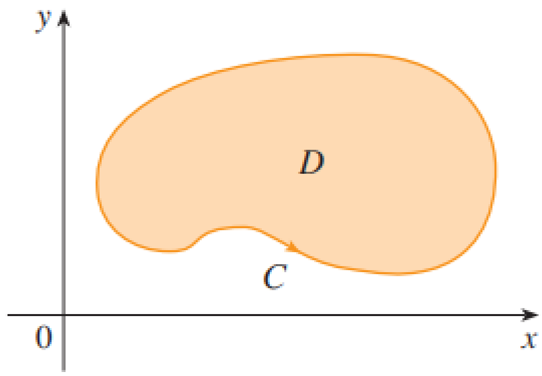
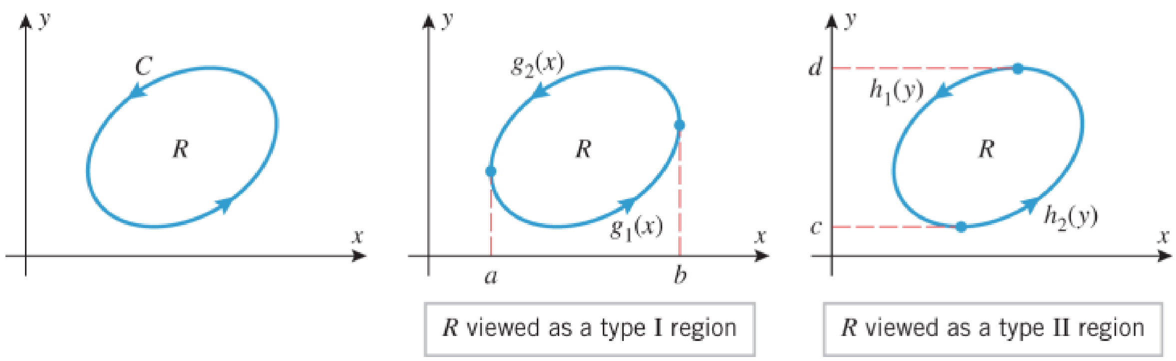
Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the region R bounded by C. We assume that R consists of all points inside C as well as all points on C.




**15.4.1 THEOREM (*Green's Theorem*)** *Let $R$ be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve $C$ oriented counterclockwise. If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open set containing $R$, then*

$$\int_C f(x, y)\,dx + g(x, y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \tag{1}$$




## A NOTATION FOR LINE INTEGRALS AROUND SIMPLE CLOSED CURVES

It is common practice to denote a line integral around a simple closed curve by an integral sign with a superimposed circle. With this notation Formula (1) would be written as

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Sometimes a direction arrow is added to the circle to indicate whether the integration is clockwise or counterclockwise. Thus, if we wanted to emphasize the counterclockwise direction of integration required by Theorem 15.4.1, we could express (1) as

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \tag{5}$$

**3–13** Use Green's Theorem to evaluate the integral. In each exercise, assume that the curve $C$ is oriented counterclockwise.

**3.** $\oint_C 3xy\,dx + 2xy\,dy$, where $C$ is the rectangle bounded by $x = -2$, $x = 4$, $y = 1$, and $y = 2$.

③ $\oint_C 3xy\,dx + 2xy\,dy = \int_{-2}^{4}\int_{1}^{2} (2y - 3x)\,dy\,dx$

$= 0$

$f(x,y) = 3xy$

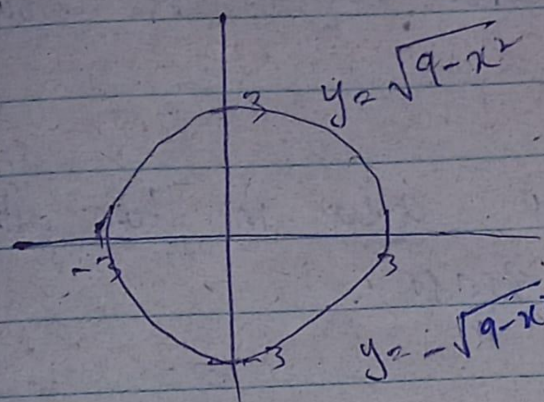
$g(x,y) = 2xy$

$\partial g/\partial x = 2y$ , $\partial f/\partial y = 3x$

**4.** $\oint_C (x^2 - y^2)\,dx + x\,dy$, where $C$ is the circle $x^2 + y^2 = 9$.

(4) $\oint_C (x^2-y^2)\,dx + x\,dy$.

$C: x^2+y^2=9$




In polar:

$$R.H.S = \int_0^{2\pi}\int_0^{3}(1+2r\sin\theta)\,r\,dr\,d\theta = 9\pi$$

$f(x,y)=x^2-y^2$, $\partial f/\partial y = -2y$

$g(x,y)=x$, $\partial g/\partial x = 1$

**6.** $\oint_C y\tan^2 x\,dx + \tan x\,dy$, where $C$ is the circle $x^2+(y+1)^2=1$.

(6)

$$f(x,y) = y\tan^2 x$$

$$g(x,y) = \tan x.$$

$$f_y = \tan^2 x$$

$$g_x = \sec^2 x.$$

$$\iint_R (\sec^2 x - \tan^2 x)\, dA$$

In Polar:

$$x^2 + (y+1)^2 = 1.$$

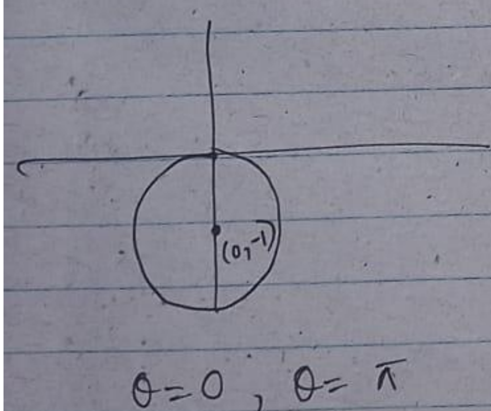
$$r^2\cos^2\theta + (r\sin\theta + 1)^2 = 1.$$

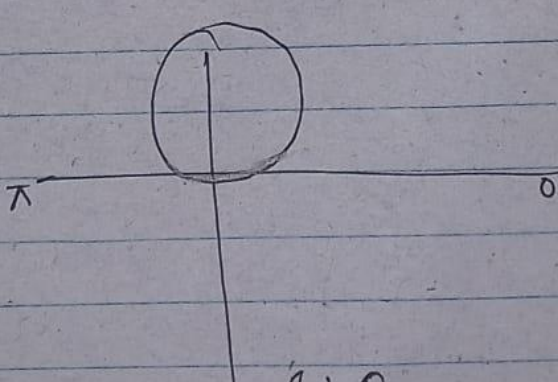
$$r^2\cos^2\theta + r^2\sin^2\theta + 2r\sin\theta + 1 = 1$$

$$r^2 + 2r\sin\theta = 0$$

$$r^2 = -2r\sin\theta$$

$$r = -2\sin\theta$$




$\theta = 0$, $\theta = \pi$, $r = 0$, $r = 2\sin\theta$

(Since it repeats)

$$\int_0^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$$

$$\int_0^{\pi} \frac{r^2}{2}\Big|_0^{2\sin\theta} d\theta$$

$$\int_0^{\pi} \frac{4\sin^2\theta}{2}\, d\theta$$

$$= \int_0^{\pi} 2\sin^2\theta\, d\theta.$$

$\therefore \sin^2\theta = \frac{1-\cos 2x}{2}$

$$= \int_0^{\pi} 2\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$\theta - \frac{\sin 2\theta}{2}\Big|_0^{\pi} = \pi$$