14.5.1 THEOREM (*Fubini's Theorem)** *Let G be the rectangular box defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d, \quad k \le z \le l$$

If f is continuous on the region G, then

$$\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^l f(x, y, z)\, dz\, dy\, dx \tag{2}$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

Six orders of integration are possible for the iterated integral in Theorem 14.5.1:

$$\begin{array}{lll} dx\,dy\,dz, & dy\,dz\,dx, & dz\,dx\,dy \\ dx\,dz\,dy, & dz\,dy\,dx, & dy\,dx\,dz \end{array}$$

▶ **Example 1** Evaluate the triple integral

$$\iiint_G 12xy^2z^3\, dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2$.

Solution. Of the six possible iterated integrals we might use, we will choose the one in (2). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\begin{aligned} \iiint_G 12xy^2z^3\, dV &= \int_{-1}^{2}\int_0^3\int_0^2 12xy^2z^3\, dz\, dy\, dx \\ &= \int_{-1}^{2}\int_0^3 \left[3xy^2z^4\right]_{z=0}^{2} dy\, dx = \int_{-1}^{2}\int_0^3 48xy^2\, dy\, dx \\ &= \int_{-1}^{2} \left[16xy^3\right]_{y=0}^{3} dx = \int_{-1}^{2} 432x\, dx \\ &= 216x^2\Big]_{-1}^{2} = 648 \end{aligned}$$ ◀

3. $\displaystyle\int_0^2 \int_{-1}^{y^2} \int_{-1}^{z} yz\, dx\, dz\, dy$

3) $\int_0^2 \int_{-1}^{y^2} \int_{-1}^{z} yz \, dx \, dz \, dy$

$= \int_0^2 \int_{-1}^{y^2} \left[xyz \right]_{-1}^{z} dz \, dy$

$= \int_0^2 \int_{-1}^{y^2} (zyz + yz) \, dz \, dy$

$= \int_0^2 \int_{-1}^{y^2} (z^2y + yz) \, dz \, dy$

$= \int_0^2 \left[\frac{z^3y}{3} + \frac{yz^2}{2} \right]_{-1}^{y^2} dy$

$$= \int_0^2 \left[\frac{y^7}{3} + \frac{y^5}{2} + \frac{y}{3} - \frac{y}{2} \right] dy$$

$$= \left[\frac{y^8}{24} + \frac{y^6}{12} + \frac{y^2}{6} - \frac{y^2}{4} \right]_0^2$$

$$= \frac{256}{24} + \frac{64}{12} + \frac{4}{6} - \frac{4}{4} = \frac{47}{3}$$