14.5 TRIPLE INTEGRALS

In the preceding sections we defined and discussed properties of double integrals for functions of two variables. In this section we will define triple integrals for functions of three variables.

$$\iiint_G f(x, y, z)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k \tag{1}$$

is called the ***triple integral*** of $f(x, y, z)$ over the region G. Conditions under which the triple integral exists are studied in advanced calculus. However, for our purposes it suffices to say that existence is ensured when f is continuous on G and the region G is not too "complicated."

EVALUATING TRIPLE INTEGRALS OVER RECTANGULAR BOXES

Just as a double integral can be evaluated by two successive single integrations, so a triple integral can be evaluated by three successive integrations. The following theorem, which we state without proof, is the analog of Theorem 14.1.3.

There are two possible orders of integration for the iterated integrals in Theorem 14.1.3:

$$dx\,dy, \quad dy\,dx$$

Six orders of integration are possible for the iterated integral in Theorem 14.5.1:

$$dx\,dy\,dz, \quad dy\,dz\,dx, \quad dz\,dx\,dy$$
$$dx\,dz\,dy, \quad dz\,dy\,dx, \quad dy\,dx\,dz$$

14.5.1 THEOREM (***Fubini's Theorem****) *Let G be the rectangular box defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d, \quad k \le z \le l$$

If f is continuous on the region G, then

$$\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^l f(x, y, z)\, dz\, dy\, dx \tag{2}$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

▶ Example 1 Evaluate the triple integral

$$\iiint_G 12xy^2z^3\,dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2$.

Solution. Of the six possible iterated integrals we might use, we will choose the one in (2). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\begin{aligned}\iiint_G 12xy^2z^3\,dV &= \int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3\,dz\,dy\,dx \\ &= \int_{-1}^{2}\int_{0}^{3} \left[3xy^2z^4\right]_{z=0}^{2}\,dy\,dx = \int_{-1}^{2}\int_{0}^{3} 48xy^2\,dy\,dx \\ &= \int_{-1}^{2} \left[16xy^3\right]_{y=0}^{3}\,dx = \int_{-1}^{2} 432x\,dx \\ &= 216x^2\Big]_{-1}^{2} = 648 \blacktriangleleft\end{aligned}$$

EXERCISE SET 14.5   C CAS

1–8 Evaluate the iterated integral.

1. $\displaystyle\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1} (x^2+y^2+z^2)\,dx\,dy\,dz$

2. $\displaystyle\int_{1/3}^{1/2}\int_{0}^{\pi}\int_{0}^{1} zx\sin xy\,dz\,dy\,dx$

3. $\displaystyle\int_{0}^{2}\int_{-1}^{y^2}\int_{-1}^{z} yz\,dx\,dz\,dy$

4. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{1}\int_{0}^{x^2} x\cos y\,dz\,dx\,dy$

5. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\int_{0}^{x} xy\,dy\,dx\,dz$

6. $\displaystyle\int_{1}^{3}\int_{x}^{x^2}\int_{0}^{\ln z} xe^y\,dy\,dz\,dx$

7. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx$

8. $\displaystyle\int_{1}^{2}\int_{z}^{2}\int_{0}^{\sqrt{3}y} \frac{y}{x^2+y^2}\,dx\,dy\,dz$