National University of Computer & Emerging Sciences, Karachi

Multivariable Calculus Assignment - 01 (CLO2)

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Question 1.[2+3+3=08 marks]

(a) Find and sketch the domain of the given functions.

(i) $f(x,y) = \sqrt{x+y} - \sqrt{x-3}$ (ii) $f(x,y) = \sqrt{\frac{1}{x^2} - \frac{1}{y^2}}$

(iii) $f(x,y,z) = \frac{1}{x+1} + \frac{1}{y-1} + \frac{1}{x+y-z}$ (iv) $f(x,y) = ln(x^2 - 8y)$

(b) Identify and sketch the level curves (or contours) for the given function.

(i) $x^2 - 4z - y = 2$ (ii) $z^2 + 4x^2 = 1 - y^2$ (iii) $2x - 6y + z = -2$

(c) Identify and sketch the level surfaces (or contours) for the given functions at the specified value of k.

$$(i)\; f(x,y,z) = x - y^2 - z^2 + 1,\; k = -3, \quad (ii)\; f(x,y,z) = \frac{3x^2 + y^2}{z^2},\; k = 9$$

$$(iii)\; f(x,y,z) = 9x^2 + 4y^2 + z^2,\; k = 4.$$

Question 2.[3+2=5 marks]

(a) Examine whether the following limits exist and find their values if they exist.

(i) $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$ (ii) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ (iii) $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2}{(x^2+y^2)^2})$

(b) Examine the following functions for continuity at the point $(0,0)$, where $f(0,0) = 0$ and $f(x,y)$ for $(x,y) \neq (0,0)$ is given by

$i)$ $\frac{xy}{\sqrt{x^2+y^2}}$ $ii)$ $\frac{xy}{x^2+y^2}$ $iii)$ $\frac{x^4-y^2}{x^4+y^2}$ $iv)$ $\frac{x^2y}{x^4+y^2}$.

Question 3.[2+1+2=5 marks]

(I) Let $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0, otherwise. Prove that

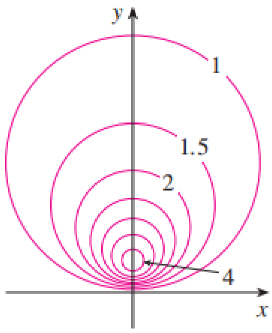
(a) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all x and y;

(b) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ and (c) $f(x,y)$ is differentiable at $(0,0)$.

(II) Suppose f is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that $f(x,y) = c$, a constant.

A contour map of a function f is shown. Use it to make a rough sketch of the graph of f.

(III)



Question 4.[2+2+2=6 marks]

(a) The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions $(1,2)$ and $(2,1)$ are 1 and 2 respectively. Find $f_x(0,0)$ and $f_y(0,0)$.

(b) Suppose $z = f(x, y)$, where $x = g(s, t)$, $y = h(s, t)$, $g(1, 2) = 3$, $g_s(1, 2) = -1$, $g_t(1, 2) = 4$, $h(1, 2) = 6$, $h_s(1, 2) = -5$, $h_t(1, 2) = 10$, $f_x(3, 6) = 7$, and $f_y(3, 6) = 8$. Find $\partial z/\partial s$ and $\partial z/\partial t$ when $s = 1$ and $t = 2$.

(c) If $z = y + f(x^2 - y^2)$, where f is differentiable, show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x$$

Question 5.[2+2+2=6 marks]

(a) Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point $(2, 3, 4)$ and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

(b) A metal plate is situated in the xy-plane and occupies the rectangle $0 \leq x \leq 10$, $0 \leq y \leq 8$, where x and y are measured in meters. The temperature at the point (x, y) in the plate is $T(x, y)$, where T is measured in degrees Celsius. Temperatures

at equally spaced points were measured and recorded in the table.

(a) Estimate the values of the partial derivatives $T_x(6, 4)$ and $T_y(6, 4)$. What are the units?

(b) Estimate the value of $D_{\mathbf{u}}T(6, 4)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$. Interpret your result.

(c) Estimate the value of $T_{xy}(6, 4)$.

x \ y	0	2	4	6	8
0	30	38	45	51	55
2	52	56	60	62	61
4	78	74	72	68	66
6	98	87	80	75	71
8	96	90	86	80	75
10	92	92	91	87	78

(d) Find a linear approximation to the temperature function $T(x, y)$ near the point $(6, 4)$. Then use
it to estimate the temperature at the point $(5, 3.8)$.

Question 6.[2+2=4 marks]

(a) The two legs of a right triangle are measured as 5 m and 12 m with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of (a) the area of the triangle and (b) the length of the hypotenuse.

(b) The length x of a side of a triangle is increasing at a rate of 3 in/s, the length y of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \pi/6$?

Question 7.[2+2+2=6 marks]

(a) Find the direction in which $f(x, y, z) = ze^{xy}$ increases most rapidly at the point $(0, 1, 2)$. What is the maximum rate of increase?

(b) Let $f(x, y, z) = |r|^{-n}$ where $r = xi + yj + zk$. Show that

$$\nabla \mathbf{f} = \frac{-\mathbf{n}\,\mathbf{r}}{|\mathbf{r}|^{\mathbf{n+2}}}$$

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(c) Find the gradient of the function $f(x, y, z) = x^2\, e^{yz^2}$.

(i) When is the directional derivative of f a maximum?

(ii) When it is minimum?

(iii) When it is zero?

(iv) When is it half of its maximum value?