15.2

Line integral: similar to single integral except that instead of integrating over an interval $[a, b]$, integrate over a curve C.

C is given in parametric equation.

$$x = x(t) \quad , \quad y = y(t) \quad , \quad a \le t \le b$$

Line integral : $\int_C f(x,y)\, ds$

ds : length of the curve

f is any function of two variables whose domain includes C.

Interpret it as an area.

Evaluating line integrals:

In 2-space:

- $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\int_C f(x,y)\, ds = \int_a^b f\left(x(t), y(t)\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where L is the formula for the length of the curve in parametric form.

In 3-space:

$x = x(t)$; $y = y(t)$, $z = z(t)$

$(a \leq t \leq b)$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, d$$

11. Let C be the curve represented by the equations

$$x = 2t, \quad y = t^2 \qquad (0 \leq t \leq 1)$$

In each part, evaluate the line integral along C.

(a) $\int_C (x - \sqrt{y})\, ds$ (b) $\int_C (x - \sqrt{y})\, dx$

$$\int_C (x-\sqrt{y})\,ds = \int_0^1 (2t-\sqrt{t^2})\sqrt{4+4t^2}\,dt$$

$$= \int_0^1 2t\sqrt{1+t^2}\,dt.$$

$\therefore \quad u = (1+t^2) \quad , \quad du = 2t\,dt$

$$= \int_0^1 u^{1/2}\,du = \frac{2}{3} u^{3/2}\Big|_0^1$$

$$= \frac{2}{3}(1+t^2)^{3/2}\Big|_0^1$$

$$= \frac{2}{3}(2\sqrt{2}-1)$$

Line integral with respect to x.

$x = 2t \quad , \quad y = t^2 \quad , \quad 0 \le t \le 1$

$dx = 2dt$

$$\int_C (x-\sqrt{y})\,dx = \int_0^1 (2t-\sqrt{t^2})\,2dt$$

$$= \int_0^1 (4t-2t)\,dt.$$

$$= \int_0^1 2t\,dt = 1$$

INTEGRATING A VECTOR FIELD ALONG A CURVE

$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ or $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} \quad \text{or} \quad d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

In 2-space:

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (f(x, y)\mathbf{i} + g(x, y)\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \int_C f(x, y)\,dx + g(x, y)\,dy$$

In 3-space:

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_C f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz\end{aligned}$$

In parametric form:

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad (a \le t \le b)$$

$$\mathbf{F}(\mathbf{r}(t)) = f(x(t), y(t))\mathbf{i} + g(x(t), y(t))\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

7–10 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the line segment C from P to Q.

7. $\mathbf{F}(x, y) = 8\mathbf{i} + 8\mathbf{j}$; $P(-4, 4)$, $Q(-4, 5)$

Vector representation of line segment that starts at $\vec{r}_0$ and ends at $\vec{r}_1$ is given by

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \quad , \quad 0 \le t \le 1$$

Here $\vec{r}_0 = P\langle -4, 4\rangle$, $\vec{r}_1 = Q\langle -4, 5\rangle$

$$\vec{r}(t) = (1-t)\langle -4, 4\rangle + t\langle -4, 5\rangle$$
$$= \langle -4+4t, 4-4t\rangle + \langle -4t, 5t\rangle$$
$$= \langle -4+4t-4t, \; 4-4t+5t\rangle$$
$$= \langle -4, \; 4+t\rangle$$
$$r(t) = -4\hat{i} + (4+t)\hat{j}$$

comparing with $r(t) = x\hat{i} + y\hat{j}$

$\Rightarrow \quad x = -4 \quad , \quad y = 4+t$

and $\quad r'(t) = \hat{j}\, dt$

$$\int_C F \cdot dr = \int_0^1 (8\hat{i} + 8\hat{j}) \cdot (j\,dt)$$

$$= \int_0^1 8\, dt = 8t\Big]_0^1 = 8$$

13. In each part, evaluate the integral

$$\int_C (3x + 2y)\,dx + (2x - y)\,dy$$

along the stated curve.

(a) The line segment from $(0, 0)$ to $(1, 1)$.

(b) The parabolic arc $y = x^2$ from $(0, 0)$ to $(1, 1)$.

a) $\vec{r_0} = \langle 0, 0 \rangle$, $\vec{r_1} = \langle 1, 1 \rangle$

$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$, $0 \leq t \leq 1$

$= (1-t)\langle 0,0 \rangle + t\langle 1,1 \rangle$

$= \langle 0, 0 \rangle + \langle t, t \rangle$

$= \langle 0+t, 0+t \rangle$

$= \langle t, t \rangle$

$\vec{r}(t) = t\hat{i} + t\hat{j}$

comparing with

$\vec{r}(t) = x\hat{i} + y\hat{j}$

$\Rightarrow$ $x = t$; $y = t$

$dx = dt$, $dy = dt$

$$\int_c (3x+2y)dx + (2x-y)dy$$

$$= \int_0^1 (3t+2t)dt + (2t-t)dt$$

$$= \int_0^1 6t\,dt = 3$$

b) $y = x^2$

let's take $x = t$, $y = t^2$

For lower limit of t:

$x = 0$, $y = 0$ $\Rightarrow$ $t = 0$

For upper limit

$x = 1$, $y = 1$ $\Rightarrow$ $t = 1$

$dx = dt$, $dy = 2t\,dt$

$$\int_C (3x+2y)\,dx + (2x-y)\,dy$$

$$= \int_0^1 (3t+2t^2)\,dt + (2t-t^2)\,2t\,dt$$

$$= \int_0^1 (3t+2t^2+4t^2-2t^3)\,dt$$

$$= \int_0^1 (3t+6t^2-2t^3)\,dt = 3$$

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Or

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \|\mathbf{r}'(t)\|\, dt$$

Similarly, if C is a curve in 3-space that is parametrized by

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (a \le t \le b)$$

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Or

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \|\mathbf{r}'(t)\|\, dt$$

19–22 Evaluate the line integral with respect to s along the curve C.

19. $\displaystyle\int_C \frac{1}{1 + x}\, ds$

$C : \mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{j} \quad (0 \le t \le 3)$

(19) $\int_C \frac{1}{1+x}\, ds$

$C: r(t) = t\hat{i} + \frac{2}{3} t^{3/2} \hat{j} \quad (0 \le t \le 3)$

$\therefore \; r(t) = x(t)\hat{i} + y(t)\hat{j}$

$x(t) = t, \; y(t) = \frac{2}{3} t^{3/2}$

$$\int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\int_0^3 \frac{1}{1+t} \sqrt{1 + \left(t^{1/2}\right)^2}\, dt$$

$$= \int_0^3 \frac{\sqrt{1+t}}{1+t}\, dt$$

$$= \int_0^3 \frac{1}{\sqrt{1+t}}\, dt = \int_0^3 (1+t)^{-1/2}\, dt = 2$$