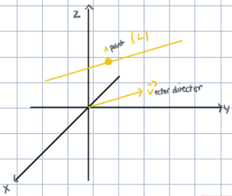
Professor Leonard Math Notes

11.5 - Lines and Panel

Same direction: same unit vector or parallel

In 3D, to define a line, you need a point and a vector director (like the slope in 2D). If you have 2 points, you can find the director vector between the points.



Since the line (L) // vector director, then $\text{Vector}_{Line} = t \cdot \vec{v}$ where t is a scalar.

Given a point $P_0(x_0, y_0, z_0)$ and any other point $P(x, y, z)$ on the line, so

$$\text{Vector}_L = \overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$$

let $\vec{v} = \langle a, b, c \rangle$ be the direction vector where a, b, c are direction #'s.

Then if you have 2 points P_0 and P, $\overrightarrow{P_0P} = t \cdot \vec{v} \iff \langle x - x_0, y - y_0, z - z_0 \rangle = t \cdot \langle a, b, c \rangle$

$\langle x - x_0, y - y_0, z - z_0 \rangle = \langle at, bt, ct \rangle \iff x - x_0 = at,\ y - y_0 = bt,\ z - z_0 = ct$

↳ $\boxed{x = at + x_0 \ ;\ y = bt + y_0 \ ;\ z = ct + z_0}$ eq. of a line 1 (parametric eq. in 3D)

$\boxed{t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$ eq. of a line 2 (symmetric equation)

This works for any point $P_0(x_0, y_0, z_0)$ and Dir. Vector $\vec{v} = \langle a, b, c \rangle$

Note: 1) If a direction number ($\langle a, b, c \rangle$) = 0, then the line lies on a plane.

Suppose that $a = 0$, then $x = x_0$ and the line stays in $x = x_0$ plane (parallel to yz-plane)

2) Lines are parallel if their direction vectors are // (scalar multiples $\vec{AB} = m\,\vec{A_2}$)

Ex: 1) Find the equation of the line containing $P(-1, -2, -1/2)$ and $Q(1, 3/2, -3)$

$\vec{v} = \overrightarrow{PQ} = \langle 1+1, 3/2 + 2, -3 + 1/2 \rangle = \langle 2, 7/2, -5/2 \rangle \cdot 2 = \langle 4, 7, -5 \rangle$ ← you can multiply here b/c we are only looking for a direct. however you can't for a distance

Need a point $P_1(-1, -2, -1/2)$ ← you can't multiply the point, and a vector $\vec{v_1} \langle 4, 7, -5 \rangle$

$\boxed{x = -1 + 4t,\ y = -2 + 7t,\ z = -\frac{1}{2} - 5t}$ ⇒ parametric eq. of the line. To find a point on the line, plug a value for t.

A $\vec{v} = \langle 4, 7, -5 \rangle$

$\boxed{\frac{x+1}{4} = \frac{y+2}{7} = \frac{z + 1/2}{-5}}$ ⇒ symmetric equation

what if "c = 0"? ⇒ $\frac{x+1}{4} = \frac{y+2}{7}$, $z = -\frac{1}{2}$ ⇒ the line is on the plane $z = -1/2$

2) Show $L_2 : \frac{x-7}{8} = \frac{y - 1/2}{14} = \frac{z+9}{-10}$ is // L_1

$P_2(7, 1/2, -9)$ and $\vec{v_2} = \langle 8, 14, -10 \rangle$; $\vec{v_2} = 2 \cdot \langle 4, 7, -5 \rangle = 2 \cdot \vec{v_1} \Rightarrow \vec{v_2} \parallel \vec{v_1}$

So, $L_1 \parallel L_2$

$L_2 : x = 7 + 8t,\ y = \frac{1}{2} + 14t,\ z = -9 - 10t$

Ex: Airplane 1 ; $L_1 : x_1 = 1 - 2t_1,\ y_1 = -1 - 3t_1,\ z_1 = -2 + t_1$ $\vec{v_1} = \langle -2, -3, 1 \rangle$

" 2 : $L_2 : x_2 = -3 + t_2,\ y_2 = -2 + 2t_2,\ z_2 = 3 - t_2$ $\vec{v_2} = \langle 1, 2, -1 \rangle$

} not parallel b/c NOT scalar multiples

Check for an intersection: $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

"x": $1 - 2t_1 = -3 + t_2$
"y": $-1 - 3t_1 = -2 + 2t_2$
"z": $-2 + t_1 = 3 - t_2$

$1 - 2t_1 = -3 + t_2$
$+\ -2 + t_1 = 3 - t_2$
$-1 - t_1 = 0 \Rightarrow t_1 = -1,\ t_2 = 6$

$\Rightarrow -1 - 3(-1) = 2$; $-2 + 2(6) = 10$; $\underbrace{2 \quad\quad 10}_{\neq}$ ⇒ "x": 3, "y": $2 \neq 10$, "z": -3 } So the planes do not intersect

↳ Lines NOT //
Lines DO NOT Intersect ⇒ Skew lines

Ex: $L_1 = \frac{x_1 - 1}{-1} = \frac{y_1 - 3}{-2} = z_1$; $L_2: \frac{x_2 - 2}{3} = \frac{y_2 - 3}{2} = z_2 - 1$

1) Are they //
2) Do they intersect?
3) Are they skew?

1) $L_1: x_1 = 1 - t_1,\ y_1 = 3 - 2t_1,\ z_1 = t_1$ $\quad L_2: x_2 = 2 + 3t_2,\ y_2 = 3 + 2t_2,\ z_2 = 1 + t_2$

$\vec{v_1}: \langle -1, -2, 1\rangle \quad \vec{v_2}: \langle 3, 2, 1\rangle \Rightarrow L_1$ NOT // L_2

2) "x": $1 - t_1 = 2 + 3t_2$
"y": $3 - 2t_1 = 3 + 2t_2$
"z": $t_1 = 1 + t_2$

$+\ \begin{aligned} 1 - t_1 &= 2 + 3t_2 \\ t_1 &= 1 + t_2 \\ \hline 1 &= 3 + 4t_2 \Rightarrow t_2 = -\tfrac{1}{2} \end{aligned}$

and $t_1 = 1/2$

$\Rightarrow$ "x": $1/2$, "y": $2 = 2$, "z": $1/2$ $\Rightarrow$ these lines intersect at $(1/2, 2, 1/2)$

Find the angle between planes (or lines) as they collide

Angle between lines = Angles between DIR. Vectors

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} \rightarrow \cos\theta = \frac{|\vec{v_1}\cdot\vec{v_2}|}{\|\vec{v_1}\|\cdot\|\vec{v_2}\|} = \frac{|-6|}{\sqrt{6}\times\sqrt{14}} = \frac{6}{\sqrt{84}} \Rightarrow \theta = \cos^{-1}\left(\frac{6}{\sqrt{84}}\right) \approx 49.1°$

(absolute value gives the acute angle)

$\vec{v_1}\cdot\vec{v_2} = -3 - 4 + 1 = -6$

$\|\vec{v_1}\| = \sqrt{6} \quad \|\vec{v_2}\| = \sqrt{14}$

Find where plane 2 (line 2) cross the coordinate planes ; $x_2: 2 + 3t_2$; $y_2 = 3 + 2t_2$, $z_2 = 1 + t_2$

xy-plane: set $z = 0 \Rightarrow 0 = 1 + t_2 \rightarrow t_2 = -1$; so $(-1, 1, 0)$ is on the xy-plane

xz-plane: set $y = 0 \Rightarrow 0 = 3 + 2t_2 \rightarrow t_2 = -3/2$; so $\left(-\frac{5}{2}, 0, -1/2\right)$

yz-plane: set $x = 0 \Rightarrow 0 = 2 + 3t_2 \rightarrow t_2 = -\frac{2}{3}$; so $\left(0, \frac{5}{3}, \frac{1}{3}\right)$

Planes in $\mathbb{R}3$

Normal: A vector $\perp$ to a plane. So given one point on a plane and a normal vector to the plane $\vec{n} = \langle a, b, c\rangle$, we can define a specific plane.

* A vector between $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$ on the plane $\overrightarrow{PP_0} = \langle x - x_0, y - y_0, z - z_0\rangle$ and a normal to the plane $\vec{n} = \langle a, b, c\rangle$

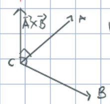
$\vec{n} \perp \overrightarrow{P_0P} \Rightarrow \vec{n}\cdot\overrightarrow{P_0P} = 0 \Leftrightarrow \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$

$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$ => Standard form of the equation of a plane

$ax - ax_0 + by - by_0 + cz - cz_0 = 0$

$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{d}$, =>

$\boxed{ax + by + cz = d}$ => General form of the eq. of a plane



With 3 points you can find 2 vectors and find the cross products

Ex: EQ of a plane with $P(3, 6, -2)$ and // to $2x + 3y - z = 4$ $\Rightarrow \vec{n_2} = \langle 2, 3, -1\rangle$

* If planes are //, they have the same normal vector

So $\vec{n_1} = \vec{n_2} = \langle 2, 3, -1\rangle$ and we have a point $P(3, 6, -2)$

↳ $2(x - 3) + 3(y - 6) - 1(z + 2) = 0$

$2x - 6 + 3y - 18 - z - 2 = 0$

$\boxed{2x + 3y - z = 26}$ => equation of the plane

Ex: Find Eq. of the plane containing $P(2, 3, -1)$; $Q(1, -2, 3)$, $R(-1, 2, 4)$

$\overrightarrow{PQ} = \langle -1, -5, 4\rangle$, $\overrightarrow{PR} = \langle -3, -1, 5\rangle$

$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -1 & -5 & 4 \\ -3 & -1 & 5 \end{vmatrix} = (-25+4)\vec{i} - (-5+12)\vec{j} + (1-15)\vec{k} = \langle -21, -7, -14 \rangle \div (-7)$

$P(2,3,-1)$ and $\vec{n} = \langle 3, 1, 2 \rangle = 3\vec{i} + 1\vec{j} + 2\vec{k}$

$3(x-2) + (y-3) + 2(z+1) = 0 \iff 3x - 6 + y - 3 + 2z + 2 = 0 \Rightarrow \boxed{3x + y + 2z = 7}$

to check your work, you try to find if your coefficients $= \langle a, b, c \rangle$ of $\vec{n}$

Ex: Find a plane with $P(3,-4,5)$ containing the line $L: \frac{x-2}{2} = \frac{y+1}{-3} = \frac{z+3}{5}$

$P(3,-4,5)$. $Q(2,-1,-3)$ is a point on the line. $\vec{b} = \langle 2, -3, 5 \rangle$ a vector direction of the line

$\vec{PQ} = \langle -1, 3, -8 \rangle = \vec{a}$.

$\vec{a} \times \vec{b} = \vec{n} = \begin{vmatrix} i & j & k \\ -1 & 3 & -8 \\ 2 & -3 & 5 \end{vmatrix} = (15-24)\vec{i} - (-5+16)\vec{j} + (3-6)\vec{k} = \langle -9, -11, -3 \rangle \div (-1) \Rightarrow \langle 9, 11, 3 \rangle$

So eq: $9(x-3) + 11(x+4) + 3(x-5) = 0$

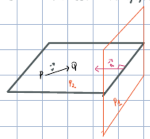
$9x - 27 + 11x + 44 + 3x - 15 = 0$

$\boxed{9x + 11x + 3x = -2} \Rightarrow$ eq. of a plane

Planes cannot be skew, they go forever.
If they are not //, they met (intersect somewhere)
When they cross, they make a line

Ex: Plane with $P(2,1,1)$ and $Q(-1,3,2)$. & $\perp$ to the plane $P_1: \underbrace{2x+3y-4z}_{} = 3$

$\vec{n_1}$ must be in the plane we need to find



$\vec{v} = \vec{PQ} = \langle -3, 2, 1 \rangle$

$\vec{n_1} = \langle 2, 3, -4 \rangle$

Then $\vec{n} = \vec{v} \times \vec{n_1} = \begin{vmatrix} i & j & k \\ -3 & 2 & 1 \\ 2 & 3 & -4 \end{vmatrix} = \langle 11, 10, 13 \rangle$

So $\vec{n} = \langle 11, 10, 13 \rangle$ and $P(2,1,1)$

eq: $11(x-2) + 10(y-1) + 13(z-1) = 0$

$11x + 10y + 13z = 45$

NOTE:
- 2 planes are // if their normal vectors are //
- " " " $\perp$ " " " " " $\perp$
- Angles between planes = Angle between normal vector

Ex: $P_1: 3x - y + 2z = 2$, $P_2: 2x + 3y + z = 4$

$\vec{n_1} = \langle 3, -1, 2 \rangle$ $\quad \vec{n_2} = \langle 2, 3, 1 \rangle$.

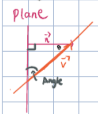
- $\vec{n_1}$ not // $\vec{n_2}$, Not scalar multiple
- $\vec{n_1} \cdot \vec{n_2} = 6 - 3 + 2 = 5 \neq 0 \Rightarrow \vec{n_1}$ not $\perp \vec{n_2} \Rightarrow$ NOT $\perp$.

So these planes are neither // nor $\perp$.

$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{\|\vec{n_1}\| \|\vec{n_2}\|} = \frac{5}{\sqrt{14} \cdot \sqrt{14}} = \frac{5}{14}$

$\theta = \cos^{-1}\left(\frac{5}{14}\right) = 69.1°$ $\Rightarrow$ So these lines met at an angle of $\theta = 69.1°$

Ex: To find the angle between a line and a plane



$\cos\theta = \frac{|\vec{n} \cdot \vec{v}|}{\|\vec{n}\| \|\vec{v}\|}$ so Angle $= \left(90° - \cos^{-1} \frac{|\vec{n} \cdot \vec{v}|}{\|\vec{n}\| \|\vec{v}\|}\right)$

NOTE: Intersection of line and plane $\Rightarrow$ "Plug" line into plane $ax + by + cz = d$ and $x = x_0 + at$, $y = y_0 + bt$ and $z = z_0 + ct$

$\hookrightarrow a(x_0 + at) + b(y_0 + bt) + c(z_0 + c_1 t)$

Ex: $P: 2x + 3y - z = 9$ and $L: x = 2 + 3t$, $y = -1 + t$, $z = 3 - 2t$

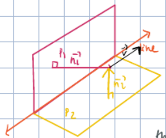
$2(2 + 3t) + 3(-1 + t) - (3 - 2t) = 9 \Rightarrow t = 1 \Rightarrow P(5, 0, 1)$

Ex: Find EQ. for line of Intersection of 2 planes

$P_1: 2x-3y+4z=3 \quad \vec{n_1}=\langle 2,-3,4\rangle \quad \vec{n_1}$ not $\parallel \vec{n_2}$ b/c $\vec{n_1} \neq t\vec{n_2}$

$P_2: x+4y-2z=7 \quad \vec{n_2}=\langle 1,4,-2\rangle \quad \vec{n_1}$ not $\perp \vec{n_2}$ bc $\vec{n_1}\cdot\vec{n_2}=2-12-8\neq 0$

Lines need a POINT and a DIRECTION VECTOR



At one point, the normals of each point will meet. $\Rightarrow$ so $\vec{n_1}\times\vec{n_2}=\vec{v}=\begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ 1 & 4 & -2 \end{vmatrix} = (6-16)\vec{i}-(-4-4)\vec{j}+(8+3)\vec{k} = \langle -10, 8, 11\rangle = \vec{v}$

For the point, set x or y or z = 0 in the system

· If $y=0 \Rightarrow \begin{cases} 2x+4z=3 \\ x-2z=7 \end{cases}$ $\quad 7+2z=\frac{3-4z}{2}$; $14+4z=3-4z$; $8z=-11 \Rightarrow z=-\frac{11}{8}$

the point lies on the xz-plane

and $x=7+2\left(\frac{-11}{8}\right)=\frac{7-22}{8}=\frac{17}{4}$

$P\left(\frac{17}{4}, 0, -\frac{11}{8}\right)$

$\boxed{L: x=\frac{17}{4}-10t,\ y=8t,\ z=-\frac{11}{8}+11t}$ $\Rightarrow$ eq. of the line

11.6 - Cylinders and surfaces

Cylinders:

1) Equations of cylinders have only 2 variables. These equations give a trace of the curve on the coordinate plane denoted by the given variables.

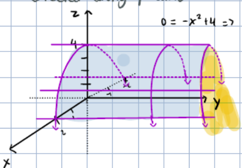
2) Curve is directed along the axis of the missing variables

3) The curve / trace DOES NOT change along the direction axis.

Ex: $z=4-x^2 \Rightarrow$ - 2 variables = cylinder
- trace is drawn on the xz plane
- directed along y-axis

$z=-x^2+4$ · parabola
· shifted up 4 on the z-axis $\Rightarrow$
· opening towards $\ominus$ z-axis

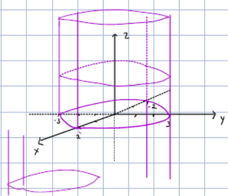
$0=-x^2+4 \Rightarrow x=\pm 2$



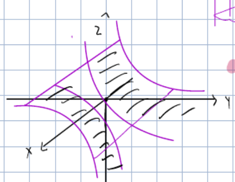
Ex: $9x^2+4y^2=36$ · 2 variables
· trace drawn on xy-plane
· projected on z axis

$\frac{x^2}{4}+\frac{y^2}{9}=1$

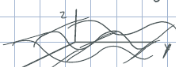
↳ ellipse on xy-plane with intercepts of $x=\pm 2$ and $y=\pm 3$



Ex: $yz=1$ → 2 variables : cylinder
↳ $z=\frac{1}{y}$ → Trace on yz-plane
→ Directed along x-axis



$z=\cos y$



General surfaces

- Have all 3 variables
- Traces occur on coord. planes and/or on planes // to coord. planes
- Still directed along an axis, but the trace changes along the axis

Step: 1) Determine the type of surface
2) Determine the direction axis
3) Find some traces on coordinate plane
4) Find at least 2 other traces along the direction axis

A) Ellipsoid : $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

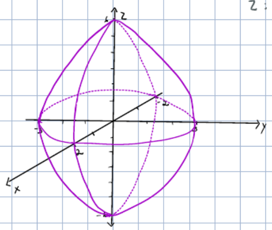
How to tell:
1) All (+)
2) All power "2"
3) Has a constant

NOTE
Standard form helps with traces.

Intercepts:
$x=\pm a$
$y=\pm b$
$z=\pm c$

Ex: $9x^2+4y^2+z^2=36$ ⇒ surface b/c 3 variables
- All (+) - All sq^2 - Has a constant ⇒ Ellipsoid

$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{36}=1$ ⇒ Intercept = $x: \pm 2$
$y: \pm 3$
$z: \pm 6$



B) One Sheet Hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

How to tell:
1) Has 1 ⊖
2) All power "2"
3) Has a constant

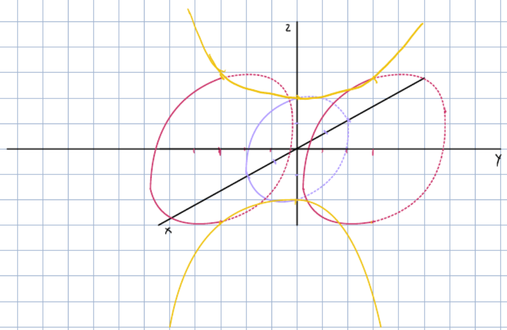
NOTE
1) Standard form helps with traces.
2) is ALWAYS directed along the axis with the ⊖!
3) Set ⊖ var := 0 AND $=\pm\sqrt{\text{DENOM}}$ to get 3 Traces (will always be circles or ellipses)

Ex: $9x^2+4y^2-z^2=36$ ⇒ All sq^2, 1 (−), const ⇒ 1 Sheet hyperboloid along y-axis

↳ $\frac{x^2}{4}-\frac{y^2}{9}+\frac{z^2}{4}=1$

Traces: $y=0$: $\frac{x^2}{4}+\frac{z^2}{4}=1$ ⇒ circle of radius $=2$ centered at $(0,0,0)$ b/c $a^2=b^2$, $\sqrt{4}$

if $y=\pm 3$, ⇒ $\frac{x^2}{4}-\frac{9}{9}+\frac{z^2}{4}=1$
↳ $\frac{x^2}{4}+\frac{z^2}{4}=2$
↳ $\frac{x^2}{8}+\frac{z^2}{8}=1$ ⇒ circle $r=2\sqrt{2}\approx 2.83$ ($\sqrt{8}$)



C) 2-Sheet Hyperboloid $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

How to tell:
1) Has 1 (+)
2) All power "2"
3) Has a constant

NOTE
1) Standard form helps with traces.
2) IS ALWAYS directed along the axis with The (+)!
3) Set (+) var = 0 DOES NOTHING!!!

→ so Set BOTH (−) var = 0 to get Axis-intercept

→ Set (+) variable = to some # divisible b DENOM.

Ex: $-y^2 + x^2 + 9z^2 + 9 = 0$

↳ $x^2 + 9z^2 - y^2 = -9 \iff -x^2 + y^2 - 9z^2 = 9 \iff \frac{-x^2}{9} + \frac{y^2}{9} - \frac{z^2}{1} = 1$

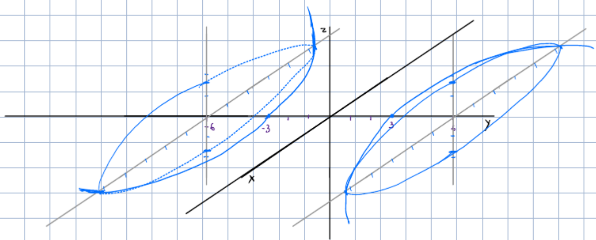
- All sq², 1(+), constant ⇒ 2 Sheet · Hyperboloid (2⊖)
- Along Y- AXIS

Traces:

For intercept: $x = z = 0 \Rightarrow \frac{y^2}{9} = 1 \Rightarrow y = \pm 3$

- If $y = \pm 6 \Rightarrow \frac{-x^2}{9} + \frac{36}{9} - \frac{z^2}{1} = 1 \Rightarrow -\frac{x^2}{9} - \frac{z^2}{1} = -3$

↳ $\frac{x^2}{9} + \frac{z^2}{1} = 3 \Rightarrow \frac{x^2}{27} + \frac{z^2}{3} = 1$ ⇒ ellipse parallel to xz plane



D) Cones $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

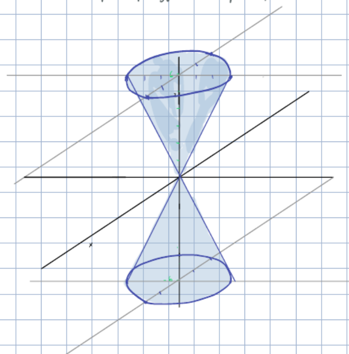
How to tell:
1) Has 1 (−)
2) All power "2"
3) NO CONSTANT

NOTE
1) Standard form helps with traces
2) Always meet at origin
3) IS ALWAYS directed along the axis with The (−)!
4) Set (−) var = 0 says you are at the origin
So plug in values for (−) VARiable that are divisible by DENOM.
↳ Circles or ellipses

Ex: $z^2 - 9x^2 - 4y^2 = 0 \Rightarrow 9x^2 + 4y^2 - z^2 = 0$ ⇒ no CONSTANT, ALL SQ²

↳ $\frac{9x^2 + 4y^2 - z^2 = 0}{36} \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{36} = 0$ along z-axis

if $z = \pm 6 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} - \frac{36}{36} = 0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$ ⇒ ellipse



E) Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$

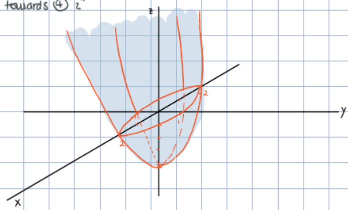
How to tell:
1) 3 variables with 2 sq²
2) sq² variables are (+)

NOTES
* Opens along axis of VAR. with degree 1
  - Coefficient of degree 1 VAR. gives DIRECTION:
    - (+) opens towards POSITIVE part of axis
    - (−) opens towards NEGATIVE part of axis
* Set Degree 1 VAR = 0 to get a Trace on a COORD. plane (If paraboloid is SHIFTED)

Ex: $z = x^2+4y-4$; we have 2 sq² VAR ⇒ Paraboloid. It is opening along z axis towards ⊕ "z"

$x^2+4y^2 = \underbrace{z+4}$
shift DOWN 4 on (z-axis).

If $z=0$ ⇒ $x^2+4y^2=4$ ⇒ $\frac{x^2}{4}+\frac{y^2}{1}=1$



Other example: $x^2+4y^2=z$,

If $z=0$ ⇒ $x^2+4y^2=0$; there is no more an ellipse

If $z=4$ ⇒ $x^2+4y^2=4$ ⇒ $\frac{x^2}{4}+\frac{y^2}{1}=1$

the shift is undone and we start at the origin

F) Hyperbolic paraboloids: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=cz$

How to tell:

1) 3 variables with 2 sq²
2) ONE SQ² has ⊖

NOTES

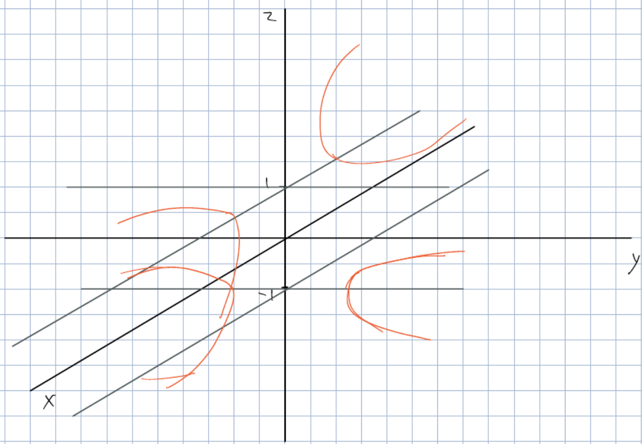
* Degree 1 VAR. Gives direction Axis
* Plug in ⊕ AND ⊖ #'s to degree 1 variable

Ex: $x^2-y^2-z=0$ ⇒ 3 variables with 2 sq².

$x^2-y^2=z$ ⇒ 2 sq², 1 (−), hyperboloid Paraboloid along "z"

set $z=1$ ⇒ $x^2-y^2=1$ ⇒ HYPERBOLA along "x"

set $z=-1$ ⇒ $x^2-y^2=-1$ ⇒ $y^2-x^2=1$ ⇒ HYPERBOLA along "y"



12.1 - Vector Functions

Parametric Eq:

$x = f(t)$, $y = g(t)$; "t" is the PARAMETER for some intervals "I" on a common domain.

For $\mathbb{R}$-3:

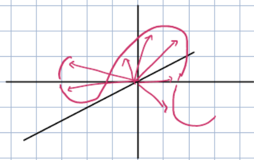
$x = f(t)$, $y = g(t)$, $z = h(t)$

But, vectors are defined by $\vec{r} =$ "x"$\vec{i}$ + "y"$\vec{j}$ + "z"$\vec{k}$

so $\boxed{\vec{r}(t) = \underbrace{f(t)}_{\text{"x"}}\vec{i} + \underbrace{g(t)}_{\text{"y"}}\vec{j} + \underbrace{h(t)}_{\text{"z"}}\vec{k}} \Rightarrow \langle f(t), g(t), h(t) \rangle$

The TERMINAL POINTS of the vector create a CURVE, then SPACE (not surface) for the domain of "t"!

* A vector function is a parametrically defined function where the TERMINAL POINTS of our vectors trace a CURVE in 3-D. The fact that "t" has a certain domain gives $\vec{r}(t)$ an orientation.



Ex: $\vec{r}(t_0) = \langle -3, 5, 2 \rangle$. Terminal point: $(-3, 5, 2)$ b/c vector starts at the origin.

Ex: $\vec{r}(t) = \langle \sqrt{t}, \frac{1}{t-1}, \ln t \rangle$

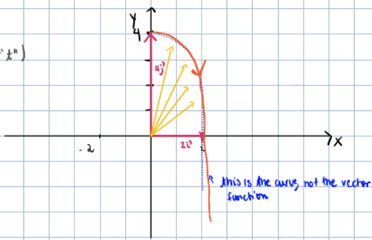
- $x = \sqrt{t}$, $y = \frac{1}{t-1}$ and $z = \ln t$
- $\hookrightarrow t \geq 0$ $\quad \hookrightarrow t \neq 1$ $\quad \hookrightarrow t > 0$ So Domain: $]0,1[\cup]1,+\infty[$

- Find vectors at $t=2$ and $t=4$

If $t=2$ (should ∈ to domain) $\Rightarrow \vec{r}(2) = \langle \sqrt{2}, 1, \ln 2 \rangle$; @ $t=4$, $\vec{r}(4) = \langle 2, \frac{1}{3}, \ln 4 \rangle$

SKETCHING

1. Identify "x", "y" and "z"
2. Use 1 or MORE components to get A CURVE or a SURFACE (get rid of "t")
   - For TWO components, sketch on a PLANE
   - For THREE components, the CURVE is on a SURFACE
3. Use values of "t" to find POINTS and ORIENTATION



Ex: $\vec{r}(t) = \sqrt{t}\,\vec{i} + (4-t)\vec{j}$, $t \geq 0$

$x = \sqrt{t}$, $y = 4 - t \Rightarrow x^2 = t$ so $y = 4 - x^2 \Rightarrow y = -x^2 + 4$

@ $t=0 \Rightarrow \vec{r}(0) = 4\vec{j}$

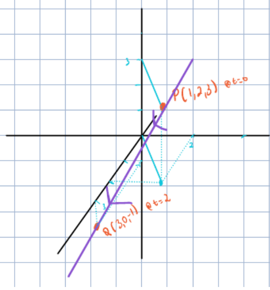
@ $t=4 \Rightarrow \vec{r}(4) = 2\vec{i}$

Ex: $\vec{r}(t) = (1+t)\vec{i} + (2-t)\vec{j} + (3-2t)\vec{k}$ Domain: $-\infty < t < +\infty$ or $t \in \mathbb{R}$

$x = 1+t \quad y = 2-t \quad z = 3-2t \Rightarrow$ parametric eq. of a line (this is the curve)

@ $t=0$; a Point $P(1,2,3)$ * use value of t to make at least one component $=0$

@ $t=2 \Rightarrow Q(3,0,-1)$



Ex: $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $t \geq 0$

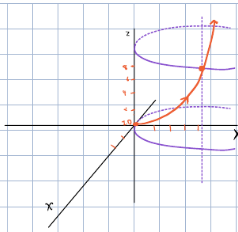
$\underbrace{x = t \quad y = t^2}$ $\quad z = t^3$

$y = x^2 \Rightarrow$ this is a cylinder along "z" with the trace on the xy-plane

* More than 1 surface is possible... Stick to ones you are familiar with
* The "unused" component gives the CURVE (that is on the surface of the cylinder)

↳ $y = x^2$ ... gives the surface

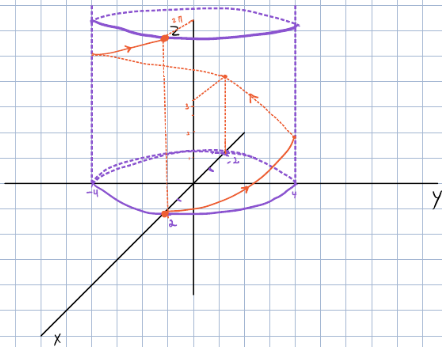
$z = t^3$ ... gives the curve on the



↗ use the 1st point of the domain

@ $t = 0 \Rightarrow \vec{r}(0) = \langle 0, 0, 0 \rangle \Rightarrow P_0(0,0,0)$

@ $t = 2$; $\vec{r}(2) = \langle 2, 4, 8 \rangle \Rightarrow P_1(2,4,8)$



Ex: $\vec{r}(t) = 2\cos t\,\vec{i} + 4\sin t\,\vec{j} + t\,\vec{k}$, $0 \leq t \leq 2\pi$

$\underbrace{x = 2\cos t \quad y = 4\sin t}$, $\quad z = t$

$\frac{x}{2} = \cos t \quad \frac{y}{4} = \sin t$

$\cos^2 t + \sin^2 t = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1$,

This is a cylinder along z-axis with the shape / trace of an ellipse on xy plane

$z = t$ is on the cylinder (the curve)

@ $t = 0$: $P(2,0,0)$

↗ The curve can't pass through the origin b/c the cylinder does not go through the cylinder. It has to be on the origin.

@ $t = \frac{\pi}{2}$: $P(0, 4, \frac{\pi}{2})$

@ $t = \pi$; $P(-2, 0, \pi)$

@ $t = 2\pi$, $P(2, 0, 2\pi)$

* If we have $\vec{r}(t) = 2\cos t\,\vec{i} + 4\sin t\,\vec{j} + 3\,\vec{k}$, $0 \leq t \leq 2\pi$

$\underbrace{x = 2\cos t \quad y = 4\sin t}$, $\quad z = 3$

$\frac{x}{2} = \cos t \quad \frac{y}{4} = \sin t$

$\cos^2 t + \sin^2 t = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1$,

This is a cylinder along z-axis with the shape / trace of an ellipse on xy plane

$z = 3$ is on the cylinder (the curve)

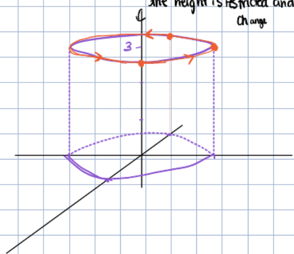
@ $t = 0$: $P(2,0,3)$

↗ The curve can't pass through the origin b/c the cylinder does through the cylinder. It has to be on the origin.

@ $t = \frac{\pi}{2}$: $P(0, 4, 3)$

@ $t = \pi$; $P(-2, 0, 3)$

@ $t = 2\pi$, $P(2, 0, 3)$



* $\vec{r}(t) = t\cos t\,\vec{i} + t\sin t\,\vec{j} + t\,\vec{k}$, $0 \leq t \leq 2\pi$

$\underbrace{x = t\cos t \quad y = t\sin t}$, $\quad z = t$ ↘ b/c $\cos^2 x + \sin^2 x = 1$

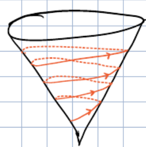
$\cos t = \frac{x}{t} \quad \sin t = \frac{y}{t} \Rightarrow \frac{x^2}{t^2} + \frac{y^2}{t^2} = 1$

$z = t \Rightarrow \frac{x^2}{z^2} + \frac{y^2}{z^2} = 1$

↳ $x^2 + y^2 = z^2$

↳ $x^2 + y^2 - z^2 = 0$

cone b/c 3 sq, 1 ⊖ and NO constant

$\Rightarrow$ 

Limits

$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle \Rightarrow$ this is a vector

Ex: $\lim_{t\to 2} \left[\sqrt{t}\,\vec{i} + \left(\frac{t^2-4}{t-2}\right)\vec{j} + \left(\frac{t}{t^2+1}\right)\vec{k} \right] =$

- $\lim_{t\to 2} \sqrt{t} = \sqrt{2}$
- $\lim_{t\to 2} \left(\frac{t^2-4}{t-2}\right) \overset{L}{=} \lim_{t\to 2} \frac{2t}{1} = 4$
- $\lim_{t\to 2} \frac{t}{t^2+1} = \frac{2}{5}$

so $\lim_{t\to 2} \left[\sqrt{t}\,\vec{i} + \left(\frac{t^2-4}{t-2}\right)\vec{j} + \left(\frac{t}{t^2+1}\right)\vec{k} \right] = \langle \sqrt{2}, 4, 2/5 \rangle$

Ex: $\lim_{t\to 0^+} \left[\cos t\,\vec{i} + \frac{\tan t}{t}\vec{j} + t\ln t\,\vec{k} \right] = \langle 1, 1, 0 \rangle = \vec{i} + \vec{j}$

- $\lim_{t\to 0^+} \cos t = 1$
- $\lim_{t\to 0^+} \frac{\tan t}{t} = \lim_{t\to 0^+} \frac{\sin t}{\cos t} \times \frac{1}{t} = \lim_{t\to 0^+} \frac{\sin t}{t} \cdot \frac{1}{\cos t} = 1$
- $\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \overset{L}{=} \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} -t = 0$

- $\lim_{t\to -\infty} \tan^{-1} = -\frac{\pi}{2}$
- $\lim_{t\to -\infty} e^{2t} = 0$

Continuity

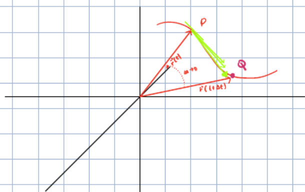
$\vec{r}(t) = \left\langle \frac{\cos t - 1}{t}, \frac{\sqrt{t}}{1+2t}, te^{-1/t} \right\rangle \Rightarrow$ continuous on $t > 0$

$\hookrightarrow t \neq 0$ $\quad \hookrightarrow t \geq 0,\ t \neq -\frac{1}{2}$ $\quad \hookrightarrow t \neq 0$

Ex: $\vec{r}(t) = \left(\frac{2t}{t^2-4}\right)\vec{i} + \sin^{-1}(t)\,\vec{j} + \sqrt[3]{t}\,\vec{k} \Rightarrow$ continuous on $t \in [-1, 1]$

$\hookrightarrow t^2-4 \neq 0,\ t \neq \pm 2$ $\quad \hookrightarrow -1 \leq t \leq 1$ $\quad \hookrightarrow -\infty < t < +\infty$

12.2 - Derivatives and Integrals of Vector Functions



$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$;

$\hookrightarrow$ this gives a tangent vector to a space curve

$\vec{r}'(t) = \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \Rightarrow \begin{cases} \text{- Tangent vector to a space curve} \\ \text{- Instantaneous velocity} \end{cases}$

$= \lim_{\Delta t\to 0} \frac{\left(f(t+\Delta t)\vec{i} + g(t+\Delta t)\vec{j} + h(t+\Delta t)\vec{k}\right) - f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}}{\Delta t}$

$= \lim_{\Delta t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\vec{i} + \ldots\vec{j} + \ldots\vec{k}$

$$\boxed{r'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}}$$

Ex: $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$

So $\vec{r}'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\vec{k} \leftarrow$ Direction vector of the TANGENT LINE to the space curve @ any "t" for $r(t)$

$\vec{r}''(t) = 2\vec{j} + 6t\,\vec{k}$

Ex: $\vec{r}(t) = \sqrt{t}\,\hat{i} + \frac{1}{t}\hat{j} + \ln t\,\hat{k}$

$\vec{r}\,'(t) = \frac{1}{2\sqrt{t}}\hat{i} - \frac{1}{t^2}\hat{j} + \frac{1}{t}\hat{k}$

$\vec{r}\,''(t) = -\frac{1}{4}t^{-\frac{3}{2}}\hat{i} + 2t^{-3}\hat{j} - t^{-2}\hat{k}$

$\Rightarrow \vec{r}\,''(t) = \frac{-1}{4\sqrt{t^3}}\hat{i} + \frac{2}{t^3}\hat{j} - \frac{1}{t^2}\hat{k}$

Ex: $\vec{r}(t) = e^{-t}\sin t\,\hat{i} + e^{-t}\cos t\,\hat{j} + \tan^{-1}t\,\hat{k}$

- $(e^{-t}\sin t)'\hat{i} = f'(t) = (-e^{-t}\sin t + e^{-t}\cos t)\hat{i} = e^{-t}(\cos t - \sin t)\hat{i}$
- $g'(t) = (e^{-t}\cos t)'\hat{j} = (-e^{-t}\cos t - e^{-t}\sin t)\hat{j} = -e^{-t}(\sin t + \cos t)\hat{j}$
- $h'(t) = (\tan^{-1}t)'\hat{k} = \left(\frac{1}{1+t^2}\right)\hat{k}$

so $\vec{r}\,'(t) = e^{-t}(\cos t - \sin t)\hat{i} - e^{-t}(\cos t + \sin t) + \frac{1}{(1+t^2)}\hat{k}$

Ex: 1. Sketch $\vec{r}(t) = \langle 4\cos t, 2\sin t\rangle$ $0 \le t \le 2\pi$

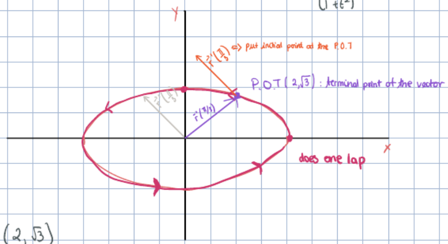
2. Find point of tangency (P.O.T) @ $t = \frac{\pi}{3}$

3. Find the tangent vector @ P.O.T

1- $\begin{cases} x = 4\cos t \text{ and } y = 2\sin t \\ \cos t = \frac{x}{4} \text{ and } \sin t = \frac{y}{2} \Rightarrow \frac{x^2}{4^2} + \frac{y^2}{2^2} = 1 \end{cases}$

this is a cylinder, an ellipse on xy plane

@ $t = 0 \Rightarrow P(4, 0)$ @ $t = \frac{\pi}{2}$, $P(0, 2)$



2- Plug in the value of t : $\vec{r}(\frac{\pi}{3}) = \langle 4\cos\frac{\pi}{3}, 2\sin\frac{\pi}{3}\rangle$
$= \langle 2, \sqrt{3}\rangle$ and P.O.T: $(2, \sqrt{3})$

3- $\vec{r}\,'(t) = -4\sin t\,\hat{i} + 2\cos t\,\hat{j}$ => this is the tangent vector

$\vec{r}\,'(\frac{\pi}{3}) = \langle -2\sqrt{3}, 1\rangle$

* The tangent vector gives orientation

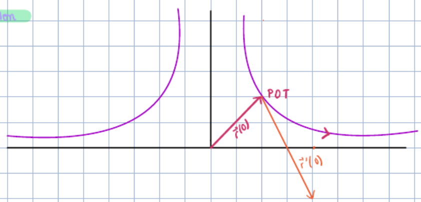
Ex: $\vec{r}(t) = \langle e^t, e^{-2t}\rangle$. Find the tangent vector @ $t = 0$

sketch: $x = e^t$ and $y = (e^{-2t}) = (e^t)^{-2} \Rightarrow y = x^{-2} = \frac{1}{x^2}$

POT: $\vec{r}(0) = \langle 1, 1\rangle \Rightarrow P(1, 1)$

$\vec{r}\,'(t) = \langle e^t, -2e^{-2t}\rangle$: tangent vector

$r'(0) = \langle 1, -2\rangle$



Unit Tangent Vectors ($T(t)$)

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ => gives the direction of tangent line

$\vec{r}(t) = 2\sin(2t)\hat{i} + 3\cos(2t)\hat{j} + 3\hat{k}$. Find POT and $T'(t)$ @ $t = \frac{\pi}{6}$

P.O.T: $\vec{r}(\frac{\pi}{6}) = \langle 2\sin\frac{2\pi}{6}, 3\cos\frac{\pi}{3}, 3\rangle = \langle\sqrt{3}, \frac{3}{2}, 3\rangle$. So POT $(\sqrt{3}, \frac{3}{2}, 3)$

$T(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{4\cos(2t)\hat{i} - 6\sin(2t)}{\sqrt{(4\cos(2t))^2 + (6\sin(2t))^2}} \Rightarrow$

$\vec{r}\,'(\pi/6) = 2\hat{i} - 3\sqrt{3}\hat{j} \Rightarrow \|\vec{r}\,'(\pi/6)\| = \sqrt{4 + 27} = \sqrt{31}$

So $T(\pi/6) = \frac{\vec{r}\,'(\pi/6)}{\|\vec{r}\,'(\pi/6)\|} = \frac{1}{\sqrt{31}}\langle 2, -3\sqrt{3}\rangle = \frac{2\hat{i} - 3\sqrt{3}\hat{j}}{\sqrt{31}} = \frac{\sqrt{31}}{31}(2\hat{i} - 3\sqrt{3}\hat{j})$

Tangent lines

- Need a P.O.T and a tangent vector at the P.O.T
  ↳ $\vec{r}\,'(t)$ @ "t"

Ex: $\vec{r}(t) = \sqrt{t+2}\,\hat{i} + \frac{1}{t+1}\hat{j} + \frac{2}{t^2+4}\hat{k}$. Find the equation of the tangent line at $t = 2$.

$\vec{r}(2) = \langle 2, \frac{1}{3}, \frac{1}{4}\rangle$ so P.O.T $= (2, \frac{1}{3}, \frac{1}{4})$

$\vec{r}\,'(t) = \frac{1}{2}(t+2)^{-1/2}\hat{i} + (-1)(t+1)^{-2} + 4t(t^2+4)^{-2} = \frac{1}{2\sqrt{t+2}}\hat{i} - \frac{1}{(t+1)^2}\hat{j} - \frac{4t}{(t^2+4)^2}\hat{k}$

so tangent vector @ $t = 2 \Rightarrow r'(2) = \langle\frac{1}{4}, -\frac{1}{9}, \frac{-1}{8}\rangle$

Lines: $\begin{cases} X = X_0 + at \\ Y = Y_0 + bt \\ Z = Z_0 + ct \end{cases}$ where X_0, Y_0, Z_0 = coordinates of P.O.T and a, b, c = coordinates of tangent vector

Parametric equations of the tangent line (L) $\Rightarrow$ L: $\begin{cases} x = 2 + \frac{1}{4}t \\ y = \frac{1}{3} - \frac{1}{9}t \\ z = \frac{1}{4} - \frac{1}{8}t \end{cases}$

Ex: $\vec{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, \sin^{-1} t \rangle$. Find the eq. of the tangent line @ $t=0$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$ so P.O.T = $(1, 0, 0)$

$\vec{r}'(t) = \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), \frac{1}{\sqrt{1-t^2}} \rangle$

$\cdot f'(t) = (e^{-t}\cos t)' = -e^{t}\sin t - e^{t}\cos t = -e^{-t}(\sin t + \cos t)$ $\cdot h'(t) = e^{-t}\cos t - e^{t}\sin t = e^{t}(\cos t - \sin t)$ $\cdot g(t) = \frac{1}{\sqrt{1-t^2}}$

$\vec{r}'(0) = \langle -1, 1, 1 \rangle$

Parametric equation of tangent line (L): $\begin{cases} X = 1 - 1t \\ Y = 1t \\ Z = 1t \end{cases}$

Integrals

Ex: 1) $\int_0^1 (t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k})\,dt = \left[\frac{t^2}{2}\hat{\imath} + \frac{t^3}{3}\hat{\jmath} + \frac{t^4}{4}\hat{k}\right]_0^1$

$= \frac{1}{2}(1-0)^2\hat{\imath} + \frac{1}{3}(1-0)^3\hat{\jmath} + \frac{1}{4}(1-0)^4\hat{k} = \frac{1}{2}\hat{\imath} + \frac{1}{3}\hat{\jmath} + \frac{1}{4}\hat{k}$

2) $\int \left(\underset{1}{t(\cos t)}\hat{\imath} + \underset{2}{t\sin(t^2)}\hat{\jmath} - \underset{3}{te^{t^2}}\hat{k}\right) dt$

1. $\int t\cos t\,dt$ $\quad u = t$, $du = dt$ and $v = \int \cos t\,dt = \sin t$

$\hookrightarrow = t\sin t - \int \sin t\,dt = (t\sin t + \cos t + C_1)\hat{\imath}$

2. $\int t\sin(t^2)\hat{\jmath}\,dt = \left(-\frac{1}{2}\cos(t^2) + C_2\right)\hat{\jmath}$

3. $-\int te^{t^2}\,dt = \left(-\frac{1}{2}e^{t^2} + C_3\right)\hat{k}$

So $\int \left(\underset{1}{t(\cos t)}\hat{\imath} + \underset{2}{t\sin(t^2)}\hat{\jmath} - \underset{3}{te^{t^2}}\hat{k}\right) dt = (t\sin t + \cos t)\hat{\imath} - \frac{1}{2}\cos(t^2)\hat{\jmath} - \frac{1}{2}e^{t^2}\hat{k} + \overbrace{(C_1\hat{\imath} + C_2\hat{\jmath} + C_3\hat{k})}^{C}$

Ex: $\int \left(\frac{1}{1+t^2}\hat{\imath} + \frac{t}{1+2t^2}\hat{\jmath} - \frac{1}{\sqrt{1-t^2}}\hat{k}\right) dt = \tan^{-1}t\,\hat{\imath} + \frac{\ln(1+2t^2)}{4}\hat{\jmath} - \sin^{-1}t\,\hat{k} + C$

$\cdot \int \frac{t}{1+2t^2}\,dt$ $\quad u = 1+2t^2$, $\frac{du}{dt} = 4t \Rightarrow \frac{du}{4} = t\,dt$ $\quad // \int \frac{1}{u}\frac{du}{4} = \frac{\ln u}{4} + C = \frac{\ln(1+2t^2)}{4} + C$

Ex: 1) $\vec{r}'(t) = 2\hat{\imath} + 4t\hat{\jmath} - 6t^2\hat{k}$ (a tangent) and $\vec{r}(0) = \hat{\imath} + \hat{k}$

$\vec{r}(t) = \int (2\hat{\imath} + 4t\hat{\jmath} - 6t^2\hat{k})\,dt = 2t\hat{\imath} + 2t^2\hat{\jmath} - 2t^3\hat{k} + C$

$\vec{r}(0) = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} + C = \hat{\imath} + \hat{k} \Rightarrow C = \hat{\imath} + \hat{k}$

So $\vec{r}(t) = 2t\hat{\imath} + 2t^2\hat{\jmath} - 2t^3\hat{k} + \hat{\imath} + \hat{k}$

$\vec{r}(t) = (2t+1)\hat{\imath} + 2t^2\hat{\jmath} + (1-2t^3)\hat{k}$

2) Ex: $\vec{r}''(t) = \sqrt{t}\,\hat{\imath} + \sec^2 t\,\hat{\jmath} + e^t\hat{k}$ $\quad \vec{r}'(0) = \hat{\imath} + \hat{k}$, $\vec{r}(0) = 2\hat{\imath} + \hat{\jmath} - \hat{k}$

$\vec{r}'(t) = \int (t^{1/2}\hat{\imath} + \sec^2 t\,\hat{\jmath} + e^t\hat{k})\,dt$

$\vec{r}'(t) = \frac{2}{3}t^{3/2}\hat{\imath} + \tan t\,\hat{\jmath} + e^t\hat{k} + C$

$\vec{r}'(0) = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k} + C = \hat{\imath} + \hat{k} \Rightarrow C = 1\hat{\imath}$

so $\vec{r}'(t) = \left(\frac{2}{3}t^{3/2} + 1\right)\hat{\imath} + \tan t\,\hat{\jmath} + e^t\hat{k}$

$\vec{r}(t) = \int \left(\left(\frac{2}{3}t^{3/2} + 1\right)\hat{\imath} + \tan t\,\hat{\jmath} + e^t\hat{k}\right) dt$

$\vec{r}(t) = \left(\frac{4}{15}t^{5/2} + t\right)\hat{\imath} - \ln|\cos t|\,\hat{\jmath} + e^t\hat{k} + C$

$\left(\int \tan t\,dt = \int \frac{\sin t}{\cos t}\,dt \Rightarrow u = \cos t, \frac{du}{dt} = -\sin t \Rightarrow du = -\sin t\,dt \Rightarrow \int \tan t\,dt = -\int \frac{1}{u}\,du = -\ln|u| = -\ln(\cos t)\right)$

$\vec{r}(0) = 0\hat{\imath} - 0\hat{\jmath} + 1\hat{k} + C = 2\hat{\imath} + \hat{\jmath} - \hat{k} \Rightarrow C = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$

$\vec{r}(t) = \left(\frac{4}{15}t^{5/2} + t + 2\right)\hat{\imath} + \left(1 - \ln|\cos(t)|\right)\hat{\jmath} + (e^t - 2)\hat{k}$

12.3 - Arc lenght and Arc lenght parameterization

Arc lenght for parametric equations:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt \quad \text{for } t \in [a,b]$$

For 3-D:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt \quad \text{for } t \in [a,b]$$

$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$

$\vec{r}'(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k}$ ← tangent vector to this curve

$\|\vec{r}'(t)\| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$ ← magnitude of the tangent vector

So $\boxed{L = \int_a^b \|\vec{r}'(t)\|\, dt}$ ⟹ 1) Can give arc length "L" if $t \in [a,b]$
2) Can give arc length function (has "t") $\underline{s(t)}$

Ex: Find Arclength: $\vec{r}(t) = 4\sin t\,\hat{\imath} + 3t\,\hat{\jmath} + 4\cos t\,\hat{k}$ on $0 \le t \le 2\pi$

$\vec{r}'(t) = 4\cos t\,\hat{\imath} + 3\hat{\jmath} - 4\sin t\,\hat{k}$

$\|\vec{r}'(t)\| = \sqrt{16\cos^2 t + 9 + 16\sin^2 t} = \sqrt{16(\underbrace{\cos^2 t + \sin^2 t}_{1}) + 9} = \sqrt{16+9} = 5$

$L = \int_0^{2\pi} 5\, dt = [5t]_0^{2\pi} = 5(2\pi) = 10\pi$

Ex: $\vec{r}(t) = \langle e^t\cos t,\ e^t \sin t\ \ e^t\rangle$ find arclenght for $t \in [0, 2\pi]$

$\vec{r}'(t) = \langle e^t(\cos t - \sin t),\ e^t(\sin t + \cos t),\ e^t\rangle$

$\|r'(t)\| = \sqrt{[e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2 + (e^t)^2}$

$= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^t(\sin t + \cos t)^2 + e^{2t}}$

$= \sqrt{e^{2t}[(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1]}$

$= e^t\sqrt{\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t + 1}$

$= e^t\sqrt{\cos^2 t + \cos^2 t + \sin^2 t + \sin^2 t - 2\cos t\sin t + 2\cos t\sin t + 1}$

$= e^t\sqrt{2\cos^2 t + 2\sin^2 t + 1}$

$= e^t\sqrt{\underbrace{2+1}_{2\times 1 = 2}} = \sqrt{3}\, e^t$

So $L = \int_0^{2\pi} \sqrt{3}\, e^t\, dt = [\sqrt{3}\, e^t]_0^{2\pi} = \sqrt{3}\, e^{2\pi} - \sqrt{3}\, \underbrace{e^0}_{1} = \sqrt{3}\, e^{2\pi} - \sqrt{3} = \sqrt{3}(e^{2\pi} - 1)$

For "smooth curves" we can reparameterize by arclength ... why?
- #1- We can walk the curve
- #2- Some formulas are way easier
- #3- $\vec{r}'(s)$ will always have a magnitude of 1 ⟹ it's the unit tangent vector

Ex: $\vec{r}(t) = (1+t)\hat{\imath} + (1+2t)\hat{\jmath} + 3t\hat{k}$ with $t \ge 0$

1- Find arc length function ($s(t)$ b/c you have $\underline{t \ge 0}$)

$\vec{r}'(t) = \langle 1, 2, 3\rangle$, $\|\vec{r}'(t)\| = \sqrt{1+4+9} = \sqrt{14}$

$s(t) = \int_a^t \|\vec{r}'(u)\|\, du$

"a" is where "t" begins; replace "t" with "u"

$s(t) = \int_0^t \sqrt{14}\, du = [\sqrt{14}\, u]_0^t = \sqrt{14}\, t$ This gives $t = \dfrac{s(t)}{\sqrt{14}}$

So $\vec{r}(s) = \left(1 + \frac{s}{\sqrt{14}}\right)\hat{\imath} + \left(1 + \frac{2s}{\sqrt{14}}\right)\hat{\jmath} + \frac{3s}{\sqrt{14}}\hat{k}, \quad s \geq 0$

Ex: $\vec{r}(t) = e^t \cos t\, \hat{\imath} + e^t \sin t\, \hat{\jmath} + e^t\, \hat{k}, \quad t \geq 0$

$\|\vec{r}\,'(t)\| = \sqrt{3}\, e^t$ and $s(t) = \int_a^t \|\vec{r}\,'(u)\|\, du$

$$s(t) = \int_0^t \sqrt{3}\, e^u\, du = \left[\sqrt{3}\, e^u\right]_0^t = \sqrt{3}e^t - \sqrt{3} = \sqrt{3}\,(e^t - 1) = s(t)$$

$$s = \sqrt{3}\,(e^t - 1) \Rightarrow \frac{s}{\sqrt{3}} = e^t - 1 \Rightarrow \frac{\sqrt{3}}{3}s = e^t - 1 \text{ and } e^t = \frac{\sqrt{3}}{3}s + 1 \;;\; t = \ln\left(\frac{\sqrt{3}}{3}s + 1\right)$$

$$\vec{r}(s) = \left(\frac{\sqrt{3}}{3}s + 1\right)\cos\left(\ln\left(\frac{\sqrt{3}}{3}s + 1\right)\right)\hat{\imath} + \left(\frac{\sqrt{3}}{3}s + 1\right)\sin\left[\ln\left(\frac{\sqrt{3}}{3}s + 1\right)\right]\hat{\jmath} + \left(\frac{\sqrt{3}}{3}s + 1\right)\hat{k}$$

Ex: $\vec{r}(t) = t^2\,\hat{\imath} + t\cos t\,\hat{\jmath} + t\sin t\,\hat{k}$, Find L on $t \in [0,1]$

$$L = \int_a^b \|\vec{r}\,'(t)\|\, dt$$

$\vec{r}\,'(t) = 2t\,\hat{\imath} + (\cos t - t\sin t)\,\hat{\jmath} + (\sin t + t\cos t)\,\hat{k}$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$$

$$= \sqrt{4t^2 + \cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t}$$

$$= \sqrt{4t^2 + \underbrace{\cos^2 t + \sin^2 t}_{1} + t^2\underbrace{(\sin^2 t + \cos^2 t)}_{1}}$$

$$= \sqrt{4t^2 + 1 + t^2} = \sqrt{5t^2 + 1}$$

$$L = \int_0^1 \sqrt{5t^2 + 1}\, dt = \int_0^1 \sqrt{(\sqrt{5}t)^2 + 1^2}$$

$\tan\theta = \sqrt{5}t \rightarrow t = \frac{\sqrt{5}}{5}\tan\theta$

$dt = \frac{\sqrt{5}}{5}\sec^2\theta\, d\theta$

$\sqrt{5t^2+1}$, $\sqrt{5}t$, θ

$$= \int_0^1 \sqrt{\tan^2\theta + 1} \cdot \frac{\sqrt{5}}{5}\sec^2\theta\, d\theta =$$

$$= \int_0^1 \sqrt{\sec^2\theta} \cdot \frac{\sqrt{5}}{5}\, d\theta = \frac{\sqrt{5}}{5}\int_0^1 \sec^3\theta\, d\theta = \frac{\sqrt{5}}{5}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right]$$

$$L = \frac{\sqrt{5}}{5}\left[\frac{1}{2}\sqrt{5t^2+1}\,\sqrt{5}\,t + \frac{1}{2}\ln\left|\sqrt{5t^2+1} + \sqrt{5}\,t\right|\right]_0^1$$

$$L = \frac{\sqrt{5}}{10}\left[\sqrt{6}\cdot\sqrt{5} + \ln\left|\sqrt{6} + \sqrt{5}\right| - 0\right] = \frac{\sqrt{5}}{10}\left[\sqrt{30} + \ln\left|\sqrt{6} + \sqrt{5}\right|\right] \Rightarrow \text{unit tangent vector}$$

TNB Frames: Frenet-Serret Frames

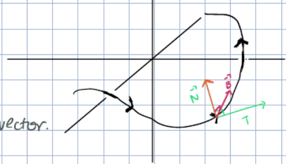
At every point, the TNB frame gives us the **unit vectors**...

- $\vec{T}$angent → direction particle is heading → where "Nose" of airplane is pointing
- $\vec{N}$ormal → direction particle is turning → where "TAIL-FIN" is pointing
- $\vec{B}$inormal → direction of particle's twist → where "wings" are pointing

} these are unit vectors which are mutually orthogonal (just like $\hat{\imath}, \hat{\jmath}, \hat{k}$ but with a moving particle

How to find TNB:

- $\vec{T}$: unit tangent ; $\vec{T}(s) = \vec{r}\,'(s)$ or $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$



- $\vec{N}$: unit normal $\Rightarrow$ $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$

  ↑ a unit vector $\perp$ to $\vec{T}$

* every vector with a constant length is $\perp$ to its tangent vector.
($\vec{T}(t)$ is a UNIT vector ... constant length)

Proof: Let $\vec{v}$ have a constant length.

$\vec{v}\cdot\vec{v} = \|\vec{v}\|^2 \rightarrow \vec{v}\cdot\vec{v} = C^2$, "C" a constant

$$\frac{d}{dt}\left[\vec{v}\cdot\vec{v}\right] = \frac{d}{dt}\left[C^2\right] \Rightarrow \underbrace{\vec{v}\,'\cdot\vec{v} + \vec{v}\cdot\vec{v}\,'}_{2(\vec{v}\cdot\vec{v}\,')} = 0 \Rightarrow \vec{v}\cdot\vec{v}\,' = 0 \Rightarrow \vec{v} \perp \vec{v}\,'$$

So $\vec{T} \perp \vec{T}'$ and $\vec{T} \perp \frac{\vec{T}'}{\|\vec{T}'\|} \Rightarrow \vec{T} \perp \vec{N}$

- $\vec{B}$: unit binormal $\Rightarrow \vec{B} = \vec{T} \times \vec{N}$

12.3-12.5 : TNB frames, Curvature, Torsion

TNB frames describes how a particle on A space curve (Vector Function) is heading ($\vec{T}$), turning ($\vec{N}$) and Twisting ($\vec{B}$)

$\vec{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$, $\vec{N} = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$ and $\vec{B} = \vec{T} \times \vec{N}$ or $\vec{B} = \frac{\vec{r}' \times \vec{r}''}{\|\vec{r}' \times \vec{r}''\|}$

Curvature: a measure of a curve's failure to be a line. The more "curvy" the function, the larger the curvature.

Curvature is really arclength compared to heading ($\vec{T}$). (How $\vec{T}$ is changing with regard to arclength)

$$K = \left\| \frac{dT}{ds} \right\| = \|\vec{T}'(s)\| \quad \text{where } \vec{T}(s) = \vec{r}'(s)$$

$$\vec{T}(t) = \frac{d\vec{T}}{dt} \Rightarrow \frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$

$$\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} \Rightarrow \left\|\frac{d\vec{T}}{ds}\right\| = \frac{\left\|\frac{d\vec{T}}{dt}\right\|}{\left\|\frac{ds}{dt}\right\|} \rightarrow K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

$$S(t) = \int \|\vec{r}'(t)\|\,dt \Rightarrow \frac{d}{dt}(s(t)) = \frac{d}{dt}\left(\int_a^e \|\vec{r}'(t)\|\,dt\right) \Rightarrow \frac{ds}{dt} = \|\vec{r}'(t)\|$$

$$K = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3}$$ << use for polynomials when you just have to find the curvature

Torsion (τ)

$$\tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{\|\vec{r}' \times \vec{r}''\|^2}$$, is a measure of a curve's failure to be contained in a plane

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$

osculating circle

at any point on a curve, there will be a circle that fits that curve "best".
- it JUST touches the curves, "kiss the curve"
- at the point: (points)
  - Same tangent
  - Same normal
  - normal through the center of the circle

- the plane that contains the circle (osculating plane) must contain $\vec{T}$ and $\vec{N}$

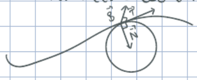
$\vec{B}$ is normal for the osculating plane

* The radius of osc. circle is called the radius of curvature.

$$\rho = \frac{1}{K}$$

* The plane that contains $\vec{N}, \vec{B}$ is called the normal plane and contains ALL vectors orthogonal to $\vec{T}$ ($\vec{T}$ is NORMAL)

Ex: Find $\vec{T}, \vec{N}, \vec{B}, K, \rho$, eq. of osc. plane and eq. of normal plane @ $t = \frac{\pi}{2}$
for $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\hat{k}$



$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$; $\vec{r}\,'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$

so $\vec{T}(t) = \dfrac{-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}\left(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}\right)$

$\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$; $\vec{T}\,'(t) = \dfrac{\sqrt{2}}{2}\left(-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}\right)$

$\|\vec{T}\,'(t)\| = \dfrac{\sqrt{2}}{2}\underbrace{\sqrt{\cos^2 t + \sin^2 t}}_{1} = \dfrac{\sqrt{2}}{2}$

so $\vec{N}(t) = \dfrac{\cancel{\sqrt{2}/2}\left(-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}\right)}{\cancel{\sqrt{2}/2}}$

$\vec{N}(t) = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$

$\vec{B} = \vec{T}(t) \times \vec{N}(t) = \dfrac{\sqrt{2}}{2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \dfrac{\sqrt{2}}{2}\left(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + (\sin^2 t + \cos^2 t)\hat{k}\right)$
$= \dfrac{\sqrt{2}}{2}\left(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + 1\hat{k}\right)$

$\vec{B} = \dfrac{\sqrt{2}}{2}\left(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + 1\hat{k}\right)$

$K = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \dfrac{\sqrt{2}/2}{\sqrt{2}} = \dfrac{\cancel{\sqrt{2}}}{2} \times \dfrac{1}{\cancel{\sqrt{2}}} = \dfrac{1}{2}$

$\rho = \dfrac{1}{K} = 2$

$\vec{T}(t) = \dfrac{\sqrt{2}}{2}\left(-\sin t + \cos t\,\hat{\jmath} + \hat{k}\right)$

$\vec{N}(t) = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$

$\vec{B}(t) = \dfrac{\sqrt{2}}{2}\left(\sin t\,\hat{\imath} - \cos t\,\hat{\jmath} + \hat{k}\right)$

Point: $\vec{r}\left(\frac{\pi}{2}\right) = 0\hat{\imath} + 1\hat{\jmath} + \frac{\pi}{2}\hat{k} \Rightarrow P\left(0, 1, \pi/2\right)$

$\vec{T}\left(\frac{\pi}{2}\right) = \dfrac{\sqrt{2}}{2}\left(-1\hat{\imath} + 0\hat{\jmath} + \hat{k}\right)$

$\vec{N}(\pi/2) = 0\hat{\imath} - 1\hat{\jmath} \Rightarrow P(0, -1, 0)$

$\vec{B}(\pi/2) = \dfrac{\sqrt{2}}{2}\left(\hat{\imath} + \hat{k}\right)$

* $\vec{B} = \vec{T} \times \vec{N} = \dfrac{\sqrt{2}}{2}\left(-\hat{\imath} + \hat{k}\right) \times \left(-\hat{\jmath}\right) = \dfrac{\sqrt{2}}{2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \dfrac{\sqrt{2}}{2}\left(\hat{\imath} + \hat{k}\right)$

So to find $\vec{B}$ ONLY, you can plug in the value of t in $\vec{T}$ and $\vec{N}$ to make it easier.

* OSCULATING plane:
  - contains $\vec{T}$ and $\vec{N}$, $\vec{B}$ is the normal vector)

$P\left(0, 1, \pi/2\right)$ is a point and $\vec{v_n} = \vec{B} = \dfrac{\sqrt{2}}{2}\left(\hat{\imath} + \hat{k}\right)$
or $\vec{v_n} = \hat{\imath} + \hat{k}$ b/c its a scalar multiple of $\vec{B}$

Eq: $(x - 0) + \left(z - \frac{\pi}{2}\right) = 0$

Eq: $x + z = \dfrac{\pi}{2}$

Normal Plane:

- contains $\vec{N}$ and $\vec{B}$, $\vec{T}$ is the normal vector

$P(0, 1, \pi/2)$ and $\vec{v_n} = \vec{T} = \frac{\sqrt{2}}{2}(-\hat{i} + \hat{k})$ or $\vec{v_n} = \langle -1, 0, 1 \rangle$

Eq: $-1(x - 0) + 1(z - \frac{\pi}{2}) = 0$

$-x + z = \pi/2$

Ex: $\vec{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle$, $\vec{T}, \vec{N}, K$

$\vec{T}(t) = \frac{\vec{r}'(t)}{\|r'(t)\|} = \frac{e^t(\cos t - \sin t)\hat{i} + e^t(\sin t + \cos t)\hat{j} + e^t \hat{k}}{\sqrt{3}e^t} = \frac{(\cos t - \sin t)\hat{i} + (\sin t + \cos t)\hat{j} + \hat{k}}{\sqrt{3}}$

- $r'(t) = \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t \rangle$
- $\|r'(t)\| = \sqrt{3}\, e^t$

So $\vec{T}'(t) = \frac{\sqrt{3}}{3}\langle (\cos t - \sin t), (\sin t + \cos t), 1 \rangle$

$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

- $\vec{T}'(t) = \frac{\sqrt{3}}{3}\langle (-\sin t - \cos t), (\cos t - \sin t), 0 \rangle$
- $\|\vec{T}'(t)\| = \frac{\sqrt{3}}{3}\sqrt{\underbrace{(\sin t - \cos t)^2 + (\cos t - \sin t)^2}_{\substack{\sin^2 t + \cancel{2\cos t \sin t} + \cos^2 t + \cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t \\ 2}}}$

$= \frac{\sqrt{3}}{3}(\sqrt{2}) = \frac{\sqrt{6}}{3}$

So $\vec{N}(t) = \frac{\frac{\sqrt{3}}{3}\langle (-\sin t - \cos t), (\cos t - \sin t), 0 \rangle}{\sqrt{6}/3}$

$\vec{N}(t) = \frac{\sqrt{2}}{2}\langle (-\sin t - \cos t), (\cos t - \sin t), 0 \rangle$

$K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\frac{\sqrt{6}}{3}}{\sqrt{3}e^t} = \frac{\sqrt{6}}{3} \times \frac{1}{\sqrt{3}e^t} = \frac{\sqrt{2}}{3e^t}$

Ex: Find eq. of osculating plane with curvature

To $\vec{r}(t) = t\hat{i} + \hat{j} + t^2\hat{k}$ @ $t = 0$

$\left.\begin{array}{l} \vec{r}'(t) = \hat{i} + 2t\hat{k} \\ \|\vec{r}'(t)\| = \sqrt{1 + 4t^2} \end{array}\right\}$ $\vec{T} = \frac{\hat{i} + 2t\hat{k}}{\sqrt{1 + 4t^2}} = \frac{1}{\sqrt{1+4t^2}}\hat{i} + \frac{2t}{\sqrt{1+4t^2}}\hat{k}$

$\vec{T}'(t) = \left(-\frac{1}{2}(1 + 4t^2)^{-3/2} \cdot 8t\right)\hat{i} + \left(2(1+4t^2)^{-1/2} - 2t(-1/2)(1+4t^2)^{-3/2} \cdot 8t\right)\hat{k}$

$= \frac{-4t}{(1 + 4t^2)^{3/2}}\hat{i} + \left(\frac{2}{(1+4t^2)^{1/2}} - \frac{8t^2}{(1+4t^2)^{3/2}}\right)\hat{k}$

$T'(t) = \frac{-4t}{(1 + 4t^2)^{3/2}}\hat{i} + \frac{2}{(1+4t^2)^{3/2}}\hat{k}$

STOP !!! If you are given $t =$ "a", plug it in now.

@ $t = 0 \Rightarrow \vec{r}(0) = P(0, 1, 0)$

$\vec{T}(0) = \hat{i}$ and $\vec{T}'(0) = 2\hat{k}$; $\|\vec{T}'(0)\| = \sqrt{2^2} = 2$

so $\vec{N}(0) = \frac{2\hat{k}}{2} = \hat{k}$

$\vec{B}(0) = \vec{N}(0) \times \vec{T}(0) = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, 1, 0 \rangle$

$= \hat{i} \times \hat{k} = \hat{j}$

$\|\vec{r}'(0)\| = 1$ so $k = \frac{\|\vec{T}'(0)\|}{\|\vec{r}'(0)\|} = \frac{2}{1} = 2$

Eq of plane : $1(y-1) = 0 \Rightarrow \boxed{y = 1}$

BUT if you ONLY need curvature and you have a polynomial

$k = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$

$\vec{r}'(t) = \langle 1, 0, 2t \rangle$ and $\|\vec{r}'(t)\| = \sqrt{1+4t^2}$

$r''(t) = 2$

$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} i & j & k \\ 1 & 0 & 2t \\ 0 & 0 & 2 \end{vmatrix} = -2\hat{j}$

$\|\vec{r}'(t) \times \vec{r}''(t)\| = 2$

$k = \frac{2}{(1+4t^2)^{3/2}}$

@ $t = 0$ $k = 2$

Ex: Find curvature of $y = x^3 + 1$

$y = f(x) \to \langle t, f(t), 0 \rangle$

$x = g(y) \to \langle g(t), t, 0 \rangle$

$x = f(t), y = g(t) \to \langle f(t), g(t), 0 \rangle$

$r = f(\theta) \to \langle f(t)\cos t, f(t)\sin t, 0 \rangle$

$\to \vec{r}(t) = \langle t, t^3+1, 0 \rangle = t\hat{i} + (t^3+1)\hat{j}$

$\vec{r}'(t) = \hat{i} + 3t^2\hat{j}$ $\|\vec{r}'(t)\| = \sqrt{1 + 9t^4}$

$\vec{r}''(t) = 6t\hat{j}$ $\vec{r}'(t) \times \vec{r}''(t) = 6t\,\hat{k}$

$\|\vec{r}'(t) \times \vec{r}''(t)\| = 6t$

so $k = \frac{6t}{(1+9t^4)^{3/2}}$

$k' = \frac{6 - 270t^4}{(1+9t^4)^{5/2}} = 0$

12.4 - Velocity and acceleration

for a vector function, $\vec{r}(t)$

Velocity : $\vec{v}(t) = \vec{r}'(t)$ acceleration : $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$ speed : $\|\vec{v}(t)\| = \|\vec{r}'(t)\|$

Ex: Find velocity, acceleration and speed of a particle described by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ @ $t = 1$

- $\vec{v}(t) = \vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ so $\vec{v}(1) = \langle 1, 2, 3 \rangle$
- $\vec{a}(t) = \vec{r}''(t) = \langle 0, 2, 6t \rangle$ so $\vec{a}(1) = \langle 0, 2, 6 \rangle$
- $\|\vec{v}(1)\| = \sqrt{1+4+9} = \sqrt{14}$

Ex: $\vec{r}(t) = \langle \sqrt{t}, t^2, e^{2t} \rangle$

- $\vec{v}(t) = \vec{r}'(t) = \langle \frac{1}{2\sqrt{t}}, 2t, 2e^{2t} \rangle$
- $\|\vec{v}(t)\| = \sqrt{1/4\,(t)^{-1} + 4t^2 + 4e^{4t}} = \sqrt{\frac{1}{4t} + 4t^2 + 4e^{4t}}$
- $\vec{a}(t) = \vec{r}''(t) = \langle -1/4\,(t)^{-3/2}, 2, 4e^{2t} \rangle$

Ex: $\vec{a}(t) = e^t\hat{\imath} + e^{-t}\hat{k}$, $\vec{v}(0) = \hat{\imath} + 2\hat{\jmath}$ => initial velocity and $\vec{r}(0) = 3\hat{\imath}+\hat{\jmath}+2\hat{k}$ => initial vector

$\vec{r}\,''(t)$, $\vec{r}\,'(0)$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int (e^t\hat{\imath} + e^{-t}\hat{k})\,dt = e^t\hat{\imath} - e^{-t}\hat{k} + C_1$$

$\vec{v}(t) = e^t\hat{\imath} - e^{-t}\hat{k} + C_1$ but $\vec{v}(0) = \langle 1, 2, 0\rangle$,

so $e^0\hat{\imath} + 0\hat{\jmath} - e^0\hat{k} + C_1 = \hat{\imath} + 2\hat{\jmath}$ => $\cancel{1\hat{\imath}} - 1\hat{k} + C = \cancel{1\hat{\imath}} + 2\hat{\jmath}$ => $C = 2\hat{\jmath} + 1\hat{k}$

so $\vec{v}(t) = e^t\hat{\imath} - e^{-t}\hat{k} + 2\hat{\jmath} - 1\hat{k} = \langle e^t, 2, (e^{-t}+1)\rangle$

$$\vec{r}(t) \int \vec{v}(t)\,dt = \int (e^t\hat{\imath} + 2\hat{\jmath} + (1-e^{-t})\hat{k})\,dt = e^t\hat{\imath} + 2t\hat{\jmath} + (t + e^{-t})\hat{k} + C$$

$\vec{r}(0) = 1\hat{\imath} + 0\hat{\jmath} + 1\hat{k} + C = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ => $C = 2\hat{\imath} + \hat{\jmath} + \hat{k}$

so $\vec{r}(t) = (e^t + 2)\hat{\imath} + (2t+1)\hat{\jmath} + (t + e^{-t} + 1)\hat{k}$

NOTE: We can define acceleration as a vector with $\vec{T}$ and $\vec{N}$.

let $\|\vec{v}(t)\| = v$ (speed)

$$\vec{a} = \underbrace{v'}_{\text{tangental component of acceleration}}\vec{T} + \underbrace{K v^2}_{\text{normal component of acceleration}}\vec{N} \quad \text{or} \quad \vec{a} = \frac{\vec{r}\,'\cdot\vec{r}\,''}{\|\vec{r}\,'\|}\vec{T} + \frac{\|\vec{r}\,'\times\vec{r}\,''\|}{\|\vec{r}\,'\|}\vec{N}$$

$v = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$, $v' = \|\vec{r}\,'(t)\|'$, $v^2 = \|\vec{r}\,'(t)\|^2$

Projectile motion

$$\vec{r}(t) = (\vec{v}_0 \cos\alpha)t\,\hat{\imath} + \left[h + (\vec{v}_0 \sin\alpha)t - \frac{1}{2}gt^2\right]\hat{\jmath}$$

$\vec{V}_0$ = initial velocity h = heigth

α = angle of inclination g = acceleration of gravity.

Ex: You shoot a riffle from a 200 Ft cliff with an angle of 30° and a muzzle velocity of 1500 Ft/sec.

$V_0 = 1500$ Ft/s , $\alpha = \pi/6$, $h = 200$ Ft , $g = 9.8\,m.s^{-2}$ or $32\,Ft.s^{-2}$

$$\vec{r}(t) = \left(1500\cos\frac{\pi}{6}\right)t\,\hat{\imath} + \left(200 + 1500\sin\frac{\pi}{6} - \frac{1}{2}32t^2\right)\hat{\jmath}$$

$$\vec{r}(t) = \underbrace{750\sqrt{3}}_{x}\,t\,\hat{\imath} + \underbrace{(200 + 750t - 16t^2)}_{y}\hat{\jmath}$$

x = range y = height

1) When will the bullet hit the ground?

we set $y = 0$ => $t = 47.1$ sec.

2) Maximum range?

$x = 750\sqrt{3}(t)$ we set $x(47.1) = 750\sqrt{3}(47.1) = 61.185$ Ft

3) Max height

set $y' = 0$ and replace value of t in y.

$y' = 750 - 32t = 0$ => $t = \frac{375}{16}$

$y\left(\frac{375}{16}\right) = 8989$ Ft

4)

4) Impact velocity => speed when it hits the ground

$\vec{v}(t) = \vec{r}\,'(t) = 750\sqrt{3}\,\hat{\imath} + (750 - 32t)\hat{\jmath}$

$\vec{v}(47.1) = 1503.6$ Ft/s

$\|\vec{v}(47.1)\| = 1503.6^2$

25.10.2020

CHAP 13: MULTIVARIABLE FUNCTIONS

13.1- Intro to Multivariable functions (Domain, Sketching, Level Curves)

To graph a function, you must have 1 Dimension more than the # of independent variables.

$f(x) = x+1 \rightarrow y = x+1$, there is one INDEPENDENT variable so graphed in 2D

$g(x,y) = x^2+y^2 \rightarrow z = x^2+y^2$, 3D, so there is at most 1 DEPENDENT variable

$h(x,y,z) = \dfrac{x^2+y^2}{z-3} \rightarrow w = \dfrac{x^2+y^2}{z-3}$, 3 independent variables so $\Rightarrow$ 4D

To graph the DOMAIN of a function, you must have the same direction as the independent variable.

$f(x)$, D: 1-D
$g(x,y)$, D: 2-D
$h(x,y,z)$, D: 3-D

Ex: $f(x,y,z) = \sqrt{x^2+2y^2+3z^2}$

- 3 IND var (x,y,z) + 1 DEP var (w) = 4 variables so the function needs 4D to be graphed
- 3D Domain and 4D Graph

$f(0,2,-1) = \sqrt{0^2+2(2)^2+3(-1)^2} = \sqrt{11}$

$f(u,u-1,u+1) = \sqrt{u^2+2(u-1)^2+3(u+1)^2} = \sqrt{u^2+2(u^2-2u+1)+3(u^2+2u+1)} = \sqrt{u^2+2u^2-4u+2+3u^2+6u+3}$
$= \sqrt{6u^2+2u+5}$

Ex: $f(x,y) = \dfrac{xy}{x-y} \longrightarrow x-y \neq 0$, $x \neq y$ Domain D: $\{(x,y) \mid x \neq y\}$ and Range:

include all IND variable

such that

include 1 var (DEP. variable)

Range: the output called "z". R: $\{z \mid -\infty < z < +\infty\}$

Ex: $g(x,y) = \sqrt{4-x^2-y^2}$ Surface: 3D, Domain: 2D

$\hookrightarrow 4-x^2-y^2 \geq 0 \Rightarrow 0 \leq x^2+y^2 \leq 4$

b/c sq²

$\sqrt{4-(4)}$ x^2+y^2 $\sqrt{4-0}$

So Domain D: $\{(x,y) \mid x^2+y^2 \leq 4\}$ and Range R: $\{z \mid 0 \leq z \leq 2\}$

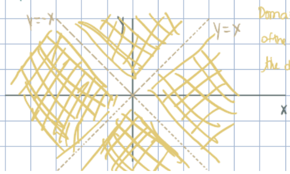
To graph DOMAIN, you must have an AXIS for each IND. variable.

- $f(x) = \dfrac{1}{\sqrt{x}}$ D: $\underset{0}{|}\!\longrightarrow x$ ← can state domain this way when we only have one IND. var

$x > 0$

- $f(x,y) = \dfrac{xy}{x^2-y^2}$ D: $\{(x,y) \mid y \neq x, y \neq -x\}$

$x^2-y^2 \neq 0 \Rightarrow y^2 \neq x^2 \Rightarrow y \neq \pm x$

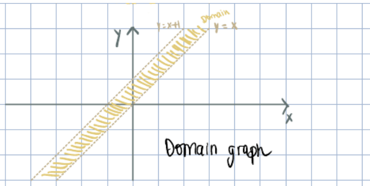


Domain is any point of the plane EXCEPT the dotted lines

Ex: $f(x,y) = \dfrac{\ln(y-x)}{\sqrt{x-y+1}}$ Surface: 3D Domain: 2D

- $y - x > 0 \rightarrow y > x$
- $x - y + 1 > 0 \rightarrow y < x+1$

$D: \left\{ (x,y) \mid y > x \text{ and } y < x+1 \right\}$



Ex: $f(x,y,z) = \sqrt{9 - x^2 - y^2 - z^2}$ $9 - (x^2+y^2-z^2) \geq 0$

Surface: 4D Domain: 3D $x^2+y^2+z^2 \leq 9$

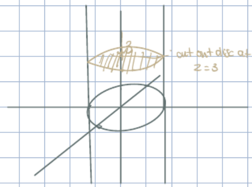
↳ inside of a sphere with a radius 3 centered at the origin

$D: \left\{ (x,y,z) \mid 0 \leq x^2+y^2+z^2 \leq 9 \right\}$

Ex: $f(x,y,z) = \dfrac{\sqrt{4-x^2-y^2}}{z-3}$ 3 IND var.

- $z - 3 \neq 0 \rightarrow z \neq 3$
- $4 - (x^2+y^2) \geq 0 \rightarrow x^2+y^2 \leq 4$

$D: \left\{ (x,y,z) \mid x^2+y^2 \leq 4 \text{ and } z \neq 3 \right\}$



How to GRAPH:

1- Set $f(x,y) = z$
2- Try to get a surface you know.
3- OR use a computer

Ex: $f(x,y) = 6 - 2x + 3y$ Graph: 3D, Domain: 2D

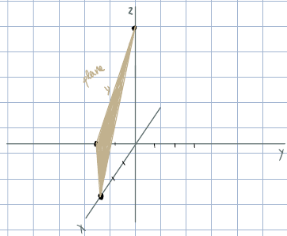
$z = 6 - 2x + 3y \Rightarrow 2x + 3y + z = 6$ is a plane with

a $\vec{n} = \langle 2, 3, 1 \rangle$

@ y and z = 0, $2x = 6 \Rightarrow x = 3$

@ z and x = 0 ; $-3y = 6 \Rightarrow y = -2$

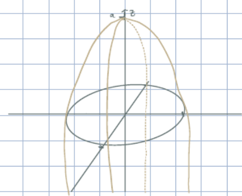
@ x and y = 0 , $z = 6$



Ex: $f(x,y) = 9 - x^2 - y^2$

$z = 9 - x^2 - y^2$

$x^2 + y^2 = -z + 9 \Rightarrow$ Paraboloid along "z" opening towards ⊖"z" and shifted +9 on "z"



Ex: $g(x,y) = \dfrac{1}{2}\sqrt{36 - 9x^2 - 36y^2}$ Domain: 2D Surface: 3D

- $36 - 9x^2 - y^2 \geq 0 \iff 9x^2 + y^2 \leq 36$

$D: \left\{ (x,y) \mid 0 \leq 9x^2 + y^2 \leq 36 \right\}$

$z = \dfrac{1}{2}\sqrt{36 - 9x^2 - 36y^2} \iff (2z)^2 = \left(\sqrt{36 - 9x^2 - 36y^2}\right)^2$

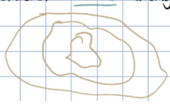
$4z^2 = 36 - 9x^2 - 36y^2 \iff 9x^2 + 36y^2 + 4z^2 = 36$

$\dfrac{x^2}{4} + \dfrac{y^2}{1} + \dfrac{z^2}{9} = 1 \Rightarrow$ ELLIPSOID

LEVELS CURVES

The shape we get when a plane intersects over surface at different levels along the axis of the dependent variable.

- A map of level curves is a CONTOUR PLOT



To find level curves, set $f = $ "k"

Ex: $f(x,y) = \sqrt{16-x^2-y^2}$
$k = \sqrt{16-x^2-y^2}$
$k^2 = 16-x^2-y^2$
$x^2+y^2 = 16-k^2 \Rightarrow$
these are circles with max rad $=4$ that get smaller as $k=4$

Ex: $f(x,y) = y^2 - x^2$
along z ↳ $k = y^2 - x^2$
$\frac{y^2}{k} - \frac{x^2}{k} = 1 \Rightarrow$ hyperbolas along "x" or "y" depending on "k"

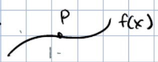
Ex: $f(x,y) = \ln(x+y)$
$k = \ln(x+y)$
$e^k = x+y \Rightarrow y = e^k - x \Rightarrow$ lines with slope (-1) with y-Intercept of e^k.

Ex: $f(x,y,z) = 2x+4y-3z+1$
$k = 2x+4y-3z+1$
$2x+4y-3z = k-1 \Rightarrow$ planes with a normal $\vec{n} = \langle 2,4,-3\rangle$

Ex: $f(x,y,z) = x^2+y^2-z^2$
$k = x^2+y^2-z^2$
- $k>0$: 1-Sheet hyperboloid
- $k=0$: Cone
- $k<0$: 2-Sheet hyperboloid

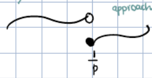
13.2 - Limits and Continuity of Multivariable Functions (with Squeeze Theorem)

- $\lim_{x\to a} f(x) = L$



∗ we travel along the curve, not the x-axis; lim does not exist if the height of the function as we approach a point is ≠ when coming from both sides

- $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$



What about $\lim_{(x,y)\to(a,b)} f(x,y) = L$

1-variable ⇒ this is a curve that has a certain path. So there are 2 directions to approach the point.
2-variables ⇒ we have a surface. There are ∞ # paths along the surface that approaches the point
To prove that the limit exists, we have to show that along ALL PATHS, we approach the same point.
↳ - We can use the SQUEEZE THEOREM.
- We can also prove that a limit D.N.E (does not exist) (2 paths giving 2 ≠ heights when getting to (x,y))
↳ By showing that along 2 paths, we get a ≠ value ↳ $f(x,y)$ as we approach the same point ↳ (x,y)

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{2x^2+y}$ @ Show that the lim D.N.E. To do so, ALWAYS pick $y=0$ or $x=0$

Along $x=0$: traveling along the surface directly over/under "y"-Axis.

$$\lim_{y\to 0} \frac{0^2-y^2}{2(0)^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1$$

Along $y=0$ (over x-axis); $\lim_{x\to 0} \frac{x^2-0^2}{2(x^2)+(0)^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

$-1 \neq \frac{1}{2}$ => Therefore the limit does not exists at $(0,0)$

How To Do So

#1- Try $x=0$ and $y=0$ paths 1st

#2- If $\uparrow$ don't work, choose other paths with at least $x=0$ or $y=0$

a- Be certain that the point (a,b) is ACTUALLY ON YOUR PATH

b- Try to substitute so degrees of numerator and denominator are =

c- Always use either $x=0$ or $y=0$ as 1 path.

Ex: $\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2}$ Prove D.N.E

Along $x=0$: $\lim_{y\to 0} \frac{0}{y^2} = 0$

Along $y=0$: $\lim_{x\to 0} \frac{0}{3x^2} = 0$

* $y=f(x)$ or $x=f(y)$

Along "$y=x$" => $\lim_{x\to 0} \frac{3x\,x}{3x^2+x^2} = \lim_{x\to 0} \frac{3x^2}{4x^2} = \frac{3}{4}$

$0 \neq 3/4$, so lim D.N.E @ $(0,0)$

Ex: $\lim_{(x,y)\to(0,0)} \frac{xy^3 \cos x}{2x^2+y^6}$, Prove D.N.E

Along $x=0$: $\lim_{y\to 0} \frac{0}{y^6} = 0$

Along $y=0$: $\lim_{x\to 0} \frac{0}{2x^2} = 0$

Along $x=y^3$: $\lim_{y\to 0} \frac{y^3 \cdot y^3 \cos y^3}{2(y^3)^2+y^6} = \lim_{y\to 0} \frac{y^6 \cos(y^3)}{3y^6} = \lim_{y\to 0} \frac{\cos 0}{3} = \frac{1}{3}$

since $0 \neq 1/3$ => lim DNE @ $(0,0)$

Ex: $\lim_{(x,y)\to(1,0)} \frac{2xy-2y}{x^2+y^2-2x+1}$, prove lim DNE

Along $y=0$: $\lim_{x\to 1} \frac{0}{x^2-2x+1} = 0$

Along $x=1$: $\lim_{y\to 0} \frac{2y-2y}{1+y^2-2+1} = \lim_{y\to 0} \frac{0}{y^2} = 0$

$\lim_{(x,y)\to(1,0)} \frac{2y(x-1)}{y^2+(x-1)^2}$

Along $x-1=y$: $\lim_{y\to 0} \frac{2y\cdot y}{y^2+y^2} = 1$

$0 \neq 1$, so lim D N E @ $(1,0)$

Ex: Show $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$ DNE:

- For 3-var path are now PARAMETRIC "t"
- Always choose an AXIS as 1st path

Along x-axis: set $y=0$ and $z=0$: $\lim_{x\to 0} \frac{0}{x^2} = 0$

Along C: $x=t, y=t, z=t$: $\lim_{t\to 0} \frac{t^2+t^2+t^2}{t^2+t^2+t^2} = 1$

$0 \neq 1$ so lim DNE @ $(0,0,0)$

Ex: $\lim_{(x,y,z)\to(0,0,0)} \frac{xz^2+2y^2}{x^2+2y^2+z^4}$

Along x axis: $\lim_{x\to 0} \frac{0}{x^2} = 0$

Along C: $x=t^2$, $y=t^2$, $z=t$

$\lim_{t\to 0} \frac{t^2 t^2 + 2(t^2)^2}{(t^2)^2+2(t^2)^2+t^4} = \lim_{t\to 0} \frac{3t^4}{4t^4} = \frac{3}{4}$

$0 \neq \frac{3}{4}$ so lim DNE @ $(0,0,0)$

Ex: • $\lim_{(x,y)\to(1,-2)} \frac{3xy}{2x^2-y^2} = 3$

• $\lim_{(x,y)\to(0^+,0^+)} \frac{e^{\sqrt{x+y}}}{x+y-1} = -1$

• $\lim_{(x,y,z)\to(0,3,1)} e^{\sin(\pi x)} + \ln(\cos\pi(y-z)) =$

$= e^{\sin(0)} + \ln(\cos\pi(2)) = e^0 + \ln(\underbrace{\cos 2\pi}_{1}) = e^0 + \underbrace{\ln 1}_{0} = 1$

• $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ $\qquad r^2 = x^2+y^2$ if $x = r\cos\theta$ and $y = r\sin\theta$

$(x,y)\to(0,0)$

$r\to 0^+$

$\lim_{r\to 0^+} \frac{r^3\cos^3\theta + r\sin^3\theta}{r^2}$

$\lim_{r\to 0^+} \frac{\cancel{r^3}(\cos^3\theta+\sin^3\theta)}{\cancel{r^2}} = \lim_{r\to 0^+} r(\cos^3\theta+\sin^3\theta) = 0$ so lim exists at $(0,0)$

• $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = 0\cdot(\infty) = \overset{\text{indefinite form}}{I.F.}$ $\qquad$ let $x^2+y^2 = r$

$\lim_{r\to 0^+} r^2 \ln r^2 = \lim_{r\to 0^+} \frac{\ln r^2}{\frac{1}{r^2}} \overset{L}{=} \lim_{r\to 0^+} \frac{\frac{1}{r^2}}{\frac{-2}{r^3}} = \lim_{r\to 0^+} \frac{1}{r^2} \times \frac{r^3}{-2} = 0$

CONTINUITY

A function is continuous at ANY point on the region for which it is defined (its domain).

1- Polynomial are continuous EVERYWHERE
2- Rational functions are continuous where DENOM $\neq 0$
3- Continuity holds for compositions

Ex: ① $f(x,y) = \frac{x^3+xy+y^3}{x^2+y^2+1}$ continuous on all points

$f(x,y) = \frac{x^3+xy+y^3}{x^2+y^2}$ continuous on $\{(x,y) \mid (x,y) \neq (0,0)\}$

② $f(x,y) = \sqrt{x}\, e^{\frac{x}{y}}$ continuous on $\{(x,y) \mid x \geq 0 \text{ and } y \neq 0\}$

$y \neq 0$; $x \geq 0$

③ $f(x,y,z) = \frac{xyz}{x^2+y^2+z^2-4}$ cont. on $\{(x,y,z) \mid x^2+y^2+z^2 \neq 4\}$

④ $f(x,y) = x^2+xy+y^2$, $g(t) = t\cos t + \sin t$

↳ CONT on all (x,y) ↳ CONT on all t

$h(x,y) = g(f(x,y))$ CONT. on all (x,y)

⑤ $f(x,y) = x-2y+3$, $g(t) = \sqrt{t} + \frac{1}{t}$

↳ CONT on all (x,y) ↳ CONT on $\{t \mid t > 0\}$

$h(x,y) = g(f(x,y)) \Rightarrow t = x-2y+3$ so $0 < x-2y+3$ and $x-2y > -3$

↳ CONT on $\{(x,y) \mid x-2y > -3\}$

⑥ $f(x,y) = x\tan y$, $g(t) = \cos t$

↳ CONT on $\{(x,y) \mid y \neq \frac{\pi}{2} + k\pi\}$ ↳ cont on all "y"

$h(x,y) = g(f(x,y))$ cont $\{(x,y) \mid y \neq \frac{\pi}{2} + k\pi\}$

Ex: ① $\lim\limits_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$

$$\frac{5x^2y}{x^2+y^2} \longrightarrow \left|\frac{5x^2y}{x^2+y^2}\right| = \frac{5x^2|y|}{x^2+y^2}$$

$$\frac{5x^2}{x^2} = 5 \Rightarrow \frac{5x^2}{x^2+y^2} \leq 5 \Rightarrow \frac{5x^2|y|}{x^2+y^2} \leq 5|y| \Leftrightarrow$$

$$0 \leq \frac{5x^2|y|}{x^2+y^2} \leq 5|y|$$

$\lim\limits_{(x,y)\to(0,0)} 0 = 0$ $\lim\limits_{(x,y)\to(0,0)} 5|y| = 0$

so $\lim\limits_{(x,y)\to(0,0)} \frac{5x^2|y|}{x^2+y^2} = 0 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \left|\frac{5x^2y}{x^2+y^2}\right| = 0 =$

$$\left|\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}\right| = 0$$

13.3 - Derivatives of multivariable functions

What does the derivative of a multi. function look like?

$f(x,y)$ is a surface in R-3.

- is it the slope of the tangent to the surface at a point? ⇒ Too ambiguous → ∞ Tangents

To find the slope of a Tangent line to a surface at a point, we must give the tangent line a DIRECTION.

- Fully directional derivatives come later. For now, we restrict our derivat. to: In the "x-direction" or in the "y-direction".

To find the slope of a Tan. line in "x-direction", we must contain the tan line in a plane parallel to xz plane (contains x-axis)

- Requires "y" to be held constant. (ex: $y = 3 \to$ plane // xz plane) and makes certain tangent line is in the xz plane).
- For "y-Direction", hold "x" constant ⇒ forces tan. line to be in a plane // to the yz plane
- We don't have to worry about z because its not an independent variable.

The idea of treating a variable as a constant and thereby insuring that the tan. line is in the direction of the other variable is called a PARTIAL DERIVATIVE.

Notation: for $f(x,y) = z$

$\frac{\partial f}{\partial x}$: holds "y" constant (plane), so it gives the slope tan. line to the surface at a point in x-direction with respect to "x"

$\frac{\partial f}{\partial y}$: holds "x" constant, gives the slope of TAN. line to the surface at a point in y-direction W/R/T "y"

Note: $\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x$ and $\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y = z_y$

Ex: ① $f(x,y) = 2x^2y^3 - 3x^2y + 2x^2 + 3y^2 + 1$

$\frac{\partial f}{\partial x} = 4x^2y^3 - 6xy + 4x$ and $\frac{\partial f}{\partial y} = 6y^2x^2 - 3x^2 + 6y$

↳ slope of TAN line to surface in x-direction

② $f(x,y) = e^x \cos y + e^y \sin x$

$f_x = e^x \cos y + e^y \cos x$ and $f_y = -e^x \sin y + e^y \sin x$

③ $z = xe^{xy^2}$

x $z_x = e^{xy^2} + x\,xy^2 e^{xy^2} = e^{xy^2} + x^2y^2e^{xy^2} = e^{xy^2}(1 + x^2y^2)$ false

$z_y = 2x^2ye^{xy^2}$

$z_x = \frac{\partial}{\partial x}[x] \cdot e^{xy^2} + x\frac{\partial}{\partial x}[e^{xy^2}] = e^{xy^2} + x \cdot e^{xy^2} \cdot \frac{\partial}{\partial x}[xy^2] = e^{xy^2} + xe^{xy^2} \cdot y^2$

$z_x = e^{xy^2}(1 + xy^2)$

$z_y = xe^{xy^2} \cdot \frac{\partial}{\partial y}[xy^2] = xe^{xy^2} \cdot 2xy = 2x^2ye^{xy^2}$

④ $z = y^x$

$\frac{\partial z}{\partial x} = y^x \cdot \ln|y|$ and $\frac{\partial z}{\partial y} = xy^{x-1}$

⑤ $g(x,y) = x^2 \cosh\left(\frac{x}{y}\right)$

$g_x = x^2 \cdot \frac{\partial}{\partial x}\left[\cosh\left(\frac{x}{y}\right)\right] + \frac{\partial}{\partial x}[x^2] \cdot \cosh\left(\frac{x}{y}\right) = x^2 \cdot \left(-\sin h\left(\frac{x}{y}\right)\right) \cdot \frac{\partial}{\partial x}\left[h\left(\frac{x}{y}\right)\right] + 2x \cdot \cos h\left(\frac{x}{y}\right)$

$= -x^2 \sin h\left(\frac{x}{y}\right) \cdot \frac{h}{y} + 2x \cdot \cosh\left(\frac{x}{y}\right) = -x^2 \cdot \frac{h}{y} \cdot \sin h\left(\frac{x}{y}\right) + 2x \cos h\left(\frac{x}{y}\right)$ x false

$g_x = 2x\cosh\left(\frac{x}{y}\right) + x^2 \cdot \sinh\left(\frac{x}{y}\right) \cdot \frac{1}{y}$ $\quad \cosh \neq \cos(h)$

$g_y = x^2 \sinh\left(\frac{x}{y}\right) \cdot \left(\frac{-x}{y^2}\right) = \frac{-x^3}{y^2} \sinh\left(\frac{x}{y}\right)$

⑥ $w = 2x^3 + 3xy + 2yz - z^2$

$w_x = 6x^2 + 3y$, $w_y = 3x + 2z$, $w_z = 2y - 2z$

⑦ $f(x,y,z,w) = \frac{xw^2}{y + \sin(zw)}$

$f_x = \frac{w^2}{y + \sin(zw)}$

$f_w = \frac{(y + \sin(zw)) \cdot \frac{\partial}{\partial w}[xw^2] - xw^2 \cdot \frac{\partial}{\partial w}[y + \sin(zw)]}{(y + \sin(zw))^2}$

$= \frac{y + \sin(zw) \cdot (2xw) - xw^2 \cdot z\cos(zw)}{(y + \sin(zw))^2}$

$= \frac{2xw(y + \sin(zw)) - xw^2 z \cos(zw)}{(y + \sin(zw))^2}$

Implicit derivates

"z" is the implicitly defined variable, not "y". $\frac{\partial z}{\partial x \cdot \partial y}$

- Both sides - $z = f(x,y)$, $x \cdot z$ or $y \cdot z \Rightarrow$ product rule
  ↳ take derivatives

Ex: ① $x^2y + xz + yz^2 = 8$

$\frac{\partial z}{\partial x}$: $\frac{\partial}{\partial x}[x^2y + \underbrace{xz + yz^2}] = \frac{\partial}{\partial y}[8] = 0$

↳ $2xy + \frac{\partial}{\partial x}[x] \cdot z + x\frac{\partial}{\partial x}[z] + 2yz \cdot \frac{\partial}{\partial x}[z] = 0$

$$2xy + z + x \cdot \frac{\partial z}{\partial x} + 2yz\,\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(x + 2yz\right) = -2xy - z \Rightarrow \frac{\partial z}{\partial x} = \frac{-2xy - z}{x + 2yz}$$

$$\frac{\partial z}{\partial y} : \frac{\partial}{\partial y}\left[x^2y + xz + yz^2\right] = \frac{\partial}{\partial y}[8] = 0$$

$$\Rightarrow x^2 + x \cdot \frac{\partial z}{\partial y} + y \cdot 2z \cdot \frac{\partial}{\partial y} z + z^2 = 0$$

$$\Rightarrow x\frac{\partial z}{\partial y} + 2zy\frac{\partial z}{\partial y} = -x^2 - z^2 \Rightarrow \frac{\partial z}{\partial y} = \frac{-x^2 - z^2}{x + 2zy}$$

② $2\cos(x+2y) + \sin(yz) - 1 = 0$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial x}\left[2\cos(x+2y) + \sin(yz) - 1\right] = \frac{\partial}{\partial x}[0] = 0$$

$$\Rightarrow -2\sin(x+2y) \cdot 1 + \cos(yz)\,\cancel{\frac{\partial}{\partial x}[yz]}^{\,y \cdot \frac{\partial z}{\partial x}} = 0$$

$$\frac{\partial z}{\partial y} : -4\sin(x+2y) + \cos(yz)\left(z + y\frac{\partial z}{\partial y}\right) = 0$$

Higher derivatives

$z = f(x,y)$

- $\frac{\partial f}{\partial x}$
  - $\frac{\partial^2 f}{\partial x^2}$, f_{xx}, z_{xx}
  - $\frac{\partial^2 f}{\underbrace{\partial y}_{\text{2nd time}}\underbrace{\partial x}_{\text{1st time}}}$, f_{xy}, z_{xy}
- $\frac{\partial f}{\partial y}$
  - $\frac{\partial^2 f}{\partial y^2}$, f_{yy}, z_{yy}
  - $\frac{\partial^2 f}{\partial x \partial y}$, f_{yx}, z_{yx}

Ex: ① $g(x,y) = x^3y^2 + xy^3 - 2x + 3y + 1$

- $\frac{\partial g}{\partial x} = 3x^2y^2 + y^3 - 2$ $\quad \frac{\partial^2 g}{\partial x^2} = 6xy^2$ $\quad \frac{\partial^2 g}{\partial y \partial x} = 6yx^2 + 3y^2$
- $\frac{\partial g}{\partial y} = 2yx^3 + 3y^2x + 3$ $\quad \frac{\partial^2 g}{\partial y^2} = 2x^3 + 6yx$ $\quad \frac{\partial^2 g}{\partial x \partial y} = 6yx^2 + 3y^2$

② $f(x,y) = x\sin^2 y + y^2\cos x$

$f_x = \sin^2 y - y^2 \sin x$

$\quad f_{xy} = 2\cos y \sin y - 2y\sin x$

$f_y = x \cdot (2\cos y \sin y) + 2y\cos x$

$\quad f_{yx} = 2\cos y \sin y - 2y \sin x$

Note: For any function that is continuous on a region, mixed partial derivatives are <u>equal</u>. (just make sure you have the same letters)

$f_{xy} = f_{yx}$ and $g_{xzy} = g_{yxz} = g_{zxy} \neq g_{xyx}$

Ex: ① $f(x,y,z) = x^2y^3 - y^2z^3$

- $f_x = 2xy^3$ $\quad f_{xy} = 6xy^2$ $\quad f_{xyz} = 0$
- $f_y = 3x^2y^2 - 2yz^3$ $\quad f_{yz} = -6yz^2$ $\quad f_{yzx} = 0$

② $h(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$h_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x^2} = \frac{-y}{x^2+y^2} = -y(x^2+y^2)^{-1}$$

$$h_{xx} = y(x^2+y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$h_{xy} = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$h_y = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$

$$h_{yy} = -x(x^2+y^2)^{-2} \cdot 2y$$

$$h_{yy} = \frac{-2xy}{(x^2+y^2)^2}$$

Note: $h_{xx}+h_{yy}=0$. "h" is called a LAPLACE equation

Lagrange Multiplier

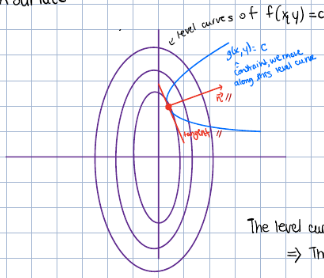
Idea: Function

$f(x,y) = z$ ← variable ⇒ a surface

A surface

Constraint

$g(x,y) = c$ ← constant ⇒ A level curve to a surface $g(x,y) = z$



The intersection gives constrained maxima and minima.

The normals of level curves are scalar multiples.

Normal L.C — normal L.C

$f(x,y) = k \cdot g(x,y)$

so $\vec{\nabla} f(x,y) = \lambda \cdot \nabla g(x,y)$

$\nabla f(x,y,z) = \lambda \cdot \nabla g(x,y,z)$

The level curves have parallel tangents.

⇒ The level curves have parallel normals.

Normal to L.C of a surface — Normal to L.C of constraint

$f(x,y) = K \cdot g(x,y)$

Ex: $f(x,y) = xy$ with constraint of $2x+3y=6$

Goal: Find λ in $\nabla f(x,y) = \lambda \cdot \nabla g(x,y)$ ← maxima and minima will be found as solutions to this

- create a $g(x,y)$ such that $2x+3y$ is a level curve: $g(x,y) = 2x+3y$
- normal to level curves is given by $\vec{\nabla} f(x,y)$.

$\nabla f(x,y) = \langle y, x \rangle$ and $\nabla g(x,y) = \langle 2, 3 \rangle$

So $y\hat{i} + x\hat{j} = \lambda(2\hat{i} + 3\hat{j})$

$$\begin{cases} y = 2\lambda & (1) \\ x = 3\lambda & (2) \\ 2x+3y=6 & (3) \end{cases}$$

constraint → (3)

· Solve for λ and plug it in.

$\Rightarrow \lambda = \frac{1}{2}, \; x = \frac{3}{2}, \; y = 1$

1 critical point: $(\frac{3}{2}, 1)$

$f(\frac{3}{2}, 1) = \frac{3}{2}$ (this is a max)

Ex: $f(x,y,z) = x + 2y - 2z$ and constraint: $x^2+2y^2+4z^2=1 \Rightarrow g(x,y,z) = x^2+2y^2+4z^2$

$\vec{\nabla} f(x,y,z) = 1\hat{i} + 2\hat{j} - 2\hat{k}$ and $\nabla g(x,y,z) = 2x\hat{i} + 4y\hat{j} + 8z\hat{k}$

$\vec{\nabla} f(x,y,z) = \lambda \cdot \vec{\nabla} g(x,y,z)$

$(1\hat{i} + 2\hat{j} - 2\hat{k}) = \lambda \cdot (2x\hat{i} + 4y\hat{j} + 8z\hat{k})$

$$\begin{cases} 1 = 2x\lambda & x = \frac{1}{2\lambda} \\ 2 = 4y\lambda & \Rightarrow y = \frac{1}{2\lambda} \\ -2 = 8z\lambda & z = -\frac{1}{4\lambda} \\ x^2+2y^2+4z^2=1 & \Rightarrow \lambda^2 = 1 \text{ and } \lambda = \pm 1 \end{cases}$$

at $\lambda = 1$, $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = -\frac{1}{4}$ — $f(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}) = 2$ max

$\lambda = -1$, $x = -\frac{1}{2}$, $y = -\frac{1}{2}$, $z = \frac{1}{4}$ — $f(-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{4}) = -2$ min

Ex: $f(x,y) = x^2+xy+y^2$ constraint: $x^2+y^2=8 \Rightarrow g(x,y) = x^2+y^2$

$\nabla f(x,y) = (2x+y)\hat{i} + (2y+x)\hat{j}$; $\nabla g(x,y) = 2x\hat{i} + 2y\hat{j}$

$(2x+y)\hat{i} + (2y+x)\hat{j} = \lambda(2x\hat{i} + 2y\hat{j})$

$$\begin{cases} 2x+y = 2x\lambda & \rightarrow y = 2x(\lambda-1) \\ 2y+x = 2y\lambda & \rightarrow x = 2y(\lambda-1) = 2(2x(\lambda-1))(\lambda-1) = 4x(\lambda-1)^2 \Rightarrow 4x(\lambda-1)^2 - x = 0 \Rightarrow x(4(\lambda-1)^2-1) = 0 \\ x^2+y^2 = 8 & \rightarrow \end{cases}$$

so $\cancel{x=0}$ ← irrelevant b/c $0^2+0^2 \neq 8$ or $4(\lambda-1)^2 = 0$

$\lambda = \frac{3}{2}$ or $\lambda = \frac{1}{2}$

at $\lambda = \frac{3}{2}$, $y = x$ so $x = \pm 2$. if $x=2$, $y=2$; if $x=-2$, $y=-2$

at $\lambda = \frac{1}{2}$, $y = x$ so $x = \pm 2$, if $x=2$, $y=-2$; if $x=-2$, $y=2$

4 points:
$f(2,2) = 12$, $f(-2,-2) = 12$ } max
$f(2,-2) = 4$, $f(-2,2) = 4$ } min