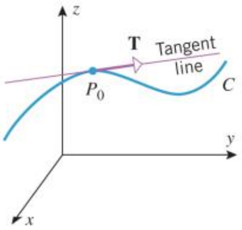
13.7 TANGENT PLANES AND NORMAL VECTORS

In this section we will discuss tangent planes to surfaces in three-dimensional space. We will be concerned with three main questions: What is a tangent plane? When do tangent planes exist? How do we find equations of tangent planes?

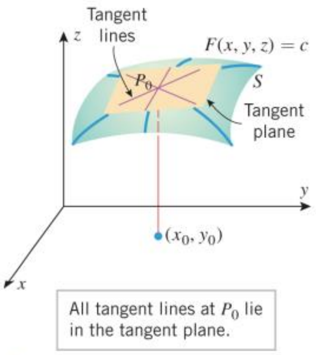
TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES $F(x, y, z) = c$

We begin by considering the problem of finding tangent planes to level surfaces of a function $F(x, y, z)$. These surfaces are represented by equations of the form $F(x, y, z) = c$. We will assume that F has continuous first-order partial derivatives, since this has an important geometric consequence. Fix c, and suppose that $P_0(x_0, y_0, z_0)$ satisfies the equation $F(x, y, z) = c$. In advanced courses it is proved that if F has continuous first-order partial derivatives, and if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then near P_0 the graph of $F(x, y, z) = c$ is indeed a "surface" rather than some possibly exotic-looking set of points in 3-space.



▲ Figure 13.7.1

We will base our concept of a tangent plane to a level surface S: $F(x, y, z) = c$ on the more elementary notion of a tangent line to a curve C in 3-space (Figure 13.7.1). Intuitively, we would expect a tangent plane to S at a point P_0 to be composed of the tangent lines at P_0 of all curves on S that pass through P_0 (Figure 13.7.2). Suppose C is a curve on S through P_0 that is parametrized by $x = x(t)$, $y = y(t)$, $z = z(t)$ with $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$. The tangent line l to C through P_0 is then parallel to the vector



▲ Figure 13.7.2

$$\mathbf{r}' = x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}$$

where we assume that $\mathbf{r}' \neq \mathbf{0}$ (Definition 12.2.7). Since C is on the surface $F(x, y, z) = c$, we have

$$c = F(x(t), y(t), z(t)) \tag{1}$$

Computing the derivative at t_0 of both sides of (1), we have by the chain rule that

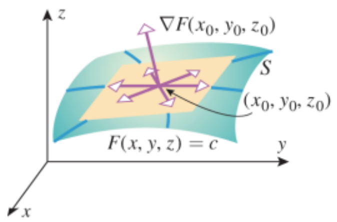
$$0 = F_x(x_0, y_0, z_0)x'(t_0) + F_y(x_0, y_0, z_0)y'(t_0) + F_z(x_0, y_0, z_0)z'(t_0)$$

We can write this equation in vector form as

$$0 = (F_x(x_0, y_0, z_0)\mathbf{i} + F_y(x_0, y_0, z_0)\mathbf{j} + F_z(x_0, y_0, z_0)\mathbf{k}) \cdot (x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k})$$

or

$$0 = \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}' \tag{2}$$



▲ Figure 13.7.3

It follows that if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to line l. Therefore, the tangent line l to C at P_0 is contained in the plane through P_0 with normal vector $\nabla F(x_0, y_0, z_0)$. Since C was *arbitrary*, we conclude that the same is true for any curve on S through P_0 (Figure 13.7.3). Thus, it makes sense to define the tangent plane to S at P_0 to be the plane through P_0 whose normal vector is

$$\mathbf{n} = \nabla F(x_0, y_0, z_0) = \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\rangle$$

Using the point-normal form [see Formula (3) in Section 11.6], we have the following definition.

Definition 13.7.1 can be viewed as an extension of Theorem 13.6.6 from curves to surfaces.

13.7.1 DEFINITION Assume that $F(x, y, z)$ has continuous first-order partial derivatives and that $P_0(x_0, y_0, z_0)$ is a point on the level surface S: $F(x, y, z) = c$. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\mathbf{n} = \nabla F(x_0, y_0, z_0)$ is a ***normal vector*** to S at P_0 and the ***tangent plane*** to S at P_0 is the plane with equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad (3)$$

The line through the point P_0 parallel to the normal vector $\mathbf{n}$ is perpendicular to the tangent plane (3). We will call this the ***normal line***, or sometimes more simply the ***normal*** to the surface $F(x, y, z) = c$ at P_0. It follows that this line can be expressed parametrically as

$$x = x_0 + F_x(x_0, y_0, z_0)t, \quad y = y_0 + F_y(x_0, y_0, z_0)t, \quad z = z_0 + F_z(x_0, y_0, z_0)t \quad (4)$$

Example 1 Consider the ellipsoid $x^2 + 4y^2 + z^2 = 18$.

(a) Find an equation of the tangent plane to the ellipsoid at the point $(1, 2, 1)$.

(b) Find parametric equations of the line that is normal to the ellipsoid at the point $(1, 2, 1)$.

(c) Find the acute angle that the tangent plane at the point $(1, 2, 1)$ makes with the xy-plane.

Solution (a). We apply Definition 13.7.1 with $F(x, y, z) = x^2 + 4y^2 + z^2$ and $(x_0, y_0, z_0) = (1, 2, 1)$. Since

$$\nabla F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)\rangle = \langle 2x, 8y, 2z\rangle$$

we have

$$\mathbf{n} = \nabla F(1, 2, 1) = \langle 2, 16, 2\rangle$$

Hence, from (3) the equation of the tangent plane is

$$2(x - 1) + 16(y - 2) + 2(z - 1) = 0 \quad \text{or} \quad x + 8y + z = 18$$

Solution (b). Since $\mathbf{n} = \langle 2, 16, 2\rangle$ at the point $(1, 2, 1)$, it follows from (4) that parametric equations for the normal line to the ellipsoid at the point $(1, 2, 1)$ are

$$x = 1 + 2t, \quad y = 2 + 16t, \quad z = 1 + 2t$$

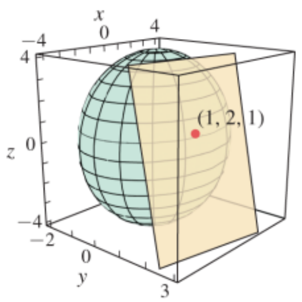


Figure 13.7.4

Solution (c). To find the acute angle θ between the tangent plane and the xy-plane, we will apply Formula (9) of Section 11.6 with $\mathbf{n}_1 = \mathbf{n} = \langle 2, 16, 2\rangle$ and $\mathbf{n}_2 = \langle 0, 0, 1\rangle$. This yields

$$\cos\theta = \frac{|\langle 2, 16, 2\rangle \cdot \langle 0, 0, 1\rangle|}{\|\langle 2, 16, 2\rangle\|\,\|\langle 0, 0, 1\rangle\|} = \frac{2}{(2\sqrt{66})(1)} = \frac{1}{\sqrt{66}}$$

Thus,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{66}}\right) \approx 83^\circ$$

(Figure 13.7.4). ◄

TANGENT PLANES TO SURFACES OF THE FORM $z = f(x, y)$

To find a tangent plane to a surface of the form $z = f(x, y)$, we can use Equation (3) with the function $F(x, y, z) = z - f(x, y)$.

▶ **Example 2** Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2y$ at the point $(2, 1, 4)$.

Solution. Let $F(x, y, z) = z - x^2y$. Then $F(x, y, z) = 0$ on the surface, so we can find the find the gradient of F at the point $(2, 1, 4)$:

$$\nabla F(x, y, z) = -2xy\mathbf{i} - x^2\mathbf{j} + \mathbf{k}$$
$$\nabla F(2, 1, 4) = -4\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

From (3) the tangent plane has equation

$$-4(x-2) - 4(y-1) + 1(z-4) = 0 \quad \text{or} \quad -4x - 4y + z = -8$$

and the normal line has equations

$$x = 2 - 4t, \qquad y = 1 - 4t, \qquad z = 4 + t \;◀$$

Suppose that $f(x, y)$ is differentiable at a point (x_0, y_0) and that $z_0 = f(x_0, y_0)$. It can be shown that the procedure of Example 2 can be used to find the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0). This yields an alternative equation for a tangent plane to the graph of a differentiable function.

EXERCISE SET 13.7 C CAS

1. Consider the ellipsoid $x^2 + y^2 + 4z^2 = 12$.
   (a) Find an equation of the tangent plane to the ellipsoid at the point $(2, 2, 1)$.
   (b) Find parametric equations of the line that is normal to the ellipsoid at the point $(2, 2, 1)$.
   (c) Find the acute angle that the tangent plane at the point $(2, 2, 1)$ makes with the xy-plane.
2. Consider the surface $xz - yz^3 + yz^2 = 2$.
   (a) Find an equation of the tangent plane to the surface at the point $(2, -1, 1)$.
   (b) Find parametric equations of the line that is normal to the surface at the point $(2, -1, 1)$.
   (c) Find the acute angle that the tangent plane at the point $(2, -1, 1)$ makes with the xy-plane.

3–12 Find an equation for the tangent plane and parametric equations for the normal line to the surface at the point P.

3. $x^2 + y^2 + z^2 = 25;\ P(-3, 0, 4)$
4. $x^2y - 4z^2 = -7;\ P(-3, 1, -2)$
5. $x^2 - xyz = 56;\ P(-4, 5, 2)$
6. $z = x^2 + y^2;\ P(2, -3, 13)$
7. $z = 4x^3y^2 + 2y;\ P(1, -2, 12)$
8. $z = \frac{1}{2}x^7y^{-2};\ P(2, 4, 4)$
9. $z = xe^{-y};\ P(1, 0, 1)$
10. $z = \ln\sqrt{x^2 + y^2};\ P(-1, 0, 0)$
11. $z = e^{3y}\sin 3x;\ P(\pi/6, 0, 1)$
12. $z = x^{1/2} + y^{1/2};\ P(4, 9, 5)$