

MVC

Assignment : 01

23K-2001

BCS-2J

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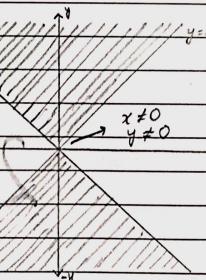
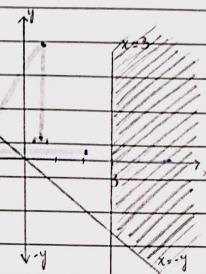
Question #1:

a. i. $f(x,y) = \sqrt{x+y} - \sqrt{x-3}$

Sol:

$$\begin{aligned}\sqrt{x+y} &\geq 0 \\ x+y &\geq 0\end{aligned}$$

$$\begin{aligned}\sqrt{x-3} &\geq 0 \\ x-3 &\geq 0 \\ x &\geq 3\end{aligned}$$



ii. $f(x,y) = \sqrt{\frac{1}{x^2} - \frac{1}{y^2}}$ []

Sol:

$$\sqrt{\frac{1}{x^2} - \frac{1}{y^2}} \geq 0 \quad \text{OR} \quad \frac{1}{x^2} - \frac{1}{y^2} \geq 0$$

$$\rightarrow y^2 - x^2 \geq 0, \quad x^2 y^2 \neq 0$$

$$(y-x)(y+x) \geq 0$$

$$y-x \geq 0, \quad y+x \geq 0$$

$$y \geq x, \quad y \geq -x$$

100% 18/18

iv. $f(x,y) = \ln(x^2 - 8y)$

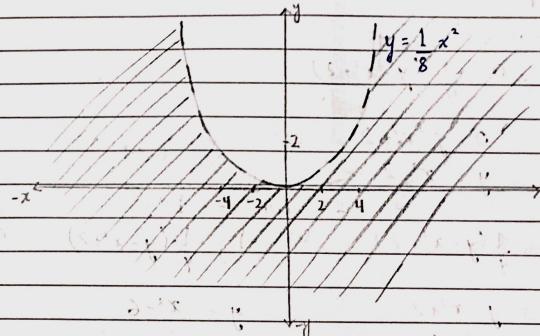
Sol:

$$x^2 - 8y > 0$$

 $x^2 > 8y \rightarrow \text{parabola:}$

$y < \frac{1}{8}x^2 \quad a=2, \text{ end points:}$

$(-2a, a), (2a, a) \rightarrow (-4, 2), (4, 2)$

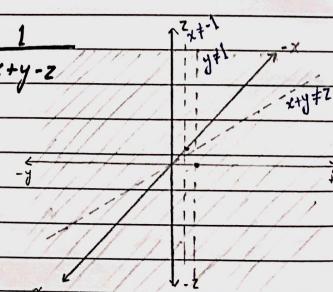


iii. $f(x,y,z) = \frac{1}{x+1} + \frac{1}{y-1} + \frac{1}{z-y}$

Sol:

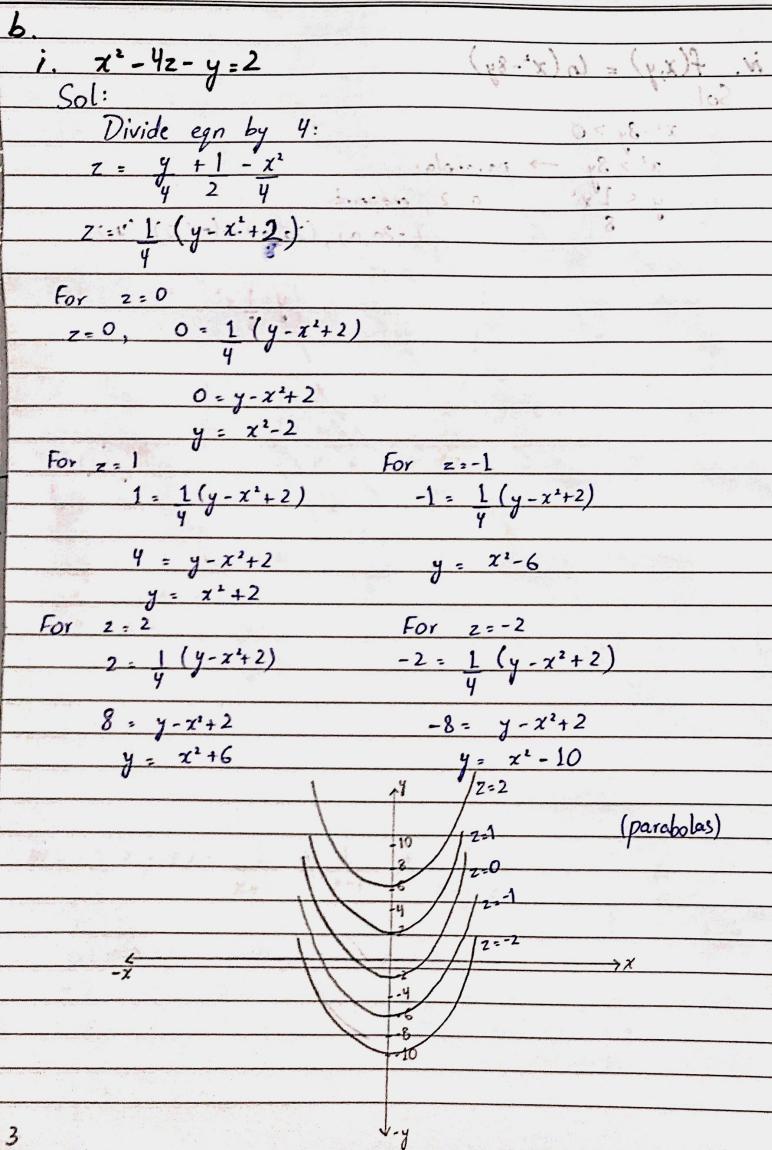
$$x+1 \neq 0, \quad y-1 \neq 0, \quad x+y-z \neq 0$$

$$x \neq -1, \quad y \neq 1$$

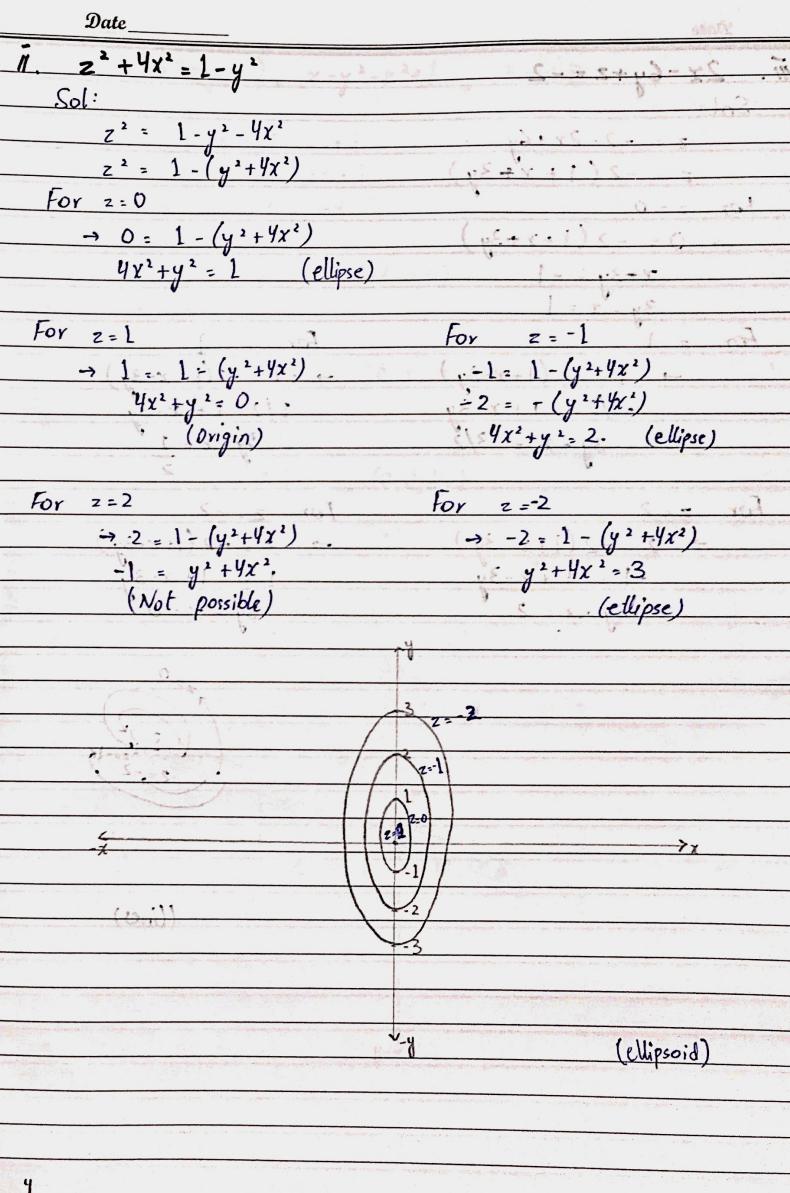


(calculator)

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$$\text{III. } 2x - 6y + z = -2$$

Sol:

$$z = -2 - 2x + 6y$$

$$z = -2(1 + x - 3y)$$

For $z = 0$

$$\rightarrow 0 = -2(1 + x - 3y)$$

$$x - 3y = -1$$

$$3y - x = 1$$

For $z = 1$

$$\rightarrow 1 = -2(1 + x - 3y)$$

$$-1/2 = 1 + x - 3y$$

$$3y - x = 3/2$$

$$z = 1 = x + 3y$$

$$(x + 3y) - 1 = 0$$

$$0 = 0$$

$$(x + 3y) - 1 = 0$$

$$x + 3y = 1$$

For $z = -1$

$$\rightarrow -1 = -2(1 + x - 3y)$$

$$1/2 = 1 + x - 3y$$

$$3y - x = \frac{1}{2}$$

For $z = 2$

$$\rightarrow 2 = -2(1 + x - 3y)$$

$$-1 = 1 + x - 3y$$

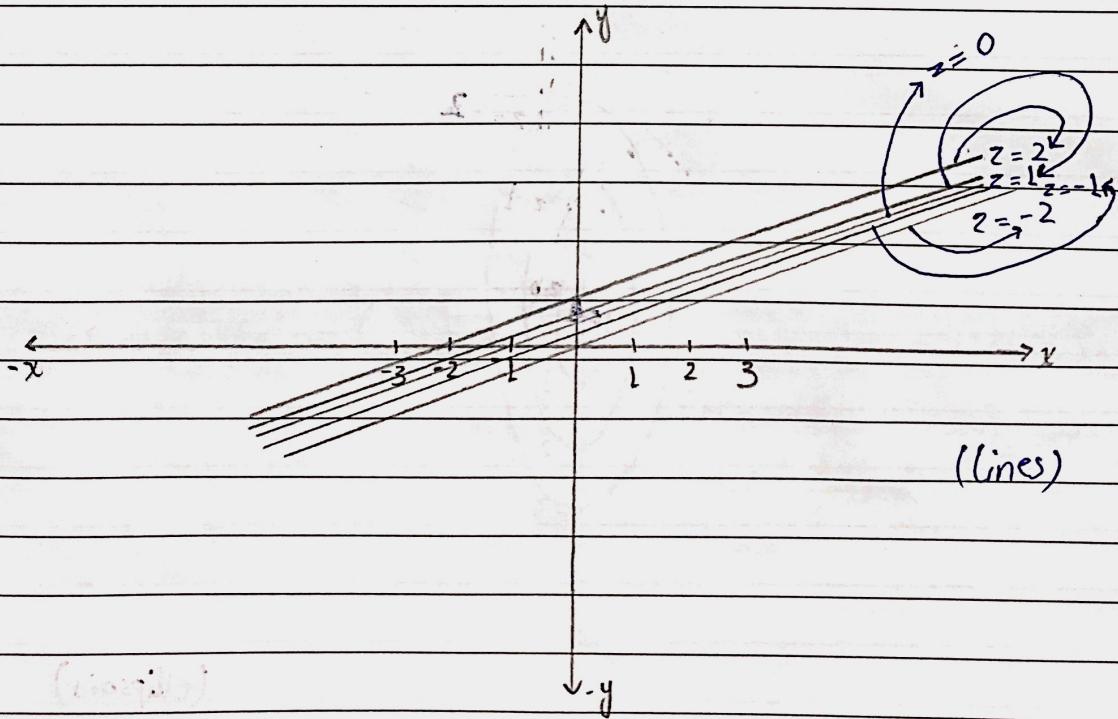
$$3y - x = 2$$

For $z = -2$

$$\rightarrow -2 = -2(1 + x - 3y)$$

$$1 = 1 + x - 3y$$

$$3y - x = 0$$



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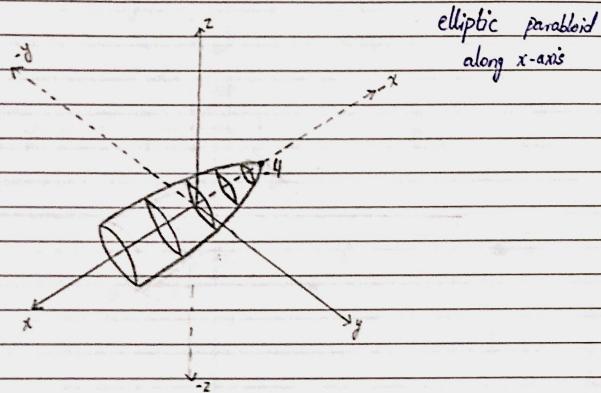
i. $f(x,y,z) = x^2 - y^2 - z^2 + 1$; $k = -3$
 $\Rightarrow k = x^2 - y^2 - z^2 + 1$
 $-3 = x^2 - y^2 - z^2 + 1$
 $-4 = x^2 - y^2 - z^2 \Rightarrow y^2 + z^2 = 4 + x^2$

By traces:

xy -plane:
 $-4 = x^2 - y^2$
 $x^2 = y^2 + 4$ (parabola)

xz -plane:
 $-4 = x^2 - z^2$
 $x^2 = z^2 + 4$ (parabola)

yz -plane:
 $-4 = -y^2 - z^2$
 $y^2 + z^2 = 4$ (ellipse)



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ii. $f(x,y,z) = \frac{3x^2 + y^2}{z^2}$; $k = 9$

Sol:

$$\Rightarrow K = \frac{3x^2 + y^2}{z^2}$$

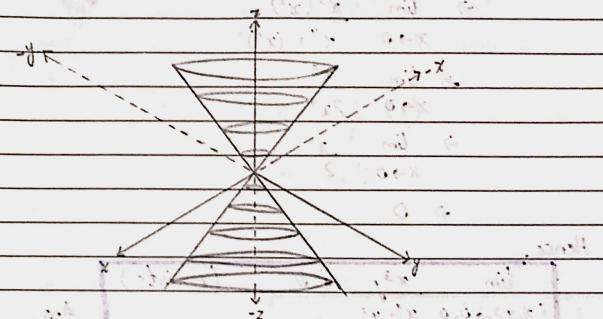
$$Kz^2 = 3x^2 + y^2$$

$$9z^2 = 3x^2 + y^2$$

$$z^2 = \frac{x^2}{3} + \frac{y^2}{9}$$

$$(cone)$$

$$c = \pm 1, a = \pm \sqrt{3}, b = \pm 3$$



iii. $f(x,y,z) = 9x^2 + 4y^2 + z^2$; $k = 4$

Sol:

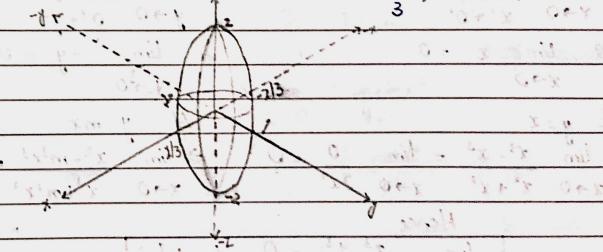
$$\Rightarrow k = 9x^2 + 4y^2 + z^2$$

$$4 = 9x^2 + 4y^2 + z^2$$

$$1 = \frac{x^2}{4/9} + \frac{y^2}{1} + \frac{z^2}{4}$$

$$(ellipsoid)$$

$$a = \pm 2, b = \pm 1, c = \pm \sqrt{2}$$



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Question #2: $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y}{x^4+y^2}$

a.

$$\text{i. } \lim_{(x,y) \rightarrow (0,0)} \frac{x^3y}{x^4+y^2}$$

Sol:

At x-axis, $y=0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3(0)}{x^4+(0)^2} = 0 \quad \Rightarrow \lim_{y \rightarrow 0} \frac{0^3(y)}{0^4+y^2} = 0$$

At $y=x^2$:

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3(x^2)}{x^4+(x^2)^2}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^5}{2x^4}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x}{2}$$

$$\Rightarrow 0$$

Hence,

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y}{x^4+y^2} = 0 \quad (\text{exists})}$$

Ans.

$$\text{ii. } \lim_{(x,y) \rightarrow (0,0)} \frac{x^3-y^3}{x^2+y^2}$$

Sol:

At $y=0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3-0^3}{x^2+0^2} = \lim_{y \rightarrow 0} \frac{0^3-y^3}{0^2+y^2}$$

$$\Rightarrow \lim_{x \rightarrow 0} x = 0 \quad \Rightarrow \lim_{y \rightarrow 0} -y = 0$$

At $y=x$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3-x^3}{x^2+x^2} = \lim_{x \rightarrow 0} \frac{0}{2x^2} = 0 \quad \Rightarrow \lim_{x \rightarrow 0} \frac{x^3-m^3x^3}{x^2+m^2x^2} = \lim_{x \rightarrow 0} \frac{x(1-m^3)}{1+m^2}$$

Hence,

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^3-y^3}{x^2+y^2} = 0 \quad (\text{exists})}$$

Ans.

$$\text{iii. } \lim_{(x,y) \rightarrow (0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$$

Sol:

At $y=0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1-\cos(x^2+0)}{(x^2+0)^2}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1-\cos x^2}{x^4}$$

Apply L-Hopital Rule

$$\Rightarrow \lim_{x \rightarrow 0} \frac{0+2x\sin x^2}{4x^3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 \cdot \sin x^2}{2 \cdot x^2}$$

$$\Rightarrow \frac{1}{2}(1)$$

$$\Rightarrow 1/2 \quad \text{Hence,} \Rightarrow 1/2$$

At $x=0$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{1-\cos(0+y^2)}{(0+y^2)^2}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{1-\cos y^2}{y^4}$$

Apply L-Hopital Rule

$$\Rightarrow \lim_{y \rightarrow 0} \frac{0+2y\sin y^2}{4y^3}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{1 \cdot \sin y^2}{2 \cdot y^2}$$

$$\Rightarrow \frac{1}{2}(1)$$

$$\Rightarrow 1/2 \quad \text{Hence,} \Rightarrow 1/2$$

At $y=x$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1-\cos(x^2+x^2)}{(x^2+x^2)^2}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1-\cos 2x^2}{4x^4}$$

Apply L-Hopital Rule

$$\Rightarrow \lim_{x \rightarrow 0} \frac{0+4x\sin 2x^2}{16x^3}$$

$$\Rightarrow 1/2 \quad \text{OR by polar coordinates: } (r, \theta)$$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{1-\cos 2r^2}{4r^4} \quad \text{let } r^2 = x^2+y^2, \text{ at } r \rightarrow 0^+$$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{1-\cos r^2}{r^4} \quad \Rightarrow \lim_{r \rightarrow 0} \frac{1-\cos r^2}{r^4} \quad \text{Apply L-hopital}$$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{2\sin r^2}{4r^3} \quad \Rightarrow \lim_{r \rightarrow 0} \frac{2\sin r^2}{4r^3}$$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{1}{2} \quad \text{Hence,} \Rightarrow \frac{1}{2}$$

Hence,

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} = 1}$$

(exists)

Ans.

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Question #2: $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$

a.

$$\text{i. } \lim_{(x,y) \rightarrow (0,0)} \frac{x^3y}{x^4+y^4}$$

Sol:

At $x\text{-axis}, y=0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3(0)}{x^4+0^4} = 0$$

At $y\text{-axis}, x=0$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{0^3(y)}{0^4+y^4} = 0$$

At $y = x^2$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3(x^2)}{x^4+(x^2)^4}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^5}{2x^8}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x}{2}$$

$$\Rightarrow 0$$

Hence,

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y}{x^4+y^4} = 0 \quad (\text{exists})}$$

Ans.

$$\text{ii. } \lim_{(x,y) \rightarrow (0,0)} \frac{x^3-y^3}{x^2+y^2}$$

Sol:

At $y=0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3-0^3}{x^2+0^2} = \lim_{x \rightarrow 0} \frac{x^3}{x^2} = \lim_{x \rightarrow 0} x = 0$$

At $x=0$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{0^3-y^3}{0^2+y^2} = \lim_{y \rightarrow 0} \frac{-y^3}{y^2} = \lim_{y \rightarrow 0} -y = 0$$

At $y=x$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3-x^3}{x^2+x^2} = \lim_{x \rightarrow 0} \frac{0}{2x^2} = 0$$

At $y=mx$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3-m^3x^3}{x^2+m^2x^2} = \lim_{x \rightarrow 0} \frac{x^3(1-m^3)}{x^2(1+m^2)} = 0$$

Hence,

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^3-y^3}{x^2+y^2} = 0 \quad (\text{exists})}$$

Ans.

$$\text{iii. } \lim_{(x,y) \rightarrow (0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$$

Sol:

At $y=0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1-\cos(x^2+0^2)}{(x^2+0^2)^2} = \lim_{x \rightarrow 0} \frac{1-\cos x^2}{x^4}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1-\cos x^2}{x^4}$$

Apply L-Hopital Rule

$$\Rightarrow \lim_{x \rightarrow 0} \frac{0+2x \sin x^2}{4x^3} = \lim_{x \rightarrow 0} \frac{2x \sin x^2}{4x^3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 \cdot \sin x^2}{2 \cdot x^2}$$

$$\Rightarrow \frac{1}{2}(1)$$

$$\Rightarrow 1/2$$

At $x=0$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{1-\cos(0+y^2)}{(0+y^2)^2} = \lim_{y \rightarrow 0} \frac{1-\cos y^2}{y^4}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{1-\cos y^2}{y^4}$$

Apply L-Hopital Rule

$$\Rightarrow \lim_{y \rightarrow 0} \frac{0+2y \sin y^2}{4y^3} = \lim_{y \rightarrow 0} \frac{2y \sin y^2}{4y^3}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{1 \cdot \sin y^2}{2 \cdot y^2}$$

$$\Rightarrow \frac{1}{2}(1)$$

$$\Rightarrow 1/2$$

At $y=x$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1-\cos(x^2+x^2)}{(x^2+x^2)^2} = \lim_{x \rightarrow 0} \frac{1-\cos 2x^2}{4x^4}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1-\cos 2x^2}{4x^4}$$

Apply L-Hopital Rule

$$\Rightarrow \lim_{x \rightarrow 0} \frac{0+4x \sin 2x^2}{16x^3} = \lim_{x \rightarrow 0} \frac{4x \sin 2x^2}{16x^3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 \cdot \sin 2x^2}{2 \cdot 2x^2} = \lim_{x \rightarrow 0} \frac{1-\cos 2x^2}{4x^4}$$

$$\Rightarrow \frac{1}{2}(1)$$

Hence,

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} = \frac{1}{2}}$$

(exists)

Ans.

OR by polar coordinates: (r, θ)

$$\Rightarrow \lim_{r \rightarrow 0} \frac{1-\cos 2r^2}{4r^4} \quad \text{let } r^2 = x^2+y^2, \text{ at } r \rightarrow 0^+$$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{1-\cos 2r^2}{4r^4} \quad \Rightarrow \lim_{r \rightarrow 0} \frac{1-\cos r^2}{r^4} \quad \text{, Apply L-hopital}$$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{2r \sin r^2}{4r^3} = \lim_{r \rightarrow 0} \frac{r \sin r^2}{2r^2}$$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{1}{2} \frac{\sin r^2}{r^2} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}(1) = \frac{1}{2} \quad (\text{limit exists})$$

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Question #3:**Part I:**

a. $f_x(0, y) = -y$ Prove when:
 $f_y(x, 0) = x$
 $\therefore f(x, y) = \frac{xy}{x^2+y^2}$

$$\Rightarrow f(x, y) = \frac{x^3y - xy^3}{x^2+y^2}$$

$$f_x(0, y) = (x^2+y^2)(3xy - y^3) - (x^3y - xy^3)(2x)$$

$$f_x(0, y) = 3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3$$

$$f_x(0, y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_x(0, y) = (0)y + 4(0)y^3 - y^5$$

$$f_x(0, y) = -y$$

Proved.

$$\Rightarrow f_y(x, y) = (x^2+y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)$$

$$f_y(x, y) = x^5 - 3x^3y^2 + x^3y - 3xy^3 - 2x^3y^2 + 2xy^4$$

$$f_y(x, 0) = \frac{x^5 - 0 + 0 - 0 - 0 + 0}{(x^2+0^2)^2}$$

$$f_y(x, 0) = \frac{x^5}{x^4}$$

$$f_y(x, 0) = x$$

Proved.

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b. $f_{xy}(0, 0) = -1$ $f_{yx}(0, 0) = 1$

By part(a):

$$f_x(0, y) = -y$$

By f_y :

$$f_{xy}(0, y) = -1$$

 $f_{yx}(x, 0) = x$

$$f_{xy}(0, 0) = -1$$

 $f_{yx}(x, 0) = 1$

$$f_{xy}(0, 0) = -1$$

 $f_{yx}(0, 0) = 1$ Proved.c. Prove: $f(x, y)$ is differentiable at $(0, 0)$:By polar coordinates: $r \rightarrow 0^+$

$$x = r\cos\theta, y = r\sin\theta, r^2 = x^2+y^2$$

$$\Rightarrow \lim_{r \rightarrow 0^+} (r\cos\theta)(r\sin\theta) \cdot \left(\frac{r\cos^2\theta - r^2\sin^2\theta}{r^2} \right)$$

$$\Rightarrow \lim_{r \rightarrow 0^+} r^2 \cos\theta \sin\theta (\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \lim_{r \rightarrow 0^+} r^2 (\cos^3\theta \sin\theta - \cos\theta \sin^3\theta)$$

Apply limit

$$\Rightarrow = 0$$

Hence limit exists and since all first-order partial derivatives of f exist, $f(x, y)$ is differentiable at $(0, 0)$ **Part II:**

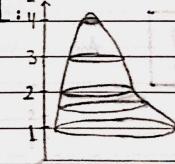
$f_x = 0$ implies that f does not change whenever x changes
 similarly $f_y = 0$ implies that f does not change whenever y changes
 so, f does not depend on xy implying that is a constant.

We can prove by integrating the partial derivatives:

$$\int f_x(x, y) dx = \int 0 dx \quad \& \quad \int f_y(x, y) dy = \int 0 dy$$

$$f(x, y) = 0 + C$$

$$f(x, y) = 0 + C$$

Hence $f(x, y) = c \rightarrow$ (constant) Proved!**Part III:**

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Question #4:

a.

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{v}$$

$$D_u f(0,0) = \nabla f(0,0)(1,2) = 1$$

$$D_v f(0,0) = \nabla f(0,0)(2,1) = 2$$

$$f_x(0,0) = ? , f_y(0,0) = ?$$

$$\Rightarrow \langle 1, 2 \rangle \cdot \langle f_x(0,0), f_y(0,0) \rangle = 1 \quad \text{--- (i)}$$

$$\Rightarrow \langle 2, 1 \rangle \cdot \langle f_x(0,0), f_y(0,0) \rangle = 2 \quad \text{--- (ii)}$$

From eqn(i):

$$f_x + 2f_y = 1 \quad \text{--- (A)}$$

From eqn(ii)

$$2f_x + f_y = 2 \quad \text{--- (B)} \quad (\text{Multiply whole by 2})$$

$$4f_x + 2f_y = 4$$

Subtract (A), (B)

$$f_x + 2f_y = 1$$

$$4f_x + 2f_y = 4$$

$$\Rightarrow -3f_x + 0 = -3$$

$$\therefore f_x = 1 \quad \text{Ans.}$$

from (A):

$$\Rightarrow 1 + 2f_y = 1$$

$$2f_y = 0$$

$$f_y = 0 \quad \text{Ans.}$$

$$f_x(0,0) = 1$$

$$f_y(0,0) = 0$$

Ans.

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b. $z = f(x, y)$

$x = g(s, t)$, $y = h(s, t)$

$g(1, 2) = 3$

$g_s(1, 2) = -1$, $g_t(1, 2) = 4$

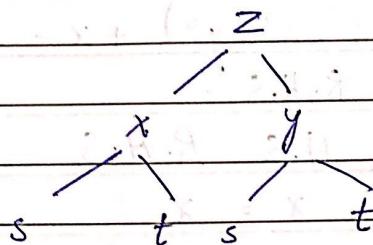
$(x - 3)^2 + y^2 = 2^2$

$h(1, 2) = 6$

$h_s(1, 2) = -5$, $h_t(1, 2) = 10$

$f_x(3, 6) = 7$ & $f_y(3, 6) = 8$

$\frac{\partial z}{\partial s} \Big|_{s=1, t=2} : ?$, $\frac{\partial z}{\partial t} \Big|_{s=1, t=2} : ?$



$\therefore \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$

$\frac{\partial z}{\partial s} \Big|_{(1,2)} = f_x(3, 6)g_s(1, 2) + f_y(3, 6)h_s(1, 2)$

$\frac{\partial z}{\partial s} \Big|_{(1,2)} = (7)(-1) + (8)(-5)$

$\frac{\partial z}{\partial s} \Big|_{(1,2)} = -7 - 40$

$\boxed{\frac{\partial z}{\partial s} \Big|_{(1,2)} = -47}$ Ans.

$\therefore \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$

$\frac{\partial z}{\partial t} \Big|_{(1,2)} = f_x(3, 6)g_t(1, 2) + f_y(3, 6)h_t(1, 2)$

$\frac{\partial z}{\partial t} \Big|_{(1,2)} = 7(4) + 8(10)$

$\boxed{\frac{\partial z}{\partial t} \Big|_{(1,2)} = 108}$ Ans.

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c. $z = y + f(x^2 - y^2)$

Prove: $(\frac{\partial z}{\partial x})_y = y$, $(\frac{\partial z}{\partial y})_x = x$

$$y \frac{dz}{dx} + x \frac{dz}{dy} = x$$

$\text{at } z = (x, y)_z$, $\frac{dz}{dx} = (x, y)_y$

$y = (x, y)_x$, $1 = (x, y)_y$

L.H.S:

$$\Rightarrow y [0 + f'(x^2 - y^2) \cdot 2x] + x [1 + f'(x^2 - y^2)(-2y)]$$

$$\Rightarrow 2xy f'(x^2 - y^2) + x - 2xy f'(x^2 - y^2)$$

$$\Rightarrow x = \text{R.H.S}$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

$$x = x$$

Hence Proved!

Question # 5:

a. $f(x, y, z) = x^3 \sqrt{y^2 + z^2}$ at $(2, 3, 4)$

Find linear approximation

estimate:

$$(1.98)^3 \sqrt{(3.01)^2 + (3.97)^2}$$

Sol:

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f_z(x_0, y_0)(z - z_0)$$

$$\Rightarrow = 2^3 \sqrt{3^2 + 4^2} + 3x_0^2 \sqrt{y_0^2 + z_0^2}(x - 2) + \frac{x_0^3 \cdot 2y_0}{2\sqrt{y_0^2 + z_0^2}} (-3 + y_0) + \frac{x_0^3 \cdot 2z_0}{2\sqrt{y_0^2 + z_0^2}} (z - 4)$$

$$\Rightarrow = 8\sqrt{25} + 3(2^2) \sqrt{3^2 + 4^2}(x - 2) + \frac{8(3)}{\sqrt{3^2 + 4^2}} (y - 3) + \frac{8(4)}{\sqrt{3^2 + 4^2}} (z - 4)$$

$$\text{at } (x, y, z) = (1.98, 3.01, 3.97)$$

$$\Rightarrow = 40 + 60(1.98 - 2) + \frac{24(3.01 - 3)}{5} + \frac{32(3.97 - 4)}{5}$$

$$\boxed{L = 38.656} \quad \text{Ans.}$$

$$f(1.98, 3.01, 3.97) = 38.672 \approx L = 38.656 \quad \text{Ans.}$$

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b. $0 \leq x \leq 10$

x, y : meters

$0 \leq y \leq 8$

$T(x, y)$: celsius

i. $T_x(6, 4) : ?$

$T_y(6, 4) : ?$

$\therefore T_x(6, 4) = \lim_{\Delta v \rightarrow 0} \frac{T(6 + \Delta v, 4) - T(6, 4)}{\Delta v}$

For $\Delta v = +2$

$T_x(6, 4) \approx T(6 + 2, 4) - T(6, 4)$

$T_x(6, 4) \approx \frac{86 - 80}{2}$

$T_x(6, 4) \approx 3$

For $\Delta v = -2$

$T_x(6, 4) \approx T(6 - 2, 4) - T(6, 4)$

$T_x(6, 4) \approx 72 - 80$

$T_x(6, 4) \approx 4$

$T_x(6, 4) \approx \frac{3 + 4}{2} = 3.5 \text{ } ^\circ\text{C/m}$

Ans.

$\therefore T_y(6, 4) = \lim_{\Delta v \rightarrow 0} \frac{T(6, 4 + \Delta v) - T(6, 4)}{\Delta v}$

$\Rightarrow T_y(6, 4) \approx T(6, 4 + 2) - T(6, 4) \quad \left. \begin{array}{l} \\ 2 \end{array} \right\} \Rightarrow T_y(6, 4) \approx T(6, 4 - 2) - T(6, 4) \quad \left. \begin{array}{l} \\ -2 \end{array} \right\}$

$T_y(6, 4) \approx \frac{75 - 80}{2}$

$T_y(6, 4) \approx \frac{87 - 80}{2}$

$T_y(6, 4) \approx -2.5$

$T_y(6, 4) \approx -3.5$

$T_y(6, 4) \approx \frac{-2.5 - 3.5}{2} = -3 \text{ } ^\circ\text{C/m}$

Ans.

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$$\text{ii. } D_u T(6,4) ; u = (i+j)/\sqrt{2}$$

$$\therefore D_u T(6,4) = f_x(6,4)u_i + f_y(6,4)u_j$$

From previous part: we get $f_x \approx f_y$ at $(6,4)$

$$\Rightarrow D_u T(6,4) = 3.5(1/\sqrt{2}) + (-3)(1/\sqrt{2})$$

$$D_u T(6,4) = \frac{1}{2\sqrt{2}}$$

$$D_u T(6,4) = 0.353 \text{ } ^\circ\text{C/m} \quad \text{Ans.}$$

We conclude that if we move a small distance from the point $(6,4)$ in the direction of u , the function $T(x,y)$ (i.e. temperature) will increase at a rate of $0.353 \text{ } ^\circ\text{C/m}$

iii. $T_{xy}(6,4)$

From part(i) we have,

$$T_x(6,4) \approx 3.5 \text{ } ^\circ\text{C/m}$$

$$\because T_{xy}(6,4) = \lim_{\Delta v \rightarrow 0} \frac{T_x(6,4+\Delta v) - T_x(6,4)}{\Delta v}$$

at $\Delta v = 2$

$$T_{xy}(6,4) \approx \frac{T_x(6,6) - T_x(6,4)}{2} \quad \begin{array}{c} \text{at } \Delta v = 2 \\ \Delta v \end{array} \quad T_{xy}(6,4) \approx \frac{T_x(6,2) - T_x(6,4)}{-2} \quad \begin{array}{c} \text{at } \Delta v = -2 \\ \Delta v \end{array}$$

For $T_x(6,6)$:

[Right-hand]

$$\Rightarrow T_x(6,6) \approx \frac{T(6+2,6) - T(6,6)}{2} \quad \begin{array}{c} \text{[left-hand]} \\ \Rightarrow T_x(6,6) \approx \frac{T(6-2,6) - T(6,6)}{2} \end{array}$$

$$T_x(6,6) \approx \frac{80 - 75}{2} \quad \begin{array}{c} \text{[left-hand]} \\ \Rightarrow T_x(6,6) \approx \frac{68 - 75}{2} \end{array}$$

$$T_x(6,6) \approx 2.5 \quad \begin{array}{c} \text{[left-hand]} \\ \Rightarrow T_x(6,6) \approx 3.5 \end{array}$$

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$$\Rightarrow T_x(6,6) \approx 2.5 + 3.5 = 3 \text{ } ^\circ\text{C/m}$$

For $T_x(6,2)$:

[Right-hand]

$$\Rightarrow T_x(6,2) \approx \frac{T(6+2,2) - T(6,2)}{2} \quad \begin{array}{c} \text{[left-hand]} \\ \Rightarrow T_x(6,2) \approx \frac{T(6-2,2) - T(6,2)}{-2} \end{array}$$

$$T_x(6,2) \approx \frac{90 - 87}{2}$$

$$T_x(6,2) \approx 1.5$$

$$T_x(6,2) \approx 6.5$$

$$\Rightarrow T_x(6,2) \approx \frac{1.5 + 6.5}{2} = 4 \text{ } ^\circ\text{C/m}$$

Now,

$$(A) \Rightarrow T_{xy}(6,4) \approx \frac{3 - 3.5}{2} \quad (B) \Rightarrow T_{xy}(6,4) \approx \frac{4 - 3.5}{-2}$$

$$T_{xy}(6,4) \approx -0.25$$

$$T_{xy}(6,4) \approx -0.25$$

$$\Rightarrow T_{xy}(6,4) \approx \frac{-0.25 - 0.25}{2} = -0.25 \text{ } ^\circ\text{C/m}^2 \quad \text{Ans.}$$

iv. linear approximation of $T(x,y)$ near $(6,4)$:

Sol:

$$L = T(x_0, y_0) + T_x(x_0, y_0)(x - x_0) + T_y(x_0, y_0)(y - y_0)$$

From previous parts,

$$L = T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4)$$

$$L = 80 + (3.5)(5-6) + (-3)(3.8-4) ; \quad (x, y) = (5, 3.8)$$

$$L = 80 + 3.5(5-6) - 3(3.8-4)$$

$$L = 77.1 \text{ } ^\circ\text{C} \quad \text{Ans.}$$

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Question #6:

a. $x = 5\text{m}$ $\Delta x = 0.002\text{ m}$
 $y = 12\text{m}$ $\Delta y = 0.002\text{ m}$

i. Max error in area:

$$\because A = \frac{1}{2}xy \rightarrow dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$$

$$\Rightarrow dA = \frac{1}{2}ydx + \frac{1}{2}xdy$$

$$dA = \frac{1}{2}(12)(0.002) + \frac{1}{2}(5)(0.002)$$

$$dA = 0.017\text{ m}^2 \quad \text{Ans.}$$

ii. Max error in hypotenuse:

By pythagorus theorem:

$$H = \sqrt{x^2+y^2}$$

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy$$

$$dH = 2x \frac{dx}{\partial x} + 2y \frac{dy}{\partial y}$$

$$dH = 2\sqrt{x^2+y^2} \cdot \frac{2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$dH = 5(0.002) + 12(0.002)$$

$$dH = \sqrt{5^2+12^2} \cdot 0.002$$

$$dH = 2.615 \times 10^{-3}$$

$$dH = 0.002615\text{ m} \quad \text{Ans.}$$

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b. $\frac{dx}{dt} = +3 \text{ in/s}$, $\frac{dy}{dt} = -2 \text{ in/s}$, $\frac{d\theta}{dt} = 0.05 \text{ rad/s}$

$$\left[\frac{dA}{dt} \right]_{\begin{array}{l} x=40 \\ y=50 \\ \theta=\pi/6 \end{array}} : ? \quad \therefore A = \frac{1}{2} xy \sin \theta$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x} \frac{dx}{dt} + \frac{\partial A}{\partial y} \frac{dy}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} y \sin \theta (3) + \frac{1}{2} x \sin \theta (-2) + \frac{1}{2} xy \cos \theta (0.05)$$

$$\frac{dA}{dt} = \frac{3}{2} (50) \sin \frac{\pi}{6} - 40 \sin \frac{\pi}{6} + \frac{(40)(50) \cos \frac{\pi}{6}}{2} (0.05)$$

$$\frac{dA}{dt} = 3.25 \left(\frac{1}{2}\right) - 40 \left(\frac{1}{2}\right) + 50 \left(\frac{\sqrt{3}}{2}\right)$$

$$\left[\frac{dA}{dt} \right]_{\begin{array}{l} x=40 \\ y=50 \\ \theta=\pi/6 \end{array}} = 60.801 \text{ in}^2/\text{sec} \quad \text{Ans.}$$

Question #7:

a. $f(x, y, z) = ze^{xy}$; $P(0, 1, 2)$
direction: ? $|\nabla F|$: ?

Sol:

$$\nabla F = f_x(x, y, z) \hat{i} + f_y(x, y, z) \hat{j} + f_z(x, y, z) \hat{k}$$

$$\nabla F = (ze^{xy} \cdot y) \hat{i} + (ze^{xy} \cdot x) \hat{j} + e^{xy} \hat{k}$$

$$\nabla F(0, 1, 2) = 2e^0(1) \hat{i} + 2e^0(0) \hat{j} + e^0 \hat{k}$$

$$\nabla F(0, 1, 2) = 2 \hat{i} + \hat{k}$$

Direction:

$$2\hat{i} + \hat{k} \text{ OR } (2, 0, 1) \quad \text{Ans.}$$

$$|\nabla F| = \sqrt{2^2 + 0^2 + 1^2}$$

$$|\nabla F| = \sqrt{5} \quad \text{max rate} \quad \text{Ans.}$$

of increase.

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b. $f(x, y, z) = |x|^n$; $\vec{r} = xi + yj + zk$

Prove:

$$\nabla f = -nr$$

$$|r|^{n+2}$$

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

Sol:

$$\Rightarrow f(x, y, z) = (\sqrt{x^2 + y^2 + z^2})^{-n}$$

$$f(x, y, z) = (x^2 + y^2 + z^2)^{-n/2}$$

$$\therefore \nabla F = f_x(x, y, z)\hat{i} + f_y(x, y, z)\hat{j} + f_z(x, y, z)\hat{k}$$

$$\rightarrow f_x(x, y, z) = -\frac{n}{2} (x^2 + y^2 + z^2)^{-n/2-1} \cdot (2x)$$

$$f_x(x, y, z) = -\frac{nx}{(x^2 + y^2 + z^2)^{(n+2)/2}}$$

$$\rightarrow f_y(x, y, z) = -\frac{n}{2} (x^2 + y^2 + z^2)^{-n/2-1} (2y)$$

$$f_y(x, y, z) = -\frac{ny}{(x^2 + y^2 + z^2)^{(n+2)/2}}$$

$$\rightarrow f_z(x, y, z) = -\frac{n}{2} (x^2 + y^2 + z^2)^{-n/2-1} (2z)$$

$$f_z(x, y, z) = -\frac{nz}{(x^2 + y^2 + z^2)^{(n+2)/2}}$$

$$\rightarrow \nabla F = -\frac{nx}{(x^2 + y^2 + z^2)^{(n+2)/2}}\hat{i} - \frac{ny}{(x^2 + y^2 + z^2)^{(n+2)/2}}\hat{j} - \frac{nz}{(x^2 + y^2 + z^2)^{(n+2)/2}}\hat{k}$$

$$\nabla F = -n(xi + yj + zk)$$

$$[(x^2 + y^2 + z^2)^{1/2}]^{(n+2)}$$

∇F	$= -nr$
	$ r ^{n+2}$

Hence proved!

$$\therefore r = xi + yj + zk$$

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

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c. $f(x, y, z) = x^2 e^{yz^2}$; $\nabla f = ?$

Sol:

$$\Rightarrow \nabla f = f_x(x, y, z)\hat{i} + f_y(x, y, z)\hat{j} + f_z(x, y, z)\hat{k}$$

$$\rightarrow f_x(x, y, z) = 2x e^{yz^2}$$

$$\rightarrow f_y(x, y, z) = x^2 e^{yz^2} \cdot z^2$$

$$\rightarrow f_z(x, y, z) = x^2 e^{yz^2} \cdot 2yz$$

Now,

$$\Rightarrow \nabla f = 2x e^{yz^2} \hat{i} + x^2 z^2 e^{yz^2} \hat{j} + 2x^2 yz e^{yz^2} \hat{k}$$

Ans.

$$\nabla f = \langle 2x e^{yz^2}, x^2 z^2 e^{yz^2}, 2x^2 yz e^{yz^2} \rangle$$

i. When is the directional derivative maximum:

Ans:

$D_u f(x, y, z)$ is maximum when ∇f and u are in same direction i.e. $\theta = 0^\circ$

$$D_u f(x, y, z) = \nabla f \cdot u$$

$$D_u f(x, y, z) = |\nabla f| |u| \cos \theta$$

$$D_u f(x, y, z) = |\nabla f| (1) \cos 0^\circ$$

$$D_u f(x, y, z) = |\nabla f| \quad (\text{max})$$

ii. When is $D_u f$ minimum:

Ans:

$D_u f(x, y, z)$ is minimum when ∇f and u are antiparallel i.e. $\theta = 180^\circ$

$$D_u f(x, y, z) = |\nabla f| |u| \cos 180^\circ$$

$$D_u f(x, y, z) = -|\nabla f| \quad (\text{min})$$

iii. When is $D_u f$ equal to zero:

Ans: $D_u f(x, y, z)$ is zero when ∇f and u are perpendicular i.e. $\theta = 90^\circ$

$$D_u f(x, y, z) = |\nabla f| |u| \cos 90^\circ$$

$$D_u f(x, y, z) = 0 \quad (\text{zero})$$

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iv. When is $D_u f(x, y, z)$ half of its maximum value:

Ans:

$$D_u f(x, y, z) = \frac{1}{2} D_{\max}$$

$$|\nabla f| |u| \cos \theta = \frac{1}{2} |\nabla f|$$

$$(1) \cos \theta = \frac{1}{2}$$

$$\cos \theta = 1/2$$

$$\theta = \cos^{-1}(1/2)$$

$$\boxed{\theta = \frac{\pi}{3}} \quad \text{or} \quad \theta = 60^\circ$$

$D_u f(x, y, z)$ is half of its maximum value when θ between ∇f and u is 60° .