Vector field:

In 2-space a vector field is a function $\mathbf{F}$ that assigns to each point (x,y) a two dimensional vector $F(x,y)$.

$$F(x,y) = f(x,y)\hat{i} + g(x,y)\hat{j}$$

In 3-space:

$$F(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$$

Compact Notation:

Identify (x,y) as radius vector $r = x\hat{i} + y\hat{j}$ and (x,y,z) as $r = x\hat{i} + y\hat{j} + z\hat{k}$. Then vector field in either 2-space or 3-space can be written as $\mathbf{F}(\vec{r})$.

Gradient Field:

Let ϕ be a function of three variables then the gradient of ϕ is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

~~Also~~ This defines a vector field in 3-space called the gradient field of ϕ. Similarly in 2-space.

where $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

is a del operator.

Divergence: tells about the fluid flows toward or away from a point.

Curl: tells the rotational properties of the fluid at a point.

15.1.4 DEFINITION If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then we define the ***divergence of*** $\mathbf{F}$, written div $\mathbf{F}$, to be the function given by

$$\operatorname{div}\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{7}$$

$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

15.1.5 DEFINITION If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then we define the ***curl of* F**, written curl **F**, to be the vector field given by

$$\text{curl } \mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} \tag{8}$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

THE LAPLACIAN ∇^2

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

17–22 Find div **F** and curl **F**.

17. $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2\mathbf{j} + yz\mathbf{k}$

$$\text{div}(F) = \nabla \cdot F = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x^2\hat{i} - 2\hat{j} + yz\hat{k}\right)$$

$$\text{div}(F) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(-2) + \frac{\partial}{\partial z}(yz)$$

$$= 2x + 0 + y \Rightarrow \text{div}(F) = 2x + y$$

$$\mathrm{Curl}(F) = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & -2 & yz \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(yz) - \partial/\partial z(-2)\right\}$$

$$- \hat{j}\left\{\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(x^2)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}(-2) - \frac{\partial}{\partial y}(x^2)\right\}$$

$$\mathrm{Curl}(F) = \hat{i}\{z - 0\} - \hat{j}\{0 - 0\} + \hat{k}\{0 - 0\}$$

$$\mathrm{Curl}(F) = z\hat{i}.$$

23–24 Find $\nabla \cdot (\mathbf{F} \times \mathbf{G})$.

23. $\mathbf{F}(x, y, z) = 2x\mathbf{i} + \mathbf{j} + 4y\mathbf{k}$
$\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$

First find $F \times G$.

$$F \times G = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2x & 1 & 4y \\ x & y & -z \end{vmatrix}$$

$$F \times G = \hat{i}\{-z-4y^2\} - \hat{j}\{-2xz-4xy\} + \hat{k}\{2xy-x\}$$

$$F \times G = \hat{i}(-z-4y^2) + \hat{j}(2xz+4xy) + \hat{k}(2xy-x)$$

$$\nabla \cdot (F \times G) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[\hat{i}(-z-4y^2) + \hat{j}(2xz+4xy) + \hat{k}(2xy-x)\right]$$

$$= \frac{\partial}{\partial x}(-z-4y^2) + \frac{\partial}{\partial y}(2xz+4xy) + \frac{\partial}{\partial z}(2xy-x)$$

$$= 0 + 4x + 0$$

$$\nabla \cdot (F \times G) = 4x.$$

$$\nabla \cdot (F \times G) = \operatorname{div}(F \times G) = 4x.$$

25–26 Find $\nabla \cdot (\nabla \times \mathbf{F})$.

25. $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos(x - y)\mathbf{j} + z\mathbf{k}$

(25) First find $(\nabla \times F)$ or curl(F)

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin x & \cos(x-y) & z \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(\cos(x-y))\right\}$$

$$-\hat{j}\left\{\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(\sin x)\right\}$$

$$+\hat{k}\left\{\frac{\partial}{\partial x}(\cos(x-y)) - \frac{\partial}{\partial y}(\sin x)\right\}$$

$$= \hat{i}\{0\} - \hat{j}\{0\} + \hat{k}\{-\sin(x-y) - 0\}$$

$$\nabla \times F = -\sin(x-y)\hat{k}$$

$$\nabla \cdot (\nabla \times F) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(-\sin(x-y)\hat{k}\right)$$

$$= \frac{\partial}{\partial z}\left(-\sin(x-y)\right) = 0$$

$$\nabla \cdot (\nabla \times F) = \mathrm{div}(\mathrm{curl}(F)) = 0$$

27–28 Find $\nabla \times (\nabla \times \mathbf{F})$.

27. $\mathbf{F}(x, y, z) = xy\,\mathbf{j} + xyz\,\mathbf{k}$

(27) First find $\nabla \times F$ or curl(F)

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xy & xyz \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(xy)\right\}$$

$$-\hat{j}\left\{\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(0)\right\}$$

$$+\hat{k}\left\{\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(0)\right\}$$

$$= \hat{i}\{xz - 0\} - \hat{j}\{yz - 0\} + \hat{k}\{y - 0\}$$

$$\nabla \times F = xz\,\hat{i} - yz\,\hat{j} + y\,\hat{k}$$

$$\nabla \times (\nabla \times F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & -yz & y \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(-yz)\right\}$$

$$-\hat{j}\left\{\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial z}(xz)\right\}$$

$$+\hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(xz)\right\}$$

$$= \hat{i}\{1 + y\} - \hat{j}\{0 - x\} + \hat{k}\{0 - 0\}$$

$$\nabla \times (\nabla \times F) = (1 + y)\hat{i} + x\hat{j}$$