15.2 LINE INTEGRALS

In earlier chapters we considered three kinds of integrals in rectangular coordinates: single integrals over intervals, double integrals over two-dimensional regions, and triple integrals over three-dimensional regions. In this section we will discuss integrals along curves in two- or three-dimensional space.

LINE INTEGRALS

The first goal of this section is to define what it means to integrate a function along a curve. To motivate the definition we will consider the problem of finding the mass of a very thin wire whose linear density function (mass per unit length) is known. We assume that we can model the wire by a smooth curve C between two points P and Q in 3-space (Figure 15.2.1). Given any point (x, y, z) on C, we let $f(x, y, z)$ denote the corresponding value of the density function. To compute the mass of the wire, we proceed as follows:

▲ **Figure 15.2.1** A bent thin wire modeled by a smooth curve

15.2.1 DEFINITION If C is a smooth curve in 2-space or 3-space, then the ***line integral of f with respect to s along C*** is

$$\int_C f(x, y)\, ds = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k \qquad \text{2-space} \qquad (3)$$

or

$$\int_C f(x, y, z)\, ds = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k \qquad \text{3-space} \qquad (4)$$

provided this limit exists and does not depend on the choice of partition or on the choice of sample points.

Although the term "curve integrals" is more descriptive, the integrals in Definition 15.2.1 are called "line integrals" for historical reasons.

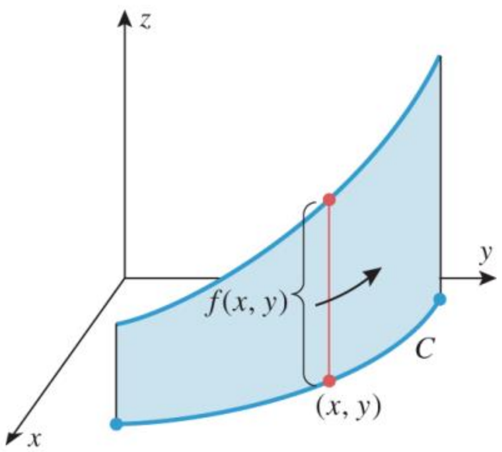
It is usually impractical to evaluate line integrals directly from Definition 15.2.1. However, the definition is important in the application and interpretation of line integrals. For example:

- If C is a curve in 3-space that models a thin wire, and if $f(x, y, z)$ is the linear density function of the wire, then it follows from (2) and Definition 15.2.1 that the mass M of the wire is given by

$$M = \int_C f(x, y, z)\, ds \tag{5}$$

  That is, to obtain the mass of a thin wire, we integrate the linear density function over the smooth curve that models the wire.

- If C is a smooth curve of arc length L, and f is identically 1, then it immediately follows from Definition 15.2.1 that



▲ **Figure 15.2.3**

EVALUATING LINE INTEGRALS

Except in simple cases, it will not be feasible to evaluate a line integral directly from (3) or (4). However, we will now show that it is possible to express a line integral as an ordinary definite integral, so that no special methods of evaluation are required. For example, suppose that C is a curve in the xy-plane that is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad (a \le t \le b)$$

Therefore, if C is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad (a \le t \le b)$$

then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\|\mathbf{r}'(t)\|\, dt \tag{9}$$

Similarly, if C is a curve in 3-space that is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad (a \le t \le b)$$

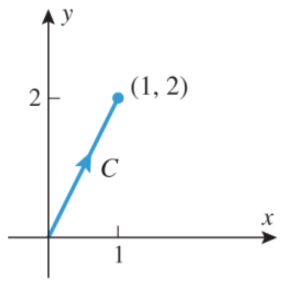
then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\|\mathbf{r}'(t)\|\, dt$$

▶ **Example 1** Using the given parametrization, evaluate the line integral $\int_C (1 + xy^2)\, ds$.

(a) $C : \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} \quad (0 \le t \le 1)$ (see Figure 15.2.6*a*)

(b) $C : \mathbf{r}(t) = (1 - t)\mathbf{i} + (2 - 2t)\mathbf{j} \quad (0 \le t \le 1)$ (see Figure 15.2.6*b*)



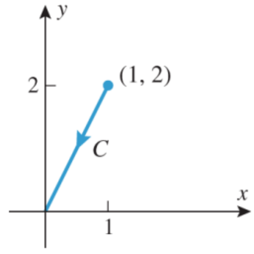
(*a*)

Solution (a). Since $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}$, we have $\|\mathbf{r}'(t)\| = \sqrt{5}$ and it follows from Formula (9) that

$$\begin{aligned}\int_C (1 + xy^2)\, ds &= \int_0^1 [1 + t(2t)^2]\sqrt{5}\, dt \\ &= \int_0^1 (1 + 4t^3)\sqrt{5}\, dt \\ &= \sqrt{5}\left[t + t^4\right]_0^1 = 2\sqrt{5}\end{aligned}$$

Solution (*b*). Since $\mathbf{r}'(t) = -\mathbf{i} - 2\mathbf{j}$, we have $\|\mathbf{r}'(t)\| = \sqrt{5}$ and it follows from Formula (9) that

$$\int_C (1 + xy^2)\, ds = \int_0^1 [1 + (1-t)(2-2t)^2]\sqrt{5}\, dt$$
$$= \int_0^1 [1 + 4(1-t)^3]\sqrt{5}\, dt$$
$$= \sqrt{5}\left[t - (1-t)^4\right]_0^1 = 2\sqrt{5}$$



(*b*)

Note that the integrals in parts (a) and (b) of Example 1 agree, even though the corresponding parametrizations of C have opposite orientations. This illustrates the important result that the value of a line integral of f with respect to s along C *does not depend on an orientation of* C. (This is because Δs_k is always positive; therefore, it does not matter in which *direction* along C we list the partition points of the curve in Definition 15.2.1.) Later in this section we will discuss line integrals that are defined only for oriented curves.

Formula (9) has an alternative expression for a curve C in the xy-plane that is given by parametric equations

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

In this case, we write (9) in the expanded form

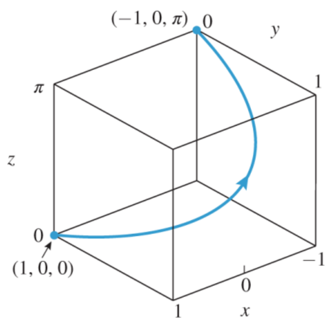
$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{11}$$

Similarly, if C is a curve in 3-space that is parametrized by

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (a \le t \le b)$$

then we write (10) in the form

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{12}$$



▲ **Figure 15.2.7**

▶ **Example 2** Evaluate the line integral $\int_C (xy + z^3)\, ds$ from $(1, 0, 0)$ to $(-1, 0, \pi)$ along the helix C that is represented by the parametric equations

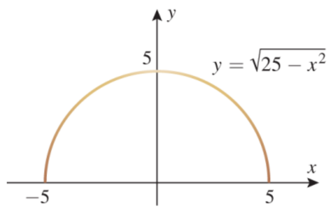
$$x = \cos t, \quad y = \sin t, \quad z = t \qquad (0 \le t \le \pi)$$

(Figure 15.2.7).

Solution. From (12)

$$\begin{aligned}
\int_C (xy + z^3)\, ds &= \int_0^\pi (\cos t \sin t + t^3)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^\pi (\cos t \sin t + t^3)\sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt \\
&= \sqrt{2}\int_0^\pi (\cos t \sin t + t^3)\, dt \\
&= \sqrt{2}\left[\frac{\sin^2 t}{2} + \frac{t^4}{4}\right]_0^\pi = \frac{\sqrt{2}\pi^4}{4} \blacktriangleleft
\end{aligned}$$

If $\delta(x, y)$ is the linear density function of a wire that is modeled by a smooth curve C in the xy-plane, then an argument similar to the derivation of Formula (5) shows that the mass of the wire is given by $\int_C \delta(x, y)\, ds$.



▲ **Figure 15.2.8**

▶ **Example 3** Suppose that a semicircular wire has the equation $y = \sqrt{25 - x^2}$ and that its mass density is $\delta(x, y) = 15 - y$ (Figure 15.2.8). Physically, this means the wire has a maximum density of 15 units at the base $(y = 0)$ and that the density of the wire decreases linearly with respect to y to a value of 10 units at the top $(y = 5)$. Find the mass of the wire.

Solution. The mass M of the wire can be expressed as the line integral

$$M = \int_C \delta(x, y)\, ds = \int_C (15 - y)\, ds$$

along the semicircle C. To evaluate this integral we will express C parametrically as

$$x = 5\cos t, \quad y = 5\sin t \qquad (0 \le t \le \pi)$$

Thus, it follows from (11) that

$$\begin{aligned}
M = \int_C (15 - y)\, ds &= \int_0^\pi (15 - 5\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^\pi (15 - 5\sin t)\sqrt{(-5\sin t)^2 + (5\cos t)^2}\, dt \\
&= 5\int_0^\pi (15 - 5\sin t)\, dt \\
&= 5\left[15t + 5\cos t\right]_0^\pi \\
&= 75\pi - 50 \approx 185.6 \text{ units of mass} \blacktriangleleft
\end{aligned}$$

LINE INTEGRALS WITH RESPECT TO *x*, *y*, AND *z*

The basic procedure for evaluating these line integrals is to find parametric equations for C, say

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (a \le t \le b)$$

in which the orientation of C is in the direction of increasing t, and then express the integrand in terms of t. For example,

$$\int_C f(x, y, z)\, dz = \int_a^b f(x(t), y(t), z(t))z'(t)\, dt$$

[Such a formula is easy to remember—just substitute for x, y, and z using the parametric equations and recall that $dz = z'(t)\, dt$.]

▶ **Example 5** Evaluate $\int_C 3xy\,dy$, where C is the line segment joining $(0,0)$ and $(1,2)$ with the given orientation.

(a) Oriented from $(0,0)$ to $(1,2)$ as in Figure 15.2.6a.

(b) Oriented from $(1,2)$ to $(0,0)$ as in Figure 15.2.6b.

Solution (a). Using the parametrization

$$x = t, \quad y = 2t \qquad (0 \le t \le 1)$$

we have

$$\int_C 3xy\,dy = \int_0^1 3(t)(2t)(2)\,dt = \int_0^1 12t^2\,dt = 4t^3\Big]_0^1 = 4$$

Solution (b). Using the parametrization

$$x = 1 - t, \quad y = 2 - 2t \qquad (0 \le t \le 1)$$

we have

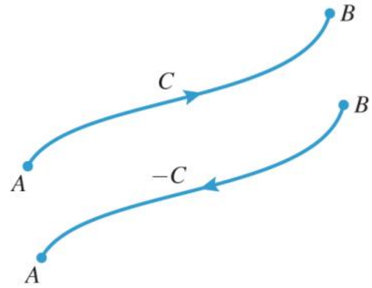
$$\int_C 3xy\,dy = \int_0^1 3(1-t)(2-2t)(-2)\,dt = \int_0^1 -12(1-t)^2\,dt = 4(1-t)^3\Big]_0^1 = -4$$ ◀

In Example 5, note that reversing the orientation of the curve changed the sign of the line integral. This is because reversing the orientation of a curve changes the sign of Δx_k in definition (16). Thus, unlike line integrals of functions with respect to s along C, reversing the orientation of C changes the sign of a line integral with respect to x, y, and z. If C is a smooth oriented curve, we will let $-C$ denote the oriented curve consisting of the same points as C but with the opposite orientation (Figure 15.2.10). We then have

$$\boxed{\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx} \quad \text{and} \quad \boxed{\int_{-C} g(x,y)\,dy = -\int_C g(x,y)\,dy} \qquad (18\text{–}19)$$

while

$$\boxed{\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds} \qquad (20)$$



▲ **Figure 15.2.10**

▶ Example 6 Evaluate

$$\int_C 2xy\,dx + (x^2 + y^2)\,dy$$

along the circular arc C given by $x = \cos t, y = \sin t\,(0 \le t \le \pi/2)$ (Figure 15.2.11).

Solution. We have

$$\int_C 2xy\,dx = \int_0^{\pi/2} (2\cos t\,\sin t)\left[\frac{d}{dt}(\cos t)\right]\,dt$$

$$= -2\int_0^{\pi/2} \sin^2 t\cos t\,dt = -\frac{2}{3}\sin^3 t\Big]_0^{\pi/2} = -\frac{2}{3}$$

$$\int_C (x^2 + y^2)\,dy = \int_0^{\pi/2} (\cos^2 t + \sin^2 t)\left[\frac{d}{dt}(\sin t)\right]\,dt$$

$$= \int_0^{\pi/2} \cos t\,dt = \sin t\Big]_0^{\pi/2} = 1$$

Thus, from (21)

$$\int_C 2xy\,dx + (x^2 + y^2)\,dy = \int_C 2xy\,dx + \int_C (x^2 + y^2)\,dy$$

$$= -\frac{2}{3} + 1 = \frac{1}{3} \blacktriangleleft$$

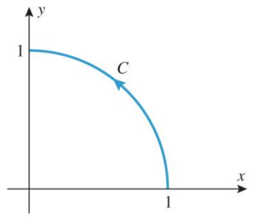


Figure 15.2.11

▶ Example 7 Evaluate

$$\int_C (3x^2 + y^2)\,dx + 2xy\,dy$$

along the circular arc C given by $x = \cos t, y = \sin t\,(0 \le t \le \pi/2)$ (Figure 15.2.11).

Solution. From (23) we have

$$\int_C (3x^2 + y^2)\,dx + 2xy\,dy = \int_0^{\pi/2} [(3\cos^2 t + \sin^2 t)(-\sin t) + 2(\cos t)(\sin t)(\cos t)]\,dt$$

$$= \int_0^{\pi/2} (-3\cos^2 t\sin t - \sin^3 t + 2\cos^2 t\sin t)\,dt$$

$$= \int_0^{\pi/2} (-\cos^2 t - \sin^2 t)(\sin t)\,dt = \int_0^{\pi/2} -\sin t\,dt$$

$$= \cos t\Big]_0^{\pi/2} = -1 \blacktriangleleft$$

INTEGRATING A VECTOR FIELD ALONG A CURVE

15.2.2 DEFINITION If $\mathbf{F}$ is a continuous vector field and C is a smooth oriented curve, then the ***line integral of* F *along C*** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} \tag{28}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

▶ Example 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \cos x\mathbf{i} + \sin x\mathbf{j}$ and where C is the given oriented curve.

(a) $C : \mathbf{r}(t) = -\frac{\pi}{2}\mathbf{i} + t\mathbf{j} \quad (1 \le t \le 2)$ (see Figure 15.2.12a)

(b) $C : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \quad (-1 \le t \le 2)$ (see Figure 15.2.12b)

Solution (a). Using (29) we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_1^2 (-\mathbf{j}) \cdot \mathbf{j}\, dt = \int_1^2 (-1)\, dt = -1$$

Solution (b). Using (29) we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_{-1}^2 (\cos t\mathbf{i} + \sin t\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})\, dt \\
&= \int_{-1}^2 (\cos t + 2t \sin t)\, dt = (-2t \cos t + 3 \sin t)\Big]_{-1}^{2} \\
&= -2 \cos 1 - 4 \cos 2 + 3(\sin 1 + \sin 2) \approx 5.83629 \blacktriangleleft
\end{aligned}$$

WORK AS A LINE INTEGRAL

15.2.3 DEFINITION Suppose that under the influence of a continuous force field $\mathbf{F}$ a particle moves along a smooth curve C and that C is oriented in the direction of motion of the particle. Then the ***work performed by the force field*** on the particle is

$$\int_C \mathbf{F} \cdot d\mathbf{r} \tag{34}$$

EXERCISE SET 15.2 Graphing Utility C CAS

7–10 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the line segment C from P to Q.

7. $\mathbf{F}(x, y) = 8\mathbf{i} + 8\mathbf{j}$; $P(-4, 4), Q(-4, 5)$

8. $\mathbf{F}(x, y) = 2\mathbf{i} + 5\mathbf{j}$; $P(1, -3), Q(4, -3)$

9. $\mathbf{F}(x, y) = 2x\mathbf{j}$; $P(-2, 4), Q(-2, 11)$

10. $\mathbf{F}(x, y) = -8x\mathbf{i} + 3y\mathbf{j}$; $P(-1, 0), Q(6, 0)$

11. Let C be the curve represented by the equations

$$x = 2t, \quad y = t^2 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\displaystyle\int_C (x - \sqrt{y})\, ds$ (b) $\displaystyle\int_C (x - \sqrt{y})\, dx$

(c) $\displaystyle\int_C (x - \sqrt{y})\, dy$

12. Let C be the curve represented by the equations

$$x = t, \quad y = 3t^2, \quad z = 6t^3 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\displaystyle\int_C xyz^2\, ds$ (b) $\displaystyle\int_C xyz^2\, dx$

(c) $\displaystyle\int_C xyz^2\, dy$ (d) $\displaystyle\int_C xyz^2\, dz$

13. In each part, evaluate the integral

$$\int_C (3x + 2y)\, dx + (2x - y)\, dy$$

along the stated curve.

(a) The line segment from $(0, 0)$ to $(1, 1)$.
(b) The parabolic arc $y = x^2$ from $(0, 0)$ to $(1, 1)$.
(c) The curve $y = \sin(\pi x/2)$ from $(0, 0)$ to $(1, 1)$.
(d) The curve $x = y^3$ from $(0, 0)$ to $(1, 1)$.

14. In each part, evaluate the integral

$$\int_C y\, dx + z\, dy - x\, dz$$

along the stated curve.

(a) The line segment from $(0, 0, 0)$ to $(1, 1, 1)$.
(b) The twisted cubic $x = t$, $y = t^2$, $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$.
(c) The helix $x = \cos \pi t$, $y = \sin \pi t$, $z = t$ from $(1, 0, 0)$ to $(-1, 0, 1)$.

19–22 Evaluate the line integral with respect to s along the curve C.

19. $\displaystyle\int_C \frac{1}{1+x}\,ds$

$C: \mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{j} \quad (0 \le t \le 3)$

20. $\displaystyle\int_C \frac{x}{1+y^2}\,ds$

$C: x = 1 + 2t,\ y = t \quad (0 \le t \le 1)$

21. $\displaystyle\int_C 3x^2yz\,ds$

$C: x = t,\ y = t^2,\ z = \frac{2}{3}t^3 \quad (0 \le t \le 1)$

22. $\displaystyle\int_C \frac{e^{-z}}{x^2+y^2}\,ds$

$C: \mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k} \quad (0 \le t \le 2\pi)$