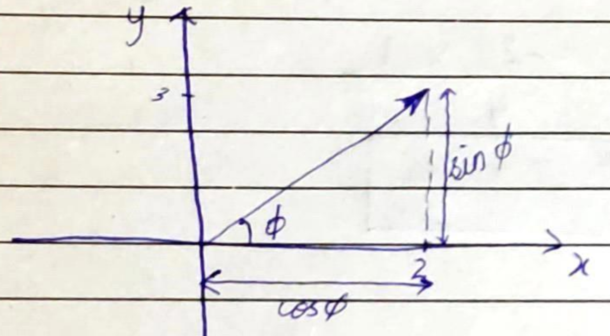
→ Directional Derivatives



$|v| = \sqrt{2^2 + 3^2} = \sqrt{13}$

If $f(x,y)$ is a function of x and y, if $u = u_1 i + u_2 j$ is a unit vector, then the direction derivative of f in the direction of u at (x_0, y_0) is denoted by $D_u f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + u_1 s,\ y_0 + u_2 s)\right]_{s=0}$

provided that derivatives exist.

If (x,y,z) is a function of $x, y,$ and z, if $u = u_1 i + u_2 j + u_3 k$, then

$$D_u f(x_0, y_0, z_0) = \frac{d}{ds}\left[f(x_0 + u_1 s,\ y_0 + u_2 s,\ z_0 + u_3 s)\right]_{s=0}$$

THM: a) If $f(x,y)$ is differentiable at (x_0, y_0) and if $u = u_1 i + u_2 j$ is a unit vector, then the directional derivative at this point is denoted by $\boxed{D_u f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2}$

b)

b) If $f(x,y,z)$ is differentiable at (x_0, y_0, z_0) and if $u = u_1 i + u_2 j + u_3 k$, is a unit vector, then

$$\boxed{D_u f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)u_1 + f_y(x_0,y_0,z_0)u_2 + f_z(x_0,y_0,z_0)u_3}$$

Ex 13-6

Q9-18

Find the directional derivative of f at P in the direction of a

Q16) $f(x,y,z) = y - \sqrt{x^2+z^2}$, $P(-3,1,4)$

$d = 2i - 2j - k$

$a = 2i - 2j - k$

$|a| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3$

$\frac{a}{|a|} = \frac{2i - 2j - k}{3}$

$= \frac{2}{3} i - \frac{2}{3} j - \frac{1}{3} k$

using formula

$D_a f(-3,1,4) = f_x(-3,1,4)a_1 + f_y(-3,1,4)a_2 + f_z(-3,1,4)a_3$

$f_x = \frac{-\cancel{2}x}{\cancel{2}\sqrt{x^2+z^2}} = \frac{-x}{\sqrt{x^2+z^2}} = +\frac{3}{5}$

$f_y = 1$

$f_z = \frac{-\cancel{2}z}{\cancel{2}\sqrt{x^2+z^2}} = \frac{-z}{\sqrt{x^2+z^2}} = -\frac{4}{5}$

$D_a = f(-3,1,4) = \left(\frac{2}{3} \times \frac{3}{5}\right) + \left(1 \times \frac{-2}{3}\right) + \left(\frac{-4}{5} \times \frac{-1}{3}\right)$ Q 33-46

$\boxed{D_a \cdot f(-3,1,4) = 0}$

4th March '24

Directional derivative in the direction of vector u when ϕ is the angle of u with positive x-axis.

$$\boxed{D_u f(x_0, y_0) = f_x(x_0,y_0)\cos\phi + f_y(x_0,y_0)\sin\phi}$$

Q.19-22 (13.6)

Gradient of function f:

a) If f is a function of x and y, then the gradient of f is defined by

$$\boxed{\nabla f(x,y) = f_x(x,y)i + f_y(x,y)j}$$

b) If f is a function of ~~f~~ x, y & z, then

$$\boxed{\nabla f(x,y,z) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k}$$

The directional derivative in terms of gradient is in the form

$$\boxed{D_u f(x_0,y_0,z_0) = \nabla f(x_0,y_0,z_0) \cdot u}$$

where $u = u_1 i + u_2 j + u_3 k$

Q45: $f(x,y,z) = y\ln(x+y+2)$, $(-3,4,0)$

Find the gradient of at the indicated point using formula:

$$\nabla f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)i + f_y(x_0,y_0,z_0)j + f_z(x_0,y_0,z_0)k$$

$$f_x = \frac{y}{x+y+z} = 4$$

$$f_y = \frac{y}{x+y+2} + \ln(x+y+2) = 4$$

$$f_z = \frac{y}{x+y+z} = 4$$

$$\boxed{\nabla f(-3,4,0) = 4i + 4j + 4k}$$

→ Properties of Gradient:

1) At (x,y), the surface $z = f(x,y)$ has its maximum slope in the direction of the gradient and the maximum slope is $\|\nabla f(x,y)\|$

2) At (x,y) the surface $z = f(x,y)$ has its minimum slope in the direction that is opposite to the gradient and its minimum slope is $-\|\nabla f(x,y)\|$

THM 13.6.5:

Let f be a function of either two variables or three variables and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$ respectively. Assume that f is differentiable at P.

a) If $\nabla f = 0$ at P, then all directional derivative of f at P are zero.

b) If $\nabla f \neq 0$ at P, then among all possible directional derivative of f at P, the ~~direction~~ derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivatives at P is $\|\nabla f\|$.

c) If ∇f at P has the smallest value, the value, the value of this smallest directional derivative at P is $-\|\nabla f\|$

Q53-60

Find a unit vector in the direction in which f increases most rapidly at P and find the rate of change of f at P in that direction

59) $f(x,y,z) = \dfrac{x}{z} + \dfrac{z}{y^2}$, $P(1,2,-2)$

find $\nabla f(1,2,-2)$

$$\nabla f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)i + f_y(x_0,y_0,z_0)j + f_z(x_0,y_0,z_0)k$$

$f_x = 1/z =$

$f_y = \dfrac{-2z}{y^3} =$

$f_z = \dfrac{-x}{z^2} + 1$

Ex 13.6 Q61-66

Find the unit vector in the direction in which f decreases most rapidly at P, and find the rate of change of f at P in that direction

Q66: $f(x,y,z) = 4e^{xy}\cos z$, $P(0,1,\pi/4)$

A unit vector in the direction in which f decreases most rapidly at P is the direction opposite to gradient i.e.

$$\frac{-\nabla f(x_0,y_0,z_0)}{\|\nabla f(x_0,y_0,z_0)\|}$$

using formula for gradient

$$\nabla f(0,1,\pi/4) = f_x(0,1,\pi/4)i + f_y(0,1,\pi/4)j + f_z(0,1,\pi/4)k$$

$$f_x = 4y\cos z\, e^{xy} = 4(1)\cos(\pi/4)e^{(0)(1)} = 2\sqrt{2}$$

$$f_y = 4x\cos z\, e^{xy} = 4(0)\cos(\pi/4)e^{(0)(1)} = 0$$

$$f_z = -4e^{xy}\sin z = -4e^{(0)(1)}\sin(\pi/4) = -2\sqrt{2}$$

$$\nabla f(0,1,\pi/4) = 2\sqrt{2}\,i + 0j - 2\sqrt{2}\,k$$

$$\boxed{\nabla f(0,1,\pi/4) = 2\sqrt{2}(i-k)}$$

magnitude:

$$\|\nabla f(0,1,\pi/4)\| = 4$$

$$\Rightarrow -\frac{2\sqrt{2}(i-k)}{4} = -\frac{\sqrt{2}}{2}(i-k)$$

Also, the rate of change of f at P in the rapid decrease direction is

$$\boxed{-\|\nabla f(0,1,\pi/4)\| = -4} \text{ Ans}$$

→ Tangent Planes and normal vectors.

line for tangent:

$$y = y_0 + \frac{dy}{dx}(x - x_0)$$

$y = f(x)$, $y_0 = f(x_0)$, $f'(x_0)$

$$y_0 = f(x_0)\ f_x(x - x_0) + f_y(y - y_0)$$

Assume that $f(x,y,z) = c$ has continuous first order partial derivative and that $P_0(x_0,y_0,z_0)$ is a point on the level surface.

$S: f(x,y,z) = c$ If $\nabla f(x_0,y_0,z_0)$ ~~is a normal vector~~ $\neq 0$, then $n = \nabla f(x_0,y_0,z_0)$ is a normal vector to S at P_0 and the tangent plane to S at P_0 is the plane ~~whose~~ with equation $f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)$

Q3-12 **Ex 13.7**

Find an equation for the tangent plane and parametric equations for the normal line to the surface at the point P:

Q4 $x^2y - 4z^2 = -7$, $P(-3,1,-2)$

using formula for tangent plane

$F_x(-3,1,-2)(x+3) + F_y(-3,1,-2)(y-1) + F_z(-3,1,-2)(z+2) = 0$

$F_x = 2xy = -6$

$F_y = x^2 = 9$

$F_z = -8z = 16$

I $\Rightarrow -6(x+3) + 9(y-1) + 16(z+2) = 0$

$= \boxed{-6x + 9y + 16z + 5 = 0}$

→ For parametric equation of normal lines to the surface:

we have,

$n = \nabla f(-3,1,-2)$ is the normal vector to the given surface.

$\nabla f(-3,1,-2) = F_x(-3,1,-2)\,i + F_y(-3,1,-2)\,j + F_z(-3,1,-2)\,k$

$\boxed{n = \nabla f(-3,1,-2) = -6i + 9j + 16k}$

the parametric equations of a normal lines are

$x = x_0 + at \rightarrow x = -3 - 6t$

$y = y_0 + bt \rightarrow y = 1 + 9t$

$z = z_0 + ct \rightarrow z = -2 + 16t$

→ Tangent Planes to surface of the form $z = f(x,y)$

If $f(x,y)$ is differentiable at the point (x_0,y_0), then the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0, y_0, f(x_0,y_0))$ is the plane

$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

Q10: $z = \ln(x^2+y^2)$, $P(-1,0,0)$

using formula for explicit function

$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f_x = \dfrac{2x}{x^2+y^2} = -2$

$f_y = \dfrac{2y}{x^2+y^2} = 0$

$z = 0 - 2(x+1) + 0(y-0)$

$z = -2x - 2$

Parametric equations of normal line, we have

$n = \nabla f(-1,0)$ is the normal vector to the surface $n = f_x i + f_y j = -2i$

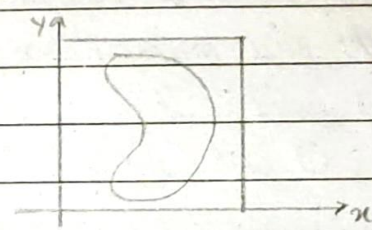
$x = x_0 + at \rightarrow \boxed{x = -1 - 2t}$

$y = y_0 + bt \rightarrow \boxed{y = 0}$

→ Maxima and Minima of Function of ~~th~~ two variables:

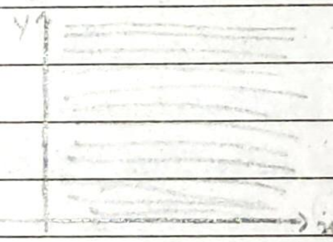
$f_x = \frac{1}{xy}$ $(x,y) \in R^2$

$f'(a) = 0$



a bounded set in 2-space

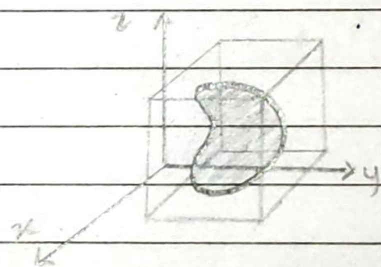
→ Relative Extreme:



for $x \geq 0$ & $y \geq 0$

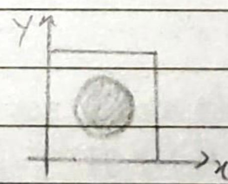
an unbounded set in 2-space

a) A function of two variables is said to have relative maximum at point (x_0, y_0) if there is a disk centred at (x_0, y_0) such that $f(x_0, y_0) \geq f(x,y)$ for all points (x,y) that lie inside the disk



a bounded set in 3-space

b) A function of two variables is said to have relative minimum at point (x_0, y_0) if there is a disk centred at (x_0, y_0) such that ~~the value of~~ $f(x_0, y_0) \leq f(x,y)$ for all points (x,y) that lie inside the disk.



a bounded set in 2-space

$\boxed{f(x_0, y_0) \geq f(x,y) \rightarrow \text{for all } (x,y) \text{ in the domain}}$

→ **Bounded Sets:** ① A set of points in two dimension/space is called bounded if the entire set can be contained within some rectangle, and is called unbounded. If there is no rectangle that contains all the points of the set

② A set of points in 3-space is bounded if the entire set can be contained within some box and is unbounded otherwise

→ Extreme Value Theorem:

If $f(x,y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and absolute minimum

→ Finding extreme values:

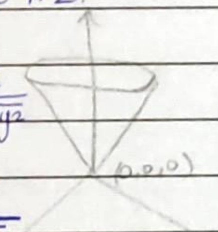
If f has a relative extreme at a point (x_0, y_0) and if the first order PD of f exists at this point, then

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0$$

Critical point:

Definition: A point (x_0, y_0) in the domain of a function $f(x,y)$ is called a critical point of $f(x,y)$ if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both PD do not exist at (x_0, y_0).

Example #2:



$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad z = \sqrt{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$(x_0, y_0) = (0,0) \rightarrow$ provides absolute min value of $f(x,y)$.

→ The 2nd partial test:

Let f be a function of two variables with continuous second-order PD in some disk centred at a critical point (x_0, y_0) and let

$$\boxed{D = \{f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0)\} - \{f_{xy}(x_0,y_0)\}^2}$$

(a) If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, then f has a relative minimum at (x_0, y_0).

(b) If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, then f has a relative maximum at (x_0, y_0)

(c) If $D < 0$, then f has a saddle point at (x_0, y_0)

(d) If $D = 0$, then no conclusion can be drawn.

Ex 13.8

Q1, 2, 9-20, 31-36

Q1-4, 9-20, 31-40

~~Q.12 $f(x,y) = xy - x^3 - y^2$~~

~~Step 1~~ Locate all relative maxima, relative minima and saddle point if any.

Q12: $f(x,y) = xy - x^3 - y^2$

Step 1: For critical points,

$f_x = y - 3x^2 \rightarrow \boxed{f_{xx} = -6x}$

$f_y = x - 2y \rightarrow \boxed{f_{yy} = -2}$

put $f_x = 0$ & $f_y = 0$

$y - 3x^2 = 0$ — (1) $\qquad \boxed{f_{xy} = 1}$

$x - 2y = 0$ — (2)

$x = 2y$

substituting in eq (1):

$y - 3(2y)^2 = 0$

$y - 12y^2 = 0$

$y(1 - 12y) = 0$

$y = 0 \qquad y = \frac{1}{12}$

$x = 2(0) \qquad x = 2(1/12) \rightarrow \left(\frac{1}{6}, \frac{1}{12}\right)$

$x = 0 \qquad x = \frac{1}{6}$

$\therefore \left(\frac{1}{6}, \frac{1}{12}\right)$ are C.P for given function

Step 2: Finding relative extreme:

we have ~~D = $f_{xx}(x_0,y_0)$~~

$D = \{f_{xx}(x_0,y_0) f_{yy}(x_0,y_0)\} - \{f_{xy}(x_0,y_0)\}^2$

when $(x_0,y_0) = (0,0)$

$f_{xx}(0,0) = 0$

$f_{yy}(0,0) = -2$

$f_{xy}(0,0) = 1$

Now,

$D = \{(0)(-2)\} - \{(1)\}^2$

$D = -1$

$D < 0 \rightarrow \therefore (0,0)$ is the saddle point of the function.

$\boxed{f(0,0) = 0}$

When $(x_0,y_0) = (1/6, 1/12)$,

$f_{xx}(1/6, 1/12) = -1$

$f_{yy}(1/6, 1/12) = -2$

$f_{xy}(1/6, 1/12) = 1$

$D = (-1 \times -2) - (1)^2$

$D = 1$

$D > 0 \rightarrow$

but $f_{xx}(1/6, 1/12) < 0$, hence $(1/6, 1/12)$ is the relative maximum point for given function.

$f(1/6, 1/12) = \frac{1}{72} - \frac{1}{416} - \frac{1}{194}$

$= \boxed{\frac{1}{432}}$

Q17 $f(x,y) = e^x \sin y$

Step 1: for C.P

$f_x = \sin y \, e^x$

$f_y = e^x \cos y$

put $f_x = 0$ & $f_y = 0$

~~$ab = 0 \Rightarrow a = 0 \Rightarrow b = 0$~~

$e^x \sin y = 0 \quad e^x \cos y = 0$

since $e^x \neq 0$ for any value of x,

$\Rightarrow e^x \sin y = 0 \rightarrow \sin y = 0$

$\Rightarrow e^x \cos y = 0 \rightarrow \cos y = 0$

$\therefore$ there is no value of y such that $\sin y = 0$ and $\cos y = 0$

$\Rightarrow$ no extreme value.

Q19 $f(x,y) = e^{-(x^2+y^2+2x)}$

Step 1: For C.P.

$f_x = -2x - 2 \cdot e^{-(x^2+y^2+2x)}$

$f_y = -2y \cdot e^{(-x^2+y^2+2x)}$

put $f_x = 0$ and $f_y = 0$

$-2(x+1)e^{-(x^2+y^2+2x)} = 0 \Rightarrow \boxed{x = -1}$

$-2y e^{-(x^2+y^2+2x)} = 0 \Rightarrow \boxed{y = 0}$

$\Rightarrow (-1,0)$ is the critical point for $f(x,y)$.

Step 2: Finding relative extreme

$(x_0,y_0) = (-1,0)$

$D = \{f_{xx}(-1,0) \cdot f_{yy}(-1,0)\} - \{f_{xy}(-1,0)\}^2$

$f_{xx} = -2e^{-(x^2+y^2+2x)} + 4(x+1)^2 e^{-(x^2+y^2+2x)}$

$\boxed{f_{xx} = (4(x+1)^2 - 2) e^{-(x^2+y^2+2x}}$

$$\boxed{f_{yy} = (4y^2-2)e^{-(x^2+y^2+2x)}}$$

$$\boxed{f_{xy} = 4(x+1)y\,e^{-(x^2+y^2+2x)}}$$

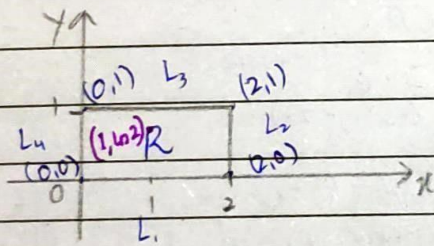
→THM 13.8.7:

Finding absolute extrema (pg # 876).
eg: 5 and 6

Ex 13.8 (Q31-40)

Find the absolute extrema of the given function on indicated closed and bounded set R:

Q.34: $f(x,y) = xe^{y} - x^2 - e^{y}$; R is the rectangular region with vertices (0,0), (0,1), (2,1), (2,0)



Step 1: For C.P:

$f_x = e^y - 2x$

$f_y = xe^y - e^y$

put f_x and $f_y = 0$

$e^y - 2x = 0$ $\qquad$ $xe^y - e^y = 0$

$e^y - 2(1) = 0$ $\qquad$ $e^y(x-1) = 0$

$e^y = 2$ $\qquad$ $x - 1 = 0$

$y = \ln 2 = 0.693$ $\qquad$ $x = 1$

$(1, \ln 2)$ is the critical point for the given function.

Step 2: For boundary points

→along L_1: (0,0) & (2,0)

$y = 0,\quad 0 \leq x \leq 2$

$f(x,0) = x - x^2 - 1$

$f_x = 1 - 2x$

$0 = 1 - 2x$

$\boxed{x = 1/2}$

$f''(x) = -2$

$f''(x) < 0$

$\Rightarrow x = 1/2$ is relative -ve, so maximum

$\boxed{f(1/2, 0) = -3/4}$

along L_2: (2,0) & (2,1)

$x = 2,\quad 0 \leq y \leq 1$

$f(2,y) = 2e^y - 4 - e^y$

$f_y = 2e^y - e^y$

$f_y = e^y$

$0 = e^y \rightarrow$ D.N.E

so extreme occurs at (2,0) and (2,1)

$\boxed{f(2,0) = -3}$, $\boxed{f(2,1) = e-4}$

along L_3: (2,1) & (0,1)

$y = 1\quad 0 \leq x \leq 2$

$f(x,1) = xe - x^2 - e$

$f_x = e - 2x$

$0 = e - 2x$

$\boxed{x = e/2}$

$f''(x) = -2$

$f''(x) < 0$

$\Rightarrow x = e/2$ is relative maximum.

$\boxed{f(e/2, 1) = \frac{e^2}{4} - e}$ $\qquad$ $\boxed{f(0,1) = e}$

along L_y: $(0,1)$ and $(0,0)$

~~f(0,y)~~ $x=0$, $0 \leq y \leq 1$

$f(0,y) = -e^y$

$f_y = -e^y$

$0 = -e^y \rightarrow$ D.N.E

$\boxed{f(0,0) = -1}$

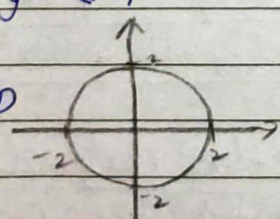
Step 3.

points	values
$(1, \ln 2)$	-1
$(1/2, 0)$	$-3/4 \rightarrow$ absolute maximum
$(0,0)$	-1
$(2,0)$	$-3 \rightarrow$ absolute minimum
$(2,1)$	$e-4$
$(e/2, 1)$	$e^2/4 - e$
$(0,1)$	$-e$

Q.35: Locate all absolute extrema over the bounded set R:

$f(x,y) = x^2+2y^2-x$; R is the disk $x^2+y^2 \leq 4$

Step 1: CP



$\boxed{f(1/2, 0) = -\frac{1}{4}}$

$f_x = 2x-1 \quad f_y = 4y$

put $f_x = 0$ & $f_y = 0$

$\Rightarrow 2x-1=0 \qquad \Rightarrow 4y=0$

$x = 1/2 \qquad y=0$

$\therefore (1/2, 0)$ is the critical point and it lies inside the set R.

Step 2: For boundary points:

$-2 \leq x \leq 2$

Along $x^2+y^2=4$

$y^2 = 4-x^2 \rightarrow y = \pm\sqrt{4-x^2}$

$f(x, \pm\sqrt{4-x^2}) = x^2 + 2(4-x^2) - x$

whole function is now a one-variable func.

$u(x) = 8-x^2-x$

$u'(x) = -2x-1$

$0 = -2x-1$

$\boxed{x = -1/2}$

$u''(x) = -2 < 0$ at $x = -1/2$

$\Rightarrow$ it is the relative maximum.

when $x = -1/2$,

$y^2 = 4 - 1/4 = 15/4$

$\boxed{y = \pm\frac{\sqrt{15}}{2}}$

$(-1/2, \sqrt{15}/2)$, $(-1/2, -\sqrt{15}/2)$ are critical points

$f(-\frac{1}{2}, \pm\sqrt{15}/2) = \frac{1}{4} + 2\left(\frac{\sqrt{15}}{2}\right)^2 + \frac{1}{2}$

$\boxed{f\left(-\frac{1}{2}, \pm\frac{\sqrt{15}}{2}\right) = \frac{33}{4}}$

when $x=2$, $y=0 \Rightarrow (2,0)$

when $x=-2$, $y=0 \Rightarrow (-2,0)$

$(2,0)$ & $(-2,0)$ are corner points / boundary points.

$\boxed{f(2,0) = 2}$ $\boxed{f(-2,0) = 6}$

(x,y)	$f(x,y)$
$\left(\frac{1}{2},0\right)$	$-\frac{1}{4}$ $\rightarrow$ absolute minimum
$\left(\frac{-1}{2}, \pm\frac{\sqrt{15}}{2}\right)$	$\frac{33}{4}$ $\rightarrow$ absolute maximum
$(2,0)$	2
$(-2,0)$	6

Q.39: Find all points on the portion of the plane $x+y+z=5$ in the first octant at which $f(x,y,z)=xy^2z^2$ has a ~~minimu~~ maximum value:

$\cancel{x+y+z}$ Due to first octant

$x\geq 0,\ y\geq 0,\ z\geq 0$

for maximum value of $f(x,y,z)=xy^2z^2$, we must have $x>0, y>0, z>0$

we have

$$x+y+z=5$$

$$z=5-x-y$$

$$u(x,y)=xy^2(5-x-y)^2$$

Step 1: For critical points.

$$u_x=-2xy^2(5-x-y)+y^2(5-x-y)^2$$

$$=(5-x-y-2x)\cdot y^2(5-x-y)^2$$

$$\boxed{u_x=y^2(5-x-y)(5-y-3x)}$$

$$u_y=2xy(5-x-y)^2-2xy^2(5-x-y)$$

$$=2xy(5-x-y)[5-x-y-y]$$

$$\boxed{u_y=2xy(5-x-y)(5-x-2y)}$$

put $u_x=0$ and $u_y=0$

$$y^2(5-x-y)(5-y-3x)=0 \quad \text{—(1)}$$

$$2xy(5-x-y)(5-x-2y)=0 \quad \text{—(2)}$$

since $x>0,\ y>0,\ z>0$

$$5-y-3x=0$$

$$5-x-2y=0$$

$$\boxed{x=1},\ \boxed{y=2}$$

$(1,2)$ is a critical point

we have $D=\{f_{xx}(1,2)\cdot f_{yy}(1,2)\}-\{f_{xy}(1,2)\}^2$

~~$f_{xx}=y^2(5-x$~~

$$u_x=y^2(25-5y-15x-5x+xy+3x^2-5y+y^2+3xy)$$

$$u_x=y^2(25-10y-20x+3x^2+y^2+4xy)$$

$$u_x=25y^2-10y^3+y^4+4xy^3+3x^2y^2-20xy^2$$

$$\boxed{u_{xx}=4y^3+6y^2x-20y^2}$$

$$u_y=2xy(25-25x-10y-5x+x^2+2xy-5y+xy+2y^2)$$

$$u_y=50xy-50x^2y-20xy^2-10x^2y+2x^3y+4x^2y^2-10xy^2+2x^2y^2+4xy^3$$

$$u_y=50xy-60x^2y-30xy^2+6x^2y^2+2x^3y+4xy^3$$

$$\boxed{u_{yy}=50x-60x^2-60xy+12x^2y+2x^3+12xy^2}$$

$$u_x=25y^2-10y^3+y^4+4xy^3+3x^2y^2-20xy^2$$

$$\boxed{u_{xy}=50y-30y^2+4y^3+12xy^2+6x^2y-40xy}$$

→LAGRANGE MULTIPLIER:

Extremum Problems with Constraints:

Two/three variables extremum problems with one constraint:

Suppose we have to maximize/minimize a function $f(x,y)$ subject to constraint curve $g(x,y)=0$

- For a point (x_0,y_0) on the graph of constraint curve at which $f(x,y)$ is as large as possible
- We conclude that vectors $\nabla f(x_0,y_0)$ and $\nabla g(x_0,y_0)$ must be parallel, that is $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$ for some scalar λ

→ THM 13.9.3:

Constrainted extremum principle for two variables and one constraint:

Let f and g be functions of two variables with continuous first order partial derivative on some open set containing the constrained curve $g(x,y)=0$, and assume that $\nabla g \neq 0$ at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point (x_0,y_0) on the constrainted curve at which the gradient vectors $\nabla f(x_0,y_0)$ and $\nabla g(x_0,y_0)$ are parallel i.e. there is some number λ such that $\boxed{\nabla f(x_0,y_0)=\lambda \nabla g(x_0,y_0)}$

Ex 13.9

QS-12

example 2, 3 and 4.

Use lagrange multiplier to find the maximum and minimum values of f subject to the constraint.

Q7: $f(x,y) = 4x^3+y^2$; $2x^2+y^2=1$

Here, conditional curve is

$g(x,y) = 2x^2+y^2-1$

$\nabla f = \lambda \Delta g$ — ①

For ∇f:

$\nabla f = f_x i + f_x j$

$= 12x^2 i + 2y j$

For ∇g:

$\nabla g = f_x i + f_x j$

$= 4x i + 2y j$

substituting ∇f and ∇g in eq①:

$12x^2 i + 12y j = \lambda(4x i + 2y j)$

$12x^2 = 4\lambda x$ — ①

$2y = \cancel{2}\, 2\lambda y$ — ②

$\cancel{2}y = \cancel{2}\lambda y$

$\boxed{\lambda = 1}$

$\boxed{y=0}$

$12x^2 - 4x = 0$

$4x(3x-1) = 0$

$\boxed{x=0}$, $\boxed{x=1/3}$

we have $g(x,y) = 2x^2+y^2-1$

when $x=0$

$0+y^2=1 \Rightarrow \boxed{y=\pm 1}$

$\boxed{(x_0,y_0) = (0,\pm 1)}$

when $x = 1/3$

$2(1/3)^2 + y^2 - 1 \Rightarrow y^2 = 1 - 2/9 = 7/9$

$\boxed{y = \pm \frac{\sqrt{7}}{3}}$

$\boxed{(x_0, y_0) = (1/3, \pm \sqrt{7}/3)}$

when $y = 0$

$2x^2 + 0 = 1 \Rightarrow x = \pm 1/\sqrt{2}$

$\boxed{(x_0, y_0) = (\pm \frac{1}{\sqrt{2}}, 0)}$

points (x, y)	$f(x, y)$
$(1/3, \pm \sqrt{7}/3)$	
$(0, \pm 1)$	
$(1/\sqrt{2}, 0)$	

Use Lagrange multiplier to find the maximum and minimum values of f subject to given constraint

Q.12: $f(x, y, z) = x^4 + y^4 + z^4 : x^2 + y^2 + z^2 = 1$

we have

$\nabla f = \lambda \nabla g$ — ①

for ∇f.

$\nabla f = f_x i + f_y j + f_z k$

$\nabla f = 4x^3 i + 4y^3 j + 4z^3 k$

$\nabla g = g_x i + g_y j + g_z k$

$\nabla g = 2xi + 2yj + 2zk$

$4x^3 i + 4y^3 j + 4z^3 k = \lambda(2xi + 2yj + 2zk)$

comparing both sides:

$4x^3 = \lambda 2x \qquad 4y^3 = \lambda 2y \qquad 4z^3 = \lambda 2z$

$\lambda = 2x^2 \qquad \lambda = 2y^2 \qquad \lambda = 2z^2$

$x^2 = y^2 \qquad x^2 = z^2$

we have

$g(x, y, z) = x^2 + y^2 + z^2 - 1$

substituting value of x:

$x^2 + x^2 + x^2 = 1$

$\boxed{x = \pm \frac{1}{\sqrt{3}}}$ $\boxed{y = \pm \frac{1}{\sqrt{3}}}$

$\boxed{z = \pm \frac{1}{\sqrt{3}}}$

all values coming from all the points is the same and the maximum value because function has all even powers.

$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ have eight possible points and $\boxed{f\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right) = \frac{1}{3}}$ is absolute maximum over $g(x,y,z)$.

Q.11: $f(x,y,z) = xyz$; $g(x,y,z) = x^2+y^2+z^2-1$

we have

$\nabla f = \lambda \nabla g$ — ①

for ∇f:

$\nabla f = f_x i + f_y j + f_z k$

$\nabla f = yz i + xz j + xy k$

$\nabla g = g_x i + g_y j + g_z k$

$\nabla g = 2x i + 2y j + 2z k$

$yz i + xz j + xy k = \lambda(2x i + 2y j + 2z k)$

comparing both sides:

$yz = \lambda 2x \qquad xz = \lambda 2y \qquad xy = \lambda 2z$

$\lambda = \frac{yz}{2x} \qquad \lambda = \frac{xz}{2y} \qquad \lambda = \frac{xy}{2z}$

$\frac{yz}{2x} = \frac{xz}{2y} \qquad \frac{yz}{2x} = \frac{xy}{2z}$

$\boxed{y^2 = x^2} \qquad \boxed{z^2 = x^2}$

we have $g(x,y,z) = x^2+y^2+z^2-1$

substituting values:

$x^2+x^2+x^2 = 1$

$\boxed{x = \pm\frac{1}{\sqrt{3}}} \qquad \boxed{y = \pm\frac{1}{\sqrt{3}}}$

$\boxed{z = \pm\frac{1}{\sqrt{3}}}$

$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ have eight possible points and ~~$f\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right) = \frac{1}{3\sqrt{3}}$~~

$$\left.\begin{aligned}
f(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) &= \frac{1}{3\sqrt{3}} \\
f(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}) &= 1/3\sqrt{3} \\
f(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}) &= 1/3\sqrt{3} \\
f(1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}) &= 1/3\sqrt{3}
\end{aligned}\right\} \text{maximum values}$$

$$\left.\begin{aligned}
f(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}) &= -1/3\sqrt{3} \\
f(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) &= -1/3\sqrt{3} \\
f(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}) &= -1/3\sqrt{3} \\
f(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}) &= -1/3\sqrt{3}
\end{aligned}\right\} \text{minimum values}$$