + \frac{rf''(r) - f'(r)}{r^3}\mathbf{r} \cdot \mathbf{r} = 2\frac{f'(r)}{r} + f''(r)$.

45. $f(r) = 1/r^3, f'(r) = -3/r^4$, $\text{div}(\mathbf{r}/r^3) = 3(1/r^3) + r(-3/r^4) = 0$.

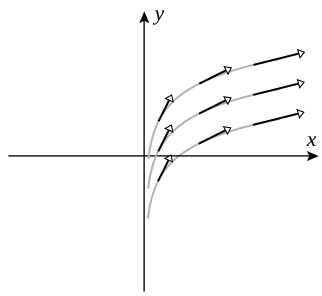
46. Multiply $3f(r) + rf'(r) = 0$ through by r^2 to obtain $3r^2 f(r) + r^3 f'(r) = 0$, $d[r^3 f(r)]/dr = 0$, $r^3 f(r) = C, f(r) = C/r^3$, so $\mathbf{F} = C\mathbf{r}/r^3$ (an inverse-square field).

47. **(a)** At the point (x, y) the slope of the line along which the vector $-y\mathbf{i} + x\mathbf{j}$ lies is $-x/y$; the slope of the tangent line to C at (x, y) is dy/dx, so $dy/dx = -x/y$.

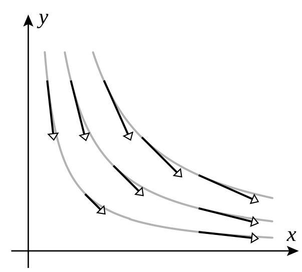
(b) $ydy = -xdx,\ y^2/2 = -x^2/2 + K_1,\ x^2 + y^2 = K.$



48. $dy/dx = x,\ \ y = x^2/2 + K$.



49. $dy/dx = 1/x,\ \ y = \ln x + K$.



50. $dy/dx = -y/x,\ (1/y)dy = (-1/x)dx,\ \ln y = -\ln x + K_1,\ y = e^{K_1}e^{-\ln x} = K/x$.

Exercise Set 15.2

1. (a) $\displaystyle\int_C ds =$ length of line segment $= 1$. **(b)** 0, because $\sin xy = 0$ along C.

2. (a) $\displaystyle\int_C ds =$ length of line segment $= 2$. **(b)** 0, because x is constant and $dx = 0$.

3. Since $\mathbf{F}$ and $\mathbf{r}$ are parallel, $\mathbf{F}\cdot\mathbf{r} = \|\mathbf{F}\|\|\mathbf{r}\|$, and since $\mathbf{F}$ is constant, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C d(\mathbf{F}\cdot\mathbf{r}) = \int_C d(\|\mathbf{F}\|\|\mathbf{r}\|) = \sqrt{2}\int_{-4}^{4}\sqrt{2}dt = 16$.

4. $\displaystyle\int_C \mathbf{F}\cdot\mathbf{r} = 0$, since $\mathbf{F}$ is perpendicular to the curve.

5. By inspection the tangent vector in part (a) is given by $\mathbf{T} = \mathbf{j}$, so $\mathbf{F}\cdot\mathbf{T} = \mathbf{F}\cdot\mathbf{j} = \sin x$ on C. But $x = -\pi/2$ on C, thus $\sin x = -1$, $\mathbf{F}\cdot\mathbf{T} = -1$ and $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (-1)ds$.

6. (a) Let α be the angle between $\mathbf{F}$ and $\mathbf{T}$. Since $\|\mathbf{F}\| = 1$, $\cos\alpha = \|\mathbf{F}\|\|\mathbf{T}\|\cos\alpha = \mathbf{F}\cdot\mathbf{T}$, and $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds =$

$\int_C \cos\alpha(s)\,ds$. From Figure 15.2.12(b) it is apparent that α is close to zero on most of the parabola, thus $\cos\alpha \approx 1$ though $\cos\alpha \le 1$. Hence $\int_C \cos\alpha(s)\,ds \le \int_C ds$ and the first integral is close to the second.

(b) From Example 8(b) $\int_C \cos\alpha\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} \approx 5.83629$, and $\int_C ds = \int_{-1}^{2} \sqrt{1+(2t)^2}\,dt \approx 6.125726619$.

7. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_4^5 (8\cdot 0 + 8\cdot 1)\,dt = 8$.

8. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^4 (2\cdot 1 + 5\cdot 0)\,dt = 6$.

9. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_4^{11} (0\cdot 0 + 2(-2)\cdot 1)\,dt = -28$.

10. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{6} (-8(-t)\cdot(-1) + 3(0)\cdot 0)\,dt = -140$.

11. (a) $ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt$, so $\int_0^1 (2t-\sqrt{t^2})\sqrt{4+4t^2}\,dt = \int_0^1 2t\sqrt{1+t^2}\,dt = \dfrac{2}{3}(1+t^2)^{3/2}\Big]_0^1 = \dfrac{2}{3}(2\sqrt{2}-1)$.

(b) $\int_0^1 (2t-\sqrt{t^2})2\,dt = 1$. **(c)** $\int_0^1 (2t-\sqrt{t^2})2t\,dt = \dfrac{2}{3}$.

12. (a) $\int_0^1 t(3t^2)(6t^3)^2\sqrt{1+36t^2+324t^4}\,dt = \dfrac{864}{5}$. **(b)** $\int_0^1 t(3t^2)(6t^3)^2\,dt = \dfrac{54}{5}$.

(c) $\int_0^1 t(3t^2)(6t^3)^2 6t\,dt = \dfrac{648}{11}$. **(d)** $\int_0^1 t(3t^2)(6t^3)^2 18t^2\,dt = 162$.

13. (a) $C: x=t,\ y=t,\ 0\le t\le 1$; $\int_0^1 6t\,dt = 3$.

(b) $C: x=t,\ y=t^2,\ 0\le t\le 1$; $\int_0^1 (3t+6t^2-2t^3)dt = 3$.

(c) $C: x=t,\ y=\sin(\pi t/2),\ 0\le t\le 1$; $\int_0^1 [3t+2\sin(\pi t/2)+\pi t\cos(\pi t/2)-(\pi/2)\sin(\pi t/2)\cos(\pi t/2)]dt = 3$.

(d) $C: x=t^3,\ y=t,\ 0\le t\le 1$; $\int_0^1 (9t^5+8t^3-t)dt = 3$.

14. (a) $C: x=t, y=t, z=t, 0\le t\le 1$; $\int_0^1 (t+t-t)\,dt = \dfrac{1}{2}$.

(b) $C: x=t, y=t^2, z=t^3, 0\le t\le 1$; $\int_0^1 (t^2+t^3(2t)-t(3t^2))\,dt = -\dfrac{1}{60}$.

(c) $C: x=\cos\pi t, y=\sin\pi t, z=t, 0\le t\le 1$; $\int_0^1 (-\pi\sin^2\pi t + \pi t\cos\pi t - \cos\pi t)\,dt = -\dfrac{\pi}{2} - \dfrac{2}{\pi}$.

15. False, a line integral is independent of the orientation of the curve.

16. False, it's a scalar.

17. True, see (26).

18. True, since ∇f is normal to C; see (30).

19. $\displaystyle\int_0^3 \frac{\sqrt{1+t}}{1+t}\,dt = \int_0^3 (1+t)^{-1/2}\,dt = 2.$

20. $\displaystyle\sqrt{5}\int_0^1 \frac{1+2t}{1+t^2}\,dt = \sqrt{5}(\pi/4 + \ln 2).$

21. $\displaystyle\int_0^1 3(t^2)(t^2)(2t^3/3)(1+2t^2)\,dt = 2\int_0^1 t^7(1+2t^2)\,dt = 13/20.$

22. $\displaystyle\frac{\sqrt{5}}{4}\int_0^{2\pi} e^{-t}\,dt = \sqrt{5}(1-e^{-2\pi})/4.$

23. $\displaystyle\int_0^{\pi/4} (8\cos^2 t - 16\sin^2 t - 20\sin t\cos t)dt = 1-\pi.$

24. $\displaystyle\int_{-1}^1 \left(\frac{2}{3}t - \frac{2}{3}t^{5/3} + t^{2/3}\right)dt = 6/5.$

25. $C: x = (3-t)^2/3,\ y = 3-t,\ 0 \le t \le 3;\ \displaystyle\int_0^3 \frac{1}{3}(3-t)^2 dt = 3.$

26. $C: x = t^{2/3},\ y = t,\ -1 \le t \le 1;\ \displaystyle\int_{-1}^1 \left(\frac{2}{3}t^{2/3} - \frac{2}{3}t^{1/3} + t^{7/3}\right)dt = 4/5.$

27. $C: x = \cos t,\ y = \sin t,\ 0 \le t \le \pi/2;\ \displaystyle\int_0^{\pi/2} (-\sin t - \cos^2 t)dt = -1-\pi/4.$

28. $C: x = 3-t,\ y = 4-3t,\ 0 \le t \le 1;\ \displaystyle\int_0^1 (-37 + 41t - 9t^2)dt = -39/2.$

29. $\displaystyle\int_0^1 (-3)e^{3t}dt = 1 - e^3.$

30. $\displaystyle\int_0^{\pi/2} (\sin^2 t\cos t - \sin^2 t\cos t + t^4(2t))\,dt = \frac{\pi^6}{192}.$

31. **(a)** $\displaystyle\int_0^{\ln 2} \left(e^{3t}+e^{-3t}\right)\sqrt{e^{2t}+e^{-2t}}\,dt = \frac{63}{64}\sqrt{17} + \frac{1}{4}\ln(4+\sqrt{17}) - \frac{1}{4}\tanh^{-1}\left(\frac{1}{17}\sqrt{17}\right).$

(b) $\displaystyle\int_0^{\pi/2} \left[e^t\sin t\cos t - (\sin t - t)\sin t + (1+t^2)\right]\,dt = \frac{1}{24}\pi^3 + \frac{1}{5}e^{\pi/2} + \frac{1}{4}\pi + \frac{6}{5}.$

32. **(a)** $\displaystyle\int_0^{\pi/2} \cos^{21} t\sin^9 t\sqrt{(-3\cos^2 t\sin t)^2 + (3\sin^2 t\cos t)^2}\,dt = 3\int_0^{\pi/2} \cos^{22} t\sin^{10} t\,dt = \frac{61{,}047}{4{,}294{,}967{,}296}\pi.$

(b) $\displaystyle\int_1^e \left(t^5 \ln t + 7t^2(2t) + t^4(\ln t)\frac{1}{t}\right) dt = \frac{5}{36}e^6 + \frac{59}{16}e^4 - \frac{491}{144}.$

33. **(a)** $C_1 : (0,0)$ to $(1,0); x = t, y = 0, 0 \le t \le 1$, $C_2 : (1,0)$ to $(0,1); x = 1-t, y = t, 0 \le t \le 1$, $C_3 : (0,1)$ to $(0,0); x = 0, y = 1-t, 0 \le t \le 1$, $\displaystyle\int_0^1 (0)dt + \int_0^1 (-1)dt + \int_0^1 (0)dt = -1.$

(b) $C_1 : (0,0)$ to $(1,0); x = t, y = 0, 0 \le t \le 1$, $C_2 : (1,0)$ to $(1,1); x = 1, y = t, 0 \le t \le 1$, $C_3 : (1,1)$ to $(0,1); x = 1-t, y = 1, 0 \le t \le 1$, $C_4 : (0,1)$ to $(0,0); x = 0, y = 1-t, 0 \le t \le 1$, $\displaystyle\int_0^1 (0)dt + \int_0^1 (-1)dt + \int_0^1 (-1)dt + \int_0^1 (0)dt = -2.$

34. **(a)** $C_1 : (0,0)$ to $(1,1); x = t, y = t, 0 \le t \le 1$, $C_2 : (1,1)$ to $(2,0); x = 1+t, y = 1-t, 0 \le t \le 1$, $C_3 : (2,0)$ to $(0,0); x = 2-2t, y = 0, 0 \le t \le 1$, $\displaystyle\int_0^1 (0)dt + \int_0^1 2dt + \int_0^1 (0)dt = 2.$

(b) $C_1 : (-5,0)$ to $(5,0); x = t, y = 0, -5 \le t \le 5$, $C_2 : x = 5\cos t, y = 5\sin t, 0 \le t \le \pi$, $\displaystyle\int_{-5}^5 (0)dt + \int_0^\pi (-25)dt = -25\pi.$

35. $C_1 : x = t, y = z = 0, 0 \le t \le 1$, $\displaystyle\int_0^1 0\,dt = 0$; $C_2 : x = 1, y = t, z = 0, 0 \le t \le 1$, $\displaystyle\int_0^1 (-t)\,dt = -\frac{1}{2}$; $C_3 : x = 1, y = 1, z = t, 0 \le t \le 1$, $\displaystyle\int_0^1 3\,dt = 3$; $\displaystyle\int_C x^2 z\,dx - yx^2\,dy + 3\,dz = 0 - \frac{1}{2} + 3 = \frac{5}{2}.$

36. $C_1 : (0,0,0)$ to $(1,1,0); x = t, y = t, z = 0, 0 \le t \le 1$, $C_2 : (1,1,0)$ to $(1,1,1); x = 1, y = 1, z = t, 0 \le t \le 1$, $C_3 : (1,1,1)$ to $(0,0,0); x = 1-t, y = 1-t, z = 1-t, 0 \le t \le 1$, $\displaystyle\int_0^1 (-t^3)dt + \int_0^1 3\,dt + \int_0^1 -3dt = -1/4.$

37. $\displaystyle\int_0^\pi (0)dt = 0.$

38. $\displaystyle\int_0^1 (e^{2t} - 4e^{-t})dt = e^2/2 + 4e^{-1} - 9/2.$

39. $\displaystyle\int_0^1 e^{-t}dt = 1 - e^{-1}$

40. $\displaystyle\int_0^{\pi/2} (7\sin^2 t\cos t + 3\sin t\cos t)dt = 23/6.$

41. Represent the circular arc by $x = 3\cos t, y = 3\sin t, 0 \le t \le \pi/2$. $\displaystyle\int_C x\sqrt{y}ds = 9\sqrt{3}\int_0^{\pi/2} \sqrt{\sin\ t}\cos t\ dt = 6\sqrt{3}.$

42. $\delta(x,y) = k\sqrt{x^2+y^2}$ where k is the constant of proportionality,
$\displaystyle\int_C k\sqrt{x^2+y^2}ds = k\int_0^1 e^t(\sqrt{2}e^t)\ dt = \sqrt{2}k\int_0^1 e^{2t}\ dt = (e^2-1)k/\sqrt{2}.$

43. $\displaystyle\int_C \frac{kx}{1+y^2}ds = 15k\int_0^{\pi/2} \frac{\cos t}{1+9\sin^2 t}\ dt = 5k\tan^{-1} 3.$

44. $\delta(x,y,z) = kz$ where k is the constant of proportionality, $\displaystyle\int_C k\,z\,ds = \int_1^4 k(4\sqrt{t})(2+1/t)\ dt = 136k/3.$

45. $C : x = t^2,\ y = t,\ 0 \le t \le 1;\ W = \int_0^1 3t^4 dt = 3/5.$

46. $W = \int_1^3 (t^2 + 1 - 1/t^3 + 1/t)dt = 92/9 + \ln 3.$

47. $W = \int_0^1 (t^3 + 5t^6)dt = 27/28.$

48. $C_1 : (0,0,0)$ to $(1,3,1)$; $x = t,\ y = 3t,\ z = t,\ 0 \le t \le 1,\ \ C_2 : (1,3,1)$ to $(2,-1,4)$; $x = 1+t,\ y = 3-4t,\ z = 1+3t,$ $0 \le t \le 1,\quad W = \int_0^1 (4t + 8t^2)dt + \int_0^1 (-11 - 17t - 11t^2)dt = -37/2.$

49. $C : x = 4\cos t, y = 4\sin t, 0 \le t \le \pi/2,\quad \int_0^{\pi/2} \left(-\frac{1}{4}\sin t + \cos t\right) dt = 3/4.$

50. $C_1 : (0,3)$ to $(6,3); x = 6t, y = 3, 0 \le t \le 1,\ C_2 : (6,3)$ to $(6,0); x = 6, y = 3 - 3t, 0 \le t \le 1,\quad \int_0^1 \frac{6}{36t^2+9}dt +$ $\int_0^1 \frac{-12}{36 + 9(1-t)^2}dt = \frac{1}{3}\tan^{-1} 2 - \frac{2}{3}\tan^{-1}(1/2).$

51. Represent the parabola by $x = t, y = t^2, 0 \le t \le 2.$ $\int_C 3x\, ds = \int_0^2 3t\sqrt{1+4t^2}\ dt = (17\sqrt{17} - 1)/4.$

52. Represent the semicircle by $x = 2\cos t, y = 2\sin t, 0 \le t \le \pi.$ $\int_C x^2\, y\, ds = \int_0^\pi 16\cos^2 t\sin t\ dt = 32/3.$

53. **(a)** $2\pi rh = 2\pi(1)2 = 4\pi.$

(b) $S = \int_C z(t)\, dt$, since the average height is 2.

(c) $C : x = \cos t, y = \sin t, 0 \le t \le 2\pi; S = \int_0^{2\pi} (2 + (1/2)\sin 3t)\, dt = 4\pi.$

54. $C : x = a\cos t, y = -a\sin t, 0 \le t \le 2\pi,\quad \int_C \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{-a^2\cos^2 t - a^2\sin^2 t}{a^2}\, dt = -\int_0^{2\pi} dt = -2\pi.$

55. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (\lambda t^2(1-t), t - \lambda t(1-t))\cdot(1, \lambda - 2\lambda t)\, dt = -\lambda/12$, $W = 1$ when $\lambda = -12$.

56. The force exerted by the farmer is $\mathbf{F} = \left(150 + 20 - \frac{1}{10}z\right)\mathbf{k} = \left(170 - \frac{3}{4\pi}t\right)\mathbf{k}$, so $\mathbf{F}\cdot d\mathbf{r} = \left(170 - \frac{1}{10}z\right) dz$, and $W = \int_0^{60}\left(170 - \frac{1}{10}z\right) dz = 10{,}020$ foot-pounds. Note that the functions $x(z), y(z)$ are irrelevant.

57. **(a)** From (8), $\Delta s_k = \int_{t_{k-1}}^{t_k} \|\mathbf{r}'(t)\|\, dt$, thus $m\Delta t_k \le \Delta s_k \le M\Delta t_k$ for all k. Obviously $\Delta s_k \le M(\max\Delta t_k)$, and since the right side of this inequality is independent of k, it follows that $\max\Delta s_k \le M(\max\Delta t_k)$. Similarly $m(\max\Delta t_k) \le \max\Delta s_k$.

(b) This follows from $\max\Delta t_k \le \frac{1}{m}\max\Delta s_k$ and $\max\Delta s_k \le M\max\Delta t_k$.

Exercise Set 15.3

1. $\partial x/\partial y = 0 = \partial y/\partial x$, conservative so $\partial\phi/\partial x = x$ and $\partial\phi/\partial y = y$, $\phi = x^2/2 + k(y)$, $k'(y) = y$, $k(y) = y^2/2 + K$, $\phi = x^2/2 + y^2/2 + K$.

2. $\partial(3y^2)/\partial y = 6y = \partial(6xy)/\partial x$, conservative so $\partial\phi/\partial x = 3y^2$ and $\partial\phi/\partial y = 6xy$, $\phi = 3xy^2 + k(y)$, $6xy + k'(y) = 6xy$, $k'(y) = 0$, $k(y) = K$, $\phi = 3xy^2 + K$.

3. $\partial(x^2y)/\partial y = x^2$ and $\partial(5xy^2)/\partial x = 5y^2$, not conservative.

4. $\partial(e^x \cos y)/\partial y = -e^x \sin y = \partial(-e^x \sin y)/\partial x$, conservative so $\partial\phi/\partial x = e^x \cos y$ and $\partial\phi/\partial y = -e^x \sin y$, $\phi = e^x \cos y + k(y)$, $-e^x \sin y + k'(y) = -e^x \sin y$, $k'(y) = 0$, $k(y) = K$, $\phi = e^x \cos y + K$.

5. $\partial(\cos y + y\cos x)/\partial y = -\sin y + \cos x = \partial(\sin x - x\sin y)/\partial x$, conservative so $\partial\phi/\partial x = \cos y + y\cos x$ and $\partial\phi/\partial y = \sin x - x\sin y$, $\phi = x\cos y + y\sin x + k(y)$, $-x\sin y + \sin x + k'(y) = \sin x - x\sin y$, $k'(y) = 0$, $k(y) = K$, $\phi = x\cos y + y\sin x + K$.

6. $\partial(x\ln y)/\partial y = x/y$ and $\partial(y\ln x)/\partial x = y/x$, not conservative.

7. **(a)** Let $C : x(t) = 1+2t, y = 4-3t, 0 \le t \le 1$. Then $I = \displaystyle\int_C (2xy^3\,dx + (1+3x^2y^2)\,dy) = \int_0^1 [2(1+2t)(4-3t)^3)2 + (1+3(1+2t)^2(4-3t)^2)(-3)]dt = -58$.

(b) Let $C_1 : x(t) = 1, y(t) = 4-3t, 0 \le t \le 1$ and $C_2 : x(t) = 2t-1, y = 1, 1 \le t \le 2$. Then $I_1 = \int_0^1 [(1+3(4-3t)^2](-3) = -66$ and $I_2 = \int_1^2 2(2t-1)2\,dt = 8$, so $I = I_1 + I_2 = -66 + 8 = -58$.

8. **(a)** $\partial(y\sin x)/\partial y = \sin x = \partial(-\cos x)/\partial x$, independent of path.

(b) $C_1 : x = \pi t$, $y = 1-2t$, $0 \le t \le 1$; $\displaystyle\int_0^1 (\pi\sin\pi t - 2\pi t\sin\pi t + 2\cos\pi t)dt = 0$.

(c) $\partial\phi/\partial x = y\sin x$ and $\partial\phi/\partial y = -\cos x$, $\phi = -y\cos x + k(y)$, $-\cos x + k'(y) = -\cos x$, $k'(y) = 0$, $k(y) = K$, $\phi = -y\cos x + K$. Let $K = 0$ to get $\phi(\pi, -1) - \phi(0, 1) = (-1) - (-1) = 0$.

9. $\partial(3y)/\partial y = 3 = \partial(3x)/\partial x$, $\phi = 3xy$, $\phi(4, 0) - \phi(1, 2) = -6$.

10. $\partial(e^x \sin y)/\partial y = e^x \cos y = \partial(e^x \cos y)/\partial x$, $\phi = e^x \sin y$, $\phi(1, \pi/2) - \phi(0, 0) = e$.

11. $\partial(2xe^y)/\partial y = 2xe^y = \partial(x^2e^y)/\partial x$, $\phi = x^2e^y$, $\phi(3, 2) - \phi(0, 0) = 9e^2$.

12. $\partial(3x - y + 1)/\partial y = -1 = \partial[-(x + 4y + 2)]/\partial x$, $\phi = 3x^2/2 - xy + x - 2y^2 - 2y$, $\phi(0, 1) - \phi(-1, 2) = 11/2$.

13. $\partial(2xy^3)/\partial y = 6xy^2 = \partial(3x^2y^2)/\partial x$, $\phi = x^2y^3$, $\phi(-1, 0) - \phi(2, -2) = 32$.

14. $\partial(e^x \ln y - e^y/x)/\partial y = e^x/y - e^y/x = \partial(e^x/y - e^y \ln x)/\partial x$, $\phi = e^x \ln y - e^y \ln x$, $\phi(3, 3) - \phi(1, 1) = 0$.

15. $\phi = x^2y^2/2$, $W = \phi(0, 0) - \phi(1, 1) = -1/2$.

16. $\phi = x^2y^3, W = \phi(4, 1) - \phi(-3, 0) = 16$.

17. $\phi = e^{xy}$, $W = \phi(2, 0) - \phi(-1, 1) = 1 - e^{-1}$.

18. $\phi = e^{-y}\sin x, W = \phi(-\pi/2, 0) - \phi(\pi/2, 1) = -1 - 1/e$.

19. False, the integral must be zero for *all* closed curves C.

20. True, Theorem 15.3.3.

21. True; if $\nabla\phi$ is constant then ϕ is linear.

22. True, Theorem 15.3.3.

23. $\partial(e^y + ye^x)/\partial y = e^y + e^x = \partial(xe^y + e^x)/\partial x$ so $\mathbf{F}$ is conservative, $\phi(x, y) = xe^y + ye^x$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(0, \ln 2) - \phi(1, 0) = \ln 2 - 1$.

24. $\partial(2xy)/\partial y = 2x = \partial(x^2 + \cos y)/\partial x$ so $\mathbf{F}$ is conservative, $\phi(x, y) = x^2y + \sin y$ so $\int_C \mathbf{F} \cdot dr = \phi(\pi, \pi/2) - \phi(0, 0) = \pi^3/2 + 1$.

25. $\mathbf{F} \cdot d\mathbf{r} = [(e^y + ye^x)\mathbf{i} + (xe^y + e^x)\mathbf{j}] \cdot [(\pi/2)\cos(\pi t/2)\mathbf{i} + (1/t)\mathbf{j}]dt = \left(\frac{\pi}{2}\cos(\pi t/2)(e^y + ye^x) + (xe^y + e^x)/t\right) dt$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \left(\frac{\pi}{2}\cos(\pi t/2)\left(t + (\ln t)e^{\sin(\pi t/2)}\right) + \left(\sin(\pi t/2) + \frac{1}{t}e^{\sin(\pi t/2)}\right)\right) dt = \ln 2 - 1 \approx -0.306853$.

26. $\mathbf{F} \cdot d\mathbf{r} = \left(2t^2\cos(t/3) + [t^2 + \cos(t\cos(t/3))](\cos(t/3) - (t/3)\sin(t/3))\right) dt$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \left(2t^2\cos(t/3) + [t^2 + \cos(t\cos(t/3))](\cos(t/3) - (t/3)\sin(t/3))\right) dt = 1 + \pi^3/2.$$

27. No; a closed loop can be found whose tangent everywhere makes an angle $< \pi$ with the vector field, so the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$, and by Theorem 15.3.2 the vector field is not conservative.

28. The vector field is constant, say $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$, so let $\phi(x, y) = ax + by$ and $\mathbf{F}$ is conservative.

29. Let $\mathbf{r}(t)$ be a parametrization of the circle C. Then by Theorem 15.3.2(b), $\int_C \mathbf{F}d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{r}'(t)\, dt = 0$. Let $h(t) = \mathbf{F}(x, y) \cdot \mathbf{r}'(t)$. Then h is continuous. We must find two points at which $h = 0$. If $h(t) = 0$ everywhere on the circle, then we are done; otherwise there are points at which h is nonzero, say $h(t_1) > 0$. Then there is a small interval around t_1 on which the integral of h is positive. (Let $\epsilon = h(t_1)/2$. Since $h(t)$ is continuous there exists $\delta > 0$ such that for $|t - t_1| < \delta$, $h(t) > \epsilon/2$. Then $\int_{t_1-\delta}^{t_1+\delta} h(t)\, dt \geq (2\delta)\epsilon/2 > 0$.) Since $\int_C h = 0$, there are points on the circle where $h < 0$, say $h(t_2) < 0$. Now consider the parametrization $h(\theta)$, $0 \leq \theta \leq 2\pi$. Let $\theta_1 < \theta_2$ correspond to the points above where $h > 0$ and $h < 0$. Then by the Intermediate Value Theorem on $[\theta_1, \theta_2]$ there must be a point where $h = 0$, say $h(\theta_3) = 0, \theta_1 < \theta_3 < \theta 2$. To find a second point where $h = 0$, assume that h is a periodic function with period 2π (if need be, extend the definition of h). Then $h(t_2 - 2\pi) = h(t_2) < 0$. Apply the Intermediate Value Theorem on $[t_2 - 2\pi, t_1]$ to find an additional point θ_4 at which $h = 0$. The reader should prove that θ_3 and θ_4 do indeed correspond to distinct points on the circle.

30. The function $\mathbf{F} \cdot \mathbf{r}'(t)$ is not necessarily continuous since the tangent to the square has obvious discontinuities. For a counterexample to the result, let the square have vertices at $(0, 0), (0, 1), (1, 1), (1, 0)$. Let $\Phi(x, y) = xy + x + y$ and let $\mathbf{F} = \nabla\Phi = (y + 1)\mathbf{i} + (x + 1)\mathbf{j}$. Then $\mathbf{F}$ is conservative , but on the bottom side of the square, where $y = 0$, $\mathbf{F} \cdot \mathbf{r}' = -\mathbf{F} \cdot \mathbf{j} = -x - 1 \leq 1 < 0$. On the top edge $\mathbf{F} \cdot \mathbf{r}' = \mathbf{F} \cdot \mathbf{j} = x + 1 \geq 1 > 0$. Similarly for the other two sides of the square. Thus at no point is $\mathbf{F} \cdot \mathbf{r}' = 0$.

31. If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$ and hence $f = \frac{\partial\phi}{\partial x}, g = \frac{\partial\phi}{\partial y}$, and $h = \frac{\partial\phi}{\partial z}$. Thus $\frac{\partial f}{\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ and $\frac{\partial g}{\partial x} = \frac{\partial^2\phi}{\partial x\partial y}, \frac{\partial f}{\partial z} = \frac{\partial^2\phi}{\partial z\partial x}$ and $\frac{\partial h}{\partial x} = \frac{\partial^2\phi}{\partial x\partial z}, \frac{\partial g}{\partial z} = \frac{\partial^2\phi}{\partial z\partial y}$ and $\frac{\partial h}{\partial y} = \frac{\partial^2\phi}{\partial y\partial z}$. The result follows from the equality of mixed second partial derivatives.

32. Let $f(x,y,z) = yz, g(x,y,z) = xz,\ h(x,y,z) = yx^2$, then $\partial f/\partial z = y, \partial h/\partial x = 2xy \neq \partial f/\partial z$, thus by Exercise 31, $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ is not conservative, and by Theorem 15.3.2, $\int_C yz\,dx + xz\,dy + yx^2\,dz$ is not independent of the path.

33. $\dfrac{\partial}{\partial y}(h(x)[x\sin y + y\cos y]) = h(x)[x\cos y - y\sin y + \cos y]$, $\dfrac{\partial}{\partial x}(h(x)[x\cos y - y\sin y]) = h(x)\cos y + h'(x)[x\cos y - y\sin y]$, equate these two partial derivatives to get $(x\cos y - y\sin y)(h'(x) - h(x)) = 0$ which holds for all x and y if $h'(x) = h(x)$, $h(x) = Ce^x$ where C is an arbitrary constant.

34. **(a)** $\dfrac{\partial}{\partial y}\dfrac{cx}{(x^2+y^2)^{3/2}} = -\dfrac{3cxy}{(x^2+y^2)^{-5/2}} = \dfrac{\partial}{\partial x}\dfrac{cy}{(x^2+y^2)^{3/2}}$ when $(x,y) \neq (0,0)$, so by Theorem 15.3.3, $\mathbf{F}$ is conservative. Set $\partial\phi/\partial x = cx/(x^2+y^2)^{-3/2}$, then $\phi(x,y) = -c(x^2+y^2)^{-1/2} + k(y), \partial\phi/\partial y = cy/(x^2+y^2)^{-3/2} + k'(y)$, so $k'(y) = 0$. Thus $\phi(x,y) = -\dfrac{c}{(x^2+y^2)^{1/2}}$ is a potential function.

(b) curl $\mathbf{F} = \mathbf{0}$ is similar to part (a), so $\mathbf{F}$ is conservative. Let $\phi(x,y,z) = \displaystyle\int \frac{cx}{(x^2+y^2+z^2)^{3/2}}\,dx = -c(x^2+y^2+z^2)^{-1/2} + k(y,z)$. As in part (a), $\partial k/\partial y = \partial k/\partial z = 0$, so $\phi(x,y,z) = -c/(x^2+y^2+z^2)^{1/2}$ is a potential function for $\mathbf{F}$.

35. **(a)** See Exercise 34, $c = 1$; $W = \displaystyle\int_P^Q \mathbf{F}\cdot d\mathbf{r} = \phi(3,2,1) - \phi(1,1,2) = -\frac{1}{\sqrt{14}} + \frac{1}{\sqrt{6}}$.

(b) C begins at $P(1,1,2)$ and ends at $Q(3,2,1)$ so the answer is again $W = -\dfrac{1}{\sqrt{14}} + \dfrac{1}{\sqrt{6}}$.

(c) The circle is not specified, but cannot pass through $(0,0,0)$, so Φ is continuous and differentiable on the circle. Start at any point P on the circle and return to P, so the work is $\Phi(P) - \Phi(P) = 0$. C begins at, say, $(3,0)$ and ends at the same point so $W = 0$.

36. **(a)** $\mathbf{F}\cdot d\mathbf{r} = \left(y\dfrac{dx}{dt} - x\dfrac{dy}{dt}\right)dt$ for points on the circle $x^2+y^2 = 1$, so $C_1: x = \cos t, y = \sin t, 0 \le t \le \pi$, $\displaystyle\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_0^\pi(-\sin^2 t - \cos^2 t)\,dt = -\pi$, $C_2: x = \cos t, y = -\sin t, 0 \le t \le \pi$, $\displaystyle\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_0^\pi(\sin^2 t + \cos^2 t)\,dt = \pi$.

(b) $\dfrac{\partial f}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}, \dfrac{\partial g}{\partial x} = -\dfrac{y^2-x^2}{(x^2+y^2)^2} = \dfrac{\partial f}{\partial y}$.

(c) The circle about the origin of radius 1, which is formed by traversing C_1 and then traversing C_2 in the reverse direction, does not lie in an open simply connected region inside which $\mathbf{F}$ is continuous, since $\mathbf{F}$ is not defined at the origin, nor can it be defined there in such a way as to make the resulting function continuous there.

37. If C is composed of smooth curves $C_1, C_2, \ldots, C_n$ and curve C_i extends from (x_{i-1}, y_{i-1}) to (x_i, y_i) then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \sum_{i=1}^n \int_{C_i}\mathbf{F}\cdot d\mathbf{r} = \sum_{i=1}^n [\phi(x_i,y_i) - \phi(x_{i-1},y_{i-1})] = \phi(x_n,y_n) - \phi(x_0,y_0)$, where (x_0,y_0) and (x_n,y_n) are the endpoints of C.

38. $\displaystyle\int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = 0$, but $\displaystyle\int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = -\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$ so $\displaystyle\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$, thus $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.

39. Let C_1 be an arbitrary piecewise smooth curve from (a,b) to a point (x,y_1) in the disk, and C_2 the vertical line segment from (x,y_1) to (x,y). Then $\phi(x,y) = \displaystyle\int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_{(a,b)}^{(x,y_1)}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$. The first term does not depend on y; hence $\dfrac{\partial\phi}{\partial y} = \dfrac{\partial}{\partial y}\displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \dfrac{\partial}{\partial y}\int_{C_2} f(x,y)dx + g(x,y)dy$. However, the line integral with respect to x is zero along C_2, so $\dfrac{\partial\phi}{\partial y} = \dfrac{\partial}{\partial y}\displaystyle\int_{C_2} g(x,y)\ dy$. Express C_2 as $x = x, y = t$ where t varies from y_1 to y, then $\dfrac{\partial\phi}{\partial y} = \dfrac{\partial}{\partial y}\displaystyle\int_{y_1}^{y} g(x,t)\ dt = g(x,y)$.

Exercise Set 15.4

1. $\displaystyle\iint_R (2x-2y)dA = \int_0^1\int_0^1 (2x-2y)dy\,dx = 0$; for the line integral, on $x=0, y^2\,dx = 0, x^2\,dy = 0$; on $y=0, y^2\,dx = x^2\,dy = 0$; on $x=1, y^2\,dx + x^2\,dy = dy$; and on $y=1, y^2\,dx + x^2\,dy = dx$, hence $\displaystyle\oint_C y^2\,dx + x^2\,dy = \int_0^1 dy + \int_1^0 dx = 1-1 = 0$.

2. $\displaystyle\iint_R (1-1)dA = 0$; for the line integral let $x = \cos t, y = \sin t$, $\displaystyle\oint_C y\,dx + x\,dy = \int_0^{2\pi}(-\sin^2 t + \cos^2 t)dt = 0$.

3. $\displaystyle\int_{-2}^4\int_1^2 (2y-3x)dy\ dx = 0$.

4. $\displaystyle\int_0^{2\pi}\int_0^3 (1+2r\sin\theta)r\,dr\,d\theta = 9\pi$.

5. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2} (-y\cos x + x\sin y)dy\,dx = 0$.

6. $\displaystyle\iint_R (\sec^2 x - \tan^2 x)dA = \iint_R dA = \pi$.

7. $\displaystyle\iint_R [1-(-1)]dA = 2\iint_R dA = 8\pi$.

8. $\displaystyle\int_0^1\int_{x^2}^x (2x-2y)dy\,dx = 1/30$.

9. $\displaystyle\iint_R \left(-\frac{y}{1+y} - \frac{1}{1+y}\right) dA = -\iint_R dA = -4$.

10. $\displaystyle\int_0^{\pi/2}\int_0^4 (-r^2)r\,dr\,d\theta = -32\pi$.

11. $\displaystyle\iint_R \left(-\frac{y^2}{1+y^2} - \frac{1}{1+y^2}\right) dA = -\iint_R dA = -1$.

12. $\displaystyle\iint_R (\cos x\cos y - \cos x\cos y)dA = 0$.

13. $\displaystyle\int_0^1\int_{x^2}^{\sqrt{x}} (y^2-x^2)dy\,dx = 0$.

14. (a) $\int_0^2 \int_{x^2}^{2x} (-6x+2y)dy\,dx = -56/15.$ **(b)** $\int_0^2 \int_{x^2}^{2x} 6y\,dy\,dx = 64/5.$

15. False, Green's Theorem applies to closed curves in the plane.

16. False, the first partial derivatives need not even exist.

17. True, it is the area of the region bounded by C.

18. True by Green's Theorem.

19. (a) $C: x = \cos t, y = \sin t, 0 \le t \le 2\pi;\ \oint_C = \int_0^{2\pi} \left(e^{\sin t}(-\sin t) + \sin t \cos t e^{\cos t}\right) dt \approx -3.550999378;$

$$\iint_R \left[\frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}e^y\right] dA = \iint_R [ye^x - e^y]\,dA = \int_0^{2\pi}\int_0^1 \left[r\sin\theta e^{r\cos\theta} - e^{r\sin\theta}\right] r\,dr\,d\theta \approx -3.550999378.$$

(b) $C_1: x = t, y = t^2, 0 \le t \le 1;\ \int_{C_1} [e^y\,dx + ye^x\,dy] = \int_0^1 \left[e^{t^2} + 2t^3 e^t\right] dt \approx 2.589524432,$

$C_2: x = t^2, y = t, 0 \le t \le 1;\ \int_{C_2} [e^y\,dx + ye^x\,dy] = \int_0^1 \left[2te^t + te^{t^2}\right] dt = \frac{e+3}{2} \approx 2.859140914.$

$\int_{C_1} - \int_{C_2} \approx -0.269616482;\ \iint_R = \int_0^1 \int_{x^2}^{\sqrt{x}} [ye^x - e^y]\,dy\,dx \approx -0.269616482.$

20. (a) $\oint_C x\,dy = \int_0^{2\pi} ab\cos^2 t\,dt = \pi ab.$ **(b)** $\oint_C -y\,dx = \int_0^{2\pi} ab\sin^2 t\,dt = \pi ab.$

21. $A = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi}(3a^2\sin^4\phi\cos^2\phi + 3a^2\cos^4\phi\sin^2\phi)d\phi = \frac{3}{2}a^2\int_0^{2\pi}\sin^2\phi\cos^2\phi\,d\phi =$
$= \frac{3}{8}a^2\int_0^{2\pi}\sin^2 2\phi\,d\phi = 3\pi a^2/8.$

22. $C_1 : (0,0)$ to $(a,0); x = at, y = 0, 0 \le t \le 1$, $C_2 : (a,0)$ to $(0,b); x = a - at, y = bt, 0 \le t \le 1$, $C_3 :$ $(0,b)$ to $(0,0); x = 0, y = b - bt, 0 \le t \le 1$, $A = \oint_C x\,dy = \int_0^1 (0)dt + \int_0^1 ab(1-t)dt + \int_0^1 (0)dt = \frac{1}{2}ab.$

23. $C_1 : (0,0)$ to $(a,0)$; $x = at$, $y = 0$, $0 \le t \le 1$, $C_2 : (a,0)$ to $(a\cos t_0, b\sin t_0)$; $x = a\cos t$, $y = b\sin t$, $0 \le t \le t_0$, $C_3 : (a\cos t_0, b\sin t_0)$ to $(0,0)$; $x = -a(\cos t_0)t$, $y = -b(\sin t_0)t$, $-1 \le t \le 0$, $A = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^1 (0)\,dt +$ $\frac{1}{2}\int_0^{t_0} ab\,dt + \frac{1}{2}\int_{-1}^0 (0)\,dt = \frac{1}{2}ab\,t_0.$

24. $C_1 : (0,0)$ to $(a,0)$; $x = at$, $y = 0$, $0 \le t \le 1$, $C_2 : (a,0)$ to $(a\cosh t_0, b\sinh t_0)$; $x = a\cosh t$, $y = b\sinh t$, $0 \le t \le t_0$, $C_3 : (a\cosh t_0, b\sinh t_0)$ to $(0,0)$; $x = -a(\cosh t_0)t$, $y = -b(\sinh t_0)t$, $-1 \le t \le 0$, $A = \frac{1}{2}\oint_C -y\,dx + x\,dy =$ $\frac{1}{2}\int_0^1 (0)\,dt + \frac{1}{2}\int_0^{t_0} ab\,dt + \frac{1}{2}\int_{-1}^0 (0)\,dt = \frac{1}{2}ab\,t_0.$

25. $\operatorname{curl}\mathbf{F}(x,y,z) = (g_x - f_y)\mathbf{k}$, since f and g are independent of z. Thus $\iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA = \iint_R (g_x - f_y)\,dA =$ $\int_C f(x,y)\,dx + g(x,y)\,dy = \int_C \mathbf{F}\cdot d\mathbf{r}$ by Green's Theorem.

26. The boundary of the region R in Figure Ex-22 is $C = C_1 - C_2$, so by Green's Theorem, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$, since $f_y = g_x$. Thus $\int_{C_1} = \int_{C_2}$.

27. Let C_1 denote the graph of $g(x)$ from left to right, and C_2 the graph of $f(x)$ from left to right. On the vertical sides $x =$ const, and so $dx = 0$ there. Thus the area between the curves is $A(R) = \iint\limits_R dA = -\int_C y\,dx = -\int_{C_1} g(x)\,dx + \int_{C_2} f(x)\,dx = -\int_a^b g(x)\,dx + \int_a^b f(x)\,dx = \int_a^b (f(x) - g(x))\,dx.$

28. Let $A(R_1)$ denote the area of the region R_1 bounded by C and the lines $y = y_0, y = y_1$ and the y-axis. Then by Formula (6) $A(R_1) = \int_C x\,dy$, since the integrals on the top and bottom are zero ($dy = 0$ there), and $x = 0$ on the y-axis. Similarly, $A(R_2) = \int_{-C} y\,dx = -\int_C y\,dx$, where R_2 is the region bounded by C, $x = x_0, x = x_1$ and the x-axis.

(a) R_1 **(b)** R_2 **(c)** $\displaystyle\int_C y\,dx + x\,dy = A(R_1) + A(R_2) = x_1y_1 - x_0y_0.$

(d) Let $\phi(x, y) = xy$. Then $\nabla\phi \cdot d\mathbf{r} = y\,dx + x\,dy$ and thus by the Fundamental Theorem $\int_C y\,dx + x\,dy = \phi(x_1, y_1) - \phi(x_0, y_0) = x_1y_1 - x_0y_0$.

(e) $\displaystyle\int_{t_0}^{t_1} x(t)\frac{dy}{dt}\,dt = x(t_1)y(t_1) - x(t_0)y(t_0) - \int_{t_0}^{t_1} y(t)\frac{dx}{dt}\,dt$ which is equivalent to $\displaystyle\int_C y\,dx + x\,dy = x_1y_1 - x_0y_0.$

29. $\displaystyle W = \iint\limits_R y\,dA = \int_0^\pi \int_0^5 r^2 \sin\theta\,dr\,d\theta = 250/3.$

30. We cannot apply Green's Theorem on the region enclosed by the closed curve C, since $\mathbf{F}$ does not have first order partial derivatives at the origin. However, the curve C_{x_0}, consisting of $y = x_0^3/4, x_0 \le x \le 2; x = 2, x_0^3/4 \le y \le 2$; and $y = x^3/4, x_0 \le x \le 2$ encloses a region R_{x_0} in which Green's Theorem does hold, and $\displaystyle W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \lim_{x_0 \to 0^+} \oint_{C_{x_0}} \mathbf{F} \cdot d\mathbf{r} = \lim_{x_0 \to 0^+} \iint\limits_{R_{x_0}} \nabla \cdot \mathbf{F}\,dA = \lim_{x_0 \to 0^+} \int_{x_0}^2 \int_{x_0^3/4}^{x^3/4} \left(\frac{1}{2}x^{-1/2} - \frac{1}{2}y^{-1/2}\right) dy\,dx = \lim_{x_0 \to 0^+} \left(-\frac{18}{35}\sqrt{2} - \frac{\sqrt{2}}{4}x_0^3 + x_0^{3/2} + \frac{3}{14}x_0^{7/2} - \frac{3}{10}x_0^{5/2}\right) = -\frac{18}{35}\sqrt{2}.$

31. $\displaystyle\oint_C y\,dx - x\,dy = \iint\limits_R (-2)dA = -2\int_0^{2\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta = -3\pi a^2.$

32. $\displaystyle\bar{x} = \frac{1}{A}\iint\limits_R x\,dA$, but $\displaystyle\oint_C \frac{1}{2}x^2 dy = \iint\limits_R x\,dA$ from Green's Theorem so $\displaystyle\bar{x} = \frac{1}{A}\oint_C \frac{1}{2}x^2 dy = \frac{1}{2A}\oint_C x^2 dy$. Similarly, $\displaystyle\bar{y} = -\frac{1}{2A}\oint_C y^2 dx.$

33. $\displaystyle A = \int_0^1 \int_{x^3}^x dy\,dx = \frac{1}{4}$; $C_1 : x = t, y = t^3, 0 \le t \le 1$, $\displaystyle\int_{C_1} x^2\,dy = \int_0^1 t^2(3t^2)\,dt = \frac{3}{5}$, $C_2 : x = t, y = t, 0 \le t \le 1$; $\displaystyle\int_{C_2} x^2\,dy = \int_0^1 t^2\,dt = \frac{1}{3}$, $\displaystyle\oint_C x^2\,dy = \int_{C_1} - \int_{C_2} = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}, \bar{x} = \frac{8}{15}$, $\displaystyle\int_C y^2\,dx = \int_0^1 t^6\,dt - \int_0^1 t^2\,dt = \frac{1}{7} - \frac{1}{3} = -\frac{4}{21}, \bar{y} = \frac{8}{21}$, centroid $\left(\frac{8}{15}, \frac{8}{21}\right)$.

34. $A = \dfrac{a^2}{2}; C_1 : x = t, y = 0, 0 \le t \le a, C_2 : x = a - t, y = t, 0 \le t \le a; C_3 : x = 0, y = a - t, 0 \le t \le a; \displaystyle\int_{C_1} x^2\,dy = 0, \int_{C_2} x^2\,dy = \int_0^a (a-t)^2\,dt = \frac{a^3}{3}, \int_{C_3} x^2\,dy = 0, \oint_C x^2\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{a^3}{3},\ \bar{x} = \frac{a}{3}; \int_C y^2\,dx = 0 - \int_0^a t^2\,dt + 0 = -\frac{a^3}{3},\ \bar{y} = \frac{a}{3}$, centroid $\left(\dfrac{a}{3}, \dfrac{a}{3}\right)$.

35. $\bar{x} = 0$ from the symmetry of the region, $C_1 : (a, 0)$ to $(-a, 0)$ along $y = \sqrt{a^2 - x^2}$; $x = a\cos t$, $y = a\sin t$, $0 \le t \le \pi$, $C_2 : (-a, 0)$ to $(a, 0)$; $x = t$, $y = 0$, $-a \le t \le a$, $A = \pi a^2/2$, $\quad \bar{y} = -\dfrac{1}{2A}\left[\displaystyle\int_0^\pi -a^3\sin^3 t\,dt + \int_{-a}^a (0)dt\right] = -\dfrac{1}{\pi a^2}\left(-\dfrac{4a^3}{3}\right) = \dfrac{4a}{3\pi}$; centroid $\left(0, \dfrac{4a}{3\pi}\right)$.

36. $A = \dfrac{ab}{2}; C_1 : x = t, y = 0,\ 0 \le t \le a, C_2 : x = a, y = t,\ 0 \le t \le b; C_3 : x = a - at, y = b - bt,\ 0 \le t \le 1; \displaystyle\int_{C_1} x^2\,dy = 0, \int_{C_2} x^2\,dy = \int_0^b a^2\,dt = ba^2, \int_{C_3} x^2\,dy = \int_0^1 a^2(1-t)^2(-b)\,dt = -\frac{ba^2}{3}, \oint_C x^2\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{2ba^2}{3},\ \bar{x} = \frac{2a}{3}; \int_C y^2\,dx = 0 + 0 - \int_0^1 ab^2(1-t)^2\,dt = -\frac{ab^2}{3},\ \bar{y} = \frac{b}{3}$, centroid $\left(\dfrac{2a}{3}, \dfrac{b}{3}\right)$.

37. From Green's Theorem, the given integral equals $\displaystyle\iint_R (1 - x^2 - y^2)dA$ where R is the region enclosed by C. The value of this integral is maximum if the integration extends over the largest region for which the integrand $1 - x^2 - y^2$ is nonnegative so we want $1 - x^2 - y^2 \ge 0$, $x^2 + y^2 \le 1$. The largest region is that bounded by the circle $x^2 + y^2 = 1$ which is the desired curve C.

38. **(a)** $C : x = a + (c - a)t,\ y = b + (d - b)t,\ 0 \le t \le 1,\ \displaystyle\int_C -y\,dx + x\,dy = \int_0^1 (ad - bc)dt = ad - bc.$

(b) Let C_1, C_2, and C_3 be the line segments from (x_1, y_1) to (x_2, y_2), (x_2, y_2) to (x_3, y_3), and (x_3, y_3) to (x_1, y_1), then if C is the entire boundary consisting of C_1, C_2, and C_3. $A = \dfrac{1}{2}\displaystyle\int_C -y\,dx + x\,dy = \frac{1}{2}\sum_{i=1}^3 \int_{C_i} -y\,dx + x\,dy$
$= \dfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)].$

(c) $A = \dfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)].$

(d) $A = \dfrac{1}{2}[(0 - 0) + (6 + 8) + (0 + 2) + (0 - 0)] = 8.$

39. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (x^2 + y)\,dx + (4x - \cos y)\,dy = 3\iint_R dA = 3(25 - 2) = 69.$

40. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (e^{-x} + 3y)\,dx + x\,dy = -2\iint_R dA = -2[\pi(4)^2 - \pi(2)^2] = -24\pi.$

Exercise Set 15.5

1. R is the annular region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$;

$$\iint_\sigma z^2 dS = \iint_R (x^2 + y^2)\sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1}\,dA = \sqrt{2}\iint_R (x^2 + y^2)dA = \sqrt{2}\int_0^{2\pi}\int_1^2 r^3 dr\,d\theta = \frac{15}{2}\pi\sqrt{2}.$$

2. $z = 1 - x - y$, R is the triangular region enclosed by $x + y = 1$, $x = 0$ and $y = 0$;
$$\iint_\sigma xy\, dS = \iint_R xy\sqrt{3}\, dA = \sqrt{3}\int_0^1 \int_0^{1-x} xy\, dy\, dx = \frac{\sqrt{3}}{24}.$$

3. Let $\mathbf{r}(u,v) = \cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}, 0 \le u \le \pi, 0 \le v \le 1$. Then $\mathbf{r}_u = -\sin u\mathbf{i} + \cos u\mathbf{k}, \mathbf{r}_v = \mathbf{j}, \mathbf{r}_u \times \mathbf{r}_v = -\cos u\mathbf{i} - \sin u\mathbf{k}, \|\mathbf{r}_u \times \mathbf{r}_v\| = 1$, $\displaystyle\iint_\sigma x^2 y\, dS = \int_0^1 \int_0^\pi v\cos^2 u\, du\, dv = \pi/4$.

4. $z = \sqrt{4 - x^2 - y^2}$, R is the circular region enclosed by $x^2 + y^2 = 3$; $\displaystyle\iint_\sigma (x^2 + y^2)z\, dS =$
$$= \iint_R (x^2 + y^2)\sqrt{4 - x^2 - y^2}\sqrt{\frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1}\, dA = \iint_R 2(x^2 + y^2)dA = 2\int_0^{2\pi}\int_0^{\sqrt{3}} r^3 dr\, d\theta = 9\pi.$$

5. If we use the projection of σ onto the xz-plane then $y = 1 - x$ and R is the rectangular region in the xz-plane enclosed by $x = 0$, $x = 1$, $z = 0$ and $z = 1$; $\displaystyle\iint_\sigma (x - y - z)dS = \iint_R (2x - 1 - z)\sqrt{2}dA = \sqrt{2}\int_0^1\int_0^1 (2x - 1 - z)dz\, dx = -\sqrt{2}/2$.

6. R is the triangular region enclosed by $2x + 3y = 6$, $x = 0$, and $y = 0$; $\displaystyle\iint_\sigma (x + y)dS = \iint_R (x + y)\sqrt{14}\, dA =$
$$\sqrt{14}\int_0^3 \int_0^{(6-2x)/3} (x + y)dy\, dx = 5\sqrt{14}.$$

7. There are six surfaces, parametrized by projecting onto planes:
$\sigma_1 : z = 0$; $0 \le x \le 1$, $0 \le y \le 1$ (onto xy-plane), $\sigma_2 : x = 0$; $0 \le y \le 1$, $0 \le z \le 1$ (onto yz-plane),
$\sigma_3 : y = 0$; $0 \le x \le 1$, $0 \le z \le 1$ (onto xz-plane), $\sigma_4 : z = 1$; $0 \le x \le 1$, $0 \le y \le 1$ (onto xy-plane),
$\sigma_5 : x = 1$; $0 \le y \le 1$, $0 \le z \le 1$ (onto yz-plane), $\sigma_6 : y = 1$; $0 \le x \le 1$, $0 \le z \le 1$ (onto xz-plane).
By symmetry the integrals over σ_1, σ_2 and σ_3 are equal, as are those over σ_4, σ_5 and σ_6, and $\displaystyle\iint_{\sigma_1} (x + y + z)dS =$
$\displaystyle\int_0^1\int_0^1 (x + y)dx\, dy = 1$; $\displaystyle\iint_{\sigma_4} (x + y + z)dS = \int_0^1\int_0^1 (x + y + 1)dx\, dy = 2$, thus, $\displaystyle\iint_\sigma (x + y + z)dS = 3\cdot 1 + 3\cdot 2 = 9$.

8. Let $\mathbf{r}(\phi,\theta) = a\sin\phi\cos\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$; $\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = a^2\sin\phi$, $x^2 + y^2 = a^2\sin^2\phi$,
$$\iint_\sigma f(x,y,z) = \int_0^{2\pi}\int_0^\pi a^4\sin^3\phi\, d\phi\, d\theta = \frac{8}{3}\pi a^4.$$

9. True by definition of the integral.

10. False, it could be more on one part and less on another; or $f = 1 + g$ where the integral of g is zero.

11. False, it's the total mass of the lamina.

12. True, Theorem 15.5.3.

13. **(a)** The integral is improper because the function $z(x,y)$ is not differentiable when $x^2 + y^2 = 1$.

(b) Fix r_0, $0 < r_0 < 1$. Then $z + 1 = \sqrt{1 - x^2 - y^2} + 1$, and $\displaystyle\iint_{\sigma_{r_0}} (z + 1)\, dS =$

$$= \iint_{\sigma_{r_0}} (\sqrt{1-x^2-y^2}+1)\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\,dx\,dy = \int_0^{2\pi}\int_0^{r_0}(\sqrt{1-r^2}+1)\frac{1}{\sqrt{1-r^2}}r\,dr\,d\theta =$$

$= 2\pi\left(1+\frac{1}{2}r_0^2-\sqrt{1-r_0^2}\right)$, and, after passing to the limit as $r_0 \to 1$, $\iint_\sigma (z+1)\,dS = 3\pi$.

(c) Let $\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\mathbf{i}+\sin\phi\sin\theta\mathbf{j}+\cos\phi\mathbf{k}, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/2$; $\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = \sin\phi$, $\iint_\sigma (1+\cos\phi)\,dS =$ $\int_0^{2\pi}\int_0^{\pi/2}(1+\cos\phi)\sin\phi\,d\phi\,d\theta = 2\pi\int_0^{\pi/2}(1+\cos\phi)\sin\phi\,d\phi = 3\pi$.

14. (a) The function $z(x,y)$ is not differentiable at the origin (in fact it's partial derivatives are unbounded there).

(b) R is the circular region enclosed by $x^2+y^2=1$; $\iint_\sigma \sqrt{x^2+y^2+z^2}\,dS =$

$= \iint_R \sqrt{2(x^2+y^2)}\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\,dA = \lim_{r_0\to 0^+} 2\iint_{R'} \sqrt{x^2+y^2}\,dA$, where R' is the annular region enclosed by $x^2+y^2=1$ and $x^2+y^2=r_0^2$ with r_0 slightly larger than 0 because $\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}$ is not defined for $x^2+y^2=0$, so $\iint_\sigma \sqrt{x^2+y^2+z^2}\,dS = \lim_{r_0\to 0^+} 2\int_0^{2\pi}\int_{r_0}^1 r^2 dr\,d\theta = \lim_{r_0\to 0^+}\frac{4\pi}{3}(1-r_0^3) = \frac{4\pi}{3}$.

(c) The cone is contained in the locus of points satisfying $\phi = \pi/4$, so it can be parametrized with spherical coordinates ρ,θ: $\mathbf{r}(\rho,\theta) = \frac{1}{\sqrt{2}}\rho\cos\theta\mathbf{i}+\frac{1}{\sqrt{2}}\rho\sin\theta\mathbf{j}+\frac{1}{\sqrt{2}}\rho\mathbf{k}$, $0 \le \theta \le 2\pi$, $r < \rho \le \sqrt{2}$. Then $\mathbf{r}_\rho = \frac{1}{\sqrt{2}}\cos\theta\mathbf{i} + \frac{1}{\sqrt{2}}\sin\theta\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$, $\mathbf{r}_\theta = -\frac{1}{\sqrt{2}}\rho\sin\theta\mathbf{i}+\frac{1}{\sqrt{2}}\rho\cos\theta\,\mathbf{j}$, $\mathbf{r}_\rho\times\mathbf{r}_\theta = \frac{\rho}{2}(-\cos\theta\mathbf{i}-\sin\theta\mathbf{j}+\mathbf{k})$ and $\|\mathbf{r}_\rho\times\mathbf{r}_\theta\| = \frac{1}{\sqrt{2}}\rho$, and thus $\iint_{\sigma_r} f(x,y,z)\,dS = \lim_{r\to 0}\int_0^{2\pi}\int_r^{\sqrt{2}}\rho\frac{1}{\sqrt{2}}\rho\,d\rho\,d\theta = \lim_{r\to 0} 2\pi\frac{1}{\sqrt{2}}\frac{1}{3}\rho^3\Big]_r^{\sqrt{2}} = \lim_{r\to 0}\frac{\sqrt{2}}{3}\left(2\sqrt{2}-r^3\right)\pi = \frac{4\pi}{3}$.

15. (a) Subdivide the right hemisphere $\sigma \cap \{x > 0\}$ into patches, each patch being as small as desired (i). For each patch there is a corresponding patch on the left hemisphere $\sigma \cap \{x < 0\}$ which is the reflection through the yz-plane. Condition (ii) now follows.

(b) Use the patches in Part (a) and the function $f(x,y,z) = x^n$ to define the sum in Definition 15.5.1. The patches of the sum divide into two classes, each the negative of the other since n is odd. Thus the sum adds to zero. Since x^n is a continuous function the limit exists and must also be zero, $\int_\sigma x^n\,dS = 0$.

16. Since g is independent of x it is convenient to say that g is an even function of x, and hence $f(x,y)g(x,y)$ is a continuous odd function of x. Following the argument in Exercise 15, the sum again breaks into two classes, consisting of pairs of patches with the opposite sign. Thus the sum is zero and $\int_\sigma fg\,dS = 0$.

17. (a) Permuting the variables x,y,z by sending $x\to y\to z\to x$ will leave the integrals equal, through symmetry in the variables.

(b) $\iint_\sigma (x^2+y^2+z^2)\,dS$ = surface area of sphere, so each integral contributes one third, i.e. $\iint_\sigma x^2\,dS =$ $\frac{1}{3}\left[\iint_\sigma x^2\,dS + \iint_\sigma y^2\,dS + \iint_\sigma z^2\,dS\right]$.

(c) Since σ has radius 1, $\iint\limits_\sigma dS$ is the surface area of the sphere, which is 4π, therefore $\iint\limits_\sigma x^2\,dS = \frac{4}{3}\pi$.

18. $\iint_\sigma (x-y)^2\,dS = \iint_\sigma x^2\,dS - \iint_\sigma 2xy\,dS + \int_\sigma y^2\,dS = \frac{4}{3}\pi + 0 + \frac{4}{3}\pi = \frac{8}{3}\pi$. The middle integral is zero by Exercise 15 as the integrand is an odd function of x.

19. **(a)** $\frac{\sqrt{29}}{16}\int_0^6\int_0^{(12-2x)/3} xy(12-2x-3y)dy\,dx.$

(b) $\frac{\sqrt{29}}{4}\int_0^3\int_0^{(12-4z)/3} yz(12-3y-4z)dy\,dz.$

(c) $\frac{\sqrt{29}}{9}\int_0^3\int_0^{6-2z} xz(12-2x-4z)dx\,dz.$

20. **(a)** $a\int_0^a\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$ **(b)** $a\int_0^a\int_0^{\sqrt{a^2-z^2}} z\,dy\,dz$ **(c)** $a\int_0^a\int_0^{\sqrt{a^2-z^2}} \frac{xz}{\sqrt{a^2-x^2-z^2}}dx\,dz$

21. $18\sqrt{29}/5$.

22. $a^4/3$.

23. $\int_0^4\int_1^2 y^3z\sqrt{4y^2+1}\,dy\,dz;\ \frac{1}{2}\int_0^4\int_1^4 xz\sqrt{1+4x}\,dx\,dz.$

24. $a\int_0^9\int_{a/\sqrt{5}}^{a/\sqrt{2}} \frac{x^2y}{\sqrt{a^2-y^2}}\,dy\,dx,\ a\int_{a/\sqrt{2}}^{2a/\sqrt{5}}\int_0^9 x^2dx\,dz$

25. $391\sqrt{17}/15 - 5\sqrt{5}/3$.

26. The region $R: 3x^2+2y^2=5$ is symmetric in y. The integrand is $x^2yz\,dS = x^2y(5-3x^2-2y^2)\sqrt{1+36x^2+16y^2}\,dy\,dx$, which is odd in y, hence $\iint\limits_\sigma x^2yz\,dS = 0$.

27. $z=\sqrt{4-x^2}$, $\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{4-x^2}}$, $\frac{\partial z}{\partial y}=0$; $\iint\limits_\sigma \delta_0 dS = \delta_0\iint\limits_R \sqrt{\frac{x^2}{4-x^2}+1}\,dA = 2\delta_0\int_0^4\int_0^1 \frac{1}{\sqrt{4-x^2}}dx\,dy = \frac{4}{3}\pi\delta_0.$

28. $z=\frac{1}{2}(x^2+y^2)$, R is the circular region enclosed by $x^2+y^2=8$; $\iint\limits_\sigma \delta_0 dS = \delta_0\iint\limits_R\sqrt{x^2+y^2+1}\,dA = \delta_0\int_0^{2\pi}\int_0^{\sqrt{8}}\sqrt{r^2+1}\,r\,dr\,d\theta = \frac{52}{3}\pi\delta_0.$

29. $z=4-y^2$, R is the rectangular region enclosed by $x=0$, $x=3$, $y=0$ and $y=3$; $\iint\limits_\sigma y\,dS = \iint\limits_R y\sqrt{4y^2+1}\,dA = \int_0^3\int_0^3 y\sqrt{4y^2+1}\,dy\,dx = \frac{1}{4}(37\sqrt{37}-1).$

30. R is the annular region enclosed by $x^2+y^2=1$ and $x^2+y^2=16$; $\iint\limits_\sigma x^2z\,dS =$

$$= \iint_R x^2\sqrt{x^2+y^2}\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\,dA = \sqrt{2}\iint_R x^2\sqrt{x^2+y^2}\,dA = \sqrt{2}\int_0^{2\pi}\int_1^4 r^4\cos^2\theta\,dr\,d\theta = \frac{1023\sqrt{2}}{5}\pi.$$

31. $M = \iint_\sigma \delta(x,y,z)dS = \iint_\sigma \delta_0 dS = \delta_0 \iint_\sigma dS = \delta_0 S.$

32. $\delta(x,y,z) = |z|$; use $z = \sqrt{a^2-x^2-y^2}$, let R be the circular region enclosed by $x^2+y^2=a^2$, and σ the hemisphere above R. By the symmetry of both the surface and the density function with respect to the xy-plane we have $M = 2\iint_\sigma z\,dS = 2\iint_R \sqrt{a^2-x^2-y^2}\sqrt{\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}+1}\,dA = \lim_{r_0\to a^-} 2a\iint_{R_{r_0}} dA$ where R_{r_0} is the circular region with radius r_0 that is slightly less than a. But $\iint_{R_{r_0}} dA$ is simply the area of the circle with radius r_0 so $M = \lim_{r_0\to a^-} 2a(\pi r_0^2) = 2\pi a^3$.

33. By symmetry $\bar{x} = \bar{y} = 0$. $\iint_\sigma dS = \iint_R \sqrt{x^2+y^2+1}\,dA = \int_0^{2\pi}\int_0^{\sqrt{8}} \sqrt{r^2+1}\,r\,dr\,d\theta = \frac{52\pi}{3}$, $\iint_\sigma z\,dS = \iint_R z\sqrt{x^2+y^2+1}\,dA = \frac{1}{2}\iint_R (x^2+y^2)\sqrt{x^2+y^2+1}\,dA = \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{8}} r^3\sqrt{r^2+1}\,dr\,d\theta = \frac{596\pi}{15}$, so $\bar{z} = \frac{596\pi/15}{52\pi/3} = \frac{149}{65}$. The centroid is $(\bar{x},\bar{y},\bar{z}) = (0,0,149/65)$.

34. By symmetry $\bar{x} = \bar{y} = 0$. $\iint_\sigma dS = \iint_R \frac{2}{\sqrt{4-x^2-y^2}}dA = 2\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r}{\sqrt{4-r^2}}dr\,d\theta = 4\pi$, $\iint_\sigma z\,dS = \iint_R 2\,dA = (2)(\text{area of circle of radius } \sqrt{3}) = 6\pi$, so $\bar{z} = \frac{6\pi}{4\pi} = \frac{3}{2}$. The centroid is $(\bar{x},\bar{y},\bar{z}) = (0,0,3/2)$.

35. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i} + \sin v\mathbf{j} + 3\mathbf{k}, \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$, $\|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = \sqrt{10}u$; $3\sqrt{10}\iint_R u^4\sin v\cos v\,dA = 3\sqrt{10}\int_0^{\pi/2}\int_1^2 u^4\sin v\cos v\,du\,dv = 93/\sqrt{10}$.

36. $\partial\mathbf{r}/\partial u = \mathbf{j}, \partial\mathbf{r}/\partial v = -2\sin v\mathbf{i} + 2\cos v\mathbf{k}$, $\|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = 2$; $8\iint_R \frac{1}{u}dA = 8\int_0^{2\pi}\int_1^3 \frac{1}{u}du\,dv = 16\pi\ln 3$.

37. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i} + \sin v\mathbf{j} + 2u\mathbf{k}, \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$, $\|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = u\sqrt{4u^2+1}$; $\iint_R u\,dA =$

$$= \int_0^\pi\int_0^{\sin v} u\,du\,dv = \pi/4.$$

38. $\partial\mathbf{r}/\partial u = 2\cos u\cos v\mathbf{i} + 2\cos u\sin v\mathbf{j} - 2\sin u\mathbf{k}$, $\partial\mathbf{r}/\partial v = -2\sin u\sin v\mathbf{i} + 2\sin u\cos v\mathbf{j}$; $\|\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v\| = 4\sin u$; $4\iint_R e^{-2\cos u}\sin u\,dA = 4\int_0^{2\pi}\int_0^{\pi/2} e^{-2\cos u}\sin u\,du\,dv = 4\pi(1-e^{-2})$.

39. $\partial z/\partial x = -2xe^{-x^2-y^2}, \partial z/\partial y = -2ye^{-x^2-y^2}$, $(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1 = 4(x^2+y^2)e^{-2(x^2+y^2)} + 1$; use polar coordinates to get $M = \int_0^{2\pi}\int_0^3 r^2\sqrt{4r^2e^{-2r^2}+1}\,dr\,d\theta \approx 57.895751$.

40. **(b)** $A = \iint_{\sigma} dS = \int_0^{2\pi}\int_{-1}^{1} \frac{1}{2}\sqrt{40u\cos(v/2) + u^2 + 4u^2\cos^2(v/2) + 100}\,du\,dv \approx 62.93768644$; $\bar{x} = \frac{1}{A}\iint_{\sigma} x\,dS \approx 0.016302$; $\bar{y} = \bar{z} = 0$ by symmetry.

Exercise Set 15.6

1. **(a)** Zero. **(b)** Zero. **(c)** Positive. **(d)** Negative. **(e)** Zero. **(f)** Zero.

2. 0; the vector field is constant, so when we compute the flux, for any given contribution on one face of the cube, we will get the same contribution with a minus sign on the opposite face. (Whatever "flows in" on one side "flows out" on the opposite, so the total flux is 0.)

3. The vector field is constant $-5\mathbf{i}$ on a square like this, $\mathbf{n} = \mathbf{i}$, so $\iint_R \mathbf{F}\cdot\mathbf{n}\,dS = -5A(R) = -80$.

4. $\mathbf{n} = -\mathbf{j}$, so $\mathbf{F}\cdot\mathbf{n} = -1$, $\iint_R \mathbf{F}\cdot\mathbf{n}\,dS = -A(R) = -25$.

5. $\mathbf{n} = \mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = 5$, so the flux is $5A(R) = 5\cdot 6 = 30$.

6. $\mathbf{n} = \mathbf{j}$, so $\mathbf{F}\cdot\mathbf{n} = 3$, so the flux is $3A(R) = 3\cdot 25\pi = 75\pi$.

7. $\mathbf{n} = \mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = 8$, so the flux is $8A(R) = 8\cdot 25\pi = 200\pi$.

8. **(a)** $\mathbf{n} = -\cos v\mathbf{i} - \sin v\mathbf{j}$. **(b)** Inward, by inspection.

9. $\mathbf{n} = \mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = x$, $\iint_R \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^2\int_0^2 x\,dx\,dy = 4$.

10. $\mathbf{n} = \mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = 2x$, $\iint_R \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^3\int_0^2 2x\,dx\,dy = 12$.

11. $\mathbf{n} = -z_x\mathbf{i} - z_y\mathbf{j} + \mathbf{k}$, $\iint_R \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (2x^2 + 2y^2 + 2(1 - x^2 - y^2))\,dS = \int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = 2\pi$.

12. $\mathbf{n} = -\mathbf{j}$, so $\mathbf{F}\cdot\mathbf{n} = -(x + e^{-1})$, $\iint_R \mathbf{F}\cdot\mathbf{n}\,dS = -\int_0^4\int_0^2 x + e^{-1}\,dx\,dz = -8 - 8/e$.

13. R is the annular region enclosed by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$; $\iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS =$

$$= \iint_R \left(-\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + 2z\right) dA = \iint_R \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_1^2 r^2 dr\,d\theta = \frac{14\pi}{3}.$$

14. R is the circular region enclosed by $x^2 + y^2 = 4$; $\iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (2y^2 - 1)dA = \int_0^{2\pi}\int_0^2 (2r^2\sin^2\theta - 1)r\,dr\,d\theta = 4\pi$.

15. R is the circular region enclosed by $x^2+y^2-y=0$; $\iint\limits_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_{R}(-x)dA = 0$ since the region R is symmetric across the y-axis.

16. With $z=\dfrac{1}{2}(6-6x-3y)$, R is the triangular region enclosed by $2x+y=2$, $x=0$, and $y=0$; $\iint\limits_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS =$ $\iint\limits_{R}\left(3x^2+\dfrac{3}{2}yx+zx\right)dA = 3\iint\limits_{R} x\,dA = 3\int_0^1\int_0^{2-2x} x\,dy\,dx = 1.$

17. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i}+\sin v\mathbf{j}-2u\mathbf{k}, \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i}+u\cos v\mathbf{j}$, $\partial\mathbf{r}/\partial u\times\partial\mathbf{r}/\partial v = 2u^2\cos v\mathbf{i}+2u^2\sin v\mathbf{j}+u\mathbf{k}$; $\iint\limits_{R}(2u^3+u)\,dA = \int_0^{2\pi}\int_1^2(2u^3+u)du\,dv = 18\pi.$

18. $\partial\mathbf{r}/\partial u = \mathbf{k}, \partial\mathbf{r}/\partial v = -2\sin v\mathbf{i}+\cos v\mathbf{j}$, $\partial\mathbf{r}/\partial u\times\partial\mathbf{r}/\partial v = -\cos v\mathbf{i}-2\sin v\mathbf{j}$; $\iint\limits_{R}(2\sin^2 v - e^{-\sin v}\cos v)\,dA =$ $\int_0^{2\pi}\int_0^5(2\sin^2 v - e^{-\sin v}\cos v)du\,dv = 10\pi.$

19. $\partial\mathbf{r}/\partial u = \cos v\mathbf{i}+\sin v\mathbf{j}+2\mathbf{k}, \partial\mathbf{r}/\partial v = -u\sin v\mathbf{i}+u\cos v\mathbf{j}$, $\partial\mathbf{r}/\partial u\times\partial\mathbf{r}/\partial v = -2u\cos v\mathbf{i}-2u\sin v\mathbf{j}+u\mathbf{k}$; $\iint\limits_{R} u^2\,dA =$ $\int_0^{\pi}\int_0^{\sin v} u^2du\,dv = 4/9.$

20. $\partial\mathbf{r}/\partial u = 2\cos u\cos v\mathbf{i}+2\cos u\sin v\mathbf{j}-2\sin u\mathbf{k}$, $\partial\mathbf{r}/\partial v = -2\sin u\sin v\mathbf{i}+2\sin u\cos v\mathbf{j}$; $\partial\mathbf{r}/\partial u\times\partial\mathbf{r}/\partial v = 4\sin^2 u\cos v\mathbf{i}+$ $4\sin^2 u\sin v\mathbf{j}+4\sin u\cos u\mathbf{k}$; $\iint\limits_{R} 8\sin u\,dA = 8\int_0^{2\pi}\int_0^{\pi/3}\sin u\,du\,dv = 8\pi.$

21. In each part, divide σ into the six surfaces

$\sigma_1: x=-1$ with $|y|\le 1$, $|z|\le 1$, and $\mathbf{n}=-\mathbf{i}$, $\sigma_2: x=1$ with $|y|\le 1$, $|z|\le 1$, and $\mathbf{n}=\mathbf{i}$,

$\sigma_3: y=-1$ with $|x|\le 1$, $|z|\le 1$, and $\mathbf{n}=-\mathbf{j}$, $\sigma_4: y=1$ with $|x|\le 1$, $|z|\le 1$, and $\mathbf{n}=\mathbf{j}$,

$\sigma_5: z=-1$ with $|x|\le 1$, $|y|\le 1$, and $\mathbf{n}=-\mathbf{k}$, $\sigma_6: z=1$ with $|x|\le 1$, $|y|\le 1$, and $\mathbf{n}=\mathbf{k}$,

(a) $\iint\limits_{\sigma_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_{\sigma_1}dS = 4$, $\iint\limits_{\sigma_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_{\sigma_2}dS = 4$, and $\iint\limits_{\sigma_i}\mathbf{F}\cdot\mathbf{n}\,dS = 0$ for $i=3,4,5,6$, so $\iint\limits_{\sigma}\mathbf{F}\cdot\mathbf{n}\,dS = 4+4+0+0+0+0=8.$

(b) $\iint\limits_{\sigma_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_{\sigma_1}dS = 4$, similarly $\iint\limits_{\sigma_i}\mathbf{F}\cdot\mathbf{n}\,dS = 4$ for $i=2,3,4,5,6$, so $\iint\limits_{\sigma}\mathbf{F}\cdot\mathbf{n}\,dS = 4+4+4+4+4+4 =$ 24.

(c) $\iint\limits_{\sigma_1}\mathbf{F}\cdot\mathbf{n}\,dS = -\iint\limits_{\sigma_1}dS = -4$, $\iint\limits_{\sigma_2}\mathbf{F}\cdot\mathbf{n}\,dS = 4$, similarly $\iint\limits_{\sigma_i}\mathbf{F}\cdot\mathbf{n}\,dS = -4$ for $i=3,5$ and $\iint\limits_{\sigma_i}\mathbf{F}\cdot\mathbf{n}\,dS =$ 4 for $i=4,6$, so $\iint\limits_{\sigma}\mathbf{F}\cdot\mathbf{n}\,dS = -4+4-4+4-4+4=0.$

22. Decompose σ into a top σ_1 (the disk) and a bottom σ_2 (the portion of the paraboloid). Then $\mathbf{n}_1 = \mathbf{k}$, $\iint\limits_{\sigma_1} \mathbf{F} \cdot \mathbf{n}_1\, dS =$ $-\iint\limits_{\sigma_1} y\, dS = -\int_0^{2\pi}\int_0^1 r^2 \sin\theta\, dr\, d\theta = 0, \mathbf{n}_2 = (2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k})/\sqrt{1+4x^2+4y^2}$, $\iint\limits_{\sigma_2} \mathbf{F} \cdot \mathbf{n}_2\, dS =$ $= \iint\limits_{\sigma_2} \dfrac{y(2x^2+2y^2+1)}{\sqrt{1+4x^2+4y^2}}\, dS = 0$, because the surface σ_2 is symmetric with respect to the xy-plane and the integrand is an odd function of y. Thus the flux is 0.

23. False, the Möbius strip is not orientable.

24. False, the flux is a scalar.

25. False, the value can be zero if as much liquid passes in the negative direction as in the positive.

26. True, it is $\iint\limits_{\sigma} \mathbf{n} \cdot \mathbf{n}\, dS = \iint\limits_{\sigma} dS = A(\sigma)$.

27. The surface is parametrized by $x = u\cos v, y = u\sin v, z = u, 1 \le u \le 2,\ 0 \le v \le 2\pi$. $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} = -(u\cos v\mathbf{i} + u\sin v\mathbf{j} - u\mathbf{k})$; $\iint\limits_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{R} u\,(\cos v + \sin v - 1)\, dA = \int_0^{2\pi}\int_1^2 (\cos v + \sin v - 1)\, u\, du\, dv = -3\pi$.

28. The surface is parametrized by $x = 4\cos u, y = 4\sin u, z = v, 0 \le u \le 2\pi,\ -2 \le v \le 2$. $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} = (4\cos u\mathbf{i} + 4\sin u\mathbf{j})$; $\iint\limits_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{R} 12\cos u + 28\sin u dA = \int_0^{2\pi}\int_{-2}^2 12\cos u + 28\sin u\, dv\, du = 0$.

29. **(a)** $\mathbf{n} = \dfrac{1}{\sqrt{3}}[\mathbf{i} + \mathbf{j} + \mathbf{k}],\ V = \int_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1\int_0^{1-x} (2x - 3y + 1 - x - y)\, dy\, dx = 0\ \text{m}^3/\text{s}$.

(b) $m = 0 \cdot 806 = 0$ kg/s.

30. **(a)** Let $x = 3\sin\phi\cos\theta, y = 3\sin\phi\sin\theta, z = 3\cos\phi,\ \mathbf{n} = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, so $V = \int_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS =$ $\iint\limits_{A} 9\sin\phi\,(-3\sin^2\phi\sin\theta\cos\theta + 3\sin\phi\cos\phi\sin\theta + 9\sin\phi\cos\phi\cos\theta)\, dA = \int_0^{2\pi}\int_0^3 3\sin\phi\cos\theta(-\sin\phi\sin\theta + 4\cos\phi)\, r\, dr\, d\theta = 0\ \text{m}^3$.

(b) $\dfrac{dm}{dt} = 0 \cdot 1060 = 0$ kg/s.

31. **(a)** $G(x,y,z) = x - g(y,z),\ \nabla G = \mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} - \dfrac{\partial g}{\partial z}\mathbf{k}$, apply Theorem 15.6.3: $\iint\limits_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint\limits_{R} \mathbf{F} \cdot \left(\mathbf{i} - \dfrac{\partial x}{\partial y}\mathbf{j} - \dfrac{\partial x}{\partial z}\mathbf{k}\right) dA$, if σ is oriented by front normals, and $\iint\limits_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint\limits_{R} \mathbf{F} \cdot \left(-\mathbf{i} + \dfrac{\partial x}{\partial y}\mathbf{j} + \dfrac{\partial x}{\partial z}\mathbf{k}\right) dA$, if σ is oriented by back normals, where R is the projection of σ onto the yz-plane.

(b) R is the semicircular region in the yz-plane enclosed by $z = \sqrt{1-y^2}$ and $z = 0$; $\iint\limits_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{R} (-y -$

$2yz + 16z)dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}(-y - 2yz + 16z)dz\,dy = \dfrac{32}{3}.$

32. **(a)** $G(x,y,z) = y - g(x,z)$, $\nabla G = -\dfrac{\partial g}{\partial x}\mathbf{i} + \mathbf{j} - \dfrac{\partial g}{\partial z}\mathbf{k}$, apply Theorem 15.6.3: $\iint\limits_R \mathbf{F}\cdot\left(\dfrac{\partial y}{\partial x}\mathbf{i} - \mathbf{j} + \dfrac{\partial y}{\partial z}\mathbf{k}\right)dA$, σ oriented by left normals, and $\iint\limits_R \mathbf{F}\cdot\left(-\dfrac{\partial y}{\partial x}\mathbf{i} + \mathbf{j} - \dfrac{\partial y}{\partial z}\mathbf{k}\right)dA$, σ oriented by right normals, where R is the projection of σ onto the xz-plane.

(b) R is the semicircular region in the xz-plane enclosed by $z = \sqrt{1-x^2}$ and $z = 0$; $\iint\limits_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R (-2x^2 + (x^2+z^2) - 2z^2)dA = -\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}(x^2+z^2)dz\,dx = -\dfrac{\pi}{4}.$

33. **(a)** On the sphere, $\|\mathbf{r}\| = a$ so $\mathbf{F} = a^k\mathbf{r}$ and $\mathbf{F}\cdot\mathbf{n} = a^k\mathbf{r}\cdot(\mathbf{r}/a) = a^{k-1}\|\mathbf{r}\|^2 = a^{k-1}a^2 = a^{k+1}$, hence $\iint\limits_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = a^{k+1}\iint\limits_\sigma dS = a^{k+1}(4\pi a^2) = 4\pi a^{k+3}$.

(b) If $k = -3$, then $\iint\limits_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi$.

34. Let $\mathbf{r} = \sin u\cos v\mathbf{i} + \sin u\sin v\mathbf{j} + \cos u\mathbf{k}, \mathbf{r}_u\times\mathbf{r}_v = \sin^2 u\cos v\mathbf{i} + \sin^2 u\sin v\mathbf{j} + \sin u\cos u\mathbf{k}, \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v) = a^2\sin^3 u\cos^2 v + \dfrac{1}{a}\sin^3 u\sin^2 v + a\sin u\cos^3 u$,

$\iint\limits_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\left(a^2\sin^3 u\cos^2 v + \dfrac{1}{a}\sin^3 u\sin^2 v + a\sin u\cos^3 u\right)du\,dv = \dfrac{4}{3a}\int_0^{\pi}(a^3\cos^2 v + \sin^2 v)\,dv = \dfrac{4\pi}{3}\left(a^2 + \dfrac{1}{a}\right) = 3\pi$ if $a = \dfrac{1}{2}, \dfrac{-1\pm\sqrt{33}}{4}$.

35. Let $\mathbf{r} = a\sin u\cos v\mathbf{i} + a\sin u\sin v\mathbf{j} + a\cos u\mathbf{k}, \mathbf{r}_u\times\mathbf{r}_v = a^2\sin^2 u\cos v\mathbf{i} + a^2\sin^2 u\sin v\mathbf{j} + a^2\sin u\cos u\mathbf{k}, \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v) = \left(\dfrac{6}{a}+1\right)a^3\sin^3 u\cos^2 v - 4a^4\sin^3 u\sin^2 v + a^5\sin u\cos^2 u$,

$\iint\limits_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\left(\left(\dfrac{6}{a}+1\right)a^3\sin^3 u\cos^2 v - 4a^4\sin^3 u\sin^2 v + a^5\sin u\cos^2 u\right)du\,dv =$

$= \dfrac{4}{3}a^2(6 + a - 4a^2 + a^3)\pi = 0$ if $a = -1, 2, 3$. We discard $a = -1$ because a is the radius of the sphere.

Exercise Set 15.7

1. $\sigma_1 : x = 0, \mathbf{F}\cdot\mathbf{n} = -x = 0, \iint\limits_{\sigma_1}(0)dA = 0,$ $\quad\sigma_2 : x = 1, \mathbf{F}\cdot\mathbf{n} = x = 1, \iint\limits_{\sigma_2}(1)dA = 1,$

$\sigma_3 : y = 0, \mathbf{F}\cdot\mathbf{n} = -y = 0, \iint\limits_{\sigma_3}(0)dA = 0,$ $\quad\sigma_4 : y = 1, \mathbf{F}\cdot\mathbf{n} = y = 1, \iint\limits_{\sigma_4}(1)dA = 1,$

$\sigma_5 : z = 0, \mathbf{F}\cdot\mathbf{n} = -z = 0, \iint\limits_{\sigma_5}(0)dA = 0,$ $\quad\sigma_6 : z = 1, \mathbf{F}\cdot\mathbf{n} = z = 1, \iint\limits_{\sigma_6}(1)dA = 1.$

$\iint\limits_\sigma \mathbf{F}\cdot\mathbf{n} = 3;\ \iiint\limits_G \operatorname{div}\mathbf{F}dV = \iiint\limits_G 3dV = 3.$

2. Let $\mathbf{r} = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + \cos u\mathbf{k}, \mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}, \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 5\sin^2 u \sin v + 7\sin u \cos u$,

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi} 5\sin^2 u \sin v + 7\sin u\cos u\,du\,dv = 0;\ \iiint_G \text{div }\mathbf{F}dV = \iiint_G 0dV = 0.$$

3. $\sigma_1 : z = 1, \mathbf{n} = \mathbf{k}, \mathbf{F}\cdot\mathbf{n} = z^2 = 1, \iint_{\sigma_1}(1)dS = \pi,\ \sigma_2 : \mathbf{n} = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}, \mathbf{F}\cdot\mathbf{n} = 4x^2 - 4x^2y^2 - x^4 - 3y^4, \iint_{\sigma_2}\mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^1 \left[4r^2\cos^2\theta - 4r^4\cos^2\theta\sin^2\theta - r^4\cos^4\theta - 3r^4\sin^4\theta\right] r\,dr\,d\theta = \frac{\pi}{3}$;

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \frac{4\pi}{3};\ \iiint_G \text{div }\mathbf{F}dV = \iiint_G (2+z)dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 (2+z)dz\,r\,dr\,d\theta = 4\pi/3.$$

4. $\sigma_1 : x = 0, \mathbf{F}\cdot\mathbf{n} = -xy = 0, \iint_{\sigma_1}(0)dA = 0,\quad \sigma_2 : x = 2, \mathbf{F}\cdot\mathbf{n} = xy = 2y, \iint_{\sigma_2}(2y)dA = 8,$

$\sigma_3 : y = 0, \mathbf{F}\cdot\mathbf{n} = -yz = 0, \iint_{\sigma_3}(0)dA = 0,\quad \sigma_4 : y = 2, \mathbf{F}\cdot\mathbf{n} = yz = 2z, \iint_{\sigma_4}(2z)dA = 8,$

$\sigma_5 : z = 0, \mathbf{F}\cdot\mathbf{n} = -xz = 0, \iint_{\sigma_5}(0)dA = 0,\quad \sigma_6 : z = 2, \mathbf{F}\cdot\mathbf{n} = xz = 2x, \iint_{\sigma_6}(2x)dA = 8.$

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n} = 24;\ \iiint_G \text{div }\mathbf{F}dV = \iiint_G (y+z+x)dV = 24.$$

5. False, it equates a surface integral with a volume integral.

6. True, the integral of a positive function is positive.

7. True, see the subsection "Sources and Sinks".

8. False, see Theorem 15.7.2.

9. G is the rectangular solid; $\iiint_G \text{div }\mathbf{F}\,dV = \int_0^2\int_0^1\int_0^3 (2x-1)\,dx\,dy\,dz = 12.$

10. G is the spherical solid enclosed by σ; $\iiint_G \text{div }\mathbf{F}\,dV = \iiint_G 0\,dV = 0\iiint_G dV = 0.$

11. G is the cylindrical solid; $\iiint_G \text{div }\mathbf{F}\,dV = 3\iiint_G dV = (3)(\text{volume of cylinder}) = (3)[\pi a^2(1)] = 3\pi a^2.$

12. G is the solid bounded by $z = 1 - x^2 - y^2$ and the xy-plane; $\iiint_G \text{div }\mathbf{F}\,dV = 3\iiint_G dV = 3\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = \frac{3\pi}{2}$.

13. G is the cylindrical solid; $\iiint_G \text{div }\mathbf{F}\,dV = 3\iiint_G (x^2+y^2+z^2)dV = 3\int_0^{2\pi}\int_0^2\int_0^3 (r^2+z^2)r\,dz\,dr\,d\theta = 180\pi.$

14. G is the tetrahedron; $\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = \iiint\limits_G x\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx = \dfrac{1}{24}$.

15. G is the hemispherical solid bounded by $z = \sqrt{4 - x^2 - y^2}$ and the xy-plane; $\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = 3 \iiint\limits_G (x^2 + y^2 + z^2) dV = 3 \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta = \dfrac{192\pi}{5}$.

16. G is the hemispherical solid; $\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = 5 \iiint\limits_G z\, dV = 5 \int_0^{2\pi} \int_0^{\pi/2} \int_0^a \rho^3 \sin\phi \cos\phi\, d\rho\, d\phi\, d\theta = \dfrac{5\pi a^4}{4}$.

17. G is the conical solid; $\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = 2 \iiint\limits_G (x + y + z) dV = 2 \int_0^{2\pi} \int_0^1 \int_r^1 (r\cos\theta + r\sin\theta + z) r\, dz\, dr\, d\theta = \dfrac{\pi}{2}$.

18. G is the solid bounded by $z = 2x$ and $z = x^2 + y^2$; $\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = \iiint\limits_G dV = 2 \int_0^{\pi/2} \int_0^{2\cos\theta} \int_{r^2}^{2r\cos\theta} r\, dz\, dr\, d\theta = \dfrac{\pi}{2}$.

19. From the examples we see that we need to integrate the divergence of the vector field over the whole volume. The former is $\operatorname{div} \mathbf{F} = 4x^2$. Finding the volume is more difficult. We are looking for all the points in the x-y plane that lie above the plane $z = 0$, and that lie i) underneath the surface $z = 4 - x^2$, (this surface passes through the parabola $z = 4 - x^2$ in the x-z plane and has no restriction on y); and ii) lie underneath the plane $z = 5 - y$ together with the condition $y \geq 0$. First we observe that $-2 \leq x \leq 2$. Given such a value of x, we note that $0 \leq y \leq 5$. Finally, $0 \leq z \leq \min(5 - y, 4 - x^2)$. And which is the minimum? We check the boundary, where one z equals the other z, i.e. $4 - x^2 = 5 - y$, or the parabola $y = x^2 + 1$ in the x-y plane. Thus $\iiint\limits_G \operatorname{div} \mathbf{F}\, dV = 4 \iiint\limits_G x^2 dV = \int_{-2}^2 \int_0^{1+x^2} \int_0^{4-x^2} 4x^2 dz\, dy\, dx + \int_{-2}^2 \int_{x^2+1}^5 \int_0^{5-y} 4x^2\, dz\, dy\, dx = \dfrac{4608}{35}$.

20. $\iint\limits_\sigma \mathbf{r} \cdot \mathbf{n}\, dS = \iiint\limits_G \operatorname{div} \mathbf{r}\, dV = 3 \iiint\limits_G dV = 3\text{vol}(G)$.

21. $\iint\limits_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = 3[\pi(3^2)(5)] = 135\pi$.

22. $\iint\limits_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_G \operatorname{div} \mathbf{F}\, dV = \iiint\limits_G 0\, dV = 0$; since the vector field is constant, the same amount enters as leaves.

23. **(a)** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\operatorname{div} \mathbf{F} = 3$. **(b)** $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$, $\operatorname{div} \mathbf{F} = -3$.

24. **(a)** The flux through any cylinder whose axis is the z-axis is positive by inspection; by the Divergence Theorem, this says that the divergence cannot be negative at the origin, else the flux through a small enough cylinder would also be negative (impossible), hence the divergence at the origin must be ≥ 0.

(b) Similar to part (a), ≤ 0.

25. $0 = \iiint\limits_R \operatorname{div} \mathbf{F}\, dV = \iint\limits_\sigma \mathbf{F} \cdot \mathbf{n}\, dS$. Let σ_1 denote that part of σ on which $\mathbf{F} \cdot \mathbf{n} > 0$ and let σ_2 denote the part where $\mathbf{F}\cdot\mathbf{n}<0$. If $\iint\limits_{\sigma_1} \mathbf{F}\cdot\mathbf{n}>0$ then the integral over σ_2 is negative (and equal in magnitude). Thus the boundary between σ_1 and σ_2 is infinite, hence $\mathbf{F}$ and $\mathbf{n}$ are perpendicular on an infinite set.

26. No; the argument in Exercise 25 rests on the assumption that $\mathbf{F}\cdot\mathbf{n}$ is continuous, which may not be true on a cube because the tangent jumps from one value to the next. Let $\phi(x,y,z)=xy+xz+yz+x+y+z$, so $\mathbf{F}=\nabla\phi=(y+z+1)\mathbf{i}+(x+z+1)\mathbf{j}+(x+y+1)\mathbf{k}$. On each side of the cube we must show $\mathbf{F}\cdot\mathbf{n}\neq 0$. On the face where $x=0$, for example, $\mathbf{F}\cdot\mathbf{n}=-(y+z+1)\le -1<0$, and on the face where $x=1, \mathbf{F}\cdot\mathbf{n}=y+z+1\ge 1>0$. The other faces can be treated in a similar manner.

27. $\iint\limits_{\sigma} \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS=\iiint\limits_{G}\text{div}(\text{curl } \mathbf{F})dV=\iiint\limits_{G}(0)dV=0.$

28. $\iint\limits_{\sigma} \nabla f\cdot\mathbf{n}\,dS=\iiint\limits_{G}\text{div }(\nabla f)dV=\iiint\limits_{G}\nabla^2 f dV.$

29. $\iint\limits_{\sigma}(f\nabla g)\cdot\mathbf{n}\,dS=\iiint\limits_{G}\text{div }(f\nabla g)dV=\iiint\limits_{G}(f\nabla^2 g+\nabla f\cdot\nabla g)dV$ by Exercise 31, Section 15.1.

30. $\iint\limits_{\sigma}(f\nabla g)\cdot\mathbf{n}\,dS=\iiint\limits_{G}(f\nabla^2 g+\nabla f\cdot\nabla g)dV$ by Exercise 29; $\iint\limits_{\sigma}(g\nabla f)\cdot\mathbf{n}\,dS=\iiint\limits_{G}(g\nabla^2 f+\nabla g\cdot\nabla f)dV$ by interchanging f and g; subtract to obtain the result.

31. Since $\mathbf{v}$ is constant, $\nabla\cdot\mathbf{v}=\mathbf{0}$. Let $\mathbf{F}=f\mathbf{v}$; then div$\mathbf{F}=(\nabla f)\mathbf{v}$ and by the Divergence Theorem $\iint\limits_{\sigma} f\mathbf{v}\cdot\mathbf{n}\,dS=$ $\iint\limits_{\sigma}\mathbf{F}\cdot\mathbf{n}\,dS=\iiint\limits_{G}\text{div}\mathbf{F}\,dV=\iiint\limits_{G}(\nabla f)\cdot\mathbf{v}\,dV.$

32. Let $\mathbf{r}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$ so that, for $\mathbf{r}\neq\mathbf{0}$, $\mathbf{F}(x,y,z)=\mathbf{r}/||\mathbf{r}||^k=\dfrac{u}{(u^2+v^2+w^2)^{k/2}}\mathbf{i}+\dfrac{v}{(u^2+v^2+w^2)^{k/2}}\mathbf{j}+$ $\dfrac{w}{(u^2+v^2+w^2)^{k/2}}\mathbf{k}$. Now $\dfrac{\partial\mathbf{F}_1}{\partial u}=\dfrac{u^2+v^2+w^2-ku^2}{(u^2+v^2+w^2)^{(k/2)+1}}$; similarly for $\partial\mathbf{F}_2/\partial v, \partial\mathbf{F}_3/\partial w$, so that div $\mathbf{F}=\dfrac{3(u^2+v^2+w^2)-k(u^2+v^2+w^2)}{(u^2+v^2+w^2)^{(k/2)+1}}=0$ if and only if $k=3$.

33. div $\mathbf{F}=0$; no sources or sinks.

34. div $\mathbf{F}=y-x$; sources where $y>x$, sinks where $y<x$.

35. div $\mathbf{F}=3x^2+3y^2+3z^2$; sources at all points except the origin, no sinks.

36. div $\mathbf{F}=3(x^2+y^2+z^2-1)$; sources outside the sphere $x^2+y^2+z^2=1$, sinks inside the sphere $x^2+y^2+z^2=1$.

37. Let σ_1 be the portion of the paraboloid $z=1-x^2-y^2$ for $z\ge 0$, and σ_2 the portion of the plane $z=0$ for $x^2+y^2\le 1$. Then $\iint\limits_{\sigma_1}\mathbf{F}\cdot\mathbf{n}\,dS=\iint\limits_{R}\mathbf{F}\cdot(2x\mathbf{i}+2y\mathbf{j}+\mathbf{k})\,dA=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2x[x^2y-(1-x^2-y^2)^2]+2y(y^3-x)+(2x+2-3x^2-3y^2))\,dy\,dx=3\pi/4$; $z=0$ and $\mathbf{n}=-\mathbf{k}$ on σ_2 so $\mathbf{F}\cdot\mathbf{n}=1-2x$, $\iint\limits_{\sigma_2}\mathbf{F}\cdot\mathbf{n}\,dS=\iint\limits_{\sigma_2}(1-2x)dS=\pi$. Thus $\iint\limits_{\sigma}\mathbf{F}\cdot\mathbf{n}\,dS=3\pi/4+\pi=7\pi/4$. But div $\mathbf{F}=2xy+3y^2+3$, so $\iiint\limits_{G}\text{div }\mathbf{F}\,dV=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2}(2xy+3y^2+3)\,dz\,dy\,dx=7\pi/4.$

Exercise Set 15.8

1. If σ is oriented with upward normals then C consists of three parts parametrized as $C_1 : \mathbf{r}(t) = (1-t)\mathbf{i} + t\mathbf{j}$ for $0 \le t \le 1$, $C_2 : \mathbf{r}(t) = (1-t)\mathbf{j} + t\mathbf{k}$ for $0 \le t \le 1$, $C_3 : \mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{k}$ for $0 \le t \le 1$. $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3t-1)dt = \frac{1}{2}$, so $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$. curl $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $z = 1 - x - y$, R is the triangular region in the xy-plane enclosed by $x + y = 1$, $x = 0$, and $y = 0$; $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 3 \iint_R dA = (3)(\text{area of } R) = (3)\left[\frac{1}{2}(1)(1)\right] = \frac{3}{2}$.

2. If σ is oriented with upward normals then C can be parametrized as $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$ for $0 \le t \le 2\pi$. $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (\sin^2 t \cos t - \cos^2 t \sin t)dt = 0$; curl $\mathbf{F} = \mathbf{0}$ so $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_\sigma 0\, dS = 0$.

3. If σ is oriented with upward normals then C can be parametrized as $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$ for $0 \le t \le 2\pi$. $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 0\, dt = 0$; curl $\mathbf{F} = \mathbf{0}$ so $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_\sigma 0\, dS = 0$.

4. If σ is oriented with upward normals then C can be parametrized as $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$ for $0 \le t \le 2\pi$. $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (9\sin^2 t + 9\cos^2 t)dt = 9\int_0^{2\pi} dt = 18\pi$. curl $\mathbf{F} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, R is the circular region in the xy-plane enclosed by $x^2 + y^2 = 9$; $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (-4x + 4y + 2)dA = \int_0^{2\pi}\int_0^3 (-4r\cos\theta + 4r\sin\theta + 2)r\, dr\, d\theta = 18\pi$.

5. Take σ as the part of the plane $z = 0$ for $x^2 + y^2 \le 1$ with $\mathbf{n} = \mathbf{k}$; curl $\mathbf{F} = -3y^2\mathbf{i} + 2z\mathbf{j} + 2\mathbf{k}$, $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 2\iint_\sigma dS = (2)(\text{area of circle}) = (2)[\pi(1)^2] = 2\pi$.

6. curl $\mathbf{F} = x\mathbf{i} + (x - y)\mathbf{j} + 6xy^2\mathbf{k}$; $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (x - y - 6xy^2)dA = \int_0^1\int_0^3 (x - y - 6xy^2)dy\, dx = -30$.

7. C is the boundary of R and curl $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, so $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{r} = \iint_R \text{curl}\,\mathbf{F} \cdot \mathbf{n}\, dS = \iint_R 4x + 6y + 4\, dA = 4(\text{area of } R) = 16\pi$.

8. curl $\mathbf{F} = -4\mathbf{i} - 6\mathbf{j} + 6y\mathbf{k}$, $z = y/2$ oriented with upward normals, R is the triangular region in the xy-plane enclosed by $x + y = 2$, $x = 0$, and $y = 0$; $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (3 + 6y)dA = \int_0^2\int_0^{2-x} (3 + 6y)dy\, dx = 14$.

9. curl $\mathbf{F} = x\mathbf{k}$, take σ as part of the plane $z = y$ oriented with upward normals, R is the circular region in the xy-plane enclosed by $x^2 + y^2 - y = 0$; $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R x\, dA = \int_0^\pi\int_0^{\sin\theta} r^2\cos\theta\, dr\, d\theta = 0$.

10. curl $\mathbf{F} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$, $z = 1 - x - y$ oriented with upward normals, R is the triangular region in the xy-plane enclosed by $x+y=1$, $x=0$ and $y=0$; $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (-y-z-x)dA = -\iint_R dA = -\frac{1}{2}(1)(1) = -\frac{1}{2}$.

11. curl $\mathbf{F} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, take σ as the part of the plane $z=0$ with $x^2+y^2 \le a^2$ and $\mathbf{n}=\mathbf{k}$; $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_\sigma dS =$ area of circle $= \pi a^2$.

12. curl $\mathbf{F} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, take σ as the part of the plane $z = 1/\sqrt{2}$ with $x^2+y^2 \le 1/2$ and $\mathbf{n}=\mathbf{k}$; $\displaystyle\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_\sigma dS =$ area of circle $= \dfrac{\pi}{2}$.

13. True, Theorem 15.8.1.

14. False, Green's Theorem is a special case of Stokes's Theorem.

15. False, the circulation is $\displaystyle\int_C \mathbf{F}\cdot \mathbf{T}\, ds$.

16. True, Theorem 15.8.1.

17. **(a)** Take σ as the part of the plane $2x+y+2z=2$ in the first octant, oriented with downward normals; curl $\mathbf{F} = -x\mathbf{i}+(y-1)\mathbf{j}-\mathbf{k}$, $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_\sigma (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \iint_R \left(x - \frac{1}{2}y + \frac{3}{2}\right) dA = \int_0^1 \int_0^{2-2x} \left(x - \frac{1}{2}y + \frac{3}{2}\right) dy\, dx = \frac{3}{2}$.

(b) At the origin curl $\mathbf{F} = -\mathbf{j}-\mathbf{k}$ and with $\mathbf{n}=\mathbf{k}$, curl $\mathbf{F}(0,0,0)\cdot\mathbf{n} = (-\mathbf{j}-\mathbf{k})\cdot\mathbf{k} = -1$.

(c) The rotation of $\mathbf{F}$ has its maximum value at the origin about the unit vector in the same direction as curl $\mathbf{F}(0,0,0)$ so $\mathbf{n} = -\dfrac{1}{\sqrt{2}}\mathbf{j} - \dfrac{1}{\sqrt{2}}\mathbf{k}$.

18. **(a)** Using the hint, the orientation of the curve C with respect to the surface σ_1 is the opposite of the orientation of C with respect to the surface σ_2. Thus in the expressions $\displaystyle\iint_{\sigma_1} (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \int_C \mathbf{F}\cdot\mathbf{T}\, dS$ and $\displaystyle\iint_{\sigma_2} (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \oint_C \mathbf{F}\cdot\mathbf{T}\, dS$, the two line integrals have oppositely oriented tangents $\mathbf{T}$. Hence $\displaystyle\iint_\sigma (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \iint_{\sigma_1} (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS + \iint_{\sigma_2} (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \mathbf{0}$.

(b) The flux of the curl field through the boundary of a solid is zero.

19. **(a)** The flow is independent of z and has no component in the direction of $\mathbf{k}$, and so by inspection the only nonzero component of the curl is in the direction of $\mathbf{k}$. However both sides of (9) are zero, as the flow is orthogonal to the curve C_a. Thus the curl is zero.

(b) Since the flow appears to be tangential to the curve C_a, it seems that the right hand side of (9) is nonzero, and thus the curl is nonzero, and points in the positive z-direction.

20. (a) The only nonzero vector component of the vector field is in the direction of $\mathbf{i}$, and it increases with y and is independent of x. Thus the curl of F is nonzero, and points in the positive z-direction. Alternatively, let $\mathbf{F} = f\mathbf{i}$, and let C be the circle of radius ϵ with positive orientation. Then $\mathbf{T} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$, and $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = -\epsilon\int_0^{2\pi} f(\epsilon,\theta)\sin\theta\,d\theta = -\epsilon\int_0^{\pi} f(\epsilon,\theta)\sin\theta\,d\theta - \epsilon\int_{-\pi}^{0} f(\epsilon,\theta)\sin\theta\,d\theta = -\epsilon\int_0^{\pi}(f(\epsilon,\theta) - f(-\epsilon,\theta))\sin\theta\,d\theta < 0$, because from the picture $f(\epsilon,\theta) > f(\epsilon,-\theta)$ for $0 < \theta < \pi$. Thus, from (9), the curl is nonzero and points in the negative z-direction.

(b) By inspection the vector field is constant, and thus its curl is zero.

21. Since $\mathbf{F}$ is conservative, if C is any closed curve then $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$. But $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ from (30) of Section 15.2. In equation (9) the direction of $\mathbf{n}$ is arbitrary, so for any fixed curve C_a the integral $\int_{C_a} \mathbf{F}\cdot\mathbf{T}ds = 0$. Thus curl $\mathbf{F}(P_0)\cdot\mathbf{n} = 0$. But $\mathbf{n}$ is arbitrary, so we conclude that curl $\mathbf{F} = \mathbf{0}$.

22. Since $\oint_C \mathbf{E}\cdot\mathbf{r}d\mathbf{r} = \iint_\sigma \text{curl}\,\mathbf{E}\cdot\mathbf{n}\,dS$, it follows that $\iint_\sigma \text{curl}\,\mathbf{E}\cdot\mathbf{n}dS = -\iint_\sigma \dfrac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{n}dS$. This relationship holds for any surface σ, hence curl $\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$.

23. Parametrize C by $x = \cos t, y = \sin t, 0 \le t \le 2\pi$. But $\mathbf{F} = x^2y\mathbf{i} + (y^3 - x)\mathbf{j} + (2x-1)\mathbf{k}$ along C so $\oint_C \mathbf{F}\cdot d\mathbf{r} = -5\pi/4$. Since curl $\mathbf{F} = (-2z-2)\mathbf{j} + (-1-x^2)\mathbf{k}$, $\iint_\sigma (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R (\text{curl}\,\mathbf{F})\cdot(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [2y(2x^2+2y^2-4) - 1 - x^2]\,dy\,dx = -5\pi/4$.

Chapter 15 Review Exercises

2. (b) $c\dfrac{\mathbf{r}-\mathbf{r}_0}{\|\mathbf{r}-\mathbf{r}_0\|^3}$. **(c)** $c\dfrac{(x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$

3. $\mathbf{v} = (1-x)\mathbf{i} + (2-y)\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{(1-x)^2 + (2-y)^2}$, $\mathbf{u} = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v} = \dfrac{1-x}{\sqrt{(1-x)^2+(2-y)^2}}\mathbf{i} + \dfrac{2-y}{\sqrt{(1-x)^2+(2-y)^2}}\mathbf{j}$.

4. $\dfrac{-2y}{(x-y)^2}\mathbf{i} + \dfrac{2x}{(x-y)^2}\mathbf{j}$.

5. $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

6. div $\mathbf{F} = \dfrac{y^2-x^2}{(x^2+y^2)^2} + \dfrac{x^2-y^2}{(x^2+y^2)^2} + \dfrac{1}{(x^2+y^2)} = \dfrac{1}{x^2+y^2}$, the level surface of div $\mathbf{F} = 1$ is the cylinder about the z-axis of radius 1.

7. (a) $\displaystyle\int_a^b \left[f(x(t),y(t))\frac{dx}{dt} + g(x(t),y(t))\frac{dy}{dt}\right]dt$. **(b)** $\displaystyle\int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$.

8. (a) $M = \displaystyle\int_C \delta(x,y,z)\,ds$. **(b)** $L = \displaystyle\int_C ds$.

11. $s = \theta, x = \cos\theta, y = \sin\theta$, $\displaystyle\int_0^{2\pi}(\cos\theta - \sin\theta)\,d\theta = 0$, also follows from odd function rule.

12. $\displaystyle\int_0^{2\pi}[\cos t(-\sin t)+t\cos t-2\sin^2 t]\,dt=0+0-2\pi=-2\pi.$

13. $\displaystyle\int_1^2\left(\frac{t}{2t}-2\frac{2t}{t}\right)dt=\int_1^2\left(-\frac{7}{2}\right)dt=-\frac{7}{2}.$

14. $\mathbf{r}(t)=t\mathbf{i}+t^2\mathbf{j}, W=\displaystyle\int_C\mathbf{F}\cdot\mathbf{r}=\int_0^1[(t^2)^2+t(t^2)(2t)]\,dt=\frac{3}{5}.$

16. By the Fundamental Theorem of Line Integrals, $\displaystyle\int_C\nabla f\cdot d\mathbf{r}=f(1,2,-1)-f(0,0,0)=-4.$

17. **(a)** If $h(x)\mathbf{F}$ is conservative, then $\dfrac{\partial}{\partial y}(yh(x))=\dfrac{\partial}{\partial x}(-2xh(x))$, or $h(x)=-2h(x)-2xh'(x)$ which has the general solution $x^3h(x)^2=C_1, h(x)=Cx^{-3/2}$, so $C\dfrac{y}{x^{3/2}}\mathbf{i}-C\dfrac{2}{x^{1/2}}\mathbf{j}$ is conservative, with potential function $\phi=-2Cy/\sqrt{x}$.

(b) If $g(y)\mathbf{F}(x,y)$ is conservative then $\dfrac{\partial}{\partial y}(yg(y))=\dfrac{\partial}{\partial x}(-2xg(y))$, or $g(y)+yg'(y)=-2g(y)$, with general solution $g(y)=C/y^3$, so $\mathbf{F}=C\dfrac{1}{y^2}\mathbf{i}-C\dfrac{2x}{y^3}\mathbf{j}$ is conservative, with potential function Cx/y^2.

18. **(a)** $f_y-g_x=e^{xy}+xye^{xy}-e^{xy}-xye^{xy}=0$ so the vector field is conservative.

(b) $\phi_x=ye^{xy}-1, \phi=e^{xy}-x+k(x), \phi_y=xe^{xy}$, let $k(x)=0; \phi(x,y)=e^{xy}-x$.

(c) $W=\displaystyle\int_C\mathbf{F}\cdot d\mathbf{r}=\phi(x(8\pi),y(8\pi))-\phi(x(0),y(0))=\phi(8\pi,0)-\phi(0,0)=-8\pi.$

21. Let O be the origin, P the point with polar coordinates $\theta=\alpha, r=f(\alpha)$, and Q the point with polar coordinates $\theta=\beta, r=f(\beta)$. Let $C_1: O$ to P; $x=t\cos\alpha,\ y=t\sin\alpha,\ 0\le t\le f(\alpha), -y\dfrac{dx}{dt}+x\dfrac{dy}{dt}=0$; $C_2: P$ to Q; $x=f(t)\cos t,\ y=f(t)\sin t,\ \alpha\le\theta\le\beta, -y\dfrac{dx}{dt}+x\dfrac{dy}{dt}=f(t)^2$; $C_3: Q$ to O; $x=-t\cos\beta,\ y=-t\sin\beta,\ -f(\beta)\le t\le 0, -y\dfrac{dx}{dt}+x\dfrac{dy}{dt}=0$. $A=\dfrac{1}{2}\displaystyle\oint_C -y\,dx+x\,dy=\frac{1}{2}\int_\alpha^\beta f(t)^2\,dt$; set $t=\theta$ and $r=f(\theta)=f(t), A=\dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta.$

22. **(a)** $\displaystyle\int_C f(x)\,dx+g(y)\,dy=\iint_R\left(\frac{\partial}{\partial x}g(y)-\frac{\partial}{\partial y}f(x)\right)dA=0.$

(b) $W=\displaystyle\int_C\mathbf{F}\cdot d\mathbf{r}=\int_C f(x)\,dx+g(y)\,dy=0$, so the work done by the vector field around any simple closed curve is zero. The field is conservative.

23. $\displaystyle\iint_\sigma f(x,y,z)dS=\iint_R f(x(u,v),y(u,v),z(u,v))\|\mathbf{r}_u\times\mathbf{r}_v\|\,du\,dv.$

24. Cylindrical coordinates $\mathbf{r}(\theta,z)=\cos\theta\mathbf{i}+\sin\theta\mathbf{j}+z\mathbf{k}, 0\le\theta\le 2\pi, 0\le z\le 1, \mathbf{r}_\theta=-\sin\theta\mathbf{i}+\cos\theta\mathbf{j}, \mathbf{r}_z=\mathbf{k}, \|\mathbf{r}_\theta\times\mathbf{r}_z\|=\|\mathbf{r}_\theta\|\|\mathbf{r}_z\|\sin(\pi/2)=1$; by Theorem 15.5.1, $\displaystyle\iint_\sigma z\,dS=\int_0^{2\pi}\int_0^1 z\,dz\,d\theta=\pi.$

25. Yes; by imagining a normal vector sliding around the surface it is evident that the surface has two sides.

27. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + (1 - x^2 - y^2)\mathbf{k}, \mathbf{r}_x \times \mathbf{r}_y = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$, $\mathbf{F} = x\mathbf{i} + y\mathbf{i} + 2z\mathbf{k}$, $\Phi = \iint_R \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA = \iint_R (2x^2 + 2y^2 + 2(1 - x^2 - y^2))\, dA = 2A = 2\pi$.

28. $\mathbf{r} = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}$, $\dfrac{\partial\mathbf{r}}{\partial\phi} \times \dfrac{\partial\mathbf{r}}{\partial\theta} = \sin^2\phi\cos\theta\mathbf{i} + \sin^2\phi\sin\theta\mathbf{j} + \sin\phi\cos\phi\mathbf{k}$, $\Phi = \iint_\sigma \mathbf{F} \cdot \left(\dfrac{\partial\mathbf{r}}{\partial\phi} \times \dfrac{\partial\mathbf{r}}{\partial\theta}\right) dA = \int_0^{2\pi}\int_0^{\pi} (\sin^3\phi\cos^2\theta + 2\sin^3\phi\sin^2\theta + 3\sin\phi\cos^2\phi)\, d\phi\, d\theta = 8\pi$. We can also obtain this result by the Divergence Theorem, simply by multiplying 6 (the constant divergence of the vector field) with $4\pi/3$ (the volume of the unit sphere).

30. $D_{\mathbf{n}}\phi = \mathbf{n} \cdot \nabla\phi$, so $\iint_\sigma D_{\mathbf{n}}\phi\, dS = \iint_\sigma \mathbf{n} \cdot \nabla\phi\, dS = \iiint_G \nabla \cdot (\nabla\phi)\, dV = \iiint_G \left[\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}\right] dV$.

31. By Exercise 30, $\iint_\sigma D_{\mathbf{n}} f\, dS = -\iiint_G [f_{xx} + f_{yy} + f_{zz}]\, dV = -6\iiint_G dV = -6\text{vol}(G) = -8\pi$.

32. C is defined by $\mathbf{r}(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + \mathbf{k}, 0 \le \theta \le 2\pi, \mathbf{r}'(\theta) = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \mathbf{T} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$. By Stokes' Theorem $\iint_\sigma (\text{curl}\mathbf{F}) \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{2\pi} -\sin\theta(1 - \sin\theta) + \cos\theta(\cos\theta + 1)\, d\theta = 2\pi$.

33. A computation of curl $\mathbf{F}$ shows that curl $\mathbf{F} = \mathbf{0}$ if and only if the three given equations hold. Moreover the equations hold if $\mathbf{F}$ is conservative, so it remains to show that $\mathbf{F}$ is conservative if curl $\mathbf{F} = \mathbf{0}$. Let C by any simple closed curve in the region. Since the region is simply connected, there is a piecewise smooth, oriented surface σ in the region with boundary C. By Stokes' Theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_\sigma 0\, dS = 0$. By the 3-space analog of Theorem 15.3.2, $\mathbf{F}$ is conservative.

34. (a) Conservative, $\phi(x, y, z) = xz^2 - e^{-y}$. **(b)** Not conservative, $f_y \neq g_x$.

35. (a) Conservative, $\phi(x, y, z) = -\cos x + yz$. **(b)** Not conservative, $f_z \neq h_x$.

36. (a) $\mathbf{F}(x, y, z) = \dfrac{qQ(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{4\pi\epsilon_0(x^2 + y^2 + z^2)^{3/2}}$.

(b) $\mathbf{F} = \nabla\phi$, where $\phi = -\dfrac{qQ}{4\pi\epsilon_0(x^2 + y^2 + z^2)^{1/2}}$, so $W = \phi(3, 1, 5) - \phi(3, 0, 0) = \dfrac{qQ}{4\pi\epsilon_0}\left(\dfrac{1}{3} - \dfrac{1}{\sqrt{35}}\right)$.

Chapter 15 Making Connections

1. Using Newton's Second Law of Motion followed by Theorem 12.6.2 we have Work $= \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C m\,\mathbf{a} \cdot \mathbf{T}\, ds = m\int_C \left[\dfrac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\dfrac{ds}{dt}\right)^2\mathbf{N}\right] \cdot \mathbf{T}\, ds$. But $\mathbf{N} \cdot \mathbf{T} = 0$ and $v = \dfrac{ds}{dt}$ so $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = m\int_C \dfrac{d^2s}{dt^2}\, ds = m\int_C \left(\dfrac{dv}{dt}\right) ds = m\int_a^b v(t)\left(\dfrac{dv}{dt}\right) dt = m\int_a^b \dfrac{d}{dt}\left(\dfrac{1}{2}(v(t))^2\right) dt = \dfrac{1}{2}m[v(b)]^2 - \dfrac{1}{2}m[v(a)]^2$, which is the change in kinetic energy of the particle.

2. Work performed with a 'constrained' motion and under the influence of $\mathbf{F}$ is equal to $\int_C (\mathbf{F} + \mathbf{S}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} =$

$\int_C \mathbf{F} \cdot \mathbf{T} ds$, because $\int_C \mathbf{S} \cdot d\mathbf{r} = 0$ by normality of $\mathbf{S}$ and C. Now proceed as in Exercise 1.

3. $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \nabla\phi(x, y, z) \cdot \langle x'(t), y'(t), z'(t)\rangle \, dt = \phi(x(t_1), y(t_1), z(t_1)) - \phi(x(t_0), y(t_0), z(t_0))$, which is the change in potential energy of the particle, or the negative of the kinetic energy of the particle. Note that we have used Theorem 15.3.3. For constrained motions the following calculations apply: work $= \int_C (\mathbf{F} + \mathbf{S}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}$ and continue as before.

4. The equation of motion can be expressed with $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, with $x''(t) = 0, y''(t) = -g$. Integrating, we get (recall child starts at rest, so $(x(0), y(0)) = (x_0, y_0), (x'(0), y'(0)) = (0, 0)$), $\mathbf{r}'(t) = -gt\mathbf{j}, \mathbf{r}(t) = x_0\mathbf{i} + (y_0 - gt^2/2)\mathbf{j}$. So the motion ends when $y_0 = gt^2/2, t = \sqrt{2y_0/g}$. Finally, from the picture $y_0 = \ell \sin\theta$, so $t = \sqrt{2\ell\sin\theta/g}, v = gt = \sqrt{2g\ell\sin\theta}$.