Multiple Integrals

Exercise Set 14.1

1. $\displaystyle\int_0^1\int_0^2 (x+3)\,dy\,dx = \int_0^1 (2x+6)\,dx = 7.$

3. $\displaystyle\int_2^4\int_0^1 x^2y\,dx\,dy = \int_2^4 \frac{1}{3}y\,dy = 2.$

5. $\displaystyle\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\,dy\,dx = \int_0^{\ln 3} e^x\,dx = 2.$

7. $\displaystyle\int_{-1}^0\int_2^5 dx\,dy = \int_{-1}^0 3\,dy = 3.$

9. $\displaystyle\int_0^1\int_0^1 \frac{x}{(xy+1)^2}\,dy\,dx = \int_0^1 \left(1-\frac{1}{x+1}\right)dx = 1-\ln 2.$

11. $\displaystyle\int_0^{\ln 2}\int_0^1 xy\,e^{y^2x}\,dy\,dx = \int_0^{\ln 2} \frac{1}{2}(e^x-1)\,dx = \frac{1-\ln 2}{2}.$

13. $\displaystyle\int_{-1}^1\int_{-2}^2 4xy^3\,dy\,dx = \int_{-1}^1 0\,dx = 0.$

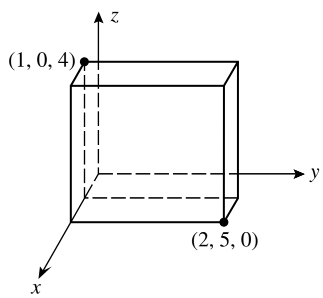
15. $\displaystyle\int_0^1\int_2^3 x\sqrt{1-x^2}\,dy\,dx = \int_0^1 x(1-x^2)^{1/2}\,dx = \frac{1}{3}.$

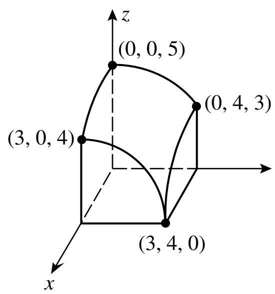
17. (a) $x_k^* = k/2 - 1/4, k = 1,2,3,4; y_l^* = l/2-1/4, l = 1,2,3,4,$ $\displaystyle\iint_R f(x,y)\,dx\,dy \approx \sum_{k=1}^4\sum_{l=1}^4 f(x_k^*,y_l^*)\Delta A_{kl} = \sum_{k=1}^4\sum_{l=1}^4\left[\left(\frac{k}{2}-\frac{1}{4}\right)^2+\left(\frac{l}{2}-\frac{1}{4}\right)\right]\left(\frac{1}{2}\right)^2 = \frac{37}{4}.$

(b) $\displaystyle\int_0^2\int_0^2 (x^2+y)\,dx\,dy = \frac{28}{3}$; the error is $\displaystyle\left|\frac{37}{4}-\frac{28}{3}\right| = \frac{1}{12}.$

19. The solid is a rectangular box with sides of length 1, 5, and 4, so its volume is $1\cdot 5\cdot 4 = 20$;

$\displaystyle\int_0^5\int_1^2 4\,dx\,dy = \int_0^5 4x\Big]_{x=1}^2 dy = \int_0^5 4\,dy = 20.$





21.

23. False. ΔA_k represents the area of such a region.

25. False. $\displaystyle\iint_R f(x,y)\,dA = \int_1^5\int_2^4 f(x,y)\,dy\,dx$.

27. $\displaystyle\iint_R f(x,y)\,dA = \int_a^b\left[\int_c^d g(x)h(y)\,dy\right]dx = \int_a^b g(x)\left[\int_c^d h(y)\,dy\right]dx = \left[\int_a^b g(x)\,dx\right]\left[\int_c^d h(y)\,dy\right]$.

29. $\displaystyle V = \int_3^5\int_1^2 (2x+y)\,dy\,dx = \int_3^5\left(2x+\frac{3}{2}\right)dx = 19$.

31. $\displaystyle V = \int_0^2\int_0^3 x^2\,dy\,dx = \int_0^2 3x^2\,dx = 8$.

33. $\displaystyle\int_0^{1/2}\int_0^{\pi} x\cos(xy)\cos^2\pi x\,dy\,dx = \int_0^{1/2}\cos^2\pi x\sin(xy)\Big]_0^{\pi}dx = \int_0^{1/2}\cos^2\pi x\sin\pi x\,dx = -\frac{1}{3\pi}\cos^3\pi x\Big]_0^{1/2} = \frac{1}{3\pi}$.

35. $\displaystyle f_{\text{ave}} = \frac{1}{48}\int_0^6\int_0^8 xy^2\,dx\,dy = \frac{1}{48}\int_0^6\left(\frac{1}{2}x^2y^2\Big]_{x=0}^{x=8}\right)dy = \frac{1}{48}\int_0^6 32y^2\,dy = 48$.

37. $\displaystyle f_{\text{ave}} = \frac{2}{\pi}\int_0^{\pi/2}\int_0^1 y\sin xy\,dx\,dy = \frac{2}{\pi}\int_0^{\pi/2}\left(-\cos xy\Big]_{x=0}^{x=1}\right)dy = \frac{2}{\pi}\int_0^{\pi/2}(1-\cos y)\,dy = 1-\frac{2}{\pi}$.

39. $\displaystyle T_{\text{ave}} = \frac{1}{2}\int_0^1\int_0^2\left(10-8x^2-2y^2\right)dy\,dx = \frac{1}{2}\int_0^1\left(\frac{44}{3}-16x^2\right)dx = \left(\frac{14}{3}\right)^{\circ}\text{C}$.

41. 1.381737122

43. The first integral equals 1/2, the second equals $-1/2$. This does not contradict Theorem 14.1.3 because the integrand is not continuous at $(x,y)=(0,0)$; if $f(x,y)=\dfrac{y-x}{(x+y)^3}$, then $\displaystyle\lim_{x\to 0} f(x,0) = \lim_{x\to 0}\frac{-1}{x^2}\to -\infty$.

45. If R is a rectangular region defined by $a \le x \le b$, $c \le y \le d$, then the volume given in equation (5) can be written as an iterated integral: $V = \iint\limits_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$. The inner integral, $\int_c^d f(x,y)\,dy$, is the area $A(x)$ of the cross-section with x-coordinate x of the solid enclosed between R and the surface $z = f(x,y)$. So $V = \int_a^b A(x)\,dx$, as found in Section 6.2.

Exercise Set 14.2

1. $\int_0^1 \int_{x^2}^x xy^2\,dy\,dx = \int_0^1 \frac{1}{3}(x^4 - x^7)\,dx = \frac{1}{40}$.

3. $\int_0^3 \int_0^{\sqrt{9-y^2}} y\,dx\,dy = \int_0^3 y\sqrt{9-y^2}\,dy = 9$.

5. $\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^3} \sin(y/x)\,dy\,dx = \int_{\sqrt{\pi}}^{\sqrt{2\pi}} [-x\cos(x^2) + x]\,dx = \frac{\pi}{2}$.

7. $\int_0^1 \int_0^x y\sqrt{x^2-y^2}\,dy\,dx = \int_0^1 \frac{1}{3}x^3\,dx = \frac{1}{12}$.

9. **(a)** $\int_0^2 \int_0^{x^2} f(x,y)\,dy\,dx$. **(b)** $\int_0^4 \int_{\sqrt{y}}^2 f(x,y)\,dx\,dy$.

11. **(a)** $\int_1^2 \int_{-2x+5}^3 f(x,y)\,dy\,dx + \int_2^4 \int_1^3 f(x,y)\,dy\,dx + \int_4^5 \int_{2x-7}^3 f(x,y)\,dy\,dx$.

(b) $\int_1^3 \int_{(5-y)/2}^{(y+7)/2} f(x,y)\,dx\,dy$.

13. **(a)** $\int_0^2 \int_0^{x^2} xy\,dy\,dx = \int_0^2 \frac{1}{2}x^5\,dx = \frac{16}{3}$.

(b) $\int_1^3 \int_{(5-y)/2}^{(y+7)/2} xy\,dx\,dy = \int_1^3 (3y^2 + 3y)\,dy = 38$.

15. **(a)** $\int_4^8 \int_{16/x}^x x^2 dy\,dx = \int_4^8 (x^3 - 16x)\,dx = 576$.

(b) $\int_2^4 \int_{16/y}^8 x^2\,dx\,dy + \int_4^8 \int_y^8 x^2\,dx\,dy = \int_4^8 \left[\frac{512}{3} - \frac{4096}{3y^3}\right] dy + \int_4^8 \frac{512 - y^3}{3}\,dy = \frac{640}{3} + \frac{1088}{3} = 576$.

17. **(a)** $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3x - 2y)\,dy\,dx = \int_{-1}^1 6x\sqrt{1-x^2}\,dx = 0$.

(b) $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (3x - 2y)\,dx\,dy = \int_{-1}^1 -4y\sqrt{1-y^2}\,dy = 0$.

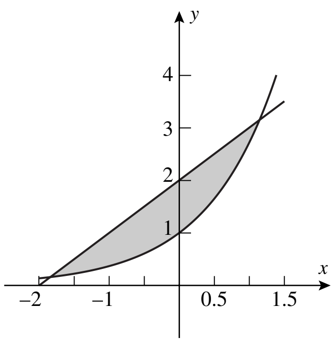
19. $\int_0^4 \int_0^{\sqrt{y}} x(1+y^2)^{-1/2}\,dx\,dy = \int_0^4 \frac{1}{2}y(1+y^2)^{-1/2}\,dy = \frac{\sqrt{17}-1}{2}$.

21. $\displaystyle\int_0^2 \int_{y^2}^{6-y} xy\,dx\,dy = \int_0^2 \frac{1}{2}(36y - 12y^2 + y^3 - y^5)\,dy = \frac{50}{3}.$

23. $\displaystyle\int_0^1 \int_{x^3}^{x} (x-1)\,dy\,dx = \int_0^1 (-x^4 + x^3 + x^2 - x)\,dx = -\frac{7}{60}.$

25. $\displaystyle\int_0^2 \int_0^{y^2} \sin(y^3)\,dx\,dy = \int_0^2 y^2 \sin(y^3)\,dy = \frac{1-\cos 8}{3}.$

27. (a)



(b) $(-1.8414, 0.1586), (1.1462, 3.1462)$.

(c) $\displaystyle\iint_R x\,dA \approx \int_{-1.8414}^{1.1462} \int_{e^x}^{x+2} x\,dy\,dx = \int_{-1.8414}^{1.1462} x(x+2-e^x)\,dx \approx -0.4044.$

(d) $\displaystyle\iint_R x\,dA \approx \int_{0.1586}^{3.1462} \int_{y-2}^{\ln y} x\,dx\,dy = \int_{0.1586}^{3.1462} \left[\frac{\ln^2 y}{2} - \frac{(y-2)^2}{2}\right] dy \approx -0.4044.$

29. $\displaystyle A = \int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx = \int_0^{\pi/4} (\cos x - \sin x)\,dx = \sqrt{2} - 1.$

31. $\displaystyle A = \int_{-3}^{3} \int_{1-y^2/9}^{9-y^2} dx\,dy = \int_{-3}^{3} 8\left(1 - \frac{y^2}{9}\right) dy = 32.$

33. False. The expression on the right side doesn't make sense. To evaluate an integral of the form $\displaystyle\int_{x^2}^{2x} g(y)\,dy$, x must have a fixed value. But then we can't use x as a variable in defining $\displaystyle g(y) = \int_0^1 f(x,y)\,dx$.

35. False. For example, if $f(x,y) = x$ then $\displaystyle\iint_R f(x,y)\,dA = \int_{-1}^{1} \int_{x^2}^{1} x\,dy\,dx = \int_{-1}^{1} xy\Big]_{y=x^2}^{1} dx = \int_{-1}^{1} x(1-x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{-1}^{1} = 0$, but $\displaystyle 2\int_0^1 \int_{x^2}^{1} x\,dy\,dx = \int_0^1 xy\Big]_{y=x^2}^{1} dx = \int_0^1 x(1-x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{4}.$

37. $\displaystyle\int_0^4 \int_0^{6-3x/2} \left(3 - \frac{3x}{4} - \frac{y}{2}\right) dy\,dx = \int_0^4 \left[\left(3 - \frac{3x}{4}\right)\left(6 - \frac{3x}{2}\right) - \frac{1}{4}\left(6 - \frac{3x}{2}\right)^2\right] dx = 12.$

39. $\displaystyle V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (3-x)\,dy\,dx = \int_{-3}^{3} \left(6\sqrt{9-x^2} - 2x\sqrt{9-x^2}\right) dx = 27\pi.$

41. $V = \int_0^3 \int_0^2 (9x^2 + y^2)\, dy\, dx = \int_0^3 \left(18x^2 + \frac{8}{3}\right) dx = 170.$

43. $V = \int_{-3/2}^{3/2} \int_{-\sqrt{9-4x^2}}^{\sqrt{9-4x^2}} (y+3)\, dy\, dx = \int_{-3/2}^{3/2} 6\sqrt{9-4x^2}\, dx = \frac{27\pi}{2}.$

45. $V = 4\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \frac{8}{3}\int_0^1 (1-x^2)^{3/2}\, dx = \frac{\pi}{2}.$

47. $\int_0^{\sqrt{2}} \int_{y^2}^2 f(x,y)\, dx\, dy.$

49. $\int_1^{e^2} \int_{\ln x}^2 f(x,y)\, dy\, dx.$

51. $\int_0^{\pi/2} \int_0^{\sin x} f(x,y)\, dy\, dx.$

53. $\int_0^4 \int_0^{y/4} e^{-y^2}\, dx\, dy = \int_0^4 \frac{1}{4} y e^{-y^2}\, dy = \frac{1 - e^{-16}}{8}.$

55. $\int_0^2 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^2 x^2 e^{x^3}\, dx = \frac{e^8 - 1}{3}.$

57. **(a)** $\int_0^4 \int_{\sqrt{x}}^2 \sin\left(\pi y^3\right) dy\, dx$; the inner integral is non-elementary.

$\int_0^2 \int_0^{y^2} \sin\left(\pi y^3\right) dx\, dy = \int_0^2 y^2 \sin\left(\pi y^3\right) dy = -\frac{1}{3\pi}\cos\left(\pi y^3\right)\Big]_0^2 = 0.$

(b) $\int_0^1 \int_{\sin^{-1} y}^{\pi/2} \sec^2(\cos x)\, dx\, dy$; the inner integral is non-elementary.

$\int_0^{\pi/2} \int_0^{\sin x} \sec^2(\cos x)\, dy\, dx = \int_0^{\pi/2} \sec^2(\cos x) \sin x\, dx = \tan 1.$

59. The region is symmetric with respect to the y-axis, and the integrand is an odd function of x, hence the answer is zero.

61. Area of triangle is $1/2$, so $f_{\text{ave}} = 2\int_0^1 \int_x^1 \frac{1}{1+x^2}\, dy\, dx = 2\int_0^1 \left[\frac{1}{1+x^2} - \frac{x}{1+x^2}\right] dx = \frac{\pi}{2} - \ln 2.$

63. $T_{\text{ave}} = \frac{1}{A(R)} \iint\limits_R (5xy + x^2)\, dA$. The diamond has corners $(\pm 2, 0), (0, \pm 4)$ and thus has area $A(R) = 4\frac{1}{2}2(4) = 16\text{m}^2$. Since $5xy$ is an odd function of x (as well as y), $\iint\limits_R 5xy\, dA = 0$. Since x^2 is an even function of both x and y, $T_{\text{ave}} = \frac{4}{16} \iint\limits_{\substack{R\\ x,y>0}} x^2\, dA = \frac{1}{4}\int_0^2 \int_0^{4-2x} x^2\, dy\, dx = \frac{1}{4}\int_0^2 (4-2x)x^2\, dx = \frac{1}{4}\left[\frac{4}{3}x^3 - \frac{1}{2}x^4\right]_0^2 = \left(\frac{2}{3}\right)^\circ \text{C}.$

65. $y = \sin x$ and $y = x/2$ intersect at $x = 0$ and $x = a \approx 1.895494$, so $V = \int_0^a \int_{x/2}^{\sin x} \sqrt{1+x+y}\, dy\, dx \approx 0.676089.$

67. See Example 7. Given an iterated integral $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, draw the type II region R defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$. If R is also a type I region, try to determine the numbers a and b and functions $g_1(x)$ and $g_2(x)$ such that R is also described by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$. Then $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$. This isn't always possible: R may not be a type I region. Even if it is, it may not be possible to find formulas for $g_1(x)$ and $g_2(x)$.

Exercise Set 14.3

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{2}\sin^2\theta\cos\theta\,d\theta = \frac{1}{6}.$

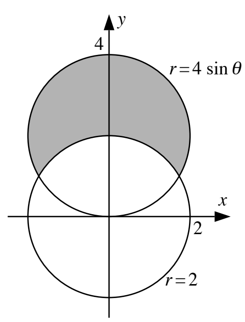
3. $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta} r^2\,dr\,d\theta = \int_0^{\pi/2} \frac{a^3}{3}\sin^3\theta\,d\theta = \frac{2}{9}a^3.$

5. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\,dr\,d\theta = \int_0^{\pi} \frac{1}{3}(1-\sin\theta)^3\cos\theta\,d\theta = 0.$

7. $\displaystyle A = \int_0^{2\pi}\int_0^{1-\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2}.$

9. $\displaystyle A = \int_{\pi/4}^{\pi/2}\int_{\sin 2\theta}^{1} r\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{2}(1-\sin^2 2\theta)\,d\theta = \frac{\pi}{16}.$

11. $\displaystyle A = \int_{\pi/6}^{5\pi/6}\int_2^{4\sin\theta} f(r,\theta)\,r\,dr\,d\theta.$



13. $\displaystyle V = 8\int_0^{\pi/2}\int_1^3 r\sqrt{9-r^2}\,dr\,d\theta.$

15. $\displaystyle V = 2\int_0^{\pi/2}\int_0^{\cos\theta} (1-r^2)r\,dr\,d\theta.$

17. $\displaystyle V = 8\int_0^{\pi/2}\int_1^3 r\sqrt{9-r^2}\,dr\,d\theta = \frac{128}{3}\sqrt{2}\int_0^{\pi/2} d\theta = \frac{64}{3}\sqrt{2}\pi.$

19. $\displaystyle V = 2\int_0^{\pi/2}\int_0^{\cos\theta} (1-r^2)r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}(2\cos^2\theta - \cos^4\theta)\,d\theta = \frac{5\pi}{32}.$

21. $\displaystyle V = \int_0^{\pi/2}\int_0^{3\sin\theta} r^2\sin\theta\,dr\,d\theta = 9\int_0^{\pi/2}\sin^4\theta\,d\theta = \frac{27\pi}{16}.$

23. $\int_0^{2\pi}\int_0^3 \sin(r^2)r\,dr\,d\theta = \frac{1}{2}(1-\cos 9)\int_0^{2\pi} d\theta = \pi(1-\cos 9)$.

25. $\int_0^{\pi/4}\int_0^2 \frac{1}{1+r^2}r\,dr\,d\theta = \frac{1}{2}\ln 5\int_0^{\pi/4} d\theta = \frac{\pi}{8}\ln 5$.

27. $\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{\pi}{8}$.

29. $\int_0^{\pi/2}\int_0^{2\cos\theta} r^2 dr\,d\theta = \frac{8}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta = \frac{16}{9}$.

31. $\int_0^{\pi/2}\int_0^a \frac{r}{(1+r^2)^{3/2}}\,dr\,d\theta = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{1+a^2}}\right)$.

33. $\int_0^{\pi/4}\int_0^2 \frac{r}{\sqrt{1+r^2}}\,dr\,d\theta = \frac{\pi}{4}(\sqrt{5}-1)$.

35. True. It can be defined by the inequalities $0 \le \theta \le 2\pi$, $0 \le r \le 2$.

37. False. The integrand in the iterated integral should be multiplied by r: $\iint\limits_R f(r,\theta)\,dA = \int_0^{\pi/2}\int_1^2 f(r,\theta)\,r\,dr\,d\theta$.

39. $\int_{\tan^{-1}(1/3)}^{\tan^{-1}(2)}\int_0^2 r^3\cos^2\theta\,dr\,d\theta = 4\int_{\tan^{-1}(1/3)}^{\tan^{-1}(2)}\cos^2\theta\,d\theta = 2\int_{\tan^{-1}(1/3)}^{\tan^{-1}(2)}(1+\cos(2\theta))\,d\theta = \Big[2\theta+2\cos\theta\sin\theta\Big]_{\tan^{-1}(1/3)}^{\tan^{-1}(2)} =$
$2\tan^{-1}(2)+2\cdot\frac{1}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}}-2\tan^{-1}(1/3)-2\cdot\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}} = 2\left(\tan^{-1}(2)-\tan^{-1}(1/3)\right)+\frac{1}{5} = 2\tan^{-1}(1)+\frac{1}{5} = \frac{\pi}{2}+\frac{1}{5}$.

41. **(a)** $V = 8\int_0^{\pi/2}\int_0^a \frac{c}{a}(a^2-r^2)^{1/2}\,r\,dr\,d\theta = -\frac{4c}{3a}\pi(a^2-r^2)^{3/2}\Big]_0^a = \frac{4}{3}\pi a^2 c$.

(b) $V \approx \frac{4}{3}\pi(6378.1370)^2 6356.5231\ \text{km}^3 \approx 1.0831682\cdot 10^{12}\ \text{km}^3 = 1.0831682\cdot 10^{21}\ \text{m}^3$.

43. $A = 4\int_0^{\pi/4}\int_0^{a\sqrt{2\cos 2\theta}} r\,dr\,d\theta = 4a^2\int_0^{\pi/4}\cos 2\theta\,d\theta = 2a^2$.

Exercise Set 14.4

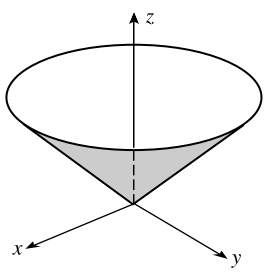
1. $z=\sqrt{9-y^2}$, $z_x = 0$, $z_y = -y/\sqrt{9-y^2}$, $z_x^2+z_y^2+1 = 9/(9-y^2)$, $S = \int_0^2\int_{-3}^3 \frac{3}{\sqrt{9-y^2}}\,dy\,dx = \int_0^2 3\pi\,dx = 6\pi$.

3. $z^2 = 4x^2+4y^2$, $2zz_x = 8x$ so $z_x = 4x/z$; similarly $z_y = 4y/z$ so $z_x^2+z_y^2+1 = (16x^2+16y^2)/z^2+1 = 5$, $S = \int_0^1\int_{x^2}^x \sqrt{5}\,dy\,dx = \sqrt{5}\int_0^1(x-x^2)\,dx = \frac{\sqrt{5}}{6}$.

5. $z^2 = x^2+y^2$, $z_x = x/z$, $z_y = y/z$, $z_x^2+z_y^2+1 = (x^2+y^2)/z^2+1 = 2$, $S = \iint\limits_R \sqrt{2}\,dA = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\sqrt{2}\,r\,dr\,d\theta =$
$4\sqrt{2}\int_0^{\pi/2}\cos^2\theta\,d\theta = \sqrt{2}\pi$.

7. $z_x = y$, $z_y = x$, $z_x^2 + z_y^2 + 1 = x^2 + y^2 + 1$, $S = \iint\limits_R \sqrt{x^2+y^2+1}\, dA = \int_0^{\pi/6}\int_0^3 r\sqrt{r^2+1}\, dr\, d\theta = \frac{1}{3}(10\sqrt{10} - 1)\int_0^{\pi/6} d\theta = \frac{\pi}{18}(10\sqrt{10}-1)$.

9. On the sphere, $z_x = -x/z$ and $z_y = -y/z$ so $z_x^2 + z_y^2 + 1 = (x^2+y^2+z^2)/z^2 = 16/(16 - x^2 - y^2)$. The planes $z = 1$ and $z = 2$ intersect the sphere along the circles $x^2 + y^2 = 15$ and $x^2 + y^2 = 12$, so $S = \iint\limits_R \frac{4}{\sqrt{16 - x^2 - y^2}} dA = \int_0^{2\pi}\int_{\sqrt{12}}^{\sqrt{15}} \frac{4r}{\sqrt{16 - r^2}}\, dr\, d\theta = 4\int_0^{2\pi} d\theta = 8\pi$.

11. **(a)**


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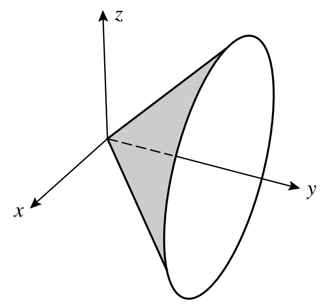
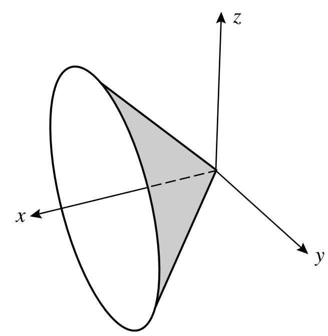


(c)


13. **(a)** $x = u, y = v, z = \frac{5}{2} + \frac{3}{2}u - 2v$. **(b)** $x = u, y = v, z = u^2$.

15. **(a)** $x = \sqrt{5}\cos u, y = \sqrt{5}\sin u, z = v; 0 \le u \le 2\pi, 0 \le v \le 1$.

(b) $x = 2\cos u, y = v, z = 2\sin u; 0 \le u \le 2\pi, 1 \le v \le 3$.

17. $x = u, y = \sin u \cos v, z = \sin u \sin v$.

19. $x = r\cos\theta, y = r\sin\theta, z = \frac{1}{1+r^2}$.

21. $x = r\cos\theta, y = r\sin\theta, z = 2r^2\cos\theta\sin\theta$.

23. $x = r\cos\theta, y = r\sin\theta, z = \sqrt{9 - r^2}; r \le \sqrt{5}$.

25. $x = \frac{1}{2}\rho\cos\theta, y = \frac{1}{2}\rho\sin\theta, z = \frac{\sqrt{3}}{2}\rho$.

27. $z = x - 2y$; a plane.

29. $(x/3)^2 + (y/2)^2 = 1; 2 \le z \le 4$; part of an elliptic cylinder.

31. $(x/3)^2 + (y/4)^2 = z^2; 0 \le z \le 1$; part of an elliptic cone.

33. **(a)** I: $x = r\cos\theta$, $y = r\sin\theta$, $z = r$, $0 \le r \le 2$; II: $x = u$, $y = v$, $z = \sqrt{u^2+v^2}$; $0 \le u^2 + v^2 \le 4$.

35. **(a)** $0 \le u \le 3$, $0 \le v \le \pi$. **(b)** $0 \le u \le 4$, $-\pi/2 \le v \le \pi/2$.

37. **(a)** $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. **(b)** $0 \le \phi \le \pi$, $0 \le \theta \le \pi$.

39. $u = 1$, $v = 2$, $\mathbf{r}_u \times \mathbf{r}_v = -2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$; $2x + 4y - z = 5$.

41. $u = 0$, $v = 1$, $\mathbf{r}_u \times \mathbf{r}_v = 6\mathbf{k}$; $z = 0$.

43. $\mathbf{r}_u \times \mathbf{r}_v = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j} + \dfrac{1}{2}\mathbf{k}$; $x - y + \dfrac{1}{\sqrt{2}}z = \dfrac{\pi\sqrt{2}}{8}$.

45. $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$, $\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{4u^2+1}$; $S = \displaystyle\int_0^{2\pi}\int_1^2 u\sqrt{4u^2+1}\,du\,dv = \dfrac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$.

47. False. For example, if $f(x,y) = 1$ then the surface has the same area as R, $\displaystyle\iint_R dA$, not $\displaystyle\iint_R \sqrt{2}\,dA$.

49. True, as explained before Definition 14.4.1.

51. $\mathbf{r}(u,v) = a\cos u\sin v\mathbf{i} + a\sin u\sin v\mathbf{j} + a\cos v\mathbf{k}$, $\|\mathbf{r}_u \times \mathbf{r}_v\| = a^2\sin v$, $S = \displaystyle\int_0^{\pi}\int_0^{2\pi} a^2\sin v\,du\,dv = 2\pi a^2\int_0^{\pi}\sin v\,dv = 4\pi a^2$.

53. $z_x = \dfrac{h}{a}\dfrac{x}{\sqrt{x^2+y^2}}$, $z_y = \dfrac{h}{a}\dfrac{y}{\sqrt{x^2+y^2}}$, $z_x^2 + z_y^2 + 1 = \dfrac{h^2x^2+h^2y^2}{a^2(x^2+y^2)} + 1 = \dfrac{a^2+h^2}{a^2}$, $S = \displaystyle\int_0^{2\pi}\int_0^a \frac{\sqrt{a^2+h^2}}{a}\,r\,dr\,d\theta = \frac{1}{2}a\sqrt{a^2+h^2}\int_0^{2\pi}d\theta = \pi a\sqrt{a^2+h^2}$.

55. $\mathbf{r}_u = -(a+b\cos v)\sin u\,\mathbf{i} + (a+b\cos v)\cos u\,\mathbf{j}$, $\mathbf{r}_v = -b\sin v\cos u\,\mathbf{i} - b\sin v\sin u\,\mathbf{j} + b\cos v\,\mathbf{k}$, $\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a+b\cos v)$; $S = \displaystyle\int_0^{2\pi}\int_0^{2\pi} b(a+b\cos v)\,du\,dv = 4\pi^2ab$.

57. $z = -1$ when $v \approx 0.27955$, $z = 1$ when $v \approx 2.86204$, $\|\mathbf{r}_u \times \mathbf{r}_v\| = |\cos v|$; $S \approx \displaystyle\int_0^{2\pi}\int_{0.27955}^{2.86204} |\cos v|\,dv\,du \approx 9.099$.

59. $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 + \left(\dfrac{z}{c}\right)^2 = 1$, ellipsoid.

61. $-\left(\dfrac{x}{a}\right)^2 - \left(\dfrac{y}{b}\right)^2 + \left(\dfrac{z}{c}\right)^2 = 1$, hyperboloid of two sheets.

Exercise Set 14.5

1. $\displaystyle\int_{-1}^1\int_0^2\int_0^1 (x^2+y^2+z^2)\,dx\,dy\,dz = \int_{-1}^1\int_0^2 (1/3+y^2+z^2)\,dy\,dz = \int_{-1}^1 (10/3+2z^2)\,dz = 8$.

3. $\displaystyle\int_0^2\int_{-1}^{y^2}\int_{-1}^{z} yz\,dx\,dz\,dy = \int_0^2\int_{-1}^{y^2}(yz^2+yz)\,dz\,dy = \int_0^2\left(\frac{1}{3}y^7 + \frac{1}{2}y^5 - \frac{1}{6}y\right)dy = \frac{47}{3}$.

5. $\displaystyle\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^x xy\,dy\,dx\,dz = \int_0^3\int_0^{\sqrt{9-z^2}}\frac{1}{2}x^3dx\,dz = \int_0^3\frac{1}{8}(81-18z^2+z^4)\,dz = \frac{81}{5}$.

7. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{4-x^2}}[2x(4-x^2)-2xy^2]\,dy\,dx = \int_0^2\frac{4}{3}x(4-x^2)^{3/2}\,dx = \frac{128}{15}$.

9. $\displaystyle\int_0^{\pi}\int_0^1\int_0^{\pi/6} xy\sin yz\,dz\,dy\,dx = \int_0^{\pi}\int_0^1 x[1-\cos(\pi y/6)]\,dy\,dx = \int_0^{\pi}(1-3/\pi)x\,dx = \frac{\pi(\pi-3)}{2}$.

11. $\displaystyle\int_0^{\sqrt{2}}\int_0^x\int_0^{2-x^2} xyz\,dz\,dy\,dx = \int_0^{\sqrt{2}}\int_0^x \frac{1}{2}xy(2-x^2)^2 dy\,dx = \int_0^{\sqrt{2}} \frac{1}{4}x^3(2-x^2)^2\,dx = \frac{1}{6}.$

13. $\displaystyle\int_0^3\int_1^2\int_{-2}^1 \frac{\sqrt{x+z^2}}{y}\,dz\,dy\,dx \approx 9.425.$

15. $\displaystyle V = \int_0^4\int_0^{(4-x)/2}\int_0^{(12-3x-6y)/4} dz\,dy\,dx = \int_0^4\int_0^{(4-x)/2} \frac{1}{4}(12-3x-6y)\,dy\,dx = \int_0^4 \frac{3}{16}(4-x)^2\,dx = 4.$

17. $\displaystyle V = 2\int_0^2\int_{x^2}^4\int_0^{4-y} dz\,dy\,dx = 2\int_0^2\int_{x^2}^4 (4-y)\,dy\,dx = 2\int_0^2 \left(8-4x^2+\frac{1}{2}x^4\right)dx = \frac{256}{15}.$

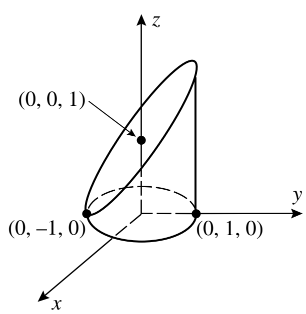
19. The projection of the curve of intersection onto the xy-plane is $x^2+y^2=1$,

(a) $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} f(x,y,z)\,dz\,dy\,dx.$ **(b)** $\displaystyle\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4x^2+y^2}^{4-3y^2} f(x,y,z)\,dz\,dx\,dy.$

21. Let $f(x,y,z)=1$ in Exercise 19(a). $\displaystyle V = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} dz\,dy\,dx = 4\int_0^1\int_0^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} dz\,dy\,dx.$

23. $\displaystyle V = 2\int_{-3}^3\int_0^{\sqrt{9-x^2}/3}\int_0^{x+3} dz\,dy\,dx.$

25. **(a)**



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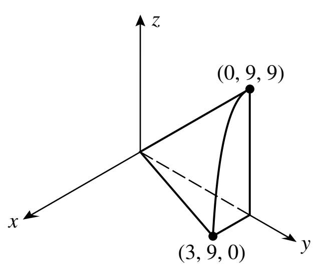

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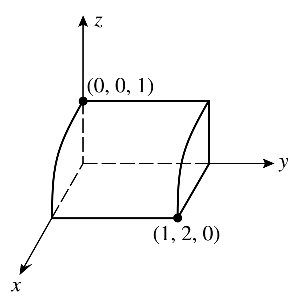


27. True, by changing the order of integration in Theorem 14.5.1.

29. False. The middle integral (with respect to y) should be $\displaystyle\int_0^{\sqrt{1-x^2}}$.

31. $\displaystyle\int_a^b\int_c^d\int_k^\ell f(x)g(y)h(z)dz\,dy\,dx = \int_a^b\int_c^d f(x)g(y)\left[\int_k^\ell h(z)\,dz\right]dy\,dx = \left[\int_a^b f(x)\left[\int_c^d g(y)\,dy\right]dx\right]\left[\int_k^\ell h(z)\,dz\right] =$
$\displaystyle\left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)dy\right]\left[\int_k^\ell h(z)\,dz\right].$

33. $\displaystyle V = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx = 1/6,\ f_{\text{ave}} = 6\int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z)\,dz\,dy\,dx = \frac{3}{4}.$

35. The volume $V = \dfrac{3\pi}{\sqrt{2}}$, and thus

$$r_{\text{ave}} = \frac{\sqrt{2}}{3\pi}\iiint_G \sqrt{x^2+y^2+z^2}\,dV = \frac{\sqrt{2}}{3\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}\int_{5x^2+5y^2}^{6-7x^2-y^2}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx \approx 3.291.$$

37. **(a)** $\displaystyle\int_0^a\int_0^{b(1-x/a)}\int_0^{c(1-x/a-y/b)} dz\,dy\,dx,\ \int_0^b\int_0^{a(1-y/b)}\int_0^{c(1-x/a-y/b)} dz\,dx\,dy,$

$\displaystyle\int_0^c\int_0^{a(1-z/c)}\int_0^{b(1-x/a-z/c)} dy\,dx\,dz,\ \int_0^a\int_0^{c(1-x/a)}\int_0^{b(1-x/a-z/c)} dy\,dz\,dx,\ \int_0^c\int_0^{b(1-z/c)}\int_0^{a(1-y/b-z/c)} dx\,dy\,dz,$

$\displaystyle\int_0^b\int_0^{c(1-y/b)}\int_0^{a(1-y/b-z/c)} dx\,dz\,dy.$

(b) Use the first integral in part (a) to get $\displaystyle\int_0^a\int_0^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx = \int_0^a \frac{1}{2}bc\left(1-\frac{x}{a}\right)^2 dx = \frac{1}{6}abc.$

39. **(a)** $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^5 f(x,y,z)\,dz\,dy\,dx$ **(b)** $\displaystyle\int_0^9\int_0^{3-\sqrt{x}}\int_y^{3-\sqrt{x}} f(x,y,z)\,dz\,dy\,dx$

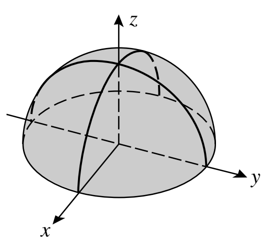
(c) $\displaystyle\int_0^2\int_0^{4-x^2}\int_y^{8-y} f(x,y,z)\,dz\,dy\,dx$

41. See discussion after Theorem 14.5.2.

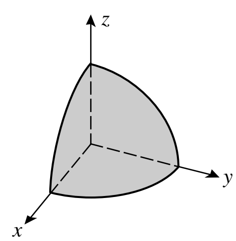
Exercise Set 14.6

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}(1-r^2)r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{8}\,d\theta = \frac{\pi}{4}.$

3. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{4}\sin\phi\cos\phi\,d\phi\,d\theta = \int_0^{\pi/2}\frac{1}{8}\,d\theta = \frac{\pi}{16}.$



5. $f(r,\theta,z) = z$



7. $f(\rho,\theta,\phi) = \rho\cos\phi$

9. $\displaystyle V = \int_0^{2\pi}\int_0^3\int_{r^2}^9 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r(9-r^2)\,dr\,d\theta = \int_0^{2\pi}\frac{81}{4}d\theta = \frac{81\pi}{2}.$

11. $r^2+z^2=20$ intersects $z=r^2$ in a circle of radius 2; the volume consists of two portions, one inside the cylinder $r=2$ and one outside that cylinder: $\displaystyle V = \int_0^{2\pi}\int_0^2\int_{-\sqrt{20-r^2}}^{r^2} r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_2^{\sqrt{20}}\int_{-\sqrt{20-r^2}}^{\sqrt{20-r^2}} r\,dz\,dr\,d\theta =$ $\displaystyle\int_0^{2\pi}\int_0^2 r\left(r^2+\sqrt{20-r^2}\right)dr\,d\theta + \int_0^{2\pi}\int_2^{\sqrt{20}} 2r\sqrt{20-r^2}\,dr\,d\theta = \frac{4}{3}(10\sqrt{5}-13)\int_0^{2\pi}d\theta + \frac{128}{3}\int_0^{2\pi}d\theta = \frac{152}{3}\pi + \frac{80}{3}\pi\sqrt{5}.$

13. $\displaystyle V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^4 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\frac{64}{3}\sin\phi\,d\phi\,d\theta = \frac{32}{3}\int_0^{2\pi}d\theta = \frac{64\pi}{3}.$

15. In spherical coordinates the sphere and the plane $z=a$ are $\rho=2a$ and $\rho=a\sec\phi$, respectively. They intersect at $\phi=\pi/3$, $\displaystyle V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{a\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2a}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta =$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}a^3 \sec^3\phi \sin\phi \, d\phi \, d\theta + \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \frac{8}{3}a^3 \sin\phi \, d\phi \, d\theta = \frac{1}{2}a^3 \int_0^{2\pi} d\theta + \frac{4}{3}a^3 \int_0^{2\pi} d\theta = \frac{11\pi a^3}{3}.$$

17. $\displaystyle\int_0^{\pi/2}\int_0^a\int_0^{a^2-r^2} r^3\cos^2\theta \, dz \, dr \, d\theta = \int_0^{\pi/2}\int_0^a (a^2r^3 - r^5)\cos^2\theta \, dr \, d\theta = \frac{1}{12}a^6 \int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{\pi a^6}{48}.$

19. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{8}} \rho^4\cos^2\phi\sin\phi \, d\rho \, d\phi \, d\theta = \frac{32\pi}{15}(2\sqrt{2}-1).$

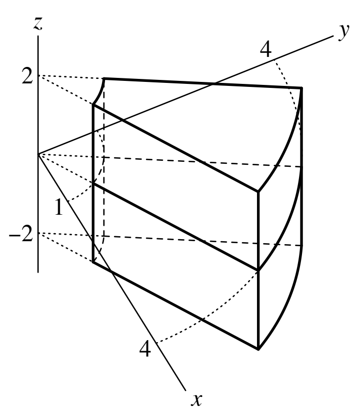
21. False. The factor r^2 should be just r.

23. True. The region is described by $0 \le \phi \le \pi/4$, $0 \le \theta \le 2\pi$, $1 \le \rho \le 3$, so the volume is $\displaystyle\iiint_G 1 \, dV = \int_0^{\pi/4}\int_0^{2\pi}\int_1^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$

25. **(a)** $\displaystyle\int_{-2}^2\int_1^4\int_{\pi/6}^{\pi/3} \frac{r\tan^3\theta}{\sqrt{1+z^2}} \, d\theta \, dr \, dz = \left(\int_{-2}^2 \frac{1}{\sqrt{1+z^2}} \, dz\right)\left(\int_1^4 r \, dr\right)\left(\int_{\pi/6}^{\pi/3} \tan^3\theta \, d\theta\right) =$

$$= 2\ln(2+\sqrt{5}) \cdot \frac{15}{2} \cdot \left(\frac{4}{3} - \frac{1}{2}\ln 3\right) = \frac{5}{2}(8 - 3\ln 3)\ln(2+\sqrt{5}) \approx 16.97774195.$$

(b) G is the cylindrical wedge $\pi/6 \le \theta \le \pi/3$, $1 \le r \le 4$, $-2 \le z \le 2$. Since $dx \, dy \, dz = dV = r \, d\theta \, dr \, dz$, the integrand in rectangular coordinates is $\displaystyle\frac{1}{r} \cdot \frac{r\tan^3\theta}{\sqrt{1+z^2}} = \frac{(y/x)^3}{\sqrt{1+z^2}}$, so $\displaystyle f(x,y,z) = \frac{y^3}{x^3\sqrt{1+z^2}}.$



27. **(a)** $\displaystyle V = 2\int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta = \frac{4\pi a^3}{3}.$ **(b)** $\displaystyle V = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = \frac{4\pi a^3}{3}.$

29. $\displaystyle V = \int_0^{\pi/2}\int_{\pi/6}^{\pi/3}\int_0^2 \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/2}\int_{\pi/6}^{\pi/3} \frac{8}{3}\sin\phi \, d\phi \, d\theta = \frac{4}{3}(\sqrt{3}-1)\int_0^{\pi/2} d\theta = \frac{2\pi}{3}(\sqrt{3}-1).$

31. The fact that none of the limits involves θ means that the solid is obtained by rotating a region in the xz-plane about the z-axis, between two angles θ_1 and θ_2. If the integral is expressed in cylindrical coordinates, then the plane region must be either a type I region or a type II region (with the role of y replaced by z); see Definition 14.2.1. If the integral is expressed in spherical coordinates, then the plane region may be a simple polar region (with the roles of θ and r replaced by ϕ and ρ); see Definition 14.3.1. Or it may be described by inequalities of the form $\rho_1 \le \rho \le \rho_2$, $\phi_1(\rho) \le \phi \le \phi_2(\rho)$ for some numbers $\rho_1 \le \rho_2$ and functions $\phi_1(\rho) \le \phi_2(\rho)$.

Exercise Set 14.7

1. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 4 \\ 3 & -5 \end{vmatrix} = -17.$

3. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos u & -\sin v \\ \sin u & \cos v \end{vmatrix} = \cos u \cos v + \sin u \sin v = \cos(u - v).$

5. $x = \dfrac{2}{9}u + \dfrac{5}{9}v,\ y = -\dfrac{1}{9}u + \dfrac{2}{9}v;\ \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2/9 & 5/9 \\ -1/9 & 2/9 \end{vmatrix} = \dfrac{1}{9}.$

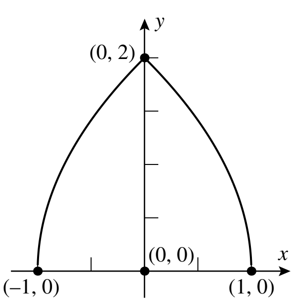
7. $x = \dfrac{\sqrt{u+v}}{\sqrt{2}},\ y = \dfrac{\sqrt{v-u}}{\sqrt{2}};\ \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{1}{2\sqrt{2}\sqrt{u+v}} & \dfrac{1}{2\sqrt{2}\sqrt{u+v}} \\ -\dfrac{1}{2\sqrt{2}\sqrt{v-u}} & \dfrac{1}{2\sqrt{2}\sqrt{v-u}} \end{vmatrix} = \dfrac{1}{4\sqrt{v^2-u^2}}.$

9. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} = 5.$

11. $y = v,\ x = \dfrac{u}{y} = \dfrac{u}{v},\ z = w - x = w - \dfrac{u}{v};\ \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1 & 0 \\ -1/v & u/v^2 & 1 \end{vmatrix} = \dfrac{1}{v}.$

13. False. It is the <u>area</u> of the parallelogram.

15. False. The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$



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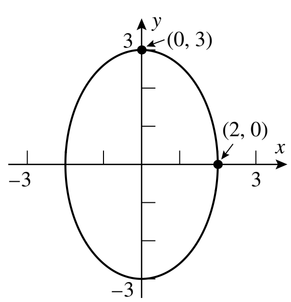


19.

21. $x = \dfrac{1}{5}u + \dfrac{2}{5}v,\ y = -\dfrac{2}{5}u + \dfrac{1}{5}v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{5};\ \dfrac{1}{5}\iint\limits_S \dfrac{u}{v}\, dA_{uv} = \dfrac{1}{5}\int_1^3 \int_1^4 \dfrac{u}{v}\, du\, dv = \dfrac{3}{2}\ln 3.$

23. $x = u + v,\ y = u - v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = -2$; the boundary curves of the region S in the uv-plane are $v = 0, v = u$, and $u = 1$ so $2\iint\limits_S \sin u \cos v\, dA_{uv} = 2\int_0^1 \int_0^u \sin u \cos v\, dv\, du = 1 - \dfrac{1}{2}\sin 2.$

25. $x = 3u,\ y = 4v,\ \dfrac{\partial(x,y)}{\partial(u,v)} = 12$; S is the region in the uv-plane enclosed by the circle $u^2 + v^2 = 1$. Use polar coordinates to obtain $\iint\limits_S 12\sqrt{u^2+v^2}(12)\, dA_{uv} = 144\int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta = 96\pi.$

27. Let S be the region in the uv-plane bounded by $u^2+v^2=1$, so $u=2x, v=3y, x=u/2, y=v/3$, $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1/2 & 0 \\ 0 & 1/3 \end{vmatrix}=1/6$, use polar coordinates to get $\dfrac{1}{6}\iint\limits_S \sin(u^2+v^2)\,dA_{uv}=\dfrac{1}{6}\displaystyle\int_0^{\pi/2}\int_0^1 r\sin r^2\,dr\,d\theta=$
$=\dfrac{\pi}{24}(-\cos r^2)\Big]_0^1=\dfrac{\pi}{24}(1-\cos 1)$.

29. $x=u/3, y=v/2, z=w$, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=1/6$; S is the region in uvw-space enclosed by the sphere $u^2+v^2+w^2=36$, so $\iiint\limits_S \dfrac{u^2}{9}\dfrac{1}{6}dV_{uvw}=\dfrac{1}{54}\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^6(\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\dfrac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^6\rho^4\sin^3\phi\cos^2\theta\,d\rho\,d\phi\,d\theta=$ $\dfrac{192\pi}{5}$.

31. $u=\theta=\begin{cases}\cot^{-1}(x/y) & \text{if } y\neq 0\\ 0 & \text{if } y=0,\ x>0\\ \pi & \text{if } y=0,\ x<0\end{cases}$, $v=r=\sqrt{x^2+y^2}$. Other answers are possible.

33. $u=\dfrac{3}{7}x-\dfrac{2}{7}y$, $v=-\dfrac{1}{7}x+\dfrac{3}{7}y$. Other answers are possible.

35. Let $u=y-4x, v=y+4x$, then $x=\dfrac{1}{8}(v-u), y=\dfrac{1}{2}(v+u)$ so $\dfrac{\partial(x,y)}{\partial(u,v)}=-\dfrac{1}{8}$; $\dfrac{1}{8}\iint\limits_S\dfrac{u}{v}\,dA_{uv}=\dfrac{1}{8}\displaystyle\int_2^5\int_0^2\dfrac{u}{v}\,du\,dv=$ $\dfrac{1}{4}\ln\dfrac{5}{2}$.

37. Let $u=x-y, v=x+y$, then $x=\dfrac{1}{2}(v+u), y=\dfrac{1}{2}(v-u)$ so $\dfrac{\partial(x,y)}{\partial(u,v)}=\dfrac{1}{2}$; the boundary curves of the region S in the uv-plane are $u=0, v=u$, and $v=\pi/4$; thus $\dfrac{1}{2}\iint\limits_S\dfrac{\sin u}{\cos v}\,dA_{uv}=\dfrac{1}{2}\displaystyle\int_0^{\pi/4}\int_0^v\dfrac{\sin u}{\cos v}\,du\,dv=\dfrac{1}{2}\left[\ln(\sqrt{2}+1)-\dfrac{\pi}{4}\right]$.

39. Let $u=\dfrac{y}{x}$, $v=\dfrac{x}{y^2}$, then $x=\dfrac{1}{u^2v}$, $y=\dfrac{1}{uv}$ so $\dfrac{\partial(x,y)}{\partial(u,v)}=\dfrac{1}{u^4v^3}$; $\iint\limits_S\dfrac{1}{u^4v^3}\,dA_{uv}=\displaystyle\int_1^4\int_1^2\dfrac{1}{u^4v^3}\,du\,dv=\dfrac{35}{256}$.

41. $x=u$, $y=\dfrac{w}{u}$, $z=v+\dfrac{w}{u}$, $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=-\dfrac{1}{u}$; $\iiint\limits_S\dfrac{v^2w}{u}\,dV_{uvw}=\displaystyle\int_2^4\int_0^1\int_1^3\dfrac{v^2w}{u}\,du\,dv\,dw=2\ln 3$.

43. **(b)** $\dfrac{\partial(x,y)}{\partial(u,v)}\cdot\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}x_u & x_v\\ y_u & y_v\end{vmatrix}\cdot\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}=\begin{vmatrix}x_uu_x+x_vv_x & x_uu_y+x_vv_y\\ y_uu_x+y_vv_x & y_uu_y+y_vv_y\end{vmatrix}=\begin{vmatrix}x_x & x_y\\ y_x & y_y\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1$.

45. $\dfrac{\partial(u,v)}{\partial(x,y)}=8xy$ so $\dfrac{\partial(x,y)}{\partial(u,v)}=\dfrac{1}{8xy}$; $xy\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|=xy\cdot\dfrac{1}{8xy}=\dfrac{1}{8}$ so $\dfrac{1}{8}\iint\limits_S dA_{uv}=\dfrac{1}{8}\displaystyle\int_9^{16}\int_1^4 du\,dv=\dfrac{21}{8}$.

47. Set $u=x+y+2z, v=x-2y+z, w=4x+y+z$, then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}1 & 1 & 2\\ 1 & -2 & 1\\ 4 & 1 & 1\end{vmatrix}=18$, and $V=\iiint\limits_R dx\,dy\,dz=$ $\displaystyle\int_{-6}^6\int_{-2}^2\int_{-3}^3\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw=6\cdot4\cdot12\cdot\dfrac{1}{18}=16$.

49. The main motivation is to change the region of integration to one that has a simple description in either rectangular, polar, cylindrical, or spherical coordinates.

Exercise Set 14.8

1. $M = \int_0^1 \int_0^{\sqrt{x}} (x+y)\,dy\,dx = \dfrac{13}{20}$, $M_x = \int_0^1 \int_0^{\sqrt{x}} (x+y)y\,dy\,dx = \dfrac{3}{10}$, $M_y = \int_0^1 \int_0^{\sqrt{x}} (x+y)x\,dy\,dx = \dfrac{19}{42}$, $\overline{x} = \dfrac{M_y}{M} = \dfrac{190}{273}$, $\overline{y} = \dfrac{M_x}{M} = \dfrac{6}{13}$; the mass is $\dfrac{13}{20}$ and the center of gravity is at $\left(\dfrac{190}{273}, \dfrac{6}{13}\right)$.

3. $M = \int_0^{\pi/2} \int_0^a r^3 \sin\theta\cos\theta\,dr\,d\theta = \dfrac{a^4}{8}$, $\overline{x} = \overline{y}$ from the symmetry of the density and the region, $M_y = \int_0^{\pi/2} \int_0^a r^4 \sin\theta\cos^2\theta\,dr\,d\theta = \dfrac{a^5}{15}$, $\overline{x} = \dfrac{8a}{15}$; mass $\dfrac{a^4}{8}$, center of gravity $\left(\dfrac{8a}{15}, \dfrac{8a}{15}\right)$.

5. $M = \iint\limits_R \delta(x,y)\,dA = \int_0^1 \int_0^1 |x+y-1|\,dx\,dy = \int_0^1 \left[\int_0^{1-x} (1-x-y)\,dy + \int_{1-x}^1 (x+y-1)\,dy\right] dx = \dfrac{1}{3}$. $\overline{x} = 3\int_0^1 \int_0^1 x\delta(x,y)\,dy\,dx = 3\int_0^1 \left[\int_0^{1-x} x(1-x-y)\,dy + \int_{1-x}^1 x(x+y-1)\,dy\right] dx = \dfrac{1}{2}$. By symmetry, $\overline{y} = \dfrac{1}{2}$ as well; center of gravity $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

7. $V = 1, \overline{x} = \int_0^1 \int_0^1 \int_0^1 x\,dz\,dy\,dx = \dfrac{1}{2}$, similarly $\overline{y} = \overline{z} = \dfrac{1}{2}$; centroid $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$.

9. True. This is the definition of "centroid"; see Section 6.7.

11. False. The coordinates are the first moments about the y- and x-axes, divided by the mass.

13. Let $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ in formulas (11) and (12).

15. $\overline{x} = \overline{y}$ from the symmetry of the region, $A = \int_0^{\pi/2} \int_0^{\sin 2\theta} r\,dr\,d\theta = \dfrac{\pi}{8}$, $\overline{x} = \dfrac{1}{A}\int_0^{\pi/2} \int_0^{\sin 2\theta} r^2\cos\theta\,dr\,d\theta = \dfrac{8}{\pi} \cdot \dfrac{16}{105} = \dfrac{128}{105\pi}$; centroid $\left(\dfrac{128}{105\pi}, \dfrac{128}{105\pi}\right)$.

17. $\overline{y} = 0$ from the symmetry of the region, $A = \dfrac{1}{2}\pi a^2$, $\overline{x} = \dfrac{1}{A}\int_{-\pi/2}^{\pi/2} \int_0^a r^2\cos\theta\,dr\,d\theta = \dfrac{1}{A}\dfrac{2}{3}a^3 = \dfrac{4a}{3\pi}$; centroid $\left(\dfrac{4a}{3\pi}, 0\right)$.

19. $\overline{x} = \overline{y} = \overline{z}$ from the symmetry of the region, $V = 1/6$, $\overline{x} = \dfrac{1}{V}\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\,dz\,dy\,dx = 6 \cdot \dfrac{1}{24} = \dfrac{1}{4}$; centroid $\left(\dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4}\right)$.

21. $\overline{x} = 1/2$ and $\overline{y} = 0$ from the symmetry of the region, $V = \int_0^1 \int_{-1}^1 \int_{y^2}^1 dz\,dy\,dx = \dfrac{4}{3}$, $\overline{z} = \dfrac{1}{V}\iiint\limits_G z\,dV = \dfrac{3}{4} \cdot \dfrac{4}{5} = \dfrac{3}{5}$; centroid $\left(\dfrac{1}{2}, 0, \dfrac{3}{5}\right)$.

23. $\overline{x} = \overline{y} = \overline{z}$ from the symmetry of the region, $V = \pi a^3/6$, $\overline{x} = \dfrac{1}{V}\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x\,dz\,dy\,dx = \dfrac{1}{V}\int_0^a \int_0^{\sqrt{a^2-x^2}} x\sqrt{a^2-x^2-y^2}\,dy\,dx = \dfrac{1}{V}\int_0^{\pi/2} \int_0^a r^2\sqrt{a^2-r^2}\cos\theta\,dr\,d\theta = \dfrac{6}{\pi a^3} \cdot \dfrac{\pi a^4}{16} = \dfrac{3a}{8}$; this gives us the centroid $\left(\frac{3a}{8}, \frac{3a}{8}, \frac{3a}{8}\right)$.

25. $M = \int_0^a \int_0^a \int_0^a (a-x)\,dz\,dy\,dx = \frac{a^4}{2}$, $\overline{y} = \overline{z} = \frac{a}{2}$ from the symmetry of density and region, $\overline{x} = \frac{1}{M}\int_0^a \int_0^a \int_0^a x(a-x)\,dz\,dy\,dx = \frac{2}{a^4}\cdot\frac{a^5}{6} = \frac{a}{3}$; mass $\frac{a^4}{2}$, center of gravity $\left(\frac{a}{3}, \frac{a}{2}, \frac{a}{2}\right)$.

27. $M = \int_{-1}^1 \int_0^1 \int_0^{1-y^2} yz\,dz\,dy\,dx = \frac{1}{6}$, $\overline{x} = 0$ by the symmetry of density and region, $\overline{y} = \frac{1}{M}\iiint\limits_G y^2 z\,dV = 6\cdot\frac{8}{105} = \frac{16}{35}$, $\overline{z} = \frac{1}{M}\iiint\limits_G yz^2\,dV = 6\cdot\frac{1}{12} = \frac{1}{2}$; mass $\frac{1}{6}$, center of gravity $\left(0, \frac{16}{35}, \frac{1}{2}\right)$.

29. **(a)** $M = \int_0^1 \int_0^1 k(x^2+y^2)\,dy\,dx = \frac{2k}{3}$, $\overline{x} = \overline{y}$ from the symmetry of density and region, $\overline{x} = \frac{1}{M}\iint\limits_R kx(x^2+y^2)dA = \frac{3}{2k}\cdot\frac{5k}{12} = \frac{5}{8}$; center of gravity $\left(\frac{5}{8}, \frac{5}{8}\right)$.

(b) $\overline{y} = 1/2$ from the symmetry of density and region, $M = \int_0^1 \int_0^1 kx\,dy\,dx = \frac{k}{2}$, $\overline{x} = \frac{1}{M}\iint\limits_R kx^2\,dA = \frac{2}{k}\cdot\frac{k}{3} = \frac{2}{3}$, center of gravity $\left(\frac{2}{3}, \frac{1}{2}\right)$.

31. $V = \iiint\limits_G dV = \int_0^\pi \int_0^{\sin x} \int_0^{1/(1+x^2+y^2)} dz\,dy\,dx \approx 0.666633$, $\overline{x} = \frac{1}{V}\iiint\limits_G x\,dV \approx 1.177406$, $\overline{y} = \frac{1}{V}\iiint\limits_G y\,dV \approx 0.353554$, $\overline{z} = \frac{1}{V}\iiint\limits_G z\,dV \approx 0.231557$.

33. $M = \int_0^{2\pi} \int_0^3 \int_r^3 (3-z)r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^3 \frac{1}{2}r(3-r)^2\,dr\,d\theta = \frac{27}{8}\int_0^{2\pi} d\theta = \frac{27\pi}{4}$.

35. $M = \int_0^{2\pi} \int_0^\pi \int_0^a k\rho^3 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} \int_0^\pi \frac{1}{4}ka^4 \sin\phi\,d\phi\,d\theta = \frac{1}{2}ka^4 \int_0^{2\pi} d\theta = \pi k a^4$.

37. $\bar{x} = \bar{y} = 0$ from the symmetry of the region, $V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 (r\sqrt{2-r^2} - r^3)\,dr\,d\theta = \frac{\pi}{6}(8\sqrt{2}-7)$, $\bar{z} = \frac{1}{V}\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} zr\,dz\,dr\,d\theta = \frac{6}{(8\sqrt{2}-7)\pi}\cdot\frac{7\pi}{12} = \frac{7}{16\sqrt{2}-14}$; centroid $\left(0, 0, \frac{7}{16\sqrt{2}-14}\right)$.

39. $\bar{y} = 0$ from the symmetry of the region, $V = 2\int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{r^2} r\,dz\,dr\,d\theta = 3\pi/2$, $\bar{x} = \frac{2}{V}\int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{r^2} r^2\cos\theta\,dz\,dr\,d\theta \frac{4}{3\pi}(\pi) = 4/3$, $\bar{z} = \frac{1}{V}\int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{r^2} zr\,dz\,dr\,d\theta = \frac{4}{3\pi}(5\pi/6) = 10/9$; centroid $(4/3, 0, 10/9)$.

41. $\bar{x} = \bar{y} = \bar{z}$ from the symmetry of the region, $V = \pi a^3/6$, $\bar{z} = \frac{1}{V}\int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^3 \cos\phi \sin\phi\,d\rho\,d\phi\,d\theta = \frac{6}{\pi a^3}\cdot\frac{\pi a^4}{16} = \frac{3a}{8}$; centroid $\left(\frac{3a}{8}, \frac{3a}{8}, \frac{3a}{8}\right)$.

43. $M = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{4}\sin\phi\, d\phi\, d\theta = \frac{1}{8}(2-\sqrt{2})\int_0^{2\pi} d\theta = \frac{\pi}{4}(2-\sqrt{2})$.

45. $\bar{x} = \bar{y} = 0$ from the symmetry of density and region, $M = \int_0^{2\pi}\int_0^1\int_0^r zr\, dz\, dr\, d\theta = \frac{\pi}{4}$,

$\bar{z} = \frac{1}{M}\int_0^{2\pi}\int_0^1\int_0^r z^2 r\, dz\, dr\, d\theta = \frac{4}{\pi}\cdot\frac{2\pi}{15} = \frac{8}{15}$; center of gravity $\left(0, 0, \frac{8}{15}\right)$.

47. $\bar{x} = \bar{z} = 0$ from the symmetry of the region, $V = 54\pi/3 - 16\pi/3 = 38\pi/3$, $\bar{y} = \frac{1}{V}\int_0^{\pi}\int_0^{\pi}\int_2^3 \rho^3 \sin^2\phi \sin\theta\, d\rho\, d\phi\, d\theta =$ $\frac{1}{V}\int_0^{\pi}\int_0^{\pi}\frac{65}{4}\sin^2\phi\sin\theta\, d\phi\, d\theta = \frac{1}{V}\int_0^{\pi}\frac{65\pi}{8}\sin\theta\, d\theta = \frac{3}{38\pi}\cdot\frac{65\pi}{4} = \frac{195}{152}$; centroid $\left(0, \frac{195}{152}, 0\right)$.

49. $I_x = \int_0^a\int_0^b y^2\delta\, dy\, dx = \frac{\delta a b^3}{3}$, $I_y = \int_0^a\int_0^b x^2\delta\, dy\, dx = \frac{\delta a^3 b}{3}$, $I_z = I_x + I_y = \frac{\delta ab(a^2+b^2)}{3}$.

51. $I_z = \int_0^{2\pi}\int_0^a\int_0^h r^2\delta\, r\, dz\, dr\, d\theta = \delta\int_0^{2\pi}\int_0^a\int_0^h r^3 dz\, dr\, d\theta = \frac{1}{2}\delta\pi a^4 h$.

53. $I_z = \int_0^{2\pi}\int_{a_1}^{a_2}\int_0^h r^2\delta\, r\, dz\, dr\, d\theta = \delta\int_0^{2\pi}\int_{a_1}^{a_2}\int_0^h r^3\, dz\, dr\, d\theta = \frac{1}{2}\delta\pi h(a_2^4 - a_1^4)$.

55. **(a)** The solid generated by R_k as it revolves about L is a cylinder of height Δy_k and radius $x_k^* + \frac{1}{2}\Delta x_k$ from which a cylinder of height Δy_k and radius $x_k^* - \frac{1}{2}\Delta x_k$ has been removed, so its volume is $\pi(x_k^* + \frac{1}{2}\Delta x_k)^2\Delta y_k - \pi(x_k^* - \frac{1}{2}\Delta x_k)^2\Delta y_k = 2\pi x_k^*\Delta x_k\Delta y_k = 2\pi x_k^*\Delta A_k$.

(b) From part (a), $V = \iint_R 2\pi x\, dA = 2\pi\iint_R x\, dA$. From equation (13), this equals $2\pi\cdot\overline{x}\cdot$[area of R].

57. $\overline{x} = k$ so $V = \pi ab\cdot 2\pi k = 2\pi^2 abk$.

59. The region generates a cone of volume $\frac{1}{3}\pi ab^2$ when it is revolved about the x-axis, the area of the region is $\frac{1}{2}ab$ so $\frac{1}{3}\pi ab^2 = \frac{1}{2}ab\cdot 2\pi\overline{y}$, $\overline{y} = \frac{b}{3}$. A cone of volume $\frac{1}{3}\pi a^2 b$ is generated when the region is revolved about the y-axis so $\frac{1}{3}\pi a^2 b = \frac{1}{2}ab\cdot 2\pi\overline{x}$, $\overline{x} = \frac{a}{3}$. The centroid is $\left(\frac{a}{3}, \frac{b}{3}\right)$.

61. It is the point P in the plane of the lamina such that the lamina will balance on any knife-edge passing through P. (If P is in the lamina, then the lamina will also balance on a point of support at P.)

Chapter 14 Review Exercises

3. **(a)** $\iint_R dA$ **(b)** $\iiint_G dV$ **(c)** $\iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$

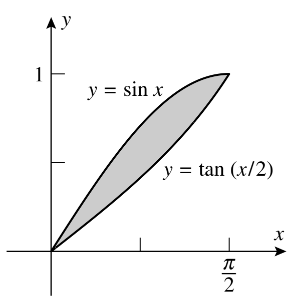
5. $\int_0^1\int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\, dx\, dy$

7. **(a)** The transformation sends $(1,0)$ to (a,c) and $(0,1)$ to (b,d). There are two possibilities: either $(a,c)=(2,1)$ and $(b,d)=(1,2)$ or $(a,c)=(1,2)$ and $(b,d)=(2,1)$. So either $a=2$, $b=1$, $c=1$, $d=2$ or $a=1$, $b=2$, $c=2$, $d=1$.

(b) For either transformation in part (a), $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = 3$, so the area is $\displaystyle\iint_R dA = \int_0^1\int_0^1 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^1\int_0^1 3\,du\,dv = 3$. The diagonals of R cut it into 4 congruent right triangles. One of these has vertices $(0,0)$, $\left(\dfrac{3}{2},\dfrac{3}{2}\right)$, and $(2,1)$, so its bases have lengths $\dfrac{3}{2}\sqrt{2}$ and $\dfrac{1}{2}\sqrt{2}$ and its area is $\dfrac{1}{2}\cdot\dfrac{3}{2}\sqrt{2}\cdot\dfrac{1}{2}\sqrt{2}=\dfrac{3}{4}$; hence R has area $4\cdot\dfrac{3}{4}=3$.

9. $\displaystyle\int_{1/2}^{1} 2x\cos(\pi x^2)\,dx = \frac{1}{\pi}\sin(\pi x^2)\Big]_{1/2}^{1} = -\frac{1}{\sqrt{2}\,\pi}$.

11. $\displaystyle\int_0^1\int_{2y}^{2} e^x e^y\,dx\,dy$

13. 

15. $\displaystyle 2\int_0^8\int_0^{y^{1/3}} x^2\sin y^2\,dx\,dy = \frac{2}{3}\int_0^8 y\sin y^2\,dy = -\frac{1}{3}\cos y^2\Big]_0^8 = \frac{1}{3}(1-\cos 64)\approx 0.20271.$

17. $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2xy}{x^2+y^2}$, and $r=2a\sin\theta$ is the circle $x^2+(y-a)^2=a^2$, so $\displaystyle\int_0^a\int_{a-\sqrt{a^2-x^2}}^{a+\sqrt{a^2-x^2}} \frac{2xy}{x^2+y^2}dy\,dx = \int_0^a x\left[\ln\left(a+\sqrt{a^2-x^2}\right)-\ln\left(a-\sqrt{a^2-x^2}\right)\right]dx = a^2.$

19. $\displaystyle\int_0^2\int_{(y/2)^{1/3}}^{2-y/2} dx\,dy = \int_0^2\left(2-\frac{y}{2}-\left(\frac{y}{2}\right)^{1/3}\right)dy = \left(2y-\frac{y^2}{4}-\frac{3}{2}\left(\frac{y}{2}\right)^{4/3}\right)\Big]_0^2 = \frac{3}{2}.$

21. $\displaystyle\int_0^{2\pi}\int_0^2\int_{r^4}^{16} r^2\cos^2\theta\, r\,dz\,dr\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^2 r^3(16-r^4)\,dr = 32\pi.$

23. **(a)** $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^a (\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^a \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta.$

(b) $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}a/2}\int_{r/\sqrt{3}}^{\sqrt{a^2-r^2}} r^2\,dz\,r dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}a/2}\int_{r/\sqrt{3}}^{\sqrt{a^2-r^2}} r^3\,dz\,dr\,d\theta.$

(c) $\displaystyle\int_{-\sqrt{3}a/2}^{\sqrt{3}a/2}\int_{-\sqrt{(3a^2/4)-x^2}}^{\sqrt{(3a^2/4)-x^2}}\int_{\sqrt{x^2+y^2}/\sqrt{3}}^{\sqrt{a^2-x^2-y^2}} (x^2+y^2)\,dz\,dy\,dx.$

25. $V = \int_0^{2\pi}\int_0^{a/\sqrt{3}}\int_{\sqrt{3}r}^{a} r\,dz\,dr\,d\theta = 2\pi\int_0^{a/\sqrt{3}} r(a-\sqrt{3}r)\,dr = \dfrac{\pi a^3}{9}$.

27. The triangular region R is described by $0 \le x \le 1$, $-x \le y \le x$. Hence $S = \iint_R \sqrt{z_x^2+z_y^2+1}\,dA = \int_0^1\int_{-x}^{x}\sqrt{(4x)^2+3^2+1}\,dy\,dx = \int_0^1\int_{-x}^{x}\sqrt{16x^2+10}\,dy\,dx = \int_0^1 2x\sqrt{16x^2+10}\,dx = \dfrac{1}{24}(16x^2+10)^{3/2}\Big]_0^1 = \dfrac{1}{12}(13\sqrt{26}-5\sqrt{10}) \approx 4.20632$.

29. $(\mathbf{r}_u \times \mathbf{r}_v)\Big|_{\substack{u=1\\v=2}} = \langle -2,-4,1\rangle$, tangent plane $2x+4y-z=5$.

33. **(a)** Add u and w to get $x = \ln(u+w) - \ln 2$; subtract w from u to get $y = \dfrac{1}{2}u - \dfrac{1}{2}w$, substitute these values into $v = y+2z$ to get $z = -\dfrac{1}{4}u + \dfrac{1}{2}v + \dfrac{1}{4}w$. Hence $x_u = \dfrac{1}{u+w}$, $x_v = 0$, $x_w = \dfrac{1}{u+w}$; $y_u = \dfrac{1}{2}$, $y_v = 0$, $y_z = -\dfrac{1}{2}$; $z_u = -\dfrac{1}{4}$, $z_v = \dfrac{1}{2}$, $z_w = \dfrac{1}{4}$, and thus $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{2(u+w)}$.

(b) $V = \iiint_G dV = \int_1^3\int_1^2\int_0^4 \dfrac{1}{2(u+w)}\,dw\,dv\,du = \dfrac{1}{2}(7\ln 7 - 5\ln 5 - 3\ln 3) = \dfrac{1}{2}\ln\dfrac{823543}{84375} \approx 1.139172308$.

35. $A = \int_{-4}^{4}\int_{y^2/4}^{2+y^2/8} dx\,dy = \int_{-4}^{4}\left(2-\dfrac{y^2}{8}\right)dy = \dfrac{32}{3}$; $\bar{y}=0$ by symmetry;
$\int_{-4}^{4}\int_{y^2/4}^{2+y^2/8} x\,dx\,dy = \int_{-4}^{4}\left(2+\dfrac{1}{4}y^2-\dfrac{3}{128}y^4\right)dy = \dfrac{256}{15}$, $\bar{x} = \dfrac{3}{32}\dfrac{256}{15} = \dfrac{8}{5}$; centroid $\left(\dfrac{8}{5},0\right)$.

37. $V = \dfrac{1}{3}\pi a^2 h, \bar{x} = \bar{y} = 0$ by symmetry, $\int_0^{2\pi}\int_0^a\int_0^{h-rh/a} rz\,dz\,dr\,d\theta = \pi\int_0^a rh^2\left(1-\dfrac{r}{a}\right)^2 dr = \dfrac{\pi a^2 h^2}{12}$, centroid $\left(0,0,\dfrac{h}{4}\right)$.

Chapter 14 Making Connections

1. **(a)** $I^2 = \left[\int_0^{+\infty} e^{-x^2}\,dx\right]\left[\int_0^{+\infty} e^{-y^2}\,dy\right] = \int_0^{+\infty}\left[\int_0^{+\infty} e^{-x^2}dx\right]e^{-y^2}dy = \int_0^{+\infty}\int_0^{+\infty} e^{-x^2}e^{-y^2}\,dx\,dy =$
$= \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy.$

(b) $I^2 = \int_0^{\pi/2}\int_0^{+\infty} e^{-r^2} r\,dr\,d\theta = \dfrac{1}{2}\int_0^{\pi/2} d\theta = \dfrac{\pi}{4}$.

(c) Since $I > 0$, $I = \sqrt{\dfrac{\pi}{4}} = \dfrac{\sqrt{\pi}}{2}$.

3. **(a)** 1.173108605 **(b)** $\int_0^{\pi}\int_0^1 re^{-r^4}\,dr\,d\theta = \pi\int_0^1 re^{-r^4}\,dr \approx 1.173108605$.

5. **(a)** Let S_1 be the set of points (x,y,z) which satisfy the equation $x^{2/3}+y^{2/3}+z^{2/3} = a^{2/3}$, and let S_2 be the set of points (x,y,z) where $x = a(\sin\phi\cos\theta)^3, y = a(\sin\phi\sin\theta)^3, z = a\cos^3\phi$, $0 \le \phi \le \pi, 0 \le \theta < 2\pi$. If (x,y,z) is a point of S_2 then $x^{2/3}+y^{2/3}+z^{2/3} = a^{2/3}[(\sin\phi\cos\theta)^3 + (\sin\phi\sin\theta)^3 + \cos^3\phi] = a^{2/3}$, so (x,y,z) belongs to S_1. If (x, y, z) is a point of S_1 then $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$. Let $x_1 = x^{1/3}, y_1 = y^{1/3}, z_1 = z^{1/3}, a_1 = a^{1/3}$. Then $x_1^2 + y_1^2 + z_1^2 = a_1^2$, so in spherical coordinates $x_1 = a_1 \sin\phi\cos\theta, y_1 = a_1 \sin\phi\sin\theta, z_1 = a_1\cos\phi$, with $\theta = \tan^{-1}\left(\dfrac{y_1}{x_1}\right) = \tan^{-1}\left(\dfrac{y}{x}\right)^{1/3}, \phi = \cos^{-1}\dfrac{z_1}{a_1} = \cos^{-1}\left(\dfrac{z}{a}\right)^{1/3}$. Then $x = x_1^3 = a_1^3(\sin\phi\cos\theta)^3 = a(\sin\phi\cos\theta)^3$, similarly $y = a(\sin\phi\sin\theta)^3, z = a\cos\phi$ so (x, y, z) belongs to S_2. Thus $S_1 = S_2$.

(b) Let $a = 1$ and $\mathbf{r} = (\cos\theta\sin\phi)^3\,\mathbf{i} + (\sin\theta\sin\phi)^3\,\mathbf{j} + \cos^3\phi\,\mathbf{k}$, then $S = 8\displaystyle\int_0^{\pi/2}\int_0^{\pi/2} \|\mathbf{r}_\theta \times \mathbf{r}_\phi\|\,d\phi\,d\theta =$ $72\displaystyle\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\cos\theta\sin^4\phi\cos\phi\sqrt{\cos^2\phi + \sin^2\phi\sin^2\theta\cos^2\theta}\,d\theta\,d\phi \approx 4.4506$.