National University of Computer & Emerging Sciences
MT-2008 Multivariable Calculus

Limits and Continuity

LIMITS ALONG CURVES

For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left. For functions of two or three variables there are infinitely many different curves along which one point can approach another (Figure 13.2.1). Our first objective in this section is to define the limit of $f(x, y)$ as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).

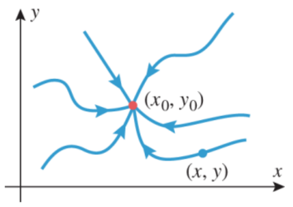


Figure 13.2.1

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y) \to (x_0,y_0) \\ (\text{along } C)}} f(x, y) \quad \text{and} \quad \lim_{\substack{(x,y,z) \to (x_0,y_0,z_0) \\ (\text{along } C)}} f(x, y, z)$$

are defined by

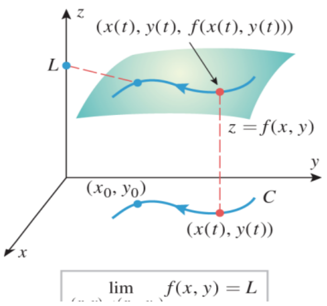
In words. Formulas (1) and (2) state that a limit of a function f along a parametric curve can be obtained by substituting the parametric equations for the curve into the formula for the function and then computing the limit of the resulting function of one variable at the appropriate point.

$$\lim_{\substack{(x,y) \to (x_0,y_0) \\ (\text{along } C)}} f(x, y) = \lim_{t \to t_0} f(x(t), y(t)) \tag{1}$$

$$\lim_{\substack{(x,y,z) \to (x_0,y_0,z_0) \\ (\text{along } C)}} f(x, y, z) = \lim_{t \to t_0} f(x(t), y(t), z(t)) \tag{2}$$

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

A geometric interpretation of the limit along a curve for a function of two variables is shown in Figure 13.2.2: As the point $(x(t), y(t))$ moves along the curve C in the xy-plane toward (x_0, y_0), the point $(x(t), y(t), f(x(t), y(t)))$ moves directly above it along the graph of $z = f(x, y)$ with $f(x(t), y(t))$ approaching the limiting value L. In the figure we followed a common practice of omitting the zero z-coordinate for points in the xy-plane.



Example 1 Figure 13.2.3a shows a computer-generated graph of the function

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of $\frac{1}{2}$ for $y = -x$, except at $(0, 0)$ where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$
(d) the line $y = -x$ (e) the parabola $y = x^2$

Solution (a). The x-axis has parametric equations $x = t$, $y = 0$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y) \to (0,0) \\ (\text{along } y=0)}} f(x, y) = \lim_{t \to 0} f(t, 0) = \lim_{t \to 0} \left(-\frac{0}{t^2}\right) = \lim_{t \to 0} 0 = 0$$

which is consistent with Figure 13.2.3b.

Solution (b). The y-axis has parametric equations $x=0, y=t$, with $(0,0)$ corresponding to $t=0$, so

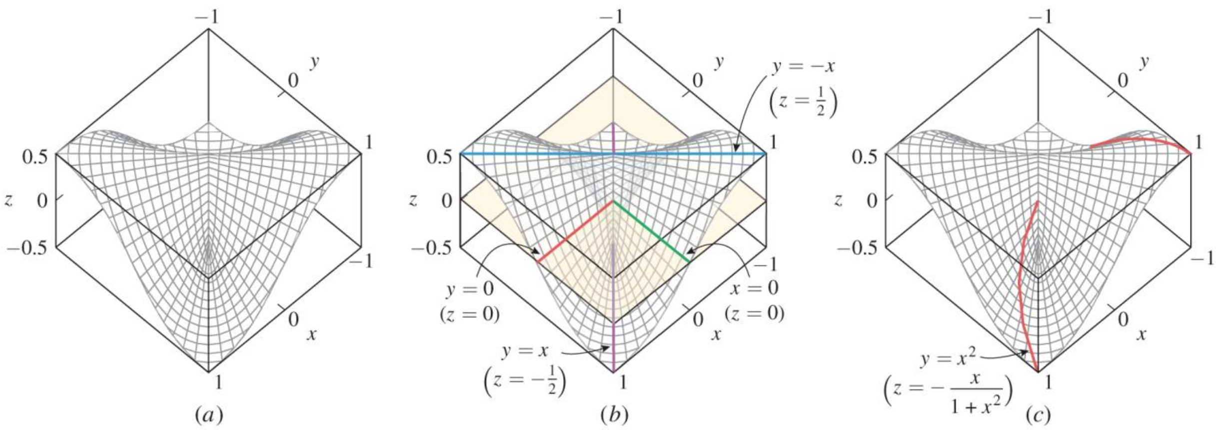
$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 13.2.3*b*.

Solution (c). The line $y=x$ has parametric equations $x=t, y=t$, with $(0,0)$ corresponding to $t=0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.



▲ **Figure 13.2.3**

Solution (d). The line $y=-x$ has parametric equations $x=t, y=-t$, with $(0,0)$ corresponding to $t=0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=-x)}} f(x,y) = \lim_{t\to 0} f(t,-t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

Solution (e). The parabola $y=x^2$ has parametric equations $x=t, y=t^2$, with $(0,0)$ corresponding to $t=0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x^2)}} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2+t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1+t^2}\right) = 0$$

This is consistent with Figure 13.2.3*c*, which shows the parametric curve

$$x=t, \quad y=t^2, \quad z=-\frac{t}{1+t^2}$$

superimposed on the surface. ◀

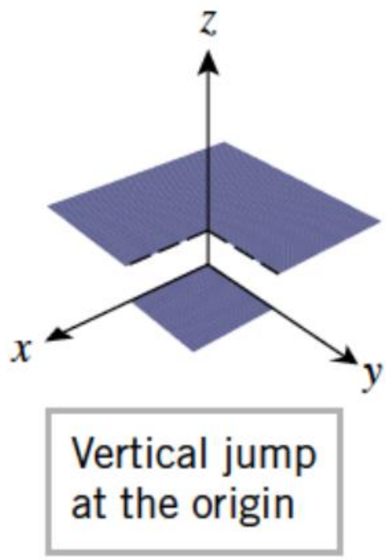
13.2.2 THEOREM

(*a*) *If $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$, then $f(x,y)\to L$ as $(x,y)\to(x_0,y_0)$ along any smooth curve.*

(*b*) *If the limit of $f(x,y)$ fails to exist as $(x,y)\to(x_0,y_0)$ along some smooth curve, or if $f(x,y)$ has different limits as $(x,y)\to(x_0,y_0)$ along two different smooth curves, then the limit of $f(x,y)$ does not exist as $(x,y)\to(x_0,y_0)$.*

CONTINUITY

Stated informally, a function of one variable is continuous if its graph is an unbroken curve without jumps or holes. To extend this idea to functions of two variables, imagine that the graph of $z = f(x, y)$ is formed from a thin sheet of clay that has been molded into peaks and valleys. We will regard f as being continuous if the clay surface has no jumps, tears, or holes. For example, the function graphed in Figure 13.2.8 fails to be continuous because its graph exhibits a vertical jump at the origin.



13.2.3 DEFINITION A function $f(x, y)$ is said to be ***continuous at*** $(\boldsymbol{x_0}, \boldsymbol{y_0})$ if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is ***continuous on D***, and if f is continuous at every point in the xy-plane, then we say that f is ***continuous everywhere***.

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

LIMITS AT DISCONTINUITIES

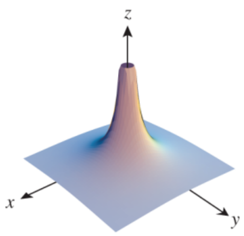
Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x,y)\to(0,0)$ along any smooth curve (Figure 13.2.9). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0\cdot\infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.



$z = \dfrac{1}{x^2+y^2}$

▲ **Figure 13.2.9**

▶ Example 7 Find $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

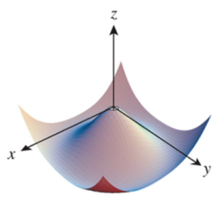
Solution. Let (r,θ) be polar coordinates of the point (x,y) with $r\ge 0$. Then we have

$$x = r\cos\theta,\quad y = r\sin\theta,\quad r^2 = x^2+y^2$$

Moreover, since $r\ge 0$ we have $r=\sqrt{x^2+y^2}$, so that $r\to 0^+$ if and only if $(x,y)\to(0,0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2\ln r^2\\ &= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} && \boxed{\text{This converts the limit to an indeterminate form of type } \infty/\infty.}\\ &= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} && \boxed{\text{L'Hôpital's rule}}\\ &= \lim_{r\to 0^+} (-r^2) = 0 \quad ◀\end{aligned}$$

REMARK The graph of $f(x,y)=(x^2+y^2)\ln(x^2+y^2)$ in Example 7 is a surface with a hole (sometimes called a *puncture*) at the origin (Figure 13.2.10). We can remove this discontinuity by *defining* $f(0,0)$ to be 0. (See Exercises 39 and 40, which also deal with the notion of a "removable" discontinuity.)



$z = (x^2+y^2)\ln(x^2+y^2)$

▲ **Figure 13.2.10**

CONTINUITY AT BOUNDARY POINTS

Recall that in our study of continuity for functions of one variable, we first defined continuity at a point, then continuity on an open interval, and then, by using one-sided limits, we extended the notion of continuity to include endpoints of the interval. Similarly, for functions of two variables one can extend the notion of continuity of $f(x,y)$ to the boundary of its domain by modifying Definition 13.2.1 appropriately so that (x,y) is restricted to approach (x_0,y_0) through points lying wholly in the domain of f. We will omit the details.

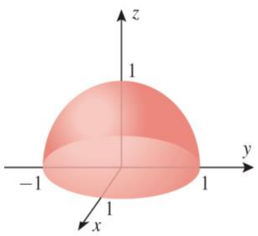
▶ Example 8 The graph of the function $f(x,y)=\sqrt{1-x^2-y^2}$ is the upper hemisphere shown in Figure 13.2.11, and the natural domain of f is the closed unit disk

$$x^2+y^2\le 1$$

The graph of f has no jumps, tears or holes, so it passes our "intuitive test" of continuity. In this case the continuity at a point (x_0,y_0) on the boundary reflects the fact that

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt{1-x^2-y^2} = \sqrt{1-x_0^2-y_0^2} = 0$$

when (x,y) is restricted to points on the closed unit disk $x^2+y^2\le 1$. It follows that f is continuous on its domain. ◀



$z = \sqrt{1-x^2-y^2}$

▲ **Figure 13.2.11**