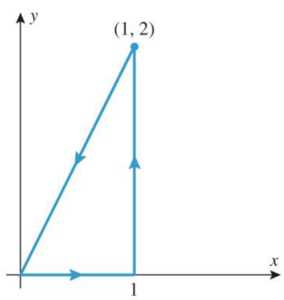
15.4 GREEN'S THEOREM

In this section we will discuss a remarkable and beautiful theorem that expresses a double integral over a plane region in terms of a line integral around its boundary.

GREEN'S THEOREM

15.4.1 THEOREM (*Green's Theorem*) *Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented counterclockwise. If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open set containing R, then*

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \qquad (1)$$



▲ Figure 15.4.3

► Example 1 Use Green's Theorem to evaluate

$$\int_C x^2 y\,dx + x\,dy$$

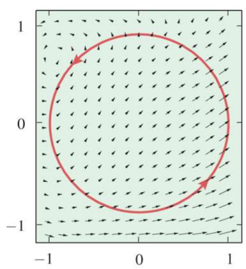
along the triangular path shown in Figure 15.4.3.

Solution. Since $f(x, y) = x^2 y$ and $g(x, y) = x$, it follows from (1) that

$$\int_C x^2 y\,dx + x\,dy = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^2 y)\right] dA = \int_0^1 \int_0^{2x} (1 - x^2)\,dy\,dx$$

$$= \int_0^1 (2x - 2x^3)\,dx = \left[x^2 - \frac{x^4}{2}\right]_0^1 = \frac{1}{2}$$

This agrees with the result obtained in Example 10 of Section 15.2, where we evaluated the line integral directly. Note how much simpler this solution is. ◄



▲ Figure 15.4.4

► Example 2 Find the work done by the force field

$$\mathbf{F}(x, y) = (e^x - y^3)\mathbf{i} + (\cos y + x^3)\mathbf{j}$$

on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counterclockwise direction (Figure 15.4.4).

Solution. The work W performed by the field is

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (e^x - y^3)\,dx + (\cos y + x^3)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3)\right] dA \qquad \boxed{\text{Green's Theorem}}$$

$$= \iint_R (3x^2 + 3y^2)\,dA = 3\iint_R (x^2 + y^2)\,dA$$

$$= 3\int_0^{2\pi}\int_0^1 (r^2) r\,dr\,d\theta = \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3\pi}{2} \quad ◄$$

We converted to polar coordinates.

EXERCISE SET 15.4 C CAS

3–13 Use Green's Theorem to evaluate the integral. In each exercise, assume that the curve C is oriented counterclockwise.

3. $\oint_C 3xy\,dx + 2xy\,dy$, where C is the rectangle bounded by $x = -2$, $x = 4$, $y = 1$, and $y = 2$.

4. $\oint_C (x^2 - y^2)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 9$.

5. $\oint_C x\cos y\,dx - y\sin x\,dy$, where C is the square with vertices $(0, 0)$, $(\pi/2, 0)$, $(\pi/2, \pi/2)$, and $(0, \pi/2)$.

6. $\oint_C y\tan^2 x\,dx + \tan x\,dy$, where C is the circle $x^2 + (y + 1)^2 = 1$.

7. $\oint_C (x^2 - y)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 4$.

8. $\oint_C (e^x + y^2)\,dx + (e^y + x^2)\,dy$, where C is the boundary of the region between $y = x^2$ and $y = x$.

9. $\oint_C \ln(1 + y)\,dx - \dfrac{xy}{1 + y}\,dy$, where C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$.

10. $\oint_C x^2y\,dx - y^2x\,dy$, where C is the boundary of the region in the first quadrant, enclosed between the coordinate axes and the circle $x^2 + y^2 = 16$.