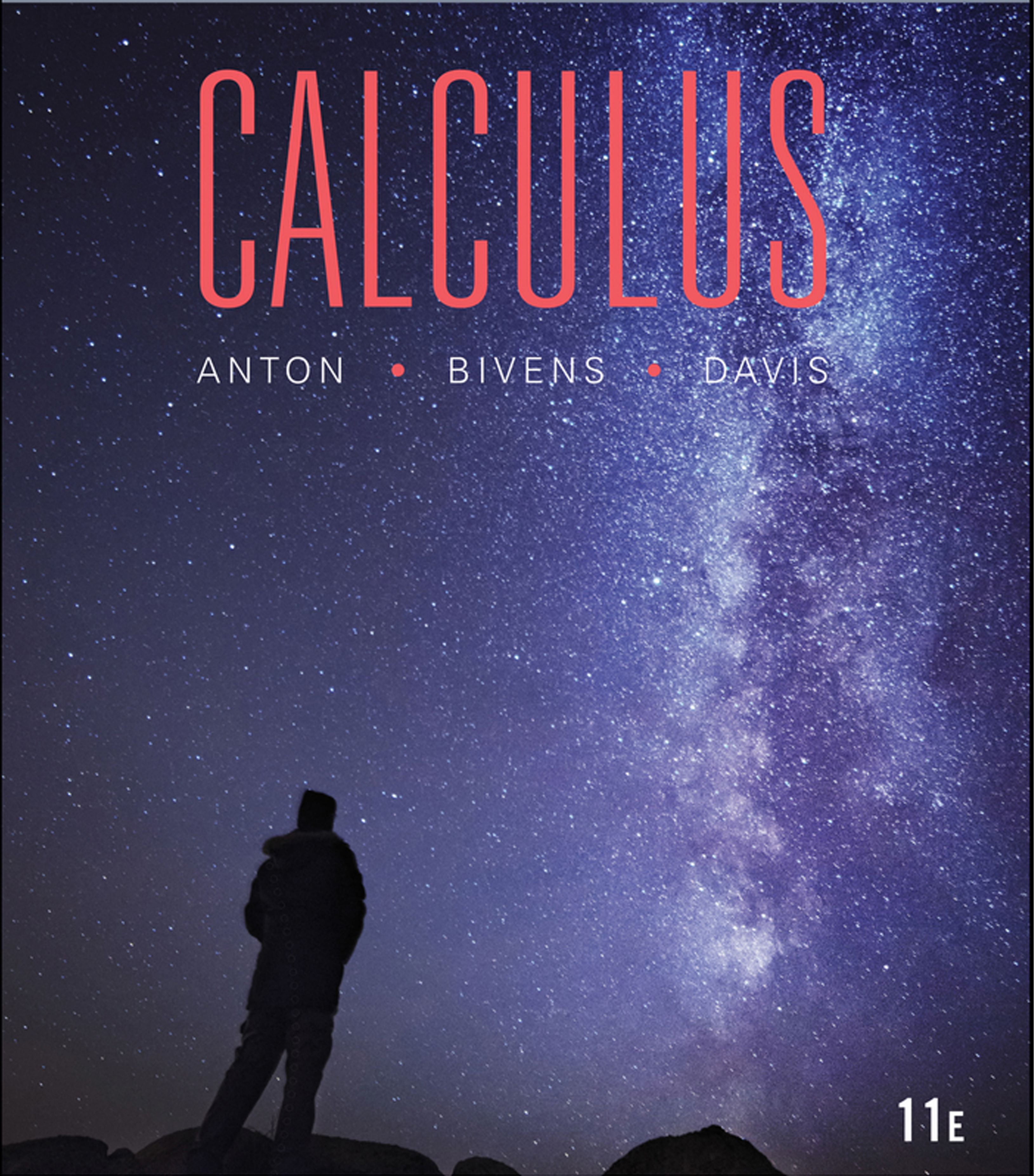
CALCULUS

ANTON • BIVENS • DAVIS



11E

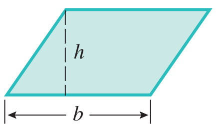
Wiley Binder Version

WILEY

GEOMETRY FORMULAS

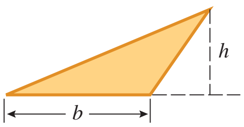
A = area, S = lateral surface area, V = volume, h = height, B = area of base, r = radius, l = slant height, C = circumference, s = arc length

Parallelogram



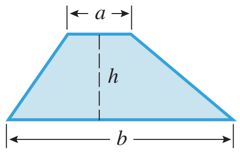
$A = bh$

Triangle



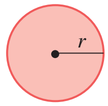
$A = \frac{1}{2}bh$

Trapezoid



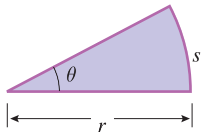
$A = \frac{1}{2}(a+b)h$

Circle



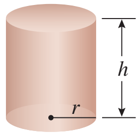
$A = \pi r^2$, $C = 2\pi r$

Sector



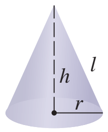
$A = \frac{1}{2}r^2\theta$, $s = r\theta$
(θ in radians)

Right Circular Cylinder



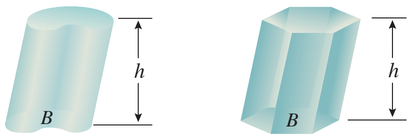
$V = \pi r^2 h$, $S = 2\pi rh$

Right Circular Cone



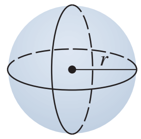
$V = \frac{1}{3}\pi r^2 h$, $S = \pi rl$

Any Cylinder or Prism with Parallel Bases



$V = Bh$

Sphere



$V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$

ALGEBRA FORMULAS

THE QUADRATIC FORMULA	THE BINOMIAL FORMULA
The solutions of the quadratic equation $ax^2+bx+c=0$ are $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$	$(x+y)^n = x^n + nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n$ $(x-y)^n = x^n - nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 - \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots \pm nxy^{n-1} \mp y^n$

TABLE OF INTEGRALS

BASIC FUNCTIONS

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$

2. $\displaystyle\int \frac{du}{u} = \ln|u| + C$

3. $\displaystyle\int e^u\,du = e^u + C$

4. $\displaystyle\int \sin u\,du = -\cos u + C$

5. $\displaystyle\int \cos u\,du = \sin u + C$

6. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$

7. $\displaystyle\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

8. $\displaystyle\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

9. $\displaystyle\int \tan^{-1} u\,du = u\tan^{-1} u - \ln\sqrt{1+u^2} + C$

10. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$

11. $\displaystyle\int \ln u\,du = u\ln u - u + C$

12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

13. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln\left|\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}u\right)\right| + C$

14. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C = \ln|\tan\tfrac{1}{2}u| + C$

15. $\displaystyle\int \cot^{-1} u\,du = u\cot^{-1} u + \ln\sqrt{1+u^2} + C$

16. $\displaystyle\int \sec^{-1} u\,du = u\sec^{-1} u - \ln|u + \sqrt{u^2-1}| + C$

17. $\displaystyle\int \csc^{-1} u\,du = u\csc^{-1} u + \ln|u + \sqrt{u^2-1}| + C$

RECIPROCALS OF BASIC FUNCTIONS

18. $\displaystyle\int \frac{1}{1 \pm \sin u}\, du = \tan u \mp \sec u + C$

19. $\displaystyle\int \frac{1}{1 \pm \cos u}\, du = -\cot u \pm \csc u + C$

20. $\displaystyle\int \frac{1}{1 \pm \tan u}\, du = \tfrac{1}{2}(u \pm \ln|\cos u \pm \sin u|) + C$

21. $\displaystyle\int \frac{1}{\sin u \cos u}\, du = \ln|\tan u| + C$

22. $\displaystyle\int \frac{1}{1 \pm \cot u}\, du = \tfrac{1}{2}(u \mp \ln|\sin u \pm \cos u|) + C$

23. $\displaystyle\int \frac{1}{1 \pm \sec u}\, du = u + \cot u \mp \csc u + C$

24. $\displaystyle\int \frac{1}{1 \pm \csc u}\, du = u - \tan u \pm \sec u + C$

25. $\displaystyle\int \frac{1}{1 \pm e^u}\, du = u - \ln(1 \pm e^u) + C$

POWERS OF TRIGONOMETRIC FUNCTIONS

26. $\displaystyle\int \sin^2 u\, du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$

27. $\displaystyle\int \cos^2 u\, du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$

28. $\displaystyle\int \tan^2 u\, du = \tan u - u + C$

29. $\displaystyle\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

30. $\displaystyle\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

31. $\displaystyle\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

32. $\displaystyle\int \cot^2 u\, du = -\cot u - u + C$

33. $\displaystyle\int \sec^2 u\, du = \tan u + C$

34. $\displaystyle\int \csc^2 u\, du = -\cot u + C$

35. $\displaystyle\int \cot^n u\, du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du$

36. $\displaystyle\int \sec^n u\, du = \frac{1}{n-1}\sec^{n-2} u \tan u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$

37. $\displaystyle\int \csc^n u\, du = -\frac{1}{n-1}\csc^{n-2} u \cot u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du$

PRODUCTS OF TRIGONOMETRIC FUNCTIONS

38. $\displaystyle\int \sin mu \sin nu\, du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

39. $\displaystyle\int \cos mu \cos nu\, du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

40. $\displaystyle\int \sin mu \cos nu\, du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)} + C$

41. $\displaystyle\int \sin^m u \cos^n u\, du = -\frac{\sin^{m-1} u \cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} u \cos^n u\, du$

$$= \frac{\sin^{m+1} u \cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \sin^m u \cos^{n-2} u\, du$$

PRODUCTS OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

42. $\displaystyle\int e^{au} \sin bu\, du = \frac{e^{au}}{a^2+b^2}(a \sin bu - b \cos bu) + C$

43. $\displaystyle\int e^{au} \cos bu\, du = \frac{e^{au}}{a^2+b^2}(a \cos bu + b \sin bu) + C$

POWERS OF *u* MULTIPLYING OR DIVIDING BASIC FUNCTIONS

44. $\displaystyle\int u \sin u\, du = \sin u - u \cos u + C$

45. $\displaystyle\int u \cos u\, du = \cos u + u \sin u + C$

46. $\displaystyle\int u^2 \sin u\, du = 2u \sin u + (2 - u^2)\cos u + C$

47. $\displaystyle\int u^2 \cos u\, du = 2u \cos u + (u^2 - 2)\sin u + C$

48. $\displaystyle\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1} \cos u\, du$

49. $\displaystyle\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1} \sin u\, du$

50. $\displaystyle\int u^n \ln u\, du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

51. $\displaystyle\int u e^u\, du = e^u(u-1) + C$

52. $\displaystyle\int u^n e^u\, du = u^n e^u - n\int u^{n-1} e^u\, du$

53. $\displaystyle\int u^n a^u\, du = \frac{u^n a^u}{\ln a} - \frac{n}{\ln a}\int u^{n-1} a^u\, du + C$

54. $\displaystyle\int \frac{e^u\, du}{u^n} = -\frac{e^u}{(n-1)u^{n-1}} + \frac{1}{n-1}\int \frac{e^u\, du}{u^{n-1}}$

55. $\displaystyle\int \frac{a^u\, du}{u^n} = -\frac{a^u}{(n-1)u^{n-1}} + \frac{\ln a}{n-1}\int \frac{a^u\, du}{u^{n-1}}$

56. $\displaystyle\int \frac{du}{u \ln u} = \ln|\ln u| + C$

POLYNOMIALS MULTIPLYING BASIC FUNCTIONS

57. $\displaystyle\int p(u)e^{au}\, du = \frac{1}{a}p(u)e^{au} - \frac{1}{a^2}p'(u)e^{au} + \frac{1}{a^3}p''(u)e^{au} - \cdots$ [signs alternate: $+ - + - \cdots$]

58. $\displaystyle\int p(u)\sin au\, du = -\frac{1}{a}p(u)\cos au + \frac{1}{a^2}p'(u)\sin au + \frac{1}{a^3}p''(u)\cos au - \cdots$ [signs alternate in pairs after first term: $+ + - - + + - - \cdots$]

59. $\displaystyle\int p(u)\cos au\, du = \frac{1}{a}p(u)\sin au + \frac{1}{a^2}p'(u)\cos au - \frac{1}{a^3}p''(u)\sin au - \cdots$ [signs alternate in pairs: $+ + - - + + - - \cdots$]

RATIONAL FUNCTIONS CONTAINING POWERS OF $a + bu$ IN THE DENOMINATOR

60. $$\int \frac{u\,du}{a+bu} = \frac{1}{b^2}[bu - a\ln|a+bu|] + C$$

61. $$\int \frac{u^2\,du}{a+bu} = \frac{1}{b^3}\left[\frac{1}{2}(a+bu)^2 - 2a(a+bu) + a^2\ln|a+bu|\right] + C$$

62. $$\int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left[\frac{a}{a+bu} + \ln|a+bu|\right] + C$$

63. $$\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left[bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right] + C$$

64. $$\int \frac{u\,du}{(a+bu)^3} = \frac{1}{b^2}\left[\frac{a}{2(a+bu)^2} - \frac{1}{a+bu}\right] + C$$

65. $$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

66. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

67. $$\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} + \frac{1}{a^2}\ln\left|\frac{u}{a+bu}\right| + C$$

RATIONAL FUNCTIONS CONTAINING $a^2 \pm u^2$ IN THE DENOMINATOR $(a > 0)$

68. $$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

69. $$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$$

70. $$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

71. $$\int \frac{bu+c}{a^2+u^2}\,du = \frac{b}{2}\ln(a^2+u^2) + \frac{c}{a}\tan^{-1}\frac{u}{a} + C$$

INTEGRALS OF $\sqrt{a^2+u^2}$, $\sqrt{a^2-u^2}$, $\sqrt{u^2-a^2}$ AND THEIR RECIPROCALS $(a > 0)$

72. $$\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln(u+\sqrt{u^2+a^2}) + C$$

73. $$\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$$

74. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

75. $$\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u+\sqrt{u^2+a^2}) + C$$

76. $$\int \frac{du}{\sqrt{u^2-a^2}} = \ln|u+\sqrt{u^2-a^2}| + C$$

77. $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a^2-u^2}$ OR ITS RECIPROCAL

78. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

79. $$\int \frac{\sqrt{a^2-u^2}\,du}{u} = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

80. $$\int \frac{\sqrt{a^2-u^2}\,du}{u^2} = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

81. $$\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

82. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

83. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{u^2 \pm a^2}$ OR THEIR RECIPROCALS

84. $$\int u\sqrt{u^2+a^2}\,du = \frac{1}{3}(u^2+a^2)^{3/2} + C$$

85. $$\int u\sqrt{u^2-a^2}\,du = \frac{1}{3}(u^2-a^2)^{3/2} + C$$

86. $$\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$$

87. $$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

88. $$\int \frac{\sqrt{u^2-a^2}\,du}{u} = \sqrt{u^2-a^2} - a\sec^{-1}\left|\frac{u}{a}\right| + C$$

89. $$\int \frac{\sqrt{u^2+a^2}\,du}{u} = \sqrt{u^2+a^2} - a\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$$

90. $$\int \frac{du}{u^2\sqrt{u^2\pm a^2}} = \mp\frac{\sqrt{u^2\pm a^2}}{a^2u} + C$$

91. $$\int u^2\sqrt{u^2+a^2}\,du = \frac{u}{8}(2u^2+a^2)\sqrt{u^2+a^2} - \frac{a^4}{8}\ln(u+\sqrt{u^2+a^2}) + C$$

92. $$\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$$

93. $$\int \frac{\sqrt{u^2+a^2}}{u^2}\,du = -\frac{\sqrt{u^2+a^2}}{u} + \ln(u+\sqrt{u^2+a^2}) + C$$

94. $$\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln|u+\sqrt{u^2-a^2}| + C$$

95. $$\int \frac{u^2}{\sqrt{u^2+a^2}}\,du = \frac{u}{2}\sqrt{u^2+a^2} - \frac{a^2}{2}\ln(u+\sqrt{u^2+a^2}) + C$$

96. $$\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$$

INTEGRALS CONTAINING $(a^2+u^2)^{3/2}$, $(a^2-u^2)^{3/2}$, $(u^2-a^2)^{3/2}$ $(a > 0)$

97. $$\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$$

98. $$\int \frac{du}{(u^2\pm a^2)^{3/2}} = \pm\frac{u}{a^2\sqrt{u^2\pm a^2}} + C$$

99. $$\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$

100. $$\int (u^2+a^2)^{3/2}\,du = \frac{u}{8}(2u^2+5a^2)\sqrt{u^2+a^2} + \frac{3a^4}{8}\ln(u+\sqrt{u^2+a^2}) + C$$

101. $$\int (u^2-a^2)^{3/2}\,du = \frac{u}{8}(2u^2-5a^2)\sqrt{u^2-a^2} + \frac{3a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a+bu}$ OR ITS RECIPROCAL

102. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$

103. $\displaystyle\int u^2\sqrt{a+bu}\,du = \frac{2}{105b^3}(15b^2u^2-12abu+8a^2)(a+bu)^{3/2}+C$

104. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2u^n(a+bu)^{3/2}}{b(2n+3)} - \frac{2an}{b(2n+3)}\int u^{n-1}\sqrt{a+bu}\,du$

105. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C$

106. $\displaystyle\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(3b^2u^2-4abu+8a^2)\sqrt{a+bu}+C$

107. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2an}{b(2n+1)}\int\frac{u^{n-1}\,du}{\sqrt{a+bu}}$

108. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right|+C & (a>0) \\[2ex] \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bu}{-a}}+C & (a<0) \end{cases}$

109. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int\frac{du}{u^{n-1}\sqrt{a+bu}}$

110. $\displaystyle\int \frac{\sqrt{a+bu}\,du}{u} = 2\sqrt{a+bu}+a\int\frac{du}{u\sqrt{a+bu}}$

111. $\displaystyle\int \frac{\sqrt{a+bu}\,du}{u^n} = -\frac{(a+bu)^{3/2}}{a(n-1)u^{n-1}} - \frac{b(2n-5)}{2a(n-1)}\int\frac{\sqrt{a+bu}\,du}{u^{n-1}}$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{2au-u^2}$ OR ITS RECIPROCAL

112. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

113. $\displaystyle\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2}+\frac{a^3}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

114. $\displaystyle\int \frac{\sqrt{2au-u^2}\,du}{u} = \sqrt{2au-u^2}+a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

115. $\displaystyle\int \frac{\sqrt{2au-u^2}\,du}{u^2} = -\frac{2\sqrt{2au-u^2}}{u}-\sin^{-1}\left(\frac{u-a}{a}\right)+C$

116. $\displaystyle\int \frac{du}{\sqrt{2au-u^2}} = \sin^{-1}\left(\frac{u-a}{a}\right)+C$

117. $\displaystyle\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au}+C$

118. $\displaystyle\int \frac{u\,du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2}+a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

119. $\displaystyle\int \frac{u^2\,du}{\sqrt{2au-u^2}} = -\frac{(u+3a)}{2}\sqrt{2au-u^2}+\frac{3a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

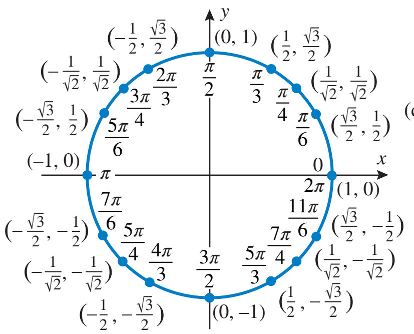
INTEGRALS CONTAINING $(2au-u^2)^{3/2}$

120. $\displaystyle\int \frac{du}{(2au-u^2)^{3/2}} = \frac{u-a}{a^2\sqrt{2au-u^2}}+C$

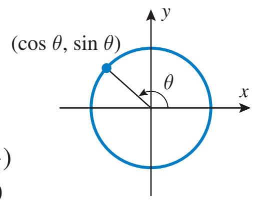
121. $\displaystyle\int \frac{u\,du}{(2au-u^2)^{3/2}} = \frac{u}{a\sqrt{2au-u^2}}+C$

THE WALLIS FORMULA

122. $\displaystyle\int_0^{\pi/2}\sin^n u\,du = \int_0^{\pi/2}\cos^n u\,du = \frac{1\cdot 3\cdot 5\cdot\,\cdots\,\cdot(n-1)}{2\cdot 4\cdot 6\cdot\,\cdots\,\cdot n}\cdot\frac{\pi}{2}\quad\left(\begin{array}{l} n\text{ an even} \\ \text{integer and} \\ n\geq 2\end{array}\right)$ or $\displaystyle\frac{2\cdot 4\cdot 6\cdot\,\cdots\,\cdot(n-1)}{3\cdot 5\cdot 7\cdot\,\cdots\,\cdot n}\quad\left(\begin{array}{l} n\text{ an odd} \\ \text{integer and} \\ n\geq 3\end{array}\right)$



TRIGONOMETRY REVIEW



PYTHAGOREAN IDENTITIES

$\sin^2\theta+\cos^2\theta=1 \qquad \tan^2\theta+1=\sec^2\theta \qquad 1+\cot^2\theta=\csc^2\theta$

SIGN IDENTITIES

$\sin(-\theta)=-\sin\theta \qquad \cos(-\theta)=\cos\theta \qquad \tan(-\theta)=-\tan\theta$

$\csc(-\theta)=-\csc\theta \qquad \sec(-\theta)=\sec\theta \qquad \cot(-\theta)=-\cot\theta$

COMPLEMENT IDENTITIES

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \quad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \quad \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$

$\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta \quad \sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta \quad \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$

SUPPLEMENT IDENTITIES

$\sin(\pi-\theta)=\sin\theta \qquad \cos(\pi-\theta)=-\cos\theta \qquad \tan(\pi-\theta)=-\tan\theta$

$\csc(\pi-\theta)=\csc\theta \qquad \sec(\pi-\theta)=-\sec\theta \qquad \cot(\pi-\theta)=-\cot\theta$

$\sin(\pi+\theta)=-\sin\theta \qquad \cos(\pi+\theta)=-\cos\theta \qquad \tan(\pi+\theta)=\tan\theta$

$\csc(\pi+\theta)=-\csc\theta \qquad \sec(\pi+\theta)=-\sec\theta \qquad \cot(\pi+\theta)=\cot\theta$

ADDITION FORMULAS

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\displaystyle\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\displaystyle\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

DOUBLE-ANGLE FORMULAS

$\sin 2\alpha=2\sin\alpha\cos\alpha \qquad \cos 2\alpha=2\cos^2\alpha-1$

$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha \qquad \cos 2\alpha=1-2\sin^2\alpha$

HALF-ANGLE FORMULAS

$\displaystyle\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}$

Library of Congress, Geography and Map Division

Three-dimensional surfaces have high points and low points that are analogous to the peaks and valleys of a mountain range. In this chapter we will use derivatives to locate these points and to study other features of such surfaces.

13

PARTIAL DERIVATIVES

In this chapter we will extend many of the basic concepts of calculus to functions of two or more variables, commonly called functions of several variables. These concepts include limits and continuity, differentiability, tangents, rates of change, and extreme values. Although many of these ideas extend in a natural way to functions of several variables, such functions also possess unique features that will suggest the development of some new mathematical concepts and tools.

13.1 FUNCTIONS OF TWO OR MORE VARIABLES

In previous sections we studied real-valued functions of a real variable and vector-valued functions of a real variable. In this section we will consider real-valued functions of two or more real variables.

NOTATION AND TERMINOLOGY

There are many familiar formulas in which a given variable depends on two or more other variables. For example, the area A of a triangle depends on the base length b and height h by the formula $A = \frac{1}{2}bh$; the volume V of a rectangular box depends on the length l, the width w, and the height h by the formula $V = lwh$; and the arithmetic average $\bar{x}$ of n real numbers, $x_1, x_2, \ldots, x_n$, depends on those numbers by the formula

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

Thus, we say that

A is a function of the two variables b and h;

V is a function of the three variables l, w, and h;

$\bar{x}$ is a function of the n variables $x_1, x_2, \ldots, x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example, the expression

$$z = f(x, y)$$

means that z is a function of x and y in the sense that a unique value of the dependent variable z is determined by specifying values for the independent variables x and y. Similarly,

$$w = f(x, y, z)$$

expresses w as a function of x, y, and z, and

$$u = f(x_1, x_2, \ldots, x_n)$$

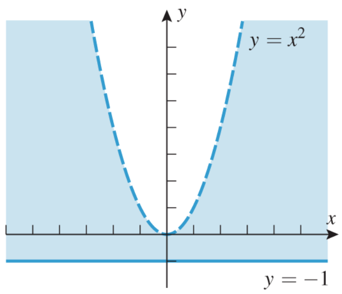
expresses u as a function of $x_1, x_2, \ldots, x_n$.

As with functions of one variable, the independent variables of a function of two or more variables may be restricted to lie in some set D, which we call the ***domain*** of f. Sometimes the domain will be determined by physical restrictions on the variables. If the function is defined by a formula and if there are no physical restrictions or other restrictions stated explicitly, then it is understood that the domain consists of all points for which the formula yields a real value for the dependent variable. We call this the ***natural domain*** of the function. The following definitions summarize this discussion.

By extension, one can define the notion of "n-dimensional space" in which a "point" is a sequence of n real numbers $(x_1, x_2, \ldots, x_n)$, and a function of n real variables is a rule that assigns a unique real number $f(x_1, x_2, \ldots, x_n)$ to each point in some set in this space.

13.1.1 DEFINITION A ***function f of two variables***, x and y, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy-plane.

13.1.2 DEFINITION A ***function f of three variables***, x, y, and z, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D in three-dimensional space.



The solid boundary line is included in the domain, while the dashed boundary is not included in the domain.

Figure 13.1.1

Example 1 Let $f(x, y) = \sqrt{y+1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the natural domain of f.

Solution. By substitution,

$$f(e, 0) = \sqrt{0+1} + \ln(e^2 - 0) = \sqrt{1} + \ln(e^2) = 1 + 2 = 3$$

To find the natural domain of f, we note that $\sqrt{y+1}$ is defined only when $y \geq -1$, while $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. Thus, the natural domain of f consists of all points in the xy-plane for which $-1 \leq y < x^2$. To sketch the natural domain, we first sketch the parabola $y = x^2$ as a "dashed" curve and the line $y = -1$ as a solid curve. The natural domain of f is then the region lying above or on the line $y = -1$ and below the parabola $y = x^2$ (Figure 13.1.1). ◀

Example 2 Let

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

Find $f\left(0, \frac{1}{2}, -\frac{1}{2}\right)$ and the natural domain of f.

Solution. By substitution,

$$f\left(0, \tfrac{1}{2}, -\tfrac{1}{2}\right) = \sqrt{1 - (0)^2 - \left(\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

Because of the square root sign, we must have $0 \leq 1 - x^2 - y^2 - z^2$ in order to have a real value for $f(x, y, z)$. Rewriting this inequality in the form

$$x^2 + y^2 + z^2 \leq 1$$

we see that the natural domain of f consists of all points on or within the sphere

$$x^2 + y^2 + z^2 = 1 \quad ◀$$

FUNCTIONS DESCRIBED BY TABLES

Sometimes it is useful to represent a function of two variables in table form, rather than as an explicit formula. For example, the U.S. National Weather Service uses the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16} \tag{1}$$

to model the wind chill index W (in °F) as a function of the temperature T (in °F) and the wind speed v (in mi/h) for wind speeds greater than 3 mi/h. This formula is sufficiently complex that it is difficult to get an intuitive feel for the relationship between the variables.

The wind chill index is that temperature (in °F) which would produce the same sensation on exposed skin at a wind speed of 3 mi/h as the temperature and wind speed combination in current weather conditions.

Table 13.1.1

WIND SPEED v (mi/h) \ TEMPERATURE T (°F)	20	25	30	35
5	13	19	25	31
15	6	13	19	25
25	3	9	16	23
35	0	7	14	21
45	−2	5	12	19

One can get a clearer sense of the relationship by selecting sample values of T and v and constructing a table, such as Table 13.1.1, in which we have rounded the values of W to the nearest integer. For example, if the temperature is 30°F and the wind speed is 5 mi/h, it feels as if the temperature is 25°F. If the wind speed increases to 15 mi/h, the temperature then feels as if it has dropped to 19°F. Note that in this case, an increase in wind speed of 10 mi/h causes a 6°F decrease in the wind chill index. To estimate wind chill values not displayed in the table, we can use ***linear interpolation***. For example, suppose that the temperature is 30°F and the wind speed is 7 mi/h. A reasonable estimate for the drop in the wind chill index from its value when the wind speed is 5 mi/h would be $\frac{2}{10} \cdot 6°\text{F} = 1.2°\text{F}$. (Why?) The resulting estimate in wind chill would then be $25° - 1.2° = 23.8°\text{F}$.

In some cases, tables for functions of two variables arise directly from experimental data, in which case one must either work directly with the table or else use some technique to construct a formula that models the data in the table. Such modeling techniques are developed in statistics and numerical analysis texts.

GRAPHS OF FUNCTIONS OF TWO VARIABLES

Recall that for a function f of one variable, the graph of $f(x)$ in the xy-plane was defined to be the graph of the equation $y = f(x)$. Similarly, if f is a function of two variables, we define the ***graph*** of $f(x, y)$ in xyz-space to be the graph of the equation $z = f(x, y)$. In general, such a graph will be a surface in 3-space.

Example 3 In each part, describe the graph of the function in an xyz-coordinate system.

(a) $f(x, y) = 1 - x - \frac{1}{2}y$ (b) $f(x, y) = \sqrt{1 - x^2 - y^2}$

(c) $f(x, y) = -\sqrt{x^2 + y^2}$

Solution (a). By definition, the graph of the given function is the graph of the equation

$$z = 1 - x - \tfrac{1}{2}y$$

which is a plane. A triangular portion of the plane can be sketched by plotting the intersections with the coordinate axes and joining them with line segments (Figure 13.1.2*a*).

Solution (b). By definition, the graph of the given function is the graph of the equation

$$z = \sqrt{1 - x^2 - y^2} \tag{2}$$

After squaring both sides, this can be rewritten as

$$x^2 + y^2 + z^2 = 1$$

which represents a sphere of radius 1, centered at the origin. Since (2) imposes the added condition that $z \geq 0$, the graph is just the upper hemisphere (Figure 13.1.2*b*).

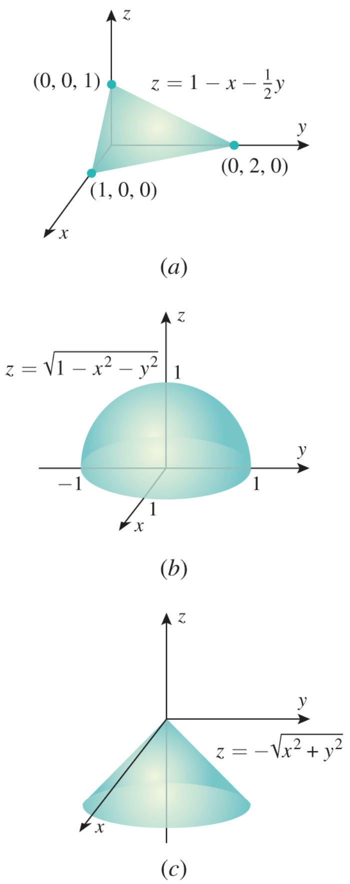
Solution (c). The graph of the given function is the graph of the equation

$$z = -\sqrt{x^2 + y^2} \tag{3}$$

After squaring, we obtain

$$z^2 = x^2 + y^2$$

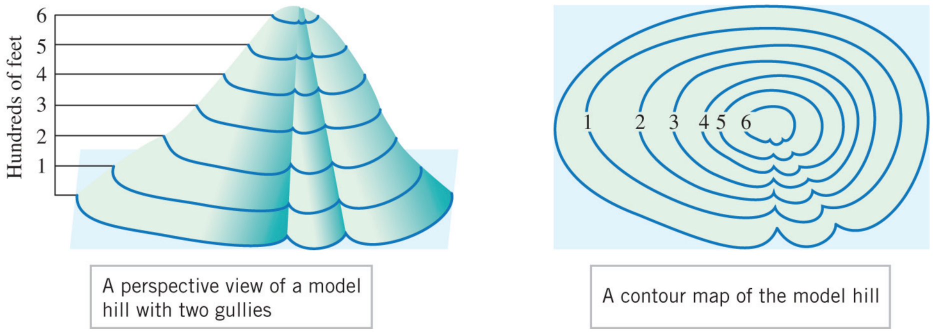
which is the equation of a circular cone (see Table 11.7.1). Since (3) imposes the condition that $z \leq 0$, the graph is just the lower nappe of the cone (Figure 13.1.2*c*). ◄



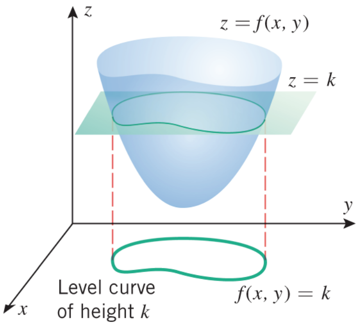
▲ **Figure 13.1.2**

LEVEL CURVES

A topographic (or contour) map represents a three-dimensional landscape, such as a mountain range, by two-dimensional contour lines or curves of constant elevation. Consider, for example, the model hill and its contour map shown in Figure 13.1.3. The contour map is constructed by passing planes of constant elevation through the hill, projecting the resulting contours onto a flat surface, and labeling the contours with their elevations. In Figure 13.1.3, note how the two gullies appear as indentations in the contour lines and how the curves are close together on the contour map where the hill has a steep slope and become more widely spaced where the slope is gradual.



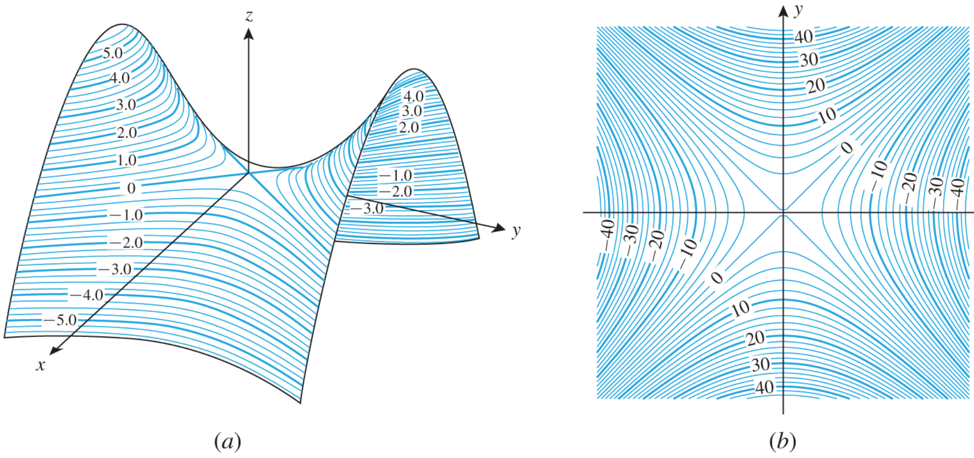
▲ **Figure 13.1.3**



▲ **Figure 13.1.4**

Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the xy-plane is called the ***level curve of height k*** or the ***level curve with constant k*** (Figure 13.1.4). A set of level curves for $z = f(x, y)$ is called a ***contour plot*** or ***contour map*** of f.

▶ Example 4 The graph of the function $f(x, y) = y^2 - x^2$ in xyz-space is the hyperbolic paraboloid (saddle surface) shown in Figure 13.1.5*a*. The level curves have equations of the form $y^2 - x^2 = k$. For $k > 0$ these curves are hyperbolas opening along lines parallel to the y-axis; for $k < 0$ they are hyperbolas opening along lines parallel to the x-axis; and for $k = 0$ the level curve consists of the intersecting lines $y + x = 0$ and $y - x = 0$ (Figure 13.1.5*b*). ◀



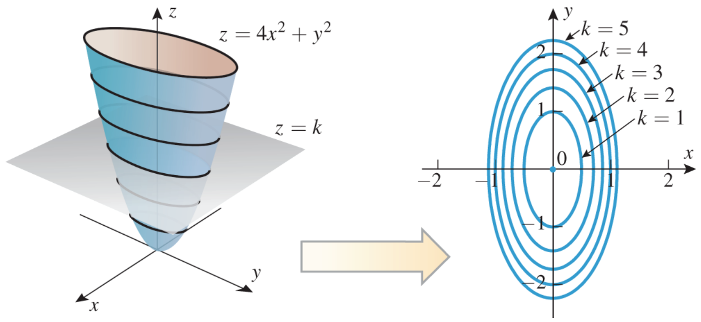
▲ **Figure 13.1.5**

▶ Example 5 Sketch the contour plot of $f(x, y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5$.

Solution. The graph of the surface $z = 4x^2 + y^2$ is the paraboloid shown in the left part of Figure 13.1.6, so we can reasonably expect the contour plot to be a family of ellipses centered at the origin. The level curve of height k has the equation $4x^2 + y^2 = k$. If $k = 0$, then the graph is the single point $(0, 0)$. For $k > 0$ we can rewrite the equation as

$$\frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

which represents a family of ellipses with x-intercepts $\pm\sqrt{k}/2$ and y-intercepts $\pm\sqrt{k}$. The contour plot for the specified values of k is shown in the right part of Figure 13.1.6. ◀



▶ **Figure 13.1.6**

In the last two examples we used a formula for $f(x, y)$ to find the contour plot of f. Conversely, if we are given a contour plot of some function, then we can use the plot to estimate values of the function.

▶ **Example 6** Let $f(r, L)$ be the monthly payment on a 5-year car loan as a function of the interest rate r and the loan amount L. Figure 13.1.7 is a contour plot of $f(r, L)$. Use this plot in each part.

(a) Estimate the monthly payment on a loan of \$3000 at an interest rate of 7%.

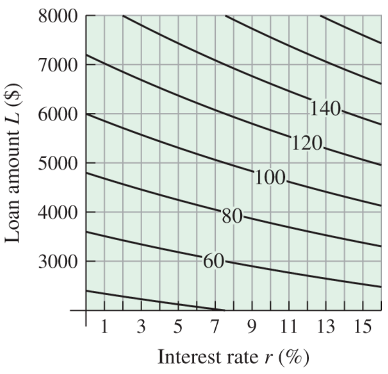
(b) Estimate the monthly payment on a loan of \$5000 at an interest rate of 3%.

(c) Estimate the loan amount if the monthly payment is \$80 and the interest rate is 3%.

Solution (a). Since the point $(7, 3000)$ appears to lie on the contour labeled 60, we estimate the monthly payment to be \$60.

Solution (b). Since the point $(3, 5000)$ appears to be midway between the contours labeled 80 and 100, we estimate the monthly payment to be \$90.

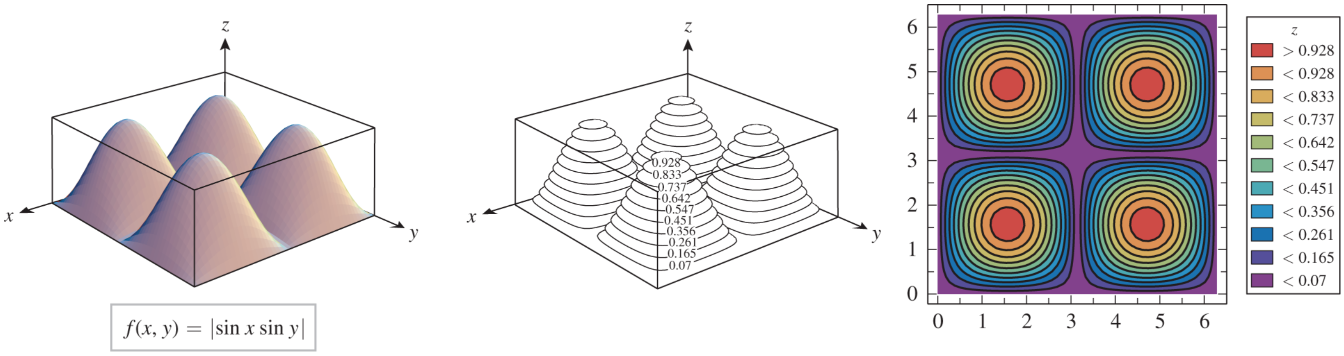
Solution (c). The vertical line $x = 3$ intersects the contour labeled 80 at a point whose L coordinate appears to be 4500. Hence, we estimate the loan amount to be \$4500. ◀



▶ **Figure 13.1.7**

CONTOUR PLOTS USING TECHNOLOGY

Except in the simplest cases, contour plots can be difficult to produce without the help of a graphing utility. Figure 13.1.8 illustrates how graphing technology can be used to display level curves. Figure 13.1.8*a* shows the graph of $f(x, y) = |\sin x \sin y|$ plotted over the domain $0 \le x \le 2\pi$, $0 \le y \le 2\pi$, and Figure 13.1.8*b* displays curves of constant elevation on the graph of f. The projections of these curves into the xy-plane are contours of f. Figure 13.1.8c illustrates how the use of color can enhance the display of these contours.



(a) (b) (c)

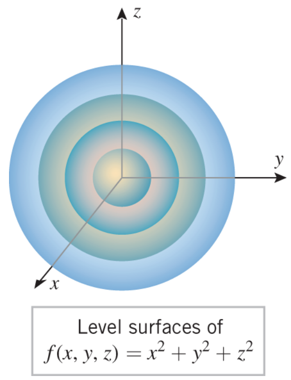
▲ **Figure 13.1.8**

WARNING

A level surface need *not* be level in the sense of being horizontal—it is simply a surface on which all values of f are the same.

LEVEL SURFACES

Observe that the graph of $y = f(x)$ is a curve in 2-space, and the graph of $z = f(x, y)$ is a surface in 3-space, so the number of dimensions required for these graphs is one greater than the number of independent variables. Accordingly, there is no "direct" way to graph a function of three variables since four dimensions are required. However, if k is a constant, then the graph of the equation $f(x, y, z) = k$ will generally be a surface in 3-space (e.g., the graph of $x^2 + y^2 + z^2 = 1$ is a sphere), which we call the ***level surface with constant k***. Some geometric insight into the behavior of the function f can sometimes be obtained by graphing these level surfaces for various values of k.



▲ **Figure 13.1.9**

▶ **Example 7** Describe the level surfaces of

(a) $f(x, y, z) = x^2 + y^2 + z^2$ (b) $f(x, y, z) = z^2 - x^2 - y^2$

Solution (a). The level surfaces have equations of the form

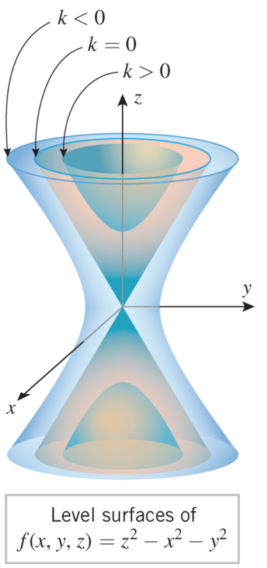
$$x^2 + y^2 + z^2 = k$$

For $k > 0$ the graph of this equation is a sphere of radius $\sqrt{k}$, centered at the origin; for $k = 0$ the graph is the single point $(0, 0, 0)$; and for $k < 0$ there is no level surface (Figure 13.1.9).

Solution (b). The level surfaces have equations of the form

$$z^2 - x^2 - y^2 = k$$

As discussed in Section 11.7, this equation represents a cone if $k = 0$, a hyperboloid of two sheets if $k > 0$, and a hyperboloid of one sheet if $k < 0$ (Figure 13.1.10). ◀



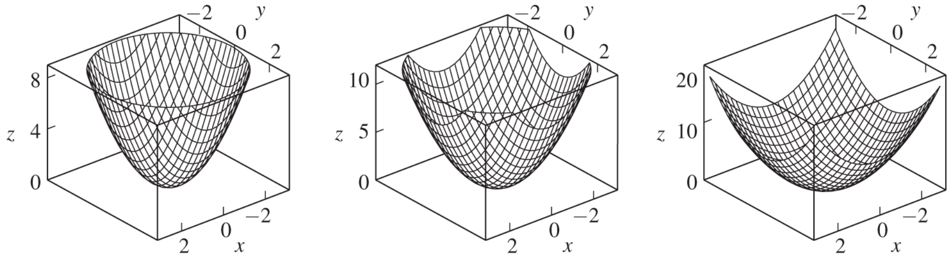
▲ **Figure 13.1.10**

GRAPHING FUNCTIONS OF TWO VARIABLES USING TECHNOLOGY

Generating surfaces with a graphing utility is more complicated than generating plane curves because there are more factors that must be taken into account. We can only touch on the ideas here, so if you want to use a graphing utility, its documentation will be your main source of information.

Graphing utilities can only show a portion of xyz-space in a viewing screen, so the first step in graphing a surface is to determine which portion of xyz-space you want to display. This region is called the ***viewing box*** or ***viewing window***. For example, Figure 13.1.11 shows the effect of graphing the paraboloid $z = x^2 + y^2$ in three different viewing windows. However, within a fixed viewing box, the appearance of the surface is also affected by the ***viewpoint***, that is, the direction from which the surface is viewed, and the distance from the viewer to the surface. For example, Figure 13.1.12 shows the graph of the paraboloid $z = x^2 + y^2$ from three different viewpoints using the first viewing box in Figure 13.1.11.

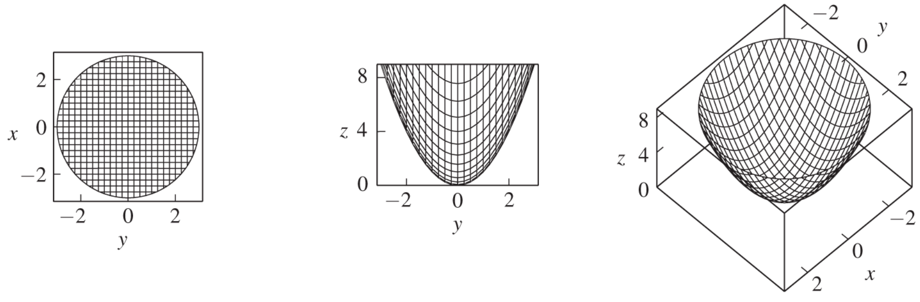
Table 13.1.2 shows six surfaces in 3-space along with their associated contour plots. Note that the level curves correspond to traces on the surface in horizontal planes.



▲ **Figure 13.1.11** Varying the viewing box.

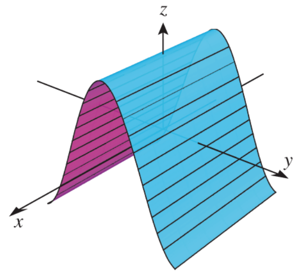
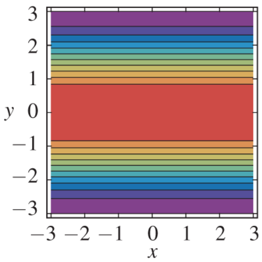
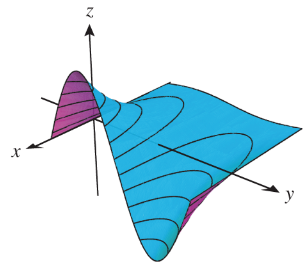
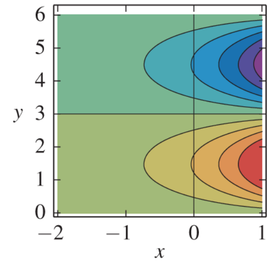
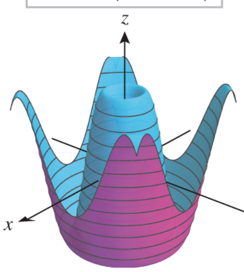
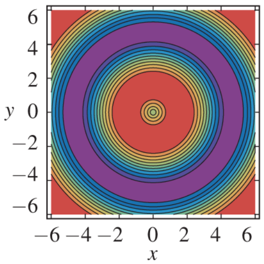
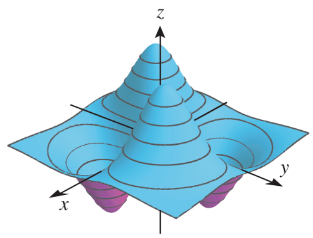
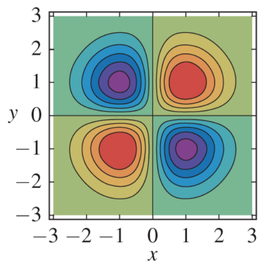
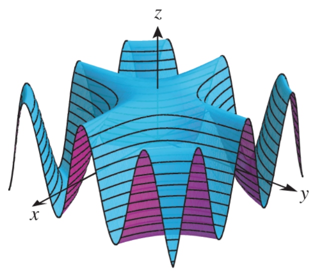
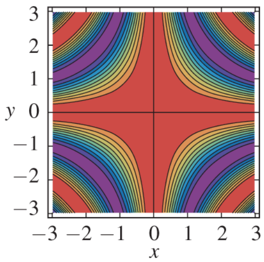
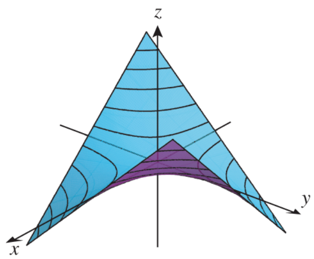
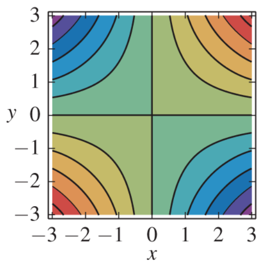
TECHNOLOGY MASTERY

If you have a graphing utility that can generate surfaces in 3-space, read the documentation and try to duplicate some of the surfaces in Figures 13.1.11 and 13.1.12 and Table 13.1.2.



▲ **Figure 13.1.12** Varying the viewpoint.

Table 13.1.2

SURFACE	CONTOUR PLOT	SURFACE	CONTOUR PLOT
$z = \cos y$ 		$z = 5e^x \sin y$ 	
$z = \sin\left(\sqrt{x^2 + y^2}\right)$ 		$z = xye^{-\frac{1}{2}(x^2+y^2)}$ 	
$z = \cos(xy)$ 		$z = xy$ 	

✓ QUICK CHECK EXERCISES 13.1 (See page 815 for answers.)

1. The domain of $f(x,y) = \ln xy$ is ______ and the domain of $g(x,y) = \ln x + \ln y$ is ______.
2. Let $f(x,y) = \dfrac{x-y}{x+y+1}$.
   (a) $f(2,1) =$ ______ (b) $f(1,2) =$ ______
   (c) $f(a,a) =$ ______ (d) $f(y+1,y) =$ ______
3. Let $f(x,y) = e^{x+y}$.
   (a) For what values of k will the graph of the level curve $f(x,y) = k$ be nonempty?
   (b) Describe the level curves $f(x,y) = k$ for the values of k obtained in part (a).
4. Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2+1}$.
   (a) For what values of k will the graph of the level surface $f(x,y,z) = k$ be nonempty?
   (b) Describe the level surfaces $f(x,y,z) = k$ for the values of k obtained in part (a).

EXERCISE SET 13.1 Graphing Utility C CAS

1–8 These exercises are concerned with functions of two variables.

1. Let $f(x,y) = x^2y + 1$. Find
   (a) $f(2,1)$ (b) $f(1,2)$ (c) $f(0,0)$
   (d) $f(1,-3)$ (e) $f(3a,a)$ (f) $f(ab, a-b)$.
2. Let $f(x,y) = x + \sqrt[3]{xy}$. Find
   (a) $f(t,t^2)$ (b) $f(x,x^2)$ (c) $f(2y^2, 4y)$.
3. Let $f(x,y) = xy + 3$. Find
   (a) $f(x+y, x-y)$ (b) $f(xy, 3x^2y^3)$.
4. Let $g(x) = x \sin x$. Find
   (a) $g(x/y)$ (b) $g(xy)$ (c) $g(x-y)$.
5. Find $F(g(x), h(y))$ if $F(x,y) = xe^{xy}$, $g(x) = x^3$, and $h(y) = 3y+1$.
6. Find $g(u(x,y), v(x,y))$ if $g(x,y) = y\sin(x^2y)$, $u(x,y) = x^2y^3$, and $v(x,y) = \pi xy$.
7. Let $f(x,y) = x + 3x^2y^2$, $x(t) = t^2$, and $y(t) = t^3$. Find
   (a) $f(x(t), y(t))$ (b) $f(x(0), y(0))$
   (c) $f(x(2), y(2))$.
8. Let $g(x,y) = ye^{-3x}$, $x(t) = \ln(t^2+1)$, and $y(t) = \sqrt{t}$. Find $g(x(t), y(t))$.

9–10 Suppose that the concentration C in mg/L of medication in a patient's bloodstream is modeled by the function $C(x,t) = 0.2x(e^{-0.2t} - e^{-t})$, where x is the dosage of the medication in mg and t is the number of hours since the beginning of administration of the medication.

9. (a) Estimate the value of $C(25,3)$ to two decimal places. Include appropriate units and interpret your answer in a physical context.
   (b) If the dosage is 100 mg, give a formula for the concentration as a function of time t.
   (c) Give a formula that describes the concentration after 1 hour in terms of the dosage x.
10. (a) Suppose that the medication in the bloodstream reaches an effective level after a half hour. Estimate how much longer the medication remains effective.
    (b) Suppose the dosage is 100 mg. Estimate the maximum concentration in the bloodstream.

11–14 Refer to Table 13.1.1 to estimate the given quantity.

11. The wind chill index when
    (a) the temperature is 25°F and the wind speed is 7 mi/h
    (b) the temperature is 28°F and the wind speed is 5 mi/h.
12. The wind chill index when
    (a) the temperature is 35°F and the wind speed is 14 mi/h
    (b) the temperature is 32°F and the wind speed is 15 mi/h.
13. The temperature when
    (a) the wind chill index is 16°F and the wind speed is 25 mi/h
    (b) the wind chill index is 6°F and the wind speed is 25 mi/h.
14. The wind speed when
    (a) the wind chill index is 7°F and the temperature is 25°F
    (b) the wind chill index is 15°F and the temperature is 30°F.
15. One method for determining relative humidity is to wet the bulb of a thermometer, whirl it through the air, and then compare the thermometer reading with the actual air temperature. If the relative humidity is less than 100%, the reading on the thermometer will be less than the temperature of the air. This difference in temperature is known as the *wet-bulb depression*. The accompanying table gives the relative humidity as a function of the air temperature and the wet-bulb depression. Use the table to complete parts (a)–(c).
    (a) What is the relative humidity if the air temperature is 20°C and the wet-bulb thermometer reads 16°C?
    (b) Estimate the relative humidity if the air temperature is 25°C and the wet-bulb depression is 3.5°C.
    (c) Estimate the relative humidity if the air temperature is 22°C and the wet-bulb depression is 5°C.

WET-BULB DEPRESSION (°C) \ AIR TEMPERATURE (°C)	15	20	25	30
3	71	74	77	79
4	62	66	70	73
5	53	59	63	67

▲ **Table Ex-15**

16. Use the table in Exercise 15 to complete parts (a)–(c).
    (a) What is the wet-bulb depression if the air temperature is 30°C and the relative humidity is 73%?

(b) Estimate the relative humidity if the air temperature is 15°C and the wet-bulb depression is 4.25°C.
(c) Estimate the relative humidity if the air temperature is 26°C and the wet-bulb depression is 3°C.

17–20 These exercises involve functions of three variables.

17. Let $f(x, y, z) = xy^2z^3 + 3$. Find
(a) $f(2, 1, 2)$ (b) $f(-3, 2, 1)$
(c) $f(0, 0, 0)$ (d) $f(a, a, a)$
(e) $f(t, t^2, -t)$ (f) $f(a + b, a - b, b)$.

18. Let $f(x, y, z) = zxy + x$. Find
(a) $f(x + y, x - y, x^2)$ (b) $f(xy, y/x, xz)$.

19. Find $F(f(x), g(y), h(z))$ if $F(x, y, z) = ye^{xyz}$, $f(x) = x^2$, $g(y) = y + 1$, and $h(z) = z^2$.

20. Find $g(u(x, y, z), v(x, y, z), w(x, y, z))$ if $g(x, y, z) = z \sin xy$, $u(x, y, z) = x^2z^3$, $v(x, y, z) = \pi xyz$, and $w(x, y, z) = xy/z$.

21–22 These exercises are concerned with functions of four or more variables.

21. (a) Let $f(x, y, z, t) = x^2y^3\sqrt{z + t}$. Find $f(\sqrt{5}, 2, \pi, 3\pi)$.
(b) Let $f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} kx_k$. Find $f(1, 1, \ldots, 1)$.

22. (a) Let $f(u, v, \lambda, \phi) = e^{u+v}\cos\lambda\tan\phi$. Find $f(-2, 2, 0, \pi/4)$.
(b) Let $f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$. Find $f(1, 2, \ldots, n)$.

23–26 Sketch the domain of f. Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included.

23. $f(x, y) = \ln(1 - x^2 - y^2)$ **24.** $f(x, y) = \sqrt{x^2 + y^2 - 4}$

25. $f(x, y) = \dfrac{1}{x - y^2}$ **26.** $f(x, y) = \ln xy$

27–28 Describe the domain of f in words.

27. (a) $f(x, y) = xe^{-\sqrt{y+2}}$
(b) $f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$
(c) $f(x, y, z) = e^{xyz}$

28. (a) $f(x, y) = \dfrac{\sqrt{4 - x^2}}{y^2 + 3}$ (b) $f(x, y) = \ln(y - 2x)$
(c) $f(x, y, z) = \dfrac{xyz}{x + y + z}$

29–32 True–False Determine whether the statement is true or false. Explain your answer.

29. If the domain of $f(x, y)$ is the xy-plane, then the domain of $f(\sin^{-1}t, \sqrt{t})$ is the interval $[0, 1]$.

30. If $f(x, y) = y/x$, then a contour $f(x, y) = m$ is the straight line $y = mx$.

31. The natural domain of $f(x, y, z) = \sqrt{1 - x^2 - y^2}$ is a disk of radius 1 centered at the origin in the xy-plane.

32. Every level surface of $f(x, y, z) = x + 2y + 3z$ is a plane.

33–42 Sketch the graph of f.

33. $f(x, y) = 3$ **34.** $f(x, y) = \sqrt{9 - x^2 - y^2}$

35. $f(x, y) = \sqrt{x^2 + y^2}$ **36.** $f(x, y) = x^2 + y^2$

37. $f(x, y) = x^2 - y^2$ **38.** $f(x, y) = 4 - x^2 - y^2$

39. $f(x, y) = \sqrt{x^2 + y^2 + 1}$ **40.** $f(x, y) = \sqrt{x^2 + y^2 - 1}$

41. $f(x, y) = y + 1$ **42.** $f(x, y) = x^2$

43–44 In each part, select the term that best describes the level curves of the function f. Choose from the terms lines, circles, noncircular ellipses, parabolas, or hyperbolas.

43. (a) $f(x, y) = 5x^2 - 5y^2$ (b) $f(x, y) = y - 4x^2$
(c) $f(x, y) = x^2 + 3y^2$ (d) $f(x, y) = 3x^2$

44. (a) $f(x, y) = x^2 - 2xy + y^2$ (b) $f(x, y) = 2x^2 + 2y^2$
(c) $f(x, y) = x^2 - 2x - y^2$ (d) $f(x, y) = 2y^2 - x$

45–46 Refer to Figure 13.1.7 in each part.

45. Suppose that \$6000 is borrowed at an interest rate of 11%.
(a) Estimate the monthly payment on the loan.
(b) If the interest rate drops to 9%, estimate how much more can be borrowed without increasing the monthly payment.

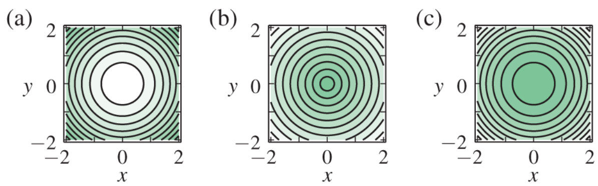
46. Suppose that \$3000 is borrowed at an interest rate of 4%.
(a) Estimate the monthly payment on the loan.
(b) If the interest rate increases to 7%, estimate how much less would need to be borrowed so as not to increase the monthly payment.

FOCUS ON CONCEPTS

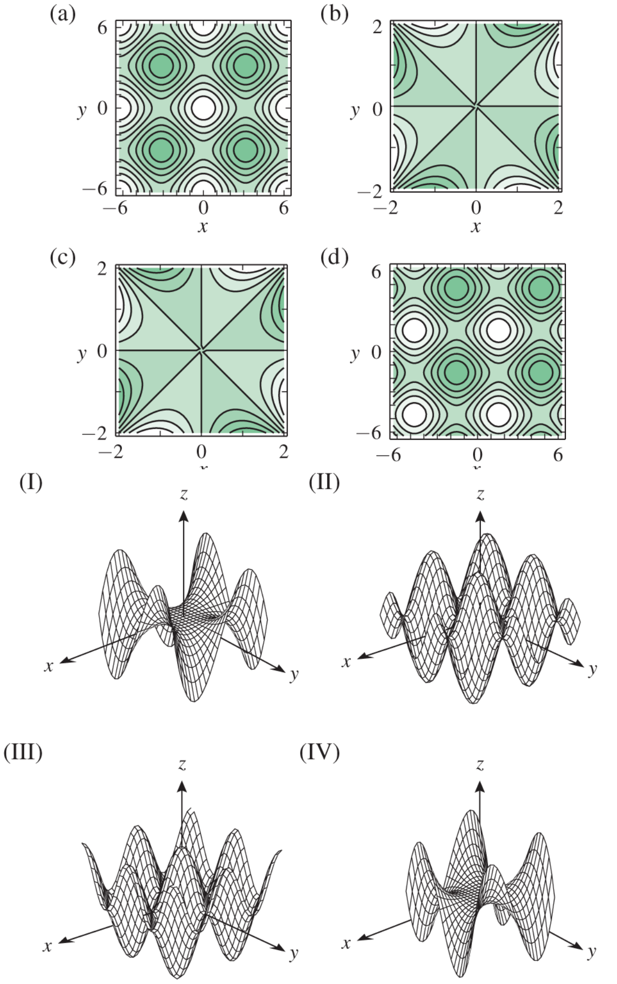
47. In each part, match the contour plot with one of the functions

$$f(x, y) = \sqrt{x^2 + y^2}, \quad f(x, y) = x^2 + y^2,$$
$$f(x, y) = 1 - x^2 - y^2$$

by inspection, and explain your reasoning. Larger values of z are indicated by lighter colors in the contour plot, and the concentric contours correspond to equally spaced values of z.



48. In each part, match the contour plot with one of the surfaces in the accompanying figure on the next page by inspection, and explain your reasoning. The larger the value of z, the lighter the color in the contour plot.



▲ **Figure Ex-48**

49. In each part, the questions refer to the contour map in the accompanying figure.

(a) Is A or B the higher point? Explain your reasoning.

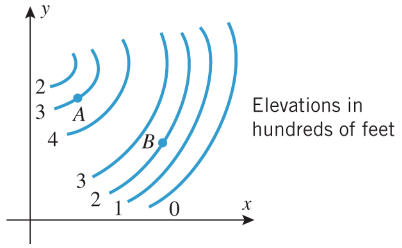
(b) Is the slope steeper at point A or at point B? Explain your reasoning.

(c) Starting at A and moving so that y remains constant and x increases, will the elevation begin to increase or decrease?

(d) Starting at B and moving so that y remains constant and x increases, will the elevation begin to increase or decrease?

(e) Starting at A and moving so that x remains constant and y decreases, will the elevation begin to increase or decrease?

(f) Starting at B and moving so that x remains constant and y decreases, will the elevation begin to increase or decrease?

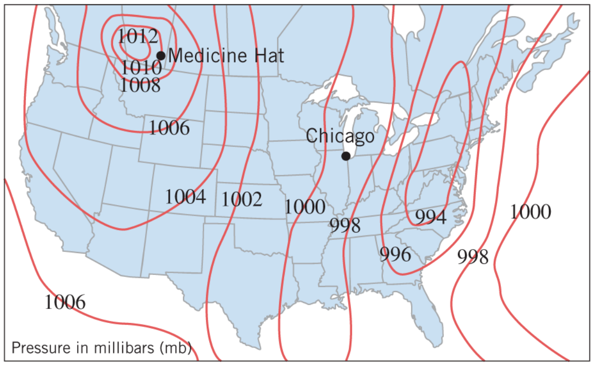


◀ **Figure Ex-49**

50. A curve connecting points of equal atmospheric pressure on a weather map is called an ***isobar***. On a typical weather map the isobars refer to pressure at mean sea level and are given in units of ***millibars*** (mb). Mathematically, isobars are level curves for the pressure function $p(x, y)$ defined at the geographic points (x, y) represented on the map. Tightly packed isobars correspond to steep slopes on the graph of the pressure function, and these are usually associated with strong winds—the steeper the slope, the greater the speed of the wind.

(a) Referring to the accompanying weather map, is the wind speed greater in Medicine Hat, Alberta or in Chicago? Explain your reasoning.

(b) Estimate the average rate of change in atmospheric pressure (in mb/mi) from Medicine Hat to Chicago, given that the distance between the two cities is approximately 1400 mi.



▲ **Figure Ex-50**

51–56 Sketch the level curve $z = k$ for the specified values of k. ■

51. $z = x^2 + y^2;\ k = 0, 1, 2, 3, 4$

52. $z = y/x;\ k = -2, -1, 0, 1, 2$

53. $z = x^2 + y;\ k = -2, -1, 0, 1, 2$

54. $z = x^2 + 9y^2;\ k = 0, 1, 2, 3, 4$

55. $z = x^2 - y^2;\ k = -2, -1, 0, 1, 2$

56. $z = y \csc x;\ k = -2, -1, 0, 1, 2$

57–60 Sketch the level surface $f(x, y, z) = k$. ■

57. $f(x, y, z) = 4x^2 + y^2 + 4z^2;\ k = 16$

58. $f(x, y, z) = x^2 + y^2 - z^2;\ k = 0$

59. $f(x, y, z) = z - x^2 - y^2 + 4;\ k = 7$

60. $f(x, y, z) = 4x - 2y + z;\ k = 1$

61–64 Describe the level surfaces in words. ■

61. $f(x, y, z) = (x - 2)^2 + y^2 + z^2$

62. $f(x, y, z) = 3x - y + 2z$

63. $f(x, y, z) = x^2 + z^2$

64. $f(x, y, z) = z - x^2 - y^2$

65. Let $f(x, y) = x^2 - 2x^3 + 3xy$. Find an equation of the level curve that passes through the point

(a) $(-1, 1)$ (b) $(0, 0)$ (c) $(2, -1)$.

66. Let $f(x, y) = ye^x$. Find an equation of the level curve that passes through the point
(a) $(\ln 2, 1)$ (b) $(0, 3)$ (c) $(1, -2)$.

67. Let $f(x, y, z) = x^2 + y^2 - z$. Find an equation of the level surface that passes through the point
(a) $(1, -2, 0)$ (b) $(1, 0, 3)$ (c) $(0, 0, 0)$.

68. Let $f(x, y, z) = xyz + 3$. Find an equation of the level surface that passes through the point
(a) $(1, 0, 2)$ (b) $(-2, 4, 1)$ (c) $(0, 0, 0)$.

69. If $T(x, y)$ is the temperature at a point (x, y) on a thin metal plate in the xy-plane, then the level curves of T are called ***isothermal curves***. All points on such a curve are at the same temperature. Suppose that a plate occupies the first quadrant and $T(x, y) = xy$.
(a) Sketch the isothermal curves on which $T = 1$, $T = 2$, and $T = 3$.
(b) An ant, initially at $(1, 4)$, wants to walk on the plate so that the temperature along its path remains constant. What path should the ant take and what is the temperature along that path?

70. If $V(x, y)$ is the voltage or potential at a point (x, y) in the xy-plane, then the level curves of V are called ***equipotential curves***. Along such a curve, the voltage remains constant. Given that

$$V(x, y) = \frac{8}{\sqrt{16 + x^2 + y^2}}$$

sketch the equipotential curves at which $V = 2.0$, $V = 1.0$, and $V = 0.5$.

71. Let $f(x, y) = x^2 + y^3$.
(a) Use a graphing utility to generate the level curve that passes through the point $(2, -1)$.
(b) Generate the level curve of height 1.

72. Let $f(x, y) = 2\sqrt{xy}$.
(a) Use a graphing utility to generate the level curve that passes through the point $(2, 2)$.
(b) Generate the level curve of height 8.

C **73.** Let $f(x, y) = xe^{-(x^2+y^2)}$.
(a) Use a CAS to generate the graph of f for $-2 \le x \le 2$ and $-2 \le y \le 2$.
(b) Generate a contour plot for the surface, and confirm visually that it is consistent with the surface obtained in part (a).
(c) Read the appropriate documentation and explore the effect of generating the graph of f from various viewpoints.

C **74.** Let $f(x, y) = \frac{1}{10}e^x \sin y$.
(a) Use a CAS to generate the graph of f for $0 \le x \le 4$ and $0 \le y \le 2\pi$.
(b) Generate a contour plot for the surface, and confirm visually that it is consistent with the surface obtained in part (a).
(c) Read the appropriate documentation and explore the effect of generating the graph of f from various viewpoints.

75. In each part, describe in words how the graph of g is related to the graph of f.
(a) $g(x, y) = f(x - 1, y)$ (b) $g(x, y) = 1 + f(x, y)$
(c) $g(x, y) = -f(x, y + 1)$

76. (a) Sketch the graph of $f(x, y) = e^{-(x^2+y^2)}$.
(b) Describe in words how the graph of the function $g(x, y) = e^{-a(x^2+y^2)}$ is related to the graph of f for positive values of a.

77. **Writing** Find a few practical examples of functions of two and three variables, and discuss how physical considerations affect their domains.

78. **Writing** Describe two different ways in which a function $f(x, y)$ can be represented geometrically. Discuss some of the advantages and disadvantages of each representation.

✓ **QUICK CHECK ANSWERS 13.1** **1.** points (x, y) in the first or third quadrants; points (x, y) in the first quadrant
2. (a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) 0 (d) $1/(2y + 2)$ **3.** (a) $k > 0$ (b) the lines $x + y = \ln k$
4. (a) $0 < k \le 1$ (b) spheres of radius $\sqrt{(1 - k)/k}$ for $0 < k < 1$, the single point $(0, 0, 0)$ for $k = 1$

13.2 LIMITS AND CONTINUITY

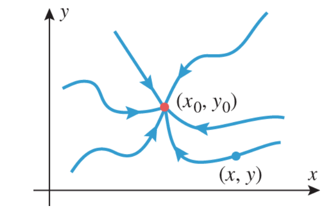
In this section we will introduce the notions of limit and continuity for functions of two or more variables. We will not go into great detail—our objective is to develop the basic concepts accurately and to obtain results needed in later sections. A more extensive study of these topics is usually given in advanced calculus.

LIMITS ALONG CURVES

For a function of one variable there are two one-sided limits at a point x_0, namely,

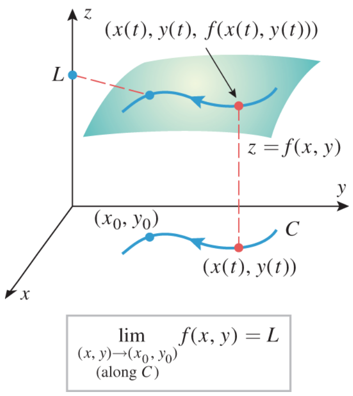
$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left. For functions of two or three variables there are infinitely many different



▲ **Figure 13.2.1**

In words, Formulas (1) and (2) state that a limit of a function f along a parametric curve can be obtained by substituting the parametric equations for the curve into the formula for the function and then computing the limit of the resulting function of one variable at the appropriate point.



▲ **Figure 13.2.2**

curves along which one point can approach another (Figure 13.2.1). Our first objective in this section is to define the limit of $f(x, y)$ as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x, y) \quad \text{and} \quad \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x, y, z)$$

are defined by

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x, y) = \lim_{t\to t_0} f(x(t), y(t)) \tag{1}$$

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x, y, z) = \lim_{t\to t_0} f(x(t), y(t), z(t)) \tag{2}$$

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

A geometric interpretation of the limit along a curve for a function of two variables is shown in Figure 13.2.2: As the point $(x(t), y(t))$ moves along the curve C in the xy-plane toward (x_0, y_0), the point $(x(t), y(t), f(x(t), y(t)))$ moves directly above it along the graph of $z = f(x, y)$ with $f(x(t), y(t))$ approaching the limiting value L. In the figure we followed a common practice of omitting the zero z-coordinate for points in the xy-plane.

▶ Example 1 Figure 13.2.3*a* shows a computer-generated graph of the function

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of $\frac{1}{2}$ for $y = -x$, except at $(0, 0)$ where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$
(d) the line $y = -x$ (e) the parabola $y = x^2$

Solution (a). The x-axis has parametric equations $x = t$, $y = 0$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=0)}} f(x, y) = \lim_{t\to 0} f(t, 0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 13.2.3*b*.

Solution (b). The y-axis has parametric equations $x = 0$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

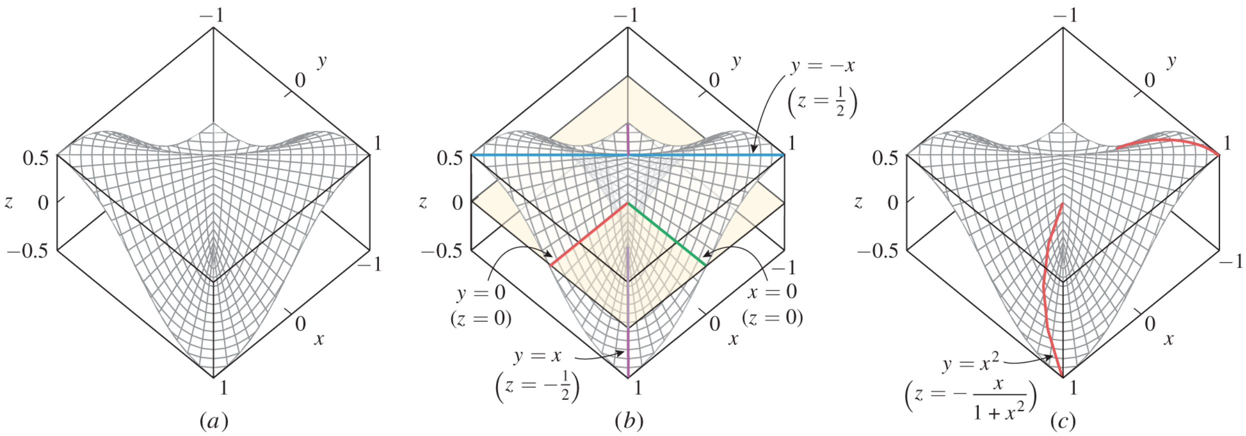
$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} f(x, y) = \lim_{t\to 0} f(0, t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 13.2.3*b*.

Solution (c). The line $y = x$ has parametric equations $x = t$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} f(x, y) = \lim_{t\to 0} f(t, t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.



▲ Figure 13.2.3

Solution (d). The line $y = -x$ has parametric equations $x = t, y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=-x)}} f(x, y) = \lim_{t\to 0} f(t, -t) = \lim_{t\to 0} \frac{t^2}{2t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

For uniformity, we have chosen the same parameter t in each part of Example 1. We could have used x or y as the parameter, according to the context. For example, part (b) could be computed using

$$\lim_{y\to 0} f(0, y)$$

and part (e) could be computed using

$$\lim_{x\to 0} f(x, x^2)$$

Solution (e). The parabola $y = x^2$ has parametric equations $x = t, y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x^2)}} f(x, y) = \lim_{t\to 0} f(t, t^2) = \lim_{t\to 0} \left(-\frac{t^3}{t^2 + t^4}\right) = \lim_{t\to 0} \left(-\frac{t}{1 + t^2}\right) = 0$$

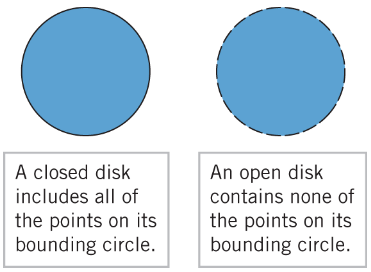
This is consistent with Figure 13.2.3*c*, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1 + t^2}$$

superimposed on the surface. ◀

OPEN AND CLOSED SETS

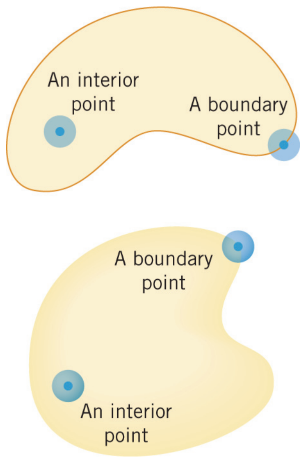
Although limits along specific curves are useful for many purposes, they do not tell the complete story about the limiting behavior of a function at a point; what is required is a limit concept that accounts for the behavior of the function in an *entire vicinity* of a point, not just along smooth curves passing through the point. For this purpose, we start by introducing some terminology.



▲ Figure 13.2.4

Let C be a circle in 2-space that is centered at (x_0, y_0) and has positive radius δ. The set of points that are enclosed by the circle, but do not lie on the circle, is called the ***open disk*** of radius δ centered at (x_0, y_0), and the set of points that lie on the circle together with those enclosed by the circle is called the ***closed disk*** of radius δ centered at (x_0, y_0) (Figure 13.2.4). Analogously, if S is a sphere in 3-space that is centered at (x_0, y_0, z_0) and has positive radius δ, then the set of points that are enclosed by the sphere, but do not lie on the sphere, is called the ***open ball*** of radius δ centered at (x_0, y_0, z_0), and the set of points that lie on the sphere together with those enclosed by the sphere is called the ***closed ball*** of radius δ centered at (x_0, y_0, z_0). Disks and balls are the two-dimensional and three-dimensional analogs of intervals on a line.

The notions of "open" and "closed" can be extended to more general sets in 2-space and 3-space. If D is a set of points in 2-space, then a point (x_0, y_0) is called an ***interior point*** of D if there is *some* open disk centered at (x_0, y_0) that contains only points of D, and (x_0, y_0) is called a ***boundary point*** of D if *every* open disk centered at (x_0, y_0) contains both points



▲ Figure 13.2.5

in D and points not in D. The same terminology applies to sets in 3-space, but in that case the definitions use balls rather than disks (Figure 13.2.5).

For a set D in either 2-space or 3-space, the set of all interior points is called the ***interior*** of D and the set of all boundary points is called the ***boundary*** of D. Moreover, just as for disks, we say that D is ***closed*** if it contains all of its boundary points and ***open*** if it contains *none* of its boundary points. The set of all points in 2-space and the set of all points in 3-space have no boundary points (why?), so by agreement they are regarded to be both open and closed.

GENERAL LIMITS OF FUNCTIONS OF TWO VARIABLES

The statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

is intended to convey the idea that the value of $f(x,y)$ can be made as close as we like to the number L by restricting the point (x,y) to be sufficiently close to (but different from) the point (x_0, y_0). This idea has a formal expression in the following definition and is illustrated in Figure 13.2.6.

When convenient, (3) can also be written as

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = L$$

or as

$$f(x,y)\to L \quad \text{as} \quad (x,y)\to(x_0,y_0)$$

13.2.1 DEFINITION Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \tag{3}$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x,y)$ satisfies

$$|f(x,y) - L| < \epsilon$$

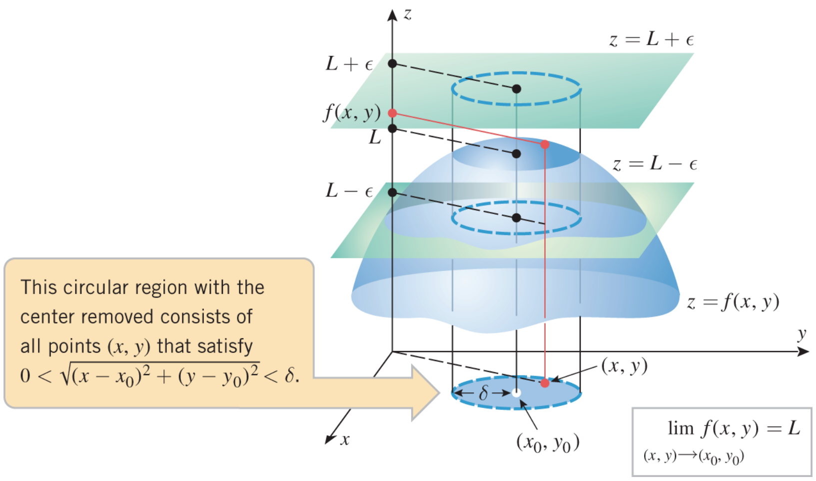
whenever the distance between (x,y) and (x_0,y_0) satisfies

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

In Figure 13.2.6, the condition

$$|f(x,y) - L| < \epsilon$$

is satisfied at each point (x,y) within the circular region. However, the fact that this condition is satisfied at the center of the circular region is not relevant to the limit.



▲ Figure 13.2.6

Another illustration of Definition 13.2.1 is shown in the "arrow diagram" of Figure 13.2.7. As in Figure 13.2.6, this figure is intended to convey the idea that the values of $f(x,y)$ can be forced within ϵ units of L on the z-axis by restricting (x,y) to lie within δ units of (x_0,y_0) in the xy-plane. We used a white dot at (x_0,y_0) to suggest that the epsilon condition need not hold at this point.

We note without proof that the standard properties of limits hold for limits along curves and for general limits of functions of two variables, so that computations involving such limits can be performed in the usual way.

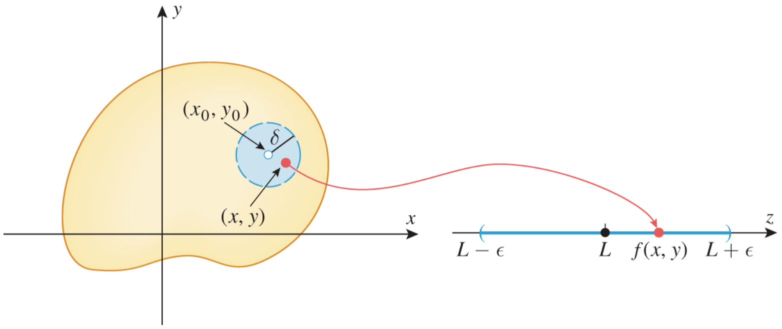


Figure 13.2.7

Example 2

$$\begin{aligned}\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] &= \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9\\ &= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3 \left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9\\ &= 5(1)^3(4)^2 - 9 = 71 \blacktriangleleft\end{aligned}$$

RELATIONSHIPS BETWEEN GENERAL LIMITS AND LIMITS ALONG SMOOTH CURVES

Stated informally, if $f(x, y)$ has limit L as (x, y) approaches (x_0, y_0), then the value of $f(x, y)$ gets closer and closer to L as the distance between (x, y) and (x_0, y_0) approaches zero. Since this statement imposes no restrictions on the direction in which (x, y) approaches (x_0, y_0), it is plausible that the function $f(x, y)$ will also have the limit L as (x, y) approaches (x_0, y_0) along *any* smooth curve C. This is the implication of the following theorem, which we state without proof.

WARNING

In general, one cannot show that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

by showing that this limit holds along a specific curve, or even some specific family of curves. The problem is there may be some other curve along which the limit does not exist or has a value different from L (see Exercise 34, for example).

13.2.2 THEOREM

(*a*) *If $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$, then $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$ along any smooth curve.*

(*b*) *If the limit of $f(x, y)$ fails to exist as $(x, y) \to (x_0, y_0)$ along some smooth curve, or if $f(x, y)$ has different limits as $(x, y) \to (x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ does not exist as $(x, y) \to (x_0, y_0)$.*

Example 3 The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2 + y^2}$$

does not exist because in Example 1 we found two different smooth curves along which this limit had different values. Specifically,

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} -\frac{xy}{x^2+y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} -\frac{xy}{x^2+y^2} = -\frac{1}{2} \blacktriangleleft$$

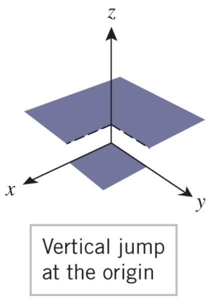


Figure 13.2.8

CONTINUITY

Stated informally, a function of one variable is continuous if its graph is an unbroken curve without jumps or holes. To extend this idea to functions of two variables, imagine that the graph of $z = f(x, y)$ is formed from a thin sheet of clay that has been molded into peaks and valleys. We will regard f as being continuous if the clay surface has no jumps, tears, or holes. For example, the function graphed in Figure 13.2.8 fails to be continuous because its graph exhibits a vertical jump at the origin.

The precise definition of continuity at a point for functions of two variables is similar to that for functions of one variable—we require the limit of the function and the value of the function to be the same at the point.

13.2.3 DEFINITION A function $f(x, y)$ is said to be ***continuous at*** (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is ***continuous on D***, and if f is continuous at every point in the xy-plane, then we say that f is ***continuous everywhere***.

The following theorem, which we state without proof, illustrates some of the ways in which continuous functions can be combined to produce new continuous functions.

13.2.4 THEOREM

(*a*) *If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).*

(*b*) *If $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x, y) = g(h(x, y))$ is continuous at (x_0, y_0).*

(*c*) *If $f(x, y)$ is continuous at (x_0, y_0), and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0.*

Example 4 Use Theorem 13.2.4 to show that the functions $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution. The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number, and therefore by part (*a*) of Theorem 13.2.4, the function $f(x, y) = 3x^2y^5$ is continuous at every point (x, y) in the xy-plane. Since $3x^2y^5$ is continuous at every point in the xy-plane and $\sin u$ is continuous at every real number u, it follows from part (*b*) of Theorem 13.2.4 that the composition $f(x, y) = \sin(3x^2y^5)$ is continuous everywhere. ◀

Theorem 13.2.4 is one of a whole class of theorems about continuity of functions in two or more variables. The content of these theorems can be summarized informally with three basic principles:

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

By using these principles and Theorem 13.2.4, you should be able to confirm that the following functions are all continuous everywhere:

$$xe^{xy} + y^{2/3}, \quad \cosh(xy^3) - |xy|, \quad \frac{xy}{1 + x^2 + y^2}$$

Example 5 Evaluate $\lim_{(x,y)\to(-1,2)} \dfrac{xy}{x^2+y^2}$.

Solution. Since $f(x,y) = xy/(x^2+y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

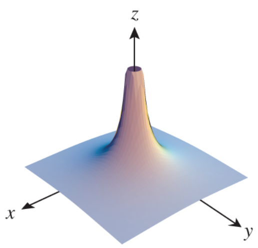
$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+(2)^2} = -\frac{2}{5} \blacktriangleleft$$

Example 6 Since the function

$$f(x,y) = \frac{x^3y^2}{1-xy}$$

is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, $f(x,y)$ is continuous everywhere except on the hyperbola $xy = 1$. ◀

LIMITS AT DISCONTINUITIES



$z = \dfrac{1}{x^2+y^2}$

▲ **Figure 13.2.9**

Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x,y)\to(0,0)$ along any smooth curve (Figure 13.2.9). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0\cdot\infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.

Example 7 Find $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

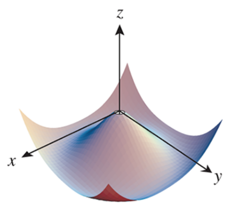
Solution. Let (r,θ) be polar coordinates of the point (x,y) with $r \ge 0$. Then we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2+y^2$$

Moreover, since $r \ge 0$ we have $r = \sqrt{x^2+y^2}$, so that $r\to 0^+$ if and only if $(x,y)\to(0,0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2\ln r^2 \\
&= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} && \boxed{\text{This converts the limit to an indeterminate form of type } \infty/\infty.} \\
&= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} && \boxed{\text{L'Hôpital's rule}} \\
&= \lim_{r\to 0^+} (-r^2) = 0 \blacktriangleleft
\end{aligned}$$

REMARK The graph of $f(x,y) = (x^2+y^2)\ln(x^2+y^2)$ in Example 7 is a surface with a hole (sometimes called a *puncture*) at the origin (Figure 13.2.10). We can remove this discontinuity by *defining* $f(0,0)$ to be 0. (See Exercises 39 and 40, which also deal with the notion of a "removable" discontinuity.)



$z = (x^2+y^2)\ln(x^2+y^2)$

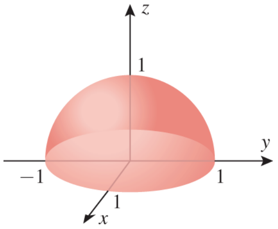
▲ **Figure 13.2.10**

CONTINUITY AT BOUNDARY POINTS

Recall that in our study of continuity for functions of one variable, we first defined continuity at a point, then continuity on an open interval, and then, by using one-sided limits, we extended the notion of continuity to include endpoints of the interval. Similarly, for functions of two variables one can extend the notion of continuity of $f(x,y)$ to the boundary of its domain by modifying Definition 13.2.1 appropriately so that (x, y) is restricted to approach (x_0, y_0) through points lying wholly in the domain of f. We will omit the details.

Example 8 The graph of the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ is the upper hemisphere shown in Figure 13.2.11, and the natural domain of f is the closed unit disk

$$x^2 + y^2 \leq 1$$



$z = \sqrt{1 - x^2 - y^2}$

Figure 13.2.11

The graph of f has no jumps, tears or holes, so it passes our "intuitive test" of continuity. In this case the continuity at a point (x_0, y_0) on the boundary reflects the fact that

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt{1 - x^2 - y^2} = \sqrt{1 - x_0^2 - y_0^2} = 0$$

when (x, y) is restricted to points on the closed unit disk $x^2 + y^2 \leq 1$. It follows that f is continuous on its domain. ◀

EXTENSIONS TO THREE VARIABLES

All of the results in this section can be extended to functions of three or more variables. For example, the distance between the points (x, y, z) and (x_0, y_0, z_0) in 3-space is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

so the natural extension of Definition 13.2.1 to 3-space is as follows:

13.2.5 DEFINITION Let f be a function of three variables, and assume that f is defined at all points within a ball centered at (x_0, y_0, z_0), except possibly at (x_0, y_0, z_0). We will write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L \tag{4}$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y, z)$ satisfies

$$|f(x, y, z) - L| < \epsilon$$

whenever the distance between (x, y, z) and (x_0, y_0, z_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta$$

As with functions of one and two variables, we define a function $f(x, y, z)$ of three variables to be continuous at a point (x_0, y_0, z_0) if the limit of the function and the value of the function are the same at this point; that is,

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$$

Although we will omit the details, the properties of limits and continuity that we discussed for functions of two variables, including the notion of continuity at boundary points, carry over to functions of three variables.

✓ QUICK CHECK EXERCISES 13.2 (See page 824 for answers.)

1. Let

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

Determine the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ along the curve C.

(a) $C: x = 0$ (b) $C: y = 0$
(c) $C: y = x$ (d) $C: y = x^2$

2. (a) $\displaystyle\lim_{(x,y)\to(3,2)} x \cos \pi y =$ ________

(b) $\displaystyle\lim_{(x,y)\to(0,1)} e^{xy^2} =$ ________

(c) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) =$ ________

3. A function $f(x, y)$ is continuous at (x_0, y_0) provided $f(x_0, y_0)$ exists and provided $f(x, y)$ has limit ________ as (x, y) approaches ________.

4. Determine all values of the constant a such that the function $f(x, y) = \sqrt{x^2 - ay^2 + 1}$ is continuous everywhere.

1-26,34, 35,
38, 39

EXERCISE SET 13.2

1–6 Use limit laws and continuity properties to evaluate the limit.

1. $\lim\limits_{(x,y)\to(1,3)} (4xy^2 - x)$

2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x - y}{\sin y - 1}$

3. $\lim\limits_{(x,y)\to(-1,2)} \dfrac{xy^3}{x + y}$

4. $\lim\limits_{(x,y)\to(1,-3)} e^{2x-y^2}$

5. $\lim\limits_{(x,y)\to(0,0)} \ln(1 + x^2y^3)$

6. $\lim\limits_{(x,y)\to(4,-2)} x\sqrt[3]{y^3 + 2x}$

7–8 Show that the limit does not exist by considering the limits as $(x, y)\to(0, 0)$ along the coordinate axes.

7. (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3}{x^2 + 2y^2}$ (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x + y}{2x^2 + y^2}$

8. (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x - y}{x^2 + y^2}$ (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\cos xy}{x^2 + y^2}$

9–12 Evaluate the limit using the substitution $z = x^2 + y^2$ and observing that $z\to0^+$ if and only if $(x, y)\to(0, 0)$.

9. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$

10. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1 - \cos(x^2 + y^2)}{x^2 + y^2}$

11. $\lim\limits_{(x,y)\to(0,0)} e^{-1/(x^2+y^2)}$

12. $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{-1/\sqrt{x^2+y^2}}}{\sqrt{x^2 + y^2}}$

13–22 Determine whether the limit exists. If so, find its value.

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - y^4}{x^2 + y^2}$

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - 16y^4}{x^2 + 4y^2}$

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{3x^2 + 2y^2}$

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1 - x^2 - y^2}{x^2 + y^2}$

17. $\lim\limits_{(x,y,z)\to(2,-1,2)} \dfrac{xz^2}{\sqrt{x^2 + y^2 + z^2}}$

18. $\lim\limits_{(x,y,z)\to(2,0,-1)} \ln(2x + y - z)$

19. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin(x^2 + y^2 + z^2)}{\sqrt{x^2 + y^2 + z^2}}$

20. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin\sqrt{x^2 + y^2 + z^2}}{x^2 + y^2 + z^2}$

21. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2 + y^2 + z^2}}$

22. $\lim\limits_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\dfrac{1}{x^2 + y^2 + z^2}\right]$

23–26 Evaluate the limits by converting to polar coordinates, as in Example 7.

23. $\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2 + y^2}\ln(x^2 + y^2)$

24. $\lim\limits_{(x,y)\to(0,0)} y\ln(x^2 + y^2)$

25. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

26. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2 + 2y^2}}$

27–28 Evaluate the limits by converting to spherical coordinates (ρ, θ, ϕ) and by observing that $\rho\to0^+$ if and only if $(x, y, z)\to(0, 0, 0)$.

27. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2 + y^2 + z^2}$

28. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin x \sin y}{\sqrt{x^2 + 2y^2 + 3z^2}}$

29–32 True–False Determine whether the statement is true or false. Explain your answer.

29. If D is an open set in 2-space or in 3-space, then every point in D is an interior point of D.

30. If $f(x, y)\to L$ as (x, y) approaches $(0, 0)$ along the x-axis, and if $f(x, y)\to L$ as (x, y) approaches $(0, 0)$ along the y-axis, then $\lim_{(x,y)\to(0,0)} f(x, y) = L$.

31. If f and g are functions of two variables such that $f + g$ and fg are both continuous, then f and g are themselves continuous.

32. If $\lim_{x\to0^+} f(x) = L \neq 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{f(x^2 + y^2)} = 0$$

FOCUS ON CONCEPTS

33. The accompanying figure shows a portion of the graph of

$$f(x, y) = \frac{x^2y}{x^4 + y^2}$$

(a) Based on the graph in the figure, does $f(x, y)$ have a limit as $(x, y)\to(0, 0)$? Explain your reasoning.

(b) Show that $f(x, y)\to0$ as $(x, y)\to(0, 0)$ along any line $y = mx$. Does this imply that $f(x, y)\to0$ as $(x, y)\to(0, 0)$? Explain.

(c) Show that $f(x, y)\to\frac{1}{2}$ as $(x, y)\to(0, 0)$ along the parabola $y = x^2$, and confirm visually that this is consistent with the graph of $f(x, y)$.

(d) Based on parts (b) and (c), does $f(x, y)$ have a limit as $(x, y)\to(0, 0)$? Is this consistent with your answer to part (a)?

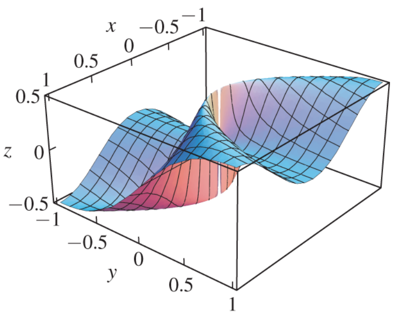


Figure Ex-33

34. (a) Show that as $(x, y)\to(0, 0)$ along any straight line $y = mx$, or along any parabola $y = kx^2$, the value of

$$\frac{x^3y}{2x^6 + y^2}$$

approaches 0.

(cont.)

(b) Show that
$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{2x^6+y^2}$$
does not exist by letting $(x,y)\to(0,0)$ along the curve $y=x^3$.

35. (a) Show that the value of
$$\frac{xyz}{x^2+y^4+z^4}$$
approaches 0 as $(x,y,z)\to(0,0,0)$ along any line $x=at$, $y=bt$, $z=ct$.

(b) Show that the limit
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^4+z^4}$$
does not exist by letting $(x,y,z)\to(0,0,0)$ along the curve $x=t^2$, $y=t$, $z=t$.

36. Find $\displaystyle\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{x^2+1}{x^2+(y-1)^2}\right]$.

1-26,34, 35, 38, 39

37. Find $\displaystyle\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{x^2-1}{x^2+(y-1)^2}\right]$.

38. Let $f(x,y)=\begin{cases}\dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 1, & (x,y)=(0,0).\end{cases}$

Show that f is continuous at $(0,0)$.

39–40 A function $f(x,y)$ is said to have a ***removable discontinuity*** at (x_0,y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists but f is not continuous at (x_0,y_0), either because f is not defined at (x_0,y_0) or because $f(x_0,y_0)$ differs from the value of the limit. Determine whether $f(x,y)$ has a removable discontinuity at $(0,0)$.

39. $f(x,y)=\dfrac{x^2}{x^2+y^2}$

The required limit does not exist, so the singularity is not removable.

40. $f(x)=\begin{cases}x^2+7y^2, & \text{if } (x,y)\neq(0,0)\\ -4, & \text{if } (x,y)=(0,0)\end{cases}$

41–48 Sketch the largest region on which the function f is continuous.

41. $f(x,y)=y\ln(1+x)$

42. $f(x,y)=\sqrt{x-y}$

43. $f(x,y)=\dfrac{x^2y}{\sqrt{25-x^2-y^2}}$

44. $f(x,y)=\ln(2x-y+1)$

45. $f(x,y)=\dfrac{y}{11x^2+3}$

46. $f(x,y)=e^{1-xy}$

47. $f(x,y)=\sin^{-1}(xy)$

48. $f(x,y)=\tan^{-1}(y-x)$

49–52 Describe the largest region on which the function f is continuous.

49. $f(x,y,z)=3x^2e^{yz}\cos(xyz)$

50. $f(x,y,z)=\ln(4-x^2-y^2-z^2)$

51. $f(x,y,z)=\dfrac{y+1}{x^2+z^2-1}$

52. $f(x,y,z)=\sin\sqrt{x^2+y^2+3z^2}$

53. **Writing** Describe the procedure you would use to determine whether or not the limit
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$
exists.

54. **Writing** In your own words, state the geometric interpretations of ϵ and δ in the definition of
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$
given in Definition 13.2.1.

✓ **QUICK CHECK ANSWERS 13.2** **1.** (a) -1 (b) 1 (c) 0 (d) 1 **2.** (a) 3 (b) 1 (c) 0 **3.** $f(x_0,y_0)$; (x_0,y_0) **4.** $a\le 0$

13.3 PARTIAL DERIVATIVES

In this section we will develop the mathematical tools for studying rates of change that involve two or more independent variables.

PARTIAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

If $z=f(x,y)$, then one can inquire how the value of z changes if y is held fixed and x is allowed to vary, or if x is held fixed and y is allowed to vary. For example, the ideal gas law in physics states that under appropriate conditions the pressure exerted by a gas is a function of the volume of the gas and its temperature. Thus, a physicist studying gases might be interested in the rate of change of the pressure if the volume is held fixed and the temperature is allowed to vary, or if the temperature is held fixed and the volume is allowed to vary. We now define a derivative that describes such rates of change.

Suppose that (x_0, y_0) is a point in the domain of a function $f(x, y)$. If we fix $y = y_0$, then $f(x, y_0)$ is a function of the variable x alone. The value of the derivative

$$\frac{d}{dx}[f(x, y_0)]$$

at x_0 then gives us a measure of the instantaneous rate of change of f with respect to x at the point (x_0, y_0). Similarly, the value of the derivative

$$\frac{d}{dy}[f(x_0, y)]$$

at y_0 gives us a measure of the instantaneous rate of change of f with respect to y at the point (x_0, y_0). These derivatives are so basic to the study of differential calculus of multivariable functions that they have their own name and notation.

13.3.1 DEFINITION If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f, then the ***partial derivative of f with respect to x*** at (x_0, y_0) [also called the ***partial derivative of z with respect to x*** at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)]\bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \tag{1}$$

Similarly, the ***partial derivative of f with respect to y*** at (x_0, y_0) [also called the ***partial derivative of z with respect to y*** at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\bigg|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \tag{2}$$

The limits in (1) and (2) show the relationship between partial derivatives and derivatives of functions of one variable. In practice, our usual method for computing partial derivatives is to hold one variable fixed and then differentiate the resulting function using the derivative rules for functions of one variable.

▶ Example 1 Find $f_x(1, 3)$ and $f_y(1, 3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

Solution. Since

$$f_x(x, 3) = \frac{d}{dx}[f(x, 3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4$$

we have $f_x(1, 3) = 54 + 4 = 58$. Also, since

$$f_y(1, y) = \frac{d}{dy}[f(1, y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2$$

we have $f_y(1, 3) = 4(3) + 2 = 14$. ◀

THE PARTIAL DERIVATIVE FUNCTIONS

Formulas (1) and (2) define the partial derivatives of a function at a specific point (x_0, y_0). However, often it will be desirable to omit the subscripts and think of the partial derivatives as functions of the variables x and y. These functions are

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \qquad f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

The following example gives an alternative way of performing the computations in Example 1.

Example 2 Find $f_x(x, y)$ and $f_y(x, y)$ for $f(x, y) = 2x^3y^2 + 2y + 4x$, and use those partial derivatives to compute $f_x(1, 3)$ and $f_y(1, 3)$.

Solution. Keeping y fixed and differentiating with respect to x yields

$$f_x(x, y) = \frac{d}{dx}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

and keeping x fixed and differentiating with respect to y yields

$$f_y(x, y) = \frac{d}{dy}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

Thus,

$$f_x(1, 3) = 6(1^2)(3^2) + 4 = 58 \quad \text{and} \quad f_y(1, 3) = 4(1^3)3 + 2 = 14$$

which agree with the results in Example 1. ◀

TECHNOLOGY MASTERY

Computer algebra systems have specific commands for calculating partial derivatives. If you have a CAS, use it to find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ in Example 2.

PARTIAL DERIVATIVE NOTATION

If $z = f(x, y)$, then the partial derivatives f_x and f_y are also denoted by the symbols

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x, y)$ at a point (x_0, y_0) are

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0}, \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

The symbol ∂ is called a partial derivative sign. It is derived from the Cyrillic alphabet.

Example 3 Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^4 \sin(xy^3)$.

Solution.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial x}(x^4) \\
&= x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3 = x^4y^3 \cos(xy^3) + 4x^3 \sin(xy^3) \\
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial y}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial y}(x^4) \\
&= x^4 \cos(xy^3) \cdot 3xy^2 + \sin(xy^3) \cdot 0 = 3x^5y^2 \cos(xy^3)
\end{aligned}$$ ◀

PARTIAL DERIVATIVES VIEWED AS RATES OF CHANGE AND SLOPES

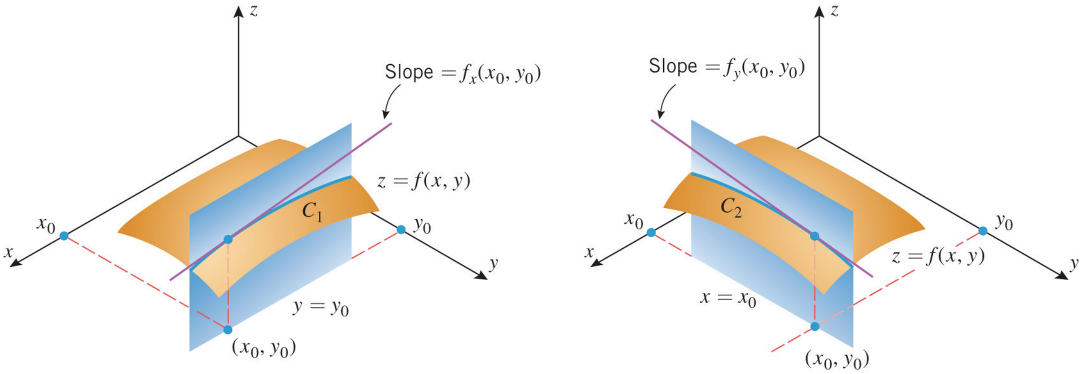
Recall that if $y = f(x)$, then the value of $f'(x_0)$ can be interpreted either as the rate of change of y with respect to x at x_0 or as the slope of the tangent line to the graph of f at x_0. Partial derivatives have analogous interpretations. To see that this is so, suppose that C_1 is the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$ and that C_2 is its intersection with the plane $x = x_0$ (Figure 13.3.1). Thus, $f_x(x, y_0)$ can be interpreted as the rate of change of z with respect to x along the curve C_1, and $f_y(x_0, y)$ can be interpreted as the rate of change of z with respect to y along the curve C_2. In particular, $f_x(x_0, y_0)$ is the rate of change of z with respect to x along the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ is the rate of change of z with respect to y along the curve C_2 at the point (x_0, y_0).

In an applied problem, the interpretations of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ must be accompanied by the proper units. See Example 4.

Example 4 Recall that the wind chill temperature index is given by the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16}$$

Compute the partial derivative of W with respect to v at the point $(T, v) = (25, 10)$ and interpret this partial derivative as a rate of change.



▲ **Figure 13.3.1**

Solution. Holding T fixed and differentiating with respect to v yields

$$\frac{\partial W}{\partial v}(T, v) = 0 + 0 + (0.4275T - 35.75)(0.16)v^{0.16-1} = (0.4275T - 35.75)(0.16)v^{-0.84}$$

Since W is in degrees Fahrenheit and v is in miles per hour, a rate of change of W with respect to v will have units $^\circ$F/(mi/h) (which may also be written as $^\circ$F·h/mi). Substituting $T = 25$ and $v = 10$ gives

$$\frac{\partial W}{\partial v}(25, 10) = (-4.01)10^{-0.84} \approx -0.58\ \frac{^\circ\text{F}}{\text{mi/h}}$$

as the instantaneous rate of change of W with respect to v at $(T, v) = (25, 10)$. We conclude that if the air temperature is a constant 25°F and the wind speed changes by a small amount from an initial speed of 10 mi/h, then the ratio of the change in the wind chill index to the change in wind speed should be about -0.58°F/(mi/h). ◄

Confirm the conclusion of Example 4 by calculating

$$\frac{W(25, 10 + \Delta v) - W(25, 10)}{\Delta v}$$

for values of Δv near 0.

Geometrically, $f_x(x_0, y_0)$ can be viewed as the slope of the tangent line to the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ can be viewed as the slope of the tangent line to the curve C_2 at the point (x_0, y_0) (Figure 13.3.1). We will call $f_x(x_0, y_0)$ the ***slope of the surface in the x-direction*** at (x_0, y_0) and $f_y(x_0, y_0)$ the ***slope of the surface in the y-direction*** at (x_0, y_0).

► Example 5 Let $f(x, y) = x^2y + 5y^3$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(1, -2)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(1, -2)$.

Solution (a). Differentiating f with respect to x with y held fixed yields

$$f_x(x, y) = 2xy$$

Thus, the slope in the x-direction is $f_x(1, -2) = -4$; that is, z is decreasing at the rate of 4 units per unit increase in x.

Solution (b). Differentiating f with respect to y with x held fixed yields

$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the y-direction is $f_y(1, -2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y. ◄

ESTIMATING PARTIAL DERIVATIVES FROM TABULAR DATA

For functions that are presented in tabular form, we can estimate partial derivatives by using adjacent entries within the table.

Table 13.3.1

TEMPERATURE T (°F)

WIND SPEED v (mi/h)	20	25	30	35
5	13	19	25	31
10	9	15	21	27
15	6	13	19	25
20	4	11	17	24

▶ Example 6 Use the values of the wind chill index function $W(T, v)$ displayed in Table 13.3.1 to estimate the partial derivative of W with respect to v at $(T, v) = (25, 10)$. Compare this estimate with the value of the partial derivative obtained in Example 4.

Solution. Since

$$\frac{\partial W}{\partial v}(25, 10) = \lim_{\Delta v \to 0} \frac{W(25, 10 + \Delta v) - W(25, 10)}{\Delta v} = \lim_{\Delta v \to 0} \frac{W(25, 10 + \Delta v) - 15}{\Delta v}$$

we can approximate the partial derivative by

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 + \Delta v) - 15}{\Delta v}$$

With $\Delta v = 5$ this approximation is

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 + 5) - 15}{5} = \frac{W(25, 15) - 15}{5} = \frac{13 - 15}{5} = -\frac{2}{5}\ \frac{°\text{F}}{\text{mi/h}}$$

and with $\Delta v = -5$ this approximation is

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 - 5) - 15}{-5} = \frac{W(25, 5) - 15}{-5} = \frac{19 - 15}{-5} = -\frac{4}{5}\ \frac{°\text{F}}{\text{mi/h}}$$

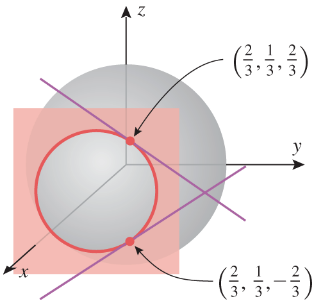
We will take the average, $-\frac{3}{5} = -0.6\,°\text{F}/(\text{mi/h})$, of these two approximations as our estimate of $(\partial W/\partial v)(25, 10)$. This is close to the value

$$\frac{\partial W}{\partial v}(25, 10) = (-4.01)10^{-0.84} \approx -0.58\ \frac{°\text{F}}{\text{mi/h}}$$

found in Example 4. ◀

IMPLICIT PARTIAL DIFFERENTIATION

▶ Example 7 Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y-direction at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ (Figure 13.3.2).



▲ **Figure 13.3.2**

Solution. The point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ lies on the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, and the point $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ lies on the lower hemisphere $z = -\sqrt{1 - x^2 - y^2}$. We could find the slopes by differentiating each expression for z separately with respect to y and then evaluating the derivatives at $x = \frac{2}{3}$ and $y = \frac{1}{3}$. However, it is more efficient to differentiate the given equation

$$x^2 + y^2 + z^2 = 1$$

implicitly with respect to y, since this will give us both slopes with one differentiation. To perform the implicit differentiation, we view z as a function of x and y and differentiate both sides with respect to y, taking x to be fixed. The computations are as follows:

$$\frac{\partial}{\partial y}[x^2 + y^2 + z^2] = \frac{\partial}{\partial y}[1]$$

$$0 + 2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

Substituting the y- and z-coordinates of the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ in this expression, we find that the slope at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ is $-\frac{1}{2}$ and the slope at $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ is $\frac{1}{2}$. ◀

Check the results in Example 7 by differentiating the functions

$$z = \sqrt{1 - x^2 - y^2}$$

and

$$z = -\sqrt{1 - x^2 - y^2}$$

directly.

▶ Example 8 Suppose that $D = \sqrt{x^2 + y^2}$ is the length of the diagonal of a rectangle whose sides have lengths x and y that are allowed to vary. Find a formula for the rate of change of D with respect to x if x varies with y held constant, and use this formula to find the rate of change of D with respect to x at the point where $x = 3$ and $y = 4$.

Solution. Differentiating both sides of the equation $D^2 = x^2 + y^2$ with respect to x yields

$$2D\frac{\partial D}{\partial x} = 2x \quad \text{and thus} \quad D\frac{\partial D}{\partial x} = x$$

Since $D = 5$ when $x = 3$ and $y = 4$, it follows that

$$5\left.\frac{\partial D}{\partial x}\right|_{x=3,y=4} = 3 \quad \text{or} \quad \left.\frac{\partial D}{\partial x}\right|_{x=3,y=4} = \frac{3}{5}$$

Thus, D is increasing at a rate of $\frac{3}{5}$ unit per unit increase in x at the point (3, 4). ◀

PARTIAL DERIVATIVES AND CONTINUITY

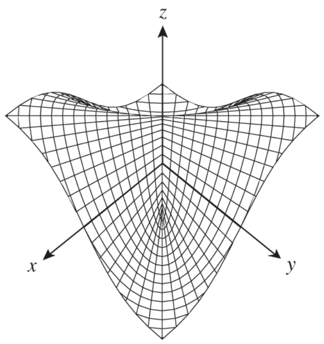
In contrast to the case of functions of a single variable, the existence of partial derivatives for a multivariable function does not guarantee the continuity of the function. This fact is shown in the following example.

▶ Example 9 Let

$$f(x,y) = \begin{cases} -\dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \tag{3}$$

(a) Show that $f_x(x,y)$ and $f_y(x,y)$ exist at all points (x, y).

(b) Explain why f is not continuous at $(0, 0)$.



▲ **Figure 13.3.3**

Solution (a). Figure 13.3.3 shows the graph of f. Note that f is similar to the function considered in Example 1 of Section 13.2, except that here we have assigned f a value of 0 at $(0, 0)$. Except at this point, the partial derivatives of f are

$$f_x(x,y) = -\frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \frac{x^2y - y^3}{(x^2+y^2)^2} \tag{4}$$

$$f_y(x,y) = -\frac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \frac{xy^2 - x^3}{(x^2+y^2)^2} \tag{5}$$

It is not evident from Formula (3) whether f has partial derivatives at $(0, 0)$, and if so, what the values of those derivatives are. To answer that question we will have to use the definitions of the partial derivatives (Definition 13.3.1). Applying Formulas (1) and (2) to (3) we obtain

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

This shows that f has partial derivatives at $(0, 0)$ and the values of both partial derivatives are 0 at that point.

Solution (b). We saw in Example 3 of Section 13.2 that

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$$

does not exist. Thus, f is not continuous at $(0, 0)$. ◀

We will study the relationship between the continuity of a function and the properties of its partial derivatives in the next section.

PARTIAL DERIVATIVES OF FUNCTIONS WITH MORE THAN TWO VARIABLES

For a function $f(x, y, z)$ of three variables, there are three ***partial derivatives***:

$$f_x(x, y, z), \quad f_y(x, y, z), \quad f_z(x, y, z)$$

The partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x. For f_y the variables x and z are held constant, and for f_z the variables x and y are held constant. If a dependent variable

$$w = f(x, y, z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \text{and} \quad \frac{\partial w}{\partial z}$$

Example 10 If $f(x, y, z) = x^3y^2z^4 + 2xy + z$, then

$$\begin{aligned}
f_x(x, y, z) &= 3x^2y^2z^4 + 2y \\
f_y(x, y, z) &= 2x^3yz^4 + 2x \\
f_z(x, y, z) &= 4x^3y^2z^3 + 1 \\
f_z(-1, 1, 2) &= 4(-1)^3(1)^2(2)^3 + 1 = -31 \quad \blacktriangleleft
\end{aligned}$$

Example 11 If $f(\rho, \theta, \phi) = \rho^2 \cos\phi \sin\theta$, then

$$\begin{aligned}
f_\rho(\rho, \theta, \phi) &= 2\rho \cos\phi \sin\theta \\
f_\theta(\rho, \theta, \phi) &= \rho^2 \cos\phi \cos\theta \\
f_\phi(\rho, \theta, \phi) &= -\rho^2 \sin\phi \sin\theta \quad \blacktriangleleft
\end{aligned}$$

In general, if $f(v_1, v_2, \ldots, v_n)$ is a function of n variables, there are n partial derivatives of f, each of which is obtained by holding $n - 1$ of the variables fixed and differentiating the function f with respect to the remaining variable. If $w = f(v_1, v_2, \ldots, v_n)$, then these partial derivatives are denoted by

$$\frac{\partial w}{\partial v_1}, \frac{\partial w}{\partial v_2}, \ldots, \frac{\partial w}{\partial v_n}$$

where $\partial w/\partial v_i$ is obtained by holding all variables except v_i fixed and differentiating with respect to v_i.

HIGHER-ORDER PARTIAL DERIVATIVES

Suppose that f is a function of two variables x and y. Since the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are also functions of x and y, these functions may themselves have partial derivatives. This gives rise to four possible ***second-order*** partial derivatives of f, which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to x.

Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to x and then with respect to y.

Differentiate first with respect to y and then with respect to x.

The last two cases are called the ***mixed second-order partial derivatives*** or the ***mixed second partials***. Also, the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are often called the ***first-order partial derivatives*** when it is necessary to distinguish them from higher-order partial derivatives. Similar conventions apply to the second-order partial derivatives of a function of three variables.

WARNING Observe that the two notations for the mixed second partials have opposite conventions for the order of differentiation. In the "∂" notation the derivatives are taken right to left, and in the "subscript" notation they are taken left to right. The conventions are logical if you insert parentheses:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Right to left. Differentiate inside the parentheses first.

$$f_{xy} = (f_x)_y$$

Left to right. Differentiate inside the parentheses first.

Example 12 Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Solution. We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

so that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3 \blacktriangleleft$$

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

Example 13 Let $f(x, y) = y^2e^x + y$. Find f_{xyy}.

Solution.

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y^2}(y^2e^x) = \frac{\partial}{\partial y}(2ye^x) = 2e^x \blacktriangleleft$$

EQUALITY OF MIXED PARTIALS

For a function $f(x, y)$ it might be expected that there would be four distinct second-order partial derivatives: f_{xx}, f_{xy}, f_{yx}, and f_{yy}. However, observe that the mixed second-order partial derivatives in Example 12 are equal. The following theorem (proved in Web Appendix L) explains why this is so.

If f is a function of three variables, then the analog of Theorem 13.3.2 holds for each pair of mixed second-order partials if we replace "open disk" by "open ball." How many second-order partials does $f(x, y, z)$ have?

13.3.2 THEOREM *Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.*

It follows from this theorem that if $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous everywhere, then $f_{xy}(x, y) = f_{yx}(x, y)$ for all values of x and y. Since polynomials are continuous everywhere, this explains why the mixed second-order partials in Example 12 are equal.

THE WAVE EQUATION

Consider a string of length L that is stretched taut between $x = 0$ and $x = L$ on an x-axis, and suppose that the string is set into vibratory motion by "plucking" it at time $t = 0$ (Figure 13.3.4*a*). The displacement of a point on the string depends on both its coordinate x and the elapsed time t, and hence is described by a function $u(x, t)$ of two variables. For a fixed value t, the function $u(x, t)$ depends on x alone, and the graph of u versus x describes the shape of the string—think of it as a "snapshot" of the string at time t (Figure 13.3.4*b*). It follows that at a fixed time t, the partial derivative $\partial u/\partial x$ represents the slope of the string at x, and the sign of the second partial derivative $\partial^2 u/\partial x^2$ tells us whether the string is concave up or concave down at x (Figure 13.3.4*c*).

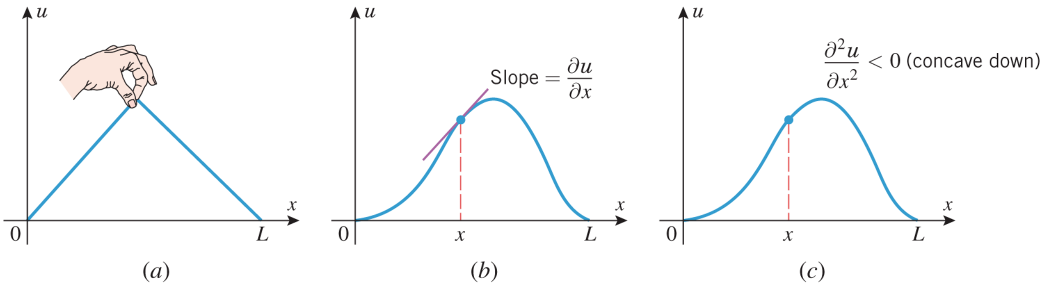


Figure 13.3.4

Ken Vander Putten/iStockphoto
The vibration of a plucked string is governed by the wave equation.

For a fixed value of x, the function $u(x, t)$ depends on t alone, and the graph of u versus t is the position versus time curve of the point on the string with coordinate x. Thus, for a fixed value of x, the partial derivative $\partial u/\partial t$ is the velocity of the point with coordinate x, and $\partial^2 u/\partial t^2$ is the acceleration of that point.

It can be proved that under appropriate conditions the function $u(x, t)$ satisfies an equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

where c is a positive constant that depends on the physical characteristics of the string. This equation, which is called the ***one-dimensional wave equation***, involves partial derivatives of the unknown function $u(x, t)$ and hence is classified as a ***partial differential equation***. Techniques for solving partial differential equations are studied in advanced courses and will not be discussed in this text.

▶ Example 14 Show that the function $u(x, t) = \sin(x - ct)$ is a solution of Equation (6).

Solution. We have

$$\frac{\partial u}{\partial x} = \cos(x - ct), \qquad \frac{\partial^2 u}{\partial x^2} = -\sin(x - ct)$$

$$\frac{\partial u}{\partial t} = -c\cos(x - ct), \qquad \frac{\partial^2 u}{\partial t^2} = -c^2 \sin(x - ct)$$

Thus, $u(x, t)$ satisfies (6). ◀

✓QUICK CHECK EXERCISES 13.3

(See page 837 for answers.)

1. Let $f(x, y) = x \sin xy$. Then $f_x(x, y) =$ ________ and $f_y(x, y) =$ ________.
2. The slope of the surface $z = xy^2$ in the x-direction at the point $(2, 3)$ is ________, and the slope of this surface in the y-direction at the point $(2, 3)$ is ________.
3. The volume V of a right circular cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.
   (a) Find a formula for the instantaneous rate of change of V with respect to r if r changes and h remains constant.
   (b) Find a formula for the instantaneous rate of change of V with respect to h if h changes and r remains constant.
4. Find all second-order partial derivatives for the function $f(x, y) = x^2y^3$.

EXERCISE SET 13.3 Graphing Utility

1. Let $f(x, y) = 3x^3y^2$. Find
   (a) $f_x(x, y)$ (b) $f_y(x, y)$ (c) $f_x(1, y)$
   (d) $f_x(x, 1)$ (e) $f_y(1, y)$ (f) $f_y(x, 1)$
   (g) $f_x(1, 2)$ (h) $f_y(1, 2)$.
2. Let $z = e^{2x} \sin y$. Find
   (a) $\partial z/\partial x$ (b) $\partial z/\partial y$ (c) $\partial z/\partial x|_{(0,y)}$
   (d) $\partial z/\partial x|_{(x,0)}$ (e) $\partial z/\partial y|_{(0,y)}$ (f) $\partial z/\partial y|_{(x,0)}$
   (g) $\partial z/\partial x|_{(\ln 2,0)}$ (h) $\partial z/\partial y|_{(\ln 2,0)}$.

3–10 Evaluate the indicated partial derivatives.

3. $z = 9x^2y - 3x^5y$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$
4. $f(x, y) = 10x^2y^4 - 6xy^2 + 10x^2$; $f_x(x, y)$, $f_y(x, y)$
5. $z = (x^2 + 5x - 2y)^8$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

1-14,17,18,
25-50

6. $f(x, y) = \dfrac{1}{xy^2 - x^2y}$; $f_x(x, y)$, $f_y(x, y)$
7. $\dfrac{\partial}{\partial p}(e^{-7p/q})$, $\dfrac{\partial}{\partial q}(e^{-7p/q})$
8. $\dfrac{\partial}{\partial x}(xe^{\sqrt{15xy}})$, $\dfrac{\partial}{\partial y}(xe^{\sqrt{15xy}})$
9. $z = \sin(5x^3y + 7xy^2)$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$
10. $f(x, y) = \cos(2xy^2 - 3x^2y^2)$; $f_x(x, y)$, $f_y(x, y)$
11. Let $f(x, y) = \sqrt{3x + 2y}$.
    (a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(4, 2)$.
    (b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(4, 2)$.
12. Let $f(x, y) = xe^{-y} + 5y$.
    (a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(3, 0)$.
    (b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(3, 0)$.
13. Let $z = \sin(y^2 - 4x)$.
    (a) Find the rate of change of z with respect to x at the point $(2, 1)$ with y held fixed.
    (b) Find the rate of change of z with respect to y at the point $(2, 1)$ with x held fixed.
14. Let $z = (x + y)^{-1}$.
    (a) Find the rate of change of z with respect to x at the point $(-2, 4)$ with y held fixed.
    (b) Find the rate of change of z with respect to y at the point $(-2, 4)$ with x held fixed.

FOCUS ON CONCEPTS

15. Use the information in the accompanying figure to find the values of the first-order partial derivatives of f at the point $(1, 2)$.

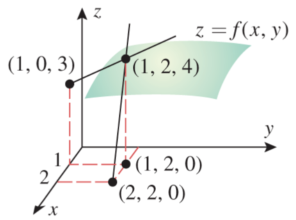


Figure Ex-15

16. The accompanying figure shows a contour plot for an unspecified function $f(x, y)$. Make a conjecture about the signs of the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$, and explain your reasoning.

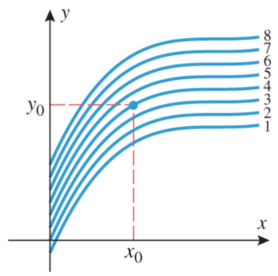


Figure Ex-16

17. Suppose that Nolan throws a baseball to Ryan and that the baseball leaves Nolan's hand at the same height at which it is caught by Ryan. It we ignore air resistance, the horizontal range r of the baseball is a function of the initial speed v of the ball when it leaves Nolan's hand and the angle θ above the horizontal at which it is thrown. Use the accompanying table and the method of Example 6 to estimate
    (a) the partial derivative of r with respect to v when $v = 80$ ft/s and $\theta = 40°$

1-14,17,18,
25-50

(b) the partial derivative of r with respect to θ when $v = 80$ ft/s and $\theta = 40°$.

SPEED v (ft/s)

ANGLE θ (degrees)	75	80	85	90
35	165	188	212	238
40	173	197	222	249
45	176	200	226	253
50	173	197	222	249

Table Ex-17

18. Use the table in Exercise 17 and the method of Example 6 to estimate
(a) the partial derivative of r with respect to v when $v = 85$ ft/s and $\theta = 45°$
(b) the partial derivative of r with respect to θ when $v = 85$ ft/s and $\theta = 45°$.

19. The accompanying figure shows the graphs of an unspecified function $f(x, y)$ and its partial derivatives $f_x(x, y)$ and $f_y(x, y)$. Determine which is which, and explain your reasoning.

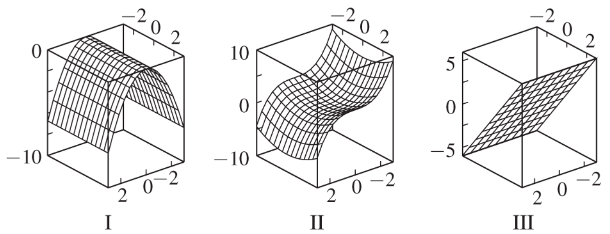


Figure Ex-19

20. What can you say about the signs of $\partial z/\partial x$, $\partial^2 z/\partial x^2$, $\partial z/\partial y$, and $\partial^2 z/\partial y^2$ at the point P in the accompanying figure? Explain your reasoning.

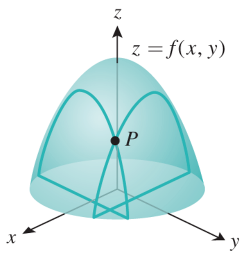


Figure Ex-20

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. If the line $y = 2$ is a contour of $f(x, y)$ through $(4, 2)$, then $f_x(4, 2) = 0$.

22. If the plane $x = 3$ intersects the surface $z = f(x, y)$ in a curve that passes through $(3, 4, 16)$ and satisfies $z = y^2$, then $f_y(3, 4) = 8$.

23. If the graph of $z = f(x, y)$ is a plane in 3-space, then both f_x and f_y are constant functions.

24. There exists a polynomial $f(x, y)$ that satisfies the equations $f_x(x, y) = 3x^2 + y^2 + 2y$ and $f_y(x, y) = 2xy + 2y$.

25–30 Find $\partial z/\partial x$ and $\partial z/\partial y$.

25. $z = 4e^{x^2 y^3}$

26. $z = \cos(x^5 y^4)$

27. $z = x^3 \ln(1 + xy^{-3/5})$

28. $z = e^{xy} \sin 4y^2$

29. $z = \dfrac{xy}{x^2 + y^2}$

30. $z = \dfrac{x^2 y^3}{\sqrt{x + y}}$

31–36 Find $f_x(x, y)$ and $f_y(x, y)$.

31. $f(x, y) = \sqrt{3x^5 y - 7x^3 y}$

32. $f(x, y) = \dfrac{x + y}{x - y}$

33. $f(x, y) = y^{-3/2} \tan^{-1}(x/y)$

34. $f(x, y) = x^3 e^{-y} + y^3 \sec\sqrt{x}$

35. $f(x, y) = (y^2 \tan x)^{-4/3}$

36. $f(x, y) = \cosh(\sqrt{x}) \sinh^2(xy^2)$

37–40 Evaluate the indicated partial derivatives.

37. $f(x, y) = 9 - x^2 - 7y^3$; $f_x(3, 1)$, $f_y(3, 1)$

38. $f(x, y) = x^2 y e^{xy}$; $\partial f/\partial x(1, 1)$, $\partial f/\partial y(1, 1)$

39. $z = \sqrt{x^2 + 4y^2}$; $\partial z/\partial x(1, 2)$, $\partial z/\partial y(1, 2)$

40. $w = x^2 \cos xy$; $\partial w/\partial x\left(\frac{1}{2}, \pi\right)$, $\partial w/\partial y\left(\frac{1}{2}, \pi\right)$

41. Let $f(x, y, z) = x^2 y^4 z^3 + xy + z^2 + 1$. Find
(a) $f_x(x, y, z)$ (b) $f_y(x, y, z)$ (c) $f_z(x, y, z)$
(d) $f_x(1, y, z)$ (e) $f_y(1, 2, z)$ (f) $f_z(1, 2, 3)$.

42. Let $w = x^2 y \cos z$. Find
(a) $\partial w/\partial x(x, y, z)$ (b) $\partial w/\partial y(x, y, z)$
(c) $\partial w/\partial z(x, y, z)$ (d) $\partial w/\partial x(2, y, z)$
(e) $\partial w/\partial y(2, 1, z)$ (f) $\partial w/\partial z(2, 1, 0)$.

43–46 Find f_x, f_y, and f_z.

43. $f(x, y, z) = z \ln(x^2 y \cos z)$

44. $f(x, y, z) = y^{-3/2} \sec\left(\dfrac{xz}{y}\right)$

1-14,17,18,
25-50

45. $f(x, y, z) = \tan^{-1}\left(\dfrac{1}{xy^2 z^3}\right)$

46. $f(x, y, z) = \cosh(\sqrt{z}) \sinh^2(x^2 yz)$

47–50 Find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$.

47. $w = ye^z \sin xz$

48. $w = \dfrac{x^2 - y^2}{y^2 + z^2}$

49. $w = \sqrt{x^2 + y^2 + z^2}$

50. $w = y^3 e^{2x+3z}$

51. Let $f(x, y, z) = y^2 e^{xz}$. Find
(a) $\partial f/\partial x|_{(1,1,1)}$ (b) $\partial f/\partial y|_{(1,1,1)}$ (c) $\partial f/\partial z|_{(1,1,1)}$.

52. Let $w = \sqrt{x^2 + 4y^2 - z^2}$. Find
(a) $\partial w/\partial x|_{(2,1,-1)}$ (b) $\partial w/\partial y|_{(2,1,-1)}$
(c) $\partial w/\partial z|_{(2,1,-1)}$.

53. Let $f(x, y) = e^x \cos y$. Use a graphing utility to graph the functions $f_x(0, y)$ and $f_y(x, \pi/2)$.

54. Let $f(x, y) = e^x \sin y$. Use a graphing utility to graph the functions $f_x(0, y)$ and $f_y(x, 0)$.

55. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$. At what rate is z changing with respect to x when the point is at $(2, 1, 7)$?

56. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $x = 2$. At what rate is z changing with respect to y when the point is at $(2, 1, 7)$?

57. A point moves along the intersection of the plane $y = 3$ and the surface $z = \sqrt{29 - x^2 - y^2}$. At what rate is z changing with respect to x when the point is at $(4, 3, 2)$?

58. Find the slope of the tangent line at $(-1, 1, 5)$ to the curve of intersection of the surface $z = x^2 + 4y^2$ and
(a) the plane $x = -1$ (b) the plane $y = 1$.

59. The volume V of a right circular cylinder is given by the formula $V = \pi r^2 h$, where r is the radius and h is the height.
(a) Find a formula for the instantaneous rate of change of V with respect to r if r changes and h remains constant.

57-65,69-100

(b) Find a formula for the instantaneous rate of change of V with respect to h if h changes and r remains constant.
(c) Suppose that h has a constant value of 4 in, but r varies. Find the rate of change of V with respect to r at the point where $r = 6$ in.
(d) Suppose that r has a constant value of 8 in, but h varies. Find the instantaneous rate of change of V with respect to h at the point where $h = 10$ in.

60. The volume V of a right circular cone is given by

$$V = \frac{\pi}{24} d^2 \sqrt{4s^2 - d^2}$$

where s is the slant height and d is the diameter of the base.
(a) Find a formula for the instantaneous rate of change of V with respect to s if d remains constant.
(b) Find a formula for the instantaneous rate of change of V with respect to d if s remains constant.
(c) Suppose that d has a constant value of 16 cm, but s varies. Find the rate of change of V with respect to s when $s = 10$ cm.
(d) Suppose that s has a constant value of 10 cm, but d varies. Find the rate of change of V with respect to d when $d = 16$ cm.

61. According to the ideal gas law, the pressure, temperature, and volume of a gas are related by $P = kT/V$, where k is a constant of proportionality. Suppose that V is measured in cubic inches (in^3), T is measured in kelvins (K), and that for a certain gas the constant of proportionality is $k = 10$ in·lb/K.
(a) Find the instantaneous rate of change of pressure with respect to temperature if the temperature is 80 K and the volume remains fixed at 50 in^3.
(b) Find the instantaneous rate of change of volume with respect to pressure if the volume is 50 in^3 and the temperature remains fixed at 80 K.

62. The temperature at a point (x, y) on a metal plate in the xy-plane is $T(x, y) = x^3 + 2y^2 + x$ degrees Celsius. Assume that distance is measured in centimeters and find the rate at which temperature changes with respect to distance if we start at the point $(1, 2)$ and move
(a) to the right and parallel to the x-axis
(b) upward and parallel to the y-axis.

63. The length, width, and height of a rectangular box are $l = 5$, $w = 2$, and $h = 3$, respectively.
(a) Find the instantaneous rate of change of the volume of the box with respect to the length if w and h are held constant.
(b) Find the instantaneous rate of change of the volume of the box with respect to the width if l and h are held constant.
(c) Find the instantaneous rate of change of the volume of the box with respect to the height if l and w are held constant.

64. The area A of a triangle is given by $A = \frac{1}{2}ab\sin\theta$, where a and b are the lengths of two sides and θ is the angle between these sides. Suppose that $a = 5$, $b = 10$, and $\theta = \pi/3$.
(a) Find the rate at which A changes with respect to a if b and θ are held constant.
(b) Find the rate at which A changes with respect to θ if a and b are held constant.
(c) Find the rate at which b changes with respect to a if A and θ are held constant.

65. The volume of a right circular cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Show that if the height remains constant while the radius changes, then the volume satisfies

$$\frac{\partial V}{\partial r} = \frac{2V}{r}$$

66. Find parametric equations for the tangent line at $(1, 3, 3)$ to the curve of intersection of the surface $z = x^2 y$ and
(a) the plane $x = 1$ (b) the plane $y = 3$.

67. (a) By differentiating implicitly, find the slope of the hyperboloid $x^2 + y^2 - z^2 = 1$ in the x-direction at the points $(3, 4, 2\sqrt{6})$ and $(3, 4, -2\sqrt{6})$.
(b) Check the results in part (a) by solving for z and differentiating the resulting functions directly.

68. (a) By differentiating implicitly, find the slope of the hyperboloid $x^2 + y^2 - z^2 = 1$ in the y-direction at the points $(3, 4, 2\sqrt{6})$ and $(3, 4, -2\sqrt{6})$.
(b) Check the results in part (a) by solving for z and differentiating the resulting functions directly.

69–72 Calculate $\partial z/\partial x$ and $\partial z/\partial y$ using implicit differentiation. Leave your answers in terms of x, y, and z. ■

69. $(x^2 + y^2 + z^2)^{3/2} = 1$ **70.** $\ln(2x^2 + y - z^3) = x$

71. $x^2 + z\sin xyz = 0$ **72.** $e^{xy}\sinh z - z^2 x + 1 = 0$

73–76 Find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$ using implicit differentiation. Leave your answers in terms of x, y, z, and w. ■

73. $(x^2 + y^2 + z^2 + w^2)^{3/2} = 4$

74. $\ln(2x^2 + y - z^3 + 3w) = z$

75. $w^2 + w\sin xyz = 1$

76. $e^{xy}\sinh w - z^2 w + 1 = 0$

77–80 Find f_x and f_y. ■

77. $f(x, y) = \int_y^x e^{t^2}\,dt$ **78.** $f(x, y) = \int_1^{xy} e^{t^2}\,dt$

79. $f(x, y) = \int_0^{x^2y^3} \sin t^3 \, dt$ **80.** $f(x, y) = \int_{x+y}^{x-y} \sin t^3 \, dt$

81. Let $z = \sqrt{x} \cos y$. Find
(a) $\partial^2 z/\partial x^2$ (b) $\partial^2 z/\partial y^2$
(c) $\partial^2 z/\partial x \partial y$ (d) $\partial^2 z/\partial y \partial x$.

57-65,69-100

82. Let $f(x, y) = 4x^2 - 2y + 7x^4y^5$. Find
(a) f_{xx} (b) f_{yy} (c) f_{xy} (d) f_{yx}.

83. Let $f(x, y) = \sin(3x^2 + 6y^2)$. Find
(a) f_{xx} (b) f_{yy} (c) f_{xy} (d) f_{yx}.

84. Let $f(x, y) = xe^{2y}$. Find
(a) f_{xx} (b) f_{yy} (c) f_{xy} (d) f_{yx}.

85–92 Confirm that the mixed second-order partial derivatives of f are the same. ■

85. $f(x, y) = 4x^2 - 8xy^4 + 7y^5 - 3$

86. $f(x, y) = \sqrt{x^2 + y^2}$ **87.** $f(x, y) = e^x \cos y$

88. $f(x, y) = e^{x-y^2}$ **89.** $f(x, y) = \ln(4x - 5y)$

90. $f(x, y) = \ln(x^2 + y^2)$

91. $f(x, y) = (x - y)/(x + y)$

92. $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$

93. Express the following derivatives in "∂" notation.
(a) f_{xxx} (b) f_{xyy} (c) f_{yyxx} (d) f_{xyyy}

94. Express the derivatives in "subscript" notation.
(a) $\dfrac{\partial^3 f}{\partial y^2 \partial x}$ (b) $\dfrac{\partial^4 f}{\partial x^4}$ (c) $\dfrac{\partial^4 f}{\partial y^2 \partial x^2}$ (d) $\dfrac{\partial^5 f}{\partial x^2 \partial y^3}$

95. Given $f(x, y) = x^3y^5 - 2x^2y + x$, find
(a) f_{xxy} (b) f_{yxy} (c) f_{yyy}.

96. Given $z = (2x - y)^5$, find
(a) $\dfrac{\partial^3 z}{\partial y \partial x \partial y}$ (b) $\dfrac{\partial^3 z}{\partial x^2 \partial y}$ (c) $\dfrac{\partial^4 z}{\partial x^2 \partial y^2}$.

97. Given $f(x, y) = y^3e^{-5x}$, find
(a) $f_{xyy}(0, 1)$ (b) $f_{xxx}(0, 1)$ (c) $f_{yyxx}(0, 1)$.

98. Given $w = e^y \cos x$, find
(a) $\left.\dfrac{\partial^3 w}{\partial y^2 \partial x}\right|_{(\pi/4,0)}$ (b) $\left.\dfrac{\partial^3 w}{\partial x^2 \partial y}\right|_{(\pi/4,0)}$

99. Let $f(x, y, z) = x^3y^5z^7 + xy^2 + y^3z$. Find
(a) f_{xy} (b) f_{yz} (c) f_{xz} (d) f_{zz}
(e) f_{zyy} (f) f_{xxy} (g) f_{zyx} (h) f_{xxyz}.

100. Let $w = (4x - 3y + 2z)^5$. Find
(a) $\dfrac{\partial^2 w}{\partial x \partial z}$ (b) $\dfrac{\partial^3 w}{\partial x \partial y \partial z}$ (c) $\dfrac{\partial^4 w}{\partial z^2 \partial y \partial x}$.

101. Show that the function satisfies ***Laplace's equation***

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

(a) $z = x^2 - y^2 + 2xy$
(b) $z = e^x \sin y + e^y \cos x$
(c) $z = \ln(x^2 + y^2) + 2 \tan^{-1}(y/x)$

102. Show that the function satisfies the ***heat equation***

$$\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2} \quad (c > 0, \text{ constant})$$

(a) $z = e^{-t} \sin(x/c)$ (b) $z = e^{-t} \cos(x/c)$

103. Show that the function $u(x, t) = \sin c\omega t \sin \omega x$ satisfies the wave equation [Equation (6)] for all real values of ω.

104. In each part, show that $u(x, y)$ and $v(x, y)$ satisfy the ***Cauchy–Riemann equations***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(a) $u = x^2 - y^2$, $v = 2xy$
(b) $u = e^x \cos y$, $v = e^x \sin y$
(c) $u = \ln(x^2 + y^2)$, $v = 2 \tan^{-1}(y/x)$

105. Show that if $u(x, y)$ and $v(x, y)$ each have equal mixed second partials, and if u and v satisfy the Cauchy–Riemann equations (Exercise 104), then u, v, and $u + v$ satisfy Laplace's equation (Exercise 101).

106. When two resistors having resistances R_1 ohms and R_2 ohms are connected in parallel, their combined resistance R in ohms is $R = R_1R_2/(R_1 + R_2)$. Show that

$$\frac{\partial^2 R}{\partial R_1^2} \frac{\partial^2 R}{\partial R_2^2} = \frac{4R^2}{(R_1 + R_2)^4}$$

107–110 Find the indicated partial derivatives. ■

107. $f(v, w, x, y) = 4v^2w^3x^4y^5$;
$\partial f/\partial v,\ \partial f/\partial w,\ \partial f/\partial x,\ \partial f/\partial y$

108. $w = r \cos st + e^u \sin ur$;
$\partial w/\partial r,\ \partial w/\partial s,\ \partial w/\partial t,\ \partial w/\partial u$

109. $f(v_1, v_2, v_3, v_4) = \dfrac{v_1^2 - v_2^2}{v_3^2 + v_4^2}$;
$\partial f/\partial v_1,\ \partial f/\partial v_2,\ \partial f/\partial v_3,\ \partial f/\partial v_4$

110. $V = xe^{2x-y} + we^{zw} + yw$;
$\partial V/\partial x,\ \partial V/\partial y,\ \partial V/\partial z,\ \partial V/\partial w$

111. Let $u(w, x, y, z) = xe^{yw} \sin^2 z$. Find
(a) $\dfrac{\partial u}{\partial x}(0, 0, 1, \pi)$ (b) $\dfrac{\partial u}{\partial y}(0, 0, 1, \pi)$
(c) $\dfrac{\partial u}{\partial w}(0, 0, 1, \pi)$ (d) $\dfrac{\partial u}{\partial z}(0, 0, 1, \pi)$
(e) $\dfrac{\partial^4 u}{\partial x \partial y \partial w \partial z}$ (f) $\dfrac{\partial^4 u}{\partial w \partial z \partial y^2}$.

112. Let $f(v, w, x, y) = 2v^{1/2}w^4x^{1/2}y^{2/3}$. Find $f_v(1, -2, 4, 8)$, $f_w(1, -2, 4, 8)$, $f_x(1, -2, 4, 8)$, and $f_y(1, -2, 4, 8)$.

113–114 Find $\partial w/\partial x_i$ for $i = 1, 2, \ldots, n$. ■

113. $w = \cos(x_1 + 2x_2 + \cdots + nx_n)$

114. $w = \left(\displaystyle\sum_{k=1}^{n} x_k\right)^{1/n}$

115–116 Describe the largest set on which Theorem 13.3.2 can be used to prove that f_{xy} and f_{yx} are equal on that set. Then confirm by direct computation that $f_{xy} = f_{yx}$ on the given set. ■

115. (a) $f(x, y) = 4x^3y + 3x^2y$ (b) $f(x, y) = x^3/y$

116. (a) $f(x, y) = \sqrt{x^2 + y^2 - 1}$
(b) $f(x, y) = \sin(x^2 + y^3)$

117. Let $f(x, y) = 2x^2 - 3xy + y^2$. Find $f_x(2, -1)$ and $f_y(2, -1)$ by evaluating the limits in Definition 13.3.1. Then check your work by calculating the derivative in the usual way.

118. Let $f(x, y) = (x^2 + y^2)^{2/3}$. Show that

$$f_x(x, y) = \begin{cases} \dfrac{4x}{3(x^2 + y^2)^{1/3}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Source: This problem, due to Don Cohen, appeared in *Mathematics and Computer Education*, Vol. 25, No. 2, 1991, p. 179.

119. Let $f(x, y) = (x^3 + y^3)^{1/3}$.
(a) Show that $f_y(0, 0) = 1$.
(b) At what points, if any, does $f_y(x, y)$ fail to exist?

120. Writing Explain how one might use the graph of the equation $z = f(x, y)$ to determine the signs of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ by inspection.

121. Writing Explain how one might use the graphs of some appropriate contours of $z = f(x, y)$ to determine the signs of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ by inspection.

✓ **QUICK CHECK ANSWERS 13.3** **1.** $\sin xy + xy \cos xy$; $x^2 \cos xy$ **2.** 9; 12 **3.** (a) $\frac{2}{3}\pi r h$ (b) $\frac{1}{3}\pi r^2$
4. $f_{xx}(x, y) = 2y^3$, $f_{yy}(x, y) = 6x^2 y$, $f_{xy}(x, y) = f_{yx}(x, y) = 6xy^2$

13.4 DIFFERENTIABILITY, DIFFERENTIALS, AND LOCAL LINEARITY

In this section we will extend the notion of differentiability to functions of two or three variables. Our definition of differentiability will be based on the idea that a function is differentiable at a point provided it can be very closely approximated by a linear function near that point. In the process, we will expand the concept of a "differential" to functions of more than one variable and define the "local linear approximation" of a function.

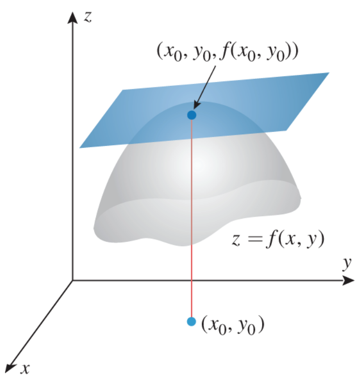
DIFFERENTIABILITY

Recall that a function f of one variable is called differentiable at x_0 if it has a derivative at x_0, that is, if the limit

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{1}$$

exists. As a consequence of (1) a differentiable function enjoys a number of other important properties:

- The graph of $y = f(x)$ has a nonvertical tangent line at the point $(x_0, f(x_0))$;
- f may be closely approximated by a linear function near x_0 (Section 2.9);
- f is continuous at x_0.



▲ **Figure 13.4.1**

Our primary objective in this section is to extend the notion of differentiability to functions of two or three variables in such a way that the natural analogs of these properties hold. For example, if a function $f(x, y)$ of two variables is differentiable at a point (x_0, y_0), we want it to be the case that

- the surface $z = f(x, y)$ has a nonvertical tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ (Figure 13.4.1);
- the values of f at points near (x_0, y_0) can be very closely approximated by the values of a linear function;
- f is continuous at (x_0, y_0).

One could reasonably conjecture that a function f of two or three variables should be called differentiable at a point if all the first-order partial derivatives of the function exist at that point. However, this condition alone is not strong enough to guarantee that the properties above hold. For instance, we saw in Example 9 of Section 13.3 that the mere existence of both first-order partial derivatives for a function is not sufficient to guarantee the continuity of the function. To determine what else we should include in our definition, it will be helpful to reexamine one of the consequences of differentiability for a *single-variable* function $f(x)$. Suppose that $f(x)$ is differentiable at $x = x_0$ and let

$$\Delta f = f(x_0 + \Delta x) - f(x_0)$$

denote the change in f that corresponds to the change Δx in x from x_0 to $x_0 + \Delta x$. We saw in Section 2.9 that

$$\Delta f \approx f'(x_0)\Delta x$$

provided Δx is close to 0. In fact, for Δx close to 0 the error $\Delta f - f'(x_0)\Delta x$ in this approximation will have magnitude much smaller than that of Δx because

$$\lim_{\Delta x \to 0} \frac{\Delta f - f'(x_0)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0) \right) = f'(x_0) - f'(x_0) = 0$$

Since the magnitude of Δx is just the distance between the points x_0 and $x_0 + \Delta x$, we see that when the two points are close together, the magnitude of the error in the approximation will be much smaller than the distance between the two points (Figure 13.4.2). The extension of this idea to functions of two or three variables is the "extra ingredient" needed in our definition of differentiability for multivariable functions.

For a function $f(x, y)$, the symbol Δf, called the ***increment*** of f, denotes the change in the value of $f(x, y)$ that results when (x, y) varies from some initial position (x_0, y_0) to some new position $(x_0 + \Delta x, y_0 + \Delta y)$; thus

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \tag{2}$$

(see Figure 13.4.3). [If a dependent variable $z = f(x, y)$ is used, then we will sometimes write Δz rather than Δf.] Let us assume that both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and (by analogy with the one-variable case) make the approximation

Show that if $f(x, y)$ is a linear function, then (3) becomes an equality.

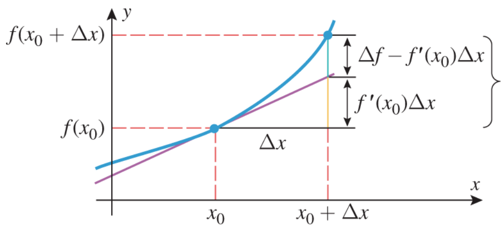
$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \tag{3}$$

For Δx and Δy close to 0, we would like the error

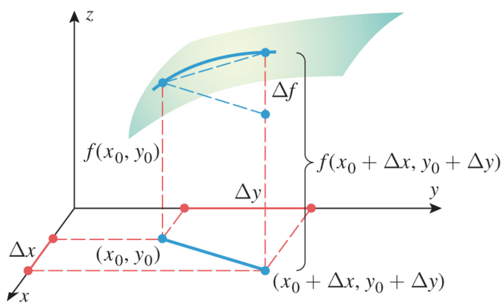
$$\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y$$

in this approximation to be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$. We can guarantee this by requiring that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$



▲ **Figure 13.4.2**



▲ **Figure 13.4.3**

Based on these ideas, we can now give our definition of differentiability for functions of two variables.

13.4.1 DEFINITION A function f of two variables is said to be ***differentiable*** at (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 \tag{4}$$

As with the one-variable case, verification of differentiability using this definition involves the computation of a limit.

▶ Example 1 Use Definition 13.4.1 to prove that $f(x, y) = x^2 + y^2$ is differentiable at $(0, 0)$.

Solution. The increment is

$$\Delta f = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = (\Delta x)^2 + (\Delta y)^2$$

Since $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, we have $f_x(0, 0) = f_y(0, 0) = 0$, and (4) becomes

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{(\Delta x)^2 + (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y)\to(0,0)} \sqrt{(\Delta x)^2 + (\Delta y)^2} = 0$$

Therefore, f is differentiable at $(0, 0)$. ◀

We now derive an important consequence of limit (4). Define a function

$$\epsilon = \epsilon(\Delta x, \Delta y) = \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \quad \text{for } (\Delta x, \Delta y) \neq (0, 0)$$

and define $\epsilon(0, 0)$ to be 0. Equation (4) then implies that

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \epsilon(\Delta x, \Delta y) = 0$$

Furthermore, it immediately follows from the definition of ϵ that

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{5}$$

In other words, if f is differentiable at (x_0, y_0), then Δf may be expressed as shown in (5), where $\epsilon \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ and where $\epsilon = 0$ if $(\Delta x, \Delta y) = (0, 0)$.

For functions of three variables we have an analogous definition of differentiability in terms of the increment

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0)$$

13.4.2 DEFINITION A function f of three variables is said to be ***differentiable*** at (x_0, y_0, z_0) provided $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, and $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z)\to(0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0 \tag{6}$$

In a manner similar to the two-variable case, we can express the limit (6) in terms of a function $\epsilon(\Delta x, \Delta y, \Delta z)$ that vanishes at $(\Delta x, \Delta y, \Delta z) = (0, 0, 0)$ and is continuous there. The details are left as an exercise for the reader.

If a function f of two variables is differentiable at each point of a region R in the xy-plane, then we say that f is ***differentiable on R***; and if f is differentiable at every point in the xy-plane, then we say that f is ***differentiable everywhere***. For a function f of three variables we have corresponding conventions.

DIFFERENTIABILITY AND CONTINUITY

Recall that we want a function to be continuous at every point at which it is differentiable. The next result shows this to be the case.

13.4.3 THEOREM *If a function is differentiable at a point, then it is continuous at that point.*

PROOF We will give the proof for $f(x, y)$, a function of two variables, since that will reveal the essential ideas. Assume that f is differentiable at (x_0, y_0). To prove that f is continuous at (x_0, y_0) we must show that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

which, on letting $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$, is equivalent to

$$\lim_{(\Delta x,\Delta y)\to(0,0)} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0)$$

By Equation (2) this is equivalent to

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \Delta f = 0$$

However, from Equation (5)

$$\begin{aligned}\lim_{(\Delta x,\Delta y)\to(0,0)} \Delta f &= \lim_{(\Delta x,\Delta y)\to(0,0)} \Big[f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \\ &\qquad + \epsilon(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2}\,\Big] \\ &= 0 + 0 + 0 \cdot 0 = 0 \quad \blacksquare\end{aligned}$$

The converse of Theorem 13.4.3 is false. For example, the function

$$f(x, y) = \sqrt{x^2 + y^2}$$

is continuous at $(0, 0)$ but is not differentiable at $(0, 0)$. Why not?

It can be difficult to verify that a function is differentiable at a point directly from the definition. The next theorem, whose proof is usually studied in more advanced courses, provides simple conditions for a function to be differentiable at a point.

13.4.4 THEOREM *If all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.*

For example, consider the function

$$f(x, y, z) = x + yz$$

Since $f_x(x, y, z) = 1$, $f_y(x, y, z) = z$, and $f_z(x, y, z) = y$ are defined and continuous everywhere, we conclude from Theorem 13.4.4 that f is differentiable everywhere.

DIFFERENTIALS

As with the one-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for a function of two variables and the approximation

$$\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z \tag{7}$$

for a function of three variables have a convenient formulation in the language of differentials. If $z = f(x, y)$ is differentiable at a point (x_0, y_0), we let

$$dz = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy \tag{8}$$

denote a new function with dependent variable dz and independent variables dx and dy. We refer to this function (also denoted df) as the ***total differential of z*** at (x_0, y_0) or as the ***total differential of f*** at (x_0, y_0). Similarly, for a function $w = f(x, y, z)$ of three variables we have the ***total differential of w*** at (x_0, y_0, z_0),

$$dw = f_x(x_0, y_0, z_0)\, dx + f_y(x_0, y_0, z_0)\, dy + f_z(x_0, y_0, z_0)\, dz \tag{9}$$

which is also referred to as the ***total differential of f*** at (x_0, y_0, z_0). It is common practice to omit the subscripts and write Equations (8) and (9) as

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy \tag{10}$$

and

$$dw = f_x(x, y, z)\,dx + f_y(x, y, z)\,dy + f_z(x, y, z)\,dz \tag{11}$$

In the two-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

can be written in the form

$$\Delta f \approx df \tag{12}$$

for $dx = \Delta x$ and $dy = \Delta y$. Equivalently, we can write approximation (12) as

$$\Delta z \approx dz \tag{13}$$

In other words, we can estimate the change Δz in z by the value of the differential dz where dx is the change in x and dy is the change in y. Furthermore, it follows from (4) that if Δx and Δy are close to 0, then the magnitude of the error in approximation (13) will be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$.

▶ Example 2 Use (13) to approximate the change in $z = xy^2$ from its value at (0.5, 1.0) to its value at (0.503, 1.004). Compare the magnitude of the error in this approximation with the distance between the points (0.5, 1.0) and (0.503, 1.004).

Solution. For $z = xy^2$ we have $dz = y^2\,dx + 2xy\,dy$. Evaluating this differential at $(x, y) = (0.5, 1.0)$, $dx = \Delta x = 0.503 - 0.5 = 0.003$, and $dy = \Delta y = 1.004 - 1.0 = 0.004$ yields

$$dz = 1.0^2(0.003) + 2(0.5)(1.0)(0.004) = 0.007$$

Since $z = 0.5$ at $(x, y) = (0.5, 1.0)$ and $z = 0.507032048$ at $(x, y) = (0.503, 1.004)$, we have

$$\Delta z = 0.507032048 - 0.5 = 0.007032048$$

and the error in approximating Δz by dz has magnitude

$$|dz - \Delta z| = |0.007 - 0.007032048| = 0.000032048$$

Since the distance between (0.5, 1.0) and $(0.503, 1.004) = (0.5 + \Delta x, 1.0 + \Delta y)$ is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.003)^2 + (0.004)^2} = \sqrt{0.000025} = 0.005$$

we have

$$\frac{|dz - \Delta z|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{0.000032048}{0.005} = 0.0064096 < \frac{1}{150}$$

Thus, the magnitude of the error in our approximation is less than $\frac{1}{150}$ of the distance between the two points. ◀

With the appropriate changes in notation, the preceding analysis can be extended to functions of three or more variables.

▶ Example 3 The length, width, and height of a rectangular box are measured with an error of at most 5%. Use a total differential to estimate the maximum percentage error that results if these quantities are used to calculate the diagonal of the box.

Solution. The diagonal D of a box with length x, width y, and height z is given by

$$D = \sqrt{x^2 + y^2 + z^2}$$

Let x_0, y_0, z_0, and $D_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$ denote the actual values of the length, width, height, and diagonal of the box. The total differential dD of D at (x_0, y_0, z_0) is given by

$$dD = \frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dx + \frac{y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dy + \frac{z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dz$$

If x, y, z, and $D=\sqrt{x^2+y^2+z^2}$ are the measured and computed values of the length, width, height, and diagonal, respectively, then

$$\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta z = z - z_0$$

and

$$\left|\frac{\Delta x}{x_0}\right| \le 0.05, \quad \left|\frac{\Delta y}{y_0}\right| \le 0.05, \quad \left|\frac{\Delta z}{z_0}\right| \le 0.05$$

We are seeking an estimate for the maximum size of $\Delta D/D_0$. With the aid of Equation (11) we have

$$\begin{aligned}\frac{\Delta D}{D_0} \approx \frac{dD}{D_0} &= \frac{1}{x_0^2+y_0^2+z_0^2}[x_0\Delta x + y_0\Delta y + z_0\Delta z] \\ &= \frac{1}{x_0^2+y_0^2+z_0^2}\left[x_0^2\frac{\Delta x}{x_0} + y_0^2\frac{\Delta y}{y_0} + z_0^2\frac{\Delta z}{z_0}\right]\end{aligned}$$

Since

$$\begin{aligned}\left|\frac{dD}{D_0}\right| &= \frac{1}{x_0^2+y_0^2+z_0^2}\left|x_0^2\frac{\Delta x}{x_0} + y_0^2\frac{\Delta y}{y_0} + z_0^2\frac{\Delta z}{z_0}\right| \\ &\le \frac{1}{x_0^2+y_0^2+z_0^2}\left(x_0^2\left|\frac{\Delta x}{x_0}\right| + y_0^2\left|\frac{\Delta y}{y_0}\right| + z_0^2\left|\frac{\Delta z}{z_0}\right|\right) \\ &\le \frac{1}{x_0^2+y_0^2+z_0^2}\left(x_0^2(0.05)+y_0^2(0.05)+z_0^2(0.05)\right) = 0.05\end{aligned}$$

we estimate the maximum percentage error in D to be 5%. ◀

LOCAL LINEAR APPROXIMATIONS

We now show that if a function f is differentiable at a point, then it can be very closely approximated by a linear function near that point. For example, suppose that $f(x,y)$ is differentiable at the point (x_0,y_0). Then approximation (3) can be written in the form

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

If we let $x = x_0 + \Delta x$ and $y = x_0 + \Delta y$, this approximation becomes

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \tag{14}$$

Show that if $f(x,y)$ is a linear function, then (14) becomes an equality.

which yields a linear approximation of $f(x,y)$. Since the error in this approximation is equal to the error in approximation (3), we conclude that for (x,y) close to (x_0,y_0), the error in (14) will be much smaller than the distance between these two points. When $f(x,y)$ is differentiable at (x_0,y_0) we let

Explain why the error in approximation (14) is the same as the error in approximation (3).

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \tag{15}$$

and refer to $L(x,y)$ as the ***local linear approximation to f at*** $(\boldsymbol{x_0}, \boldsymbol{y_0})$.

▶ Example 4 Let $L(x,y)$ denote the local linear approximation to $f(x,y)=\sqrt{x^2+y^2}$ at the point $(3,4)$. Compare the error in approximating

$$f(3.04, 3.98) = \sqrt{(3.04)^2+(3.98)^2}$$

by $L(3.04, 3.98)$ with the distance between the points $(3,4)$ and $(3.04, 3.98)$.

Solution. We have

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

with $f_x(3,4)=\frac{3}{5}$ and $f_y(3,4)=\frac{4}{5}$. Therefore, the local linear approximation to f at $(3,4)$ is given by

$$L(x,y) = 5 + \tfrac{3}{5}(x-3) + \tfrac{4}{5}(y-4)$$

Consequently,

$$f(3.04, 3.98) \approx L(3.04, 3.98) = 5 + \tfrac{3}{5}(0.04) + \tfrac{4}{5}(-0.02) = 5.008$$

Since

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$$

the error in the approximation is about $5.00819 - 5.008 = 0.00019$. This is less than $\frac{1}{200}$ of the distance

$$\sqrt{(3.04 - 3)^2 + (3.98 - 4)^2} \approx 0.045$$

between the points $(3, 4)$ and $(3.04, 3.98)$. ◀

For a function $f(x, y, z)$ that is differentiable at (x_0, y_0, z_0), the local linear approximation is

$$\begin{aligned} L(x, y, z) = {} & f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) \\ & + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \end{aligned} \tag{16}$$

We have formulated our definitions in this section in such a way that continuity and local linearity are consequences of differentiability. In Section 13.7 we will show that if a function $f(x, y)$ is differentiable at a point (x_0, y_0), then the graph of $L(x, y)$ is a nonvertical tangent plane to the graph of f at the point $(x_0, y_0, f(x_0, y_0))$.

✓QUICK CHECK EXERCISES 13.4

(See page 845 for answers.)

1. Assume that $f(x, y)$ is differentiable at (x_0, y_0) and let Δf denote the change in f from its value at (x_0, y_0) to its value at $(x_0 + \Delta x, y_0 + \Delta y)$.
   (a) $\Delta f \approx$ ________
   (b) The limit that guarantees the error in the approximation in part (a) is very small when both Δx and Δy are close to 0 is ________.
2. Compute the differential of each function.
   (a) $z = xe^{y^2}$ (b) $w = x \sin(yz)$
3. If f is differentiable at (x_0, y_0), then the local linear approximation to f at (x_0, y_0) is $L(x) =$ ________.
4. Assume that $f(1, -2) = 4$ and $f(x, y)$ is differentiable at $(1, -2)$ with $f_x(1, -2) = 2$ and $f_y(1, -2) = -3$. Estimate the value of $f(0.9, -1.950)$.

EXERCISE SET 13.4

FOCUS ON CONCEPTS

1. Suppose that a function $f(x, y)$ is differentiable at the point $(3, 4)$ with $f_x(3, 4) = 2$ and $f_y(3, 4) = -1$. If $f(3, 4) = 5$, estimate the value of $f(3.01, 3.98)$.
2. Suppose that a function $f(x, y)$ is differentiable at the point $(-1, 2)$ with $f_x(-1, 2) = 1$ and $f_y(-1, 2) = 3$. If $f(-1, 2) = 2$, estimate the value of $f(-0.99, 2.02)$.
3. Suppose that a function $f(x, y, z)$ is differentiable at the point $(1, 2, 3)$ with $f_x(1, 2, 3) = 1$, $f_y(1, 2, 3) = 2$, and $f_z(1, 2, 3) = 3$. If $f(1, 2, 3) = 4$, estimate the value of $f(1.01, 2.02, 3.03)$.
4. Suppose that a function $f(x, y, z)$ is differentiable at the point $(2, 1, -2)$, $f_x(2, 1, -2) = -1$, $f_y(2, 1, -2) = 1$, and $f_z(2, 1, -2) = -2$. If $f(2, 1, -2) = 0$, estimate the value of $f(1.98, 0.99, -1.97)$.
5. Use Definitions 13.4.1 and 13.4.2 to prove that a constant function of two or three variables is differentiable everywhere.
6. Use Definitions 13.4.1 and 13.4.2 to prove that a linear function of two or three variables is differentiable everywhere.
7. Use Definition 13.4.2 to prove that
$$f(x, y, z) = x^2 + y^2 + z^2$$
is differentiable at $(0, 0, 0)$.
8. Use Definition 13.4.2 to determine all values of r such that $f(x, y, z) = (x^2 + y^2 + z^2)^r$ is differentiable at $(0, 0, 0)$.

9–20 Compute the differential dz or dw of the function. ■

9. $z = 7x - 2y$
10. $z = e^{xy}$
11. $z = x^3y^2$
12. $z = 5x^2y^5 - 2x + 4y + 7$
13. $z = \tan^{-1} xy$
14. $z = e^{-3x} \cos 6y$
15. $w = 8x - 3y + 4z$
16. $w = e^{xyz}$
17. $w = x^3y^2z$
18. $w = 4x^2y^3z^7 - 3xy + z + 5$
19. $w = \tan^{-1}(xyz)$
20. $w = \sqrt{x} + \sqrt{y} + \sqrt{z}$

21–26 Use a total differential to approximate the change in the values of f from P to Q. Compare your estimate with the actual change in f. ■

21. $f(x, y) = x^2 + 2xy - 4x$; $P(1, 2)$, $Q(1.01, 2.04)$

22. $f(x, y) = x^{1/3}y^{1/2}$; $P(8, 9)$, $Q(7.78, 9.03)$

23. $f(x, y) = \dfrac{x + y}{xy}$; $P(-1, -2)$, $Q(-1.02, -2.04)$

24. $f(x, y) = \ln\sqrt{1 + xy}$; $P(0, 2)$, $Q(-0.09, 1.98)$

25. $f(x, y, z) = 2xy^2z^3$; $P(1, -1, 2)$, $Q(0.99, -1.02, 2.02)$

26. $f(x, y, z) = \dfrac{xyz}{x + y + z}$; $P(-1, -2, 4)$, $Q(-1.04, -1.98, 3.97)$

27–30 True–False Determine whether the statement is true or false. Explain your answer.

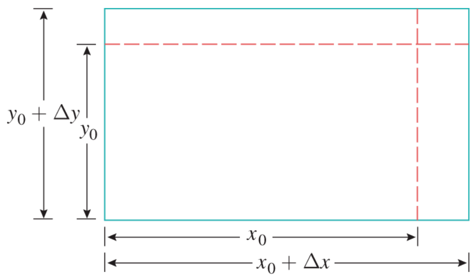
27. By definition, a function $f(x, y)$ is differentiable at (x_0, y_0) provided both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are defined.

28. For any point (x_0, y_0) in the domain of a function $f(x, y)$, we have
$$\lim_{(\Delta x, \Delta y)\to(0,0)} \Delta f = 0$$
where
$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

29. If f_x and f_y are both continuous at (x_0, y_0), then so is f.

30. The graph of a local linear approximation to a function $f(x, y)$ is a plane.

31. In the accompanying figure a rectangle with initial length x_0 and initial width y_0 has been enlarged, resulting in a rectangle with length $x_0 + \Delta x$ and width $y_0 + \Delta y$. What portion of the figure represents the increase in the area of the rectangle? What portion of the figure represents an approximation of the increase in area by a total differential?



▲ **Figure Ex-31**

32. The volume V of a right circular cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. Suppose that the height decreases from 20 in to 19.95 in and the radius increases from 4 in to 4.05 in. Compare the change in volume of the cone with an approximation of this change using a total differential.

33–40 (a) Find the local linear approximation L to the specified function f at the designated point P. (b) Compare the error in approximating f by L at the specified point Q with the distance between P and Q.

33. $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$; $P(4, 3)$, $Q(3.92, 3.01)$

34. $f(x, y) = x^{0.5}y^{0.3}$; $P(1, 1)$, $Q(1.05, 0.97)$

35. $f(x, y) = x \sin y$; $P(0, 0)$, $Q(0.003, 0.004)$

36. $f(x, y) = \ln xy$; $P(1, 2)$, $Q(1.01, 2.02)$

37. $f(x, y, z) = xyz$; $P(1, 2, 3)$, $Q(1.001, 2.002, 3.003)$

38. $f(x, y, z) = \dfrac{x + y}{y + z}$; $P(-1, 1, 1)$, $Q(-0.99, 0.99, 1.01)$

39. $f(x, y, z) = xe^{yz}$; $P(1, -1, -1)$, $Q(0.99, -1.01, -0.99)$

40. $f(x, y, z) = \ln(x + yz)$; $P(2, 1, -1)$, $Q(2.02, 0.97, -1.01)$

41. In each part, confirm that the stated formula is the local linear approximation at $(0, 0)$.
(a) $e^x \sin y \approx y$ (b) $\dfrac{2x + 1}{y + 1} \approx 1 + 2x - y$

42. Show that the local linear approximation of the function $f(x, y) = x^\alpha y^\beta$ at $(1, 1)$ is
$$x^\alpha y^\beta \approx 1 + \alpha(x - 1) + \beta(y - 1)$$

43. In each part, confirm that the stated formula is the local linear approximation at $(1, 1, 1)$.
(a) $xyz + 2 \approx x + y + z$ (b) $\dfrac{4x}{y + z} \approx 2x - y - z + 2$

44. Based on Exercise 42, what would you conjecture is the local linear approximation to $x^\alpha y^\beta z^\gamma$ at $(1, 1, 1)$? Verify your conjecture by finding this local linear approximation.

45. Suppose that a function $f(x, y)$ is differentiable at the point $(1, 1)$ with $f_x(1, 1) = 2$ and $f(1, 1) = 3$. Let $L(x, y)$ denote the local linear approximation of f at $(1, 1)$. If $L(1.1, 0.9) = 3.15$, find the value of $f_y(1, 1)$.

46. Suppose that a function $f(x, y)$ is differentiable at the point $(0, -1)$ with $f_y(0, -1) = -2$ and $f(0, -1) = 3$. Let $L(x, y)$ denote the local linear approximation of f at $(0, -1)$. If $L(0.1, -1.1) = 3.3$, find the value of $f_x(0, -1)$.

47. Suppose that a function $f(x, y, z)$ is differentiable at the point $(3, 2, 1)$ and $L(x, y, z) = x - y + 2z - 2$ is the local linear approximation to f at $(3, 2, 1)$. Find $f(3, 2, 1)$, $f_x(3, 2, 1)$, $f_y(3, 2, 1)$, and $f_z(3, 2, 1)$.

48. Suppose that a function $f(x, y, z)$ is differentiable at the point $(0, -1, -2)$ and $L(x, y, z) = x + 2y + 3z + 4$ is the local linear approximation to f at $(0, -1, -2)$. Find $f(0, -1, -2)$, $f_x(0, -1, -2)$, $f_y(0, -1, -2)$, and $f_z(0, -1, -2)$.

49–52 A function f is given along with a local linear approximation L to f at a point P. Use the information given to determine point P.

49. $f(x, y) = x^2 + y^2$; $L(x, y) = 2y - 2x - 2$

50. $f(x, y) = x^2y$; $L(x, y) = 4y - 4x + 8$

51. $f(x, y, z) = xy + z^2$; $L(x, y, z) = y + 2z - 1$

52. $f(x, y, z) = xyz$; $L(x, y, z) = x - y - z - 2$

53. The length and width of a rectangle are measured with errors of at most 3% and 5%, respectively. Use differentials to approximate the maximum percentage error in the calculated area.

54. The radius and height of a right circular cone are measured with errors of at most 1% and 4%, respectively. Use differentials to approximate the maximum percentage error in the calculated volume.

55. The period T of a simple pendulum with small oscillations is calculated from the formula $T = 2\pi\sqrt{L/g}$, where L is the length of the pendulum and g is the acceleration due to gravity. Suppose that measured values of L and g have errors of at most 0.5% and 0.1%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

56. The speed of sound ν in a medium of bulk modulus B and density ρ is

$$\nu = \sqrt{\frac{B}{\rho}}$$

Use a differential to estimate the maximum percentage change in ν produced by percentage changes in B and ρ of 0.7% and 0.3%, respectively.

57. The magnitude E of an electric field at a point outside a charged spherical shell is proportional to the total charge q of the shell and is inversely proportional to the square of the distance r from the point to the center of the shell. Suppose that measured values of q and r have errors of at most 0.2% and 0.5%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of E.

58. According to the ideal gas law, the pressure, temperature, and volume of a confined gas are related by $P = kT/V$, where k is a constant. Use differentials to approximate the percentage change in pressure if the temperature of a gas is increased 3% and the volume is increased 5%.

59. Suppose that certain measured quantities x and y have errors of at most $r\%$ and $s\%$, respectively. For each of the following formulas in x and y, use differentials to approximate the maximum possible error in the calculated result.
(a) xy (b) x/y (c) x^2y^3 (d) $x^3\sqrt{y}$

60. The total resistance R of three resistances R_1, R_2, and R_3, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Suppose that R_1, R_2, and R_3 are measured to be 100 ohms, 200 ohms, and 500 ohms, respectively, with a maximum error of 10% in each. Use differentials to approximate the maximum percentage error in the calculated value of R.

61. The area of a triangle is to be computed from the formula $A = \frac{1}{2}ab\sin\theta$, where a and b are the lengths of two sides and θ is the included angle. Suppose that a, b, and θ are measured to be 40 ft, 50 ft, and 30°, respectively. Use differentials to approximate the maximum error in the calculated value of A if the maximum errors in a, b, and θ are $\frac{1}{2}$ ft, $\frac{1}{4}$ ft, and 2°, respectively.

62. The length, width, and height of a rectangular box are measured with errors of at most $r\%$ (where r is small). Use differentials to approximate the maximum percentage error in the computed value of the volume.

63. Use Theorem 13.4.4 to prove that $f(x, y) = x^2 \sin y$ is differentiable everywhere.

64. Use Theorem 13.4.4 to prove that $f(x, y, z) = xy\sin z$ is differentiable everywhere.

65. Suppose that $f(x, y)$ is differentiable at the point (x_0, y_0) and let $z_0 = f(x_0, y_0)$. Prove that $g(x, y, z) = z - f(x, y)$ is differentiable at (x_0, y_0, z_0).

66. Suppose that Δf satisfies an equation in the form of (5), where $\epsilon(\Delta x, \Delta y)$ is continuous at $(\Delta x, \Delta y) = (0, 0)$ with $\epsilon(0, 0) = 0$. Prove that f is differentiable at (x_0, y_0).

67. Writing Discuss the similarities and differences between the definition of "differentiability" for a function of a single variable and the definition of "differentiability" for a function of two variables.

68. Writing Discuss the use of differentials in the approximation of increments and in the estimation of errors.

✓ **QUICK CHECK ANSWERS 13.4** **1.** (a) $f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$ (b) $\lim_{(\Delta x, \Delta y)\to(0,0)} \dfrac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$
2. (a) $dz = e^{y^2}dx + 2xye^{y^2}dy$ (b) $dw = \sin(yz)\,dx + xz\cos(yz)\,dy + xy\cos(yz)\,dz$ **3.** $f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
4. 3.65

13.5 THE CHAIN RULE

In this section we will derive versions of the chain rule for functions of two or three variables. These new versions will allow us to generate useful relationships among the derivatives and partial derivatives of various functions.

CHAIN RULES FOR DERIVATIVES

If y is a differentiable function of x and x is a differentiable function of t, then the chain rule for functions of one variable states that, under composition, y becomes a differentiable function of t with

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

We will now derive a version of the chain rule for functions of two variables.

Assume that $z = f(x, y)$ is a function of x and y, and suppose that x and y are in turn functions of a single variable t, say

$$x = x(t), \quad y = y(t)$$

The composition $z = f(x(t), y(t))$ then expresses z as a function of the single variable t. Thus, we can ask for the derivative dz/dt and we can inquire about its relationship to the derivatives $\partial z/\partial x$, $\partial z/\partial y$, dx/dt, and dy/dt. Letting Δx, Δy, and Δz denote the changes in x, y, and z, respectively, that correspond to a change of Δt in t, we have

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}, \quad \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \quad \text{and} \quad \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

It follows from (3) of Section 13.4 that

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \tag{1}$$

where the partial derivatives are evaluated at $(x(t), y(t))$. Dividing both sides of (1) by Δt yields

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} \tag{2}$$

Similarly, we can produce the analog of (2) for functions of three variables as follows: assume that $w = f(x, y, z)$ is a function of x, y, and z, and suppose that x, y, and z are functions of a single variable t. As above we define Δw, Δx, Δy, and Δz to be the changes in w, x, y, and z that correspond to a change of Δt in t. Then (7) in Section 13.4 implies that

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z \tag{3}$$

and dividing both sides of (3) by Δt yields

$$\frac{\Delta w}{\Delta t} \approx \frac{\partial w}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial w}{\partial z}\frac{\Delta z}{\Delta t} \tag{4}$$

Taking the limit as $\Delta t \to 0$ of both sides of (2) and (4) suggests the following results. (A complete proof of the two-variable case can be found in Web Appendix L.)

13.5.1 THEOREM (*Chain Rules for Derivatives*) *If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and*

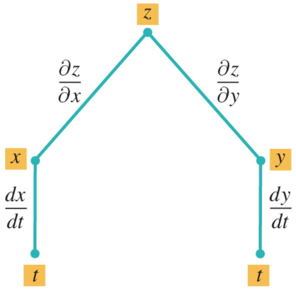
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{5}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

If each of the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ is differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \tag{6}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Figure 13.5.1

Formula (5) can be represented schematically by a "tree diagram" that is constructed as follows (Figure 13.5.1). Starting with z at the top of the tree and moving downward, join each variable by lines (or branches) to those variables on which it depends *directly*. Thus,

z is joined to x and y and these in turn are joined to t. Next, label each branch with a derivative whose "numerator" contains the variable at the top end of that branch and whose "denominator" contains the variable at the bottom end of that branch. This completes the "tree." To find the formula for dz/dt, follow the two paths through the tree that start with z and end with t. Each such path corresponds to a term in Formula (5).

Create a tree diagram for Formula (6).

Example 1 Suppose that

$$z = x^2y, \quad x = t^2, \quad y = t^3$$

Use the chain rule to find dz/dt, and check the result by expressing z as a function of t and differentiating directly.

Solution. By the chain rule [Formula (5)],

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2)\\ &= (2t^5)(2t) + (t^4)(3t^2) = 7t^6\end{aligned}$$

Alternatively, we can express z directly as a function of t,

$$z = x^2y = (t^2)^2(t^3) = t^7$$

and then differentiate to obtain $dz/dt = 7t^6$. However, this procedure may not always be convenient. ◀

Example 2 Suppose that

$$w = \sqrt{x^2+y^2+z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta$$

Use the chain rule to find $dw/d\theta$ when $\theta = \pi/4$.

Solution. From Formula (6) with θ in the place of t, we obtain

$$\begin{aligned}\frac{dw}{d\theta} &= \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}\\ &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(-\sin\theta) + \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(\cos\theta)\\ &\quad + \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(\sec^2\theta)\end{aligned}$$

When $\theta = \pi/4$, we have

$$x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad z = \tan\frac{\pi}{4} = 1$$

Confirm the result of Example 2 by expressing w directly as a function of θ.

Substituting $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$, $z = 1$, $\theta = \pi/4$ in the formula for $dw/d\theta$ yields

$$\begin{aligned}\left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(2)(2)\\ &= \sqrt{2}\end{aligned}$$ ◀

REMARK There are many variations in derivative notations, each of which gives the chain rule a different look. If $z = f(x, y)$, where x and y are functions of t, then some possibilities are

$$\frac{dz}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{df}{dt} = f_x x'(t) + f_y y'(t)$$

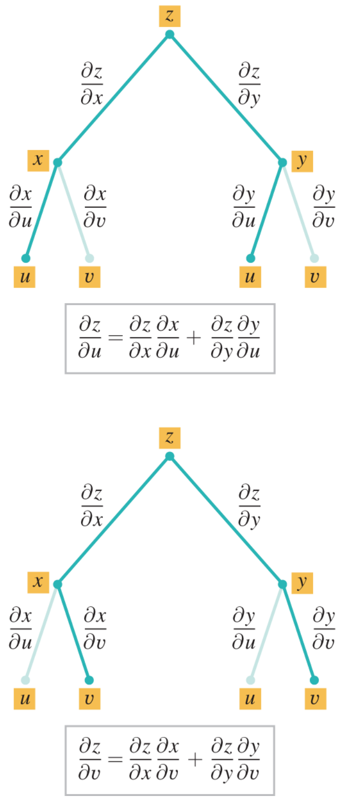
CHAIN RULES FOR PARTIAL DERIVATIVES

In Formula (5) the variables x and y are each functions of a single variable t. We now consider the case where x and y are each functions of two variables. Let $z = f(x, y)$ and suppose that x and y are functions of u and v, say

$$x = x(u, v), \quad y = y(u, v)$$

The composition $z = f(x(u, v), y(u, v))$ expresses z as a function of the two variables u and v. Thus, we can ask for the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$; and we can inquire about the relationship between these derivatives and the derivatives $\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$.

Similarly, if $w = f(x, y, z)$ and x, y, and z are each functions of u and v, then the composition $w = f(x(u, v), y(u, v), z(u, v))$ expresses w as a function of u and v. Thus we can also ask for the derivatives $\partial w/\partial u$ and $\partial w/\partial v$; and we can investigate the relationship between these derivatives, the partial derivatives $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$, and the partial derivatives of x, y, and z with respect to u and v.



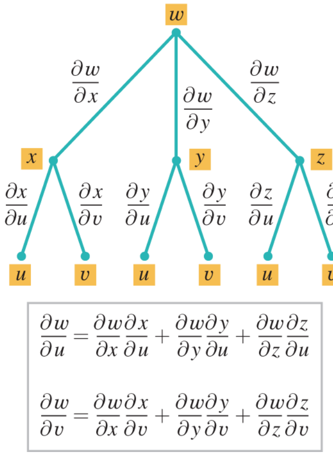
▲ **Figure 13.5.2**

13.5.2 THEOREM (*Chain Rules for Partial Derivatives*) *If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v), and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ has first-order partial derivatives at the point (u, v) given by*

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \textit{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \tag{7–8}$$

If each function $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ has first-order partial derivatives at the point (u, v), and if the function $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(u, v), y(u, v), z(u, v))$, then $w = f(x(u, v), y(u, v), z(u, v))$ has first-order partial derivatives at the point (u, v) given by

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \textit{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \tag{9–10}$$



▲ **Figure 13.5.3**

PROOF We will prove Formula (7); the other formulas are derived similarly. If v is held fixed, then $x = x(u, v)$ and $y = y(u, v)$ become functions of u alone. Thus, we are back to the case of Theorem 13.5.1. If we apply that theorem with u in place of t, and if we use ∂ rather than d to indicate that the variable v is fixed, we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad ■$$

Figures 13.5.2 and 13.5.3 show tree diagrams for the formulas in Theorem 13.5.2. As illustrated in Figure 13.5.2, the formula for $\partial z/\partial u$ can be obtained by tracing all paths through the tree that start with z and end with u, and the formula for $\partial z/\partial v$ can be obtained by tracing all paths through the tree that start with z and end with v. Figure 13.5.3 displays analogous results for $\partial w/\partial u$ and $\partial w/\partial v$.

▶ Example 3 Given that

$$z = e^{xy}, \quad x = 2u + v, \quad y = u/v$$

find $\partial z/\partial u$ and $\partial z/\partial v$ using the chain rule.

Solution.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right) = \left[2y + \frac{x}{v}\right]e^{xy}$$

$$= \left[\frac{2u}{v} + \frac{2u+v}{v}\right]e^{(2u+v)(u/v)} = \left[\frac{4u}{v} + 1\right]e^{(2u+v)(u/v)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right)$$

$$= \left[y - x\left(\frac{u}{v^2}\right)\right]e^{xy} = \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right]e^{(2u+v)(u/v)}$$

$$= -\frac{2u^2}{v^2}e^{(2u+v)(u/v)} \blacktriangleleft$$

▶ Example 4 Suppose that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

Solution. From the tree diagram and corresponding formulas in Figure 13.5.3 we obtain

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyuv)$$

and

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

If desired, we can express $\partial w/\partial u$ and $\partial w/\partial v$ in terms of u and v alone by replacing x, y, and z by their expressions in terms of u and v. ◀

OTHER VERSIONS OF THE CHAIN RULE

Although we will not prove it, the chain rule extends to functions $w = f(v_1, v_2, \ldots, v_n)$ of n variables. For example, if each v_i is a function of t, $i = 1, 2, \ldots, n$, the relevant formula is

$$\frac{dw}{dt} = \frac{\partial w}{\partial v_1}\frac{dv_1}{dt} + \frac{\partial w}{\partial v_2}\frac{dv_2}{dt} + \cdots + \frac{\partial w}{\partial v_n}\frac{dv_n}{dt} \tag{11}$$

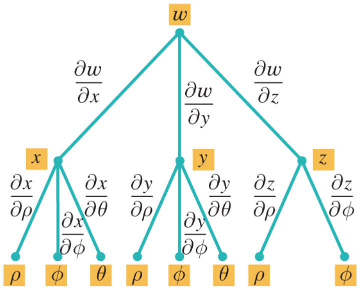
Note that (11) is a natural extension of Formulas (5) and (6) in Theorem 13.5.1.

There are infinitely many variations of the chain rule, depending on the number of variables and the choice of independent and dependent variables. A good working procedure is to use tree diagrams to derive new versions of the chain rule as needed.

▶ Example 5 Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

Use appropriate forms of the chain rule to find $\partial w/\partial \rho$ and $\partial w/\partial \theta$.



$$\frac{\partial w}{\partial \rho}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \theta}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

▲ **Figure 13.5.4**

Solution. From the tree diagram and corresponding formulas in Figure 13.5.4 we obtain

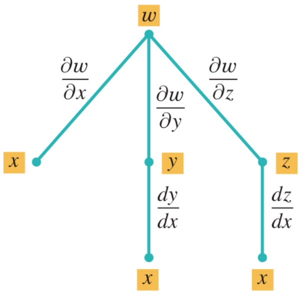
$$\begin{aligned}
\frac{\partial w}{\partial \rho} &= 2x\sin\phi\cos\theta + 2y\sin\phi\sin\theta - 2z\cos\phi \\
&= 2\rho\sin^2\phi\cos^2\theta + 2\rho\sin^2\phi\sin^2\theta - 2\rho\cos^2\phi \\
&= 2\rho\sin^2\phi(\cos^2\theta + \sin^2\theta) - 2\rho\cos^2\phi \\
&= 2\rho(\sin^2\phi - \cos^2\phi) \\
&= -2\rho\cos 2\phi \\
\frac{\partial w}{\partial \theta} &= (2x)(-\rho\sin\phi\sin\theta) + (2y)\rho\sin\phi\cos\theta \\
&= -2\rho^2\sin^2\phi\sin\theta\cos\theta + 2\rho^2\sin^2\phi\sin\theta\cos\theta \\
&= 0
\end{aligned}$$

This result is explained by the fact that w does not vary with θ. You can see this directly by expressing the variables x, y, and z in terms of ρ, ϕ, and θ in the formula for w. (Verify that $w = -\rho^2\cos 2\phi$.) ◀

▶ **Example 6** Suppose that

$$w = xy + yz, \quad y = \sin x, \quad z = e^x$$

Use an appropriate form of the chain rule to find dw/dx.



$$\frac{dw}{dx}=\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}+\frac{\partial w}{\partial z}\frac{dz}{dx}$$

▲ **Figure 13.5.5**

Solution. From the tree diagram and corresponding formulas in Figure 13.5.5 we obtain

$$\begin{aligned}
\frac{dw}{dx} &= y + (x+z)\cos x + ye^x \\
&= \sin x + (x + e^x)\cos x + e^x \sin x
\end{aligned}$$

This result can also be obtained by first expressing w explicitly in terms of x as

$$w = x\sin x + e^x \sin x$$

and then differentiating with respect to x; however, such direct substitution is not always possible. ◀

WARNING The symbol ∂z, unlike the differential dz, has no meaning of its own. For example, if we were to "cancel" partial symbols in the chain-rule formula

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

we would obtain

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial u}$$

which is false in cases where $\partial z/\partial u \neq 0$.

One of the principal uses of the chain rule for functions of a *single* variable was to compute formulas for the derivatives of compositions of functions. Theorems 13.5.1 and 13.5.2 are important not so much for the computation of formulas but because they allow us to express *relationships* among various derivatives. As an illustration, we revisit the topic of implicit differentiation.

IMPLICIT DIFFERENTIATION

Consider the special case where $z = f(x, y)$ is a function of x and y and y is a differentiable function of x. Equation (5) then becomes

$$\frac{dz}{dx}=\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx} \tag{12}$$

This result can be used to find derivatives of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = c \tag{13}$$

defines y implicitly as a differentiable function of x and we are interested in finding dy/dx. Differentiating both sides of (13) with respect to x and applying (12) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Thus, if $\partial f/\partial y \neq 0$, we obtain

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

In summary, we have the following result.

Show that the function $y = x$ is defined implicitly by the equation

$$x^2 - 2xy + y^2 = 0$$

but that Theorem 13.5.3 is not applicable for finding dy/dx.

13.5.3 THEOREM *If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x, and if $\partial f/\partial y \neq 0$, then*

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \tag{14}$$

Example 7 Given that

$$x^3 + y^2x - 3 = 0$$

find dy/dx using (14), and check the result using implicit differentiation.

Solution. By (14) with $f(x, y) = x^3 + y^2x - 3$,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

Alternatively, differentiating implicitly yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x^2 + y^2}{2yx}$$

which agrees with the result obtained by (14). ◀

The chain rule also applies to implicit partial differentiation. Consider the case where $w = f(x, y, z)$ is a function of x, y, and z and z is a differentiable function of x and y. It follows from Theorem 13.5.2 that

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \tag{15}$$

If the equation

$$f(x, y, z) = c \tag{16}$$

defines z implicitly as a differentiable function of x and y, then taking the partial derivative of each side of (16) with respect to x and applying (15) gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial f/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}$$

A similar result holds for $\partial z/\partial y$.

13.5.4 THEOREM *If the equation $f(x, y, z) = c$ defines z implicitly as a differentiable function of x and y, and if $\partial f/\partial z \neq 0$, then*

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

Example 8 Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\partial z/\partial x$ and $\partial z/\partial y$ at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Note the similarity between the expression for $\partial z/\partial y$ found in Example 8 and that found in Example 7 of Section 13.3.

Solution. By Theorem 13.5.4 with $f(x, y, z) = x^2 + y^2 + z^2$,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2x}{2z} = -\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

At the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, evaluating these derivatives gives $\partial z/\partial x = -1$ and $\partial z/\partial y = -\frac{1}{2}$. ◀

✓QUICK CHECK EXERCISES 13.5 (See page 855 for answers.)

1. Suppose that $z = xy^2$ and x and y are differentiable functions of t with $x = 1$, $y = -1$, $dx/dt = -2$, and $dy/dt = 3$ when $t = -1$. Then $dz/dt =$ ________ when $t = -1$.
2. Suppose that C is the graph of the equation $f(x, y) = 1$ and that this equation defines y implicitly as a differentiable function of x. If the point $(2, 1)$ belongs to C with $f_x(2, 1) = 3$ and $f_y(2, 1) = -1$, then the tangent line to C at the point $(2, 1)$ has slope ________.
3. A rectangle is growing in such a way that when its length is 5 ft and its width is 2 ft, the length is increasing at a rate of 3 ft/s and its width is increasing at a rate of 4 ft/s. At this instant the area of the rectangle is growing at a rate of ________.
4. Suppose that $z = x/y$, where x and y are differentiable functions of u and v such that $x = 3$, $y = 1$, $\partial x/\partial u = 4$, $\partial x/\partial v = -2$, $\partial y/\partial u = 1$, and $\partial y/\partial v = -1$ when $u = 2$ and $v = 1$. When $u = 2$ and $v = 1$, $\partial z/\partial u =$ ________ and $\partial z/\partial v =$ ________.

EXERCISE SET 13.5

1–6 Use an appropriate form of the chain rule to find dz/dt.

1. $z = 3x^2y^3$; $x = t^4, y = t^2$
2. $z = \ln(2x^2 + y)$; $x = \sqrt{t}, y = t^{2/3}$
3. $z = 3\cos x - \sin xy$; $x = 1/t, y = 3t$
4. $z = \sqrt{1 + x - 2xy^4}$; $x = \ln t, y = t$
5. $z = e^{1-xy}$; $x = t^{1/3}, y = t^3$
6. $z = \cosh^2 xy$; $x = t/2, y = e^t$

7–10 Use an appropriate form of the chain rule to find dw/dt.

7. $w = 5x^2y^3z^4$; $x = t^2, y = t^3, z = t^5$
8. $w = \ln(3x^2 - 2y + 4z^3)$; $x = t^{1/2}, y = t^{2/3}, z = t^{-2}$
9. $w = 5\cos xy - \sin xz$; $x = 1/t, y = t, z = t^3$
10. $w = \sqrt{1 + x - 2yz^4x}$; $x = \ln t, y = t, z = 4t$

FOCUS ON CONCEPTS

11. Suppose that
$$w = x^3y^2z^4; \quad x = t^2, \quad y = t + 2, \quad z = 2t^4$$
Find the rate of change of w with respect to t at $t = 1$ by using the chain rule, and then check your work by expressing w as a function of t and differentiating.
12. Suppose that
$$w = x\sin yz^2; \quad x = \cos t, \quad y = t^2, \quad z = e^t$$
Find the rate of change of w with respect to t at $t = 0$ by using the chain rule, and then check your work by expressing w as a function of t and differentiating.
13. Suppose that $z = f(x, y)$ is differentiable at the point $(4, 8)$ with $f_x(4, 8) = 3$ and $f_y(4, 8) = -1$. If $x = t^2$ and $y = t^3$, find dz/dt when $t = 2$.

14. Suppose that $w = f(x, y, z)$ is differentiable at the point $(1, 0, 2)$ with $f_x(1, 0, 2) = 1$, $f_y(1, 0, 2) = 2$, and $f_z(1, 0, 2) = 3$. If $x = t$, $y = \sin(\pi t)$, and $z = t^2 + 1$, find dw/dt when $t = 1$.

15. Explain how the product rule for functions of a single variable may be viewed as a consequence of the chain rule applied to a particular function of two variables.

16. A student attempts to differentiate the function x^x using the power rule, mistakenly getting $x \cdot x^{x-1}$. A second student attempts to differentiate x^x by treating it as an exponential function, mistakenly getting $(\ln x)x^x$. Use the chain rule to explain why the correct derivative is the sum of these two incorrect results.

17–22 Use appropriate forms of the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$.

17. $z = 8x^2y - 2x + 3y$; $x = uv, y = u - v$

18. $z = x^2 - y\tan x$; $x = u/v, y = u^2v^2$

19. $z = x/y$; $x = 2\cos u, y = 3\sin v$

20. $z = 3x - 2y$; $x = u + v\ln u, y = u^2 - v\ln v$

21. $z = e^{x^2y}$; $x = \sqrt{uv}, y = 1/v$

22. $z = \cos x \sin y$; $x = u - v, y = u^2 + v^2$

23–30 Use appropriate forms of the chain rule to find the derivatives.

23. Let $T = x^2y - xy^3 + 2$; $x = r\cos\theta$, $y = r\sin\theta$. Find $\partial T/\partial r$ and $\partial T/\partial\theta$.

24. Let $R = e^{2s-t^2}$; $s = 3\phi, t = \phi^{1/2}$. Find $dR/d\phi$.

25. Let $t = u/v$; $u = x^2 - y^2, v = 4xy^3$. Find $\partial t/\partial x$ and $\partial t/\partial y$.

26. Let $w = rs/(r^2 + s^2)$; $r = uv$, $s = u - 2v$. Find $\partial w/\partial u$ and $\partial w/\partial v$.

27. Let $z = \ln(x^2 + 1)$, where $x = r\cos\theta$. Find $\partial z/\partial r$ and $\partial z/\partial\theta$.

28. Let $u = rs^2\ln t$, $r = x^2$, $s = 4y + 1$, $t = xy^3$. Find $\partial u/\partial x$ and $\partial u/\partial y$.

29. Let $w = 4x^2 + 4y^2 + z^2, x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Find $\partial w/\partial\rho$, $\partial w/\partial\phi$, and $\partial w/\partial\theta$.

30. Let $w = 3xy^2z^3, y = 3x^2 + 2, z = \sqrt{x - 1}$. Find dw/dx.

31. Use a chain rule to find the value of $\left.\dfrac{dw}{ds}\right|_{s=1/4}$ if $w = r^2 - r\tan\theta$; $r = \sqrt{s}$, $\theta = \pi s$.

32. Use a chain rule to find the values of

$$\left.\frac{\partial f}{\partial u}\right|_{u=1,v=-2} \quad \text{and} \quad \left.\frac{\partial f}{\partial v}\right|_{u=1,v=-2}$$

if $f(x, y) = x^2y^2 - x + 2y$; $x = \sqrt{u}, y = uv^3$.

33. Use a chain rule to find the values of

$$\left.\frac{\partial z}{\partial r}\right|_{r=2,\theta=\pi/6} \quad \text{and} \quad \left.\frac{\partial z}{\partial\theta}\right|_{r=2,\theta=\pi/6}$$

if $z = xye^{x/y}$; $x = r\cos\theta, y = r\sin\theta$.

34. Use a chain rule to find $\left.\dfrac{dz}{dt}\right|_{t=3}$ if $z = x^2y$; $x = t^2, y = t + 7$.

35. Let a and b denote two sides of a triangle and let θ denote the included angle. Suppose that a, b, and θ vary with time in such a way that the area of the triangle remains constant. At a certain instant $a = 5$ cm, $b = 4$ cm, and $\theta = \pi/6$ radians, and at that instant both a and b are increasing at a rate of 3 cm/s. Estimate the rate at which θ is changing at that instant.

36. The voltage, V (in volts), across a circuit is given by Ohm's law: $V = IR$, where I is the current (in amperes) flowing through the circuit and R is the resistance (in ohms). If two circuits with resistances R_1 and R_2 are connected in parallel, then their combined resistance, R, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose that the current is 3 amperes and is increasing at 10^{-2} ampere/s, R_1 is 2 ohms and is increasing at 0.4 ohm/s, and R_2 is 5 ohms and is decreasing at 0.7 ohm/s. Estimate the rate at which the voltage is changing.

37–40 True–False Determine whether the statement is true or false. Explain your answer.

37. The symbols ∂z and ∂x are defined in such a way that the partial derivative $\partial z/\partial x$ can be interpreted as a ratio.

38. If z is a differentiable function of x_1, x_2, and x_3, and if x_i is a differentiable function of t for $i = 1, 2, 3$, then z is a differentiable function of t and

$$\frac{dz}{dt} = \sum_{i=1}^{3} \frac{\partial z}{\partial x_i}\frac{dx_i}{dt}$$

39. If z is a differentiable function of x and y, and if x and y are twice differentiable functions of t, then z is a twice differentiable function of t and

$$\frac{d^2z}{dt^2} = \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}$$

40. If $f(x, y)$ is a differentiable function of x and y, and if the line $y = x$ is a contour of f, then $f_y(t, t) = -f_x(t, t)$ for all real numbers t.

41–44 Use Theorem 13.5.3 to find dy/dx and check your result using implicit differentiation.

41. $x^2y^3 + \cos y = 0$

42. $x^3 - 3xy^2 + y^3 = 5$

43. $e^{xy} + ye^y = 1$

44. $x - \sqrt{xy} + 3y = 4$

45–48 Find $\partial z/\partial x$ and $\partial z/\partial y$ by implicit differentiation, and confirm that the results obtained agree with those predicted by the formulas in Theorem 13.5.4.

45. $x^2 - 3yz^2 + xyz - 2 = 0$

46. $\ln(1 + z) + xy^2 + z = 1$

47. $ye^x - 5\sin 3z = 3z$

48. $e^{xy}\cos yz - e^{yz}\sin xz + 2 = 0$

49. (a) Suppose that $z = f(u)$ and $u = g(x, y)$. Draw a tree diagram, and use it to construct chain rules that express $\partial z/\partial x$ and $\partial z/\partial y$ in terms of dz/du, $\partial u/\partial x$, and $\partial u/\partial y$.

(b) Show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{dz}{du}\frac{\partial^2 u}{\partial x^2} + \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{dz}{du}\frac{\partial^2 u}{\partial y^2} + \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial y}\right)^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{dz}{du}\frac{\partial^2 u}{\partial y \partial x} + \frac{d^2 z}{du^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

50. (a) Let $z = f(x^2 - y^2)$. Use the result in Exercise 49(a) to show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

(b) Let $z = f(xy)$. Use the result in Exercise 49(a) to show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

(c) Confirm the result in part (a) in the case where $z = \sin(x^2 - y^2)$.

(d) Confirm the result in part (b) in the case where $z = e^{xy}$.

51. Let f be a differentiable function of one variable, and let $z = f(x + 2y)$. Show that

$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

52. Let f be a differentiable function of one variable, and let $z = f(x^2 + y^2)$. Show that

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

53. Let f be a differentiable function of one variable, and let $w = f(u)$, where $u = x + 2y + 3z$. Show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 6\frac{dw}{du}$$

54. Let f be a differentiable function of one variable, and let $w = f(\rho)$, where $\rho = (x^2 + y^2 + z^2)^{1/2}$. Show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = \left(\frac{dw}{d\rho}\right)^2$$

55. Let $z = f(x - y, y - x)$. Show that $\partial z/\partial x + \partial z/\partial y = 0$.

56. Let f be a differentiable function of three variables and suppose that $w = f(x - y, y - z, z - x)$. Show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

57. Suppose that the equation $z = f(x, y)$ is expressed in the polar form $z = g(r, \theta)$ by making the substitution $x = r\cos\theta$ and $y = r\sin\theta$.

(a) View r and θ as functions of x and y and use implicit differentiation to show that

$$\frac{\partial r}{\partial x} = \cos\theta \quad \text{and} \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

(b) View r and θ as functions of x and y and use implicit differentiation to show that

$$\frac{\partial r}{\partial y} = \sin\theta \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

(c) Use the results in parts (a) and (b) to show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial \theta}\sin\theta$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial z}{\partial \theta}\cos\theta$$

(d) Use the result in part (c) to show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

(e) Use the result in part (c) to show that if $z = f(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

then $z = g(r, \theta)$ satisfies the equation

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = 0$$

and conversely. The latter equation is called the ***polar form of Laplace's equation***.

58. Show that the function

$$z = \tan^{-1}\frac{2xy}{x^2 - y^2}$$

satisfies Laplace's equation; then make the substitution $x = r\cos\theta$, $y = r\sin\theta$, and show that the resulting function of r and θ satisfies the polar form of Laplace's equation given in part (e) of Exercise 57.

59. (a) Show that if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations (Exercise 104, Section 13.3), and if $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

This is called the ***polar form of the Cauchy–Riemann equations***.

(b) Show that the functions

$$u = \ln(x^2 + y^2), \quad v = 2\tan^{-1}(y/x)$$

satisfy the Cauchy–Riemann equations; then make the substitution $x = r\cos\theta$, $y = r\sin\theta$, and show that the resulting functions of r and θ satisfy the polar form of the Cauchy–Riemann equations.

60. Recall from Formula (6) of Section 13.3 that under appropriate conditions a plucked string satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where c is a positive constant.

(a) Show that a function of the form $u(x, t) = f(x + ct)$ satisfies the wave equation.

(b) Show that a function of the form $u(x, t) = g(x - ct)$ satisfies the wave equation.

(c) Show that a function of the form

$$u(x, t) = f(x + ct) + g(x - ct)$$

satisfies the wave equation.

(d) It can be proved that every solution of the wave equation is expressible in the form stated in part (c). Confirm that $u(x, t) = \sin t \sin x$ satisfies the wave equation in which $c = 1$, and then use appropriate trigonometric identities to express this function in the form $f(x + t) + g(x - t)$.

61. Let f be a differentiable function of three variables, and let $w = f(x, y, z)$, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. Express $\partial w/\partial \rho$, $\partial w/\partial \phi$, and $\partial w/\partial \theta$ in terms of $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$.

62. Let $w = f(x, y, z)$ be differentiable, where $z = g(x, y)$. Taking x and y as the independent variables, express each of the following in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, $\partial z/\partial x$, and $\partial z/\partial y$.
(a) $\partial w/\partial x$ (b) $\partial w/\partial y$

63. Let $w = \ln(e^r + e^s + e^t + e^u)$. Show that
$$w_{rstu} = -6e^{r+s+t+u-4w}$$
[*Hint:* Take advantage of the relationship $e^w = e^r + e^s + e^t + e^u$.]

64. Suppose that w is a differentiable function of x_1, x_2, and x_3, and
$$x_1 = a_1y_1 + b_1y_2$$
$$x_2 = a_2y_1 + b_2y_2$$
$$x_3 = a_3y_1 + b_3y_2$$
where the a's and b's are constants. Express $\partial w/\partial y_1$ and $\partial w/\partial y_2$ in terms of $\partial w/\partial x_1$, $\partial w/\partial x_2$, and $\partial w/\partial x_3$.

65. (a) Let w be a differentiable function of x_1, x_2, x_3, and x_4, and let each x_i be a differentiable function of t. Find a chain-rule formula for dw/dt.
(b) Let w be a differentiable function of x_1, x_2, x_3, and x_4, and let each x_i be a differentiable function of v_1, v_2, and v_3. Find chain-rule formulas for $\partial w/\partial v_1$, $\partial w/\partial v_2$, and $\partial w/\partial v_3$.

66. Let $w = (x_1^2 + x_2^2 + \cdots + x_n^2)^k$, where $n \geq 2$. For what values of k does
$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \cdots + \frac{\partial^2 w}{\partial x_n^2} = 0$$
hold?

67. Derive the identity
$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))g'(x) - f(h(x))h'(x)$$
by letting $u = g(x)$ and $v = h(x)$ and then differentiating the function
$$F(u, v) = \int_v^u f(t)\,dt$$
with respect to x.

68. Prove: If f, f_x, and f_y are continuous on a circular region containing $A(x_0, y_0)$ and $B(x_1, y_1)$, then there is a point (x^*, y^*) on the line segment joining A and B such that
$$f(x_1, y_1) - f(x_0, y_0) = f_x(x^*, y^*)(x_1 - x_0) + f_y(x^*, y^*)(y_1 - y_0)$$
This result is the two-dimensional version of the Mean-Value Theorem. [*Hint:* Express the line segment joining A and B in parametric form and use the Mean-Value Theorem for functions of one variable.]

69. Prove: If $f_x(x, y) = 0$ and $f_y(x, y) = 0$ throughout a circular region, then $f(x, y)$ is constant on that region. [*Hint:* Use the result of Exercise 68.]

70. **Writing** Use differentials to give an informal justification for the chain rules for derivatives.

71. **Writing** Compare the use of the formula
$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$
with the process of implicit differentiation.

✓ **QUICK CHECK ANSWERS 13.5** **1.** -8 **2.** 3 **3.** 26 ft^2/s **4.** 1; 1

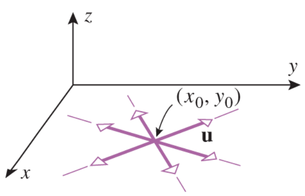
13.6 DIRECTIONAL DERIVATIVES AND GRADIENTS

The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ represent the rates of change of $f(x, y)$ in directions parallel to the x- and y-axes. In this section we will investigate rates of change of $f(x, y)$ in other directions.

DIRECTIONAL DERIVATIVES

In this section we extend the concept of a *partial* derivative to the more general notion of a *directional* derivative. We have seen that the partial derivatives of a function give the instantaneous rates of change of that function in directions parallel to the coordinate axes. Directional derivatives allow us to compute the rates of change of a function with respect to distance in *any* direction.

Suppose that we wish to compute the instantaneous rate of change of a function $f(x, y)$ with respect to distance from a point (x_0, y_0) in some direction. Since there are infinitely many different directions from (x_0, y_0) in which we could move, we need a convenient method for describing a specific direction starting at (x_0, y_0). One way to do this is to use a unit vector
$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$$
that has its initial point at (x_0, y_0) and points in the desired direction (Figure 13.6.1). This vector determines a line l in the xy-plane that can be expressed parametrically as
$$x = x_0 + su_1, \quad y = y_0 + su_2 \tag{1}$$



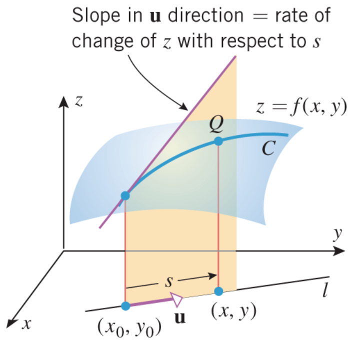
▲ Figure 13.6.1

Since u is a unit vector, s is the arc length parameter that has its reference point at (x_0, y_0) and has positive values in the direction of $\mathbf{u}$. For $s = 0$, the point (x, y) is at the reference point (x_0, y_0), and as s increases, the point (x, y) moves along l in the direction of $\mathbf{u}$. On the line l the variable $z = f(x_0 + su_1, y_0 + su_2)$ is a function of the parameter s. The value of the derivative dz/ds at $s = 0$ then gives an instantaneous rate of change of $f(x, y)$ with respect to distance from (x_0, y_0) in the direction of $\mathbf{u}$.

13.6.1 DEFINITION If $f(x, y)$ is a function of x and y, and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the ***directional derivative of f in the direction of*** $\mathbf{u}$ at (x_0, y_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0)$ and is defined by

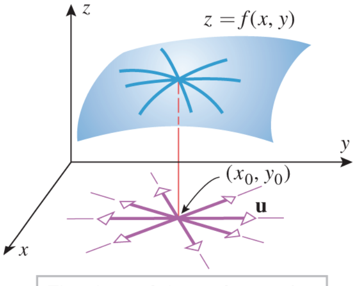
$$D_{\mathbf{u}}f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2)\right]_{s=0} \tag{2}$$

provided this derivative exists.



▲ Figure 13.6.2

Geometrically, $D_{\mathbf{u}}f(x_0, y_0)$ can be interpreted as the ***slope of the surface $z = f(x, y)$ in the direction of*** $\mathbf{u}$ at the point $(x_0, y_0, f(x_0, y_0))$ (Figure 13.6.2). Usually the value of $D_{\mathbf{u}}f(x_0, y_0)$ will depend on both the point (x_0, y_0) and the direction $\mathbf{u}$. Thus, at a fixed point the slope of the surface may vary with the direction (Figure 13.6.3). Analytically, the directional derivative represents the ***instantaneous rate of change of f(x, y) with respect to distance in the direction of*** $\mathbf{u}$ at the point (x_0, y_0).



The slope of the surface varies with the direction of **u**.

▲ Figure 13.6.3

▶ **Example 1** Let $f(x, y) = xy$. Find and interpret $D_{\mathbf{u}}f(1, 2)$ for the unit vector

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Solution. It follows from Equation (2) that

$$D_{\mathbf{u}}f(1, 2) = \frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right)\right]_{s=0}$$

Since

$$f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right) = \left(1 + \frac{\sqrt{3}s}{2}\right)\left(2 + \frac{s}{2}\right) = \frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2$$

we have

$$\begin{aligned} D_{\mathbf{u}}f(1, 2) &= \frac{d}{ds}\left[\frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2\right]_{s=0} \\ &= \left[\frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3}\right]_{s=0} = \frac{1}{2} + \sqrt{3} \end{aligned}$$

Since $\frac{1}{2} + \sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point $(1, 2)$ in the direction of $\mathbf{u}$, the function $f(x, y) = xy$ will increase by about 2.23 times the distance moved. ◀

The definition of a directional derivative for a function $f(x, y, z)$ of three variables is similar to Definition 13.6.1.

13.6.2 DEFINITION If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, and if $f(x, y, z)$ is a function of x, y, and z, then the ***directional derivative of f in the direction of* u** at (x_0, y_0, z_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0, z_0)$ and is defined by

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2, z_0 + su_3)\right]_{s=0} \tag{3}$$

provided this derivative exists.

Although Equation (3) does not have a convenient geometric interpretation, we can still interpret directional derivatives for functions of three variables in terms of instantaneous rates of change in a specified direction.

For a function that is differentiable at a point, directional derivatives exist in every direction from the point and can be computed directly in terms of the first-order partial derivatives of the function.

13.6.3 THEOREM

(*a*) *If $f(x, y)$ is differentiable at (x_0, y_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists and is given by*

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \tag{4}$$

(*b*) *If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0, z_0)$ exists and is given by*

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3 \tag{5}$$

PROOF We will give the proof of part (*a*); the proof of part (*b*) is similar and will be omitted. The function $z = f(x_0 + su_1, y_0 + su_2)$ is the composition of the function $z = f(x, y)$ with the functions

$$x = x(s) = x_0 + su_1 \quad \text{and} \quad y = y(s) = y_0 + su_2$$

As such, the chain rule in Formula (5) of Section 13.5 immediately gives

$$\begin{aligned} D_{\mathbf{u}}f(x_0, y_0) &= \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2)\right]_{s=0} \\ &= \left.\frac{dz}{ds}\right|_{s=0} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \quad \blacksquare \end{aligned}$$

We can use Theorem 13.6.3 to confirm the result of Example 1. For $f(x, y) = xy$ we have $f_x(1, 2) = 2$ and $f_y(1, 2) = 1$ (verify). With

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Equation (4) becomes

$$D_{\mathbf{u}}f(1, 2) = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2} = \sqrt{3} + \frac{1}{2}$$

which agrees with our solution in Example 1.

Recall from Formula (13) of Section 11.2 that a unit vector $\mathbf{u}$ in the xy-plane can be expressed as

$$\mathbf{u} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \tag{6}$$

where ϕ is the angle from the positive x-axis to $\mathbf{u}$. Thus, Formula (4) can also be expressed as

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)\cos\phi + f_y(x_0, y_0)\sin\phi \tag{7}$$

▶ **Example 2** Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x-axis.

Solution. The partial derivatives of f are

$$f_x(x, y) = ye^{xy}, \quad f_y(x, y) = xe^{xy}$$
$$f_x(-2, 0) = 0, \quad f_y(-2, 0) = -2$$

The unit vector $\mathbf{u}$ that makes an angle of $\pi/3$ with the positive x-axis is

$$\mathbf{u} = \cos(\pi/3)\mathbf{i} + \sin(\pi/3)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

Thus, from (7)

$$\begin{aligned} D_{\mathbf{u}}f(-2, 0) &= f_x(-2, 0)\cos(\pi/3) + f_y(-2, 0)\sin(\pi/3) \\ &= 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3} \end{aligned}$$ ◀

Note that in Example 3 we used a ***unit vector*** to specify the direction of the directional derivative. This is required in order to apply either Formula (4) or Formula (5).

▶ **Example 3** Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at the point $(1, -2, 0)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution. The partial derivatives of f are

$$\begin{array}{lll} f_x(x, y, z) = 2xy, & f_y(x, y, z) = x^2 - z^3, & f_z(x, y, z) = -3yz^2 + 1 \\ f_x(1, -2, 0) = -4, & f_y(1, -2, 0) = 1, & f_z(1, -2, 0) = 1 \end{array}$$

Since $\mathbf{a}$ is not a unit vector, we normalize it, getting

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{9}}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Formula (5) then yields

$$D_{\mathbf{u}}f(1, -2, 0) = (-4)\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3$$ ◀

THE GRADIENT

Formula (4) can be expressed in the form of a dot product as

$$\begin{aligned} D_{\mathbf{u}}f(x_0, y_0) &= (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\ &= (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u} \end{aligned}$$

Similarly, Formula (5) can be expressed as

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}) \cdot \mathbf{u}$$

In both cases the directional derivative is obtained by dotting the direction vector $\mathbf{u}$ with a new vector constructed from the first-order partial derivatives of f.

Remember that ∇f is not a product of ∇ and f. Think of ∇ as an "operator" that acts on a function f to produce the gradient ∇f.

13.6.4 DEFINITION

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \tag{8}$$

(b) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \tag{9}$$

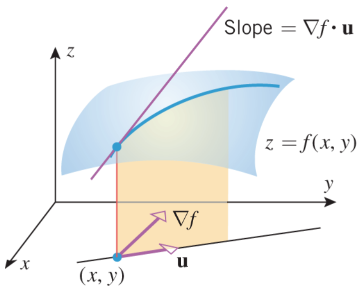
The symbol ∇ (read "del") is an inverted delta. (It is sometimes called a "nabla" because of its similarity in form to an ancient Hebrew ten-stringed harp of that name.)

Formulas (4) and (5) can now be written as

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} \tag{10}$$

and

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u} \tag{11}$$



▲ **Figure 13.6.4**

respectively. For example, using Formula (11) our solution to Example 3 would take the form

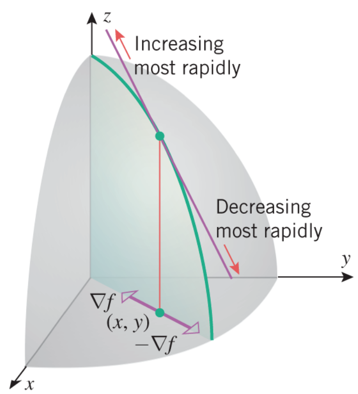
$$\begin{aligned} D_{\mathbf{u}}f(1, -2, 0) = \nabla f(1, -2, 0) \cdot \mathbf{u} &= (-4\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \left(\tfrac{2}{3}\mathbf{i} + \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}\right) \\ &= (-4)\left(\tfrac{2}{3}\right) + \tfrac{1}{3} - \tfrac{2}{3} = -3 \end{aligned}$$

Formula (10) can be interpreted to mean that the slope of the surface $z = f(x, y)$ at the point (x_0, y_0) in the direction of $\mathbf{u}$ is the dot product of the gradient with $\mathbf{u}$ (Figure 13.6.4).

PROPERTIES OF THE GRADIENT

The gradient is not merely a notational device to simplify the formula for the directional derivative; we will see that the length and direction of the gradient ∇f provide important information about the function f and the surface $z = f(x, y)$. For example, suppose that $\nabla f(x, y) \neq \mathbf{0}$, and let us use Formula (4) of Section 11.3 to rewrite (10) as

$$D_{\mathbf{u}}\, f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \|\nabla f(x, y)\|\|\mathbf{u}\|\cos\theta = \|\nabla f(x, y)\|\cos\theta \tag{12}$$



▲ **Figure 13.6.5**

where θ is the angle between $\nabla f(x, y)$ and $\mathbf{u}$. Equation (12) tells us that the maximum value of $D_{\mathbf{u}}f$ at the point (x, y) is $\|\nabla f(x, y)\|$, and this maximum occurs when $\theta = 0$, that is, when $\mathbf{u}$ is in the direction of $\nabla f(x, y)$. Geometrically, this means:

> *At (x, y), the surface $z = f(x, y)$ has its maximum slope in the direction of the gradient, and the maximum slope is $\|\nabla f(x, y)\|$.*

That is, the function $f(x, y)$ increases most rapidly in the direction of its gradient (Figure 13.6.5).

Similarly, (12) tells us that the minimum value of $D_{\mathbf{u}}f$ at the point (x, y) is $-\|\nabla f(x, y)\|$, and this minimum occurs when $\theta = \pi$, that is, when $\mathbf{u}$ is oppositely directed to $\nabla f(x, y)$. Geometrically, this means:

> *At (x, y), the surface $z = f(x, y)$ has its minimum slope in the direction that is opposite to the gradient, and the minimum slope is $-\|\nabla f(x, y)\|$.*

That is, the function $f(x, y)$ decreases most rapidly in the direction opposite to its gradient (Figure 13.6.5).

Finally, in the case where $\nabla f(x, y) = \mathbf{0}$, it follows from (12) that $D_{\mathbf{u}}f(x, y) = 0$ in all directions at the point (x, y). This typically occurs where the surface $z = f(x, y)$ has a "relative maximum," a "relative minimum," or a saddle point.

A similar analysis applies to functions of three variables. As a consequence, we have the following result.

> **13.6.5 THEOREM** *Let f be a function of either two variables or three variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that f is differentiable at P.*
>
> (*a*) *If $\nabla f = \mathbf{0}$ at P, then all directional derivatives of f at P are zero.*
>
> (*b*) *If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $\|\nabla f\|$ at P.*
>
> (*c*) *If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\|\nabla f\|$ at P.*

Example 4 Let $f(x, y) = x^2 e^y$. Find the maximum value of a directional derivative at $(-2, 0)$, and find the unit vector in the direction in which the maximum value occurs.

Solution. Since

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 2xe^y\mathbf{i} + x^2e^y\mathbf{j}$$

the gradient of f at $(-2, 0)$ is

$$\nabla f(-2, 0) = -4\mathbf{i} + 4\mathbf{j}$$

By Theorem 13.6.5, the maximum value of the directional derivative is

$$\|\nabla f(-2, 0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2, 0)$. The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2, 0)}{\|\nabla f(-2, 0)\|} = \frac{1}{4\sqrt{2}}(-4\mathbf{i} + 4\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

> What would be the minimum value of a directional derivative of
>
> $$f(x, y) = x^2e^y$$
>
> at $(-2, 0)$?

GRADIENTS ARE NORMAL TO LEVEL CURVES

We have seen that the gradient points in the direction in which a function increases most rapidly. For a function $f(x, y)$ of two variables, we will now consider how this direction of maximum rate of increase can be determined from a contour map of the function. Suppose that (x_0, y_0) is a point on a level curve $f(x, y) = c$ of f, and assume that this curve can be smoothly parametrized as

$$x = x(s), \quad y = y(s) \tag{13}$$

where s is an arc length parameter. Recall from Formula (6) of Section 12.4 that the unit tangent vector to (13) is

$$\mathbf{T} = \mathbf{T}(s) = \left(\frac{dx}{ds}\right)\mathbf{i} + \left(\frac{dy}{ds}\right)\mathbf{j}$$

Since $\mathbf{T}$ gives a direction along which f is nearly constant, we would expect the instantaneous rate of change of f with respect to distance in the direction of $\mathbf{T}$ to be 0. That is, we would expect that

$$D_{\mathbf{T}}f(x, y) = \nabla f(x, y) \cdot \mathbf{T}(s) = 0$$

To show this to be the case, we differentiate both sides of the equation $f(x, y) = c$ with respect to s. Assuming that f is differentiable at (x, y), we can use the chain rule to obtain

$$\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = 0$$

which we can rewrite as

$$\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) = 0$$

or, alternatively, as

$$\nabla f(x, y) \cdot \mathbf{T} = 0$$

Therefore, if $\nabla f(x, y) \neq \mathbf{0}$, then $\nabla f(x, y)$ should be normal to the level curve $f(x, y) = c$ at any point (x, y) on the curve.

It is proved in advanced courses that if $f(x, y)$ has continuous first-order partial derivatives, and if $\nabla f(x_0, y_0) \neq \mathbf{0}$, then near (x_0, y_0) the graph of $f(x, y) = c$ is indeed a smooth curve through (x_0, y_0). Furthermore, we also know from Theorem 13.4.4 that f will be differentiable at (x_0, y_0). We therefore have the following result.

Show that the level curves for

$$f(x, y) = x^2 + y^2$$

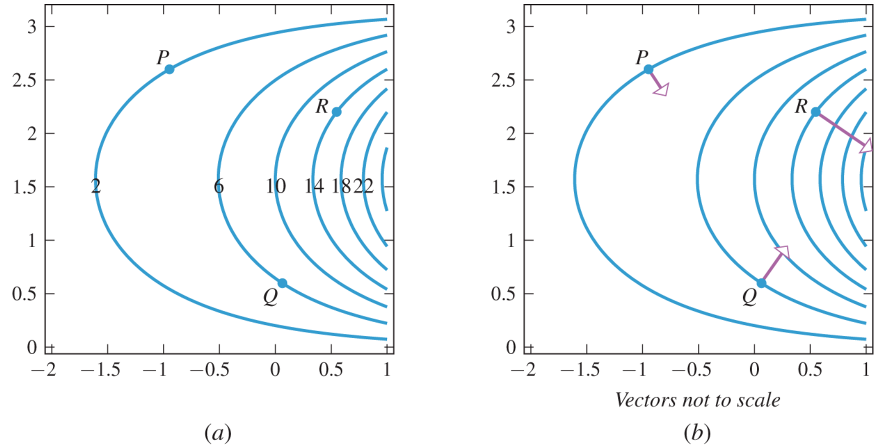
are circles and verify Theorem 13.6.6 at $(x_0, y_0) = (3, 4)$.

13.6.6 THEOREM *Assume that $f(x, y)$ has continuous first-order partial derivatives in an open disk centered at (x_0, y_0) and that $\nabla f(x_0, y_0) \neq \mathbf{0}$. Then $\nabla f(x_0, y_0)$ is normal to the level curve of f through (x_0, y_0).*

When we examine a contour map, we instinctively regard the distance between adjacent contours to be measured in a normal direction. If the contours correspond to equally spaced values of f, then the closer together the contours appear to be, the more rapidly the values of f will be changing in that normal direction. It follows from Theorems 13.6.5 and 13.6.6 that this rate of change of f is given by $\|\nabla f(x, y)\|$. Thus, the closer together the contours appear to be, the greater the length of the gradient of f.

Example 5 A contour plot of a function f is given in Figure 13.6.6*a*. Sketch the directions of the gradient of f at the points P, Q, and R. At which of these three points does the gradient vector have maximum length? Minimum length?

Solution. It follows from Theorems 13.6.5 and 13.6.6 that the directions of the gradient vectors will be as given in Figure 13.6.6*b*. Based on the density of the contour lines, we would guess that the gradient of f has maximum length at R and minimum length at P, with the length at Q somewhere in between. ◄



(*a*) (*b*)

Figure 13.6.6

REMARK

If (x_0, y_0) is a point on the level curve $f(x, y) = c$, then the slope of the surface $z = f(x, y)$ at that point in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

If $\mathbf{u}$ is tangent to the level curve at (x_0, y_0), then $f(x, y)$ is neither increasing nor decreasing in that direction, so $D_{\mathbf{u}} f(x_0, y_0) = 0$. Thus, $\nabla f(x_0, y_0)$, $-\nabla f(x_0, y_0)$, and the tangent vector $\mathbf{u}$ mark the directions of maximum slope, minimum slope, and zero slope at a point (x_0, y_0) on a level curve (Figure 13.6.7). Good skiers use these facts intuitively to control their speed by zigzagging down ski slopes—they ski across the slope with their skis tangential to a level curve to stop their downhill motion, and they point their skis down the slope and normal to the level curve to obtain the most rapid descent.

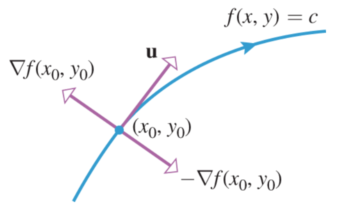


Figure 13.6.7

U. S. Air Force photo by Master Sgt. Michael Ammons

Heat-seeking missiles such as "Stinger" and "Sidewinder" use infrared sensors to measure gradients.

AN APPLICATION OF GRADIENTS

There are numerous applications in which the motion of an object must be controlled so that it moves toward a heat source. For example, in medical applications the operation of certain diagnostic equipment is designed to locate heat sources generated by tumors or infections, and in military applications the trajectories of heat-seeking missiles are controlled to seek and destroy enemy aircraft. The following example illustrates how gradients are used to solve such problems.

Example 6 A heat-seeking particle is located at the point $(2, 3)$ on a flat metal plate whose temperature at a point (x, y) is

$$T(x, y) = 10 - 8x^2 - 2y^2$$

Find an equation for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

Solution. Assume that the trajectory is represented parametrically by the equations

$$x = x(t), \quad y = y(t)$$

where the particle is at the point $(2, 3)$ at time $t = 0$. Because the particle moves in the direction of maximum temperature increase, its direction of motion at time t is in the direction of the gradient of $T(x, y)$, and hence its velocity vector $\mathbf{v}(t)$ at time t points in the direction of the gradient. Thus, there is a scalar k that depends on t such that

$$\mathbf{v}(t) = k\nabla T(x, y)$$

from which we obtain

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = k(-16x\mathbf{i} - 4y\mathbf{j})$$

Equating components yields

$$\frac{dx}{dt} = -16kx, \quad \frac{dy}{dt} = -4ky$$

and dividing to eliminate k yields

$$\frac{dy}{dx} = \frac{-4ky}{-16kx} = \frac{y}{4x}$$

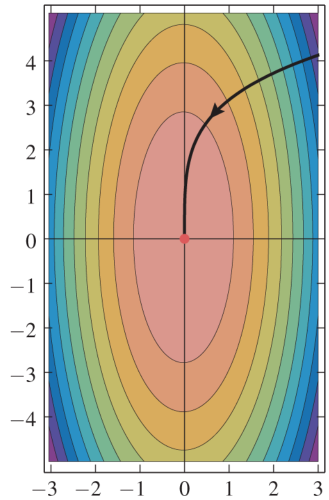
Thus, we can obtain the trajectory by solving the initial-value problem

$$\frac{dy}{dx} - \frac{y}{4x} = 0, \quad y(2) = 3$$

The differential equation is a separable first-order equation and hence can be solved by the method of separation of variables discussed in Section 8.2. We leave it for you to show that the solution of the initial-value problem is

$$y = \frac{3}{\sqrt[4]{2}}x^{1/4}$$

The graph of the trajectory and a contour plot of the temperature function are shown in Figure 13.6.8. ◀



▲ **Figure 13.6.8**

✓ QUICK CHECK EXERCISES 13.6

(See page 866 for answers.)

1. The gradient of $f(x, y, z) = xy^2z^3$ at the point $(1, 1, 1)$ is ______.

2. Suppose that the differentiable function $f(x, y)$ has the property that

$$f\left(2 + \frac{s\sqrt{3}}{2}, 1 + \frac{s}{2}\right) = 3se^s$$

The directional derivative of f in the direction of

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

at $(2, 1)$ is ______.

3. If the gradient of $f(x, y)$ at the origin is $6\mathbf{i}+8\mathbf{j}$, then the directional derivative of f in the direction of $\mathbf{a}=3\mathbf{i}+4\mathbf{j}$ at the origin is ______. The slope of the tangent line to the level curve of f through the origin at $(0,0)$ is ______.

4. If the gradient of $f(x, y, z)$ at $(1,2,3)$ is $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, then the maximum value for a directional derivative of f at $(1,2,3)$ is ______ and the minimum value for a directional derivative at this point is ______.

EXERCISE SET 13.6 Graphing Utility C CAS

1-45, 53-66

1–8 Find $D_{\mathbf{u}}f$ at P.

1. $f(x, y)=(1+xy)^{3/2};\ P(3,1);\ \mathbf{u}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$
2. $f(x, y)=\sin(5x-3y);\ P(3,5);\ \mathbf{u}=\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}$
3. $f(x, y)=\ln(1+x^2+y);\ P(0,0);$ $\mathbf{u}=-\frac{1}{\sqrt{10}}\mathbf{i}-\frac{3}{\sqrt{10}}\mathbf{j}$
4. $f(x, y)=\dfrac{cx+dy}{x-y};\ P(3,4);\ \mathbf{u}=\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{j}$
5. $f(x, y, z)=4x^5y^2z^3;\ P(2,-1,1);\ \mathbf{u}=\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$
6. $f(x, y, z)=ye^{xz}+z^2;\ P(0,2,3);\ \mathbf{u}=\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}$
7. $f(x, y, z)=\ln(x^2+2y^2+3z^2);\ P(-1,2,4);$ $\mathbf{u}=-\frac{3}{13}\mathbf{i}-\frac{4}{13}\mathbf{j}-\frac{12}{13}\mathbf{k}$
8. $f(x, y, z)=\sin xyz;\ P\left(\frac{1}{2},\frac{1}{3},\pi\right);$ $\mathbf{u}=\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$

9–18 Find the directional derivative of f at P in the direction of $\mathbf{a}$.

9. $f(x, y)=4x^3y^2;\ P(2,1);\ \mathbf{a}=4\mathbf{i}-3\mathbf{j}$
10. $f(x, y)=9x^3-2y^3;\ P(1,0);\ \mathbf{a}=\mathbf{i}-\mathbf{j}$
11. $f(x, y)=y^2\ln x;\ P(1,4);\ \mathbf{a}=-3\mathbf{i}+3\mathbf{j}$
12. $f(x, y)=e^x\cos y;\ P(0,\pi/4);\ \mathbf{a}=5\mathbf{i}-2\mathbf{j}$
13. $f(x, y)=\tan^{-1}(y/x);\ P(-2,2);\ \mathbf{a}=-\mathbf{i}-\mathbf{j}$
14. $f(x, y)=xe^y-ye^x;\ P(0,0);\ \mathbf{a}=5\mathbf{i}-2\mathbf{j}$
15. $f(x, y, z)=xy+z^2;\ P(-3,0,4);\ \mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$
16. $f(x, y, z)=y-\sqrt{x^2+z^2};\ P(-3,1,4);$ $\mathbf{a}=2\mathbf{i}-2\mathbf{j}-\mathbf{k}$
17. $f(x, y, z)=\dfrac{z-x}{z+y};\ P(1,0,-3);\ \mathbf{a}=-6\mathbf{i}+3\mathbf{j}-2\mathbf{k}$
18. $f(x, y, z)=e^{x+y+3z};\ P(-2,2,-1);\ \mathbf{a}=20\mathbf{i}-4\mathbf{j}+5\mathbf{k}$

1-45,53-66

19–22 Find the directional derivative of f at P in the direction of a vector making the counterclockwise angle θ with the positive x-axis.

19. $f(x, y)=\sqrt{xy};\ P(1,4);\ \theta=\pi/3$
20. $f(x, y)=\dfrac{x-y}{x+y};\ P(-1,-2);\ \theta=\pi/2$
21. $f(x, y)=\tan(2x+y);\ P(\pi/6,\pi/3);\ \theta=7\pi/4$
22. $f(x, y)=\sinh x\cosh y;\ P(0,0);\ \theta=\pi$
23. Find the directional derivative of
$$f(x, y)=\frac{x}{x+y}$$
at $P(1,0)$ in the direction of $Q(-1,-1)$.
24. Find the directional derivative of $f(x, y)=e^{-x}\sec y$ at $P(0,\pi/4)$ in the direction of the origin.
25. Find the directional derivative of $f(x, y)=\sqrt{xy}e^y$ at $P(1,1)$ in the direction of the negative y-axis. 1-45,53-66
26. Let
$$f(x, y)=\frac{y}{x+y}$$
Find a unit vector $\mathbf{u}$ for which $D_{\mathbf{u}}f(2,3)=0$.
27. Find the directional derivative of
$$f(x, y, z)=\frac{y}{x+z}$$
at $P(2,1,-1)$ in the direction from P to $Q(-1,2,0)$.
28. Find the directional derivative of the function
$$f(x, y, z)=x^3y^2z^5-2xz+yz+3x$$
at $P(-1,-2,1)$ in the direction of the negative z-axis.

FOCUS ON CONCEPTS

29. Suppose that $D_{\mathbf{u}}f(1,2)=-5$ and $D_{\mathbf{v}}f(1,2)=10$, where $\mathbf{u}=\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}$ and $\mathbf{v}=\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{j}$. Find
(a) $f_x(1,2)$ (b) $f_y(1,2)$
(c) the directional derivative of f at $(1,2)$ in the direction of the origin.
30. Given that $f_x(-5,1)=-3$ and $f_y(-5,1)=2$, find the directional derivative of f at $P(-5,1)$ in the direction of the vector from P to $Q(-4,3)$.
31. The accompanying figure shows some level curves of an unspecified function $f(x, y)$. Which of the three vectors shown in the figure is most likely to be ∇f? Explain.
32. The accompanying figure shows some level curves of an unspecified function $f(x, y)$. Of the gradients at P and Q, which probably has the greater length? Explain.

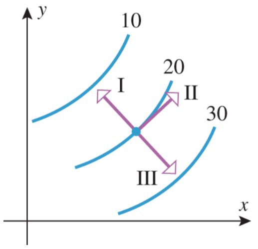


Figure Ex-31

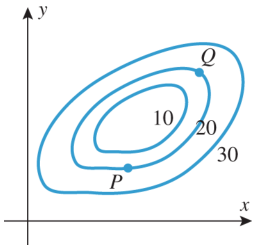


Figure Ex-32

33–40 Find ∇z or ∇w.

33. $z=\sin(7y^2-7xy)$
34. $z=7\sin(6x/y)$
35. $z=\dfrac{6x+7y}{6x-7y}$
36. $z=\dfrac{6xe^{3y}}{x+8y}$
37. $w=-x^9-y^3+z^{12}$
38. $w=xe^{8y}\sin 6z$
39. $w=\ln\sqrt{x^2+y^2+z^2}$
40. $w=e^{-5x}\sec x^2yz$

41–46 Find the gradient of f at the indicated point.

41. $f(x, y) = 5x^2 + y^4$; $(4, 2)$

42. $f(x, y) = 5\sin x^2 + \cos 3y$; $(\sqrt{\pi}/2, 0)$

1-45,53
-66

43. $f(x, y) = (x^2 + xy)^3$; $(-1, -1)$

44. $f(x, y) = (x^2 + y^2)^{-1/2}$; $(3, 4)$

45. $f(x, y, z) = y\ln(x + y + z)$; $(-3, 4, 0)$

46. $f(x, y, z) = y^2 z \tan^3 x$; $(\pi/4, -3, 1)$

47–50 Sketch the level curve of $f(x, y)$ that passes through P and draw the gradient vector at P.

47. $f(x, y) = 4x - 2y + 3$; $P(1, 2)$

48. $f(x, y) = y/x^2$; $P(-2, 2)$

49. $f(x, y) = x^2 + 4y^2$; $P(-2, 0)$

50. $f(x, y) = x^2 - y^2$; $P(2, -1)$

51. Find a unit vector $\mathbf{u}$ that is normal at $P(1, -2)$ to the level curve of $f(x, y) = 4x^2 y$ through P.

52. Find a unit vector $\mathbf{u}$ that is normal at $P(2, -3)$ to the level curve of $f(x, y) = 3x^2 y - xy$ through P.

53–60 Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f at P in that direction.

53. $f(x, y) = 4x^3 y^2$; $P(-1, 1)$

54. $f(x, y) = 3x - \ln y$; $P(2, 4)$

55. $f(x, y) = \sqrt{x^2 + y^2}$; $P(4, -3)$

56. $f(x, y) = \dfrac{x}{x + y}$; $P(0, 2)$

57. $f(x, y, z) = x^3 z^2 + y^3 z + z - 1$; $P(1, 1, -1)$

58. $f(x, y, z) = \sqrt{x - 3y + 4z}$; $P(0, -3, 0)$

59. $f(x, y, z) = \dfrac{x}{z} + \dfrac{z}{y^2}$; $P(1, 2, -2)$

60. $f(x, y, z) = \tan^{-1}\left(\dfrac{x}{y + z}\right)$; $P(4, 2, 2)$

61–66 Find a unit vector in the direction in which f decreases most rapidly at P, and find the rate of change of f at P in that direction.

61. $f(x, y) = 20 - x^2 - y^2$; $P(-1, -3)$

62. $f(x, y) = e^{xy}$; $P(2, 3)$

63. $f(x, y) = \cos(3x - y)$; $P(\pi/6, \pi/4)$

1-45,53-66

64. $f(x, y) = \sqrt{\dfrac{x - y}{x + y}}$; $P(3, 1)$

65. $f(x, y, z) = \dfrac{x + z}{z - y}$; $P(5, 7, 6)$

66. $f(x, y, z) = 4e^{xy}\cos z$; $P(0, 1, \pi/4)$

67–70 True–False Determine whether the statement is true or false. Explain your answer. In each exercise, assume that f denotes a differentiable function of two variables whose domain is the xy-plane.

67. If $\mathbf{v} = 2\mathbf{u}$, then the directional derivative of f in the direction of $\mathbf{v}$ at a point (x_0, y_0) is twice the directional derivative of f in the direction of $\mathbf{u}$ at the point (x_0, y_0).

68. If $y = x^2$ is a contour of f, then $f_x(0, 0) = 0$.

69. If $\mathbf{u}$ is a fixed unit vector and $D_{\mathbf{u}} f(x, y) = 0$ for all points (x, y), then f is a constant function.

70. If the displacement vector from (x_0, y_0) to (x_1, y_1) is a positive multiple of $\nabla f(x_0, y_0)$, then $f(x_0, y_0) \le f(x_1, y_1)$.

FOCUS ON CONCEPTS

71. Given that $\nabla f(4, -5) = 2\mathbf{i} - \mathbf{j}$, find the directional derivative of the function f at the point $(4, -5)$ in the direction of $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}$.

72. Given that $\nabla f(x_0, y_0) = \mathbf{i} - 2\mathbf{j}$ and $D_{\mathbf{u}} f(x_0, y_0) = -2$, find $\mathbf{u}$ (two answers).

73. The accompanying figure shows some level curves of an unspecified function $f(x, y)$.
(a) Use the available information to approximate the length of the vector $\nabla f(1, 2)$, and sketch the approximation. Explain how you approximated the length and determined the direction of the vector.
(b) Sketch an approximation of the vector $-\nabla f(4, 4)$.

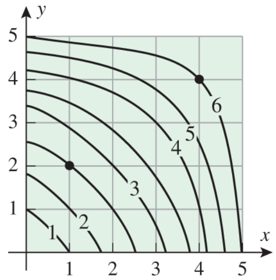


Figure Ex-73

74. The accompanying figure shows a topographic map of a hill and a point P at the bottom of the hill. Suppose that you want to climb from the point P toward the top of the hill in such a way that you are always ascending in the direction of steepest slope. Sketch the projection of your path on the contour map. This is called the ***path of steepest ascent***. Explain how you determined the path.

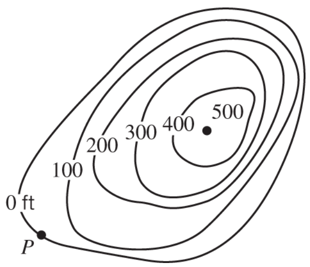


Figure Ex-74

75. Let $z = 3x^2 - y^2$. Find all points at which $\|\nabla z\| = 6$.

76. Given that $z = 3x + y^2$, find $\nabla\|\nabla z\|$ at the point $(5, 2)$.

77. A particle moves along a path C given by the equations $x = t$ and $y = -t^2$. If $z = x^2 + y^2$, find dz/ds along C at the instant when the particle is at the point $(2, -4)$.

78. The temperature (in degrees Celsius) at a point (x, y) on a metal plate in the xy-plane is

$$T(x, y) = \frac{xy}{1 + x^2 + y^2}$$

(cont.)

(a) Find the rate of change of temperature at $(1, 1)$ in the direction of $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$.
(b) An ant at $(1, 1)$ wants to walk in the direction in which the temperature drops most rapidly. Find a unit vector in that direction.

79. If the electric potential at a point (x, y) in the xy-plane is $V(x, y)$, then the ***electric intensity vector*** at the point (x, y) is $\mathbf{E} = -\nabla V(x, y)$. Suppose that $V(x, y) = e^{-2x}\cos 2y$.
(a) Find the electric intensity vector at $(\pi/4, 0)$.
(b) Show that at each point in the plane, the electric potential decreases most rapidly in the direction of the vector $\mathbf{E}$.

80. On a certain mountain, the elevation z above a point (x, y) in an xy-plane at sea level is $z = 2000 - 0.02x^2 - 0.04y^2$, where x, y, and z are in meters. The positive x-axis points east, and the positive y-axis north. A climber is at the point $(-20, 5, 1991)$.
(a) If the climber uses a compass reading to walk due west, will she begin to ascend or descend?
(b) If the climber uses a compass reading to walk northeast, will she ascend or descend? At what rate?
(c) In what compass direction should the climber begin walking to travel a level path (two answers)?

81. Given that the directional derivative of $f(x, y, z)$ at the point $(3, -2, 1)$ in the direction of $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is -5 and that $\|\nabla f(3, -2, 1)\| = 5$, find $\nabla f(3, -2, 1)$.

82. The temperature (in degrees Celsius) at a point (x, y, z) in a metal solid is

$$T(x, y, z) = \frac{xyz}{1 + x^2 + y^2 + z^2}$$

(a) Find the rate of change of temperature with respect to distance at $(1, 1, 1)$ in the direction of the origin.
(b) Find the direction in which the temperature rises most rapidly at the point $(1, 1, 1)$. (Express your answer as a unit vector.)
(c) Find the rate at which the temperature rises moving from $(1, 1, 1)$ in the direction obtained in part (b).

83. Let $r = \sqrt{x^2 + y^2}$.
(a) Show that $\nabla r = \dfrac{\mathbf{r}}{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.
(b) Show that $\nabla f(r) = f'(r)\nabla r = \dfrac{f'(r)}{r}\mathbf{r}$.

84. Use the formula in part (b) of Exercise 83 to find
(a) $\nabla f(r)$ if $f(r) = re^{-3r}$
(b) $f(r)$ if $\nabla f(r) = 3r^2\mathbf{r}$ and $f(2) = 1$.

85. Let $\mathbf{u}_r$ be a unit vector whose counterclockwise angle from the positive x-axis is θ, and let $\mathbf{u}_\theta$ be a unit vector 90° counterclockwise from $\mathbf{u}_r$. Show that if $z = f(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$, then

$$\nabla z = \frac{\partial z}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial z}{\partial \theta}\mathbf{u}_\theta$$

[*Hint:* Use part (c) of Exercise 57, Section 13.5.]

86. Prove: If f and g are differentiable, then
(a) $\nabla(f + g) = \nabla f + \nabla g$
(b) $\nabla(cf) = c\nabla f$ (c constant)
(c) $\nabla(fg) = f\nabla g + g\nabla f$
(d) $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$
(e) $\nabla(f^n) = nf^{n-1}\nabla f$.

87–88 A heat-seeking particle is located at the point P on a flat metal plate whose temperature at a point (x, y) is $T(x, y)$. Find parametric equations for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

87. $T(x, y) = 5 - 4x^2 - y^2$; $P(1, 4)$

88. $T(x, y) = 100 - x^2 - 2y^2$; $P(5, 3)$

89. Use a graphing utility to generate the trajectory of the particle together with some representative level curves of the temperature function in Exercise 87.

90. Use a graphing utility to generate the trajectory of the particle together with some representative level curves of the temperature function in Exercise 88.

C **91.** (a) Use a CAS to graph $f(x, y) = (x^2 + 3y^2)e^{-(x^2+y^2)}$.
(b) At how many points do you think it is true that $D_{\mathbf{u}}f(x, y) = 0$ for all unit vectors $\mathbf{u}$?
(c) Use a CAS to find ∇f.
(d) Use a CAS to solve the equation $\nabla f(x, y) = \mathbf{0}$ for x and y.
(e) Use the result in part (d) together with Theorem 13.6.5 to check your conjecture in part (b).

92. Prove: If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x(t), y(t))$, then

$$\frac{dz}{dt} = \nabla z \cdot \mathbf{r}'(t)$$

where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.

93. Prove: If f, f_x, and f_y are continuous on a circular region, and if $\nabla f(x, y) = \mathbf{0}$ throughout the region, then $f(x, y)$ is constant on the region. [*Hint:* See Exercise 69, Section 13.5.]

94. Prove: If the function f is differentiable at the point (x, y) and if $D_{\mathbf{u}}f(x, y) = 0$ in two nonparallel directions, then $D_{\mathbf{u}}f(x, y) = 0$ in all directions.

95. Given that the functions $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$, and $f(u, v, w)$ are all differentiable, show that

$$\nabla f(u, v, w) = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w$$

96. **Writing** Let f denote a differentiable function of two variables. Write a short paragraph that discusses the connections between directional derivatives of f and slopes of lines to the graph of f.

97. **Writing** Let f denote a differentiable function of two variables. Although we have defined what it means to say that f is differentiable, we have not defined the "derivative" of f. Write a short paragraph that discusses the merits of defining the derivative of f to be the gradient ∇f.

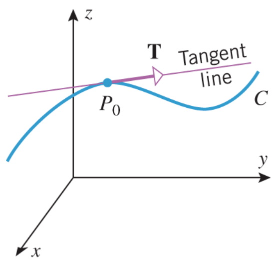
✓ **QUICK CHECK ANSWERS 13.6** **1.** $\langle 1, 2, 3\rangle$ **2.** 3 **3.** 10; $-\frac{3}{4}$ **4.** 3; -3

13.7 TANGENT PLANES AND NORMAL VECTORS

In this section we will discuss tangent planes to surfaces in three-dimensional space. We will be concerned with three main questions: What is a tangent plane? When do tangent planes exist? How do we find equations of tangent planes?

TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES *F*(*x*, *y*, *z*) = *c*

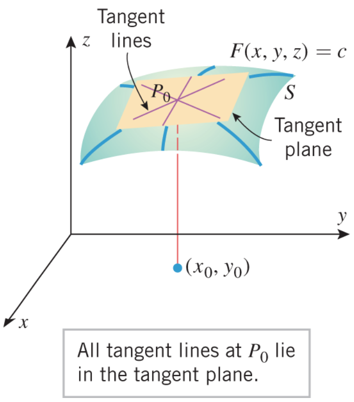
We begin by considering the problem of finding tangent planes to level surfaces of a function $F(x, y, z)$. These surfaces are represented by equations of the form $F(x, y, z) = c$. We will assume that F has continuous first-order partial derivatives, since this has an important geometric consequence. Fix c, and suppose that $P_0(x_0, y_0, z_0)$ satisfies the equation $F(x, y, z) = c$. In advanced courses it is proved that if F has continuous first-order partial derivatives, and if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then near P_0 the graph of $F(x, y, z) = c$ is indeed a "surface" rather than some possibly exotic-looking set of points in 3-space.



▲ **Figure 13.7.1**

We will base our concept of a tangent plane to a level surface S: $F(x, y, z) = c$ on the more elementary notion of a tangent line to a curve C in 3-space (Figure 13.7.1). Intuitively, we would expect a tangent plane to S at a point P_0 to be composed of the tangent lines at P_0 of all curves on S that pass through P_0 (Figure 13.7.2). Suppose C is a curve on S through P_0 that is parametrized by $x = x(t)$, $y = y(t)$, $z = z(t)$ with $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$. The tangent line l to C through P_0 is then parallel to the vector

$$\mathbf{r}' = x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}$$



▲ **Figure 13.7.2**

where we assume that $\mathbf{r}' \neq \mathbf{0}$ (Definition 12.2.7). Since C is on the surface $F(x, y, z) = c$, we have

$$c = F(x(t), y(t), z(t)) \tag{1}$$

Computing the derivative at t_0 of both sides of (1), we have by the chain rule that

$$0 = F_x(x_0, y_0, z_0)x'(t_0) + F_y(x_0, y_0, z_0)y'(t_0) + F_z(x_0, y_0, z_0)z'(t_0)$$

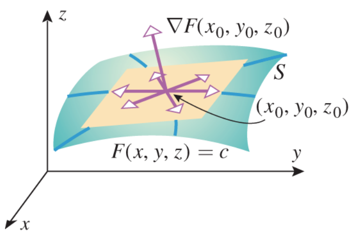
We can write this equation in vector form as

$$0 = (F_x(x_0, y_0, z_0)\mathbf{i} + F_y(x_0, y_0, z_0)\mathbf{j} + F_z(x_0, y_0, z_0)\mathbf{k}) \cdot (x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k})$$

or

$$0 = \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}' \tag{2}$$

It follows that if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to line l. Therefore, the tangent line l to C at P_0 is contained in the plane through P_0 with normal vector $\nabla F(x_0, y_0, z_0)$. Since C was *arbitrary*, we conclude that the same is true for any curve on S through P_0 (Figure 13.7.3). Thus, it makes sense to define the tangent plane to S at P_0 to be the plane through P_0 whose normal vector is



▲ **Figure 13.7.3**

$$\mathbf{n} = \nabla F(x_0, y_0, z_0) = \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\rangle$$

Using the point-normal form [see Formula (3) in Section 11.6], we have the following definition.

13.7.1 DEFINITION Assume that $F(x, y, z)$ has continuous first-order partial derivatives and that $P_0(x_0, y_0, z_0)$ is a point on the level surface $S\colon F(x, y, z) = c$. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\mathbf{n} = \nabla F(x_0, y_0, z_0)$ is a ***normal vector*** to S at P_0 and the ***tangent plane*** to S at P_0 is the plane with equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{3}$$

Definition 13.7.1 can be viewed as an extension of Theorem 13.6.6 from curves to surfaces.

The line through the point P_0 parallel to the normal vector $\mathbf{n}$ is perpendicular to the tangent plane (3). We will call this the ***normal line***, or sometimes more simply the ***normal*** to the surface $F(x, y, z) = c$ at P_0. It follows that this line can be expressed parametrically as

$$x = x_0 + F_x(x_0, y_0, z_0)t, \quad y = y_0 + F_y(x_0, y_0, z_0)t, \quad z = z_0 + F_z(x_0, y_0, z_0)t \tag{4}$$

▶ Example 1 Consider the ellipsoid $x^2 + 4y^2 + z^2 = 18$.

(a) Find an equation of the tangent plane to the ellipsoid at the point $(1, 2, 1)$.
(b) Find parametric equations of the line that is normal to the ellipsoid at the point $(1, 2, 1)$.
(c) Find the acute angle that the tangent plane at the point $(1, 2, 1)$ makes with the xy-plane.

Solution (a). We apply Definition 13.7.1 with $F(x, y, z) = x^2 + 4y^2 + z^2$ and $(x_0, y_0, z_0) = (1, 2, 1)$. Since

$$\nabla F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)\rangle = \langle 2x, 8y, 2z\rangle$$

we have

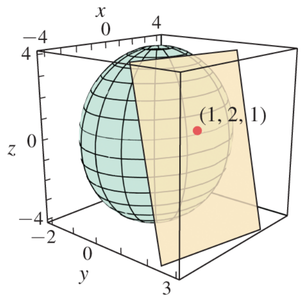
$$\mathbf{n} = \nabla F(1, 2, 1) = \langle 2, 16, 2\rangle$$

Hence, from (3) the equation of the tangent plane is

$$2(x - 1) + 16(y - 2) + 2(z - 1) = 0 \quad \text{or} \quad x + 8y + z = 18$$

Solution (b). Since $\mathbf{n} = \langle 2, 16, 2\rangle$ at the point $(1, 2, 1)$, it follows from (4) that parametric equations for the normal line to the ellipsoid at the point $(1, 2, 1)$ are

$$x = 1 + 2t, \quad y = 2 + 16t, \quad z = 1 + 2t$$



▲ Figure 13.7.4

Solution (c). To find the acute angle θ between the tangent plane and the xy-plane, we will apply Formula (9) of Section 11.6 with $\mathbf{n}_1 = \mathbf{n} = \langle 2, 16, 2\rangle$ and $\mathbf{n}_2 = \langle 0, 0, 1\rangle$. This yields

$$\cos\theta = \frac{|\langle 2, 16, 2\rangle \cdot \langle 0, 0, 1\rangle|}{\|\langle 2, 16, 2\rangle\|\,\|\langle 0, 0, 1\rangle\|} = \frac{2}{(2\sqrt{66})(1)} = \frac{1}{\sqrt{66}}$$

Thus,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{66}}\right) \approx 83^\circ$$

(Figure 13.7.4). ◀

TANGENT PLANES TO SURFACES OF THE FORM $z = f(x, y)$

To find a tangent plane to a surface of the form $z = f(x, y)$, we can use Equation (3) with the function $F(x, y, z) = z - f(x, y)$.

▶ Example 2 Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2y$ at the point $(2, 1, 4)$.

Solution. Let $F(x, y, z) = z - x^2y$. Then $F(x, y, z) = 0$ on the surface, so we can find the find the gradient of F at the point $(2, 1, 4)$:

$$\nabla F(x, y, z) = -2xy\mathbf{i} - x^2\mathbf{j} + \mathbf{k}$$
$$\nabla F(2, 1, 4) = -4\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

From (3) the tangent plane has equation

$$-4(x - 2) - 4(y - 1) + 1(z - 4) = 0 \quad \text{or} \quad -4x - 4y + z = -8$$

and the normal line has equations

$$x = 2 - 4t, \qquad y = 1 - 4t, \qquad z = 4 + t \blacktriangleleft$$

Suppose that $f(x, y)$ is differentiable at a point (x_0, y_0) and that $z_0 = f(x_0, y_0)$. It can be shown that the procedure of Example 2 can be used to find the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0). This yields an alternative equation for a tangent plane to the graph of a differentiable function.

13.7.2 THEOREM *If $f(x, y)$ is differentiable at the point (x_0, y_0), then the tangent plane to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, f(x_0, y_0))$ [or (x_0, y_0)] is the plane*

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{5}$$

PROOF Consider the function $F(x, y, z) = z - f(x, y)$. Since $F(x, y, z) = 0$ on the surface, we will apply (3) to this function. The partial derivatives of F are

$$F_x(x, y, z) = -f_x(x, y), \quad F_y(x, y, z) = -f_y(x, y), \quad F_z(x, y, z) = 1$$

Since the point at which we evaluate these derivatives lies on the surface, it will have the form $(x_0, y_0, f(x_0, y_0))$. Thus, (3) gives

$$\begin{aligned} 0 &= F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - f(x_0, y_0)) \\ &= -f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + 1(z - f(x_0, y_0)) \end{aligned}$$

which is equivalent to (5). ■

Recall from Section 13.4 that if a function $f(x, y)$ is differentiable at a point (x_0, y_0), then the local linear approximation $L(x, y)$ to f at (x_0, y_0) has the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Notice that the equation $z = L(x, y)$ is identical to that of the tangent plane to $f(x, y)$ at the point (x_0, y_0). Thus, the graph of the local linear approximation to $f(x, y)$ at the point (x_0, y_0) is the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0).

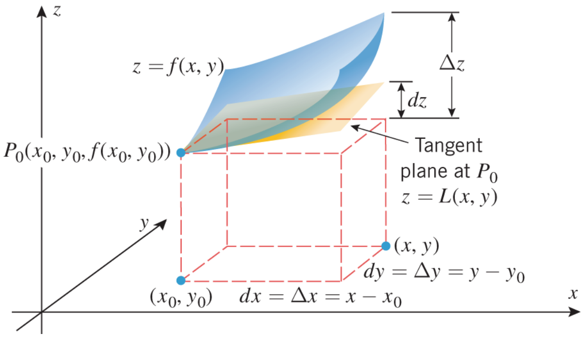
TANGENT PLANES AND TOTAL DIFFERENTIALS

Recall that for a function $z = f(x, y)$ of two variables, the approximation by differentials is

$$\Delta z = \Delta f = f(x, y) - f(x_0, y_0) \approx dz = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

> Note that the tangent plane in Figure 13.7.5 is analogous to the tangent line in Figure 13.4.2.

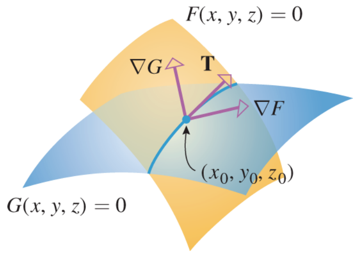
The tangent plane provides a geometric interpretation of this approximation. We see in Figure 13.7.5 that Δz is the change in z *along the surface* $z = f(x, y)$ from the point $P_0(x_0, y_0, f(x_0, y_0))$ to the point $P(x, y, f(x, y))$, and dz is the change in z *along the tangent plane* from P_0 to $Q(x, y, L(x, y))$. The small vertical displacement at (x, y) between the surface and the plane represents the error in the local linear approximation to f at (x_0, y_0). We have seen that near (x_0, y_0) this error term has magnitude much smaller than the distance between (x, y) and (x_0, y_0).



▶ **Figure 13.7.5**

USING GRADIENTS TO FIND TANGENT LINES TO INTERSECTIONS OF SURFACES

In general, the intersection of two surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ will be a curve in 3-space. If (x_0, y_0, z_0) is a point on this curve, then $\nabla F(x_0, y_0, z_0)$ will be normal to the surface $F(x, y, z) = 0$ at (x_0, y_0, z_0) and $\nabla G(x_0, y_0, z_0)$ will be normal to the surface $G(x, y, z) = 0$ at (x_0, y_0, z_0). Thus, if the curve of intersection can be smoothly parametrized, then its unit tangent vector $\mathbf{T}$ at (x_0, y_0, z_0) will be orthogonal to both $\nabla F(x_0, y_0, z_0)$ and $\nabla G(x_0, y_0, z_0)$ (Figure 13.7.6). Consequently, if

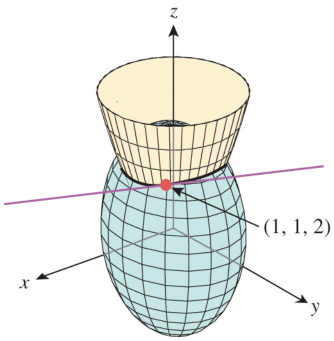


▲ **Figure 13.7.6**

$$\nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0) \neq \mathbf{0}$$

then this cross product will be parallel to $\mathbf{T}$ and hence will be tangent to the curve of intersection. This tangent vector can be used to determine the direction of the tangent line to the curve of intersection at the point (x_0, y_0, z_0).

▶ **Example 3** Find parametric equations of the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ at the point $(1, 1, 2)$ (Figure 13.7.7).



▲ **Figure 13.7.7**

Solution. We begin by rewriting the equations of the surfaces as

$$x^2 + y^2 - z = 0 \quad \text{and} \quad 3x^2 + 2y^2 + z^2 - 9 = 0$$

and we take

$$F(x, y, z) = x^2 + y^2 - z \quad \text{and} \quad G(x, y, z) = 3x^2 + 2y^2 + z^2 - 9$$

We will need the gradients of these functions at the point $(1, 1, 2)$. The computations are

$$\begin{aligned} \nabla F(x, y, z) &= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}, & \nabla G(x, y, z) &= 6x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k} \\ \nabla F(1, 1, 2) &= 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}, & \nabla G(1, 1, 2) &= 6\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \end{aligned}$$

Thus, a tangent vector at $(1, 1, 2)$ to the curve of intersection is

$$\nabla F(1, 1, 2) \times \nabla G(1, 1, 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 6 & 4 & 4 \end{vmatrix} = 12\mathbf{i} - 14\mathbf{j} - 4\mathbf{k}$$

Since any scalar multiple of this vector will do just as well, we can multiply by $\frac{1}{2}$ to reduce the size of the coefficients and use the vector of $6\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$ to determine the direction of the tangent line. This vector and the point $(1, 1, 2)$ yield the parametric equations

$$x = 1 + 6t, \quad y = 1 - 7t, \quad z = 2 - 2t \ ◀$$

✓ QUICK CHECK EXERCISES 13.7 (See page 872 for answers.)

1. Suppose that $f(1,0,-1) = 2$, and $f(x,y,z)$ is differentiable at $(1,0,-1)$ with $\nabla f(1,0,-1) = \langle 2,1,1\rangle$. An equation for the tangent plane to the level surface $f(x,y,z) = 2$ at the point $(1,0,-1)$ is ________, and parametric equations for the normal line to the level surface through the point $(1,0,-1)$ are

$x =$ ________, $y =$ ________, $z =$ ________

2. Suppose that $f(x,y)$ is differentiable at the point $(3,1)$ with $f(3,1) = 4, f_x(3,1) = 2$, and $f_y(3,1) = -3$. An equation for the tangent plane to the graph of f at the point $(3,1,4)$ is ________, and parametric equations for the normal line to the graph of f through the point $(3,1,4)$ are

$x =$ ________, $y =$ ________, $z =$ ________

3. An equation for the tangent plane to the graph of $z = x^2\sqrt{y}$ at the point $(2,4,8)$ is ________, and parametric equations for the normal line to the graph of $z = x^2\sqrt{y}$ through the point $(2,4,8)$ are

$x =$ ________, $y =$ ________, $z =$ ________

4. The sphere $x^2 + y^2 + z^2 = 9$ and the plane $x + y + z = 5$ intersect in a circle that passes through the point $(2,1,2)$. Parametric equations for the tangent line to this circle at $(2,1,2)$ are

$x =$ ________, $y =$ ________, $z =$ ________

EXERCISE SET 13.7 [C] CAS

1. Consider the ellipsoid $x^2 + y^2 + 4z^2 = 12$.
(a) Find an equation of the tangent plane to the ellipsoid at the point $(2,2,1)$.
(b) Find parametric equations of the line that is normal to the ellipsoid at the point $(2,2,1)$.
(c) Find the acute angle that the tangent plane at the point $(2,2,1)$ makes with the xy-plane.

2. Consider the surface $xz - yz^3 + yz^2 = 2$.
(a) Find an equation of the tangent plane to the surface at the point $(2,-1,1)$.
(b) Find parametric equations of the line that is normal to the surface at the point $(2,-1,1)$.
(c) Find the acute angle that the tangent plane at the point $(2,-1,1)$ makes with the xy-plane.

3–12 Find an equation for the tangent plane and parametric equations for the normal line to the surface at the point P.

3. $x^2 + y^2 + z^2 = 25$; $P(-3,0,4)$

4. $x^2y - 4z^2 = -7$; $P(-3,1,-2)$

5. $x^2 - xyz = 56$; $P(-4,5,2)$

6. $z = x^2 + y^2$; $P(2,-3,13)$

7. $z = 4x^3y^2 + 2y$; $P(1,-2,12)$

8. $z = \frac{1}{2}x^7y^{-2}$; $P(2,4,4)$

9. $z = xe^{-y}$; $P(1,0,1)$

10. $z = \ln\sqrt{x^2+y^2}$; $P(-1,0,0)$

11. $z = e^{3y}\sin 3x$; $P(\pi/6,0,1)$

12. $z = x^{1/2} + y^{1/2}$; $P(4,9,5)$

FOCUS ON CONCEPTS

13. Find all points on the surface at which the tangent plane is horizontal.
(a) $z = x^3y^2$
(b) $z = x^2 - xy + y^2 - 2x + 4y$

14. Find a point on the surface $z = 3x^2 - y^2$ at which the tangent plane is parallel to the plane $6x + 4y - z = 5$.

15. Find a point on the surface $z = 8 - 3x^2 - 2y^2$ at which the tangent plane is perpendicular to the line $x = 2 - 3t$, $y = 7 + 8t$, $z = 5 - t$.

16. Show that the surfaces

$$z = \sqrt{x^2+y^2} \quad \text{and} \quad z = \tfrac{1}{10}(x^2+y^2) + \tfrac{5}{2}$$

intersect at $(3,4,5)$ and have a common tangent plane at that point.

17. (a) Find all points of intersection of the line

$$x = -1 + t, \quad y = 2 + t, \quad z = 2t + 7$$

and the surface

$$z = x^2 + y^2$$

(b) At each point of intersection, find the cosine of the acute angle between the given line and the line normal to the surface.

18. Show that if f is differentiable and $z = xf(x/y)$, then all tangent planes to the graph of this equation pass through the origin.

19–22 True–False Determine whether the statement is true or false. Explain your answer.

19. If the tangent plane to the level surface of $F(x,y,z)$ at the point $P_0(x_0,y_0,z_0)$ is also tangent to a level surface of $G(x,y,z)$ at P_0, then $\nabla F(x_0,y_0,z_0) = \nabla G(x_0,y_0,z_0)$.

20. If the tangent plane to the graph of $z = f(x,y)$ at the point $(1,1,2)$ has equation $x - y + 2z = 4$, then $f_x(1,1) = 1$ and $f_y(1,1) = -1$.

21. If the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 2, 1)$ has equation $2x + y - z = 3$, then the local linear approximation to f at $(1, 2)$ is given by the function $L(x, y) = 1 + 2(x - 1) + (y - 2)$.

22. The normal line to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, f(x_0, y_0))$ has a direction vector given by $f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$.

23–24 Find two unit vectors that are normal to the given surface at the point P.

23. $\sqrt{\dfrac{z + x}{y - 1}} = z^2$; $P(3, 5, 1)$

24. $\sin xz - 4\cos yz = 4$; $P(\pi, \pi, 1)$

25. Show that every line that is normal to the sphere

$$x^2 + y^2 + z^2 = 1$$

passes through the origin.

26. Find all points on the ellipsoid $2x^2 + 3y^2 + 4z^2 = 9$ at which the plane tangent to the ellipsoid is parallel to the plane $x - 2y + 3z = 5$.

27. Find all points on the surface $x^2 + y^2 - z^2 = 1$ at which the normal line is parallel to the line through $P(1, -2, 1)$ and $Q(4, 0, -1)$.

28. Show that the ellipsoid $2x^2 + 3y^2 + z^2 = 9$ and the sphere

$$x^2 + y^2 + z^2 - 6x - 8y - 8z + 24 = 0$$

have a common tangent plane at the point $(1, 1, 2)$.

29. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $x^2 + 4y^2 + z^2 = 9$ at the point $(1, -1, 2)$.

30. Find parametric equations for the tangent line to the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $x + 2y + 2z = 20$ at the point $(4, 3, 5)$.

31. Find parametric equations for the tangent line to the curve of intersection of the cylinders $x^2 + z^2 = 25$ and $y^2 + z^2 = 25$ at the point $(3, -3, 4)$.

C **32.** The accompanying figure shows the intersection of the surfaces $z = 8 - x^2 - y^2$ and $4x + 2y - z = 0$.
(a) Find parametric equations for the tangent line to the curve of intersection at the point $(0, 2, 4)$.
(b) Use a CAS to generate a reasonable facsimile of the figure. You need not generate the colors, but try to obtain a similar viewpoint.

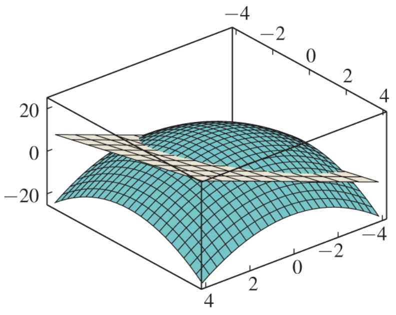


Figure Ex-32

33. Show that the equation of the plane that is tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at (x_0, y_0, z_0) can be written in the form

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$$

34. Show that the equation of the plane that is tangent to the paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

at (x_0, y_0, z_0) can be written in the form

$$z + z_0 = \frac{2x_0 x}{a^2} + \frac{2y_0 y}{b^2}$$

35. Prove: If the surfaces $z = f(x, y)$ and $z = g(x, y)$ intersect at $P(x_0, y_0, z_0)$, and if f and g are differentiable at (x_0, y_0), then the normal lines at P are perpendicular if and only if

$$f_x(x_0, y_0)g_x(x_0, y_0) + f_y(x_0, y_0)g_y(x_0, y_0) = -1$$

36. Use the result in Exercise 35 to show that the normal lines to the cones $z = \sqrt{x^2 + y^2}$ and $z = -\sqrt{x^2 + y^2}$ are perpendicular to the normal lines to the sphere $x^2 + y^2 + z^2 = a^2$ at every point of intersection (see Figure Ex-38).

37. Two surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ are said to be ***orthogonal*** at a point P of intersection if ∇f and ∇g are nonzero at P and the normal lines to the surfaces are perpendicular at P. Show that if $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$ and $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, then the surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ are orthogonal at the point (x_0, y_0, z_0) if and only if

$$f_x g_x + f_y g_y + f_z g_z = 0$$

at this point. [*Note:* This is a more general version of the result in Exercise 35.]

38. Use the result of Exercise 37 to show that the sphere $x^2 + y^2 + z^2 = a^2$ and the cone $z^2 = x^2 + y^2$ are orthogonal at every point of intersection (see the accompanying figure).

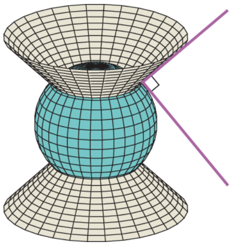


Figure Ex-38

39. Show that the volume of the solid bounded by the coordinate planes and a plane tangent to the portion of the surface $xyz = k$, $k > 0$, in the first octant does not depend on the point of tangency.

40. Writing Discuss the role of the chain rule in defining a tangent plane to a level surface.

41. Writing Discuss the relationship between tangent planes and local linear approximations for functions of two variables.

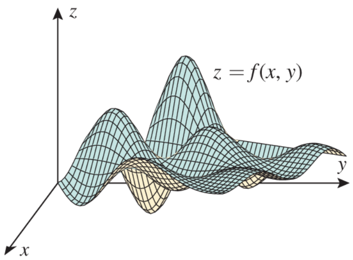
✓ **QUICK CHECK ANSWERS 13.7** **1.** $2(x-1)+y+(z+1)=0$; $x=1+2t$; $y=t$; $z=-1+t$ **2.** $z=4+2(x-3)-3(y-1)$; $x=3+2t$; $y=1-3t$; $z=4-t$ **3.** $z=8+8(x-2)+(y-4)$; $x=2+8t$; $y=4+t$; $z=8-t$ **4.** $x=2+t$; $y=1$; $z=2-t$

13.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Earlier in this text we learned how to find maximum and minimum values of a function of one variable. In this section we will develop similar techniques for functions of two variables.

EXTREMA

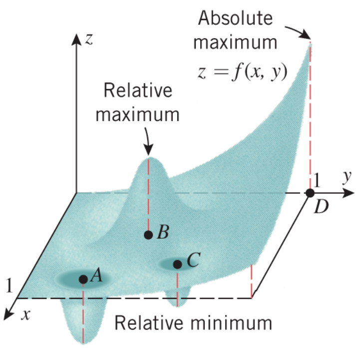
If we imagine the graph of a function f of two variables to be a mountain range (Figure 13.8.1), then the mountaintops, which are the high points in their immediate vicinity, are called *relative maxima* of f, and the valley bottoms, which are the low points in their immediate vicinity, are called *relative minima* of f.



▲ **Figure 13.8.1**

Just as a geologist might be interested in finding the highest mountain and deepest valley in an entire mountain range, so a mathematician might be interested in finding the largest and smallest values of $f(x, y)$ over the *entire* domain of f. These are called the *absolute maximum* and *absolute minimum values* of f. The following definitions make these informal ideas precise.

13.8.1 DEFINITION A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.



▲ **Figure 13.8.2**

13.8.2 DEFINITION A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a ***relative extremum*** at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an ***absolute extremum*** at (x_0, y_0).

Figure 13.8.2 shows the graph of a function f whose domain is the square region in the xy-plane whose points satisfy the inequalities $0 \leq x \leq 1, 0 \leq y \leq 1$. The function f has relative minima at the points A and C and a relative maximum at B. There is an absolute minimum at A and an absolute maximum at D.

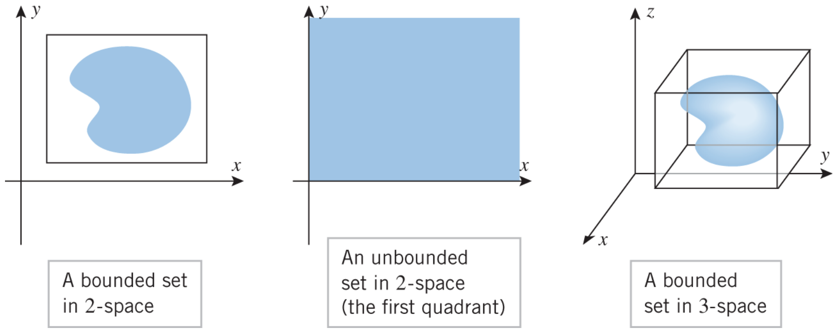
For functions of two variables we will be concerned with two important questions:

- Are there any relative or absolute extrema?
- If so, where are they located?

BOUNDED SETS

Just as we distinguished between finite intervals and infinite intervals on the real line, so we will want to distinguish between regions of "finite extent" and regions of "infinite extent" in 2-space and 3-space. A set of points in 2-space is called ***bounded*** if the entire set can be contained within some rectangle, and is called ***unbounded*** if there is no rectangle that contains all the points of the set. Similarly, a set of points in 3-space is ***bounded*** if the entire set can be contained within some box, and is unbounded otherwise (Figure 13.8.3).

Explain why any subset of a bounded set is also bounded.



▶ Figure 13.8.3

THE EXTREME-VALUE THEOREM

For functions of one variable that are continuous on a closed interval, the Extreme-Value Theorem (Theorem 3.4.2) answered the existence question for absolute extrema. The following theorem, which we state without proof, is the corresponding result for functions of two variables.

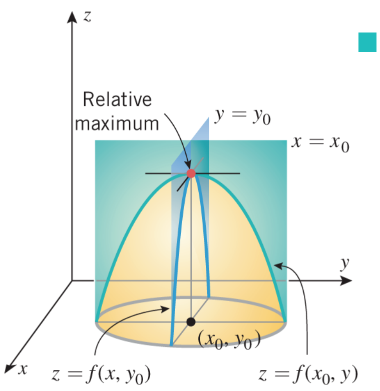
> **13.8.3 THEOREM (Extreme-Value Theorem)** *If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.*

▶ Example 1 The square region R whose points satisfy the inequalities

$$0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1$$

is a closed and bounded set in the xy-plane. The function f whose graph is shown in Figure 13.8.2 is continuous on R; thus, it is guaranteed to have an absolute maximum and minimum on R by the last theorem. These occur at points D and A that are shown in the figure. ◀

REMARK If any of the conditions in the Extreme-Value Theorem fail to hold, then there is no guarantee that an absolute maximum or absolute minimum exists on the region R. Thus, a discontinuous function on a closed and bounded set need not have any absolute extrema, and a continuous function on a set that is not closed and bounded also need not have any absolute extrema.



▲ Figure 13.8.4

FINDING RELATIVE EXTREMA

Recall that if a function g of one variable has a relative extremum at a point x_0 where g is differentiable, then $g'(x_0) = 0$. To obtain the analog of this result for functions of two variables, suppose that $f(x, y)$ has a relative maximum at a point (x_0, y_0) and that the partial derivatives of f exist at (x_0, y_0). It seems plausible geometrically that the traces of the surface $z = f(x, y)$ on the planes $x = x_0$ and $y = y_0$ have horizontal tangent lines at (x_0, y_0) (Figure 13.8.4), so

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

The same conclusion holds if f has a relative minimum at (x_0, y_0), all of which suggests the following result, which we state without formal proof.

> **13.8.4 THEOREM** *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*
>
> $$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

Recall that the *critical points* of a function f of one variable are those values of x in the domain of f at which $f'(x) = 0$ or f is not differentiable. The following definition is the analog for functions of two variables.

Explain why

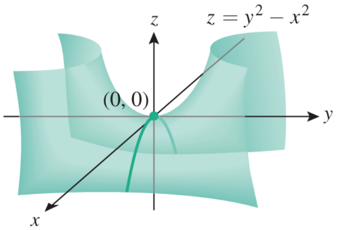
$$D_{\mathbf{u}} f(x_0, y_0) = 0$$

for all $\mathbf{u}$ if (x_0, y_0) is a critical point of f and f is differentiable at (x_0, y_0).

13.8.5 DEFINITION A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

It follows from this definition and Theorem 13.8.4 that relative extrema occur at critical points, just as for a function of one variable. However, recall that for a function of one variable a relative extremum need not occur at *every* critical point. For example, the function might have an inflection point with a horizontal tangent line at the critical point (see Figure 3.2.6). Similarly, a function of two variables need not have a relative extremum at every critical point. For example, consider the function

$$f(x, y) = y^2 - x^2$$



The function $f(x, y) = y^2 - x^2$ has neither a relative maximum nor a relative minimum at the critical point $(0, 0)$.

Figure 13.8.5

This function, whose graph is the hyperbolic paraboloid shown in Figure 13.8.5, has a critical point at $(0, 0)$, since

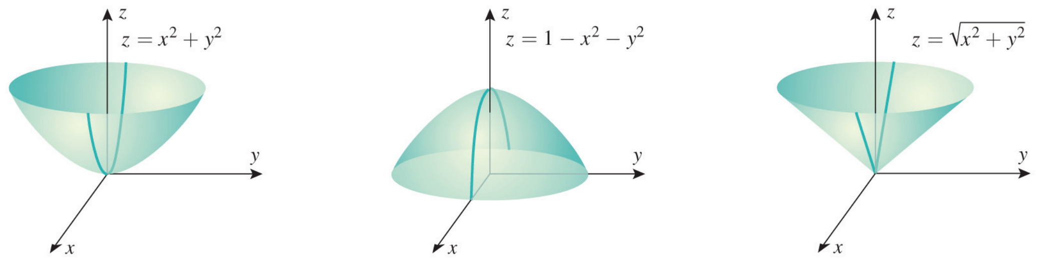
$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = 2y$$

from which it follows that

$$f_x(0, 0) = 0 \quad \text{and} \quad f_y(0, 0) = 0$$

However, the function f has neither a relative maximum nor a relative minimum at $(0, 0)$. For obvious reasons, the point $(0, 0)$ is called a *saddle point* of f. In general, we will say that a surface $z = f(x, y)$ has a ***saddle point*** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).

Example 2 The three functions graphed in Figure 13.8.6 all have critical points at $(0, 0)$. For the paraboloids, the partial derivatives at the origin are zero. You can check this algebraically by evaluating the partial derivatives at $(0, 0)$, but you can see it geometrically by observing that the traces in the xz-plane and yz-plane have horizontal tangent lines at $(0, 0)$. For the cone neither partial derivative exists at the origin because the traces in the xz-plane and the yz-plane have corners there. The paraboloid in part (a) and the cone in part (c) have a relative minimum and absolute minimum at the origin, and the paraboloid in part (b) has a relative maximum and an absolute maximum at the origin. ◀



$f_x(0, 0) = f_y(0, 0) = 0$ relative and absolute min at $(0, 0)$

(a)

$f_x(0, 0) = f_y(0, 0) = 0$ relative and absolute max at $(0, 0)$

(b)

$f_x(0, 0)$ and $f_y(0, 0)$ do not exist relative and absolute min at $(0, 0)$

(c)

Figure 13.8.6

THE SECOND PARTIALS TEST

For functions of one variable the second derivative test (Theorem 3.2.4) was used to determine the behavior of a function at a critical point. The following theorem, which is usually proved in advanced calculus, is the analog of that theorem for functions of two variables.

13.8.6 THEOREM (*The Second Partials Test*) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(*a*) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*

(*b*) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*

(*c*) *If $D < 0$, then f has a saddle point at (x_0, y_0).*

(*d*) *If $D = 0$, then no conclusion can be drawn.*

With the notation of Theorem 13.8.6, show that if $D > 0$, then $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ have the same sign. Thus, we can replace $f_{xx}(x_0, y_0)$ by $f_{yy}(x_0, y_0)$ in parts (*a*) and (*b*) of the theorem.

Example 3 Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$\begin{aligned} 6x - 2y &= 0 \\ -2x + 2y - 8 &= 0 \end{aligned}$$

Solving these for x and y yields $x = 2, y = 6$ (verify), so $(2, 6)$ is the only critical point. To apply Theorem 13.8.6 we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$

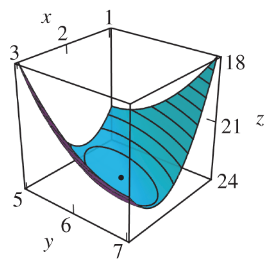
At the point $(2, 6)$ we have

$$D = f_{xx}(2, 6)f_{yy}(2, 6) - f_{xy}^2(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at $(2, 6)$ by part (*a*) of the second partials test. Figure 13.8.7 shows a graph of f in the vicinity of the relative minimum. ◀



$f(x, y) = 3x^2 - 2xy + y^2 - 8y$

Figure 13.8.7

Example 4 Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

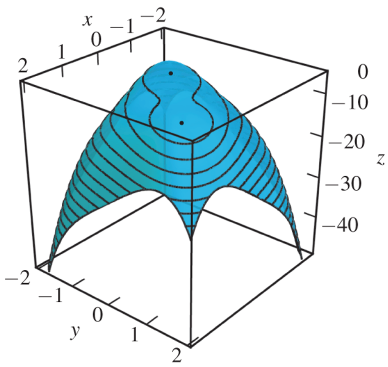
Solution. Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \tag{1}$$

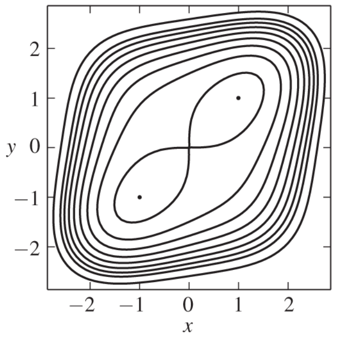
the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0, x = 1, x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0, y = 1, y = -1$. Thus, the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.



$f(x, y) = 4xy - x^4 - y^4$



▲ **Figure 13.8.8**

The "figure eight" pattern at $(0, 0)$ in the contour plot for the surface in Figure 13.8.8 is typical for level curves that pass through a saddle point. If a bug starts at the point $(0, 0, 0)$ on the surface, in how many directions can it walk and remain in the xy-plane?

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128

At the points $(1, 1)$ and $(-1, -1)$, we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At (0, 0) there is a saddle point since $D < 0$. The surface and a contour plot are shown in Figure 13.8.8. ◄

The following theorem, which is the analog for functions of two variables of Theorem 3.4.3, will lead to an important method for finding absolute extrema.

13.8.7 THEOREM *If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.*

PROOF If f has an absolute maximum at the point (x_0, y_0) in the interior of the domain of f, then f has a relative maximum at (x_0, y_0). If both partial derivatives exist at (x_0, y_0), then

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

by Theorem 13.8.4, so (x_0, y_0) is a critical point of f. If either partial derivative does not exist, then again (x_0, y_0) is a critical point, so (x_0, y_0) is a critical point in all cases. The proof for an absolute minimum is similar. ■

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

If $f(x, y)$ is continuous on a closed and bounded set R, then the Extreme-Value Theorem (Theorem 13.8.3) guarantees the existence of an absolute maximum and an absolute minimum of f on R. These absolute extrema can occur either on the boundary of R or in the interior of R, but if an absolute extremum occurs in the interior, then it occurs at a critical point by Theorem 13.8.7. Thus, we are led to the following procedure for finding absolute extrema:

Compare this procedure with that in Section 3.4 for finding the extreme values of $f(x)$ on a closed interval.

How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R

Step 1. Find the critical points of f that lie in the interior of R.

Step 2. Find all boundary points at which the absolute extrema can occur.

Step 3. Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

► **Example 5** Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7 \tag{3}$$

on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).

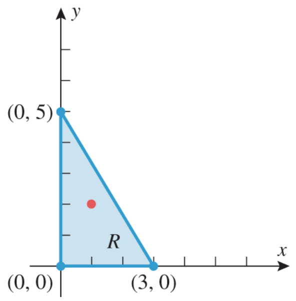


Figure 13.8.9

Solution. The region R is shown in Figure 13.8.9. We have

$$\frac{\partial f}{\partial x} = 3y - 6 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x - 3$$

so all critical points occur where

$$3y - 6 = 0 \quad \text{and} \quad 3x - 3 = 0$$

Solving these equations yields $x = 1$ and $y = 2$, so $(1, 2)$ is the only critical point. As shown in Figure 13.8.9, this critical point is in the interior of R.

Next we want to determine the locations of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we will treat separately:

The line segment between $(0, 0)$ *and* $(3, 0)$: On this line segment we have $y = 0$, so (3) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = -6x + 7, \quad 0 \le x \le 3$$

This function has no critical points because $u'(x) = -6$ is nonzero for all x. Thus the extreme values of $u(x)$ occur at the endpoints $x = 0$ and $x = 3$, which correspond to the points $(0, 0)$ and $(3, 0)$ of R.

The line segment between $(0, 0)$ *and* $(0, 5)$: On this line segment we have $x = 0$, so (3) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y + 7, \quad 0 \le y \le 5$$

This function has no critical points because $v'(y) = -3$ is nonzero for all y. Thus, the extreme values of $v(y)$ occur at the endpoints $y = 0$ and $y = 5$, which correspond to the points $(0, 0)$ and $(0, 5)$ of R.

The line segment between $(3, 0)$ *and* $(0, 5)$: In the xy-plane, an equation for this line segment is

$$y = -\tfrac{5}{3}x + 5, \quad 0 \le x \le 3 \tag{4}$$

so (3) simplifies to a function of the single variable x,

$$\begin{aligned} w(x) = f\left(x, -\tfrac{5}{3}x + 5\right) &= 3x\left(-\tfrac{5}{3}x + 5\right) - 6x - 3\left(-\tfrac{5}{3}x + 5\right) + 7 \\ &= -5x^2 + 14x - 8, \qquad 0 \le x \le 3 \end{aligned}$$

Since $w'(x) = -10x + 14$, the equation $w'(x) = 0$ yields $x = \frac{7}{5}$ as the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = \frac{7}{5}$ or at the endpoints $x = 0$ and $x = 3$. The endpoints correspond to the points $(0, 5)$ and $(3, 0)$ of R, and from (4) the critical point corresponds to $\left(\frac{7}{5}, \frac{8}{3}\right)$.

Finally, Table 13.8.1 lists the values of $f(x, y)$ at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is $f(0, 0) = 7$ and the absolute minimum value is $f(3, 0) = -11$. ◀

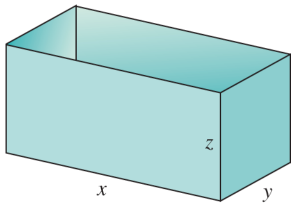
Table 13.8.1

(x, y)	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	7	-11	-8	$\frac{9}{5}$	1

▶ **Example 6** Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution. Let

$x =$ length of the box (in feet)
$y =$ width of the box (in feet)
$z =$ height of the box (in feet)
$S =$ surface area of the box (in square feet)



Two sides each have area xz.
Two sides each have area yz.
The base has area xy.

Figure 13.8.10

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area

$$S = xy + 2xz + 2yz \tag{5}$$

(Figure 13.8.10) subject to the volume requirement

$$xyz = 32 \tag{6}$$

From (6) we obtain $z = 32/xy$, so (5) can be rewritten as

$$S = xy + \frac{64}{y} + \frac{64}{x} \tag{7}$$

which expresses S as a function of two variables. The dimensions x and y in this formula must be positive, but otherwise have no limitation, so our problem reduces to finding the absolute minimum value of S over the open first quadrant: $x > 0$, $y > 0$. Because this region is neither closed nor bounded, we have no mathematical guarantee at this stage that an absolute minimum exists. However, if S has an absolute minimum value in the open first quadrant, then it must occur at a critical point of S. Thus, our next step is to find the critical points of S.

Differentiating (7) we obtain

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2} \tag{8}$$

so the coordinates of the critical points of S satisfy

$$y - \frac{64}{x^2} = 0, \quad x - \frac{64}{y^2} = 0$$

Solving the first equation for y yields

$$y = \frac{64}{x^2} \tag{9}$$

and substituting this expression in the second equation yields

$$x - \frac{64}{(64/x^2)^2} = 0$$

which can be rewritten as

$$x\left(1 - \frac{x^3}{64}\right) = 0$$

The solutions of this equation are $x = 0$ and $x = 4$. Since we require $x > 0$, the only solution of significance is $x = 4$. Substituting this value into (9) yields $y = 4$. We conclude that the point $(x, y) = (4, 4)$ is the only critical point of S in the first quadrant. Since $S = 48$ if $x = y = 4$, this suggests we try to show that the minimum value of S on the open first quadrant is 48.

It immediately follows from Equation (7) that $48 < S$ at any point in the first quadrant for which at least one of the inequalities

$$xy > 48, \quad \frac{64}{y} > 48, \quad \frac{64}{x} > 48$$

is satisfied. Therefore, to prove that $48 \le S$, we can restrict attention to the set of points in the first quadrant that satisfy the three inequalities

$$xy \le 48, \quad \frac{64}{y} \le 48, \quad \frac{64}{x} \le 48$$

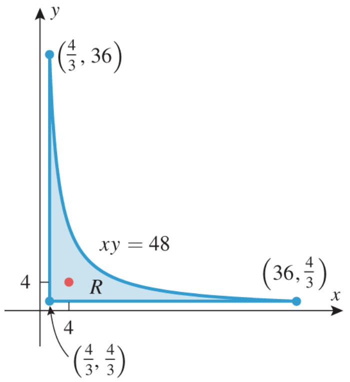


Figure 13.8.11

These inequalities can be rewritten as

$$xy \le 48, \quad y \ge \frac{4}{3}, \quad x \ge \frac{4}{3}$$

and they define a closed and bounded region R within the first quadrant (Figure 13.8.11). The function S is continuous on R, so Theorem 13.8.3 guarantees that S has an absolute minimum value somewhere on R. Since the point $(4, 4)$ lies within R, and $48 < S$ on the boundary of R (why?), the minimum value of S on R must occur at an interior point. It then follows from Theorem 13.8.7 that the mimimum value of S on R must occur at a critical point of S. Hence, the absolute minimum of S on R (and therefore on the entire open first quadrant) is $S = 48$ at the point $(4, 4)$. Substituting $x = 4$ and $y = 4$ into (6) yields $z = 2$, so the box using the least material has a height of 2 ft and a square base whose edges are 4 ft long. ◀

REMARK Fortunately, in our solution to Example 6 we were able to prove the existence of an absolute minimum of S in the first quadrant. The general problem of finding the absolute extrema of a function on an unbounded region, or on a region that is not closed, can be difficult and will not be considered in this text. However, in applied problems we can sometimes use physical considerations to deduce that an absolute extremum has been found. For example, the graph of Equation (7) in Figure 13.8.12 strongly suggests that the relative minimum at $x = 4$ and $y = 4$ is also an absolute minimum.

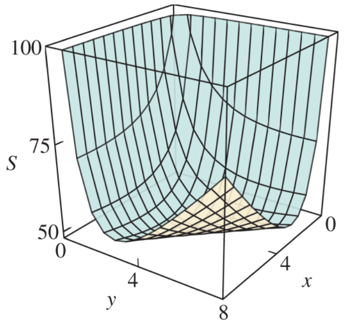


Figure 13.8.12

✓QUICK CHECK EXERCISES 13.8 (See page 883 for answers.)

1. The critical points of the function $f(x, y) = x^3 + xy + y^2$ are ______.
2. Suppose that $f(x, y)$ has continuous second-order partial derivatives everywhere and that the origin is a critical point for f. State what information (if any) is provided by the second partials test if
   (a) $f_{xx}(0,0) = 2,\ f_{xy}(0,0) = 2,\ f_{yy}(0,0) = 2$
   (b) $f_{xx}(0,0) = -2,\ f_{xy}(0,0) = 2,\ f_{yy}(0,0) = 2$
   (c) $f_{xx}(0,0) = 3,\ f_{xy}(0,0) = 2,\ f_{yy}(0,0) = 2$
   (d) $f_{xx}(0,0) = -3,\ f_{xy}(0,0) = 2,\ f_{yy}(0,0) = -2$.
3. For the function $f(x, y) = x^3 - 3xy + y^3$, state what information (if any) is provided by the second partials test at the point
   (a) $(0, 0)$ (b) $(-1, -1)$ (c) $(1, 1)$.
4. A rectangular box has total surface area of 2 ft^2. Express the volume of the box as a function of the dimensions x and y of the base of the box.

EXERCISE SET 13.8 Graphing Utility C CAS

1–2 Locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus. ■

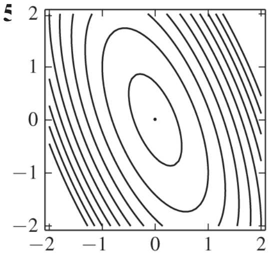
1. (a) $f(x, y) = (x - 2)^2 + (y + 1)^2$
   (b) $f(x, y) = 1 - x^2 - y^2$
   (c) $f(x, y) = x + 2y - 5$
2. (a) $f(x, y) = 1 - (x + 1)^2 - (y - 5)^2$
   (b) $f(x, y) = e^{xy}$
   (c) $f(x, y) = x^2 - y^2$

3–4 Complete the squares and locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus. ■

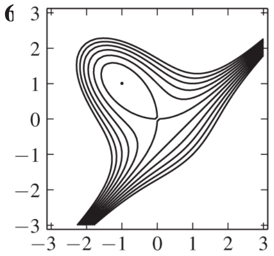
3. $f(x, y) = 13 - 6x + x^2 + 4y + y^2$
4. $f(x, y) = 1 - 2x - x^2 + 4y - 2y^2$

FOCUS ON CONCEPTS

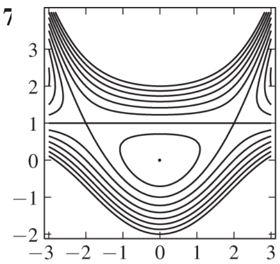
5–8 The contour plots show all significant features of the function. Make a conjecture about the number and the location of all relative extrema and saddle points, and then use calculus to check your conjecture.



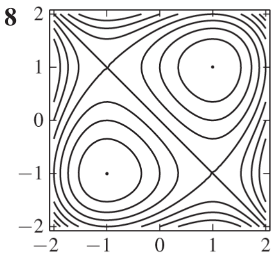
$f(x, y) = 3x^2 + 2xy + y^2$



$f(x, y) = x^3 - 3xy - y^3$



$f(x, y) = x^2 + 2y^2 - x^2y$



$f(x, y) = x^3 + y^3 - 3x - 3y$

9–20 Locate all relative maxima, relative minima, and saddle points, if any.

9. $f(x, y) = y^2 + xy + 3y + 2x + 3$

10. $f(x, y) = x^2 + xy - 2y - 2x + 1$

11. $f(x, y) = x^2 + xy + y^2 - 3x$

12. $f(x, y) = xy - x^3 - y^2$

13. $f(x, y) = x^2 + y^2 + \dfrac{2}{xy}$

14. $f(x, y) = xe^y$

15. $f(x, y) = x^2 + y - e^y$

16. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$

17. $f(x, y) = e^x \sin y$

18. $f(x, y) = y \sin x$

19. $f(x, y) = e^{-(x^2+y^2+2x)}$

20. $f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{b^3}{y} \quad (a \neq 0, b \neq 0)$

C **21.** Use a CAS to generate a contour plot of

$$f(x, y) = 2x^2 - 4xy + y^4 + 2$$

for $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$, and use the plot to approximate the locations of all relative extrema and saddle points in the region. Check your answer using calculus, and identify the relative extrema as relative maxima or minima.

C **22.** Use a CAS to generate a contour plot of

$$f(x, y) = 2y^2x - yx^2 + 4xy$$

for $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$, and use the plot to approximate the locations of all relative extrema and saddle points in the region. Check your answer using calculus, and identify the relative extrema as relative maxima or minima.

23–26 True–False Determine whether the statement is true or false. Explain your answer. In these exercises, assume that $f(x, y)$ has continuous second-order partial derivatives and that

$$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - f_{xy}^2(x, y)$$

23. If the function f is defined on the disk $x^2 + y^2 \leq 1$, then f has a critical point somewhere on this disk.

24. If the function f is defined on the disk $x^2 + y^2 \leq 1$, and if f is not a constant function, then f has a finite number of critical points on this disk.

25. If $P(x_0, y_0)$ is a critical point of f, and if f is defined on a disk centered at P with $D(x_0, y_0) > 0$, then f has a relative extremum at P.

26. If $P(x_0, y_0)$ is a critical point of f with $f(x_0, y_0) = 0$, and if f is defined on a disk centered at P with $D(x_0, y_0) < 0$, then f has both positive and negative values on this disk.

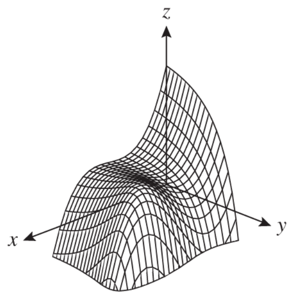
FOCUS ON CONCEPTS

27. (a) Show that the second partials test provides no information about the critical points of the function $f(x, y) = x^4 + y^4$.
(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

28. (a) Show that the second partials test provides no information about the critical points of the function $f(x, y) = x^4 - y^4$.
(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

29. Recall from Theorem 3.4.4 that if a continuous function of one variable has exactly one relative extremum on an interval, then that relative extremum is an absolute extremum on the interval. This exercise shows that this result does not extend to functions of two variables.
(a) Show that $f(x, y) = 3xe^y - x^3 - e^{3y}$ has only one critical point and that a relative maximum occurs there. (See the accompanying figure.)
(b) Show that f does not have an absolute maximum.

Source: This exercise is based on the article "The Only Critical Point in Town Test" by Ira Rosenholtz and Lowell Smylie, *Mathematics Magazine*, Vol. 58, No. 3, May 1985, pp. 149–150.

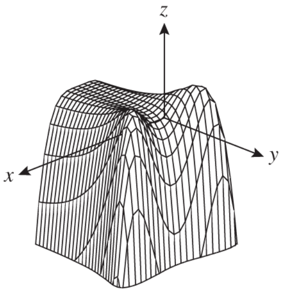


$z = 3xe^y - x^3 - e^{3y}$

Figure Ex-29

30. If f is a continuous function of one variable with two relative maxima on an interval, then there must be a relative minimum between the relative maxima. (Convince yourself of this by drawing some pictures.) The purpose of this exercise is to show that this result does not extend to functions of two variables. Show that $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$ has two relative maxima but no other critical points (see Figure Ex-30).

Source: This exercise is based on the problem "Two Mountains Without a Valley" proposed and solved by Ira Rosenholtz, *Mathematics Magazine*, Vol. 60, No. 1, February 1987, p. 48.



$z = 4x^2e^y - 2x^4 - e^{4y}$

Figure Ex-30

31–36 Find the absolute extrema of the given function on the indicated closed and bounded set R. ■

31. $f(x, y) = xy - x - 3y$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(5, 0)$.

32. $f(x, y) = xy - 2x$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(4, 0)$.

33. $f(x, y) = x^2 - 3y^2 - 2x + 6y$; R is the region bounded by the square with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$, and $(2, 0)$.

34. $f(x, y) = xe^y - x^2 - e^y$; R is the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 1)$, and $(2, 0)$.

35. $f(x, y) = x^2 + 2y^2 - x$; R is the disk $x^2 + y^2 \leq 4$.

36. $f(x, y) = xy^2$; R is the region that satisfies the inequalities $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 1$.

37. Find three positive numbers whose sum is 48 and such that their product is as large as possible.

38. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.

39. Find all points on the portion of the plane $x + y + z = 5$ in the first octant at which $f(x, y, z) = xy^2z^2$ has a maximum value.

40. Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

41. Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius a.

42. An international airline has a regulation that each passenger can carry a suitcase having the sum of its width, length, and height less than or equal to 129 cm. Find the dimensions of the suitcase of maximum volume that a passenger can carry under this regulation.

43. A closed rectangular box with a volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing 10¢ per square foot and the sides from material costing 5¢ per square foot. Find the dimensions of the box so that the cost of materials is minimized.

44. A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell $500(y - x)$ of the standard items and $45{,}000 + 500(x - 2y)$ of the deluxe each year. How should the items be priced to maximize the profit?

45. Consider the function

$$f(x, y) = 4x^2 - 3y^2 + 2xy$$

over the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$.

(a) Find the maximum and minimum values of f on each edge of the square.

(b) Find the maximum and minimum values of f on each diagonal of the square.

(c) Find the maximum and minimum values of f on the entire square.

46. Show that among all parallelograms with perimeter l, a square with sides of length $l/4$ has maximum area. [*Hint:* The area of a parallelogram is given by the formula $A = ab \sin \alpha$, where a and b are the lengths of two adjacent sides and α is the angle between them.]

47. Determine the dimensions of a rectangular box, open at the top, having volume V, and requiring the least amount of material for its construction.

48. A length of sheet metal 27 inches wide is to be made into a water trough by bending up two sides as shown in the accompanying figure. Find x and ϕ so that the trapezoid-shaped cross section has a maximum area.

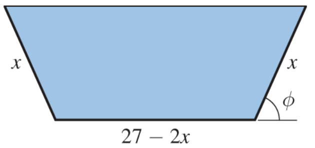


Figure Ex-48

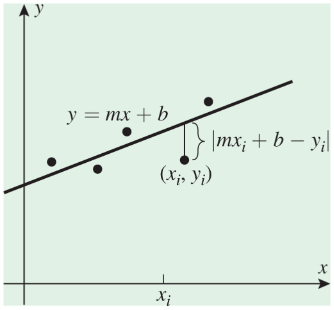
49–50 A common problem in experimental work is to obtain a mathematical relationship $y = f(x)$ between two variables x and y by "fitting" a curve to points in the plane that correspond to experimentally determined values of x and y, say

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

The curve $y = f(x)$ is called a ***mathematical model*** of the data. The general form of the function f is commonly determined by some underlying physical principle, but sometimes it is just determined by the pattern of the data. We are concerned with fitting a straight line $y = mx + b$ to data. Usually, the data will not lie on a line (possibly due to experimental error or variations in experimental conditions), so the problem is to find a line that fits the data "best" according to some criterion. One criterion for selecting the line of best fit is to choose m and b to minimize the function

$$g(m, b) = \sum_{i=1}^{n} (mx_i + b - y_i)^2$$

This is called the ***method of least squares***, and the resulting line is called the ***regression line*** or the ***least squares line of best fit***. Geometrically, $|mx_i + b - y_i|$ is the vertical distance between the data point (x_i, y_i) and the line $y = mx + b$.



These vertical distances are called the ***residuals*** of the data points, so the effect of minimizing $g(m, b)$ is to minimize the sum of the squares of the residuals. In these exercises, we will derive a formula for the regression line.

49. The purpose of this exercise is to find the values of m and b that produce the regression line.

(a) To minimize $g(m, b)$, we start by finding values of m and b such that $\partial g/\partial m = 0$ and $\partial g/\partial b = 0$. Show that these equations are satisfied if m and b satisfy the conditions

$$\left(\sum_{i=1}^{n} x_i^2\right) m + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i$$

$$\left(\sum_{i=1}^{n} x_i\right) m + nb = \sum_{i=1}^{n} y_i$$

(b) Let $\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$ denote the arithmetic average of $x_1, x_2, \ldots, x_n$. Use the fact that

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 \geq 0$$

to show that

$$n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2 \geq 0$$

with equality if and only if all the x_i's are the same.

(c) Assuming that not all the x_i's are the same, prove that the equations in part (a) have the unique solution

$$m = \frac{n\displaystyle\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\displaystyle\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right)$$

[*Note:* We have shown that g has a critical point at these values of m and b. In the next exercise we will show that g has an absolute minimum at this critical point. Accepting this to be so, we have shown that the line $y = mx + b$ is the regression line for these values of m and b.]

50. Assume that not all the x_i's are the same, so that $g(m, b)$ has a unique critical point at the values of m and b obtained in Exercise 49(c). The purpose of this exercise is to show that g has an absolute minimum value at this point.

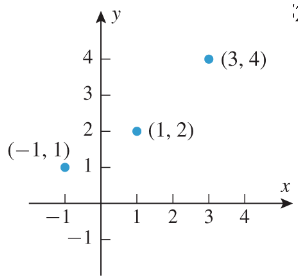
(a) Find the partial derivatives $g_{mm}(m, b)$, $g_{bb}(m, b)$, and $g_{mb}(m, b)$, and then apply the second partials test to show that g has a relative minimum at the critical point obtained in Exercise 49.

(b) Show that the graph of the equation $z = g(m, b)$ is a quadric surface. [*Hint:* See Formula (4) of Section 11.7.]

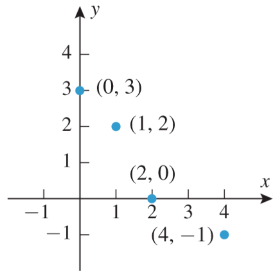
(c) It can be proved that the graph of $z = g(m, b)$ is an elliptic paraboloid. Accepting this to be so, show that this paraboloid opens in the positive z-direction, and explain how this shows that g has an absolute minimum at the critical point obtained in Exercise 49.

51–54 Use the formulas obtained in Exercise 49 to find and draw the regression line. If you have a calculating utility that can calculate regression lines, use it to check your work.

51.



52.



53.

x	1	2	3	4
y	1.5	1.6	2.1	3.0

54.

x	1	2	3	4	5
y	4.2	3.5	3.0	2.4	2.0

55. The following table shows the life expectancy by year of birth of females in the United States:

YEAR OF BIRTH	2000	2001	2002	2003	2004	2005	2006	2007
LIFE EXPECTANCY	79.3	79.4	79.5	79.6	79.9	79.9	80.2	80.4

Source: Data from *The 2011 Statistical Abstract*, the U.S. Census Bureau.

(a) Take $t = 0$ to be the year 1930, and let y be the life expectancy for birth year t. Use the regression capability of a calculating utility to find the regression line of y as a function of t.

(b) Use a graphing utility to make a graph that shows the data points and the regression line.

(c) Use the regression line to make a conjecture about the life expectancy of females born in the year 2015.

56. A company manager wants to establish a relationship between the sales of a certain product and the price. The company research department provides the following data:

PRICE (x) IN DOLLARS	\$35.00	\$40.00	\$45.00	\$48.00	\$50.00
DAILY SALES VOLUME (y) IN UNITS	80	75	68	66	63

(a) Use a calculating utility to find the regression line of y as a function of x.
(b) Use a graphing utility to make a graph that shows the data points and the regression line.
(c) Use the regression line to make a conjecture about the number of units that would be sold at a price of \$60.00.

57. If a gas is cooled with its volume held constant, then it follows from the ***ideal gas law*** in physics that its pressure drops proportionally to the drop in temperature. The temperature that, in theory, corresponds to a pressure of zero is called ***absolute zero***. Suppose that an experiment produces the following data for pressure P versus temperature T with the volume held constant:

P (KILOPASCALS)	134	142	155	160	171	184
T (°CELSIUS)	0	20	40	60	80	100

(a) Use a calculating utility to find the regression line of P as a function of T.
(b) Use a graphing utility to make a graph that shows the data points and the regression line.
(c) Use the regression line to estimate the value of absolute zero in degrees Celsius.

58. Find
(a) a continuous function $f(x, y)$ that is defined on the entire xy-plane and has no absolute extrema on the xy-plane;
(b) a function $f(x, y)$ that is defined everywhere on the rectangle $0 \le x \le 1, 0 \le y \le 1$ and has no absolute extrema on the rectangle.

59. Show that if f has a relative maximum at (x_0, y_0), then $G(x) = f(x, y_0)$ has a relative maximum at $x = x_0$ and $H(y) = f(x_0, y)$ has a relative maximum at $y = y_0$.

60. Writing Explain how to determine the location of relative extrema or saddle points of $f(x, y)$ by examining the contours of f.

61. Writing Suppose that the second partials test gives no information about a certain critical point (x_0, y_0) because $D(x_0, y_0) = 0$. Discuss what other steps you might take to determine whether there is a relative extremum at that critical point.

✓QUICK CHECK ANSWERS 13.8 **1.** $(0, 0)$ and $\left(\frac{1}{6}, -\frac{1}{12}\right)$ **2.** (a) no information (b) a saddle point at $(0, 0)$ (c) a relative minimum at $(0, 0)$ (d) a relative maximum at $(0, 0)$ **3.** (a) a saddle point at $(0, 0)$ (b) no information, since $(-1, -1)$ is not a critical point (c) a relative minimum at $(1, 1)$ **4.** $V = \dfrac{xy(1 - xy)}{x + y}$

13.9 LAGRANGE MULTIPLIERS

In this section we will study a powerful new method for maximizing or minimizing a function subject to constraints on the variables. This method will help us to solve certain optimization problems that are difficult or impossible to solve using the methods studied in the last section.

EXTREMUM PROBLEMS WITH CONSTRAINTS

In Example 6 of the last section, we solved the problem of minimizing

$$S = xy + 2xz + 2yz \tag{1}$$

subject to the constraint

$$xyz - 32 = 0 \tag{2}$$

This is a special case of the following general problem:

13.9.1 ***Three-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

We will also be interested in the following two-variable version of this problem:

13.9.2 ***Two-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

LAGRANGE MULTIPLIERS

One way to attack problems of these types is to solve the constraint equation for one of the variables in terms of the others and substitute the result into f. This produces a new function of one or two variables that incorporates the constraint and can be maximized or minimized by applying standard methods. For example, to solve the problem in Example 6 of the last section we substituted (2) into (1) to obtain

$$S = xy + \frac{64}{y} + \frac{64}{x}$$

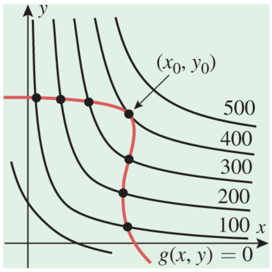
which we then minimized by finding the critical points and applying the second partials test. However, this approach hinges on our ability to solve the constraint equation for one of the variables in terms of the others. If this cannot be done, then other methods must be used. One such method, called the *method of Lagrange multipliers*, will be discussed in this section.

To motivate the method of Lagrange multipliers, suppose that we are trying to maximize a function $f(x, y)$ subject to the constraint $g(x, y) = 0$. Geometrically, this means that we are looking for a point (x_0, y_0) on the graph of the constraint curve at which $f(x, y)$ is as large as possible. To help locate such a point, let us construct a contour plot of $f(x, y)$ in the same coordinate system as the graph of $g(x, y) = 0$. For example, Figure 13.9.1a shows some typical level curves of $f(x, y) = c$, which we have labeled $c = 100, 200, 300, 400$, and 500 for purposes of illustration. In this figure, each point of intersection of $g(x, y) = 0$ with a level curve is a candidate for a solution, since these points lie on the constraint curve. Among the seven such intersections shown in the figure, the maximum value of $f(x, y)$ occurs at the intersection (x_0, y_0) where $f(x, y)$ has a value of 400. Note that at (x_0, y_0) the constraint curve and the level curve just touch and thus have a *common* tangent line at this point. Since $\nabla f(x_0, y_0)$ is normal to the level curve $f(x, y) = 400$ at (x_0, y_0), and since $\nabla g(x_0, y_0)$ is normal to the constraint curve $g(x, y) = 0$ at (x_0, y_0), we conclude that the vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ must be parallel. That is,

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{3}$$

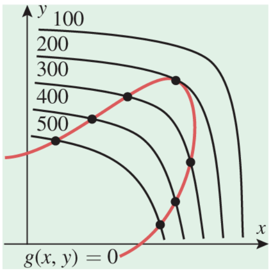
for some scalar λ. The same condition holds at points on the constraint curve where $f(x, y)$ has a minimum. For example, if the level curves are as shown in Figure 13.9.1b, then the minimum value of $f(x, y)$ occurs where the constraint curve just touches a level curve. Thus, to find the maximum or minimum of $f(x, y)$ subject to the constraint $g(x, y) = 0$, we look for points at which (3) holds—this is the method of Lagrange multipliers.

Our next objective in this section is to make the preceding intuitive argument more precise. For this purpose it will help to begin with some terminology about the problem



Maximum of $f(x, y)$ is 400

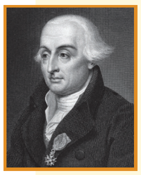
(a)



Minimum of $f(x, y)$ is 200

(b)

Figure 13.9.1

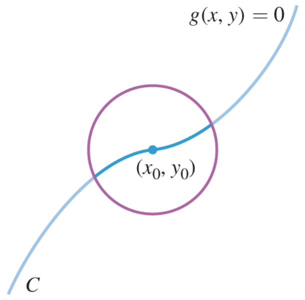


Joseph Louis Lagrange **(1736–1813)** French–Italian mathematician and astronomer. Lagrange, the son of a public official, was born in Turin, Italy. (Baptismal records list his name as Giuseppe Lodovico Lagrangia.) Although his father wanted him to be a lawyer, Lagrange was attracted to mathematics and astronomy after reading a memoir by the astronomer Halley. At age 16 he began to study mathematics on his own and by age 19 was appointed to a professorship at the Royal Artillery School in Turin. The following year Lagrange sent Euler solutions to some famous problems using new methods that eventually blossomed into a branch of mathematics called calculus of variations. These methods and Lagrange's applications of them to problems in celestial mechanics were so monumental that by age 25 he was regarded by many of his contemporaries as the greatest living mathematician.

In 1776, on the recommendations of Euler, he was chosen to succeed Euler as the director of the Berlin Academy. During his stay in Berlin, Lagrange distinguished himself not only in celestial mechanics, but also in algebraic equations and the theory of numbers. After twenty years in Berlin, he moved to Paris at the invitation of Louis XVI. He was given apartments in the Louvre and treated with great honor, even during the revolution.

Napoleon was a great admirer of Lagrange and showered him with honors—count, senator, and Legion of Honor. The years Lagrange spent in Paris were devoted primarily to didactic treatises summarizing his mathematical conceptions. One of Lagrange's most famous works is a memoir, *Mécanique Analytique*, in which he reduced the theory of mechanics to a few general formulas from which all other necessary equations could be derived.

It is an interesting historical fact that Lagrange's father speculated unsuccessfully in several financial ventures, so his family was forced to live quite modestly. Lagrange himself stated that if his family had money, he would not have made mathematics his vocation. In spite of his fame, Lagrange was always a shy and modest man. On his death, he was buried with honor in the Pantheon.



A constrained relative maximum occurs at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ on some segment of C that extends on both sides of (x_0, y_0).

Figure 13.9.2

of maximizing or minimizing a function $f(x, y)$ subject to a constraint $g(x, y) = 0$. As with other kinds of maximization and minimization problems, we need to distinguish between relative and absolute extrema. We will say that f has a ***constrained absolute maximum (minimum)*** at (x_0, y_0) if $f(x_0, y_0)$ is the largest (smallest) value of f on the constraint curve, and we will say that f has a ***constrained relative maximum (minimum)*** at (x_0, y_0) if $f(x_0, y_0)$ is the largest (smallest) value of f on some segment of the constraint curve that extends on both sides of the point (x_0, y_0) (Figure 13.9.2).

Let us assume that a constrained relative maximum or minimum occurs at the point (x_0, y_0), and for simplicity let us further assume that the equation $g(x, y) = 0$ can be smoothly parametrized as

$$x = x(s), \quad y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Thus, the quantity

$$z = f(x(s), y(s))$$

has a relative maximum or minimum at $s = 0$, and this implies that $dz/ds = 0$ at that point. From the chain rule, this equation can be expressed as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) = 0$$

where the derivatives are all evaluated at $s = 0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0$, it follows from this equation that

$$\nabla f(x_0, y_0) \cdot \mathbf{T}(0) = 0$$

which implies that the gradient is either $\mathbf{0}$ or is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x, y) = 0$ is a level curve for the function $g(x, y)$, so that if $\nabla g(x_0, y_0) \neq \mathbf{0}$, then $\nabla g(x_0, y_0)$ is normal to this curve at (x_0, y_0). It then follows that there is some scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{4}$$

This scalar is called a ***Lagrange multiplier***. Thus, the ***method of Lagrange multipliers*** for finding constrained relative extrema is to look for points on the constraint curve $g(x, y) = 0$ at which Equation (4) is satisfied for some scalar λ.

13.9.3 THEOREM (*Constrained-Extremum Principle for Two Variables and One Constraint*) *Let f and g be functions of two variables with continuous first partial derivatives on some open set containing the constraint curve $g(x, y) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Example 1 At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute maximum, and what is that maximum?

Solution. The circle $x^2 + y^2 = 1$ is a closed and bounded set and $f(x, y) = xy$ is a continuous function, so it follows from the Extreme-Value Theorem (Theorem 13.8.3) that f has an absolute maximum and an absolute minimum on the circle. To find these extrema, we will use Lagrange multipliers to find the constrained relative extrema, and then we will evaluate f at those relative extrema to find the absolute extrema.

We want to maximize $f(x, y) = xy$ subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0 \tag{5}$$

First we will look for constrained *relative* extrema. For this purpose we will need the gradients

$$\nabla f = y\mathbf{i} + x\mathbf{j} \quad \text{and} \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$$

From the formula for ∇g we see that $\nabla g = \mathbf{0}$ if and only if $x = 0$ and $y = 0$, so $\nabla g \neq \mathbf{0}$ at any point on the circle $x^2 + y^2 = 1$. Thus, at a constrained relative extremum we must have

$$\nabla f = \lambda \nabla g \quad \text{or} \quad y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j})$$

which is equivalent to the pair of equations

$$y = 2x\lambda \quad \text{and} \quad x = 2y\lambda$$

It follows from these equations that if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$. In either case we have $x^2 + y^2 = 0$, so the constraint equation $x^2 + y^2 = 1$ is not satisfied. Thus, we can assume that x and y are nonzero, and we can rewrite the equations as

$$\lambda = \frac{y}{2x} \quad \text{and} \quad \lambda = \frac{x}{2y}$$

from which we obtain

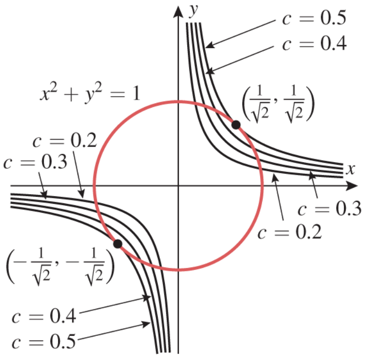
$$\frac{y}{2x} = \frac{x}{2y}$$

or

$$y^2 = x^2 \tag{6}$$

Substituting this in (5) yields

$$2x^2 - 1 = 0$$



▲ Figure 13.9.3

from which we obtain $x = \pm 1/\sqrt{2}$. Each of these values, when substituted in Equation (6), produces y-values of $y = \pm 1/\sqrt{2}$. Thus, constrained relative extrema occur at the points $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$. The values of xy at these points are as follows:

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
xy	1/2	−1/2	−1/2	1/2

Give another solution to Example 1 using the parametrization

$$x = \cos\theta, \quad y = \sin\theta$$

and the identity

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Thus, the function $f(x, y) = xy$ has an absolute maximum of $\frac{1}{2}$ occurring at the two points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Although it was not asked for, we can also see that f has an absolute minimum of $-\frac{1}{2}$ occurring at the points $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$. Figure 13.9.3 shows some level curves $xy = c$ and the constraint curve in the vicinity of the maxima. A similar figure for the minima can be obtained using negative values of c for the level curves $xy = c$. ◀

REMARK If c is a constant, then the functions $g(x, y)$ and $g(x, y) - c$ have the same gradient since the constant c drops out when we differentiate. Consequently, it is *not* essential to rewrite a constraint of the form $g(x, y) = c$ as $g(x, y) - c = 0$ in order to apply the constrained-extremum principle. Thus, in the last example, we could have kept the constraint in the form $x^2 + y^2 = 1$ and then taken $g(x, y) = x^2 + y^2$ rather than $g(x, y) = x^2 + y^2 - 1$.

▶ **Example 2** Use the method of Lagrange multipliers to find the dimensions of a rectangle with perimeter p and maximum area.

Solution. Let

x = length of the rectangle, y = width of the rectangle, A = area of the rectangle

We want to maximize $A = xy$ on the line segment

$$2x + 2y = p, \quad 0 \leq x, y \tag{7}$$

that corresponds to the perimeter constraint. This segment is a closed and bounded set, and since $f(x, y) = xy$ is a continuous function, it follows from the Extreme-Value Theorem (Theorem 13.8.3) that f has an absolute maximum on this segment. This absolute maximum must also be a constrained relative maximum since f is 0 at the endpoints of the segment and positive elsewhere on the segment. If $g(x, y) = 2x + 2y$, then we have

$$\nabla f = y\mathbf{i} + x\mathbf{j} \quad \text{and} \quad \nabla g = 2\mathbf{i} + 2\mathbf{j}$$

Noting that $\nabla g \neq \mathbf{0}$, it follows from (4) that

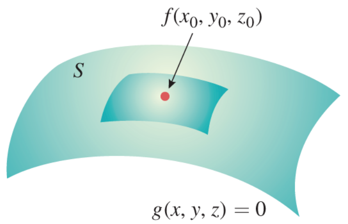
$$y\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + 2\mathbf{j})$$

at a constrained relative maximum. This is equivalent to the two equations

$$y = 2\lambda \quad \text{and} \quad x = 2\lambda$$

Eliminating λ from these equations we obtain $x = y$, which shows that the rectangle is actually a square. Using this condition and constraint (7), we obtain $x = p/4$, $y = p/4$. ◀

THREE VARIABLES AND ONE CONSTRAINT



A constrained relative maximum occurs at (x_0, y_0, z_0) if $f(x_0, y_0, z_0) \geq f(x, y, z)$ at all points of S near (x_0, y_0, z_0).

▲ **Figure 13.9.4**

The method of Lagrange multipliers can also be used to maximize or minimize a function of three variables $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$. As a rule, the graph of $g(x, y, z) = 0$ will be some surface S in 3-space. Thus, from a geometric viewpoint, the problem is to maximize or minimize $f(x, y, z)$ as (x, y, z) varies over the surface S (Figure 13.9.4). As usual, we distinguish between relative and absolute extrema. We will say that f has a ***constrained absolute maximum (minimum)*** at (x_0, y_0, z_0) if $f(x_0, y_0, z_0)$ is the largest (smallest) value of $f(x, y, z)$ on S, and we will say that f has a ***constrained relative maximum (minimum)*** at (x_0, y_0, z_0) if $f(x_0, y_0, z_0)$ is the largest (smallest) value of $f(x, y, z)$ at all points of S "near" (x_0, y_0, z_0).

The following theorem, which we state without proof, is the three-variable analog of Theorem 13.9.3.

13.9.4 THEOREM ***(Constrained-Extremum Principle for Three Variables and One Constraint)*** *Let f and g be functions of three variables with continuous first partial derivatives on some open set containing the constraint surface $g(x, y, z) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this surface. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0, z_0) on the constraint surface at which the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

▶ **Example 3** Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point $(1, 2, 2)$.

Solution. To avoid radicals, we will find points on the sphere that minimize and maximize the *square* of the distance to $(1, 2, 2)$. Thus, we want to find the relative extrema of

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 2)^2$$

subject to the constraint

$$x^2 + y^2 + z^2 = 36 \tag{8}$$

If we let $g(x, y, z) = x^2 + y^2 + z^2$, then $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Thus, $\nabla g = \mathbf{0}$ if and only if $x = y = z = 0$. It follows that $\nabla g \neq \mathbf{0}$ at any point of the sphere (8), and hence the constrained relative extrema must occur at points where

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

That is,

$$2(x - 1)\mathbf{i} + 2(y - 2)\mathbf{j} + 2(z - 2)\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$$

which leads to the equations

$$2(x-1) = 2x\lambda, \quad 2(y-2) = 2y\lambda, \quad 2(z-2) = 2z\lambda \tag{9}$$

We may assume that x, y, and z are nonzero since $x = 0$ does not satisfy the first equation, $y = 0$ does not satisfy the second, and $z = 0$ does not satisfy the third. Thus, we can rewrite (9) as

$$\frac{x-1}{x} = \lambda, \quad \frac{y-2}{y} = \lambda, \quad \frac{z-2}{z} = \lambda$$

The first two equations imply that

$$\frac{x-1}{x} = \frac{y-2}{y}$$

from which it follows that

$$y = 2x \tag{10}$$

Similarly, the first and third equations imply that

$$z = 2x \tag{11}$$

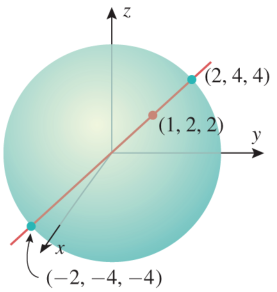
Substituting (10) and (11) in the constraint equation (8), we obtain

$$9x^2 = 36 \quad \text{or} \quad x = \pm 2$$

Substituting these values in (10) and (11) yields two points:

$$(2, 4, 4) \quad \text{and} \quad (-2, -4, -4)$$

Since $f(2, 4, 4) = 9$ and $f(-2, -4, -4) = 81$, it follows that $(2, 4, 4)$ is the point on the sphere closest to $(1, 2, 2)$, and $(-2, -4, -4)$ is the point that is farthest (Figure 13.9.5). ◀



▲ **Figure 13.9.5**

REMARK Solving nonlinear systems such as (9) usually involves trial and error. A technique that sometimes works is demonstrated in Example 3. In that example the equations were solved for a common variable (λ), and we then derived relationships between the remaining variables (x, y, and z). Substituting those relationships in the constraint equation led to the value of one of the variables, and the values of the other variables were then computed.

Next we will use Lagrange multipliers to solve the problem of Example 6 in the last section.

▶ **Example 4** Use Lagrange multipliers to determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution. With the notation introduced in Example 6 of the last section, the problem is to minimize the surface area

$$S = xy + 2xz + 2yz$$

subject to the volume constraint

$$xyz = 32 \tag{12}$$

If we let $f(x, y, z) = xy + 2xz + 2yz$ and $g(x, y, z) = xyz$, then

$$\nabla f = (y + 2z)\mathbf{i} + (x + 2z)\mathbf{j} + (2x + 2y)\mathbf{k} \quad \text{and} \quad \nabla g = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

It follows that $\nabla g \neq \mathbf{0}$ at any point on the surface $xyz = 32$, since x, y, and z are all nonzero on this surface. Thus, at a constrained relative extremum we must have $\nabla f = \lambda \nabla g$, that is,

$$(y + 2z)\mathbf{i} + (x + 2z)\mathbf{j} + (2x + 2y)\mathbf{k} = \lambda(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$$

This condition yields the three equations

$$y + 2z = \lambda yz, \quad x + 2z = \lambda xz, \quad 2x + 2y = \lambda xy$$

Because x, y, and z are nonzero, these equations can be rewritten as

$$\frac{1}{z} + \frac{2}{y} = \lambda, \quad \frac{1}{z} + \frac{2}{x} = \lambda, \quad \frac{2}{y} + \frac{2}{x} = \lambda$$

From the first two equations,

$$y = x \tag{13}$$

and from the first and third equations,

$$z = \tfrac{1}{2}x \tag{14}$$

Substituting (13) and (14) in the volume constraint (12) yields

$$\tfrac{1}{2}x^3 = 32$$

This equation, together with (13) and (14), yields

$$x = 4, \quad y = 4, \quad z = 2$$

which agrees with the result that was obtained in Example 6 of the last section. ◀

There are variations in the method of Lagrange multipliers that can be used to solve problems with two or more constraints. However, we will not discuss that topic here.

✓ QUICK CHECK EXERCISES 13.9 (See page 890 for answers.)

1. (a) Suppose that $f(x, y)$ and $g(x, y)$ are differentiable at the origin and have nonzero gradients there, and that $g(0, 0) = 0$. If the maximum value of $f(x, y)$ subject to the constraint $g(x, y) = 0$ occurs at the origin, how is the tangent line to the graph of $g(x, y) = 0$ related to the tangent line at the origin to the level curve of f through $(0, 0)$?

(b) Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable at the origin and have nonzero gradients there, and that $g(0, 0, 0) = 0$. If the maximum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at the origin, how is the tangent plane to the graph of the constraint $g(x, y, z) = 0$ related to the tangent plane at the origin to the level surface of f through $(0, 0, 0)$?

2. The maximum value of $x + y$ subject to the constraint $x^2 + y^2 = 1$ is ________.

3. The maximum value of $x + y + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$ is ________.

4. The maximum and minimum values of $2x + 3y$ subject to the constraint $x + y = 1$, where $0 \le x, 0 \le y$, are ________ and ________, respectively.

EXERCISE SET 13.9 Graphing Utility C CAS

FOCUS ON CONCEPTS

1. The accompanying figure shows graphs of the line $x + y = 4$ and the level curves of height $c = 2, 4, 6,$ and 8 for the function $f(x, y) = xy$.

(a) Use the figure to find the maximum value of the function $f(x, y) = xy$ subject to $x + y = 4$, and explain your reasoning.

(b) How can you tell from the figure that your answer to part (a) is not the minimum value of f subject to the constraint?

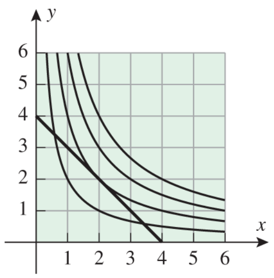
(c) Use Lagrange multipliers to check your work.

2. The accompanying figure shows the graphs of the line $3x + 4y = 25$ and the level curves of height $c = 9, 16, 25, 36,$ and 49 for the function $f(x, y) = x^2 + y^2$.

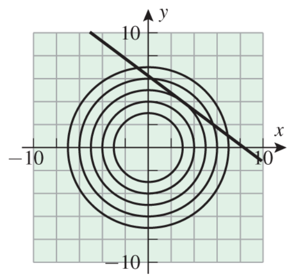
(a) Use the accompanying figure to find the minimum value of the function $f(x, y) = x^2 + y^2$ subject to $3x + 4y = 25$, and explain your reasoning.

(b) How can you tell from the accompanying figure that your answer to part (a) is not the maximum value of f subject to the constraint?

(c) Use Lagrange multipliers to check your work.



▲ **Figure Ex-1**



▲ **Figure Ex-2**

3. (a) On a graphing utility, graph the circle $x^2 + y^2 = 25$ and two distinct level curves of $f(x, y) = x^2 - y$ that just touch the circle in a single point.

(b) Use the results you obtained in part (a) to approximate the maximum and minimum values of f subject to the constraint $x^2 + y^2 = 25$.

(c) Check your approximations in part (b) using Lagrange multipliers.

C **4.** (a) If you have a CAS with implicit plotting capability, use it to graph the circle $(x-4)^2+(y-4)^2=4$ and two level curves of $f(x, y)=x^3+y^3-3xy$ that just touch the circle.
(b) Use the result you obtained in part (a) to approximate the minimum value of f subject to the constraint $(x-4)^2+(y-4)^2=4$.
(c) Confirm graphically that you have found a minimum and not a maximum.
(d) Check your approximation using Lagrange multipliers and solving the required equations numerically.

5–12 Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint. Also, find the points at which these extreme values occur.

5. $f(x, y)=xy;\ 4x^2+8y^2=16$

6. $f(x, y)=x^2-y^2;\ x^2+y^2=25$

7. $f(x, y)=4x^3+y^2;\ 2x^2+y^2=1$

8. $f(x, y)=x-3y-1;\ x^2+3y^2=16$

9. $f(x, y, z)=2x+y-2z;\ x^2+y^2+z^2=4$

10. $f(x, y, z)=3x+6y+2z;\ 2x^2+4y^2+z^2=70$

11. $f(x, y, z)=xyz;\ x^2+y^2+z^2=1$

12. $f(x, y, z)=x^4+y^4+z^4;\ x^2+y^2+z^2=1$

13–16 True–False Determine whether the statement is true or false. Explain your answer.

13. A "Lagrange multiplier" is a special type of gradient vector.

14. The extrema of $f(x, y)$ subject to the constraint $g(x, y)=0$ occur at those points for which $\nabla f=\nabla g$.

15. In the method of Lagrange mutlipliers it is necessary to solve a constraint equation $g(x, y)=0$ for y in terms of x.

16. The extrema of $f(x, y)$ subject to the constraint $g(x, y)=0$ occur at those points at which a contour of f is tangent to the constraint curve $g(x, y)=0$.

17–24 Solve using Lagrange multipliers.

17. Find the point on the line $2x-4y=3$ that is closest to the origin.

18. Find the point on the line $y=2x+3$ that is closest to $(4, 2)$.

19. Find the point on the plane $x+2y+z=1$ that is closest to the origin.

20. Find the point on the plane $4x+3y+z=2$ that is closest to $(1, -1, 1)$.

21. Find the points on the circle $x^2+y^2=45$ that are closest to and farthest from $(1, 2)$.

22. Find the points on the surface $xy-z^2=1$ that are closest to the origin.

23. Find a vector in 3-space whose length is 5 and whose components have the largest possible sum.

24. Suppose that the temperature at a point (x, y) on a metal plate is $T(x, y)=4x^2-4xy+y^2$. An ant, walking on the plate, traverses a circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

25–32 Use Lagrange multipliers to solve the indicated exercises from Section 13.8.

25. Exercise 38

26. Exercise 39

27. Exercise 40

28. Exercise 41

29. Exercise 43

30. Exercises 45(a) and (b)

31. Exercise 46

32. Exercise 47

C **33.** Let α, β, and γ be the angles of a triangle.
(a) Use Lagrange multipliers to find the maximum value of $f(\alpha, \beta, \gamma)=\cos\alpha\cos\beta\cos\gamma$, and determine the angles for which the maximum occurs.
(b) Express $f(\alpha, \beta, \gamma)$ as a function of α and β alone, and use a CAS to graph this function of two variables. Confirm that the result obtained in part (a) is consistent with the graph.

34. The accompanying figure shows the intersection of the elliptic paraboloid $z=x^2+4y^2$ and the right circular cylinder $x^2+y^2=1$. Use Lagrange multipliers to find the highest and lowest points on the curve of intersection.

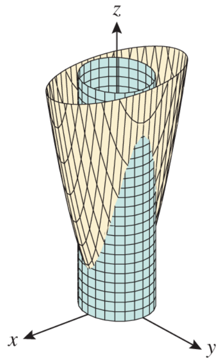


Figure Ex-34

35. Writing List a sequence of steps for solving a two-variable extremum problem with one constraint using the method of Lagrange multipliers. Interpret each step geometrically.

36. Writing Redo Example 2 using the methods of Section 3.5, and compare your solution with that of Example 2. For example, how is the perimeter constraint used in each approach?

✓ QUICK CHECK ANSWERS 13.9 **1.** (a) They are the same line. (b) They are the same plane. **2.** $\sqrt{2}$ **3.** $\sqrt{3}$ **4.** 3; 2

CHAPTER 13 REVIEW EXERCISES

Graphing Utility

1. Let $f(x, y) = e^x \ln y$. Find
(a) $f(\ln y, e^x)$ (b) $f(r + s, rs)$.

2. Sketch the domain of f using solid lines for portions of the boundary included in the domain and dashed lines for portions not included.
(a) $f(x, y) = \ln(xy - 1)$ (b) $f(x, y) = (\sin^{-1} x)/e^y$

3. Show that the level curves of the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$ are circles, and make a sketch that illustrates the difference between the contour plots of the two functions.

4. (a) In words, describe the level surfaces of the function $f(x, y, z) = a^2x^2 + a^2y^2 + z^2$, where $a > 0$.
(b) Find a function $f(x, y, z)$ whose level surfaces form a family of circular paraboloids that open in the positive z-direction.

5–6 (a) Find the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ if it exists, and (b) determine whether f is continuous at $(0, 0)$.

5. $f(x, y) = \dfrac{x^4 - x + y - x^3y}{x - y}$

6. $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

7. (a) A company manufactures two types of computer monitors: standard monitors and high resolution monitors. Suppose that $P(x, y)$ is the profit that results from producing and selling x standard monitors and y high-resolution monitors. What do the two partial derivatives $\partial P/\partial x$ and $\partial P/\partial y$ represent?
(b) Suppose that the temperature at time t at a point (x, y) on the surface of a lake is $T(x, y, t)$. What do the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$ represent?

8. Let $z = f(x, y)$.
(a) Express $\partial z/\partial x$ and $\partial z/\partial y$ as limits.
(b) In words, what do the derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ tell you about the surface $z = f(x, y)$?
(c) In words, what do the derivatives $\partial z/\partial x(x_0, y_0)$ and $\partial z/\partial y(x_0, y_0)$ tell you about the rates of change of z with respect to x and y?

9. The pressure in newtons per square meter (N/m^2) of a gas in a cylinder is given by $P = 10T/V$ with T in kelvins (K) and V in cubic meters (m^3).
(a) If T is increasing at a rate of 3 K/min with V held fixed at 2.5 m^3, find the rate at which the pressure is changing when $T = 50$ K.
(b) If T is held fixed at 50 K while V is decreasing at the rate of 3 m^3/min, find the rate at which the pressure is changing when $V = 2.5\ \text{m}^3$.

10. Find the slope of the tangent line at the point $(1, -2, -3)$ on the curve of intersection of the surface $z = 5 - 4x^2 - y^2$ with
(a) the plane $x = 1$ (b) the plane $y = -2$.

11–14 Verify the assertion.

11. If $w = \tan(x^2 + y^2) + x\sqrt{y}$, then $w_{xy} = w_{yx}$.

12. If $w = \ln(3x - 3y) + \cos(x + y)$, then $\partial^2 w/\partial x^2 = \partial^2 w/\partial y^2$.

13. If $F(x, y, z) = 2z^3 - 3(x^2 + y^2)z$, then F satisfies the equation $F_{xx} + F_{yy} + F_{zz} = 0$.

14. If $f(x, y, z) = xyz + x^2 + \ln(y/z)$, then $f_{xyzx} = f_{zxxy}$.

15. What do Δf and df represent, and how are they related?

16. If $w = x^2y - 2xy + y^2x$, find the increment Δw and the differential dw if (x, y) varies from $(1, 0)$ to $(1.1, -0.1)$.

17. Use differentials to estimate the change in the volume $V = \frac{1}{3}x^2h$ of a pyramid with a square base when its height h is increased from 2 to 2.2 m and its base dimension x is decreased from 1 to 0.9 m. Compare this to ΔV.

18. Find the local linear approximation of $f(x, y) = \sin(xy)$ at $\left(\frac{1}{3}, \pi\right)$.

19. Suppose that z is a differentiable function of x and y with

$$\frac{\partial z}{\partial x}(1, 2) = 4 \quad \text{and} \quad \frac{\partial z}{\partial y}(1, 2) = 2$$

If $x = x(t)$ and $y = y(t)$ are differentiable functions of t with $x(0) = 1$, $y(0) = 2$, $x'(0) = -\frac{1}{2}$, and (under composition) $z'(0) = 2$, find $y'(0)$.

20. In each part, use Theorem 13.5.3 to find dy/dx.
(a) $3x^2 - 5xy + \tan xy = 0$
(b) $x \ln y + \sin(x - y) = \pi$

21. Given that $f(x, y) = 0$, use Theorem 13.5.3 to express d^2y/dx^2 in terms of partial derivatives of f.

22. Let $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$.
(a) Show that

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y \partial x}\frac{dy}{dt}$$

and

$$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}$$

(b) Use the formulas in part (a) to help find a formula for d^2z/dt^2.

23. (a) How are the directional derivative and the gradient of a function related?
(b) Under what conditions is the directional derivative of a differentiable function 0?
(c) In what direction does the directional derivative of a differentiable function have its maximum value? Its minimum value?

24. In words, what does the derivative $D_{\mathbf{u}}f(x_0, y_0)$ tell you about the surface $z = f(x, y)$?

25. Find $D_{\mathbf{u}}f(-3, 5)$ for $f(x, y) = y \ln(x + y)$ if $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

26. Suppose that $\nabla f(0, 0) = 2\mathbf{i} + \frac{3}{2}\mathbf{j}$.
(a) Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(0, 0)$ is a maximum. What is this maximum value?
(b) Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(0, 0)$ is a minimum. What is this minimum value?

27. At the point $(1, 2)$, the directional derivative $D_{\mathbf{u}}f$ is $2\sqrt{2}$ toward $P_1(2, 3)$ and -3 toward $P_2(1, 0)$. Find $D_{\mathbf{u}}f(1, 2)$ toward the origin.

28. Find equations for the tangent plane and normal line to the given surface at P_0.
(a) $z = x^2e^{2y}$; $P_0(1, \ln 2, 4)$
(b) $x^2y^3z^4 + xyz = 2$; $P_0(2, 1, -1)$

29. Find all points P_0 on the surface $z = 2 - xy$ at which the normal line passes through the origin.

30. Show that for all tangent planes to the surface

$$x^{2/3} + y^{2/3} + z^{2/3} = 1$$

the sum of the squares of the x-, y-, and z-intercepts is 1.

31. Find all points on the paraboloid $z = 9x^2 + 4y^2$ at which the normal line is parallel to the line through the points $P(4, -2, 5)$ and $Q(-2, -6, 4)$.

32. Suppose the equations of motion of a particle are $x = t - 1$, $y = 4e^{-t}$, $z = 2 - \sqrt{t}$, where $t > 0$. Find, to the nearest tenth of a degree, the acute angle between the velocity vector and the normal line to the surface $(x^2/4) + y^2 + z^2 = 1$ at the points where the particle collides with the surface. Use a calculating utility with a root-finding capability where needed.

33–36 Locate all relative minima, relative maxima, and saddle points. ■

33. $f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y$

34. $f(x, y) = x^2y - 6y^2 - 3x^2$

35. $f(x, y) = x^3 - 3xy + \frac{1}{2}y^2$

36. $f(x, y) = 4x^2 - 12xy + 9y^2$

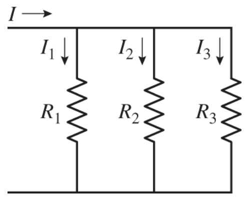
37–39 Solve these exercises two ways:
(a) Use the constraint to eliminate a variable.
(b) Use Lagrange multipliers. ■

37. Find all relative extrema of x^2y^2 subject to the constraint $4x^2 + y^2 = 8$.

38. Find the dimensions of the rectangular box of maximum volume that can be inscribed in the ellipsoid

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$$

39. As illustrated in the accompanying figure, suppose that a current I branches into currents I_1, I_2, and I_3 through resistors R_1, R_2, and R_3 in such a way that the total power dissipated in the three resistors is a minimum. Find the ratios $I_1 : I_2 : I_3$ if the power dissipated in R_i is $I_i^2R_i (i = 1, 2, 3)$ and $I_1 + I_2 + I_3 = I$.



◀ **Figure Ex-39**

40–42 In economics, a ***production model*** is a mathematical relationship between the output of a company or a country and the labor and capital equipment required to produce that output. Much of the pioneering work in the field of production models occurred in the 1920s when Paul Douglas of the University of Chicago and his collaborator Charles Cobb proposed that the output P can be expressed in terms of the labor L and the capital equipment K by an equation of the form

$$P = cL^{\alpha}K^{\beta}$$

where c is a constant of proportionality and α and β are constants such that $0 < \alpha < 1$ and $0 < \beta < 1$. This is called the ***Cobb–Douglas production model***. Typically, P, L, and K are all expressed in terms of their equivalent monetary values. These exercises explore properties of this model. ■

40. (a) Consider the Cobb–Douglas production model given by the formula $P = L^{0.75}K^{0.25}$. Sketch the level curves $P(L, K) = 1$, $P(L, K) = 2$, and $P(L, K) = 3$ in an LK-coordinate system (L horizontal and K vertical). Your sketch need not be accurate numerically, but it should show the general shape of the curves and their relative positions.
(b) Use a graphing utility to make a more extensive contour plot of the model.

41. (a) Find $\partial P/\partial L$ and $\partial P/\partial K$ for the Cobb–Douglas production model $P = cL^{\alpha}K^{\beta}$.
(b) The derivative $\partial P/\partial L$ is called the ***marginal productivity of labor***, and the derivative $\partial P/\partial K$ is called the ***marginal productivity of capital***. Explain what these quantities mean in practical terms.
(c) Show that if $\beta = 1 - \alpha$, then P satisfies the partial differential equation

$$K\frac{\partial P}{\partial K} + L\frac{\partial P}{\partial L} = P$$

42. Consider the Cobb–Douglas production model

$$P = 1000L^{0.6}K^{0.4}$$

(a) Find the maximum output value of P if labor costs \$50.00 per unit, capital costs \$100.00 per unit, and the total cost of labor and capital is set at \$200,000.
(b) How should the \$200,000 be allocated between labor and capital to achieve the maximum?

CHAPTER 13 MAKING CONNECTIONS

1. Suppose that a function $z = f(x, y)$ is expressed in polar form by making the substitutions $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$r\frac{\partial z}{\partial r} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial \theta} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

2. A function $f(x, y)$ is said to be ***homogeneous of degree n*** if $f(tx, ty) = t^n f(x, y)$ for $t > 0$. In each part, show that the function is homogeneous, and find its degree.

(a) $f(x, y) = 3x^2 + y^2$
(b) $f(x, y) = \sqrt{x^2 + y^2}$
(c) $f(x, y) = x^2y - 2y^3$
(d) $f(x, y) = \dfrac{5}{(x^2 + 2y^2)^2}$

3. Suppose that a function $f(x, y)$ is defined for all points $(x, y) \neq (0, 0)$. Prove that f is homogeneous of degree n if and only if there exists a function $g(\theta)$ that is 2π periodic such that in polar form the equation $z = f(x, y)$ becomes

$$z = r^n g(\theta)$$

for $r > 0$ and $-\infty < \theta < +\infty$.

4. (a) Use the chain rule to show that if $f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

[*Hint:* Let $u = tx$ and $v = ty$ in $f(tx, ty)$, and differentiate both sides of $f(u, v) = t^n f(x, y)$ with respect to t.]

(b) Use the results of Exercises 1 and 3 to give another derivation of the equation in part (a).

(c) Confirm that the functions in Exercise 2 satisfy the equation in part (a).

5. Suppose that a function $f(x, y)$ is defined for all points $(x, y) \neq (0, 0)$ and satisfies

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Prove that f is homogeneous of degree n. [*Hint:* Express the function $z = f(x, y)$ in polar form and use Exercise 1 to conclude that

$$r\frac{\partial z}{\partial r} - nz = 0$$

Divide both sides of this equation by r^{n+1} and interpret the left-hand side of the resulting equation as the partial derivative with respect to r of a product of two functions.]

urban_light/Depositphotos

Finding the areas of complex surfaces such as those used in the design of the Denver International Airport require integration methods studied in this chapter.

14

MULTIPLE INTEGRALS

In this chapter we will extend the concept of a definite integral to functions of two and three variables. Whereas functions of one variable are usually integrated over intervals, functions of two variables are usually integrated over regions in 2-space and functions of three variables over regions in 3-space. Calculating such integrals will require some new techniques that will be a central focus in this chapter. Once we have developed the basic methods for integrating functions of two and three variables, we will show how such integrals can be used to calculate surface areas and volumes of solids; and we will also show how they can be used to find masses and centers of gravity of flat plates and three-dimensional solids. In addition to our study of integration, we will generalize the concept of a parametric curve in 2-space to a parametric surface in 3-space. This will allow us to work with a wider variety of surfaces than previously possible and will provide a powerful tool for generating surfaces using computers and other graphing utilities.

14.1 DOUBLE INTEGRALS

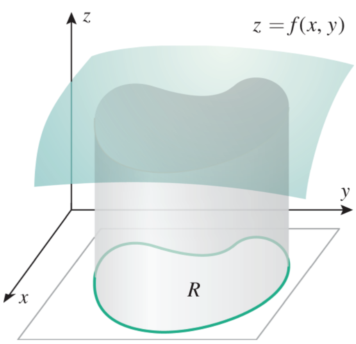
The notion of a definite integral can be extended to functions of two or more variables. In this section we will discuss the double integral, which is the extension to functions of two variables.

VOLUME

Recall that the definite integral of a function of one variable

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{1}$$

arose from the problem of finding areas under curves. [In the rightmost expression in (1), we use the "limit as $n \to +\infty$" to encapsulate the process by which we increase the number of subintervals of $[a, b]$ in such a way that the lengths of the subintervals approach zero.] Integrals of functions of two variables arise from the problem of finding volumes under surfaces.



▲ **Figure 14.1.1**

> **14.1.1 THE VOLUME PROBLEM** Given a function f of two variables that is continuous and nonnegative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R (Figure 14.1.1).

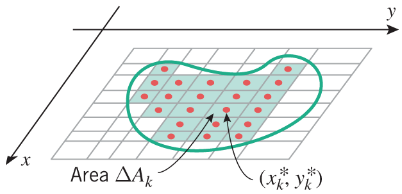
Later, we will place more restrictions on the region R, but for now we will just assume that the entire region can be enclosed within some suitably large rectangle with sides parallel to the coordinate axes. This ensures that R does not extend indefinitely in any direction.

The procedure for finding the volume V of the solid in Figure 14.1.1 will be similar to the limiting process used for finding areas, except that now the approximating elements will be rectangular parallelepipeds rather than rectangles. We proceed as follows:

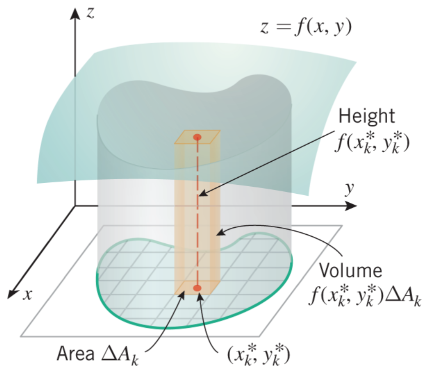
- Using lines parallel to the coordinate axes, divide the rectangle enclosing the region R into subrectangles, and exclude from consideration all those subrectangles that contain any points outside of R. This leaves only rectangles that are subsets of R

(Figure 14.1.2). Assume that there are n such rectangles, and denote the area of the kth such rectangle by ΔA_k.

- Choose any arbitrary point in each subrectangle, and denote the point in the kth subrectangle by (x_k^*, y_k^*). As shown in Figure 14.1.3, the product $f(x_k^*, y_k^*)\Delta A_k$ is the volume of a rectangular parallelepiped with base area ΔA_k and height $f(x_k^*, y_k^*)$, so the sum
$$\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$
can be viewed as an approximation to the volume V of the entire solid.
- There are two sources of error in the approximation: first, the parallelepipeds have flat tops, whereas the surface $z = f(x, y)$ may be curved; second, the rectangles that form the bases of the parallelepipeds may not completely cover the region R. However, if we repeat the above process with more and more subdivisions in such a way that both the lengths and the widths of the subrectangles approach zero, then it is plausible that the errors of both types approach zero, and the exact volume of the solid will be
$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$



▲ **Figure 14.1.2**



▲ **Figure 14.1.3**

This suggests the following definition.

Definition 14.1.2 is satisfactory for our present purposes, but some issues would have to be resolved before it could be regarded as rigorous. For example, we would have to prove that the limit actually exists and that its value does not depend on how the points $(x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_n^*, y_n^*)$ are chosen. These facts are true if the region R is not too "complicated" and if f is continuous on R. The details are beyond the scope of this text.

14.1.2 DEFINITION (***Volume Under a Surface***) If f is a function of two variables that is continuous and nonnegative on a region R in the xy-plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R is defined by
$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{2}$$
Here, $n \to +\infty$ indicates the process of increasing the number of subrectangles of the rectangle enclosing R in such a way that both the lengths and the widths of the subrectangles approach zero.

It is assumed in Definition 14.1.2 that f is nonnegative on the region R. If f is continuous on R and has both positive and negative values, then the limit
$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{3}$$
no longer represents the volume between R and the surface $z = f(x, y)$; rather, it represents a *difference* of volumes—the volume between R and the portion of the surface that is above

the xy-plane minus the volume between R and the portion of the surface below the xy-plane. We call this the ***net signed volume*** between the region R and the surface $z = f(x, y)$.

DEFINITION OF A DOUBLE INTEGRAL

As in Definition 14.1.2, the notation $n \to +\infty$ in (3) encapsulates a process in which the enclosing rectangle for R is repeatedly subdivided in such a way that both the lengths and the widths of the subrectangles approach zero. Thus, we have extended the notion conveyed by Formula (1) where the definite integral of a one-variable function is expressed as a limit of Riemann sums. By extension, the sums in (3) are also called ***Riemann sums***, and the limit of the Riemann sums is denoted by

$$\iint_R f(x, y)\,dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{4}$$

which is called the ***double integral*** of $f(x, y)$ over R.

If f is continuous and nonnegative on the region R, then the volume formula in (2) can be expressed as

$$V = \iint_R f(x, y)\,dA \tag{5}$$

If f has both positive and negative values on R, then a positive value for the double integral of f over R means that there is more volume above R than below, a negative value for the double integral means that there is more volume below R than above, and a value of zero means that the volume above R is the same as the volume below R.

EVALUATING DOUBLE INTEGRALS

Except in the simplest cases, it is impractical to obtain the value of a double integral from the limit in (4). However, we will now show how to evaluate double integrals by calculating two successive single integrals. For the rest of this section we will limit our discussion to the case where R is a rectangle; in the next section we will consider double integrals over more complicated regions.

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, ***partial integration***. The symbols

$$\int_a^b f(x, y)\,dx \quad \text{and} \quad \int_c^d f(x, y)\,dy$$

denote ***partial definite integrals***; the first integral, called the ***partial definite integral with respect to x***, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the ***partial definite integral with respect to y***, is evaluated by holding x fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function of x.

▶ Example 1

$$\int_0^1 xy^2\,dx = y^2\int_0^1 x\,dx = \frac{y^2x^2}{2}\bigg]_{x=0}^{1} = \frac{y^2}{2}$$

$$\int_0^1 xy^2\,dy = x\int_0^1 y^2\,dy = \frac{xy^3}{3}\bigg]_{y=0}^{1} = \frac{x}{3} \quad ◀$$

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y; similarly, a partial definite integral with respect to y can be integrated with respect to x. This two-stage integration process is called ***iterated*** (or ***repeated***) ***integration***. We introduce the following notation:

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy \tag{6}$$

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx \tag{7}$$

These integrals are called ***iterated integrals***.

Example 2 Evaluate

$$\text{(a)} \quad \int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx \qquad \text{(b)} \quad \int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$$

Solution (a).

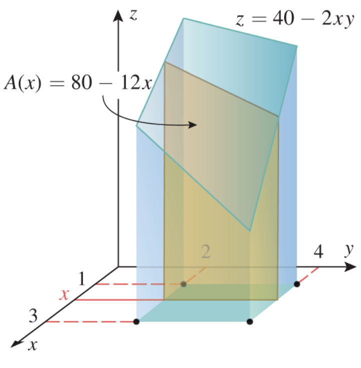
$$\begin{aligned}
\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx &= \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy\right] dx \\
&= \int_1^3 (40y - xy^2)\Big]_{y=2}^4\, dx \\
&= \int_1^3 [(160 - 16x) - (80 - 4x)]\, dx \\
&= \int_1^3 (80 - 12x)\, dx \\
&= (80x - 6x^2)\Big]_1^3 = 112
\end{aligned}$$

Solution (b).

$$\begin{aligned}
\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy &= \int_2^4 \left[\int_1^3 (40 - 2xy)\, dx\right] dy \\
&= \int_2^4 (40x - x^2y)\Big]_{x=1}^3\, dy \\
&= \int_2^4 [(120 - 9y) - (40 - y)]\, dy \\
&= \int_2^4 (80 - 8y)\, dy \\
&= (80y - 4y^2)\Big]_2^4 = 112 \blacktriangleleft
\end{aligned}$$

It is no accident that both parts of Example 2 produced the same answer. Consider the solid S bounded above by the surface $z = 40 - 2xy$ and below by the rectangle R defined by $1 \le x \le 3$ and $2 \le y \le 4$. By the method of slicing discussed in Section 5.2, the volume of S is given by

$$V = \int_1^3 A(x)\, dx$$



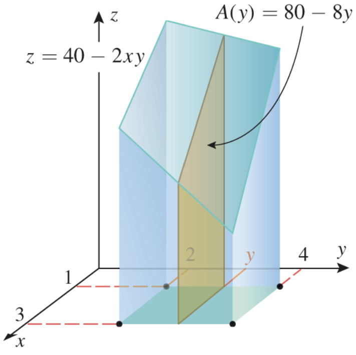
▲ **Figure 14.1.4**

where $A(x)$ is the area of a vertical cross section of S taken perpendicular to the x-axis (Figure 14.1.4). For a fixed value of x, $1 \le x \le 3$, $z = 40 - 2xy$ is a function of y, so the integral

$$A(x) = \int_2^4 (40 - 2xy)\, dy$$

represents the area under the graph of this function of y. Thus,

$$V = \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy\right] dx = \int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx$$



▲ Figure 14.1.5

We will often denote the rectangle

$$\{(x, y) : a \le x \le b, c \le y \le d\}$$

as $[a, b] \times [c, d]$ for simplicity.

is the volume of S. Similarly, by the method of slicing with cross sections of S taken perpendicular to the y-axis, the volume of S is given by

$$V = \int_2^4 A(y)\,dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\,dx\right] dy = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

(Figure 14.1.5). Thus, the iterated integrals in parts (a) and (b) of Example 2 both measure the volume of S, which by Formula (5) is the double integral of $z = 40 - 2xy$ over R. That is,

$$\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx = \iint_R (40 - 2xy)\,dA = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

The geometric argument above applies to any continuous function $f(x, y)$ that is nonnegative on a rectangle $R = [a, b] \times [c, d]$, as is the case for $f(x, y) = 40 - 2xy$ on $[1, 3] \times [2, 4]$. The conclusion that the double integral of $f(x, y)$ over R has the same value as either of the two possible iterated integrals is true even when f is negative at some points in R. We state this result in the following theorem and omit a formal proof.

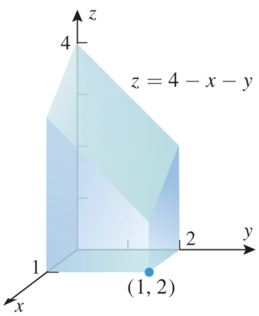
14.1.3 THEOREM (*Fubini's Theorem*) *Let R be the rectangle defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Theorem 14.1.3 allows us to evaluate a double integral over a rectangle by converting it to an iterated integral. This can be done in two ways, both of which produce the value of the double integral.



▲ Figure 14.1.6

▶ Example 3 Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 14.1.6).

Solution. The volume is the double integral of $z = 4 - x - y$ over R. Using Theorem 14.1.3, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y)\,dx\,dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\,dy\,dx \tag{8}$$

Guido Fubini (1879–1943) Italian mathematician. Fubini, the son of a mathematician, showed brilliance in mathematics as a young pupil in Venice. He entered college at the Scuola Normale Superiore di Pisa in 1896 and presented his doctoral thesis on the subject of elliptic geometry in 1900 at the young age of 20. He subsequently had teaching positions at various universities, finally settling at the University of Turin where he remained for several decades. His mathematical work was diverse, and he made major contributions to many branches of mathematics. At the outbreak of World War I he shifted his attention to the accuracy of artillery fire, and following the war he worked on other applied subjects such as electrical circuits and acoustics. In 1939, as he neared age 60 and retirement, Benito Mussolini's Fascists adopted Hitler's anti-Jewish policies, so Fubini, who was Jewish, accepted a position at Princeton University, where he stayed until his death four years later. Fubini was well liked by his colleagues at Princeton and stories about him abound. He once gave a lecture on ballistics in which he showed that if you fired a projectile of a certain shape, then under the right conditions it could double back on itself and hit your own troops. Then, tongue in cheek, he suggested that one could fool the enemy by aiming this "Fubini Gun" at one's own troops and hit the unsuspecting enemy after the projectile reversed direction.

Fubini was exceptionally short, which occasionally caused problems. The story goes that one day his worried landlady called his friends to report that he had not come home. After searching everywhere, including the area near the local lake, it was discovered that Fubini was trapped in a stalled elevator and was unable to reach any of the buttons. Fubini celebrated his rescue with a party and later left a sign in his room that said, "To my landlady: When I am not home at 6:30 at night, please check the elevator. . . ."

Using the first of these, we obtain

$$V = \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$
$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy = \int_0^2 \left(\frac{7}{2} - y\right) dy$$
$$= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5$$

You can check this result by evaluating the second integral in (8). ◀

TECHNOLOGY MASTERY

If you have a CAS with a built-in capability for computing iterated double integrals, use it to check Example 3.

Theorem 14.1.3 guarantees that the double integral in Example 4 can also be evaluated by integrating first with respect to y and then with respect to x. Verify this.

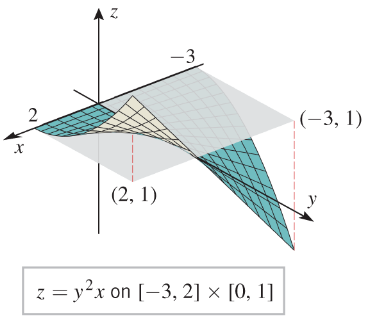
Example 4 Evaluate the double integral

$$\iint_R y^2 x\, dA$$

over the rectangle $R = \{(x, y) : -3 \le x \le 2, 0 \le y \le 1\}$.

Solution. In view of Theorem 14.1.3, the value of the double integral can be obtained by evaluating one of two possible iterated double integrals. We choose to integrate first with respect to x and then with respect to y.

$$\iint_R y^2 x\, dA = \int_0^1 \int_{-3}^2 y^2 x\, dx\, dy = \int_0^1 \left[\frac{1}{2}y^2x^2\right]_{x=-3}^{2} dy$$
$$= \int_0^1 \left(-\frac{5}{2}y^2\right) dy = -\frac{5}{6}y^3\Big]_0^1 = -\frac{5}{6} \;◀$$



▲ **Figure 14.1.7**

The integral in Example 4 can be interpreted as the net signed volume between the rectangle $[-3, 2] \times [0, 1]$ and the surface $z = y^2x$. That is, it is the volume below $z = y^2x$ and above $[0, 2] \times [0, 1]$ minus the volume above $z = y^2x$ and below $[-3, 0] \times [0, 1]$ (Figure 14.1.7).

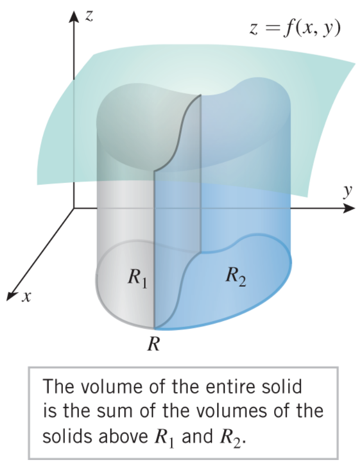
PROPERTIES OF DOUBLE INTEGRALS

To distinguish between double integrals of functions of two variables and definite integrals of functions of one variable, we will refer to the latter as ***single integrals***. Because double integrals, like single integrals, are defined as limits, they inherit many of the properties of limits. The following results, which we state without proof, are analogs of those in Theorem 4.5.4.

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad (c \text{ a constant}) \tag{9}$$

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA \tag{10}$$

$$\iint_R [f(x, y) - g(x, y)]\, dA = \iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA \tag{11}$$



▲ **Figure 14.1.8**

Figure 14.1.8 illustrates the result that if $f(x, y)$ is nonnegative on a region R, then subdividing R into two regions R_1 and R_2 has the effect of subdividing the solid between R

and $z = f(x, y)$ into two solids, the sum of whose volumes is the volume of the entire solid. This suggests the following result, which holds even if f has negative values:

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA \tag{12}$$

The proof of this result will be omitted.

✓ QUICK CHECK EXERCISES 14.1 (See page 902 for answers.)

1. The double integral is defined as a limit of Riemann sums by
$$\iint_R f(x, y)\, dA = \underline{\qquad}$$

2. The iterated integral
$$\int_1^5 \int_2^4 f(x, y)\, dx\, dy$$
integrates f over the rectangle defined by
$\underline{\qquad} \le x \le \underline{\qquad}$, $\underline{\qquad} \le y \le \underline{\qquad}$

3. Supply the missing integrand and limits of integration.
$$\int_1^5 \int_2^4 (3x^2 - 2xy + y^2)\, dx\, dy = \int_{\square}^{\square} \underline{\qquad}\, dy$$

4. The volume of the solid enclosed by the surface $z = x/y$ and the rectangle $0 \le x \le 4$, $1 \le y \le e^2$ in the xy-plane is $\underline{\qquad}$.

EXERCISE SET 14.1 C CAS

1–12 Evaluate the iterated integrals.

1. $\displaystyle\int_0^1 \int_0^2 (x + 3)\, dy\, dx$
2. $\displaystyle\int_1^3 \int_{-1}^1 (2x - 4y)\, dy\, dx$
3. $\displaystyle\int_2^4 \int_0^1 x^2 y\, dx\, dy$
4. $\displaystyle\int_{-2}^0 \int_{-1}^2 (x^2 + y^2)\, dx\, dy$
5. $\displaystyle\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\, dy\, dx$
6. $\displaystyle\int_0^2 \int_0^1 y \sin x\, dy\, dx$
7. $\displaystyle\int_{-1}^0 \int_2^5 dx\, dy$
8. $\displaystyle\int_4^6 \int_{-3}^7 dy\, dx$
9. $\displaystyle\int_0^1 \int_0^1 \frac{x}{(xy + 1)^2}\, dy\, dx$
10. $\displaystyle\int_{\pi/2}^{\pi} \int_1^2 x \cos xy\, dy\, dx$
11. $\displaystyle\int_0^{\ln 2} \int_0^1 xy e^{y^2 x}\, dy\, dx$
12. $\displaystyle\int_3^4 \int_1^2 \frac{1}{(x + y)^2}\, dy\, dx$

13–16 Evaluate the double integral over the rectangular region R.

13. $\displaystyle\iint_R 4xy^3\, dA$; $R = \{(x, y) : -1 \le x \le 1, -2 \le y \le 2\}$

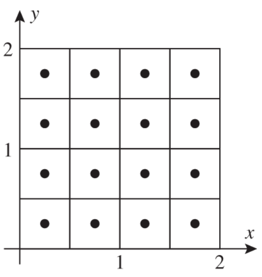
14. $\displaystyle\iint_R \frac{xy}{\sqrt{x^2 + y^2 + 1}}\, dA$;
$R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$

15. $\displaystyle\iint_R x\sqrt{1 - x^2}\, dA$; $R = \{(x, y) : 0 \le x \le 1, 2 \le y \le 3\}$

16. $\displaystyle\iint_R (x \sin y - y \sin x)\, dA$;
$R = \{(x, y) : 0 \le x \le \pi/2, 0 \le y \le \pi/3\}$

FOCUS ON CONCEPTS

17. (a) Let $f(x, y) = x^2 + y$, and as shown in the accompanying figure, let the rectangle $R = [0, 2] \times [0, 2]$ be subdivided into 16 subrectangles. Take (x_k^*, y_k^*) to be the center of the kth rectangle, and approximate the double integral of f over R by the resulting Riemann sum.
(b) Compare the result in part (a) to the exact value of the integral.



◀ Figure Ex-17

18. (a) Let $f(x, y) = x - 2y$, and as shown in Exercise 17, let the rectangle $R = [0, 2] \times [0, 2]$ be subdivided into 16 subrectangles. Take (x_k^*, y_k^*) to be the center of the kth rectangle, and approximate the double integral of f over R by the resulting Riemann sum.
(b) Compare the result in part (a) to the exact value of the integral.

19–20 Each iterated integral represents the volume of a solid. Make a sketch of the solid. Use geometry to find the volume of the solid, and then evaluate the iterated integral.

19. $\displaystyle\int_0^5 \int_1^2 4\, dx\, dy$
20. $\displaystyle\int_0^1 \int_0^1 (2 - x - y)\, dx\, dy$

21–22 Each iterated integral represents the volume of a solid. Make a sketch of the solid. (You do *not* have to find the volume.)

21. $\displaystyle\int_0^3 \int_0^4 \sqrt{25 - x^2 - y^2}\, dy\, dx$

22. $\displaystyle\int_{-2}^2 \int_{-2}^2 (x^2 + y^2)\, dx\, dy$

23–26 True–False Determine whether the statement is true or false. Explain your answer.

23. In the definition of a double integral

$$\iint_R f(x, y)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

the symbol ΔA_k represents a rectangular region within R from which the point (x_k^*, y_k^*) is taken.

24. If R is the rectangle $\{(x, y) : 1 \le x \le 4, 0 \le y \le 3\}$ and $\int_0^3 f(x, y)\, dy = 2x$, then

$$\iint_R f(x, y)\, dA = 15$$

25. If R is the rectangle $\{(x, y) : 1 \le x \le 5, 2 \le y \le 4\}$, then

$$\iint_R f(x, y)\, dA = \int_1^5 \int_2^4 f(x, y)\, dx\, dy$$

26. Suppose that for some region R in the xy-plane

$$\iint_R f(x, y)\, dA = 0$$

If R is subdivided into two regions R_1 and R_2, then

$$\iint_{R_1} f(x, y)\, dA = -\iint_{R_2} f(x, y)\, dA$$

27. In this exercise, suppose that $f(x, y) = g(x)h(y)$ and $R = \{(x, y) :\ a \le x \le b, c \le y \le d\}$. Show that

$$\iint_R f(x, y)\, dA = \left[\int_a^b g(x)\, dx\right]\left[\int_c^d h(y)\, dy\right]$$

28. Use the result in Exercise 27 to evaluate the integral

$$\int_0^{\ln 2} \int_{-1}^{1} \sqrt{e^y + 1}\, \tan x\, dx\, dy$$

by inspection. Explain your reasoning.

29–32 Use a double integral to find the volume.

29. The volume under the plane $z = 2x + y$ and over the rectangle $R = \{(x, y) : 3 \le x \le 5, 1 \le y \le 2\}$.

30. The volume under the surface $z = 3x^3 + 3x^2y$ and over the rectangle $R = \{(x, y) : 1 \le x \le 3, 0 \le y \le 2\}$.

31. The volume of the solid enclosed by the surface $z = x^2$ and the planes $x = 0$, $x = 2$, $y = 3$, $y = 0$, and $z = 0$.

32. The volume in the first octant bounded by the coordinate planes, the plane $y = 4$, and the plane $(x/3) + (z/5) = 1$.

33. Evaluate the integral by choosing a convenient order of integration:

$$\iint_R x\cos(xy)\cos^2 \pi x\, dA;\ R = \left[0, \tfrac{1}{2}\right] \times [0, \pi]$$

34. (a) Sketch the solid in the first octant that is enclosed by the planes $x = 0$, $z = 0$, $x = 5$, $z - y = 0$, and $z = -2y + 6$.
(b) Find the volume of the solid by breaking it into two parts.

35–40 The ***average value*** or ***mean value*** of a continuous function $f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ is defined as

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where $A(R) = (b - a)(d - c)$ is the area of the rectangle R (compare to Definition 4.8.1). Use this definition in these exercises.

35. Find the average value of $f(x, y) = xy^2$ over the rectangle $[0, 8] \times [0, 6]$.

36. Find the average value of $f(x, y) = x^2 + 7y$ over the rectangle $[0, 3] \times [0, 6]$.

37. Find the average value of $f(x, y) = y \sin xy$ over the rectangle $[0, 1] \times [0, \pi/2]$.

38. Find the average value of $f(x, y) = x(x^2 + y)^{1/2}$ over the rectangle $[0, 1] \times [0, 3]$.

39. Suppose that the temperature in degrees Celsius at a point (x, y) on a flat metal plate is $T(x, y) = 10 - 8x^2 - 2y^2$, where x and y are in meters. Find the average temperature of the rectangular portion of the plate for which $0 \le x \le 1$ and $0 \le y \le 2$.

40. Show that if $f(x, y)$ is constant on the rectangle $R = [a, b] \times [c, d]$, say $f(x, y) = k$, then $f_{\text{ave}} = k$ over R.

41–42 Most computer algebra systems have commands for approximating double integrals numerically. Read the relevant documentation and use a CAS to find a numerical approximation of the double integral in these exercises.

C **41.** $\displaystyle\int_0^2 \int_0^1 \sin\sqrt{x^3 + y^3}\, dx\, dy$

C **42.** $\displaystyle\int_{-1}^1 \int_{-1}^1 e^{-(x^2+y^2)}\, dx\, dy$

C **43.** Use a CAS to evaluate the iterated integrals

$$\int_0^1 \int_0^1 \frac{y - x}{(x + y)^3}\, dx\, dy \quad \text{and} \quad \int_0^1 \int_0^1 \frac{y - x}{(x + y)^3}\, dy\, dx$$

Does this contradict Theorem 14.1.3? Explain.

C **44.** Use a CAS to show that the volume V under the surface $z = xy^3 \sin xy$ over the rectangle shown in the accompanying figure is $V = 3/\pi$.

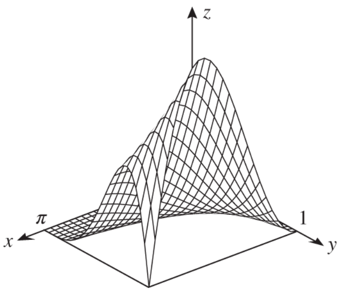


Figure Ex-44

45. Writing Discuss how computing a volume using an iterated double integral corresponds to the method of computing a volume by slicing (Section 5.2).

46. Writing Discuss how the double integral property given in Formula (12) generalizes the single integral property in Theorem 4.5.5.

✓ **QUICK CHECK ANSWERS 14.1** **1.** $\lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ **2.** $2 \le x \le 4,\ 1 \le y \le 5$ **3.** $\int_1^5 (56 - 12y + 2y^2)\,dy$ **4.** 16

14.2 DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

In this section we will show how to evaluate double integrals over regions other than rectangles.

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

Later in this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals of the following types:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx \tag{1}$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy \tag{2}$$

Note that in (1) and (2) the limits of integration in the outer integral are constants. This is consistent with the fact that the value of each iterated integral is a number that represents a net signed volume.

We begin with an example that illustrates how to evaluate such integrals.

Example 1 Evaluate

$$\text{(a)}\ \int_0^1 \int_{-x}^{x^2} y^2 x\,dy\,dx \qquad \text{(b)}\ \int_0^{\pi/3} \int_0^{\cos y} x \sin y\,dx\,dy$$

Solution (a).

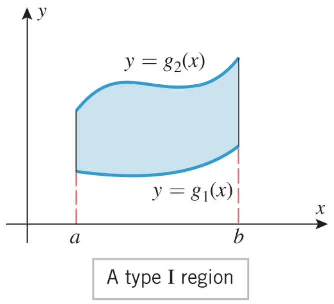
$$\int_0^1 \int_{-x}^{x^2} y^2 x\,dy\,dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x\,dy\right] dx = \int_0^1 \left.\frac{y^3 x}{3}\right]_{y=-x}^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3}\right] dx = \left(\frac{x^8}{24} + \frac{x^5}{15}\right)\Bigg]_0^1 = \frac{13}{120}$$

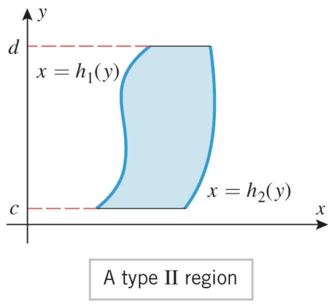
Solution (b).

$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y\,dx\,dy = \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y\,dx\right] dy = \int_0^{\pi/3} \left.\frac{x^2}{2}\sin y\right]_{x=0}^{\cos y} dy$$

$$= \int_0^{\pi/3} \left[\frac{1}{2}\cos^2 y \sin y\right] dy = -\frac{1}{6}\cos^3 y\Bigg]_0^{\pi/3} = \frac{7}{48}$$ ◀



(a)



(b)

▲ **Figure 14.2.1**

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Plane regions can be extremely complex, and the theory of double integrals over very general regions is a topic for advanced courses in mathematics. We will limit our study of double integrals to two basic types of regions, which we will call *type I* and *type II*; they are defined as follows.

14.2.1 DEFINITION

(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$ (Figure 14.2.1*a*).

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$ (Figure 14.2.1*b*).

The following theorem will enable us to evaluate double integrals over type I and type II regions using iterated integrals.

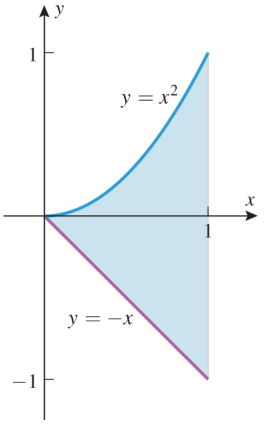
14.2.2 THEOREM

(*a*) *If R is a type I region on which $f(x, y)$ is continuous, then*

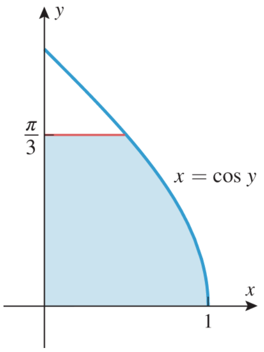
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{3}$$

(*b*) *If R is a type II region on which $f(x, y)$ is continuous, then*

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{4}$$



▲ **Figure 14.2.2**



▲ **Figure 14.2.3**

► **Example 2** Each of the iterated integrals in Example 1 is equal to a double integral over a region R. Identify the region R in each case.

Solution. Using Theorem 14.2.2, the integral in Example 1(a) is the double integral of the function $f(x, y) = y^2x$ over the type I region R bounded on the left and right by the vertical lines $x = 0$ and $x = 1$ and bounded below and above by the curves $y = -x$ and $y = x^2$ (Figure 14.2.2). The integral in Example 1(b) is the double integral of the function $f(x, y) = x \sin y$ over the type II region R bounded below and above by the horizontal lines $y = 0$ and $y = \pi/3$ and bounded on the left and right by the curves $x = 0$ and $x = \cos y$ (Figure 14.2.3). ◄

We will not prove Theorem 14.2.2, but for the case where $f(x, y)$ is nonnegative on the region R, it can be made plausible by a geometric argument that is similar to that given for Theorem 14.1.3. Since $f(x, y)$ is nonnegative, the double integral can be interpreted as the volume of the solid S that is bounded above by the surface $z = f(x, y)$ and below by the region R, so it suffices to show that the iterated integrals also represent this volume. Consider the iterated integral in (3), for example. For a fixed value of x, the function $f(x, y)$ is a function of y, and hence the integral

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

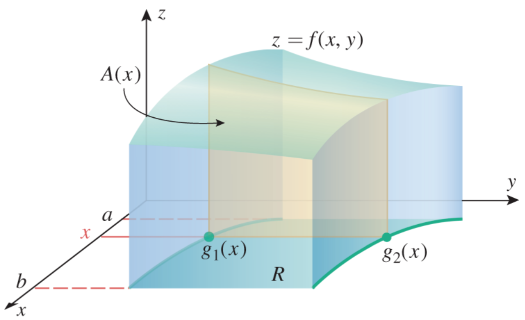
represents the area under the graph of this function of y between $y = g_1(x)$ and $y = g_2(x)$. This area, shown in yellow in Figure 14.2.4, is the cross-sectional area at x of the solid S, and hence by the method of slicing, the volume V of the solid S is

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

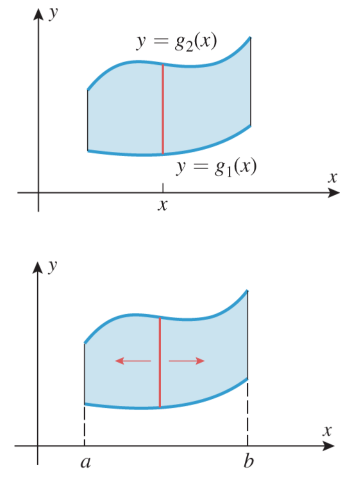
which shows that in (3) the iterated integral is equal to the double integral. Similarly, the iterated integral in (4) is equal to the corresponding double integral.

SETTING UP LIMITS OF INTEGRATION FOR EVALUATING DOUBLE INTEGRALS

To apply Theorem 14.2.2, it is helpful to start with a two-dimensional sketch of the region R. [It is not necessary to graph $f(x, y)$.] For a type I region, the limits of integration in Formula (3) can be obtained as follows:



▶ **Figure 14.2.4**



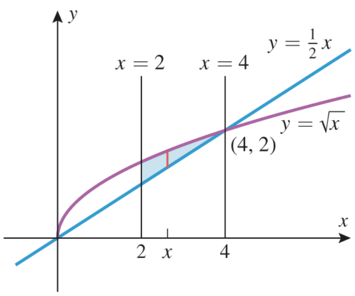
▲ **Figure 14.2.5**

> ***Determining Limits of Integration: Type I Region***
>
> **Step 1.** Since x is held fixed for the first integration, we draw a vertical line through the region R at an arbitrary fixed value x (Figure 14.2.5). This line crosses the boundary of R twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration in Formula (3).
>
> **Step 2.** Imagine moving the line drawn in Step 1 first to the left and then to the right (Figure 14.2.5). The leftmost position where the line intersects the region R is $x = a$, and the rightmost position where the line intersects the region R is $x = b$. This yields the limits for the x-integration in Formula (3).

▶ **Example 3** Evaluate

$$\iint_R xy\, dA$$

over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.



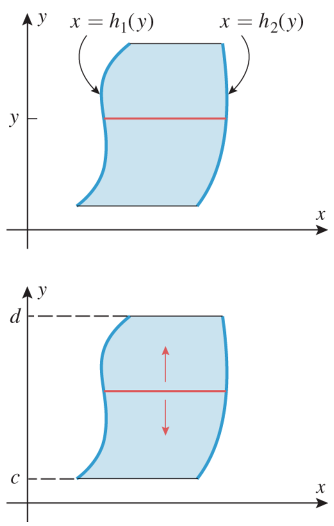
▲ **Figure 14.2.6**

Solution. We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure 14.2.6. This line meets the region R at the lower boundary $y = \frac{1}{2}x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x = 2$ and $x = 4$. Thus,

$$\iint_R xy\, dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

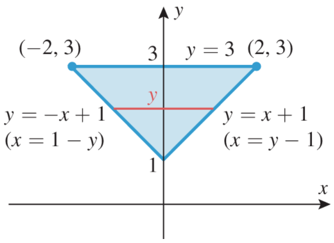
$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6} \blacktriangleleft$$

If R is a type II region, then the limits of integration in Formula (4) can be obtained as follows:



▲ **Figure 14.2.7**

> ***Determining Limits of Integration: Type II Region***
>
> **Step 1.** Since y is held fixed for the first integration, we draw a horizontal line through the region R at a fixed value y (Figure 14.2.7). This line crosses the boundary of R twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration in (4).
>
> **Step 2.** Imagine moving the line drawn in Step 1 first down and then up (Figure 14.2.7). The lowest position where the line intersects the region R is $y = c$, and the highest position where the line intersects the region R is $y = d$. This yields the y-limits of integration in (4).



▲ **Figure 14.2.8**

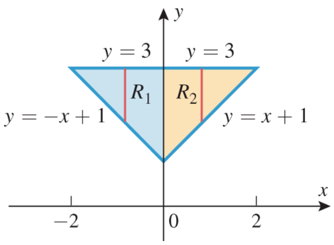
To integrate over a type II region, the left- and right-hand boundaries must be expressed in the form $x = h_1(y)$ and $x = h_2(y)$. This is why we rewrote the boundary equations

$$y = -x + 1 \quad \text{and} \quad y = x + 1$$

as

$$x = 1 - y \quad \text{and} \quad x = y - 1$$

in Example 4.



▲ **Figure 14.2.9**

▶ **Example 4** Evaluate

$$\iint_R (2x - y^2)\, dA$$

over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

Solution. We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in Figure 14.2.8. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration. Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\iint_R (2x - y^2)\, dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy$$

$$= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3} \quad ◀$$

In Example 4 we could have treated R as a type I region, but with an added complication. Viewed as a type I region, the upper boundary of R is the line $y = 3$ (Figure 14.2.9) and the lower boundary consists of two parts, the line $y = -x + 1$ to the left of the y-axis and the line $y = x + 1$ to the right of the y-axis. To carry out the integration it is necessary to decompose the region R into two parts, R_1 and R_2, as shown in Figure 14.2.9, and write

$$\iint_R (2x - y^2)\, dA = \iint_{R_1} (2x - y^2)\, dA + \iint_{R_2} (2x - y^2)\, dA$$

$$= \int_{-2}^{0} \int_{-x+1}^{3} (2x - y^2)\, dy\, dx + \int_0^2 \int_{x+1}^{3} (2x - y^2)\, dy\, dx$$

This will yield the same result that was obtained in Example 4. (Verify.)

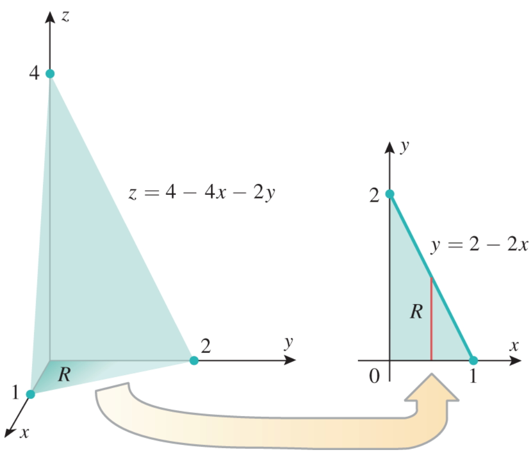
▶ **Example 5** Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution. The tetrahedron in question is bounded above by the plane

$$z = 4 - 4x - 2y \tag{5}$$

and below by the triangular region R shown in Figure 14.2.10. Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y)\, dA$$

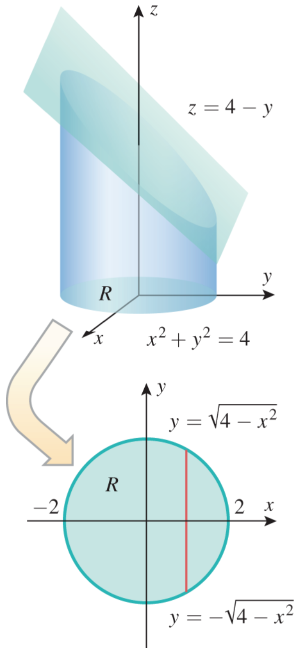


▶ **Figure 14.2.10**

The region R is bounded by the x-axis, the y-axis, and the line $y = 2 - 2x$ [set $z = 0$ in (5)], so that treating R as a type I region yields

$$V = \iint_R (4 - 4x - 2y)\, dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[4y - 4xy - y^2\right]_{y=0}^{2-2x} dx = \int_0^1 (4 - 8x + 4x^2)\, dx = \frac{4}{3} \blacktriangleleft$$



▲ Figure 14.2.11

▶ Example 6 Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution. The solid shown in Figure 14.2.11 is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi \qquad \boxed{\text{See Formula (3) of Section 7.4.}} \blacktriangleleft$$

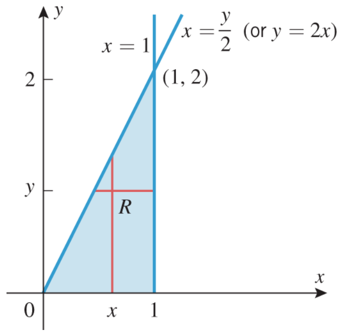
REVERSING THE ORDER OF INTEGRATION

Sometimes the evaluation of an iterated integral can be simplified by reversing the order of integration. The next example illustrates how this is done.

▶ Example 7 Since there is no elementary antiderivative of e^{x^2}, the integral

$$\int_0^2 \int_{y/2}^{1} e^{x^2}\, dx\, dy$$

cannot be evaluated by performing the x-integration first. Evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.



▲ Figure 14.2.12

Solution. For the inside integration, y is fixed and x varies from the line $x = y/2$ to the line $x = 1$ (Figure 14.2.12). For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R in Figure 14.2.12.

To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

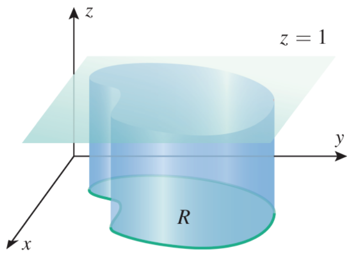
$$\int_0^2 \int_{y/2}^{1} e^{x^2}\, dx\, dy = \iint_R e^{x^2}\, dA = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 \left[e^{x^2} y\right]_{y=0}^{2x} dx$$

$$= \int_0^1 2xe^{x^2}\, dx = e^{x^2}\Big]_0^1 = e - 1 \blacktriangleleft$$

AREA CALCULATED AS A DOUBLE INTEGRAL

Although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. To see why this is so, recall that a *right cylinder* is a solid that is generated when a plane region is translated along a line that is perpendicular to the region. In Formula (2) of Section 5.2 we stated that the volume V of a right cylinder with cross-sectional area A and height h is

$$V = A \cdot h \tag{6}$$



Cylinder with base R and height 1

Figure 14.2.13

Formula (7) can be confusing because it equates an area and a volume; the formula is intended to equate only the *numerical values* of the area and volume and not the units, which must, of course, be different.

Now suppose that we are interested in finding the area A of a region R in the xy-plane. If we translate the region R upward 1 unit, then the resulting solid will be a right cylinder that has cross-sectional area A, base R, and the plane $z = 1$ as its top (Figure 14.2.13). Thus, it follows from (6) that

$$\iint_R 1\,dA = (\text{area of } R) \cdot 1$$

which we can rewrite as

$$\text{area of } R = \iint_R 1\,dA = \iint_R dA \tag{7}$$

Example 8 Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution. The region R may be treated equally well as type I (Figure 14.2.14*a*) or type II (Figure 14.2.14*b*). Treating R as type I yields

$$\begin{aligned}\text{area of } R &= \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 [y]_{y=x^2/2}^{2x}\,dx \\ &= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}\end{aligned}$$

Treating R as type II yields

$$\begin{aligned}\text{area of } R &= \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\,dy \\ &= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}\end{aligned}$$ ◀

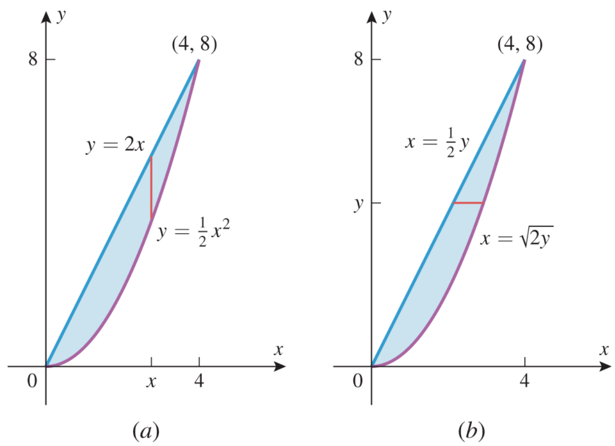


Figure 14.2.14

✓ QUICK CHECK EXERCISES 14.2

(See page 910 for answers.)

1. Supply the missing integrand and limits of integration.

(a) $\displaystyle\int_1^5 \int_2^{y/2} 6x^2y\,dx\,dy = \int_{\square}^{\square}$ ________ dy

(b) $\displaystyle\int_1^5 \int_2^{x/2} 6x^2y\,dy\,dx = \int_{\square}^{\square}$ ________ dx

2. Let R be the triangular region in the xy-plane with vertices $(0, 0)$, $(3, 0)$, and $(0, 4)$. Supply the missing portions of the integrals.

(a) Treating R as a type I region,

$$\iint_R f(x,y)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(x,y)\,\_\_\_\_\_\_$$

(b) Treating R as a type II region,

$$\iint_R f(x,y)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(x,y)\,\_\_\_\_\_\_$$

3. Let R be the triangular region in the xy-plane with vertices $(0,0)$, $(3,3)$, and $(0,4)$. Expressed as an iterated double integral, the area of R is $A(R) =$ ______.

4. The line $y = 2 - x$ and the parabola $y = x^2$ intersect at the points $(-2,4)$ and $(1,1)$. If R is the region enclosed by $y = 2 - x$ and $y = x^2$, then

$$\iint_R (1+2y)\,dA = \_\_\_\_\_\_$$

EXERCISE SET 14.2 Graphing Utility C CAS

1–8 Evaluate the iterated integral.

1-12,15-25,47-56

1. $\displaystyle\int_0^1 \int_{x^2}^{x} xy^2\,dy\,dx$ **2.** $\displaystyle\int_1^{3/2} \int_y^{3-y} y\,dx\,dy$

3. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} y\,dx\,dy$ **4.** $\displaystyle\int_{1/4}^1 \int_{x^2}^{x} \sqrt{\frac{x}{y}}\,dy\,dx$

5. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^3} \sin\frac{y}{x}\,dy\,dx$ **6.** $\displaystyle\int_{-1}^1 \int_{-x^2}^{x^2} (x^2 - y)\,dy\,dx$

7. $\displaystyle\int_0^1 \int_0^x y\sqrt{x^2 - y^2}\,dy\,dx$ **8.** $\displaystyle\int_1^2 \int_0^{y^2} e^{x/y^2}\,dx\,dy$

FOCUS ON CONCEPTS

9. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

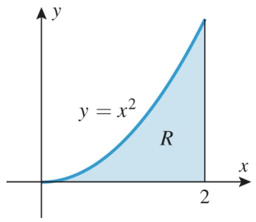
(a) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x,y)\,dy\,dx$

(b) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x,y)\,dx\,dy$

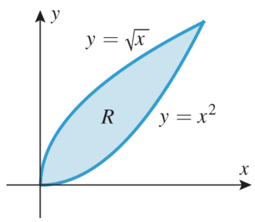
10. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x,y)\,dy\,dx$

(b) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x,y)\,dx\,dy$



▲ **Figure Ex-9**



▲ **Figure Ex-10**

11. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

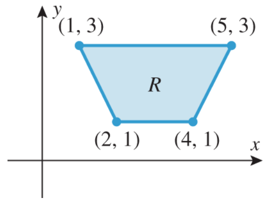
(a) $\displaystyle\iint_R f(x,y)\,dA = \int_1^2 \int_{\square}^{\square} f(x,y)\,dy\,dx + \int_2^4 \int_{\square}^{\square} f(x,y)\,dy\,dx + \int_4^5 \int_{\square}^{\square} f(x,y)\,dy\,dx$

(b) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x,y)\,dx\,dy$

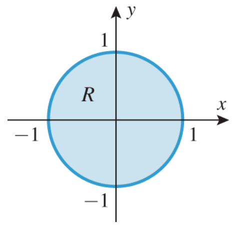
12. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x,y)\,dy\,dx$

(b) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x,y)\,dx\,dy$



▲ **Figure Ex-11**



▲ **Figure Ex-12**

13. Evaluate $\displaystyle\iint_R xy\,dA$, where R is the region in

(a) Exercise 9 (b) Exercise 11.

14. Evaluate $\displaystyle\iint_R (x+y)\,dA$, where R is the region in

(a) Exercise 10 (b) Exercise 12.

15–18 Evaluate the double integral in two ways using iterated integrals: (a) viewing R as a type I region, and (b) viewing R as a type II region.

15. $\displaystyle\iint_R x^2\,dA$; R is the region bounded by $y = 16/x$, $y = x$, and $x = 8$.

1-12,15-25

16. $\displaystyle\iint_R xy^2\,dA$; R is the region enclosed by $y = 1$, $y = 2$, $x = 0$, and $y = x$.

17. $\displaystyle\iint_R (3x - 2y)\,dA$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

18. $\displaystyle\iint_R y\,dA$; R is the region in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

19–24 Evaluate the double integral.

19. $\displaystyle\iint_R x(1+y^2)^{-1/2}\,dA$; R is the region in the first quadrant enclosed by $y = x^2$, $y = 4$, and $x = 0$.

1-12,15-25

20. $\displaystyle\iint_R x\cos y\, dA$; R is the triangular region bounded by the lines $y = x$, $y = 0$, and $x = \pi$.

21. $\displaystyle\iint_R xy\, dA$; R is the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$.

22. $\displaystyle\iint_R x\, dA$; R is the region enclosed by $y = \sin^{-1} x$, $x = 1/\sqrt{2}$, and $y = 0$.

23. $\displaystyle\iint_R (x-1)\, dA$; R is the region in the first quadrant enclosed between $y = x$ and $y = x^3$.

24. $\displaystyle\iint_R x^2\, dA$; R is the region in the first quadrant enclosed by $xy = 1$, $y = x$, and $y = 2x$.

25. Evaluate $\displaystyle\iint_R \sin(y^3)\, dA$, where R is the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$. [*Hint:* Choose the order of integration carefully.]

26. Evaluate $\displaystyle\iint_R x\, dA$, where R is the region bounded by $x = \ln y$, $x = 0$, and $y = e$.

27. (a) By hand or with the help of a graphing utility, make a sketch of the region R enclosed between the curves $y = x + 2$ and $y = e^x$.
(b) Estimate the intersections of the curves in part (a).
(c) Viewing R as a type I region, estimate $\displaystyle\iint_R x\, dA$.
(d) Viewing R as a type II region, estimate $\displaystyle\iint_R x\, dA$.

28. (a) By hand or with the help of a graphing utility, make a sketch of the region R enclosed between the curves $y = 4x^3 - x^4$ and $y = 3 - 4x + 4x^2$.
(b) Find the intersections of the curves in part (a).
(c) Find $\displaystyle\iint_R x\, dA$.

29–32 Use double integration to find the area of the plane region enclosed by the given curves.

29. $y = \sin x$ and $y = \cos x$, for $0 \le x \le \pi/4$.

30. $y^2 = -x$ and $3y - x = 4$.

1-12,15-25

31. $y^2 = 9 - x$ and $y^2 = 9 - 9x$.

32. $y = \cosh x$, $y = \sinh x$, $x = 0$, and $x = 1$.

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. $\displaystyle\int_0^1 \int_{x^2}^{2x} f(x,y)\, dy\, dx = \int_{x^2}^{2x} \int_0^1 f(x,y)\, dx\, dy$

34. If a region R is bounded below by $y = g_1(x)$ and above by $y = g_2(x)$ for $a \le x \le b$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

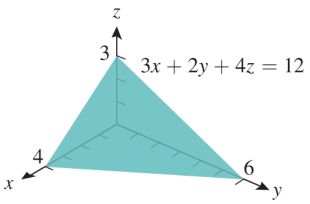
35. If R is the region in the xy-plane enclosed by $y = x^2$ and $y = 1$, then

$$\iint_R f(x,y)\, dA = 2\int_0^1 \int_{x^2}^1 f(x,y)\, dy\, dx$$

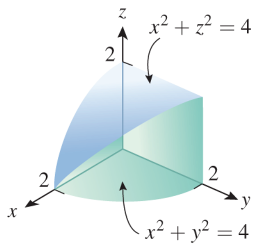
36. The area of a region R in the xy-plane is given by $\displaystyle\iint_R xy\, dA$.

37–38 Use double integration to find the volume of the solid.

37.



38.



39–44 Use double integration to find the volume of each solid.

39. The solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3 - x$.

40. The solid in the first octant bounded above by the paraboloid $z = x^2 + 3y^2$, below by the plane $z = 0$, and laterally by $y = x^2$ and $y = x$.

41. The solid bounded above by the paraboloid $z = 9x^2 + y^2$, below by the plane $z = 0$, and laterally by the planes $x = 0$, $y = 0$, $x = 3$, and $y = 2$.

42. The solid enclosed by $y^2 = x$, $z = 0$, and $x + z = 1$.

43. The wedge cut from the cylinder $4x^2 + y^2 = 9$ by the planes $z = 0$ and $z = y + 3$.

44. The solid in the first octant bounded above by $z = 9 - x^2$, below by $z = 0$, and laterally by $y^2 = 3x$.

C **45–46** Use a double integral and a CAS to find the volume of the solid.

45. The solid bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy-plane.

46. The solid in the first octant that is bounded by the paraboloid $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 4$, and the coordinate planes.

47–52 Express the integral as an equivalent integral with the order of integration reversed.

47. $\displaystyle\int_0^2 \int_0^{\sqrt{x}} f(x,y)\, dy\, dx$

48. $\displaystyle\int_0^4 \int_{2y}^8 f(x,y)\, dx\, dy$

49. $\displaystyle\int_0^2 \int_1^{e^y} f(x,y)\, dx\, dy$

50. $\displaystyle\int_1^e \int_0^{\ln x} f(x,y)\, dy\, dx$

51. $\displaystyle\int_0^1 \int_{\sin^{-1} y}^{\pi/2} f(x,y)\, dx\, dy$

52. $\displaystyle\int_0^1 \int_{y^2}^{\sqrt{y}} f(x,y)\, dx\, dy$

53–56 Evaluate the integral by first reversing the order of integration.

53. $\displaystyle\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx$

54. $\displaystyle\int_0^2 \int_{y/2}^1 \cos(x^2)\, dx\, dy$

55. $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$ **56.** $\displaystyle\int_1^3 \int_0^{\ln x} x\,dy\,dx$

C **57.** Try to evaluate the integral with a CAS using the stated order of integration, and then by reversing the order of integration.

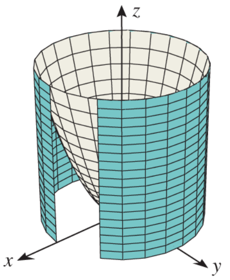
1-12,15-25

(a) $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \sin \pi y^3\,dy\,dx$

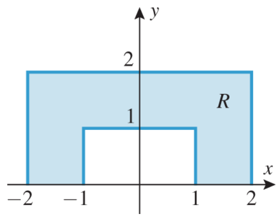
(b) $\displaystyle\int_0^1 \int_{\sin^{-1} y}^{\pi/2} \sec^2(\cos x)\,dx\,dy$

58. Use the appropriate Wallis formula (see Exercise Set 7.3) to find the volume of the solid enclosed between the circular paraboloid $z = x^2 + y^2$, the right circular cylinder $x^2 + y^2 = 4$, and the xy-plane (see the accompanying figure for cut view).

59. Evaluate $\displaystyle\iint_R xy^2\,dA$ over the region R shown in the accompanying figure.



▲ **Figure Ex-58**



▲ **Figure Ex-59**

60. Give a geometric argument to show that

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\,dx\,dy = \frac{\pi}{6}$$

61–62 The ***average value*** or ***mean value*** of a continuous function $f(x, y)$ over a region R in the xy-plane is defined as

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\,dA$$

where $A(R)$ is the area of the region R (compare to the definition preceding Exercise 35 in Section 14.1). Use this definition in these exercises. ■

61. Find the average value of $1/(1 + x^2)$ over the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$.

62. Find the average value of $f(x, y) = x^2 - xy$ over the region enclosed by $y = x$ and $y = 3x - x^2$.

63. Suppose that the temperature in degrees Celsius at a point (x, y) on a flat metal plate is $T(x, y) = 5xy + x^2$, where x and y are in meters. Find the average temperature of the diamond-shaped portion of the plate described by the inequalities $|2x + y| \le 4$ and $|2x - y| \le 4$.

64. A circular lens of radius 2 inches has thickness $1 - (r^2/4)$ inches at all points r inches from the center of the lens. Find the average thickness of the lens.

C **65.** Use a CAS to approximate the intersections of the curves $y = \sin x$ and $y = x/2$, and then approximate the volume of the solid in the first octant that is below the surface $z = \sqrt{1 + x + y}$ and above the region in the xy-plane that is enclosed by the curves.

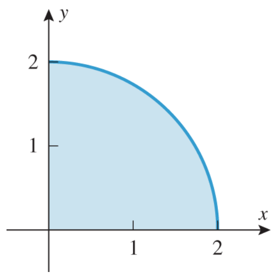
66. **Writing** Describe the steps you would follow to find the limits of integration that express a double integral over a nonrectangular region as an iterated double integral. Illustrate your discussion with an example.

67. **Writing** Describe the steps you would follow to reverse the order of integration in an iterated double integral. Illustrate your discussion with an example.

✓ **QUICK CHECK ANSWERS 14.2** **1.** (a) $\displaystyle\int_1^5 \left(\frac{1}{4}y^4 - 16y\right) dy$ (b) $\displaystyle\int_1^5 \left(\frac{3}{4}x^4 - 12x^2\right) dx$ **2.** (a) $\displaystyle\int_0^3 \int_0^{-\frac{4}{3}x+4} f(x, y)\,dy\,dx$ (b) $\displaystyle\int_0^4 \int_0^{-\frac{3}{4}y+3} f(x, y)\,dx\,dy$ **3.** $\displaystyle\int_0^3 \int_x^{-\frac{1}{3}x+4} dy\,dx$ **4.** $\displaystyle\int_{-2}^1 \int_{x^2}^{2-x} (1 + 2y)\,dy\,dx = 18.9$

14.3 DOUBLE INTEGRALS IN POLAR COORDINATES

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates. Such integrals are important for two reasons: first, they arise naturally in many applications, and second, many double integrals in rectangular coordinates can be evaluated more easily if they are converted to polar coordinates.



▲ **Figure 14.3.1**

SIMPLE POLAR REGIONS

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates. For example, the quarter-disk in Figure 14.3.1 is described in rectangular coordinates by

$$0 \le y \le \sqrt{4 - x^2}, \quad 0 \le x \le 2$$

However, in polar coordinates the region is described more simply by

$$0 \le r \le 2, \quad 0 \le \theta \le \pi/2$$

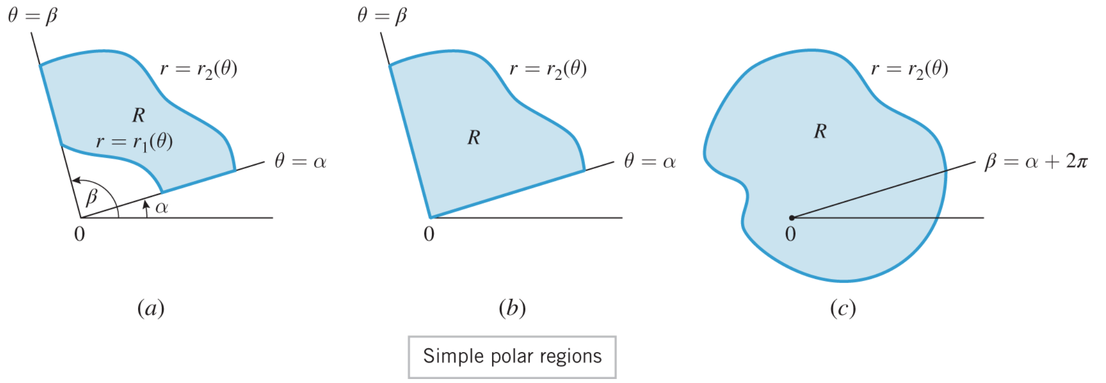
Moreover, double integrals whose integrands involve $x^2 + y^2$ also tend to be easier to evaluate in polar coordinates because this sum simplifies to r^2 when the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$ are applied.

An overview of polar coordinates can be found in Section 10.2.

Figure 14.3.2*a* shows a region R in a polar coordinate system that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$. If, as shown in the figure, the functions $r_1(\theta)$ and $r_2(\theta)$ are continuous and their graphs do not cross, then the region R is called a *simple polar region*. If $r_1(\theta)$ is identically zero, then the boundary $r = r_1(\theta)$ reduces to a point (the origin), and the region has the general shape shown in Figure 14.3.2*b*. If, in addition, $\beta = \alpha + 2\pi$, then the rays coincide, and the region has the general shape shown in Figure 14.3.2*c*. The following definition expresses these geometric ideas algebraically.

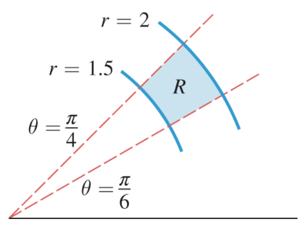
14.3.1 DEFINITION A ***simple polar region*** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \le \beta$ (ii) $\beta - \alpha \le 2\pi$ (iii) $0 \le r_1(\theta) \le r_2(\theta)$



▲ **Figure 14.3.2**

REMARK Conditions (i) and (ii) together imply that the ray $\theta = \beta$ can be obtained by rotating the ray $\theta = \alpha$ counterclockwise through an angle that is at most 2π radians. This is consistent with Figure 14.3.2. Condition (iii) implies that the boundary curves $r = r_1(\theta)$ and $r = r_2(\theta)$ can touch but cannot actually cross over one another (why?). Thus, in keeping with Figure 14.3.2, it is appropriate to describe $r = r_1(\theta)$ as the ***inner boundary*** of the region and $r = r_2(\theta)$ as the ***outer boundary***.



▲ **Figure 14.3.3**

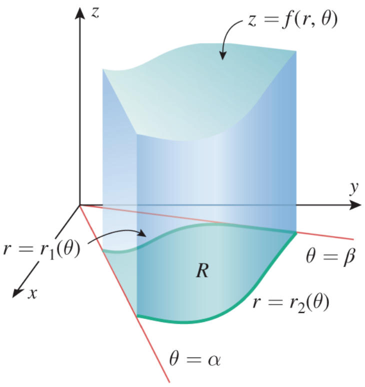
A ***polar rectangle*** is a simple polar region for which the bounding polar curves are circular arcs. For example, Figure 14.3.3 shows the polar rectangle R given by

$$1.5 \le r \le 2, \quad \frac{\pi}{6} \le \theta \le \frac{\pi}{4}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Next we will consider the polar version of Problem 14.1.1.

14.3.2 THE VOLUME PROBLEM IN POLAR COORDINATES Given a function $f(r, \theta)$ that is continuous and nonnegative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$ (Figure 14.3.4).

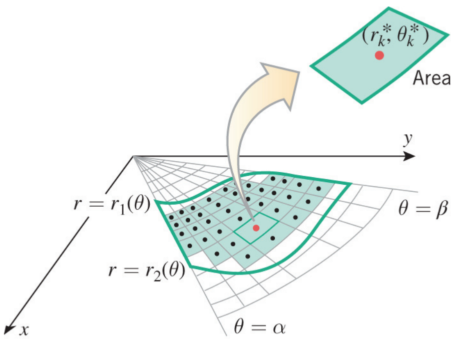


▲ **Figure 14.3.4**

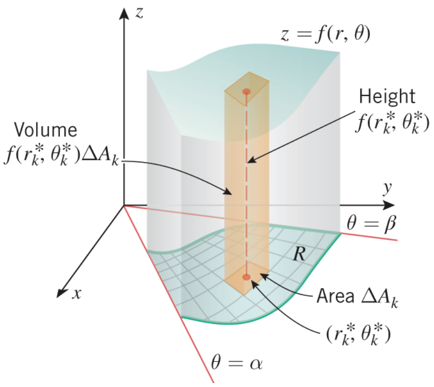
To motivate a formula for the volume V of the solid in Figure 14.3.4, we will use a limit process similar to that used to obtain Formula (2) of Section 14.1, except that here we will use circular arcs and rays to subdivide the region R into polar rectangles. As shown in Figure 14.3.5, we will exclude from consideration all polar rectangles that contain any points outside of R, leaving only polar rectangles that are subsets of R. Assume that there are n such polar rectangles, and denote the area of the kth polar rectangle by ΔA_k. Let (r_k^*, θ_k^*) be any point in this polar rectangle. As shown in Figure 14.3.6, the product $f(r_k^*, \theta_k^*)\Delta A_k$ is the volume of a solid with base area ΔA_k and height $f(r_k^*, \theta_k^*)$, so the sum

$$\sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

can be viewed as an approximation to the volume V of the entire solid.



▲ **Figure 14.3.5**



▲ **Figure 14.3.6**

If we now increase the number of subdivisions in such a way that the dimensions of the polar rectangles approach zero, then it seems plausible that the errors in the approximations approach zero, and the exact volume of the solid is

$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k \tag{1}$$

If $f(r, \theta)$ is continuous on R and has both positive and negative values, then the limit

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k \tag{2}$$

represents the net signed volume between the region R and the surface $z = f(r, \theta)$ (as with double integrals in rectangular coordinates). The sums in (2) are called ***polar Riemann sums***, and the limit of the polar Riemann sums is denoted by

$$\iint_R f(r, \theta)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k \tag{3}$$

which is called the ***polar double integral*** of $f(r, \theta)$ over R. If $f(r, \theta)$ is continuous and nonnegative on R, then the volume formula (1) can be expressed as

$$V = \iint_R f(r, \theta)\, dA \tag{4}$$

Polar double integrals are also called ***double integrals in polar coordinates*** to distinguish them from double integrals over regions in the xy-plane; the latter are called ***double integrals in rectangular coordinates***. Double integrals in polar coordinates have the usual integral properties, such as those stated in Formulas (9), (10), and (11) of Section 14.1.

EVALUATING POLAR DOUBLE INTEGRALS

In Sections 14.1 and 14.2 we evaluated double integrals in rectangular coordinates by expressing them as iterated integrals. Polar double integrals are evaluated the same way. To motivate the formula that expresses a double polar integral as an iterated integral, we will assume that $f(r, \theta)$ is nonnegative so that we can interpret (3) as a volume. To begin, let us choose the arbitrary point (r_k^*, θ_k^*) in (3) to be at the "center" of the kth polar rectangle as shown in Figure 14.3.7. Suppose also that this polar rectangle has a central angle $\Delta\theta_k$ and a "radial thickness" Δr_k. Thus, the inner radius of this polar rectangle is $r_k^* - \frac{1}{2}\Delta r_k$ and the outer radius is $r_k^* + \frac{1}{2}\Delta r_k$. Treating the area ΔA_k of this polar rectangle as the difference in area of two sectors, we obtain

$$\Delta A_k = \tfrac{1}{2}\left(r_k^* + \tfrac{1}{2}\Delta r_k\right)^2 \Delta\theta_k - \tfrac{1}{2}\left(r_k^* - \tfrac{1}{2}\Delta r_k\right)^2 \Delta\theta_k$$

which simplifies to

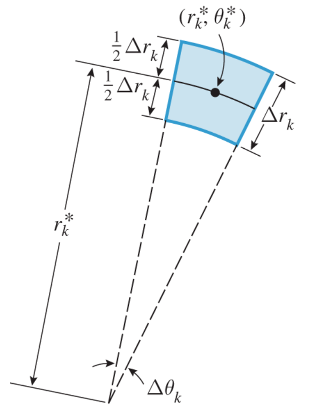
$$\Delta A_k = r_k^* \Delta r_k \Delta\theta_k \tag{5}$$

Thus, from (3) and (4)

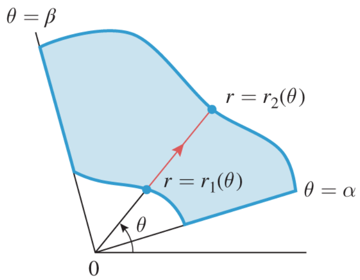
$$V = \iint_R f(r, \theta)\, dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k$$

which suggests that the volume V can be expressed as the iterated integral

$$V = \iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r\, dr\, d\theta \tag{6}$$



▲ **Figure 14.3.7**



▲ **Figure 14.3.8**

in which the limits of integration are chosen to cover the region R; that is, with θ fixed between α and β, the value of r varies from $r_1(\theta)$ to $r_2(\theta)$ (Figure 14.3.8).

Although we assumed $f(r, \theta)$ to be nonnegative in deriving Formula (6), it can be proved that the relationship between the polar double integral and the iterated integral in this formula also holds if f has negative values. Accepting this to be so, we obtain the following theorem, which we state without formal proof.

Note the extra factor of r that appears in the integrand when expressing a polar double integral as an iterated integral in polar coordinates.

14.3.3 THEOREM *If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in Figure 14.3.8, and if $f(r, \theta)$ is continuous on R, then*

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r\, dr\, d\theta \tag{7}$$

To apply this theorem you will need to be able to find the rays and the curves that form the boundary of the region R, since these determine the limits of integration in the iterated integral. This can be done as follows:

Determining Limits of Integration for a Polar Double Integral: Simple Polar Region

Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ (Figure 14.3.9*a*). This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration in (7).

Step 2. Imagine rotating the radial line from Step 1 about the origin, thus sweeping out the region R. The least angle at which the radial line intersects the region R is $\theta = \alpha$ and the greatest angle is $\theta = \beta$ (Figure 14.3.9*b*). This determines the θ-limits of integration.

▶ **Example 1** Evaluate

$$\iint_R \sin\theta\, dA$$

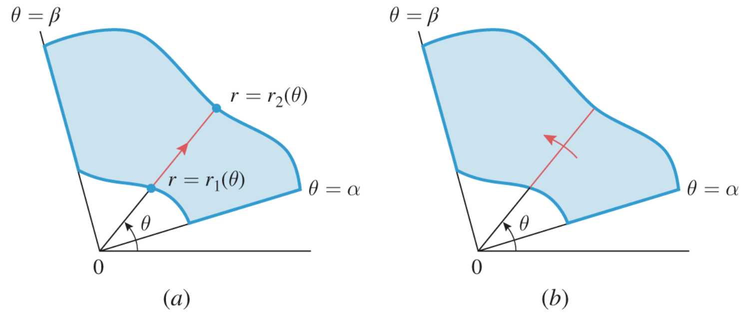


Figure 14.3.9

where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.

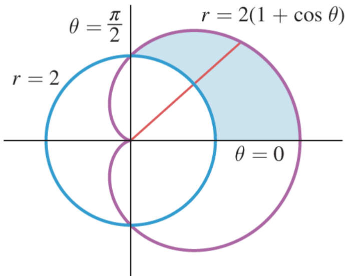


Figure 14.3.10

Solution. The region R is sketched in Figure 14.3.10. Following the two steps outlined above we obtain

$$\begin{aligned}
\iint\limits_R \sin\theta\, dA &= \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (\sin\theta) r\, dr\, d\theta \\
&= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin\theta\right]_{r=2}^{2(1+\cos\theta)} d\theta \\
&= 2\int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta]\, d\theta \\
&= 2\left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta\right]_0^{\pi/2} \\
&= 2\left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3} \blacktriangleleft
\end{aligned}$$

▶ Example 2 The sphere of radius a centered at the origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

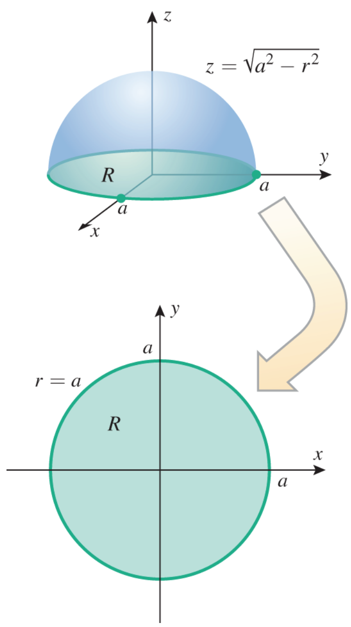


Figure 14.3.11

Solution. In cylindrical coordinates the upper hemisphere is given by the equation

$$z = \sqrt{a^2 - r^2}$$

so the volume enclosed by the entire sphere is

$$V = 2\iint\limits_R \sqrt{a^2 - r^2}\, dA$$

where R is the circular region shown in Figure 14.3.11. Thus,

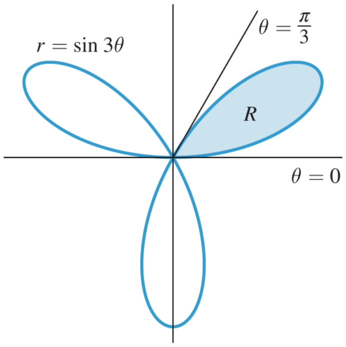
$$\begin{aligned}
V = 2\iint\limits_R \sqrt{a^2 - r^2}\, dA &= \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}(2r)\, dr\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{2}{3}(a^2 - r^2)^{3/2}\right]_{r=0}^{a} d\theta = \int_0^{2\pi} \frac{2}{3} a^3\, d\theta \\
&= \left[\frac{2}{3} a^3 \theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3 \blacktriangleleft
\end{aligned}$$

FINDING AREAS USING POLAR DOUBLE INTEGRALS

Recall from Formula (7) of Section 14.2 that the area of a region R in the xy-plane can be expressed as

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA \tag{8}$$

The argument used to derive this result can also be used to show that the formula applies to polar double integrals over regions in polar coordinates.



▲ **Figure 14.3.12**

▶ Example 3 Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.

Solution. The rose is sketched in Figure 14.3.12. We will use Formula (8) to calculate the area of the petal R in the first quadrant and multiply by 3.

$$\begin{aligned} A = 3\iint_R dA &= 3\int_0^{\pi/3}\int_0^{\sin 3\theta} r\, dr\, d\theta \\ &= \frac{3}{2}\int_0^{\pi/3} \sin^2 3\theta\, d\theta = \frac{3}{4}\int_0^{\pi/3}(1-\cos 6\theta)\, d\theta \\ &= \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{1}{4}\pi \end{aligned}$$ ◀

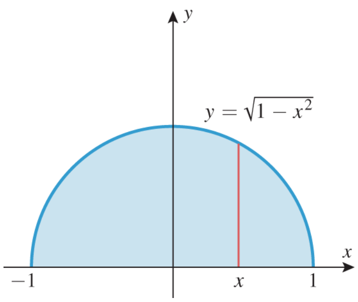
CONVERTING DOUBLE INTEGRALS FROM RECTANGULAR TO POLAR COORDINATES

Sometimes a double integral that is difficult to evaluate in rectangular coordinates can be evaluated more easily in polar coordinates by making the substitution $x = r\cos\theta$, $y = r\sin\theta$ and expressing the region of integration in polar form; that is, we rewrite the double integral in rectangular coordinates as

$$\iint_R f(x,y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, dA = \iint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta \tag{9}$$

▶ Example 4 Use polar coordinates to evaluate $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$.

Solution. In this problem we are starting with an iterated integral in rectangular coordinates rather than a double integral, so before we can make the conversion to polar coordinates we will have to identify the region of integration. Observe that for fixed x the y-integration runs from $y = 0$ to $y = \sqrt{1-x^2}$, which tells us that the lower boundary of the region is the x-axis and the upper boundary is a semicircle of radius 1 centered at the origin. From the x-integration we see that x varies from -1 to 1, so we conclude that the region of integration is as shown in Figure 14.3.13. In polar coordinates, this is the region swept out as r varies between 0 and 1 and θ varies between 0 and π. Thus,



▲ **Figure 14.3.13**

$$\begin{aligned} \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx &= \iint_R (x^2+y^2)^{3/2}\, dA \\ &= \int_0^{\pi}\int_0^1 (r^3) r\, dr\, d\theta = \int_0^{\pi} \frac{1}{5}\, d\theta = \frac{\pi}{5} \end{aligned}$$ ◀

REMARK The reason the conversion to polar coordinates worked so nicely in Example 4 is that the substitution $x = r\cos\theta$, $y = r\sin\theta$ collapsed the sum x^2+y^2 into the single term r^2, thereby simplifying the integrand. Whenever you see an expression involving x^2+y^2 in the integrand, you should consider the possibility of converting to polar coordinates.

✓ QUICK CHECK EXERCISES 14.3 (See page 917 for answers.)

1. The polar region inside the circle $r = 2\sin\theta$ and outside the circle $r = 1$ is a simple polar region given by the inequalities

$$\_\_\_\_\_\_ \le r \le \_\_\_\_\_\_, \qquad \_\_\_\_\_\_ \le \theta \le \_\_\_\_\_\_$$

2. Let R be the region in the first quadrant enclosed between the circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 100$. Supply the missing limits of integration.

$$\iint_R f(r,\theta)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(r,\theta)r\,dr\,d\theta$$

3. Let V be the volume of the solid bounded above by the hemisphere $z = \sqrt{1 - r^2}$ and bounded below by the disk enclosed within the circle $r = \sin\theta$. Expressed as a double integral in polar coordinates, $V =$ ______.

4. Express the iterated integral as a double integral in polar coordinates.

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x}\left(\frac{1}{x^2+y^2}\right)dy\,dx = \_\_\_\_\_\_$$

EXERCISE SET 14.3

1–6 Evaluate the iterated integral.

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\,dr\,d\theta$ **2.** $\displaystyle\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$

3. $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta} r^2\,dr\,d\theta$

1-10, 23-34

4. $\displaystyle\int_0^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta$

5. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\,dr\,d\theta$

6. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} r^3\,dr\,d\theta$

7–10 Use a double integral in polar coordinates to find the area of the region described.

7. The region enclosed by the cardioid $r = 1 - \cos\theta$.

8. The region enclosed by the rose $r = \sin 2\theta$.

9. The region in the first quadrant bounded by $r = 1$ and $r = \sin 2\theta$, with $\pi/4 \le \theta \le \pi/2$.

10. The region inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$.

FOCUS ON CONCEPTS

11–12 Let R be the region described. Sketch the region R and fill in the missing limits of integration.

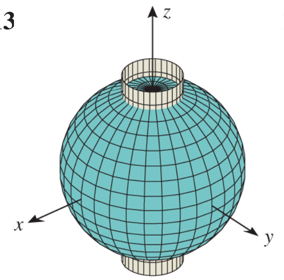
$$\iint_R f(r,\theta)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(r,\theta)r\,dr\,d\theta$$

11. The region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.

12. The region inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$.

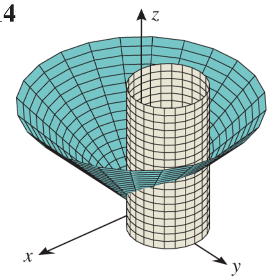
13–16 Express the volume of the solid described as a double integral in polar coordinates.

13



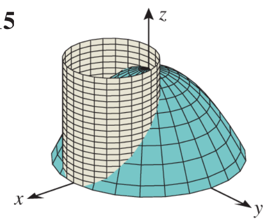
Inside of $x^2 + y^2 + z^2 = 9$
Outside of $x^2 + y^2 = 1$

14



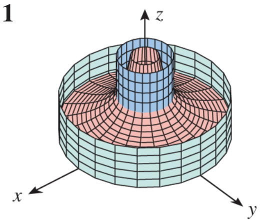
Below $z = \sqrt{x^2 + y^2}$
Inside of $x^2 + y^2 = 2y$
Above $z = 0$

15



Below $z = 1 - x^2 - y^2$
Inside of $x^2 + y^2 - x = 0$
Above $z = 0$

1



Below $z = (x^2 + y^2)^{-1/2}$
Outside of $x^2 + y^2 = 1$
Inside of $x^2 + y^2 = 9$
Above $z = 0$

17–20 Find the volume of the solid described in the indicated exercise.

17. Exercise 13 **18.** Exercise 14

19. Exercise 15 **20.** Exercise 16

21. Find the volume of the solid in the first octant bounded above by the surface $z = r\sin\theta$, below by the xy-plane, and laterally by the plane $x = 0$ and the surface $r = 3\sin\theta$.

22. Find the volume of the solid inside the surface $r^2 + z^2 = 4$ and outside the surface $r = 2\cos\theta$.

1-10, 23-34

23–26 Use polar coordinates to evaluate the double integral.

23. $\iint_R \sin(x^2+y^2)\,dA$, where R is the region enclosed by the circle $x^2+y^2=9$.

24. $\iint_R \sqrt{9-x^2-y^2}\,dA$, where R is the region in the first quadrant within the circle $x^2+y^2=9$.

25. $\iint_R \dfrac{1}{1+x^2+y^2}\,dA$, where R is the sector in the first quadrant bounded by $y=0$, $y=x$, and $x^2+y^2=4$.

26. $\iint_R 2y\,dA$, where R is the region in the first quadrant bounded above by the circle $(x-1)^2+y^2=1$ and below by the line $y=x$.

27–34 Evaluate the iterated integral by converting to polar coordinates.

27. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx$

28. $\displaystyle\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}e^{-(x^2+y^2)}\,dx\,dy$

29. $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy\,dx$

30. $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}}\cos(x^2+y^2)\,dx\,dy$

31. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{dy\,dx}{(1+x^2+y^2)^{3/2}}\quad(a>0)$

32. $\displaystyle\int_0^1\int_y^{\sqrt{y}}\sqrt{x^2+y^2}\,dx\,dy$

33. $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}}\frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$

34. $\displaystyle\int_{-4}^0\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}3x\,dy\,dx$

35–38 True–False Determine whether the statement is true or false. Explain your answer.

35. The disk of radius 2 that is centered at the origin is a polar rectangle.

36. If f is continuous and nonnegative on a simple polar region R, then the volume of the solid enclosed between R and the surface $z=f(r,\theta)$ is expressed as

$$\iint_R f(r,\theta)r\,dA$$

37. If R is the region in the first quadrant between the circles $r=1$ and $r=2$, and if f is continuous on R, then

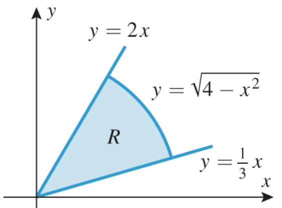
$$\iint_R f(r,\theta)\,dA=\int_0^{\pi/2}\int_1^2 f(r,\theta)\,dr\,d\theta$$

38. The area enclosed by the circle $r=\sin\theta$ is given by

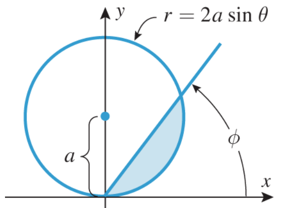
$$A=\int_0^{2\pi}\int_0^{\sin\theta}r\,dr\,d\theta$$

39. Evaluate $\iint_R x^2\,dA$ over the region R shown in the accompanying figure.

40. Show that the shaded area in the accompanying figure is $a^2\phi-\frac{1}{2}a^2\sin 2\phi$.



▲ **Figure Ex-39**



▲ **Figure Ex-40**

41. (a) Use a double integral in polar coordinates to find the volume of the oblate spheroid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1\quad(0<c<a)$$

(b) Use the result in part (a) and the World Geodetic System of 1984 (WGS-84) discussed in Exercise 54 of Section 11.7 to find the volume of the Earth in cubic meters.

42. Use polar coordinates to find the volume of the solid that is above the xy-plane, inside the cylinder $x^2+y^2-ay=0$, and inside the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$$

43. Find the area of the region enclosed by the lemniscate $r^2=2a^2\cos 2\theta$.

44. Find the area in the first quadrant that is inside the circle $r=4\sin\theta$ and outside the lemniscate $r^2=8\cos 2\theta$.

45. Writing Discuss how computing a volume of revolution using a polar double integral corresponds to the method of cylindrical shells (Section 5.3).

✓ QUICK CHECK ANSWERS 14.3 **1.** $1\le r\le 2\sin\theta,\ \pi/6\le\theta\le 5\pi/6$ **2.** $\displaystyle\int_0^{\pi/2}\int_3^{10}f(r,\theta)r\,dr\,d\theta$ **3.** $\displaystyle\int_0^{\pi}\int_0^{\sin\theta}r\sqrt{1-r^2}\,dr\,d\theta$ **4.** $\displaystyle\int_0^{\pi/4}\int_1^{\sec\theta}\frac{1}{r}\,dr\,d\theta$

14.4 SURFACE AREA; PARAMETRIC SURFACES

Earlier we showed how to find the surface area of a surface of revolution. In this section we will derive area formulas for surfaces with equations of the form $z = f(x, y)$ and for surfaces that are represented by parametric equations.

SURFACE AREA FOR SURFACES OF THE FORM $z = f(x, y)$

In Section 5.5 we showed that the expression

$$\int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

gives the area of the surface that is generated by revolving the portion of the curve $y = f(x)$ over the interval $[a, b]$ about the x-axis, assuming that f is smooth and nonnegative on the interval. We now obtain a formula for the surface area S of a surface of the form $z = f(x, y)$.

Consider a surface σ of the form $z = f(x, y)$ defined over a region R in the xy-plane (Figure 14.4.1a). We will assume that f has continuous first partial derivatives at the interior points of R. (Geometrically, this means that the surface will have a nonvertical tangent plane at each interior point of R.) We begin by subdividing R into rectangular regions by lines parallel to the x- and y-axes and by discarding any nonrectangular portions that contain points on the boundary of R. Assume that what remains are n rectangles labeled $R_1, R_2, \ldots, R_n$.

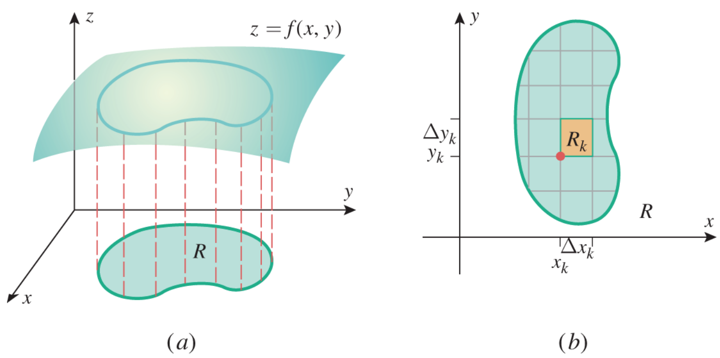


Figure 14.4.1

Let (x_k, y_k) be the lower left corner of the kth rectangle R_k, and assume that R_k has area $\Delta A_k = \Delta x_k \Delta y_k$, where Δx_k and Δy_k are the dimensions of R_k (Figure 14.4.1b). The portion of σ that lies over R_k will be some *curvilinear patch* on the surface that has a corner at $P_k(x_k, y_k, f(x_k, y_k))$; denote the area of this patch by ΔS_k (Figure 14.4.2a). To obtain an approximation of ΔS_k, we will replace σ by the tangent plane to σ at P_k. The equation of this tangent plane is

$$z = f(x_k, y_k) + f_x(x_k, y_k)(x - x_k) + f_y(x_k, y_k)(y - y_k)$$

(see Theorem 13.7.2). The portion of the tangent plane that lies over R_k will be a parallelogram τ_k. This parallelogram will have a vertex at P_k and adjacent sides determined by the vectors

$$\mathbf{q}_k = \left\langle \Delta x_k, 0, \frac{\partial z}{\partial x}\Delta x_k \right\rangle \quad \text{and} \quad \mathbf{r}_k = \left\langle 0, \Delta y_k, \frac{\partial z}{\partial y}\Delta y_k \right\rangle$$

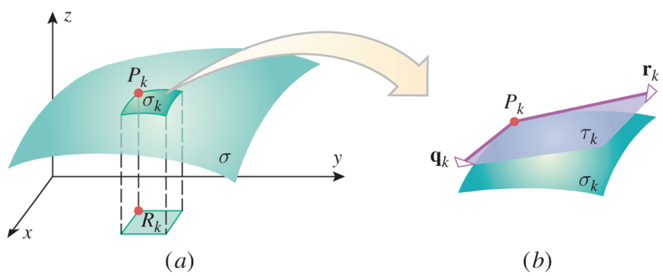


Figure 14.4.2

as illustrated in Figure 14.4.2*b*. [Here we use $\partial z/\partial x$ to represent $f_x(x_k, y_k)$ and $\partial z/\partial y$ to represent $f_y(x_k, y_k)$.]

If the dimensions of R_k are small, then τ_k should provide a good approximation to the curvilinear patch σ_k. By Theorem 11.4.5(*b*), the area of the parallelogram τ_k is the length of the cross product of $\mathbf{q}_k$ and $\mathbf{r}_k$. Thus, we expect the approximation

$$\Delta S_k \approx \text{area } \tau_k = \|\mathbf{q}_k \times \mathbf{r}_k\|$$

to be good when Δx_k and Δy_k are close to 0. Computing the cross product yields

$$\|\mathbf{q}_k \times \mathbf{r}_k\| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_k & 0 & \dfrac{\partial z}{\partial x}\Delta x_k \\ 0 & \Delta y_k & \dfrac{\partial z}{\partial y}\Delta y_k \end{vmatrix} \right\| = \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right)\Delta x_k \Delta y_k$$

so

$$\begin{aligned} \Delta S_k &\approx \left\| \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right)\Delta x_k \Delta y_k \right\| = \left\| -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k} \right\| \Delta x_k \Delta y_k \\ &= \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\Delta A_k \end{aligned} \tag{1}$$

It follows that the surface area of the entire surface can be approximated as

$$S \approx \sum_{k=1}^{n} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\Delta A_k$$

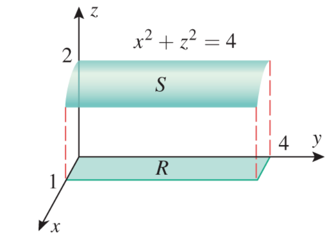
If we assume that the errors in the approximations approach zero as n increases in such a way that the dimensions of the rectangles approach zero, then it is plausible that the exact value of S is

$$S = \lim_{n \to +\infty} \sum_{k=1}^{n} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,\Delta A_k$$

or, equivalently,

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA \tag{2}$$

▶ **Example 1** Find the surface area of that portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle R in the xy-plane whose coordinates satisfy $0 \le x \le 1$ and $0 \le y \le 4$.

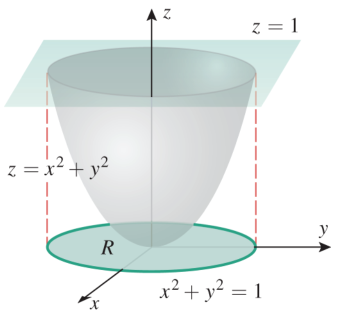


▲ **Figure 14.4.3**

Solution. As shown in Figure 14.4.3, the surface is a portion of the right circular cylinder $x^2 + z^2 = 4$. It follows from (2) that the surface area is

$$\begin{aligned} S &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA \\ &= \iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0 + 1}\,dA = \int_0^4 \int_0^1 \frac{2}{\sqrt{4-x^2}}\,dx\,dy \\ &= \underbrace{2\int_0^4 \left[\sin^{-1}\left(\tfrac{1}{2}x\right)\right]_{x=0}^{1} dy}_{\text{Formula 21 of Section 7.1}} = 2\int_0^4 \frac{\pi}{6}\,dy = \frac{4}{3}\pi \end{aligned}$$ ◀

▶ **Example 2** Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.



▲ Figure 14.4.4

Solution. The surface $z = x^2 + y^2$ is the circular paraboloid shown in Figure 14.4.4. The trace of the paraboloid in the plane $z = 1$ projects onto the circle $x^2 + y^2 = 1$ in the xy-plane, and the portion of the paraboloid that lies below the plane $z = 1$ projects onto the region R that is enclosed by this circle. Thus, it follows from (2) that the surface area is

$$S = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA$$

The expression $4x^2 + 4y^2 + 1 = 4(x^2 + y^2) + 1$ in the integrand suggests that we evaluate the integral in polar coordinates. In accordance with Formula (9) of Section 14.3, we substitute $x = r\cos\theta$ and $y = r\sin\theta$ in the integrand, replace dA by $r\,dr\,d\theta$, and find the limits of integration by expressing the region R in polar coordinates. This yields

$$\begin{aligned} S &= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\, r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_{r=0}^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{12}(5\sqrt{5}-1)\,d\theta = \frac{1}{6}\pi(5\sqrt{5}-1) \blacktriangleleft \end{aligned}$$

Some surfaces can't be described conveniently in terms of a function $z = f(x, y)$. For such surfaces, a parametric description may provide a simpler approach. We pause for a discussion of surfaces represented parametrically, with the ultimate goal of deriving a formula for the area of a parametric surface.

PARAMETRIC REPRESENTATION OF SURFACES

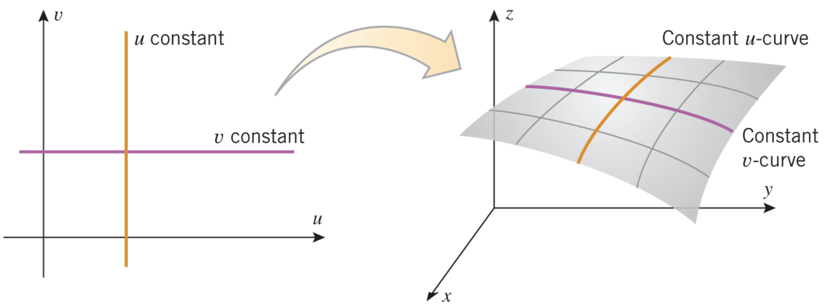
We have seen that curves in 3-space can be represented by three equations involving one parameter, say

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

Surfaces in 3-space can be represented parametrically by three equations involving two parameters, say

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v) \tag{3}$$

To visualize why such equations represent a surface, think of (u, v) as a point that varies over some region in a uv-plane. If u is held constant, then v is the only varying parameter in (3), and hence these equations represent a curve in 3-space. We call this a ***constant u-curve*** (Figure 14.4.5). Similarly, if v is held constant, then u is the only varying parameter in (3), so again these equations represent a curve in 3-space. We call this a ***constant v-curve***. By varying the constants we generate a family of u-curves and a family of v-curves that together form a surface.

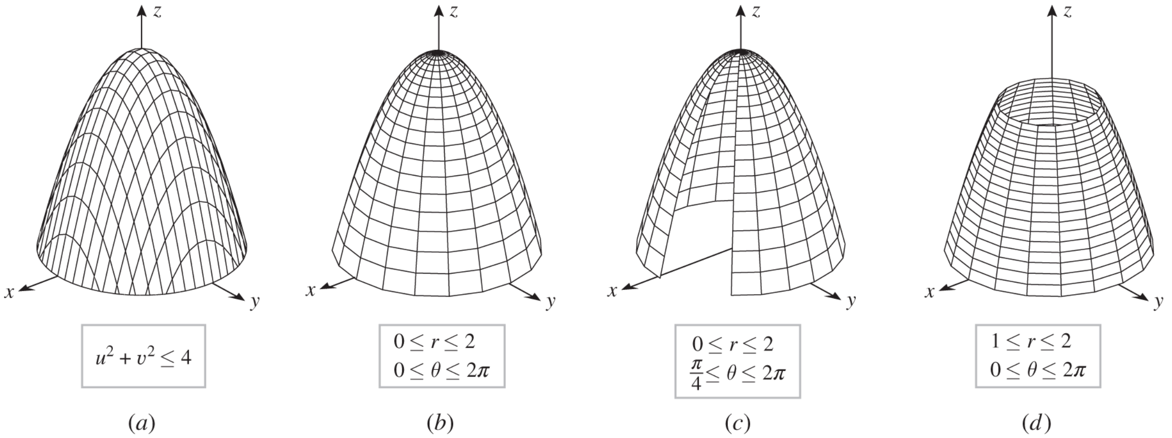


▶ Figure 14.4.5

▶ **Example 3** Consider the paraboloid $z = 4 - x^2 - y^2$. One way to parametrize this surface is to take $x = u$ and $y = v$ as the parameters, in which case the surface is represented by the parametric equations

$$x = u, \quad y = v, \quad z = 4 - u^2 - v^2 \tag{4}$$

Figure 14.4.6a shows a computer-generated graph of this surface. The constant u-curves correspond to constant x-values and hence appear on the surface as traces parallel to the yz-plane. Similarly, the constant v-curves correspond to constant y-values and hence appear on the surface as traces parallel to the xz-plane. ◀



▲ **Figure 14.4.6**

TECHNOLOGY MASTERY

If you have a graphing utility that can generate parametric surfaces, consult the relevant documentation and then try to generate the surfaces in Figure 14.4.6.

► **Example 4** The paraboloid $z = 4 - x^2 - y^2$ that was considered in Example 3 can also be parametrized by first expressing the equation in cylindrical coordinates. For this purpose, we make the substitution $x = r\cos\theta$, $y = r\sin\theta$, which yields $z = 4 - r^2$. Thus, the paraboloid can be represented parametrically in terms of r and θ as

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 4 - r^2 \tag{5}$$

A computer-generated graph of this surface for $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$ is shown in Figure 14.4.6*b*. The constant r-curves correspond to constant z-values and hence appear on the surface as traces parallel to the xy-plane. The constant θ-curves appear on the surface as traces from vertical planes through the origin at varying angles with the x-axis. Parts (c) and (d) of Figure 14.4.6 show the effect of restrictions on the parameters r and θ. ◄

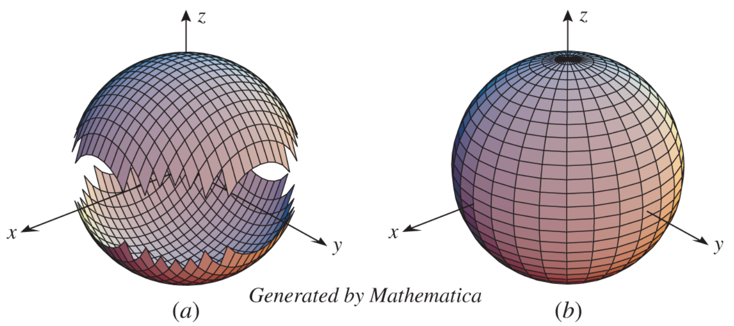
► **Example 5** One way to generate the sphere $x^2 + y^2 + z^2 = 1$ with a graphing utility is to graph the upper and lower hemispheres

$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2}$$

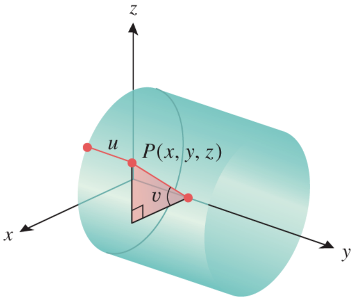
on the same screen. However, this sometimes produces a fragmented sphere (Figure 14.4.7*a*) because roundoff error sporadically produces negative values inside the radical when $1 - x^2 - y^2$ is near zero. A better graph can be generated by first expressing the sphere in spherical coordinates as $\rho = 1$ and then using the spherical-to-rectangular conversion formulas in Table 11.8.1 to obtain the parametric equations

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$$

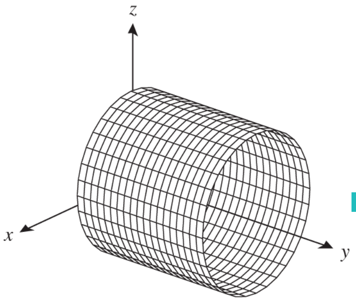
with parameters θ and ϕ. Figure 14.4.7*b* shows the graph of this parametric surface for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. In the language of cartographers, the constant ϕ-curves are the ***lines of latitude*** and the constant θ-curves are the ***lines of longitude***. ◄



► **Figure 14.4.7**



(a)



(b)

▲ Figure 14.4.8

In the exercises we will discuss formulas analogous to (6) for surfaces of revolution about other axes.

A general principle for representing surfaces of revolution parametrically is to let the variable about whose axis the curve is revolving be equal to u and let the other variables be $f(u)\cos v$ and $f(u)\sin v$.

▶ **Example 6** Find parametric equations for the portion of the right circular cylinder

$$x^2 + z^2 = 9 \quad \text{for which} \quad 0 \le y \le 5$$

in terms of the parameters u and v shown in Figure 14.4.8a. The parameter u is the y-coordinate of a point $P(x, y, z)$ on the surface, and v is the angle shown in the figure.

Solution. The radius of the cylinder is 3, so it is evident from the figure that $y = u$, $x = 3\cos v$, and $z = 3\sin v$. Thus, the surface can be represented parametrically as

$$x = 3\cos v, \quad y = u, \quad z = 3\sin v$$

To obtain the portion of the surface from $y = 0$ to $y = 5$, we let the parameter u vary over the interval $0 \le u \le 5$, and to ensure that the entire lateral surface is covered, we let the parameter v vary over the interval $0 \le v \le 2\pi$. Figure 14.4.8b shows a computer-generated graph of the surface in which u and v vary over these intervals. Constant u-curves appear as circular traces parallel to the xz-plane, and constant v-curves appear as lines parallel to the y-axis. ◀

REPRESENTING SURFACES OF REVOLUTION PARAMETRICALLY

The basic idea of Example 6 can be adapted to obtain parametric equations for surfaces of revolution. For example, suppose that we want to find parametric equations for the surface generated by revolving the plane curve $y = f(x)$ about the x-axis. Figure 14.4.9 suggests that the surface can be represented parametrically as

$$x = u, \quad y = f(u)\cos v, \quad z = f(u)\sin v \tag{6}$$

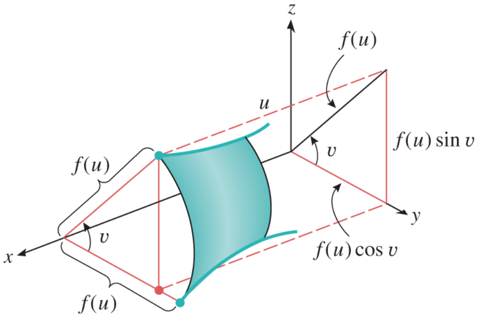
where v is the angle shown.

▶ **Example 7** Find parametric equations for the surface generated by revolving the curve $y = 1/x$ about the x-axis.

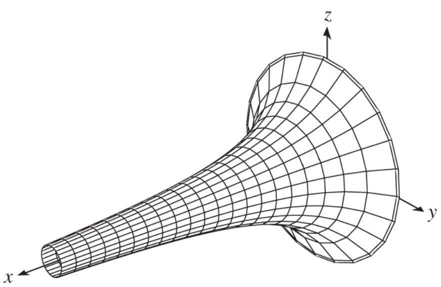
Solution. From (6) this surface can be represented parametrically as

$$x = u, \quad y = \frac{1}{u}\cos v, \quad z = \frac{1}{u}\sin v$$

Figure 14.4.10 shows a computer-generated graph of the surface in which $0.7 \le u \le 5$ and $0 \le v \le 2\pi$. This surface is a portion of Gabriel's horn, which was discussed in Exercise 55 of Section 7.8. ◀



▲ Figure 14.4.9



▲ Figure 14.4.10

VECTOR-VALUED FUNCTIONS OF TWO VARIABLES

Recall that the parametric equations

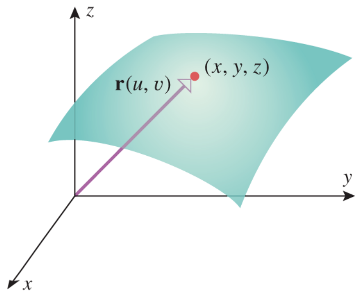
$$x = x(t), \quad y = y(t), \quad z = z(t)$$

can be expressed in vector form as

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a vector-valued function of one variable. Similarly, the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$



▲ Figure 14.4.11

can be expressed in vector form as

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

Here the function $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ is a ***vector-valued function of two variables***. We define the ***graph*** of $\mathbf{r}(u, v)$ to be the graph of the corresponding parametric equations. Geometrically, we can view $\mathbf{r}$ as a vector from the origin to a point (x, y, z) that moves over the surface $\mathbf{r} = \mathbf{r}(u, v)$ as u and v vary (Figure 14.4.11). As with vector-valued functions of one variable, we say that $\mathbf{r}(u, v)$ is ***continuous*** if each component is continuous.

▶ **Example 8** The paraboloid in Example 3 was expressed parametrically as

$$x = u, \quad y = v, \quad z = 4 - u^2 - v^2$$

These equations can be expressed in vector form as

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k} \quad ◀$$

PARTIAL DERIVATIVES OF VECTOR-VALUED FUNCTIONS

Partial derivatives of vector-valued functions of two variables are obtained by taking partial derivatives of the components. For example, if

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

then

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

These derivatives can also be written as $\mathbf{r}_u$ and $\mathbf{r}_v$ or $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$ and can be expressed as the limits

$$\frac{\partial \mathbf{r}}{\partial u} = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)}{\Delta u} = \lim_{w \to u} \frac{\mathbf{r}(w, v) - \mathbf{r}(u, v)}{w - u} \tag{7}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \lim_{\Delta v \to 0} \frac{\mathbf{r}(u, v + \Delta v) - \mathbf{r}(u, v)}{\Delta v} = \lim_{w \to v} \frac{\mathbf{r}(u, w) - \mathbf{r}(u, v)}{w - v} \tag{8}$$

▶ **Example 9** Find the partial derivatives of the vector-valued function $\mathbf{r}$ in Example 8.

Solution.

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial}{\partial u}[u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}] = \mathbf{i} - 2u\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial}{\partial v}[u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}] = \mathbf{j} - 2v\mathbf{k} \quad ◀$$

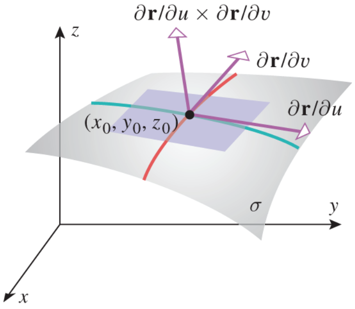
TANGENT PLANES TO PARAMETRIC SURFACES

Our next objective is to show how to find tangent planes to parametric surfaces. Let σ denote a parametric surface in 3-space, with P_0 a point on σ. We will say that a plane is ***tangent*** to σ at P_0 provided a line through P_0 lies in the plane if and only if it is a tangent line at P_0 to a curve on σ. We showed in Section 13.7 that if $z = f(x, y)$, then the graph of f has a tangent plane at a point if f is differentiable at that point. It is beyond the scope of this text to obtain precise conditions under which a parametric surface has a tangent plane at a point, so we will simply assume the existence of tangent planes at points of interest and focus on finding their equations.

Suppose that the parametric surface σ is the graph of the vector-valued function $\mathbf{r}(u, v)$ and that we are interested in the tangent plane at the point (x_0, y_0, z_0) on the surface that corresponds to the parameter values $u = u_0$ and $v = v_0$; that is,

$$\mathbf{r}(u_0, v_0) = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

If $v = v_0$ is kept fixed and u is allowed to vary, then $\mathbf{r}(u, v_0)$ is a vector-valued function of one variable whose graph is the constant v-curve through the point (u_0, v_0); similarly, if $u = u_0$ is kept fixed and v is allowed to vary, then $\mathbf{r}(u_0, v)$ is a vector-valued function of one variable whose graph is the constant u-curve through the point (u_0, v_0). Moreover, it follows from the geometric interpretation of the derivative developed in Section 12.2 that if $\partial\mathbf{r}/\partial u \neq \mathbf{0}$ at (u_0, v_0), then this vector is tangent to the constant v-curve through (u_0, v_0); and if $\partial\mathbf{r}/\partial v \neq \mathbf{0}$ at (u_0, v_0), then this vector is tangent to the constant u-curve through (u_0, v_0) (Figure 14.4.12). Thus, if $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v \neq \mathbf{0}$ at (u_0, v_0), then the vector



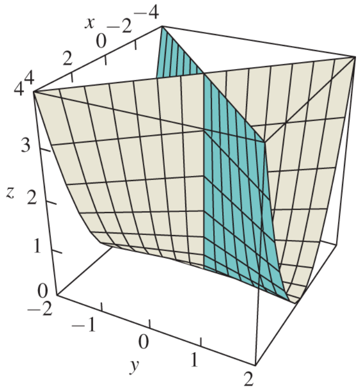
▲ Figure 14.4.12

$$\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \tag{9}$$

is orthogonal to both tangent vectors at the point (u_0, v_0) and hence is normal to the tangent plane and the surface at this point (Figure 14.4.12). Accordingly, we make the following definition.

14.4.1 DEFINITION If a parametric surface σ is the graph of $\mathbf{r} = \mathbf{r}(u, v)$, and if $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v \neq \mathbf{0}$ at a point on the surface, then the ***principal unit normal vector*** to the surface at that point is denoted by $\mathbf{n}$ or $\mathbf{n}(u, v)$ and is defined as

$$\mathbf{n} = \frac{\dfrac{\partial\mathbf{r}}{\partial u} \times \dfrac{\partial\mathbf{r}}{\partial v}}{\left\|\dfrac{\partial\mathbf{r}}{\partial u} \times \dfrac{\partial\mathbf{r}}{\partial v}\right\|} \tag{10}$$



▲ Figure 14.4.13

▶ Example 10 Find an equation of the tangent plane to the parametric surface

$$x = uv, \quad y = u, \quad z = v^2$$

at the point where $u = 2$ and $v = -1$. This surface, called ***Whitney's umbrella***, is an example of a self-intersecting parametric surface (Figure 14.4.13).

Solution. We start by writing the equations in the vector form

$$\mathbf{r} = uv\mathbf{i} + u\mathbf{j} + v^2\mathbf{k}$$

The partial derivatives of $\mathbf{r}$ are

$$\frac{\partial\mathbf{r}}{\partial u}(u, v) = v\mathbf{i} + \mathbf{j}$$

$$\frac{\partial\mathbf{r}}{\partial v}(u, v) = u\mathbf{i} + 2v\mathbf{k}$$

and at $u = 2$ and $v = -1$ these partial derivatives are

$$\frac{\partial\mathbf{r}}{\partial u}(2, -1) = -\mathbf{i} + \mathbf{j}$$

$$\frac{\partial\mathbf{r}}{\partial v}(2, -1) = 2\mathbf{i} - 2\mathbf{k}$$

Thus, from (9) and (10) a normal to the surface at this point is

$$\frac{\partial \mathbf{r}}{\partial u}(2,-1) \times \frac{\partial \mathbf{r}}{\partial v}(2,-1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 2 & 0 & -2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

Since any normal will suffice to find the tangent plane, it makes sense to multiply this vector by $-\frac{1}{2}$ and use the simpler normal $\mathbf{i}+\mathbf{j}+\mathbf{k}$. It follows from the given parametric equations that the point on the surface corresponding to $u = 2$ and $v = -1$ is $(-2, 2, 1)$, so the tangent plane at this point can be expressed in point-normal form as

$$(x+2) + (y-2) + (z-1) = 0 \quad \text{or} \quad x+y+z = 1 \blacktriangleleft$$

▶ Example 11 The sphere $x^2+y^2+z^2 = a^2$ can be expressed in spherical coordinates as $\rho = a$, and the spherical-to-rectangular conversion formulas in Table 11.8.1 can then be used to express the sphere as the graph of the vector-valued function

$$\mathbf{r}(\phi,\theta) = a\sin\phi\cos\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$$

where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$ (verify). Use this function to show that the radius vector is normal to the tangent plane at each point on the sphere.

Solution. We will show that at each point of the sphere the unit normal vector $\mathbf{n}$ is a scalar multiple of $\mathbf{r}$ (and hence is parallel to $\mathbf{r}$). We have

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$$

and hence

$$\begin{aligned} \left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| &= \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi} \\ &= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} \\ &= a^2\sqrt{\sin^2\phi} = a^2|\sin\phi| = a^2\sin\phi \end{aligned}$$

For $\phi \ne 0$ or π, it follows from (10) that

$$\mathbf{n} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} = \frac{1}{a}\mathbf{r}$$

Furthermore, the tangent planes at $\phi = 0$ or π are horizontal, to which $\mathbf{r} = \pm a\mathbf{k}$ is clearly normal. ◀

SURFACE AREA OF PARAMETRIC SURFACES

We now obtain a formula for the surface area S of a parametric surface σ. Let σ be a parametric surface whose vector equation is

$$\mathbf{r} = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

Our discussion will be analogous to the case for surfaces of the form $z = f(x,y)$. Here, R will be a region in the uv-plane that we subdivide into n rectangular regions as shown in Figure 14.4.14a. Let R_k be the kth rectangular region, and denote its area by ΔA_k. The patch σ_k is the image of R_k on σ. The patch will have a corner at $\mathbf{r}(u_k, v_k)$; denote the area of σ_k by ΔS_k (Figure 14.4.14b).

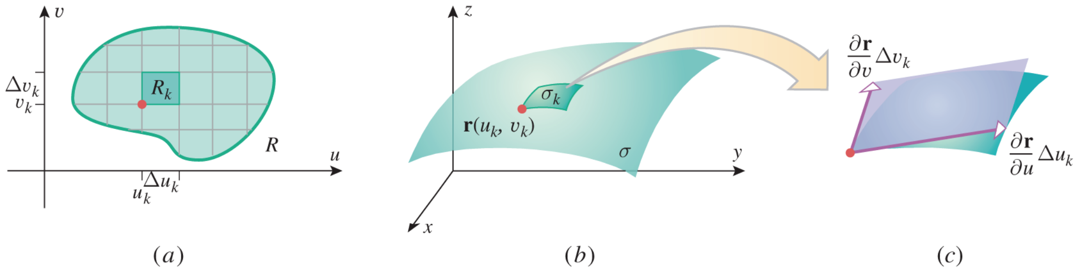


Figure 14.4.14

Recall that in the case of $z = f(x, y)$ we used the area of a parallelogram in a tangent plane to the surface to approximate the area of the patch. In the parametric case, the desired parallelogram is spanned by the tangent vectors

$$\frac{\partial \mathbf{r}}{\partial u}\Delta u_k \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}\Delta v_k$$

where the partial derivatives are evaluated at (u_k, v_k) (Figure 14.4.14c). Thus,

$$\Delta S_k \approx \left\| \frac{\partial \mathbf{r}}{\partial u}\Delta u_k \times \frac{\partial \mathbf{r}}{\partial v}\Delta v_k \right\| = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta u_k \Delta v_k = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A_k \tag{11}$$

The surface area of the entire surface is the sum of the areas ΔS_k. If we assume that the errors in the approximations in (11) approach zero as n increases in such a way that the dimensions of the rectangles R_k approach zero, then it is plausible that the exact value of S is

$$S = \lim_{n \to +\infty} \sum_{k=1}^{n} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A_k$$

or, equivalently,

$$\boxed{S = \iint_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA} \tag{12}$$

▶ Example 12 It follows from (6) that the parametric equations

$$x = u, \quad y = u\cos v, \quad z = u\sin v$$

represent the cone that results when the line $y = x$ in the xy-plane is revolved about the x-axis. Use Formula (12) to find the surface area of that portion of the cone for which $0 \le u \le 2$ and $0 \le v \le 2\pi$ (Figure 14.4.15).

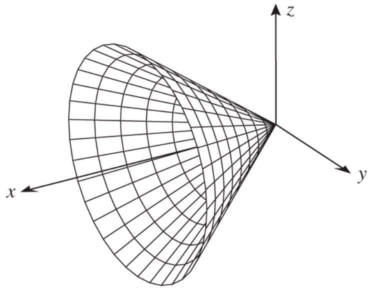


Figure 14.4.15

Solution. The surface can be expressed in vector form as

$$\mathbf{r} = u\mathbf{i} + u\cos v\,\mathbf{j} + u\sin v\,\mathbf{k} \quad (0 \le u \le 2,\ \ 0 \le v \le 2\pi)$$

Thus,

$$\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + \cos v\,\mathbf{j} + \sin v\,\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = -u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \cos v & \sin v \\ 0 & -u\sin v & u\cos v \end{vmatrix} = u\mathbf{i} - u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}$$

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{u^2 + (-u\cos v)^2 + (-u\sin v)^2} = |u|\sqrt{2} = u\sqrt{2}$$

Thus, from (12)

$$S = \iint_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA = \int_0^{2\pi} \int_0^2 \sqrt{2}u \, du \, dv = 2\sqrt{2} \int_0^{2\pi} dv = 4\pi\sqrt{2} \blacktriangleleft$$

✓ QUICK CHECK EXERCISES 14.4

(See page 930 for answers.)

1. The surface area of a surface of the form $z = f(x, y)$ over a region R in the xy-plane is given by

$$S = \iint_R \underline{\qquad\qquad} \, dA$$

2. Consider the surface represented parametrically by

$$x = 1 - u$$
$$y = (1 - u)\cos v \qquad (0 \le u \le 1, 0 \le v \le 2\pi)$$
$$z = (1 - u)\sin v$$

(a) Describe the constant u-curves.
(b) Describe the constant v-curves.

3. If

$$\mathbf{r}(u, v) = (1 - u)\mathbf{i} + [(1 - u)\cos v]\mathbf{j} + [(1 - u)\sin v]\mathbf{k}$$

then

$$\frac{\partial \mathbf{r}}{\partial u} = \underline{\qquad\qquad} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = \underline{\qquad\qquad}$$

4. If

$$\mathbf{r}(u, v) = (1 - u)\mathbf{i} + [(1 - u)\cos v]\mathbf{j} + [(1 - u)\sin v]\mathbf{k}$$

the principal unit normal to the graph of $\mathbf{r}$ at the point where $u = 1/2$ and $v = \pi/6$ is given by ________.

5. Suppose σ is a parametric surface with vector equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

If σ has no self-intersections and σ is smooth on a region R in the uv-plane, then the surface area of σ is given by

$$S = \iint_R \underline{\qquad\qquad} \, dA$$

EXERCISE SET 14.4

Graphing Utility C CAS

1–4 Express the area of the given surface as an iterated double integral, and then find the surface area. ■

1. The portion of the cylinder $y^2 + z^2 = 9$ that is above the rectangle $R = \{(x, y) : 0 \le x \le 2, -3 \le y \le 3\}$.

2. The portion of the plane $2x + 2y + z = 8$ in the first octant.

3. The portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$.

4. The portion of the surface $z = 2x + y^2$ that is above the triangular region with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.

5–10 Express the area of the given surface as an iterated double integral in polar coordinates, and then find the surface area. ■

5. The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 2x$.

6. The portion of the paraboloid $z = 1 - x^2 - y^2$ that is above the xy-plane.

7. The portion of the surface $z = xy$ that is above the sector in the first quadrant bounded by the lines $y = x/\sqrt{3}$, $y = 0$, and the circle $x^2 + y^2 = 9$.

8. The portion of the paraboloid $2z = x^2 + y^2$ that is inside the cylinder $x^2 + y^2 = 8$.

9. The portion of the sphere $x^2 + y^2 + z^2 = 16$ between the planes $z = 1$ and $z = 2$.

10. The portion of the sphere $x^2 + y^2 + z^2 = 8$ that is inside the cone $z = \sqrt{x^2 + y^2}$.

11–12 Sketch the parametric surface. ■

11. (a) $x = u,\ y = v,\ z = \sqrt{u^2 + v^2}$
(b) $x = u,\ y = \sqrt{u^2 + v^2},\ z = v$
(c) $x = \sqrt{u^2 + v^2},\ y = u,\ z = v$

12. (a) $x = u,\ y = v,\ z = u^2 + v^2$
(b) $x = u,\ y = u^2 + v^2,\ z = v$
(c) $x = u^2 + v^2,\ y = u,\ z = v$

13–14 Find a parametric representation of the surface in terms of the parameters $u = x$ and $v = y$. ■

13. (a) $2z - 3x + 4y = 5$ (b) $z = x^2$

14. (a) $z + zx^2 - y = 0$ (b) $y^2 - 3z = 5$

15. (a) Find parametric equations for the portion of the cylinder $x^2 + y^2 = 5$ that extends between the planes $z = 0$ and $z = 1$.
(b) Find parametric equations for the portion of the cylinder $x^2 + z^2 = 4$ that extends between the planes $y = 1$ and $y = 3$.

16. (a) Find parametric equations for the portion of the plane $x + y = 1$ that extends between the planes $z = -1$ and $z = 1$.
(b) Find parametric equations for the portion of the plane $y - 2z = 5$ that extends between the planes $x = 0$ and $x = 3$.

17. Find parametric equations for the surface generated by revolving the curve $y = \sin x$ about the x-axis.

18. Find parametric equations for the surface generated by revolving the curve $y - e^x = 0$ about the x-axis.

19–24 Find a parametric representation of the surface in terms of the parameters r and θ, where (r, θ, z) are the cylindrical coordinates of a point on the surface. ■

19. $z = \dfrac{1}{1 + x^2 + y^2}$ **20.** $z = e^{-(x^2+y^2)}$

21. $z = 2xy$ **22.** $z = x^2 - y^2$

23. The portion of the sphere $x^2 + y^2 + z^2 = 9$ on or above the plane $z = 2$.

24. The portion of the cone $z = \sqrt{x^2 + y^2}$ on or below the plane $z = 3$.

25. Find a parametric representation of the cone

$$z = \sqrt{3x^2 + 3y^2}$$

in terms of parameters ρ and θ, where (ρ, θ, ϕ) are spherical coordinates of a point on the surface.

26. Describe the cylinder $x^2 + y^2 = 9$ in terms of parameters θ and ϕ, where (ρ, θ, ϕ) are spherical coordinates of a point on the surface.

FOCUS ON CONCEPTS

27–32 Eliminate the parameters to obtain an equation in rectangular coordinates, and describe the surface. ■

27. $x = 2u + v,\ \ y = u - v,\ \ z = 3v$ for $-\infty < u < +\infty$ and $-\infty < v < +\infty$.

28. $x = u\cos v$, $y = u^2$, $z = u\sin v$ for $0 \le u \le 2$ and $0 \le v < 2\pi$.

29. $x = 3\sin u$, $y = 2\cos u$, $z = 2v$ for $0 \le u < 2\pi$ and $1 \le v \le 2$.

30. $x = \sqrt{u}\cos v$, $y = \sqrt{u}\sin v$, $z = u$ for $0 \le u \le 4$ and $0 \le v < 2\pi$.

31. $\mathbf{r}(u, v) = 3u\cos v\,\mathbf{i} + 4u\sin v\,\mathbf{j} + u\mathbf{k}$ for $0 \le u \le 1$ and $0 \le v < 2\pi$.

32. $\mathbf{r}(u, v) = \sin u\cos v\,\mathbf{i} + 2\sin u\sin v\,\mathbf{j} + 3\cos u\mathbf{k}$ for $0 \le u \le \pi$ and $0 \le v < 2\pi$.

33. The accompanying figure shows the graphs of two parametric representations of the cone $z = \sqrt{x^2 + y^2}$ for $0 \le z \le 2$.
(a) Find parametric equations that produce reasonable facsimiles of these surfaces.
(b) Use a graphing utility to check your answer in part (a).

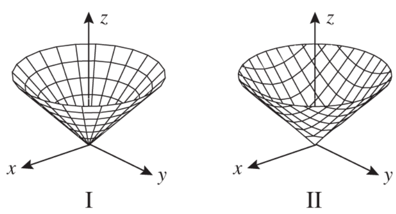


Figure Ex-33

34. The accompanying figure shows the graphs of two parametric representations of the paraboloid $z = x^2 + y^2$ for $0 \le z \le 2$.
(a) Find parametric equations that produce reasonable facsimiles of these surfaces.
(b) Use a graphing utility to check your answer in part (a).

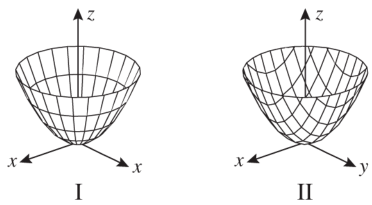
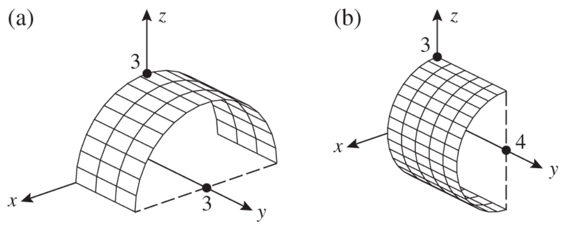
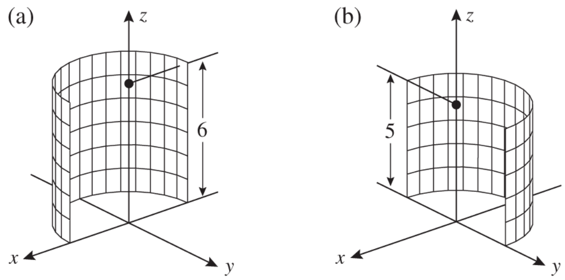


Figure Ex-34

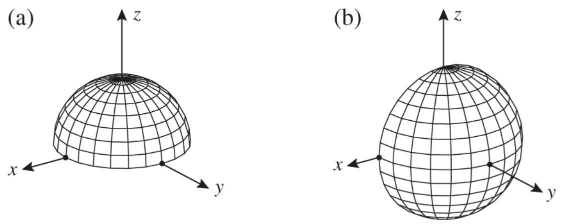
35. In each part, the figure shows a portion of the parametric surface $x = 3\cos v$, $y = u$, $z = 3\sin v$. Find restrictions on u and v that produce the surface, and check your answer with a graphing utility.



36. In each part, the figure shows a portion of the parametric surface $x = 3\cos v$, $y = 3\sin v$, $z = u$. Find restrictions on u and v that produce the surface, and check your answer with a graphing utility.

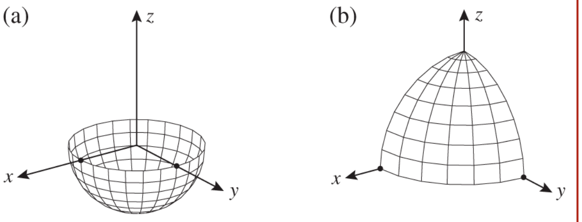


37. In each part, the figure shows a hemisphere that is a portion of the sphere $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$. Find restrictions on ϕ and θ that produce the hemisphere, and check your answer with a graphing utility.



38. In each part, the figure shows a portion of the sphere $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$. Find restrictions on ϕ and θ that produce the surface, and check your answer with a graphing utility.

(a) (b)



39–44 Find an equation of the tangent plane to the parametric surface at the stated point. ■

39. $x = u,\ y = v,\ z = u^2 + v^2;\ (1, 2, 5)$

40. $x = u^2,\ y = v^2,\ z = u + v;\ (1, 4, 3)$

41. $x = 3v \sin u,\ y = 2v \cos u,\ z = u^2;\ (0, 2, 0)$

42. $\mathbf{r} = uv\mathbf{i} + (u - v)\mathbf{j} + (u + v)\mathbf{k};\ u = 1,\ v = 2$

43. $\mathbf{r} = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k};\ u = 1/2,\ v = \pi/4$

44. $\mathbf{r} = uv\mathbf{i} + ue^v\mathbf{j} + ve^u\mathbf{k};\ u = \ln 2,\ v = 0$

45–46 Find the area of the given surface. ■

45. The portion of the paraboloid

$$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k}$$

for which $1 \le u \le 2,\ 0 \le v \le 2\pi$.

46. The portion of the cone

$$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u\mathbf{k}$$

for which $0 \le u \le 2v,\ 0 \le v \le \pi/2$.

47–50 True–False Determine whether the statement is true or false. Explain your answer. ■

47. If f has continuous first partial derivatives in the interior of a region R in the xy-plane, then the surface area of the surface $z = f(x, y)$ over R is

$$\iint_R \sqrt{[f(x, y)]^2 + 1}\, dA$$

48. Suppose that $z = f(x, y)$ has continuous first partial derivatives in the interior of a region R in the xy-plane, and set $\mathbf{q} = \langle 1, 0, \partial z/\partial x\rangle$ and $\mathbf{r} = \langle 0, 1, \partial z/\partial y\rangle$. Then the surface area of the surface $z = f(x, y)$ over R is

$$\iint_R \|\mathbf{q} \times \mathbf{r}\| dA$$

49. If $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ such that $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ are nonzero vectors at (u_0, v_0), then

$$\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}$$

is normal to the graph of $\mathbf{r} = \mathbf{r}(u, v)$ at (u_0, v_0).

50. For the function $f(x, y) = ax + by$, the area of the surface $z = f(x, y)$ over a rectangle R in the xy-plane is the product of $\|\langle 1, 0, a\rangle \times \langle 0, 1, b\rangle\|$ and the area of R.

51. Use parametric equations to derive the formula for the surface area of a sphere of radius a.

52. Use parametric equations to derive the formula for the lateral surface area of a right circular cylinder of radius r and height h.

53. The portion of the surface

$$z = \frac{h}{a}\sqrt{x^2 + y^2} \quad (a, h > 0)$$

between the xy-plane and the plane $z = h$ is a right circular cone of height h and radius a. Use a double integral to show that the lateral surface area of this cone is $S = \pi a\sqrt{a^2 + h^2}$.

54. The accompanying figure shows the ***torus*** that is generated by revolving the circle

$$(x - a)^2 + z^2 = b^2 \quad (0 < b < a)$$

in the xz-plane about the z-axis.

(a) Show that this torus can be expressed parametrically as

$$x = (a + b \cos v) \cos u$$
$$y = (a + b \cos v) \sin u$$
$$z = b \sin v$$

where u and v are the parameters shown in the figure and $0 \le u \le 2\pi,\ 0 \le v \le 2\pi$.

(b) Use a graphing utility to generate a torus.

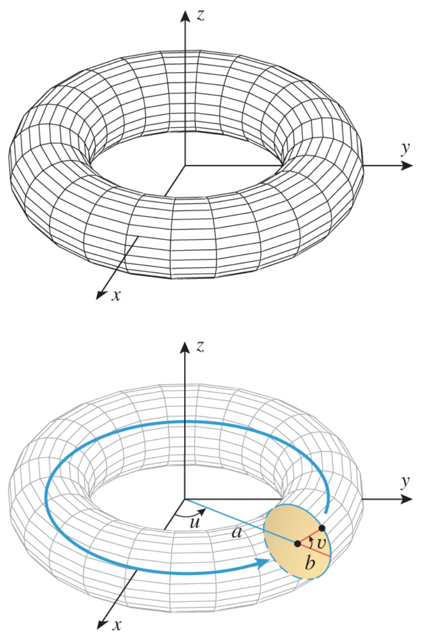


Figure Ex-54

55. Find the surface area of the torus in Exercise 54(a).

C **56.** Use a CAS to graph the ***helicoid***

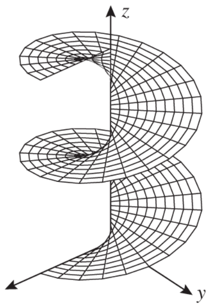
$$x = u \cos v, \quad y = u \sin v, \quad z = v$$

for $0 \le u \le 5$ and $0 \le v \le 4\pi$ (see the accompanying figure on the next page), and then use the numerical double integration operation of the CAS to approximate the surface area.

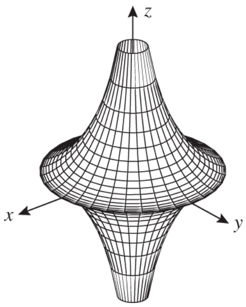
C **57.** Use a CAS to graph the ***pseudosphere***

$$x = \cos u \sin v$$
$$y = \sin u \sin v$$
$$z = \cos v + \ln\left(\tan \frac{v}{2}\right)$$

for $0 \le u \le 2\pi$, $0 < v < \pi$ (see the accompanying figure), and then use the numerical double integration operation of the CAS to approximate the surface area between the planes $z = -1$ and $z = 1$.



▲ **Figure Ex-56**



▲ **Figure Ex-57**

58. (a) Find parametric equations for the surface of revolution that is generated by revolving the curve $z = f(x)$ in the xz-plane about the z-axis.
(b) Use the result obtained in part (a) to find parametric equations for the surface of revolution that is generated by revolving the curve $z = 1/x^2$ in the xz-plane about the z-axis.
(c) Use a graphing utility to check your work by graphing the parametric surface.

59–61 The parametric equations in these exercises represent a quadric surface for positive values of a, b, and c. Identify the type of surface by eliminating the parameters u and v. Check your conclusion by choosing specific values for the constants and generating the surface with a graphing utility.

59. $x = a\cos u \cos v,\ y = b \sin u \cos v,\ z = c \sin v$

60. $x = a\cos u \cosh v,\ y = b \sin u \cosh v,\ z = c \sinh v$

61. $x = a \sinh v,\ y = b \sinh u \cosh v,\ z = c \cosh u \cosh v$

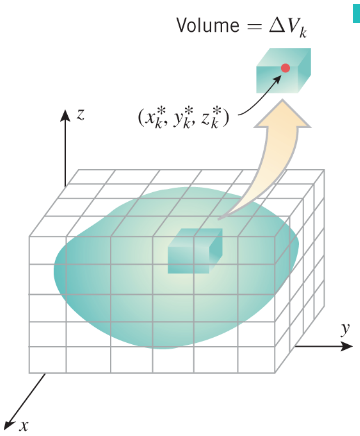
62. Writing An early popular approach to defining surface area was to take a limit of surface areas of inscribed polyhedra, but an example in which this approach fails was published in 1890 by H. A. Schwartz. Frieda Zames discusses Schwartz's example in her article "Surface Area and the Cylinder Area Paradox," *The Two-Year College Mathematics Journal*, Vol. 8, No. 4, September 1977, pp. 207–211. Read the article and write a short summary.

✓ QUICK CHECK ANSWERS 14.4 **1.** $\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}$ **2.** (a) The constant u-curves are circles of radius $1 - u$ centered at $(1 - u, 0, 0)$ and parallel to the yz-plane. (b) The constant v-curves are line segments joining the points $(1, \cos v, \sin v)$ and $(0, 0, 0)$. **3.** $\dfrac{\partial \mathbf{r}}{\partial u} = -\mathbf{i} - (\cos v)\mathbf{j} - (\sin v)\mathbf{k}$; $\dfrac{\partial \mathbf{r}}{\partial v} = -[(1-u)\sin v]\mathbf{j} + [(1-u)\cos v]\mathbf{k}$ **4.** $\dfrac{1}{\sqrt{8}}(-2\mathbf{i} + \sqrt{3}\mathbf{j} + \mathbf{k})$ **5.** $\left\|\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}\right\|$

14.5 TRIPLE INTEGRALS

In the preceding sections we defined and discussed properties of double integrals for functions of two variables. In this section we will define triple integrals for functions of three variables.

DEFINITION OF A TRIPLE INTEGRAL



▲ **Figure 14.5.1**

A single integral of a function $f(x)$ is defined over a finite closed interval on the x-axis, and a double integral of a function $f(x, y)$ is defined over a finite closed region R in the xy-plane. Our first goal in this section is to define what is meant by a *triple integral* of $f(x, y, z)$ over a closed solid region G in an xyz-coordinate system. To ensure that G does not extend indefinitely in some direction, we will assume that it can be enclosed in a suitably large box whose sides are parallel to the coordinate planes (Figure 14.5.1). In this case we say that G is a ***finite solid***.

To define the triple integral of $f(x, y, z)$ over G, we first divide the box into n "subboxes" by planes parallel to the coordinate planes. We then discard those subboxes that contain any points outside of G and choose an arbitrary point in each of the remaining subboxes. As shown in Figure 14.5.1, we denote the volume of the kth remaining subbox by ΔV_k and the point selected in the kth subbox by (x_k^*, y_k^*, z_k^*). Next we form the product

$$f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

for each subbox, then add the products for all of the subboxes to obtain the ***Riemann sum***

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

Finally, we repeat this process with more and more subdivisions in such a way that the length, width, and height of each subbox approach zero, and n approaches $+\infty$. The limit

$$\iiint_G f(x,y,z)\,dV = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k \tag{1}$$

is called the ***triple integral*** of $f(x, y, z)$ over the region G. Conditions under which the triple integral exists are studied in advanced calculus. However, for our purposes it suffices to say that existence is ensured when f is continuous on G and the region G is not too "complicated."

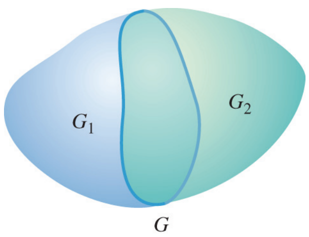
PROPERTIES OF TRIPLE INTEGRALS

Triple integrals enjoy many properties of single and double integrals:

$$\iiint_G cf(x,y,z)\,dV = c\iiint_G f(x,y,z)\,dV \quad (c \text{ a constant})$$

$$\iiint_G [f(x,y,z)+g(x,y,z)]\,dV = \iiint_G f(x,y,z)\,dV + \iiint_G g(x,y,z)\,dV$$

$$\iiint_G [f(x,y,z)-g(x,y,z)]\,dV = \iiint_G f(x,y,z)\,dV - \iiint_G g(x,y,z)\,dV$$



▲ **Figure 14.5.2**

Moreover, if the region G is subdivided into two subregions G_1 and G_2 (Figure 14.5.2), then

$$\iiint_G f(x,y,z)\,dV = \iiint_{G_1} f(x,y,z)\,dV + \iiint_{G_2} f(x,y,z)\,dV$$

We omit the proofs.

EVALUATING TRIPLE INTEGRALS OVER RECTANGULAR BOXES

Just as a double integral can be evaluated by two successive single integrations, so a triple integral can be evaluated by three successive integrations. The following theorem, which we state without proof, is the analog of Theorem 14.1.3.

There are two possible orders of integration for the iterated integrals in Theorem 14.1.3:

$$dx\,dy, \quad dy\,dx$$

Six orders of integration are possible for the iterated integral in Theorem 14.5.1:

$$dx\,dy\,dz, \quad dy\,dz\,dx, \quad dz\,dx\,dy$$
$$dx\,dz\,dy, \quad dz\,dy\,dx, \quad dy\,dx\,dz$$

14.5.1 THEOREM (*Fubini's Theorem)** *Let G be the rectangular box defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d, \quad k \le z \le l$$

If f is continuous on the region G, then

$$\iiint_G f(x,y,z)\,dV = \int_a^b \int_c^d \int_k^l f(x,y,z)\,dz\,dy\,dx \tag{2}$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

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* See the Fubini biography on p. 898.

Example 1 Evaluate the triple integral

$$\iiint_G 12xy^2z^3\,dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2$.

Solution. Of the six possible iterated integrals we might use, we will choose the one in (2). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\begin{aligned}\iiint_G 12xy^2z^3\,dV &= \int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3\,dz\,dy\,dx \\ &= \int_{-1}^{2}\int_{0}^{3} \big[3xy^2z^4\big]_{z=0}^{2}\,dy\,dx = \int_{-1}^{2}\int_{0}^{3} 48xy^2\,dy\,dx \\ &= \int_{-1}^{2} \big[16xy^3\big]_{y=0}^{3}\,dx = \int_{-1}^{2} 432x\,dx \\ &= 216x^2\big]_{-1}^{2} = 648\end{aligned}$$

EVALUATING TRIPLE INTEGRALS OVER MORE GENERAL REGIONS

Next we will consider how to evaluate triple integrals over solids that are not rectangular boxes. For the moment we will limit our discussion to solids of the type shown in Figure 14.5.3. Specifically, we will assume that the solid G is bounded above by a surface $z = g_2(x, y)$ and below by a surface $z = g_1(x, y)$ and that the projection of the solid on the xy-plane is a type I or type II region R (see Definition 14.2.1). In addition, we will assume that $g_1(x, y)$ and $g_2(x, y)$ are continuous on R and that $g_1(x, y) \le g_2(x, y)$ on R. Geometrically, this means that the surfaces may touch but cannot cross. We call a solid of this type a ***simple xy-solid***.

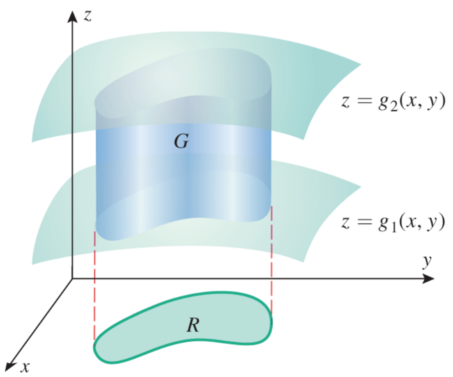


Figure 14.5.3

The following theorem, which we state without proof, will enable us to evaluate triple integrals over simple xy-solids.

14.5.2 THEOREM *Let G be a simple xy-solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$, and let R be the projection of G on the xy-plane. If $f(x, y, z)$ is continuous on G, then*

$$\iiint_G f(x, y, z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\,dz\right] dA \tag{3}$$

In (3), the first integration is with respect to z, after which a function of x and y remains. This function of x and y is then integrated over the region R in the xy-plane. To apply (3), it is helpful to begin with a three-dimensional sketch of the solid G. The limits of integration can be obtained from the sketch as follows:

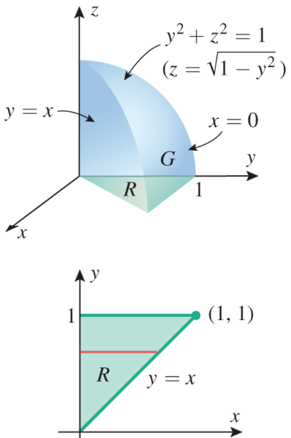
Determining Limits of Integration: Simple xy-Solid

Step 1. Find an equation $z = g_2(x, y)$ for the upper surface and an equation $z = g_1(x, y)$ for the lower surface of G. The functions $g_1(x, y)$ and $g_2(x, y)$ determine the lower and upper z-limits of integration.

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch determine the limits of integration for the double integral over R in (3).

► Example 2 Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\,dV$$



▲ **Figure 14.5.4**

Solution. The solid G and its projection R on the xy-plane are shown in Figure 14.5.4. The upper surface of the solid is formed by the cylinder and the lower surface by the xy-plane. Since the portion of the cylinder $y^2 + z^2 = 1$ that lies above the xy-plane has the equation $z = \sqrt{1 - y^2}$, and the xy-plane has the equation $z = 0$, it follows from (3) that

$$\iiint_G z\,dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dA \tag{4}$$

For the double integral over R, the x- and y-integrations can be performed in either order, since R is both a type I and type II region. We will integrate with respect to x first. With this choice, (4) yields

$$\iiint_G z\,dV = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = \int_0^1 \int_0^y \frac{1}{2}z^2\bigg]_{z=0}^{\sqrt{1-y^2}} dx\,dy$$

$$= \int_0^1 \int_0^y \frac{1}{2}(1 - y^2)\,dx\,dy = \frac{1}{2}\int_0^1 (1 - y^2)x\bigg]_{x=0}^{y} dy$$

$$= \frac{1}{2}\int_0^1 (y - y^3)\,dy = \frac{1}{2}\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{8} \quad ◄$$

TECHNOLOGY MASTERY

Most computer algebra systems have a built-in capability for computing iterated triple integrals. If you have a CAS, consult the relevant documentation and use the CAS to check Examples 1 and 2.

VOLUME CALCULATED AS A TRIPLE INTEGRAL

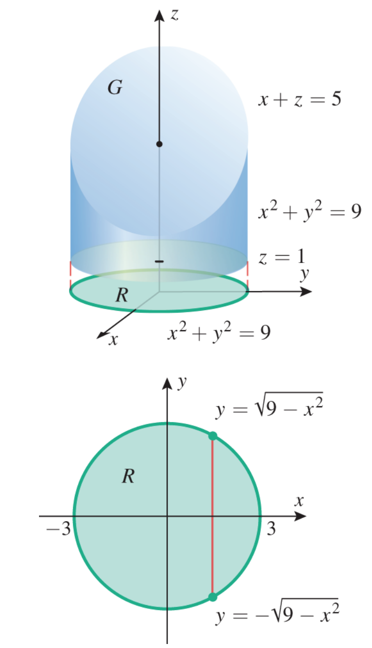
Triple integrals have many physical interpretations, some of which we will consider in Section 14.8. However, in the special case where $f(x, y, z) = 1$, Formula (1) yields

$$\iiint_G dV = \lim_{n\to+\infty} \sum_{k=1}^{n} \Delta V_k$$

which Figure 14.5.1 suggests is the volume of G; that is,

$$\text{volume of } G = \iiint_G dV \tag{5}$$

► Example 3 Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.



▲ **Figure 14.5.5**

Solution. The solid G and its projection R on the xy-plane are shown in Figure 14.5.5. The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or, equivalently, $z = 5 - x$. Thus, from (3) and (5)

$$\text{volume of } G = \iiint\limits_G dV = \iint\limits_R \left[\int_1^{5-x} dz\right] dA \tag{6}$$

For the double integral over R, we will integrate with respect to y first. Thus, (6) yields

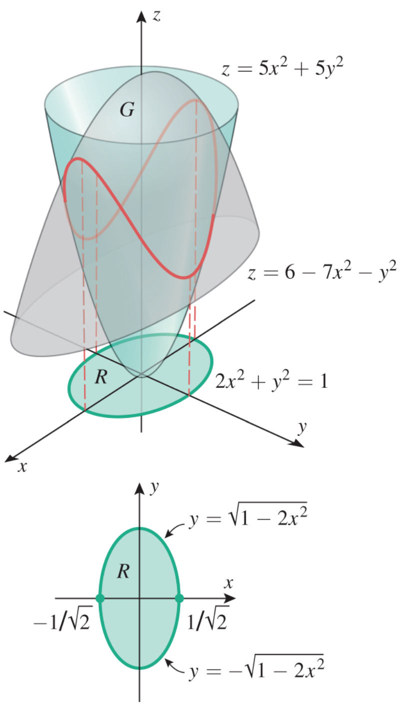
$$\begin{aligned}
\text{volume of } G &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-x} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big]_{z=1}^{5-x} dy\,dx \\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - x)\,dy\,dx = \int_{-3}^{3} (8 - 2x)\sqrt{9 - x^2}\,dx \\
&= 8\int_{-3}^{3} \sqrt{9 - x^2}\,dx - \int_{-3}^{3} 2x\sqrt{9 - x^2}\,dx \\
&= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3} 2x\sqrt{9 - x^2}\,dx \\
&= 8\left(\frac{9}{2}\pi\right) - 0 = 36\pi
\end{aligned}$$

For the first integral, see Formula (3) of Section 7.4.

The second integral is 0 because the integrand is an odd function.

▶ Example 4 Find the volume of the solid enclosed between the paraboloids

$$z = 5x^2 + 5y^2 \quad \text{and} \quad z = 6 - 7x^2 - y^2$$



▲ **Figure 14.5.6**

Solution. The solid G and its projection R on the xy-plane are shown in Figure 14.5.6. The projection R is obtained by solving the given equations simultaneously to determine where the paraboloids intersect. We obtain

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

or

$$2x^2 + y^2 = 1 \tag{7}$$

which tells us that the paraboloids intersect in a curve on the elliptic cylinder given by (7). The projection of this intersection on the xy-plane is an ellipse with this same equation. Therefore,

$$\begin{aligned}
\text{volume of } G &= \iiint\limits_G dV = \iint\limits_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\right] dA \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\,dy\,dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6 - 12x^2 - 6y^2)\,dy\,dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[6(1 - 2x^2)y - 2y^3\right]_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx \\
&= 8\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - 2x^2)^{3/2}\,dx = \frac{8}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = \frac{3\pi}{\sqrt{2}}
\end{aligned}$$

Let $x = \frac{1}{\sqrt{2}}\sin\theta$.

Use the Wallis cosine formula in Exercise 70 of Section 7.3.

INTEGRATION IN OTHER ORDERS

In Formula (3) for integrating over a simple xy-solid, the z-integration was performed first. However, there are situations in which it is preferable to integrate in a different order. For example, Figure 14.5.7a shows a ***simple xz-solid***, and Figure 14.5.7b shows a ***simple yz-solid***. For a simple xz-solid it is usually best to integrate with respect to y first, and for a simple yz-solid it is usually best to integrate with respect to x first:

$$\underset{\substack{G\\ \text{simple } xz\text{-solid}}}{\iiint} f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\,dy\right] dA \tag{8}$$

$$\underset{\substack{G\\ \text{simple } yz\text{-solid}}}{\iiint} f(x,y,z)\,dV = \iint_R \left[\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx\right] dA \tag{9}$$

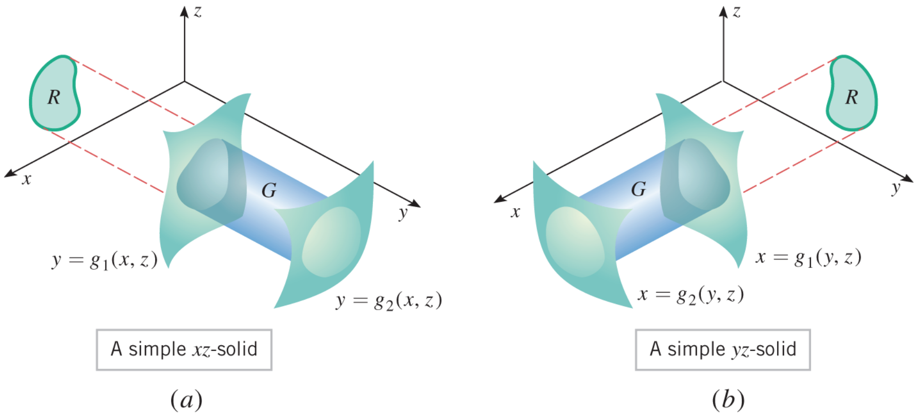


Figure 14.5.7

Sometimes a solid G can be viewed as a simple xy-solid, a simple xz-solid, and a simple yz-solid, in which case the order of integration can be chosen to simplify the computations.

Example 5 In Example 2 we evaluated

$$\iiint_G z\,dV$$

over the wedge in Figure 14.5.4 by integrating first with respect to z. Evaluate this integral by integrating first with respect to x.

Solution. The solid is bounded in the back by the plane $x = 0$ and in the front by the plane $x = y$, so

$$\iiint_G z\,dV = \iint_R \left[\int_0^y z\,dx\right] dA$$

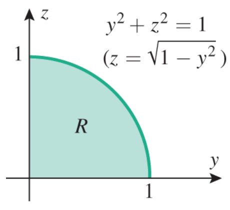


Figure 14.5.8

where R is the projection of G on the yz-plane (Figure 14.5.8). The integration over R can be performed first with respect to z and then y or vice versa. Performing the z-integration first yields

$$\iiint_G z\,dV = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^y z\,dx\,dz\,dy = \int_0^1\int_0^{\sqrt{1-y^2}} zx\Big]_{x=0}^{y} dz\,dy$$

$$= \int_0^1\int_0^{\sqrt{1-y^2}} zy\,dz\,dy = \int_0^1 \frac{1}{2}z^2y\Big]_{z=0}^{\sqrt{1-y^2}} dy = \int_0^1 \frac{1}{2}(1-y^2)y\,dy = \frac{1}{8}$$

which agrees with the result in Example 2. ◀

✓ QUICK CHECK EXERCISES 14.5 (See page 938 for answers.)

1. The iterated integral
$$\int_1^5 \int_2^4 \int_3^6 f(x, y, z)\,dx\,dz\,dy$$
integrates f over the rectangular box defined by
$\_\_\_\_ \le x \le \_\_\_\_$, $\_\_\_\_ \le y \le \_\_\_\_$, $\_\_\_\_ \le z \le \_\_\_\_$

2. Let G be the solid in the first octant bounded below by the surface $z = y + x^2$ and bounded above by the plane $z = 4$. Supply the missing limits of integration.

(a) $\displaystyle\iiint_G f(x, y, z)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} \int_{y+x^2}^{4} f(x, y, z)\,dz\,dx\,dy$

(b) $\displaystyle\iiint_G f(x, y, z)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} \int_{y+x^2}^{4} f(x, y, z)\,dz\,dy\,dx$

(c) $\displaystyle\iiint_G f(x, y, z)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x, y, z)\,dy\,dz\,dx$

3. The volume of the solid G in Quick Check Exercise 2 is ______.

EXERCISE SET 14.5 [C] CAS

1–8 Evaluate the iterated integral.

1. $\displaystyle\int_{-1}^{1} \int_0^2 \int_0^1 (x^2 + y^2 + z^2)\,dx\,dy\,dz$

2. $\displaystyle\int_{1/3}^{1/2} \int_0^{\pi} \int_0^1 zx \sin xy\,dz\,dy\,dx$

3. $\displaystyle\int_0^2 \int_{-1}^{y^2} \int_{-1}^{z} yz\,dx\,dz\,dy$

4. $\displaystyle\int_0^{\pi/4} \int_0^1 \int_0^{x^2} x \cos y\,dz\,dx\,dy$

5. $\displaystyle\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{x} xy\,dy\,dx\,dz$

6. $\displaystyle\int_1^3 \int_x^{x^2} \int_0^{\ln z} xe^y\,dy\,dz\,dx$

7. $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx$

8. $\displaystyle\int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{y}{x^2 + y^2}\,dx\,dy\,dz$

9–12 Evaluate the triple integral.

9. $\displaystyle\iiint_G xy \sin yz\,dV$, where G is the rectangular box defined by the inequalities $0 \le x \le \pi$, $0 \le y \le 1$, $0 \le z \le \pi/6$.

10. $\displaystyle\iiint_G y\,dV$, where G is the solid enclosed by the plane $z = y$, the xy-plane, and the parabolic cylinder $y = 1 - x^2$.

11. $\displaystyle\iiint_G xyz\,dV$, where G is the solid in the first octant that is bounded by the parabolic cylinder $z = 2 - x^2$ and the planes $z = 0$, $y = x$, and $y = 0$.

12. $\displaystyle\iiint_G \cos(z/y)\,dV$, where G is the solid defined by the inequalities $\pi/6 \le y \le \pi/2$, $y \le x \le \pi/2$, $0 \le z \le xy$.

[C] **13.** Use the numerical triple integral operation of a CAS to approximate
$$\iiint_G \frac{\sqrt{x + z^2}}{y}\,dV$$
where G is the rectangular box defined by the inequalities $0 \le x \le 3$, $1 \le y \le 2$, $-2 \le z \le 1$.

[C] **14.** Use the numerical triple integral operation of a CAS to approximate
$$\iiint_G e^{-x^2-y^2-z^2}\,dV$$
where G is the spherical region $x^2 + y^2 + z^2 \le 1$.

15–18 Use a triple integral to find the volume of the solid.

15. The solid in the first octant bounded by the coordinate planes and the plane $3x + 6y + 4z = 12$.

16. The solid bounded by the surface $z = \sqrt{y}$ and the planes $x + y = 1$, $x = 0$, and $z = 0$.

17. The solid bounded by the surface $y = x^2$ and the planes $y + z = 4$ and $z = 0$.

18. The wedge in the first octant that is cut from the solid cylinder $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$.

FOCUS ON CONCEPTS

19. Let G be the solid enclosed by the surfaces in the accompanying figure on the next page. Fill in the missing limits of integration.

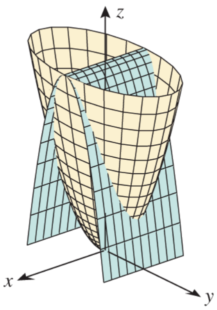
(a) $\displaystyle\iiint_G f(x, y, z)\,dV = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x, y, z)\,dz\,dy\,dx$

(b) $\displaystyle\iiint_G f(x, y, z)\,dV = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x, y, z)\,dz\,dx\,dy$

20. Let G be the solid enclosed by the surfaces in the accompanying figure on the next page. Fill in the missing limits of integration.

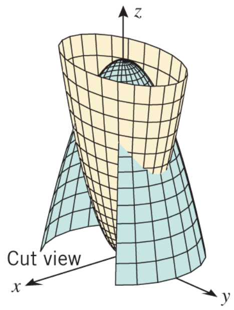
(a) $\displaystyle\iiint_G f(x, y, z)\,dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x, y, z)\,dz\,dy\,dx$

(b) $\displaystyle\iiint_G f(x, y, z)\,dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x, y, z)\,dz\,dx\,dy$



$z = 4x^2 + y^2$
$z = 4 - 3y^2$

▲ **Figure Ex-19**



$z = 3x^2 + y^2$
$z = 8 - x^2 - y^2$

▲ **Figure Ex-20**

21–24 Set up (but do not evaluate) an iterated triple integral for the volume of the solid enclosed between the given surfaces.

21. The surfaces in Exercise 19.

22. The surfaces in Exercise 20.

23. The elliptic cylinder $x^2 + 9y^2 = 9$ and the planes $z = 0$ and $z = x + 3$.

24. The cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

25–26 In each part, sketch the solid whose volume is given by the integral.

25. (a) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{y+1} dz\,dy\,dx$

(b) $\displaystyle\int_{0}^{9}\int_{0}^{y/3}\int_{0}^{\sqrt{y^2-9x^2}} dz\,dx\,dy$

(c) $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{2} dy\,dz\,dx$

26. (a) $\displaystyle\int_{0}^{3}\int_{x^2}^{9}\int_{0}^{2} dz\,dy\,dx$

(b) $\displaystyle\int_{0}^{2}\int_{0}^{2-y}\int_{0}^{2-x-y} dz\,dx\,dy$

(c) $\displaystyle\int_{-2}^{2}\int_{0}^{4-y^2}\int_{0}^{2} dx\,dz\,dy$

27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. If G is the rectangular solid that is defined by $1 \le x \le 3$, $2 \le y \le 5$, $-1 \le z \le 1$, and if $f(x, y, z)$ is continuous on G, then

$$\iiint_G f(x, y, z)\,dV = \int_{1}^{3}\int_{-1}^{1}\int_{2}^{5} f(x, y, z)\,dy\,dz\,dx$$

28. If G is a simple xy-solid and $f(x, y, z)$ is continuous on G, then the triple integral of f over G can be expressed as an iterated integral whose outermost integration is performed with respect to z.

29. If G is the portion of the unit ball in the first octant, then

$$\iiint_G f(x, y, z)\,dV = \int_{0}^{1}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2-y^2}} f(x, y, z)\,dz\,dy\,dx$$

30. If G is a simple xy-solid and

$$\text{volume of } G = \iiint_G f(x, y, z)\,dV$$

then $f(x, y, z) = 1$ at every point in G.

31. Let G be the rectangular box defined by the inequalities $a \le x \le b$, $c \le y \le d$, $k \le z \le l$. Show that

$$\iiint_G f(x)g(y)h(z)\,dV = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right]\left[\int_k^l h(z)\,dz\right]$$

32. Use the result of Exercise 31 to evaluate

(a) $\displaystyle\iiint_G xy^2 \sin z\,dV$, where G is the set of points satisfying $-1 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le \pi/2$;

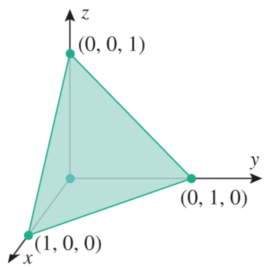
(b) $\displaystyle\iiint_G e^{2x+y-z}\,dV$, where G is the set of points satisfying $0 \le x \le 1$, $0 \le y \le \ln 3$, $0 \le z \le \ln 2$.

33–36 The ***average value*** or ***mean value*** of a continuous function $f(x, y, z)$ over a solid G is defined as

$$f_{\text{ave}} = \frac{1}{V(G)}\iiint_G f(x, y, z)\,dV$$

where $V(G)$ is the volume of the solid G (compare to the definition preceding Exercise 61 of Section 14.2). Use this definition in these exercises.

33. Find the average value of $f(x, y, z) = x + y + z$ over the tetrahedron shown in the accompanying figure.



◀ **Figure Ex-33**

34. Find the average value of $f(x, y, z) = xyz$ over the spherical region $x^2 + y^2 + z^2 \le 1$.

C **35.** Use the numerical triple integral operation of a CAS to approximate the average distance from the origin to a point in the solid of Example 4.

C **36.** Let $d(x, y, z)$ be the distance from the point (z, z, z) to the point $(x, y, 0)$. Use the numerical triple integral operation of a CAS to approximate the average value of d for

$0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$. Write a short explanation as to why this value may be considered to be the average distance between a point on the diagonal from $(0,0,0)$ to $(1,1,1)$ and a point on the face in the xy-plane for the unit cube $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$.

37. Let G be the tetrahedron in the first octant bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (a > 0, b > 0, c > 0)$$

(a) List six different iterated integrals that represent the volume of G.

(b) Evaluate any one of the six to show that the volume of G is $\frac{1}{6}abc$.

38. Use a triple integral to derive the formula for the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

FOCUS ON CONCEPTS

39–40 Express each integral as an equivalent integral in which the z-integration is performed first, the y-integration second, and the x-integration last.

39. (a) $\displaystyle\int_0^5 \int_0^2 \int_0^{\sqrt{4-y^2}} f(x,y,z)\,dx\,dy\,dz$

(b) $\displaystyle\int_0^9 \int_0^{3-\sqrt{x}} \int_0^z f(x,y,z)\,dy\,dz\,dx$

(c) $\displaystyle\int_0^4 \int_y^{8-y} \int_0^{\sqrt{4-y}} f(x,y,z)\,dx\,dz\,dy$

40. (a) $\displaystyle\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{9-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$

(b) $\displaystyle\int_0^4 \int_0^2 \int_0^{x/2} f(x,y,z)\,dy\,dz\,dx$

(c) $\displaystyle\int_0^4 \int_0^{4-y} \int_0^{\sqrt{z}} f(x,y,z)\,dx\,dz\,dy$

41. Writing The following initial steps can be used to express a triple integral over a solid G as an iterated triple integral: First project G onto one of the coordinate planes to obtain a region R, and then project R onto one of the coordinate axes. Describe how you would use these steps to find the limits of integration. Illustrate your discussion with an example.

✓QUICK CHECK ANSWERS 14.5 **1.** $3 \le x \le 6$, $1 \le y \le 5$, $2 \le z \le 4$ **2.** (a) $\displaystyle\int_0^4 \int_0^{\sqrt{4-y}} \int_{y+x^2}^4 f(x,y,z)\,dz\,dx\,dy$ (b) $\displaystyle\int_0^2 \int_0^{4-x^2} \int_{y+x^2}^4 f(x,y,z)\,dz\,dy\,dx$ (c) $\displaystyle\int_0^2 \int_{x^2}^4 \int_0^{z-x^2} f(x,y,z)\,dy\,dz\,dx$ **3.** $\dfrac{128}{15}$

14.6 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

In Section 14.3 we saw that some double integrals are easier to evaluate in polar coordinates than in rectangular coordinates. Similarly, some triple integrals are easier to evaluate in cylindrical or spherical coordinates than in rectangular coordinates. In this section we will study triple integrals in these coordinate systems.

TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

Recall that in rectangular coordinates the triple integral of a continuous function f over a solid region G is defined as

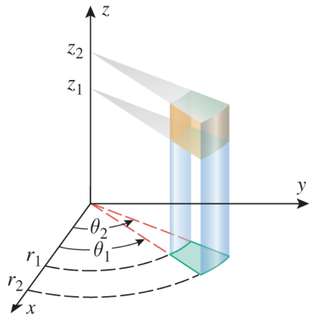
$$\iiint_G f(x,y,z)\,dV = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

where ΔV_k denotes the volume of a rectangular parallelepiped interior to G and (x_k^*, y_k^*, z_k^*) is a point in this parallelepiped (see Figure 14.5.1). Triple integrals in cylindrical and spherical coordinates are defined similarly, except that the region G is divided not into rectangular parallelepipeds but into regions more appropriate to these coordinate systems.

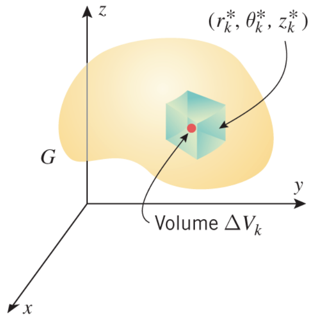
In cylindrical coordinates, the simplest equations are of the form

$$r = \text{constant}, \quad \theta = \text{constant}, \quad z = \text{constant}$$

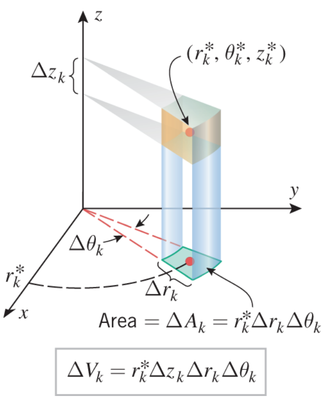
The first equation represents a right circular cylinder centered on the z-axis, the second a vertical half-plane hinged on the z-axis, and the third a horizontal plane. (See Figure 11.8.3.) These surfaces can be paired up to determine solids called ***cylindrical wedges or***



▲ **Figure 14.6.1**



▲ **Figure 14.6.2**



▲ **Figure 14.6.3**

Note the extra factor of r that appears in the integrand on converting a triple integral to an iterated integral in cylindrical coordinates.

cylindrical elements of volume. To be precise, a cylindrical wedge is a solid enclosed between six surfaces of the following form:

two cylinders (blue)	$r = r_1$,	$r = r_2$	$(r_1 < r_2)$
two vertical half-planes (yellow)	$\theta = \theta_1$,	$\theta = \theta_2$	$(\theta_1 < \theta_2)$
two horizontal planes (gray)	$z = z_1$,	$z = z_2$	$(z_1 < z_2)$

(Figure 14.6.1). The dimensions $\theta_2 - \theta_1$, $r_2 - r_1$, and $z_2 - z_1$ are called the ***central angle***, ***thickness***, and ***height*** of the wedge.

To define the triple integral over G of a function $f(r, \theta, z)$ in cylindrical coordinates we proceed as follows:

- Subdivide G into pieces by a three-dimensional grid consisting of concentric circular cylinders centered on the z-axis, half-planes hinged on the z-axis, and horizontal planes. Exclude from consideration all pieces that contain any points outside of G, thereby leaving only cylindrical wedges that are subsets of G.
- Assume that there are n such cylindrical wedges, and denote the volume of the kth cylindrical wedge by ΔV_k. As indicated in Figure 14.6.2, let $(r_k^*, \theta_k^*, z_k^*)$ be any point in the kth cylindrical wedge.
- Repeat this process with more and more subdivisions so that as n increases, the height, thickness, and central angle of the cylindrical wedges approach zero. Define

$$\iiint_G f(r, \theta, z)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*)\Delta V_k \tag{1}$$

For computational purposes, it will be helpful to express (1) as an iterated integral. Toward this end we note that the volume ΔV_k of the kth cylindrical wedge can be expressed as

$$\Delta V_k = [\text{area of base}] \cdot [\text{height}] \tag{2}$$

If we denote the thickness, central angle, and height of this wedge by Δr_k, $\Delta\theta_k$, and Δz_k, and if we choose the arbitrary point $(r_k^*, \theta_k^*, z_k^*)$ to lie above the "center" of the base (Figures 14.3.6 and 14.6.3), then it follows from (5) of Section 14.3 that the base has area $\Delta A_k = r_k^* \Delta r_k \Delta\theta_k$. Thus, (2) can be written as

$$\Delta V_k = r_k^* \Delta r_k \Delta\theta_k \Delta z_k = r_k^* \Delta z_k \Delta r_k \Delta\theta_k$$

Substituting this expression in (1) yields

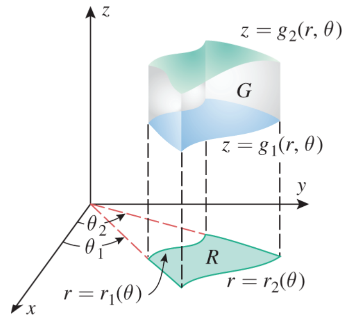
$$\iiint_G f(r, \theta, z)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) r_k^* \Delta z_k \Delta r_k \Delta\theta_k$$

which suggests that a triple integral in cylindrical coordinates can be evaluated as an iterated integral of the form

$$\iiint_G f(r, \theta, z)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r, \theta, z) r\, dz\, dr\, d\theta \tag{3}$$

In this formula the integration with respect to z is done first, then with respect to r, and then with respect to θ, but any order of integration is allowable.

The following theorem, which we state without proof, makes the preceding ideas more precise.



▲ Figure 14.6.4

14.6.1 THEOREM *Let G be a solid region whose upper surface has the equation $z = g_2(r, \theta)$ and whose lower surface has the equation $z = g_1(r, \theta)$ in cylindrical coordinates. If the projection of the solid on the xy-plane is a simple polar region R, and if $f(r, \theta, z)$ is continuous on G, then*

$$\iiint_G f(r, \theta, z)\, dV = \iint_R \left[\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\, dz\right] dA \tag{4}$$

where the double integral over R is evaluated in polar coordinates. In particular, if the projection R is as shown in Figure 14.6.4, then (4) can be written as

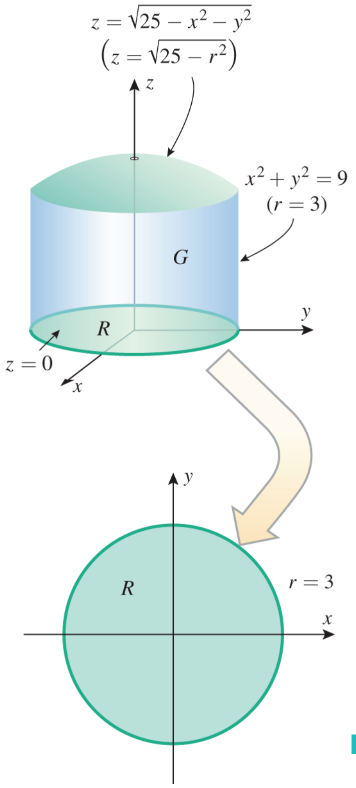
$$\iiint_G f(r, \theta, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z) r\, dz\, dr\, d\theta \tag{5}$$

The type of solid to which Formula (5) applies is illustrated in Figure 14.6.4. To apply (4) and (5) it is best to begin with a three-dimensional sketch of the solid G, from which the limits of integration can be obtained as follows:

Determining Limits of Integration: Cylindrical Coordinates

Step 1. Identify the upper surface $z = g_2(r, \theta)$ and the lower surface $z = g_1(r, \theta)$ of the solid. The functions $g_1(r, \theta)$ and $g_2(r, \theta)$ determine the z-limits of integration. (If the upper and lower surfaces are given in rectangular coordinates, convert them to cylindrical coordinates.)

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch the r- and θ-limits of integration may be obtained exactly as with double integrals in polar coordinates.



▲ Figure 14.6.5

► **Example 1** Use triple integration in cylindrical coordinates to find the volume of the solid G that is bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$.

Solution. The solid G and its projection R on the xy-plane are shown in Figure 14.6.5. In cylindrical coordinates, the upper surface of G is the hemisphere $z = \sqrt{25 - r^2}$ and the lower surface is the plane $z = 0$. Thus, from (4), the volume of G is

$$V = \iiint_G dV = \iint_R \left[\int_0^{\sqrt{25-r^2}} dz\right] dA$$

For the double integral over R, we use polar coordinates:

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^3 [rz]_{z=0}^{\sqrt{25-r^2}}\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^3 r\sqrt{25 - r^2}\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(25 - r^2)^{3/2}\right]_{r=0}^{3} d\theta \\ &= \int_0^{2\pi} \frac{61}{3}\, d\theta = \frac{122}{3}\pi \end{aligned}$$ ◄

$$\boxed{\begin{aligned} u &= 25 - r^2 \\ du &= -2r\, dr \end{aligned}}$$

CONVERTING TRIPLE INTEGRALS FROM RECTANGULAR TO CYLINDRICAL COORDINATES

Sometimes a triple integral that is difficult to integrate in rectangular coordinates can be evaluated more easily by making the substitution $x = r\cos\theta, y = r\sin\theta, z = z$ to convert it to an integral in cylindrical coordinates. Under such a substitution, a rectangular triple integral can be expressed as an iterated integral in cylindrical coordinates as

$$\iiint_G f(x,y,z)\,dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z)r\,dz\,dr\,d\theta \tag{6}$$

The order of integration on the right side of (6) can be changed, provided the limits of integration are adjusted accordingly.

Example 2 Use cylindrical coordinates to evaluate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx$$

Solution. In problems of this type, it is helpful to sketch the region of integration G and its projection R on the xy-plane. From the z-limits of integration, the upper surface of G is the paraboloid $z = 9 - x^2 - y^2$ and the lower surface is the xy-plane $z = 0$. From the x- and y-limits of integration, the projection R is the region in the xy-plane enclosed by the circle $x^2 + y^2 = 9$ (Figure 14.6.6). Thus,

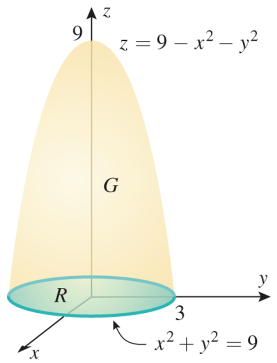


Figure 14.6.6

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx = \iiint_G x^2\,dV$$

$$= \iint_R \left[\int_0^{9-r^2} r^2\cos^2\theta\,dz\right] dA = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} (r^2\cos^2\theta)r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^3\cos^2\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \left[zr^3\cos^2\theta\right]_{z=0}^{9-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (9r^3 - r^5)\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\left[\left(\frac{9r^4}{4} - \frac{r^6}{6}\right)\cos^2\theta\right]_{r=0}^{3} d\theta$$

$$= \frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{243}{4}\int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{243\pi}{4}$$

TRIPLE INTEGRALS IN SPHERICAL COORDINATES

In spherical coordinates, the simplest equations are of the form

$$\rho = \text{ constant}, \quad \theta = \text{ constant}, \quad \phi = \text{ constant}$$

As indicated in Figure 11.8.4, the first equation represents a sphere centered at the origin and the second a half-plane hinged on the z-axis. The graph of the third equation is a right circular cone nappe with its vertex at the origin and its line of symmetry along the z-axis for $\phi \neq \pi/2$, and is the xy-plane if $\phi = \pi/2$. By a ***spherical wedge*** or ***spherical element of volume*** we mean a solid enclosed between six surfaces of the following form:

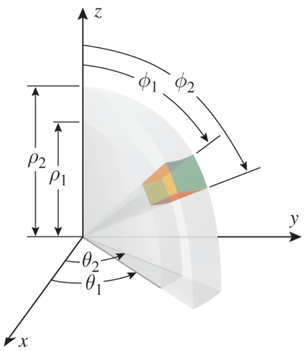


Figure 14.6.7

two spheres (green)	$\rho = \rho_1$,	$\rho = \rho_2$	$(\rho_1 < \rho_2)$
two vertical half-planes (yellow)	$\theta = \theta_1$,	$\theta = \theta_2$	$(\theta_1 < \theta_2)$
nappes of two circular cones (pink)	$\phi = \phi_1$,	$\phi = \phi_2$	$(\phi_1 < \phi_2)$

(Figure 14.6.7). We will refer to the numbers $\rho_2 - \rho_1, \theta_2 - \theta_1$, and $\phi_2 - \phi_1$ as the ***dimensions*** of a spherical wedge.

If G is a solid region in three-dimensional space, then the triple integral over G of a continuous function $f(\rho, \theta, \phi)$ in spherical coordinates is similar in definition to the triple integral in cylindrical coordinates, except that the solid G is partitioned into *spherical* wedges by a three-dimensional grid consisting of spheres centered at the origin, half-planes hinged on the z-axis, and nappes of right circular cones with vertices at the origin and lines of symmetry along the z-axis (Figure 14.6.8).

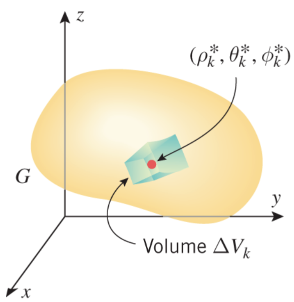


Figure 14.6.8

The defining equation of a triple integral in spherical coordinates is

$$\iiint_G f(\rho, \theta, \phi)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*)\Delta V_k \tag{7}$$

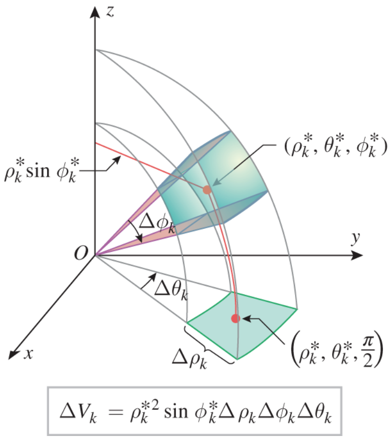
where ΔV_k is the volume of the kth spherical wedge that is interior to G, $(\rho_k^*, \theta_k^*, \phi_k^*)$ is an arbitrary point in this wedge, and n increases in such a way that the dimensions of each interior spherical wedge tend to zero.

For computational purposes, it will be desirable to express (7) as an iterated integral. In Exercise 30 we will help you to show that if the point $(\rho_k^*, \theta_k^*, \phi_k^*)$ is suitably chosen, then the volume ΔV_k in (7) can be written as

$$\Delta V_k = \rho_k^{*2} \sin \phi_k^* \Delta\rho_k \Delta\phi_k \Delta\theta_k \tag{8}$$

where $\Delta\rho_k$, $\Delta\phi_k$, and $\Delta\theta_k$ are the dimensions of the wedge (Figure 14.6.9). Substituting this in (7) we obtain

$$\iiint_G f(\rho, \theta, \phi)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*)\rho_k^{*2} \sin \phi_k^* \Delta\rho_k \Delta\phi_k \Delta\theta_k$$



▲ **Figure 14.6.9**

Note the extra factor of $\rho^2 \sin \phi$ that appears in the integrand on converting a triple integral to an iterated integral in spherical coordinates. This is analogous to the extra factor of r that appears in an iterated integral in cylindrical coordinates.

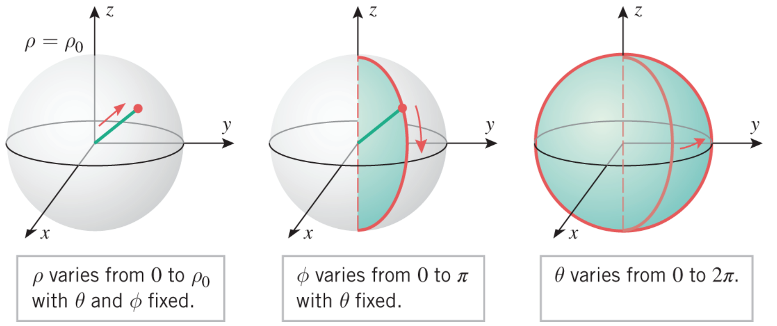
which suggests that a triple integral in spherical coordinates can be evaluated as an iterated integral of the form

$$\iiint_G f(\rho, \theta, \phi)\, dV = \underset{\substack{\text{appropriate} \\ \text{limits}}}{\iiint} f(\rho, \theta, \phi)\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \tag{9}$$

The analog of Theorem 14.6.1 for triple integrals in spherical coordinates is tedious to state, so instead we will give some examples that illustrate techniques for obtaining the limits of integration. In all of our examples we will use the same order of integration—first with respect to ρ, then ϕ, and then θ. Once you have mastered the basic ideas, there should be no trouble using other orders of integration.

Suppose that we want to integrate $f(\rho, \theta, \phi)$ over the spherical solid G enclosed by the sphere $\rho = \rho_0$. The basic idea is to choose the limits of integration so that every point of the solid is accounted for in the integration process. Figure 14.6.10 illustrates one way of doing this. Holding θ and ϕ fixed for the first integration, we let ρ vary from 0 to ρ_0. This covers a radial line from the origin to the surface of the sphere. Next, keeping θ fixed, we let ϕ vary from 0 to π so that the radial line sweeps out a fan-shaped region. Finally, we let θ vary from 0 to 2π so that the fan-shaped region makes a complete revolution, thereby sweeping out the entire sphere. Thus, the triple integral of $f(\rho, \theta, \phi)$ over the spherical solid G can be evaluated by writing

$$\iiint_G f(\rho, \theta, \phi)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{\rho_0} f(\rho, \theta, \phi)\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$



▶ **Figure 14.6.10**

Table 14.6.1 suggests how the limits of integration in spherical coordinates can be obtained for some other common solids.

Table 14.6.1

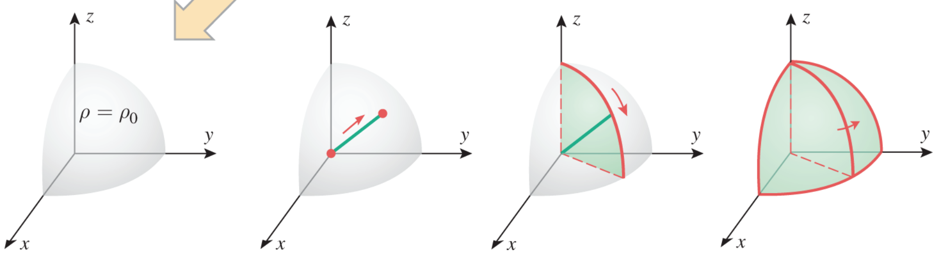
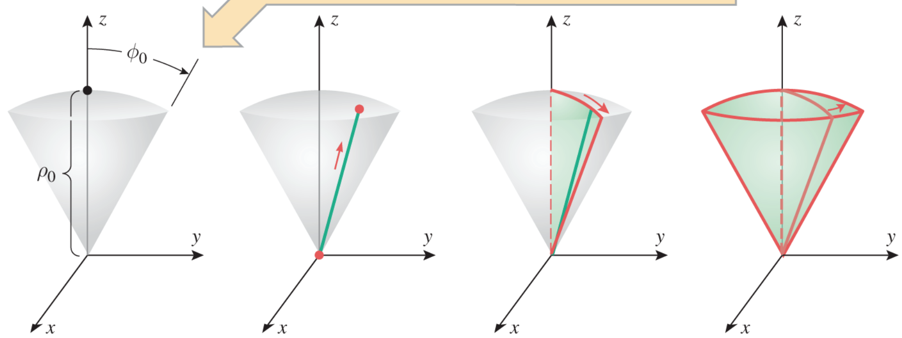
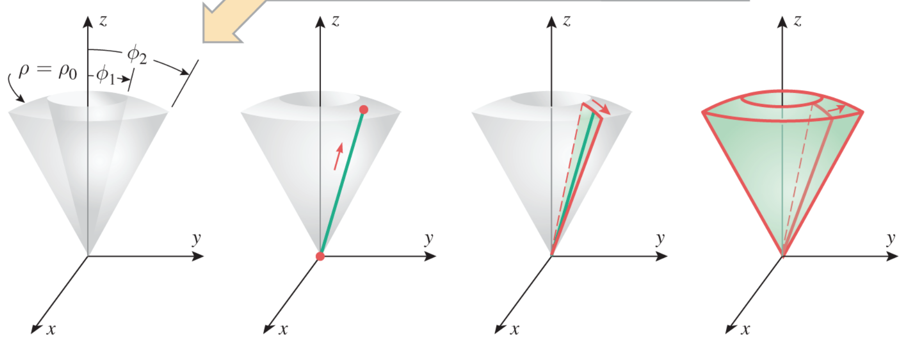
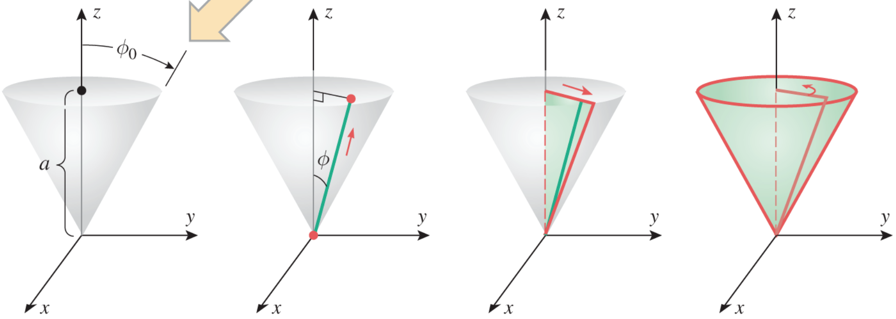
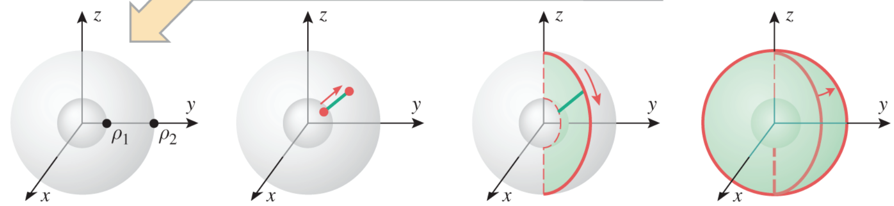
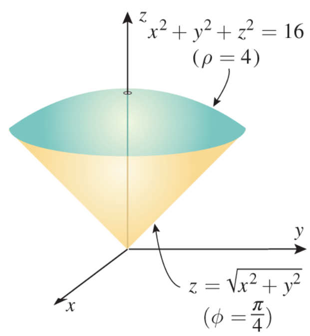
DETERMINATION OF LIMITS	INTEGRAL
This is the portion of the sphere of radius ρ_0 that lies in the first octant.  ρ varies from 0 to ρ_0 with θ and ϕ held fixed. ϕ varies from 0 to $\pi/2$ with θ held fixed. θ varies from 0 to $\pi/2$.	$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\rho_0} f(\rho, \theta, \phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$
This ice-cream-cone-shaped solid is cut from the sphere of radius ρ_0 by the cone $\phi = \phi_0$.  ρ varies from 0 to ρ_0 with θ and ϕ held fixed. ϕ varies from 0 to ϕ_0 with θ held fixed. θ varies from 0 to 2π.	$\int_0^{2\pi}\int_0^{\phi_0}\int_0^{\rho_0} f(\rho, \theta, \phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$
This solid is cut from the sphere of radius ρ_0 by two cones, $\phi = \phi_1$ and $\phi = \phi_2$, where $\phi_1 < \phi_2$.  ρ varies from 0 to ρ_0 with θ and ϕ held fixed. ϕ varies from ϕ_1 to ϕ_2 with θ held fixed. θ varies from 0 to 2π.	$\int_0^{2\pi}\int_{\phi_1}^{\phi_2}\int_0^{\rho_0} f(\rho, \theta, \phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Table 14.6.1 *(continued)*

DETERMINATION OF LIMITS	INTEGRAL
This solid is enclosed laterally by the cone $\phi = \phi_0$, where $0 < \phi_0 < \pi/2$, and on top by the horizontal plane $z = a$.  ρ varies from 0 to $a \sec \phi$ with θ and ϕ held fixed. ϕ varies from 0 to ϕ_0 with θ held fixed. θ varies from 0 to 2π.	$\int_0^{2\pi} \int_0^{\phi_0} \int_0^{a \sec \phi} f(\rho, \theta, \phi)\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
This solid is enclosed between two concentric spheres, $\rho = \rho_1$ and $\rho = \rho_2$, where $\rho_1 < \rho_2$.  ρ varies from ρ_1 to ρ_2 with θ and ϕ held fixed. ϕ varies from 0 to π with θ held fixed. θ varies from 0 to 2π.	$\int_0^{2\pi} \int_0^{\pi} \int_{\rho_1}^{\rho_2} f(\rho, \theta, \phi)\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

▶ Example 3 Use spherical coordinates to find the volume of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.



▲ **Figure 14.6.11**

Solution. The solid G is sketched in Figure 14.6.11. In spherical coordinates, the equation of the sphere $x^2 + y^2 + z^2 = 16$ is $\rho = 4$ and the equation of the cone $z = \sqrt{x^2 + y^2}$ is

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta}$$

which simplifies to

$$\rho \cos \phi = \rho \sin \phi$$

Dividing both sides of this equation by $\rho \cos \phi$ yields $\tan \phi = 1$, from which it follows that

$$\phi = \pi/4$$

Thus, it follows from the second entry in Table 14.6.1 that the volume of G is

$$\begin{aligned}
V = \iiint_G dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3}{3}\sin\phi\right]_{\rho=0}^{4} d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\frac{64}{3}\sin\phi \, d\phi \, d\theta \\
&= \frac{64}{3}\int_0^{2\pi}[-\cos\phi]_{\phi=0}^{\pi/4} d\theta = \frac{64}{3}\int_0^{2\pi}\left(1-\frac{\sqrt{2}}{2}\right) d\theta \\
&= \frac{64\pi}{3}(2-\sqrt{2}) \approx 39.26 \blacktriangleleft
\end{aligned}$$

CONVERTING TRIPLE INTEGRALS FROM RECTANGULAR TO SPHERICAL COORDINATES

Referring to Table 11.8.1, triple integrals can be converted from rectangular coordinates to spherical coordinates by making the substitution $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. The two integrals are related by the equation

$$\iiint_G f(x,y,z)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi \, d\rho \, d\phi \, d\theta \tag{10}$$

Example 4 Use spherical coordinates to evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$$

Solution. In problems like this, it is helpful to begin with a sketch of the region G of integration. From the z-limits of integration, the upper surface of G is the hemisphere $z=\sqrt{4-x^2-y^2}$ and the lower surface is the xy-plane $z=0$. From the x- and y-limits of integration, the projection of the solid G on the xy-plane is the region enclosed by the circle $x^2+y^2=4$. From this information we obtain the sketch of G in Figure 14.6.12. Thus,

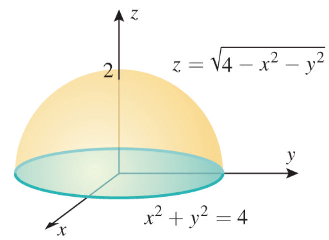


Figure 14.6.12

$$\begin{aligned}
&\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx \\
&= \iiint_G z^2\sqrt{x^2+y^2+z^2}\, dV \\
&= \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^5\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2}\frac{32}{3}\cos^2\phi\sin\phi\, d\phi\, d\theta \\
&= \frac{32}{3}\int_0^{2\pi}\left[-\frac{1}{3}\cos^3\phi\right]_{\phi=0}^{\pi/2} d\theta = \frac{32}{9}\int_0^{2\pi} d\theta = \frac{64}{9}\pi \blacktriangleleft
\end{aligned}$$

✓ QUICK CHECK EXERCISES 14.6 (See page 947 for answers.)

1. (a) The cylindrical wedge $1 \le r \le 3$, $\pi/6 \le \theta \le \pi/2$, $0 \le z \le 5$ has volume $V =$ ______.
   (b) The spherical wedge $1 \le \rho \le 3$, $\pi/6 \le \theta \le \pi/2$, $0 \le \phi \le \pi/3$ has volume $V =$ ______.

2. Let G be the solid region inside the sphere of radius 2 centered at the origin and above the plane $z=1$. In each part, supply the missing integrand and limits of integration for the iterated integral in cylindrical coordinates.
(a) The volume of G is

$$\iiint_G dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \underline{\qquad}\, dz\, dr\, d\theta$$

(b) $$\iiint_G \frac{z}{x^2+y^2+z^2}\, dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \underline{\qquad}\, dz\, dr\, d\theta$$

3. Let G be the solid region described in Quick Check Exercise 2. In each part, supply the missing integrand and limits of integration for the iterated integral in spherical coordinates.
(a) The volume of G is

$$\iiint_G dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \underline{\qquad}\, d\rho\, d\phi\, d\theta$$

(b) $$\iiint_G \frac{z}{x^2+y^2+z^2}\, dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} \underline{\qquad}\, d\rho\, d\phi\, d\theta$$

EXERCISE SET 14.6 C CAS

1–4 Evaluate the iterated integral.

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} zr\, dz\, dr\, d\theta$

2. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{r^2} r\sin\theta\, dz\, dr\, d\theta$

3. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \rho^3\sin\phi\cos\phi\, d\rho\, d\phi\, d\theta$

4. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a\sec\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \quad (a>0)$

FOCUS ON CONCEPTS

5. Sketch the region G and identify the function f so that
$$\iiint_G f(r,\theta,z)\, dV$$
corresponds to the iterated integral in Exercise 1.

6. Sketch the region G and identify the function f so that
$$\iiint_G f(r,\theta,z)\, dV$$
corresponds to the iterated integral in Exercise 2.

7. Sketch the region G and identify the function f so that
$$\iiint_G f(\rho,\theta,\phi)\, dV$$
corresponds to the iterated integral in Exercise 3.

8. Sketch the region G and identify the function f so that
$$\iiint_G f(\rho,\theta,\phi)\, dV$$
corresponds to the iterated integral in Exercise 4.

9–12 Use cylindrical coordinates to find the volume of the solid.

9. The solid enclosed by the paraboloid $z=x^2+y^2$ and the plane $z=9$.

10. The solid bounded above by the sphere $x^2+y^2+z^2=1$ and bounded below by the cone $z=\sqrt{x^2+y^2}$.

11. The solid that is inside the surface $r^2+z^2=20$ but not above the surface $z=r^2$.

12. The solid enclosed between the cone $z=(hr)/a$ and the plane $z=h$.

13–16 Use spherical coordinates to find the volume of the solid.

13. The solid bounded above by the sphere $\rho=4$ and below by the cone $\phi=\pi/3$.

14. The solid within the cone $\phi=\pi/4$ and between the spheres $\rho=1$ and $\rho=2$.

15. The solid enclosed by the sphere $x^2+y^2+z^2=4a^2$ and the planes $z=0$ and $z=a$.

16. The solid within the sphere $x^2+y^2+z^2=9$, outside the cone $z=\sqrt{x^2+y^2}$, and above the xy-plane.

17–20 Use cylindrical or spherical coordinates to evaluate the integral.

17. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{a^2-x^2-y^2} x^2\, dz\, dy\, dx \quad (a>0)$

18. $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$

19. $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2\, dz\, dx\, dy$

20. $\displaystyle\int_{-3}^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. A rectangular triple integral can be expressed as an iterated integral in cylindrical coordinates as
$$\iiint_G f(x,y,z)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z)r^2\, dz\, dr\, d\theta$$

22. If $0\le\rho_1<\rho_2$, $0\le\theta_1<\theta_2\le 2\pi$, and $0\le\phi_1<\phi_2\le\pi$, then the volume of the spherical wedge bounded by the

spheres $\rho = \rho_1$ and $\rho = \rho_2$, the half-planes $\theta = \theta_1$ and $\theta = \theta_2$, and the cones $\phi = \phi_1$ and $\phi = \phi_2$ is

$$\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

23. Let G be the solid region in 3-space between the spheres of radius 1 and 3 centered at the origin and above the cone $z = \sqrt{x^2 + y^2}$. The volume of G equals

$$\int_0^{\pi/4} \int_0^{2\pi} \int_1^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

24. If G is the solid in Exercise 23 and $f(x, y, z)$ is continuous on G, then

$$\iiint_G f(x, y, z) \, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_1^3 F(\rho, \theta, \phi) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where $F(\rho, \theta, \phi) = f(\rho \sin\phi \sin\theta, \rho \sin\phi \cos\theta, \rho \cos\phi)$.

C **25.** (a) Use a CAS to evaluate

$$\int_{-2}^{2} \int_1^4 \int_{\pi/6}^{\pi/3} \frac{r \tan^3\theta}{\sqrt{1 + z^2}} \, d\theta \, dr \, dz$$

(b) Find a function $f(x, y, z)$ and sketch a region G in 3-space so that the triple integral in rectangular coordinates

$$\iiint_G f(x, y, z) \, dV$$

matches the iterated integral in cylindrical coordinates given in part (a).

C **26.** Use a CAS to evaluate

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\cos\theta} \rho^{17} \cos\phi \cos^{19}\theta \, d\rho \, d\phi \, d\theta$$

27. Find the volume enclosed by $x^2 + y^2 + z^2 = a^2$ using
(a) cylindrical coordinates
(b) spherical coordinates.

28. Let G be the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the coordinate planes. Evaluate

$$\iiint_G xyz \, dV$$

(a) using rectangular coordinates
(b) using cylindrical coordinates
(c) using spherical coordinates.

29. Find the volume of the solid in the first octant bounded by the sphere $\rho = 2$, the coordinate planes, and the cones $\phi = \pi/6$ and $\phi = \pi/3$.

30. In this exercise we will obtain a formula for the volume of the spherical wedge illustrated in Figures 14.6.7 and 14.6.9.
(a) Use a triple integral in cylindrical coordinates to show that the volume of the solid bounded above by a sphere $\rho = \rho_0$, below by a cone $\phi = \phi_0$, and on the sides by $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$ is

$$V = \tfrac{1}{3}\rho_0^3(1 - \cos\phi_0)(\theta_2 - \theta_1)$$

[*Hint:* In cylindrical coordinates, the sphere has the equation $r^2 + z^2 = \rho_0^2$ and the cone has the equation $z = r\cot\phi_0$. For simplicity, consider only the case $0 < \phi_0 < \pi/2$.]

(b) Subtract appropriate volumes and use the result in part (a) to deduce that the volume ΔV of the spherical wedge is

$$\Delta V = \frac{\rho_2^3 - \rho_1^3}{3}(\cos\phi_1 - \cos\phi_2)(\theta_2 - \theta_1)$$

(c) Apply the Mean-Value Theorem to the functions $\cos\phi$ and ρ^3 to deduce that the formula in part (b) can be written as

$$\Delta V = \rho^{*2} \sin\phi^* \, \Delta\rho \, \Delta\phi \, \Delta\theta$$

where $\Delta\rho = \rho_2 - \rho_1$, $\Delta\phi = \phi_2 - \phi_1$, $\Delta\theta = \theta_2 - \theta_1$, and ρ^* are between ρ_1 and ρ_2, ϕ^* is between ϕ_1 and ϕ_2.

31. Writing Suppose that a triple integral is expressed in cylindrical or spherical coordinates in such a way that the outermost variable of integration is θ and none of the limits of integration involves θ. Discuss what this says about the region of integration for the integral.

✓ **QUICK CHECK ANSWERS 14.6** 1. (a) $\frac{20}{3}\pi$ (b) $\frac{13}{9}\pi$ 2. (a) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$ (b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} \frac{rz}{r^2 + z^2} \, dz \, dr \, d\theta$ 3. (a) $\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ (b) $\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^{2} \rho\cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$

14.7 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS; JACOBIANS

In this section we will discuss a general method for evaluating double and triple integrals by substitution. Most of the results in this section are very difficult to prove, so our approach will be informal and motivational. Our goal is to provide a geometric understanding of the basic principles and an exposure to computational techniques.

CHANGE OF VARIABLE IN A SINGLE INTEGRAL

To motivate techniques for evaluating double and triple integrals by substitution, it will be helpful to consider the effect of a substitution $x = g(u)$ on a single integral over an interval $[a, b]$. If g is differentiable and either increasing or decreasing, then g is one-to-one and

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,du$$

In this relationship $f(x)$ and dx are expressed in terms of u, and the u-limits of integration result from solving the equations

$$a = g(u) \quad \text{and} \quad b = g(u)$$

In the case where g is decreasing we have $g^{-1}(b) < g^{-1}(a)$, which is contrary to our usual convention of writing definite integrals with the larger limit of integration at the top. We can remedy this by reversing the limits of integration and writing

$$\int_a^b f(x)\,dx = -\int_{g^{-1}(b)}^{g^{-1}(a)} f(g(u))g'(u)\,du = \int_{g^{-1}(b)}^{g^{-1}(a)} f(g(u))|g'(u)|\,du$$

where the absolute value results from the fact that $g'(u)$ is negative. Thus, regardless of whether g is increasing or decreasing we can write

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(u))|g'(u)|\,du \tag{1}$$

where α and β are the u-limits of integration and $\alpha < \beta$.

The expression $g'(u)$ that appears in (1) is called the ***Jacobian*** of the change of variable $x = g(u)$ in honor of C. G. J. Jacobi, who made the first serious study of change of variables in multiple integrals in the mid-1800s. Formula (1) reveals three effects of the change of variable $x = g(u)$:

- The new integrand becomes $f(g(u))$ times the absolute value of the Jacobian.
- dx becomes du.
- The x-interval of integration is transformed into a u-interval of integration.

Our goal in this section is to show that analogous results hold for changing variables in double and triple integrals.

TRANSFORMATIONS OF THE PLANE

In earlier sections we considered parametric equations of three kinds:

$$x = x(t), \quad y = y(t)$$
A curve in the plane

$$x = x(t), \quad y = y(t), \quad z = z(t)$$
A curve in 3-space

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$
A surface in 3-space

Now we will consider parametric equations of the form

$$x = x(u, v), \quad y = y(u, v) \tag{2}$$

Parametric equations of this type associate points in the xy-plane with points in the uv-plane. These equations can be written in vector form as

$$\mathbf{r} = \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is a position vector in the xy-plane and $\mathbf{r}(u, v)$ is a vector-valued function of the variables u and v.

It will also be useful in this section to think of the parametric equations in (2) in terms of inputs and outputs. If we think of the pair of numbers (u, v) as an input, then the two equations, in combination, produce a unique output (x, y), and hence define a function T that associates points in the xy-plane with points in the uv-plane. This function is described by the formula

$$T(u, v) = (x(u, v), y(u, v))$$

We call T a ***transformation*** from the uv-plane to the xy-plane and (x, y) the ***image*** of (u, v) under the transformation T. We also say that T ***maps*** (u, v) into (x, y). The set R of all images in the xy-plane of a set S in the uv-plane is called the ***image of S under T***. If distinct points in the uv-plane have distinct images in the xy-plane, then T is said to be ***one-to-one***. In this case the equations in (2) define u and v as functions of x and y, say

$$u = u(x, y), \quad v = v(x, y)$$

These equations, which can often be obtained by solving (2) for u and v in terms of x and y, define a transformation from the xy-plane to the uv-plane that maps the image of (u, v) under T back into (u, v). This transformation is denoted by T^{-1} and is called the ***inverse of T*** (Figure 14.7.1).

Because there are four variables involved, a three-dimensional figure is not very useful for describing the transformation geometrically. The idea here is to use the two planes to get the four dimensions needed.

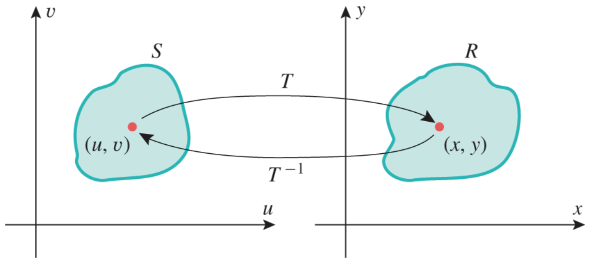


Figure 14.7.1

One way to visualize the geometric effect of a transformation T is to determine the images in the xy-plane of the vertical and horizontal lines in the uv-plane. Following the discussion in Section 14.4, sets of points in the xy-plane that are images of horizontal lines (v constant) are called ***constant v-curves***, and sets of points that are images of vertical lines (u constant) are called ***constant u-curves*** (Figure 14.7.2).

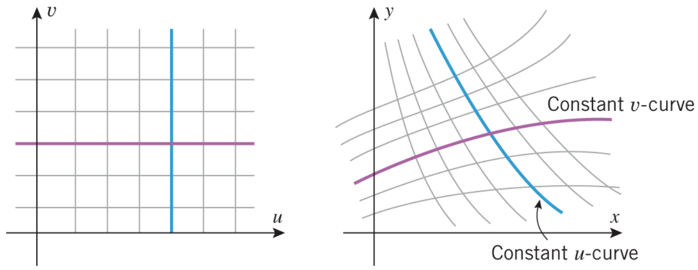
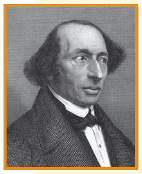


Figure 14.7.2



Carl Gustav Jacob Jacobi (1804–1851) German mathematician. Jacobi, the son of a banker, grew up in a background of wealth and culture and showed brilliance in mathematics early. He resisted studying mathematics by rote, preferring instead to learn general principles from the works of the masters, Euler and Lagrange. He entered the University of Berlin at age 16 as a student of mathematics and classical studies. However, he soon realized that he could not do both and turned fully to mathematics with a blazing intensity that he would maintain throughout his life. He received his Ph.D. in 1825 and was able to secure a position as a lecturer at the University of Berlin by giving up Judaism and becoming a Christian. However, his promotion opportunities remained limited and he moved on to the University of Königsberg. Jacobi was born to teach—he had a dynamic personality and delivered his lectures with a clarity and enthusiasm that frequently left his audience spellbound. In spite of extensive teaching commitments, he was able to publish volumes of revolutionary mathematical research that eventually made him the leading European mathematician after Gauss. His main body of research was in the area of elliptic functions, a branch of mathematics with important applications in astronomy and physics as well as in other fields of mathematics. Because of his family wealth, Jacobi was not dependent on his teaching salary in his early years. However, his comfortable world eventually collapsed. In 1840 his family went bankrupt and he was wiped out financially. In 1842 he had a nervous breakdown from overwork. In 1843 he became seriously ill with diabetes and moved to Berlin with the help of a government grant to defray his medical expenses. In 1848 he made an injudicious political speech that caused the government to withdraw the grant, eventually resulting in the loss of his home. His health continued to decline and in 1851 he finally succumbed to successive bouts of influenza and smallpox. In spite of all his problems, Jacobi was a tireless worker to the end. When a friend expressed concern about the effect of the hard work on his health, Jacobi replied, "Certainly, I have sometimes endangered my health by overwork, but what of it? Only cabbages have no nerves, no worries. And what do they get out of their perfect well-being?"

Example 1 Let T be the transformation from the uv-plane to the xy-plane defined by the equations

$$x = \tfrac{1}{4}(u + v), \quad y = \tfrac{1}{2}(u - v) \tag{3}$$

(a) Find $T(1, 3)$.

(b) Sketch the constant v-curves corresponding to $v = -2, -1, 0, 1, 2$.

(c) Sketch the constant u-curves corresponding to $u = -2, -1, 0, 1, 2$.

(d) Sketch the image under T of the square region in the uv-plane bounded by the lines $u = -2$, $u = 2$, $v = -2$, and $v = 2$.

Solution (a). Substituting $u = 1$ and $v = 3$ in (3) yields $T(1, 3) = (1, -1)$.

Solutions (b and c). In these parts it will be convenient to express the transformation equations with u and v as functions of x and y. From (3)

$$4x = u + v, \quad 2y = u - v$$

Combining these equations gives

$$4x + 2y = 2u, \quad 4x - 2y = 2v$$

or

$$2x + y = u, \quad 2x - y = v$$

Thus, the constant v-curves corresponding to $v = -2, -1, 0, 1$, and 2 are

$$2x - y = -2, \quad 2x - y = -1, \quad 2x - y = 0, \quad 2x - y = 1, \quad 2x - y = 2$$

and the constant u-curves corresponding to $u = -2, -1, 0, 1$, and 2 are

$$2x + y = -2, \quad 2x + y = -1, \quad 2x + y = 0, \quad 2x + y = 1, \quad 2x + y = 2$$

In Figure 14.7.3 the constant v-curves are shown in purple and the constant u-curves in blue.

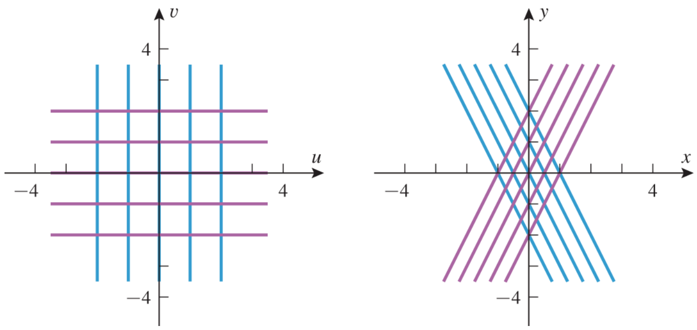


Figure 14.7.3

Solution (d). The image of a region can often be found by finding the image of its boundary. In this case the images of the boundary lines $u = -2$, $u = 2$, $v = -2$, and $v = 2$ enclose the diamond-shaped region in the xy-plane shown in Figure 14.7.4. ◀

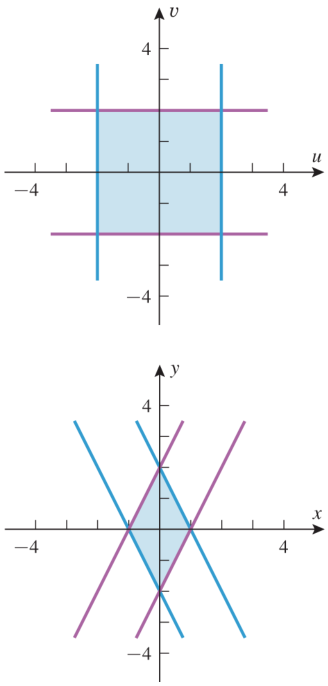


Figure 14.7.4

JACOBIANS IN TWO VARIABLES

To derive the change of variables formula for double integrals, we will need to understand the relationship between the area of a *small* rectangular region in the uv-plane and the area of its image in the xy-plane under a transformation T given by the equations

$$x = x(u, v), \quad y = y(u, v)$$

For this purpose, suppose that Δu and Δv are positive, and consider a rectangular region S in the uv-plane enclosed by the lines

$$u = u_0, \quad u = u_0 + \Delta u, \quad v = v_0, \quad v = v_0 + \Delta v$$

If the functions $x(u, v)$ and $y(u, v)$ are continuous, and if Δu and Δv are not too large, then the image of S in the xy-plane will be a region R that looks like a slightly distorted parallelogram (Figure 14.7.5). The sides of R are the constant u-curves and v-curves that correspond to the sides of S.

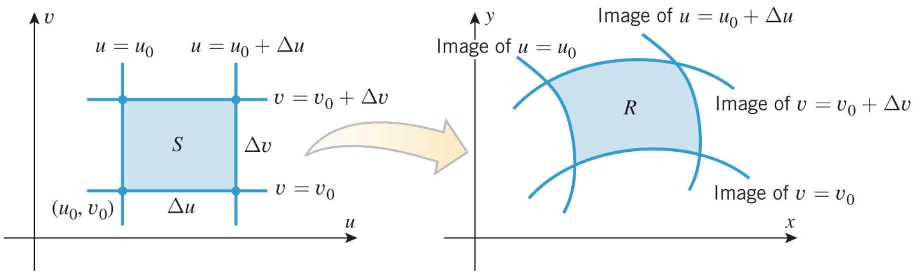


Figure 14.7.5

If we let

$$\mathbf{r} = \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j}$$

be the position vector of the point in the xy-plane that corresponds to the point (u, v) in the uv-plane, then the constant v-curve corresponding to $v = v_0$ and the constant u-curve corresponding to $u = u_0$ can be represented in vector form as

$$\mathbf{r}(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} \qquad \boxed{\text{Constant } v\text{-curve}}$$

$$\mathbf{r}(u_0, v) = x(u_0, v)\mathbf{i} + y(u_0, v)\mathbf{j} \qquad \boxed{\text{Constant } u\text{-curve}}$$

Since we are assuming Δu and Δv to be small, the region R can be approximated by a parallelogram determined by the "secant vectors"

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \tag{4}$$

$$\mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \tag{5}$$

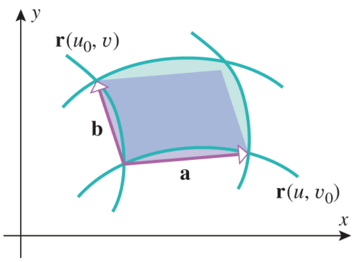


Figure 14.7.6

shown in Figure 14.7.6. A more useful approximation of R can be obtained by using Formulas (7) and (8) of Section 14.4 to approximate these secant vectors by tangent vectors as follows:

$$\begin{aligned}
\mathbf{a} &= \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}\Delta u \\
&\approx \frac{\partial \mathbf{r}}{\partial u}\Delta u = \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}\right)\Delta u \\
\mathbf{b} &= \frac{\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)}{\Delta v}\Delta v \\
&\approx \frac{\partial \mathbf{r}}{\partial v}\Delta v = \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}\right)\Delta v
\end{aligned}$$

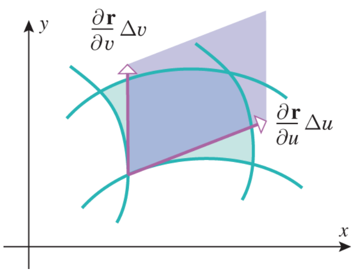


Figure 14.7.7

where the partial derivatives are evaluated at (u_0, v_0) (Figure 14.7.7). Hence, it follows that the area of the region R, which we will denote by ΔA, can be approximated by the area of the parallelogram determined by these vectors. Thus, from Theorem 11.4.5(b) we have

$$\Delta A \approx \left\|\frac{\partial \mathbf{r}}{\partial u}\Delta u \times \frac{\partial \mathbf{r}}{\partial v}\Delta v\right\| = \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\|\Delta u\,\Delta v \tag{6}$$

where the derivatives are evaluated at (u_0, v_0). Computing the cross product, we obtain

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k} \tag{7}$$

The determinant in (7) is sufficiently important that it has its own terminology and notation.

14.7.1 DEFINITION If T is the transformation from the uv-plane to the xy-plane defined by the equations $x = x(u, v)$, $y = y(u, v)$, then the ***Jacobian of T*** is denoted by $J(u, v)$ or by $\partial(x, y)/\partial(u, v)$ and is defined by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Using the notation in this definition, it follows from (6) and (7) that

$$\Delta A \approx \left\| \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k} \right\| \Delta u \, \Delta v$$

or, since **k** is a unit vector,

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v \tag{8}$$

At the point (u_0, v_0) this important formula relates the areas of the regions R and S in Figure 14.7.5; it tells us that *for small values of Δu and Δv, the area of R is approximately the absolute value of the Jacobian times the area of S*. Moreover, it is proved in advanced calculus courses that the relative error in the approximation approaches zero as $\Delta u \to 0$ and $\Delta v \to 0$.

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Our next objective is to provide a geometric motivation for the following result.

14.7.2 CHANGE OF VARIABLES FORMULA FOR DOUBLE INTEGRALS If the transformation $x = x(u, v)$, $y = y(u, v)$ maps the region S in the uv-plane into the region R in the xy-plane, and if the Jacobian $\partial(x, y)/\partial(u, v)$ is nonzero and does not change sign on S, then with appropriate restrictions on the transformation and the regions it follows that

$$\iint_R f(x, y)\, dA_{xy} = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA_{uv} \tag{9}$$

where we have attached subscripts to the dA's to help identify the associated variables.

A precise statement of conditions under which Formula (9) holds is beyond the scope of this course. Suffice it to say that the formula holds if T is a one-to-one transformation, $f(x, y)$ is continuous on R, the partial derivatives of $x(u, v)$ and $y(u, v)$ exist and are continuous on S, and the regions R and S are not complicated.

To motivate Formula (9), we proceed as follows:

- Subdivide the region S in the uv-plane into pieces by lines parallel to the coordinate axes, and exclude from consideration any pieces that contain points outside of S. This leaves only rectangular regions that are subsets of S. Assume that there are n such regions and denote the kth such region by S_k. Assume that S_k has dimensions Δu_k by Δv_k and, as shown in Figure 14.7.8*a*, let (u_k^*, v_k^*) be its "lower left corner."
- As shown in Figure 14.7.8*b*, the transformation T defined by the coordinate equations $x = x(u, v)$, $y = y(u, v)$ maps S_k into a curvilinear parallelogram R_k in the xy-plane and maps the point (u_k^*, v_k^*) into the point $(x_k^*, y_k^*) = (x(u_k^*, v_k^*),\ y(u_k^*, v_k^*))$ in R_k. Denote the area of R_k by ΔA_k.
- In rectangular coordinates the double integral of $f(x, y)$ over a region R is defined as a limit of Riemann sums in which R is subdivided into *rectangular* subregions. It is proved in advanced calculus courses that under appropriate conditions subdivisions

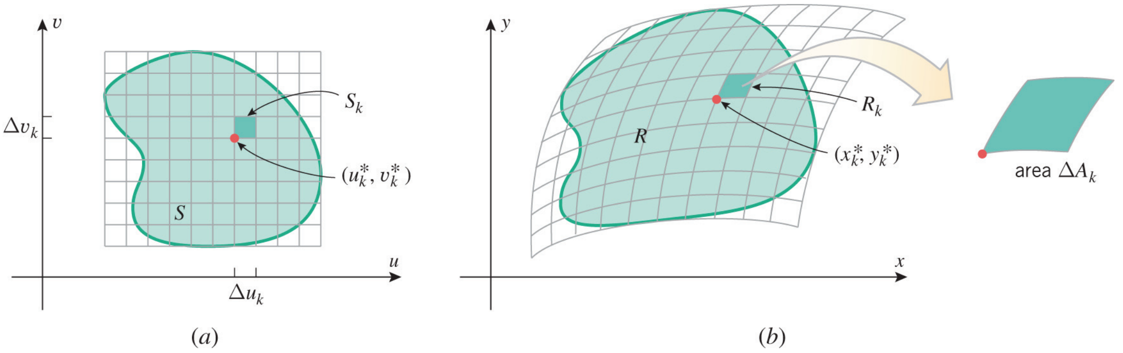
into *curvilinear* parallelograms can be used instead. Accepting this to be so, we can approximate the double integral of $f(x, y)$ over R as

$$\begin{aligned}\iint_R f(x,y)\,dA_{xy} &\approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k \\ &\approx \sum_{k=1}^{n} f(x(u_k^*, v_k^*), y(u_k^*, v_k^*)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u_k\,\Delta v_k\end{aligned}$$

where the Jacobian is evaluated at (u_k^*, v_k^*). But the last expression is a Riemann sum for the integral

$$\iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$$

so Formula (9) follows if we assume that the errors in the approximations approach zero as $n \to +\infty$.

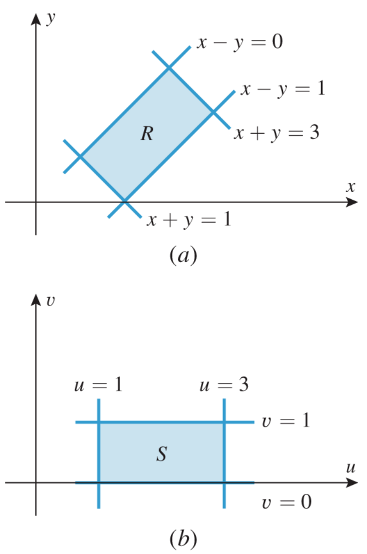


▲ **Figure 14.7.8**

▶ Example 2 Evaluate

$$\iint_R \frac{x-y}{x+y}\,dA$$

where R is the region enclosed by $x - y = 0$, $x - y = 1$, $x + y = 1$, and $x + y = 3$ (Figure 14.7.9a).



▲ **Figure 14.7.9**

Solution. This integral would be tedious to evaluate directly because the region R is oriented in such a way that we would have to subdivide it and integrate over each part separately. However, the occurrence of the expressions $x - y$ and $x + y$ in the equations of the boundary suggests that the transformation

$$u = x + y, \quad v = x - y \tag{10}$$

would be helpful, since with this transformation the boundary lines

$$x + y = 1, \quad x + y = 3, \quad x - y = 0, \quad x - y = 1$$

are constant u-curves and constant v-curves corresponding to the lines

$$u = 1, \quad u = 3, \quad v = 0, \quad v = 1$$

in the uv-plane. These lines enclose the rectangular region S shown in Figure 14.7.9b. To find the Jacobian $\partial(x, y)/\partial(u, v)$ of this transformation, we first solve (10) for x and y in terms of u and v. This yields

$$x = \tfrac{1}{2}(u + v), \quad y = \tfrac{1}{2}(u - v)$$

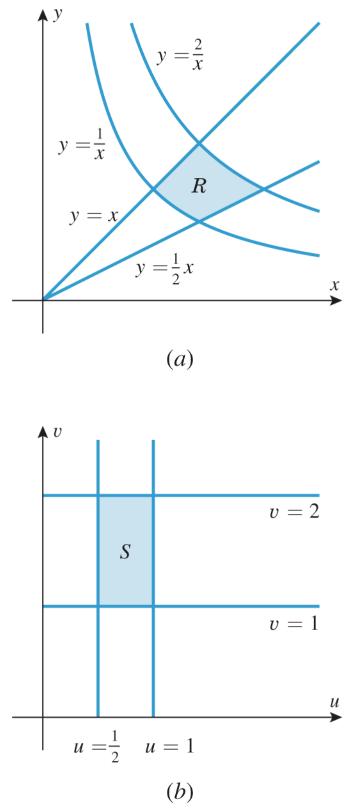
from which we obtain

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\tfrac{1}{4} - \tfrac{1}{4} = -\tfrac{1}{2}$$

Thus, from Formula (9), but with the notation dA rather than dA_{xy},

$$\begin{aligned}
\iint_R \frac{x-y}{x+y}\,dA &= \iint_S \frac{v}{u}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv} \\
&= \iint_S \frac{v}{u}\left|-\frac{1}{2}\right| dA_{uv} = \frac{1}{2}\int_0^1\int_1^3 \frac{v}{u}\,du\,dv \\
&= \frac{1}{2}\int_0^1 v\ln|u|\Big]_{u=1}^{3}\,dv \\
&= \frac{1}{2}\ln 3\int_0^1 v\,dv = \frac{1}{4}\ln 3 \blacktriangleleft
\end{aligned}$$

The underlying idea illustrated in Example 2 is to find a one-to-one transformation that maps a rectangle S in the uv-plane into the region R of integration, and then use that transformation as a substitution in the integral to produce an equivalent integral over S.



▲ **Figure 14.7.10**

▶ Example 3 Evaluate

$$\iint_R e^{xy}\,dA$$

where R is the region enclosed by the lines $y = \frac{1}{2}x$ and $y = x$ and the hyperbolas $y = 1/x$ and $y = 2/x$ (Figure 14.7.10a).

Solution. As in the last example, we look for a transformation in which the boundary curves in the xy-plane become constant v-curves and constant u-curves. For this purpose we rewrite the four boundary curves as

$$\frac{y}{x} = \frac{1}{2}, \quad \frac{y}{x} = 1, \quad xy = 1, \quad xy = 2$$

which suggests the transformation

$$u = \frac{y}{x}, \quad v = xy \tag{11}$$

With this transformation the boundary curves in the xy-plane are constant u-curves and constant v-curves corresponding to the lines

$$u = \tfrac{1}{2}, \quad u = 1, \quad v = 1, \quad v = 2$$

in the uv-plane. These lines enclose the region S shown in Figure 14.7.10b. To find the Jacobian $\partial(x,y)/\partial(u,v)$ of this transformation, we first solve (11) for x and y in terms of u and v. This yields

$$x = \sqrt{v/u}, \quad y = \sqrt{uv}$$

from which we obtain

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{2u}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2\sqrt{uv}} \\[2mm] \dfrac{1}{2}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2}\sqrt{\dfrac{u}{v}} \end{vmatrix} = -\frac{1}{4u} - \frac{1}{4u} = -\frac{1}{2u}$$

Thus, from Formula (9), but with the notation dA rather than dA_{xy},

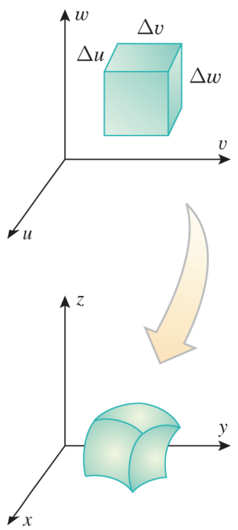
$$\begin{aligned}
\iint_R e^{xy}\,dA &= \iint_S e^{v}\left|-\frac{1}{2u}\right| dA_{uv} = \frac{1}{2}\iint_S \frac{1}{u}e^{v}\,dA_{uv} \\
&= \frac{1}{2}\int_1^2\int_{1/2}^1 \frac{1}{u}e^{v}\,du\,dv = \frac{1}{2}\int_1^2 e^{v}\ln|u|\Big]_{u=1/2}^{1}\,dv \\
&= \frac{1}{2}\ln 2\int_1^2 e^{v}\,dv = \frac{1}{2}(e^2-e)\ln 2 \blacktriangleleft
\end{aligned}$$

CHANGE OF VARIABLES IN TRIPLE INTEGRALS

Equations of the form

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w) \tag{12}$$

define a ***transformation*** T from uvw-space to xyz-space. Just as a transformation $x = x(u, v)$, $y = y(u, v)$ in two variables maps small rectangles in the uv-plane into curvilinear parallelograms in the xy-plane, so (12) maps small rectangular parallelepipeds in uvw-space into curvilinear parallelepipeds in xyz-space (Figure 14.7.11). The definition of the Jacobian of (12) is similar to Definition 14.7.1.



▲ Figure 14.7.11

14.7.3 DEFINITION If T is the transformation from uvw-space to xyz-space defined by the equations $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, then the ***Jacobian of T*** is denoted by $J(u, v, w)$ or $\partial(x, y, z)/\partial(u, v, w)$ and is defined by

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

For small values of Δu, Δv, and Δw, the volume ΔV of the curvilinear parallelepiped in Figure 14.7.11 is related to the volume $\Delta u\, \Delta v\, \Delta w$ of the rectangular parallelepiped by

$$\Delta V \approx \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \Delta u\, \Delta v\, \Delta w \tag{13}$$

which is the analog of Formula (8). Using this relationship and an argument similar to the one that led to Formula (9), we can obtain the following result.

14.7.4 CHANGE OF VARIABLES FORMULA FOR TRIPLE INTEGRALS If the transformation $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ maps the region S in uvw-space into the region R in xyz-space, and if the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ is nonzero and does not change sign on S, then with appropriate restrictions on the transformation and the regions it follows that

$$\iiint\limits_R f(x, y, z)\, dV_{xyz} = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV_{uvw} \tag{14}$$

▶ **Example 4** Find the volume of the region G enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution. The volume V is given by the triple integral

$$V = \iiint\limits_G dV$$

To evaluate this integral, we make the change of variables

$$x = au, \quad y = bv, \quad z = cw \tag{15}$$

which maps the region S in uvw-space enclosed by a sphere of radius 1 into the region G in xyz-space. This can be seen from (15) by noting that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{becomes} \quad u^2 + v^2 + w^2 = 1$$

The Jacobian of (15) is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Thus, from Formula (14), but with the notation dV rather than dV_{xyz},

$$V = \iiint\limits_G dV = \iiint\limits_S \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dV_{uvw} = abc \iiint\limits_S dV_{uvw}$$

The last integral is the volume enclosed by a sphere of radius 1, which we know to be $\frac{4}{3}\pi$. Thus, the volume enclosed by the ellipsoid is $V = \frac{4}{3}\pi abc$. ◀

Jacobians also arise in converting triple integrals in rectangular coordinates to iterated integrals in cylindrical and spherical coordinates. For example, we will ask you to show in Exercise 48 that the Jacobian of the transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

is

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

and the Jacobian of the transformation

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

is

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2\sin\phi$$

Thus, Formulas (6) and (10) of Section 14.6 can be expressed in terms of Jacobians as

$$\iiint\limits_G f(x, y, z)\, dV = \iiint\limits_{\substack{\text{appropriate} \\ \text{limits}}} f(r\cos\theta, r\sin\theta, z)\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\, dz\, dr\, d\theta \tag{16}$$

The absolute-value signs are omitted from Formulas (16) and (17) because the Jacobians are nonnegative (see the restrictions in Table 11.8.1).

$$\iiint\limits_G f(x, y, z)\, dV = \iiint\limits_{\substack{\text{appropriate} \\ \text{limits}}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}\, d\rho\, d\phi\, d\theta \tag{17}$$

✓ QUICK CHECK EXERCISES 14.7 (See page 959 for answers.)

1. Let T be the transformation from the uv-plane to the xy-plane defined by the equations

$$x = u - 2v, \quad y = 3u + v$$

(a) Sketch the image under T of the rectangle $1 \le u \le 3$, $0 \le v \le 2$.

(b) Solve for u and v in terms of x and y:

$$u = \underline{\qquad}, \quad v = \underline{\qquad}$$

2. State the relationship between R and S in the change of variables formula

$$\iint\limits_R f(x, y)\, dA_{xy} = \iint\limits_S f(x(u, v), y(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA_{uv}$$

3. Let T be the transformation in Quick Check Exercise 1.

(a) The Jacobian $\partial(x, y)/\partial(u, v)$ of T is $\underline{\qquad}$.

(b) Let R be the region in Quick Check Exercise 1(a). Fill in the missing integrand and limits of integration for the change of variables given by T.

$$\iint\limits_R e^{x+2y}\, dA = \int_{\square}^{\square}\int_{\square}^{\square} \underline{\qquad}\, du\, dv$$

4. The Jacobian of the transformation

$$x = uv, \quad y = vw, \quad z = 2w$$

is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \underline{\qquad}$$

EXERCISE SET 14.7

1–4 Find the Jacobian $\partial(x, y)/\partial(u, v)$.

1. $x = u + 4v,\ y = 3u - 5v$
2. $x = u + 2v^2,\ y = 2u^2 - v$
3. $x = \sin u + \cos v,\ y = -\cos u + \sin v$
4. $x = \dfrac{2u}{u^2 + v^2},\ y = -\dfrac{2v}{u^2 + v^2}$

5–8 Solve for x and y in terms of u and v, and then find the Jacobian $\partial(x, y)/\partial(u, v)$.

5. $u = 2x - 5y,\ v = x + 2y$
6. $u = e^x,\ v = ye^{-x}$
7. $u = x^2 - y^2,\ v = x^2 + y^2 \quad (x > 0, y > 0)$
8. $u = xy,\ v = xy^3 \quad (x > 0, y > 0)$

9–12 Find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$.

9. $x = 3u + v,\ y = u - 2w,\ z = v + w$
10. $x = u - uv,\ y = uv - uvw,\ z = uvw$
11. $u = xy,\ v = y,\ w = x + z$
12. $u = x + y + z,\ v = x + y - z,\ w = x - y + z$

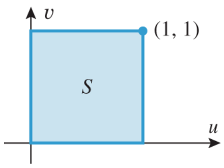
13–16 True–False Determine whether the statement is true or false. Explain your answer.

13. If $\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j}$, then evaluating $|\partial(x, y)/\partial(u, v)|$ at a point (u_0, v_0) gives the perimeter of the parallelogram generated by the vectors $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ at (u_0, v_0).
14. If $\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j}$ maps the rectangle $0 \le u \le 2$, $1 \le v \le 5$ to a region R in the xy-plane, then the area of R is given by
$$\int_1^5 \int_0^2 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$
15. The Jacobian of the transformation $x = r\cos\theta, y = r\sin\theta$ is
$$\frac{\partial(x, y)}{\partial(r, \theta)} = r^2$$
16. The Jacobian of the transformation $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$ is
$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2\sin\phi$$

FOCUS ON CONCEPTS

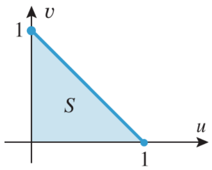
17–20 Sketch the image in the xy-plane of the set S under the given transformation.

17.



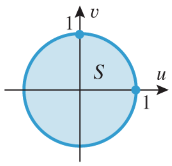
$x = u^2 - v^2$
$y = 2uv$

18.



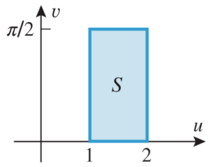
$x = 3u + 4v$
$y = 4u$

19.



$x = 2u$
$y = 3v$

20.



$x = u\cos v$
$y = u\sin v$

21. Use the transformation $u = x - 2y, v = 2x + y$ to find
$$\iint_R \frac{x - 2y}{2x + y}\, dA$$
where R is the rectangular region enclosed by the lines $x - 2y = 1, x - 2y = 4, 2x + y = 1, 2x + y = 3$.
22. Use the transformation $u = x + y, v = x - y$ to find
$$\iint_R (x - y)e^{x^2 - y^2}\, dA$$
over the rectangular region R enclosed by the lines $x + y = 0, x + y = 1, x - y = 1, x - y = 4$.
23. Use the transformation $u = \frac{1}{2}(x + y), v = \frac{1}{2}(x - y)$ to find
$$\iint_R \sin\tfrac{1}{2}(x + y)\cos\tfrac{1}{2}(x - y)\, dA$$
over the triangular region R with vertices $(0, 0)$, $(2, 0)$, $(1, 1)$.
24. Use the transformation $u = y/x, v = xy$ to find
$$\iint_R xy^3\, dA$$
over the region R in the first quadrant enclosed by $y = x$, $y = 3x, xy = 1, xy = 4$.

25–27 The transformation $x = au, y = bv$ $(a > 0, b > 0)$ can be rewritten as $x/a = u, y/b = v$, and hence it maps the circular region
$$u^2 + v^2 \le 1$$
into the elliptical region
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$$
In these exercises, perform the integration by transforming the elliptical region of integration into a circular region of integration and then evaluating the transformed integral in polar coordinates.

25. $\displaystyle\iint_R \sqrt{16x^2 + 9y^2}\, dA$, where R is the region enclosed by the ellipse $(x^2/9) + (y^2/16) = 1$.
26. $\displaystyle\iint_R e^{-(x^2 + 4y^2)}\, dA$, where R is the region enclosed by the ellipse $(x^2/4) + y^2 = 1$.

27. $\displaystyle\iint_R \sin(4x^2 + 9y^2)\, dA$, where R is the region in the first quadrant enclosed by the ellipse $4x^2 + 9y^2 = 1$ and the coordinate axes.

28. Show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.

29–30 If $a, b,$ and c are positive constants, then the transformation $x = au,\ y = bv,\ z = cw$ can be rewritten as $x/a = u$, $y/b = v$, $z/c = w$, and hence it maps the spherical region

$$u^2 + v^2 + w^2 \le 1$$

into the ellipsoidal region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

In these exercises, perform the integration by transforming the ellipsoidal region of integration into a spherical region of integration and then evaluating the transformed integral in spherical coordinates.

29. $\displaystyle\iiint_G x^2\, dV$, where G is the region enclosed by the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.

30. $\displaystyle\iiint_G (y^2 + z^2)\, dV$, where G is the region enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

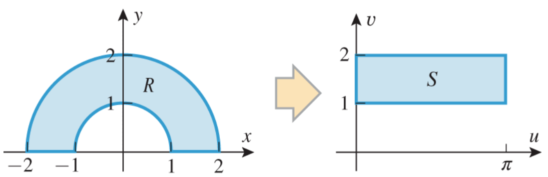
FOCUS ON CONCEPTS

31–34 Find a transformation

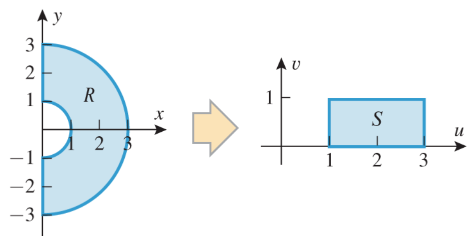
$$u = f(x, y), \quad v = g(x, y)$$

that when applied to the region R in the xy-plane has as its image the region S in the uv-plane.

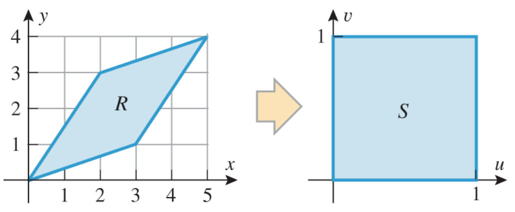
31.



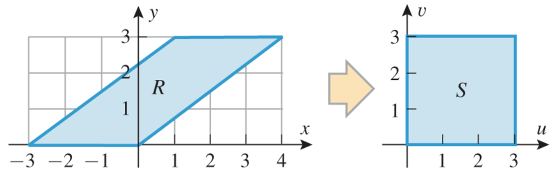
32.



33.



34.



35–38 Evaluate the integral by making an appropriate change of variables.

35. $\displaystyle\iint_R \frac{y - 4x}{y + 4x}\, dA$, where R is the region enclosed by the lines $y = 4x,\ y = 4x + 2,\ y = 2 - 4x,\ y = 5 - 4x$.

36. $\displaystyle\iint_R (x^2 - y^2)\, dA$, where R is the rectangular region enclosed by the lines $y = -x,\ y = 1 - x,\ y = x,\ y = x + 2$.

37. $\displaystyle\iint_R \frac{\sin(x - y)}{\cos(x + y)}\, dA$, where R is the triangular region enclosed by the lines $y = 0,\ y = x,\ x + y = \pi/4$.

38. $\displaystyle\iint_R e^{(y-x)/(y+x)}\, dA$, where R is the region in the first quadrant enclosed by the trapezoid with vertices $(0, 1)$, $(1, 0)$, $(0, 4)$, $(4, 0)$.

39. Use an appropriate change of variables to find the area of the region in the first quadrant enclosed by the curves $y = x,\ y = 2x,\ x = y^2,\ x = 4y^2$.

40. Use an appropriate change of variables to find the volume of the solid bounded above by the plane $x + y + z = 9$, below by the xy-plane, and laterally by the elliptic cylinder $4x^2 + 9y^2 = 36$. [*Hint:* Express the volume as a double integral in xy-coordinates, then use polar coordinates to evaluate the transformed integral.]

41. Use the transformation $u = x,\ v = z - y,\ w = xy$ to find

$$\iiint_G (z - y)^2 xy\, dV$$

where G is the region enclosed by the surfaces $x = 1$, $x = 3$, $z = y$, $z = y + 1$, $xy = 2$, $xy = 4$.

42. Use the transformation $u = xy,\ v = yz,\ w = xz$ to find the volume of the region in the first octant that is enclosed by the hyperbolic cylinders $xy = 1,\ xy = 2,\ yz = 1,\ yz = 3,\ xz = 1,\ xz = 4$.

43. (a) Verify that

$$\begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix}\begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{vmatrix}$$

(b) If $x = x(u, v),\ y = y(u, v)$ is a one-to-one transformation, then $u = u(x, y),\ v = v(x, y)$. Assuming the necessary differentiability, use the result in part (a) and the chain rule to show that

$$\frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = 1$$

44–46 The formula obtained in part (b) of Exercise 43 is useful in integration problems where it is inconvenient or impossible to solve the transformation equations $u = f(x, y)$, $v = g(x, y)$ explicitly for x and y in terms of u and v. In these exercises, use the relationship

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\partial(u, v)/\partial(x, y)}$$

to avoid solving for x and y in terms of u and v.

44. Use the transformation $u = xy$, $v = xy^4$ to find
$$\iint_R \sin(xy)\, dA$$
where R is the region enclosed by the curves $xy = \pi$, $xy = 2\pi$, $xy^4 = 1$, $xy^4 = 2$.

45. Use the transformation $u = x^2 - y^2$, $v = x^2 + y^2$ to find
$$\iint_R xy\, dA$$
where R is the region in the first quadrant that is enclosed by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 4$ and the circles $x^2 + y^2 = 9$, $x^2 + y^2 = 16$.

46. Use the transformation $u = xy$, $v = x^2 - y^2$ to find
$$\iint_R (x^4 - y^4)e^{xy}\, dA$$
where R is the region in the first quadrant enclosed by the hyperbolas $xy = 1$, $xy = 3$, $x^2 - y^2 = 3$, $x^2 - y^2 = 4$.

47. The three-variable analog of the formula derived in part (b) of Exercise 43 is
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)} = 1$$
Use this result to show that the volume V of the oblique parallelepiped that is bounded by the planes $x + y + 2z = \pm 3$, $x - 2y + z = \pm 2$, $4x + y + z = \pm 6$ is $V = 16$.

48. (a) Consider the transformation
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$
from cylindrical to rectangular coordinates, where $r \geq 0$. Show that
$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$
(b) Consider the transformation
$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$
from spherical to rectangular coordinates, where $0 \leq \phi \leq \pi$. Show that
$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin\phi$$

49. **Writing** For single-variable definite integrals, the technique of substitution was generally used to simplify the integrand. Discuss some motivations for using a change of variables in a multiple integral.

50. **Writing** Suppose that the boundary curves of a region R in the xy-plane can be described as level curves of various functions. Discuss how this information can be used to choose an appropriate change of variables for a double integral over R. Illustrate your discussion with an example.

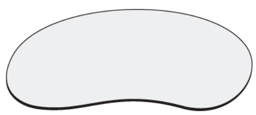
✓ **QUICK CHECK ANSWERS 14.7** **1.** (a) The image is the region in the xy-plane enclosed by the parallelogram with vertices $(1, 3)$, $(-3, 5)$, $(-1, 11)$, and $(3, 9)$. (b) $u = \frac{1}{7}(x + 2y)$, $v = \frac{1}{7}(y - 3x)$ **2.** S is a region in the uv-plane and R is the image of S in the xy-plane under the transformation $x = x(u, v)$, $y = y(u, v)$. **3.** (a) 7 (b) $\int_0^2 \int_1^3 7e^{7u}\, du\, dv$ **4.** $2vw$

14.8 CENTERS OF GRAVITY USING MULTIPLE INTEGRALS

In Section 5.7 we showed how to find the mass and center of gravity of a homogeneous lamina using single integrals. In this section we will show how double and triple integrals can be used to find the mass and center of gravity of inhomogeneous laminas and three-dimensional solids.

DENSITY AND MASS OF AN INHOMOGENEOUS LAMINA

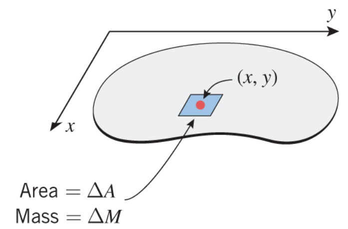
An idealized flat object that is thin enough to be viewed as a two-dimensional plane region is called a ***lamina*** (Figure 14.8.1). A lamina is called ***homogeneous*** if its composition is uniform throughout and ***inhomogeneous*** otherwise. The ***density*** of a *homogeneous* lamina was defined in Section 5.7 to be its mass per unit area. Thus, the density δ of a homogeneous lamina of mass M and area A is given by $\delta = M/A$.



The thickness of a lamina is negligible.

▲ **Figure 14.8.1**

For an inhomogeneous lamina the composition may vary from point to point, and hence an appropriate definition of "density" must reflect this. To motivate such a definition, suppose that the lamina is placed in an xy-plane. The density at a point (x, y) can be specified by a function $\delta(x, y)$, called the ***density function***, which can be interpreted as follows: Construct a small rectangle centered at (x, y) and let ΔM and ΔA be the mass and area of the portion of the lamina enclosed by this rectangle (Figure 14.8.2). If the ratio $\Delta M/\Delta A$



▲ **Figure 14.8.2**

approaches a limiting value as the dimensions (and hence the area) of the rectangle approach zero, then this limit is considered to be the density of the lamina at (x, y). Symbolically,

$$\delta(x, y) = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A} \tag{1}$$

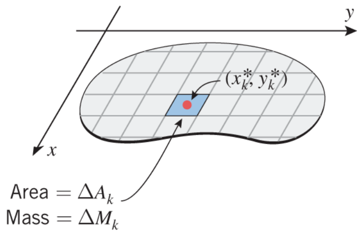
From this relationship we obtain the approximation

$$\Delta M \approx \delta(x, y)\Delta A \tag{2}$$

which relates the mass and area of a small rectangular portion of the lamina centered at (x, y). It is assumed that as the dimensions of the rectangle tend to zero, the error in this approximation also tends to zero.

The following result shows how to find the mass of a lamina from its density function.

> **14.8.1 MASS OF A LAMINA** If a lamina with a continuous density function $\delta(x, y)$ occupies a region R in the xy-plane, then its total mass M is given by
>
> $$M = \iint_R \delta(x, y)\, dA \tag{3}$$



▲ **Figure 14.8.3**

This formula can be motivated by a limiting process that can be outlined as follows: Imagine the lamina to be subdivided into rectangular pieces using lines parallel to the coordinate axes and excluding from consideration any nonrectangular parts at the boundary (Figure 14.8.3). Assume that there are n such rectangular pieces, and suppose that the kth piece has area ΔA_k. If we let (x_k^*, y_k^*) denote the center of the kth piece, then from Formula (2), the mass ΔM_k of this piece can be approximated by

$$\Delta M_k \approx \delta(x_k^*, y_k^*)\Delta A_k \tag{4}$$

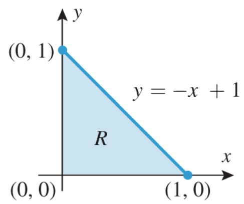
and hence the mass M of the entire lamina can be approximated by

$$M \approx \sum_{k=1}^{n} \delta(x_k^*, y_k^*)\Delta A_k$$

If we now increase n in such a way that the dimensions of the rectangles tend to zero, then it is plausible that the errors in our approximations will approach zero, so

$$M = \lim_{n \to +\infty} \sum_{k=1}^{n} \delta(x_k^*, y_k^*)\Delta A_k = \iint_R \delta(x, y)\, dA$$

▶ **Example 1** A triangular lamina with vertices (0, 0), (0, 1), and (1, 0) has density function $\delta(x, y) = xy$. Find its total mass.



▲ **Figure 14.8.4**

Solution. Referring to (3) and Figure 14.8.4, the mass M of the lamina is

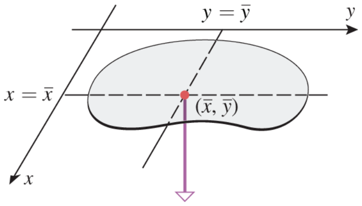
$$M = \iint_R \delta(x, y)\, dA = \iint_R xy\, dA = \int_0^1 \int_0^{-x+1} xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2}xy^2\right]_{y=0}^{-x+1} dx = \int_0^1 \left[\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right] dx = \frac{1}{24} \text{ (unit of mass)} \blacktriangleleft$$

CENTER OF GRAVITY OF AN INHOMOGENEOUS LAMINA

Recall that the ***center of gravity*** of a lamina occupying a region R in the horizontal xy-plane is the point $(\bar{x}, \bar{y})$ such that the effect of gravity on the lamina is "equivalent" to that of a single force acting at $(\bar{x}, \bar{y})$. If $(\bar{x}, \bar{y})$ is in R, then the lamina will balance horizontally on a point of support placed at $(\bar{x}, \bar{y})$. In Section 5.7 we showed how to locate the center of gravity of a homogeneous lamina. We now turn to this problem for an inhomogeneous lamina.

> **14.8.2 PROBLEM** Suppose that a lamina with a continuous density function $\delta(x, y)$ occupies a region R in a horizontal xy-plane. Find the coordinates $(\bar{x}, \bar{y})$ of the center of gravity.



▲ **Figure 14.8.5**

To motivate the solution of Problem 14.8.2, consider what happens if we try to place the lamina in Figure 14.8.5 on a knife-edge running along the line $y = \bar{y}$. Since the lamina behaves as if its entire mass is concentrated at $(\bar{x}, \bar{y})$, the lamina will be in perfect balance. Similarly, the lamina will be in perfect balance if the knife-edge runs along the line $x = \bar{x}$. To find these lines of balance we begin by reviewing some results from Section 5.7 about rotations.

Recall that if a point-mass m is located at the point (x, y), then the moment of m about $x = a$ measures the tendency of the mass to produce a rotation about the line $x = a$, and the moment of m about $y = c$ measures the tendency of the mass to produce a rotation about the line $y = c$. The moments are given by the following formulas:

$$\begin{bmatrix} \text{moment of } m \\ \text{about the} \\ \text{line } x = a \end{bmatrix} = m(x - a) \quad \text{and} \quad \begin{bmatrix} \text{moment of } m \\ \text{about the} \\ \text{line } y = c \end{bmatrix} = m(y - c) \tag{5–6}$$

If a number of point-masses are distributed throughout the xy-plane, the sum of their moments about $x = a$ is a measure of the tendency of the masses to produce a rotation of the plane (viewed as a weightless sheet) about the line $x = a$. If the sum of these moments is zero, the collective masses will produce no net rotational effect about the line. (Intuitively, this means that the plane would balance on a knife-edge along the line $x = a$. Similarly, if the sum of the moments of the masses about $y = c$ is zero, the plane would balance on a knife-edge along the line $y = c$.)

We are now ready to solve Problem 14.8.2. We imagine the lamina to be subdivided into rectangular pieces using lines parallel to the coordinate axes and excluding from consideration any nonrectangular pieces at the boundary (Figure 14.8.3). We assume that there are n such rectangular pieces and that the kth piece has area ΔA_k and mass ΔM_k. We will let (x_k^*, y_k^*) be the center of the kth piece, and we will assume that the entire mass of the kth piece is concentrated at its center. From (4), the mass of the kth piece can be approximated by

$$\Delta M_k \approx \delta(x_k^*, y_k^*)\Delta A_k$$

Since the lamina balances on the lines $x = \bar{x}$ and $y = \bar{y}$, the sum of the moments of the rectangular pieces about those lines should be close to zero; that is,

$$\sum_{k=1}^{n} (x_k^* - \bar{x})\Delta M_k = \sum_{k=1}^{n} (x_k^* - \bar{x})\delta(x_k^*, y_k^*)\Delta A_k \approx 0$$

$$\sum_{k=1}^{n} (y_k^* - \bar{y})\Delta M_k = \sum_{k=1}^{n} (y_k^* - \bar{y})\delta(x_k^*, y_k^*)\Delta A_k \approx 0$$

If we now increase n in such a way that the dimensions of the rectangles tend to zero, then it is plausible that the errors in our approximations will approach zero, so that

$$\lim_{n \to +\infty} \sum_{k=1}^{n} (x_k^* - \bar{x})\delta(x_k^*, y_k^*)\Delta A_k = 0$$

$$\lim_{n \to +\infty} \sum_{k=1}^{n} (y_k^* - \bar{y})\delta(x_k^*, y_k^*)\Delta A_k = 0$$

from which we obtain

$$\iint_R (x-\bar{x})\delta(x,y)\,dA = \iint_R x\delta(x,y)\,dA - \bar{x}\iint_R \delta(x,y)\,dA = 0 \tag{7}$$

$$\iint_R (y-\bar{y})\delta(x,y)\,dA = \iint_R y\delta(x,y)\,dA - \bar{y}\iint_R \delta(x,y)\,dA = 0 \tag{8}$$

Solving (7) and (8) respectively for $\bar{x}$ and $\bar{y}$ gives formulas for the center of gravity of a lamina:

Center of Gravity $(\bar{x},\bar{y})$ ***of a Lamina***

$$\bar{x} = \frac{\displaystyle\iint_R x\delta(x,y)\,dA}{\displaystyle\iint_R \delta(x,y)\,dA}, \qquad \bar{y} = \frac{\displaystyle\iint_R y\delta(x,y)\,dA}{\displaystyle\iint_R \delta(x,y)\,dA} \tag{9–10}$$

In both formulas the denominator is the mass M of the lamina [see (3)]. The numerator in the formula for $\bar{x}$ is denoted by M_y and is called the ***first moment of the lamina about the y-axis***; the numerator of the formula for $\bar{y}$ is denoted by M_x and is called the ***first moment of the lamina about the x-axis***. Thus, Formulas (9) and (10) can be expressed as

Alternative Formulas for Center of Gravity $(\bar{x},\bar{y})$ ***of a Lamina***

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R}\iint_R x\delta(x,y)\,dA \tag{11}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R}\iint_R y\delta(x,y)\,dA \tag{12}$$

Example 2 Find the center of gravity of the triangular lamina with vertices $(0,0)$, $(0,1)$, and $(1,0)$ and density function $\delta(x,y) = xy$.

Solution. The lamina is shown in Figure 14.8.4. In Example 1 we found the mass of the lamina to be

$$M = \iint_R \delta(x,y)\,dA = \iint_R xy\,dA = \frac{1}{24}$$

The moment of the lamina about the y-axis is

$$M_y = \iint_R x\delta(x,y)\,dA = \iint_R x^2y\,dA = \int_0^1\int_0^{-x+1} x^2y\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{y=0}^{-x+1} dx = \int_0^1 \left(\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2\right) dx = \frac{1}{60}$$

and the moment about the x-axis is

$$M_x = \iint_R y\delta(x,y)\,dA = \iint_R xy^2\,dA = \int_0^1\int_0^{-x+1} xy^2\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{3}xy^3\right]_{y=0}^{-x+1} dx = \int_0^1 \left(-\frac{1}{3}x^4 + x^3 - x^2 + \frac{1}{3}x\right) dx = \frac{1}{60}$$

From (11) and (12),

$$\bar{x} = \frac{M_y}{M} = \frac{1/60}{1/24} = \frac{2}{5}, \quad \bar{y} = \frac{M_x}{M} = \frac{1/60}{1/24} = \frac{2}{5}$$

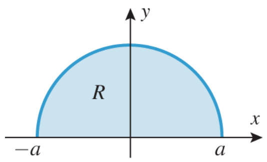
so the center of gravity is $\left(\frac{2}{5}, \frac{2}{5}\right)$. ◀

Recall that the center of gravity of a *homogeneous* lamina is called the ***centroid of the lamina*** or sometimes the ***centroid of the region R***. Because the density function δ is constant for a homogeneous lamina, the factor δ may be moved through the integral signs in (9) and (10) and canceled. Thus, the centroid $(\bar{x}, \bar{y})$ is a geometric property of the region R and is given by the following formulas:

Centroid of a Region R

$$\bar{x} = \frac{\displaystyle\iint_R x\,dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R}\iint_R x\,dA \tag{13}$$

$$\bar{y} = \frac{\displaystyle\iint_R y\,dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R}\iint_R y\,dA \tag{14}$$



▲ **Figure 14.8.6**

▶ **Example 3** Find the centroid of the semicircular region in Figure 14.8.6.

Solution. By symmetry, $\bar{x} = 0$ since the y-axis is obviously a line of balance. From (14),

$$\begin{aligned}
\bar{y} &= \frac{1}{\text{area of } R}\iint_R y\,dA = \frac{1}{\frac{1}{2}\pi a^2}\iint_R y\,dA \\
&= \frac{1}{\frac{1}{2}\pi a^2}\int_0^{\pi}\int_0^{a} (r\sin\theta)r\,dr\,d\theta \qquad \boxed{\text{Evaluating in polar coordinates}} \\
&= \frac{1}{\frac{1}{2}\pi a^2}\int_0^{\pi}\left[\frac{1}{3}r^3\sin\theta\right]_{r=0}^{a} d\theta \\
&= \frac{1}{\frac{1}{2}\pi a^2}\left(\frac{1}{3}a^3\right)\int_0^{\pi}\sin\theta\,d\theta = \frac{1}{\frac{1}{2}\pi a^2}\left(\frac{2}{3}a^3\right) = \frac{4a}{3\pi}
\end{aligned}$$

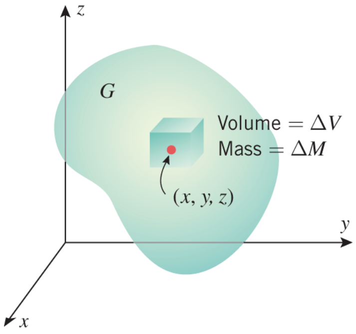
so the centroid is $\left(0, \dfrac{4a}{3\pi}\right)$. ◀

Compare the calculation in Example 3 to that of Example 3 in Section 5.7.

CENTER OF GRAVITY AND CENTROID OF A SOLID

For a three-dimensional solid G, the formulas for moments, center of gravity, and centroid are similar to those for laminas. If G is *homogeneous*, then its ***density*** is defined to be its mass per unit volume. Thus, if G is a homogeneous solid of mass M and volume V, then its density δ is given by $\delta = M/V$. If G is inhomogeneous and is in an xyz-coordinate system, then its density at a general point (x, y, z) is specified by a ***density function*** $\delta(x, y, z)$ whose value at a point can be viewed as a limit:

$$\delta(x, y, z) = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}$$



▲ **Figure 14.8.7**

where ΔM and ΔV represent the mass and volume of a rectangular parallelepiped, centered at (x, y, z), whose dimensions tend to zero (Figure 14.8.7).

Using the discussion of laminas as a model, you should be able to show that the mass M of a solid with a continuous density function $\delta(x, y, z)$ is

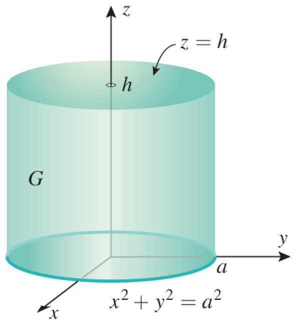
$$M = \text{mass of } G = \iiint_G \delta(x, y, z)\, dV \tag{15}$$

The formulas for center of gravity and centroid are as follows:

Center of Gravity $(\bar{x}, \bar{y}, \bar{z})$ ***of a Solid G***	***Centroid*** $(\bar{x}, \bar{y}, \bar{z})$ ***of a Solid G***
$\bar{x} = \dfrac{1}{M}\displaystyle\iiint_G x\delta(x, y, z)\, dV$	$\bar{x} = \dfrac{1}{V}\displaystyle\iiint_G x\, dV$
$\bar{y} = \dfrac{1}{M}\displaystyle\iiint_G y\delta(x, y, z)\, dV$	$\bar{y} = \dfrac{1}{V}\displaystyle\iiint_G y\, dV$
$\bar{z} = \dfrac{1}{M}\displaystyle\iiint_G z\delta(x, y, z)\, dV$	$\bar{z} = \dfrac{1}{V}\displaystyle\iiint_G z\, dV$

(16–17)

▶ Example 4 Find the mass and the center of gravity of a cylindrical solid of height h and radius a (Figure 14.8.8), assuming that the density at each point is proportional to the distance between the point and the base of the solid.



▲ **Figure 14.8.8**

Solution. Since the density is proportional to the distance z from the base, the density function has the form $\delta(x, y, z) = kz$, where k is some (unknown) positive constant of proportionality. From (15) the mass of the solid is

$$\begin{aligned}
M = \iiint_G \delta(x, y, z)\, dV &= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h kz\, dz\, dy\, dx \\
&= k\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{2}h^2\, dy\, dx \\
&= kh^2\int_{-a}^{a}\sqrt{a^2 - x^2}\, dx \\
&= \tfrac{1}{2}kh^2\pi a^2
\end{aligned}$$

Interpret the integral as the area of a semicircle.

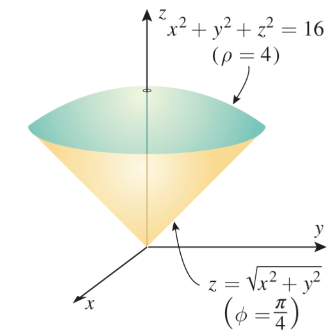
Without additional information, the constant k cannot be determined. However, as we will now see, the value of k does not affect the center of gravity.

From (16),

$$\begin{aligned}
\bar{z} = \frac{1}{M}\iiint_G z\delta(x, y, z)\, dV &= \frac{1}{\frac{1}{2}kh^2\pi a^2}\iiint_G z\delta(x, y, z)\, dV \\
&= \frac{1}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h z(kz)\, dz\, dy\, dx \\
&= \frac{k}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{3}h^3\, dy\, dx \\
&= \frac{\frac{1}{3}kh^3}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a} 2\sqrt{a^2 - x^2}\, dx \\
&= \frac{\frac{1}{3}kh^3\pi a^2}{\frac{1}{2}kh^2\pi a^2} = \frac{2}{3}h
\end{aligned}$$

Similar calculations using (16) will yield $\bar{x} = \bar{y} = 0$. However, this is evident by inspection, since it follows from the symmetry of the solid and the form of its density function that the center of gravity is on the z-axis. Thus, the center of gravity is $\left(0, 0, \frac{2}{3}h\right)$. ◀

▶ Example 5 Find the centroid of the solid G bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 16$.



▲ **Figure 14.8.9**

Solution. The solid G is sketched in Figure 14.8.9. In Example 3 of Section 14.6, spherical coordinates were used to find that the volume of G is

$$V = \frac{64\pi}{3}(2 - \sqrt{2})$$

By symmetry, the centroid $(\bar{x}, \bar{y}, \bar{z})$ is on the z-axis, so $\bar{x} = \bar{y} = 0$. In spherical coordinates, the equation of the sphere $x^2 + y^2 + z^2 = 16$ is $\rho = 4$ and the equation of the cone $z = \sqrt{x^2 + y^2}$ is $\phi = \pi/4$, so from (17) we have

$$\begin{aligned}
\bar{z} &= \frac{1}{V}\iiint_G z\,dV = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\int_0^4 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^4}{4}\cos\phi\sin\phi\right]_{\rho=0}^{4} d\phi\,d\theta \\
&= \frac{64}{V}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\cos\phi\,d\phi\,d\theta = \frac{64}{V}\int_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_{\phi=0}^{\pi/4} d\theta \\
&= \frac{16}{V}\int_0^{2\pi} d\theta = \frac{32\pi}{V} = \frac{3}{2(2 - \sqrt{2})}
\end{aligned}$$

The centroid of G is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{3}{2(2 - \sqrt{2})}\right) \approx (0, 0, 2.561) \quad ◀$$

✓ QUICK CHECK EXERCISES 14.8 *(See page 968 for answers.)*

1. The total mass of a lamina with continuous density function $\delta(x, y)$ that occupies a region R in the xy-plane is given by $M =$ ______.
2. Consider a lamina with mass M and continuous density function $\delta(x, y)$ that occupies a region R in the xy-plane. The x-coordinate of the center of gravity of the lamina is M_y/M, where M_y is called the ______ and is given by the double integral ______.
3. Let R be the region between the graphs of $y = x^2$ and $y = 2 - x$ for $0 \le x \le 1$. The area of R is $\frac{7}{6}$ and the centroid of R is ______.

EXERCISE SET 14.8 Graphing Utility C CAS

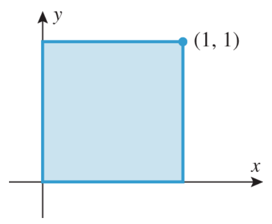
1–4 Find the mass and center of gravity of the lamina. ■

1. A lamina with density $\delta(x, y) = x + y$ is bounded by the x-axis, the line $x = 1$, and the curve $y = \sqrt{x}$.
2. A lamina with density $\delta(x, y) = y$ is bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.
3. A lamina with density $\delta(x, y) = xy$ is in the first quadrant and is bounded by the circle $x^2 + y^2 = a^2$ and the coordinate axes.
4. A lamina with density $\delta(x, y) = x^2 + y^2$ is bounded by the x-axis and the upper half of the circle $x^2 + y^2 = 1$.

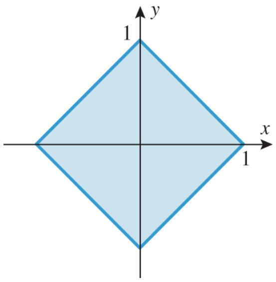
FOCUS ON CONCEPTS

5–6 For the given density function, make a conjecture about the coordinates of the center of gravity and confirm your conjecture by integrating. ■

5. $\delta(x, y) = |x + y - 1|$

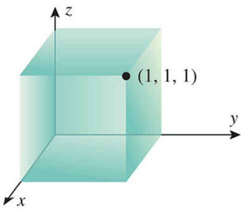


6. $\delta(x, y) = 1 + x^2 + y^2$

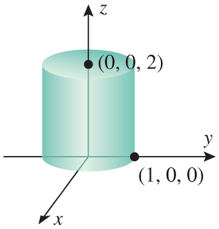


7–8 Make a conjecture about the coordinates of the centroid of the region and confirm your conjecture by integrating.

7.



8.



9–12 True–False Determine whether the statement is true or false. Explain your answer.

9. The center of gravity of a homogeneous lamina in a plane is located at the lamina's centroid.

10. The mass of a two-dimensional lamina is the product of its area and the density of the lamina at its centroid.

11. The coordinates of the center of gravity of a two-dimensional lamina are the lamina's first moments about the y- and x-axes, respectively.

12. The density of a solid in 3-space is measured in units of mass per unit area.

13. Show that in polar coordinates the formulas for the centroid $(\bar{x}, \bar{y})$ of a region R are

$$\bar{x} = \frac{1}{\text{area of } R} \iint_R r^2 \cos\theta \, dr \, d\theta$$

$$\bar{y} = \frac{1}{\text{area of } R} \iint_R r^2 \sin\theta \, dr \, d\theta$$

14–17 Use the result of Exercise 13 to find the centroid $(\bar{x}, \bar{y})$ of the region.

14. The region enclosed by the cardioid $r = a(1 + \sin\theta)$.

15. The petal of the rose $r = \sin 2\theta$ in the first quadrant.

16. The region above the x-axis and between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ $(a < b)$.

17. The region enclosed between the y-axis and the right half of the circle $x^2 + y^2 = a^2$.

18. Let R be the rectangle bounded by the lines $x = 0$, $x = 3$, $y = 0$, and $y = 2$. By inspection, find the centroid of R and use it to evaluate

$$\iint_R x \, dA \quad \text{and} \quad \iint_R y \, dA$$

19–24 Find the centroid of the solid.

19. The tetrahedron in the first octant enclosed by the coordinate planes and the plane $x + y + z = 1$.

20. The solid bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$.

21. The solid bounded by the surface $z = y^2$ and the planes $x = 0$, $x = 1$, and $z = 1$.

22. The solid in the first octant bounded by the surface $z = xy$ and the planes $z = 0$, $x = 2$, and $y = 2$.

23. The solid in the first octant that is bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and the coordinate planes.

24. The solid enclosed by the xy-plane and the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.

25–28 Find the mass and center of gravity of the solid.

25. The cube that has density $\delta(x, y, z) = a - x$ and is defined by the inequalities $0 \le x \le a$, $0 \le y \le a$, and $0 \le z \le a$.

26. The cylindrical solid that has density $\delta(x, y, z) = h - z$ and is enclosed by $x^2 + y^2 = a^2$, $z = 0$, and $z = h$.

27. The solid that has density $\delta(x, y, z) = yz$ and is enclosed by $z = 1 - y^2$ (for $y \ge 0$), $z = 0$, $y = 0$, $x = -1$, and $x = 1$.

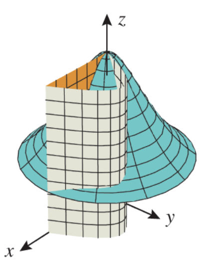
28. The solid that has density $\delta(x, y, z) = xz$ and is enclosed by $y = 9 - x^2$ (for $x \ge 0$), $x = 0$, $y = 0$, $z = 0$, and $z = 1$.

29. Find the center of gravity of the square lamina with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ if
(a) the density is proportional to the square of the distance from the origin;
(b) the density is proportional to the distance from the y-axis.

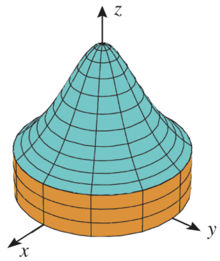
30. Find the center of gravity of the cube that is determined by the inequalities $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ if
(a) the density is proportional to the square of the distance to the origin;
(b) the density is proportional to the sum of the distances to the faces that lie in the coordinate planes.

C **31.** Use the numerical triple integral capability of a CAS to approximate the location of the centroid of the solid that is bounded above by the surface $z = 1/(1 + x^2 + y^2)$, below by the xy-plane, and laterally by the plane $y = 0$ and the surface $y = \sin x$ for $0 \le x \le \pi$ (see the accompanying figure on the next page).

32. The accompanying figure on the next page shows the solid that is bounded above by the surface $z = 1/(x^2 + y^2 + 1)$, below by the xy-plane, and laterally by the surface $x^2 + y^2 = a^2$.
(a) By symmetry, the centroid of the solid lies on the z-axis. Make a conjecture about the behavior of the z-coordinate of the centroid as $a \to 0^+$ and as $a \to +\infty$.
(b) Find the z-coordinate of the centroid, and check your conjecture by calculating the appropriate limits.
(c) Use a graphing utility to plot the z-coordinate of the centroid versus a, and use the graph to estimate the value of a for which the centroid is $(0, 0, 0.25)$.



▲ Figure Ex-31



▲ Figure Ex-32

33–34 Use cylindrical coordinates.

33. Find the mass of the solid with density $\delta(x, y, z) = 3 - z$ that is bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 3$.

34. Find the mass of a right circular cylinder of radius a and height h if the density is proportional to the distance from the base. (Let k be the constant of proportionality.)

35–36 Use spherical coordinates.

35. Find the mass of a spherical solid of radius a if the density is proportional to the distance from the center. (Let k be the constant of proportionality.)

36. Find the mass of the solid enclosed between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ if the density is $\delta(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$.

37–38 Use cylindrical coordinates to find the centroid of the solid.

37. The solid that is bounded above by the sphere

$$x^2 + y^2 + z^2 = 2$$

and below by the paraboloid $z = x^2 + y^2$.

38. The solid that is bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

39–40 Use the Wallis sine and cosine formulas:

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \quad \left(\begin{array}{c} n \text{ even} \\ \text{and} \geq 2 \end{array}\right)$$

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n} \quad \left(\begin{array}{c} n \text{ odd} \\ \text{and} \geq 3 \end{array}\right)$$

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \quad \left(\begin{array}{c} n \text{ even} \\ \text{and} \geq 2 \end{array}\right)$$

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n} \quad \left(\begin{array}{c} n \text{ odd} \\ \text{and} \geq 3 \end{array}\right)$$

39. Find the centroid of the solid bounded above by the paraboloid $z = x^2 + y^2$, below by the plane $z = 0$, and laterally by the cylinder $(x - 1)^2 + y^2 = 1$.

40. Find the mass of the solid in the first octant bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by the plane $z = 0$, and laterally by the cylinder $x^2 + y^2 = 2x$ and the plane $y = 0$, assuming the density to be $\delta(x, y, z) = z$.

41–42 Use spherical coordinates to find the centroid of the solid.

41. The solid in the first octant bounded by the coordinate planes and the sphere $x^2 + y^2 + z^2 = a^2$.

42. The solid bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \pi/3$.

43. Find the mass of the solid that is enclosed by the sphere $x^2 + y^2 + z^2 = 1$ and lies above the cone $z = \sqrt{x^2 + y^2}$ if the density is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

44. Find the center of gravity of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane, assuming the density to be $\delta(x, y, z) = x^2 + y^2 + z^2$.

45. Find the center of gravity of the solid that is bounded by the cylinder $x^2 + y^2 = 1$, the cone $z = \sqrt{x^2 + y^2}$, and the xy-plane if the density is $\delta(x, y, z) = z$.

46. Find the center of gravity of the solid hemisphere bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$ if the density is proportional to the distance from the origin.

47. Find the centroid of the solid that is enclosed by the hemispheres $y = \sqrt{9 - x^2 - z^2}$, $y = \sqrt{4 - x^2 - z^2}$, and the plane $y = 0$.

48. Suppose that the density at a point in a gaseous spherical star is modeled by the formula

$$\delta = \delta_0 e^{-(\rho/R)^3}$$

where δ_0 is a positive constant, R is the radius of the star, and ρ is the distance from the point to the star's center. Find the mass of the star.

49–50 The tendency of a lamina to resist a change in rotational motion about an axis is measured by its ***moment of inertia*** about that axis. If a lamina occupies a region R of the xy-plane, and if its density function $\delta(x, y)$ is continuous on R, then the moments of inertia about the x-axis, the y-axis, and the z-axis are denoted by I_x, I_y, and I_z, respectively, and are defined by

$$I_x = \iint_R y^2\, \delta(x, y)\, dA, \quad I_y = \iint_R x^2\, \delta(x, y)\, dA,$$

$$I_z = \iint_R (x^2 + y^2)\, \delta(x, y)\, dA$$

Use these definitions in Exercises 49 and 50.

49. Consider the rectangular lamina that occupies the region described by the inequalities $0 \leq x \leq a$ and $0 \leq y \leq b$. Assuming that the lamina has constant density δ, show that

$$I_x = \frac{\delta ab^3}{3}, \quad I_y = \frac{\delta a^3 b}{3}, \quad I_z = \frac{\delta ab(a^2 + b^2)}{3}$$

50. Consider the circular lamina that occupies the region described by the inequalities $0 \leq x^2 + y^2 \leq a^2$. Assuming that the lamina has constant density δ, show that

$$I_x = I_y = \frac{\delta \pi a^4}{4}, \quad I_z = \frac{\delta \pi a^4}{2}$$

51–54 The tendency of a solid to resist a change in rotational motion about an axis is measured by its ***moment of inertia*** about that axis. If the solid occupies a region G in an xyz-coordinate system, and if its density function $\delta(x, y, z)$ is continuous on G, then the moments of inertia about the x-axis, the y-axis, and the z-axis are denoted by I_x, I_y, and I_z, respectively, and are defined by

$$I_x = \iiint_G (y^2 + z^2)\,\delta(x, y, z)\,dV$$

$$I_y = \iiint_G (x^2 + z^2)\,\delta(x, y, z)\,dV$$

$$I_z = \iiint_G (x^2 + y^2)\,\delta(x, y, z)\,dV$$

In these exercises, find the indicated moments of inertia of the solid, assuming that it has constant density δ.

51. I_z for the solid cylinder $x^2 + y^2 \le a^2$, $0 \le z \le h$.

52. I_y for the solid cylinder $x^2 + y^2 \le a^2$, $0 \le z \le h$.

53. I_z for the hollow cylinder $a_1^2 \le x^2 + y^2 \le a_2^2$, $0 \le z \le h$.

54. I_z for the solid sphere $x^2 + y^2 + z^2 \le a^2$.

55–59 These exercises reference the ***Theorem of Pappus***: *If R is a bounded plane region and L is a line that lies in the plane of R such that R is entirely on one side of L, then the volume of the solid formed by revolving R about L is given by*

$$\textit{volume} = (\textit{area of } R)\cdot\begin{pmatrix}\textit{distance traveled}\\ \textit{by the centroid}\end{pmatrix}$$ ■

55. Perform the following steps to prove the Theorem of Pappus:

(a) Introduce an xy-coordinate system so that L is along the y-axis and the region R is in the first quadrant. Partition R into rectangular subregions in the usual way and let R_k be a typical subregion of R with center (x_k^*, y_k^*) and area $\Delta A_k = \Delta x_k \Delta y_k$. Show that the volume generated by R_k as it revolves about L is

$$2\pi x_k^* \Delta x_k \Delta y_k = 2\pi x_k^* \Delta A_k$$

(b) Show that the volume generated by R as it revolves about L is

$$V = \iint_R 2\pi x\,dA = 2\pi\cdot\bar{x}\cdot[\text{area of } R]$$

56. Use the Theorem of Pappus and the result of Example 3 to find the volume of the solid generated when the region bounded by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ is revolved about

(a) the line $y = -a$ (b) the line $y = x - a$.

57. Use the Theorem of Pappus and the fact that the area of an ellipse with semiaxes a and b is πab to find the volume of the elliptical torus generated by revolving the ellipse

$$\frac{(x-k)^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the y-axis. Assume that $k > a$.

58. Use the Theorem of Pappus to find the volume of the solid that is generated when the region enclosed by $y = x^2$ and $y = 8 - x^2$ is revolved about the x-axis.

59. Use the Theorem of Pappus to find the centroid of the triangular region with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$, where $a > 0$ and $b > 0$. [*Hint:* Revolve the region about the x-axis to obtain $\bar{y}$ and about the y-axis to obtain $\bar{x}$.]

60. It can be proved that if a bounded plane region slides along a helix in such a way that the region is always orthogonal to the helix (i.e., orthogonal to the unit tangent vector to the helix), then the volume swept out by the region is equal to the area of the region times the distance traveled by its centroid. Use this result to find the volume of the "tube" in the accompanying figure that is swept out by sliding a circle of radius $\frac{1}{2}$ along the helix

$$x = \cos t,\quad y = \sin t,\quad z = \frac{t}{4}\qquad (0 \le t \le 4\pi)$$

in such a way that the circle is always centered on the helix and lies in the plane perpendicular to the helix.

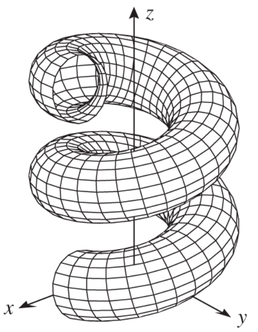


Figure Ex-60

61. **Writing** Give a physical interpretation of the "center of gravity" of a lamina.

✓ **QUICK CHECK ANSWERS 14.8** **1.** $\iint_R \delta(x, y)\,dA$ **2.** first moment about the y-axis; $\iint_R x\delta(x, y)\,dA$ **3.** $\left(\frac{5}{14}, \frac{32}{35}\right)$

CHAPTER 14 REVIEW EXERCISES

1. The double integral over a region R in the xy-plane is defined as

$$\iint_R f(x, y)\,dA = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k$$

Describe the procedure on which this definition is based.

2. The triple integral over a solid G in an xyz-coordinate system is defined as

$$\iiint_G f(x, y, z)\,dV = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\,\Delta V_k$$

Describe the procedure on which this definition is based.

3. (a) Express the area of a region R in the xy-plane as a double integral.

(b) Express the volume of a region G in an xyz-coordinate system as a triple integral.

(c) Express the area of the portion of the surface $z = f(x, y)$ that lies above the region R in the xy-plane as a double integral.

4. (a) Write down parametric equations for a sphere of radius a centered at the origin.
(b) Write down parametric equations for the right circular cylinder of radius a and height h that is centered on the z-axis, has its base in the xy-plane, and extends in the positive z-direction.

5. Let R be the region in the accompanying figure. Fill in the missing limits of integration in the iterated integral

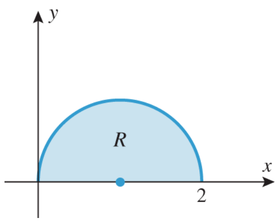
$$\int_{\square}^{\square}\int_{\square}^{\square} f(x, y)\,dx\,dy$$

over R.

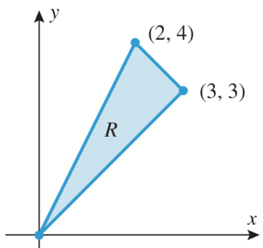
6. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration in the sum of the iterated integrals

$$\int_{0}^{2}\int_{\square}^{\square} f(x, y)\,dy\,dx + \int_{2}^{3}\int_{\square}^{\square} f(x, y)\,dy\,dx$$

over R.

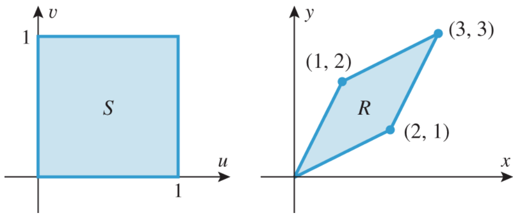


▲ **Figure Ex-5**



▲ **Figure Ex-6**

7. (a) Find constants a, b, c, and d such that the transformation $x = au + bv$, $y = cu + dv$ maps the region S in the accompanying figure into the region R.
(b) Find the area of the parallelogram R by integrating over the region S, and check your answer using a formula from geometry.



▲ **Figure Ex-7**

8. Give a geometric argument to show that

$$0 < \int_{0}^{\pi}\int_{0}^{\pi} \sin\sqrt{xy}\,dy\,dx < \pi^2$$

9–10 Evaluate the iterated integral. ■

9. $\displaystyle\int_{1/2}^{1}\int_{0}^{2x} \cos(\pi x^2)\,dy\,dx$ **10.** $\displaystyle\int_{0}^{2}\int_{-y}^{2y} xe^{y^3}\,dx\,dy$

11–12 Express the iterated integral as an equivalent integral with the order of integration reversed. ■

11. $\displaystyle\int_{0}^{2}\int_{0}^{x/2} e^x e^y\,dy\,dx$ **12.** $\displaystyle\int_{0}^{\pi}\int_{y}^{\pi} \frac{\sin x}{x}\,dx\,dy$

13–14 Sketch the region whose area is represented by the iterated integral. ■

13. $\displaystyle\int_{0}^{\pi/2}\int_{\tan(x/2)}^{\sin x} dy\,dx$

14. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{a}^{a(1+\cos\theta)} r\,dr\,d\theta \quad (a > 0)$

15–16 Evaluate the double integral. ■

15. $\displaystyle\iint_R x^2 \sin y^2\,dA$; R is the region that is bounded by $y = x^3$, $y = -x^3$, and $y = 8$.

16. $\displaystyle\iint_R (4 - x^2 - y^2)\,dA$; R is the sector in the first quadrant bounded by the circle $x^2 + y^2 = 4$ and the coordinate axes.

17. Convert to rectangular coordinates and evaluate:

$$\int_{0}^{\pi/2}\int_{0}^{2a\sin\theta} r\sin 2\theta\,dr\,d\theta$$

18. Convert to polar coordinates and evaluate:

$$\int_{0}^{\sqrt{2}}\int_{x}^{\sqrt{4-x^2}} 4xy\,dy\,dx$$

19–20 Find the area of the region using a double integral. ■

19. The region bounded by $y = 2x^3$, $2x + y = 4$, and the x-axis.

20. The region enclosed by the rose $r = \cos 3\theta$.

21. Convert to cylindrical coordinates and evaluate:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{(x^2+y^2)^2}^{16} x^2\,dz\,dy\,dx$$

22. Convert to spherical coordinates and evaluate:

$$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} \frac{1}{1 + x^2 + y^2 + z^2}\,dz\,dy\,dx$$

23. Let G be the region bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/3$. Express

$$\iiint_G (x^2 + y^2)\,dV$$

as an iterated integral in
(a) spherical coordinates (b) cylindrical coordinates
(c) rectangular coordinates.

24. Let $G = \{(x, y, z) : x^2 + y^2 \le z \le 4x\}$. Express the volume of G as an iterated integral in
(a) rectangular coordinates (b) cylindrical coordinates.

25–26 Find the volume of the solid using a triple integral. ■

25. The solid bounded below by the cone $\phi = \pi/6$ and above by the plane $z = a$.

26. The solid enclosed between the surfaces $x = y^2 + z^2$ and $x = 1 - y^2$.

27. Find the area of the portion of the surface $z = 3y + 2x^2 + 4$ that is above the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(1, -1)$.

28. Find the surface area of the portion of the hyperbolic paraboloid

$$\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + uv\mathbf{k}$$

for which $u^2 + v^2 \le 4$.

29–30 Find the equation of the tangent plane to the surface at the specified point.

29. $\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k};\ u = 1, v = 2$

30. $x = u\cosh v,\ y = u\sinh v,\ z = u^2;\ (-3, 0, 9)$

31. Suppose that you have a double integral over a region R in the xy-plane and you want to transform that integral into an equivalent double integral over a region S in the uv-plane. Describe the procedure you would use.

32. Use the transformation $u = x - 3y$, $v = 3x + y$ to find

$$\iint_R \frac{x - 3y}{(3x + y)^2}\, dA$$

where R is the rectangular region enclosed by the lines $x - 3y = 0$, $x - 3y = 4$, $3x + y = 1$, and $3x + y = 3$.

33. Let G be the solid in 3-space defined by the inequalities

$$1 - e^x \le y \le 3 - e^x,\quad 1 - y \le 2z \le 2 - y,\quad y \le e^x \le y + 4$$

(a) Using the coordinate transformation

$$u = e^x + y,\quad v = y + 2z,\quad w = e^x - y$$

calculate the Jacobian $\partial(x, y, z)/\partial(u, v, w)$. Express your answer in terms of u, v, and w.

(b) Using a triple integral and the change of variables given in part (a), find the volume of G.

34. Find the average distance from a point inside a sphere of radius a to the center. [See the definition preceding Exercise 33 of Section 14.5.]

35–36 Find the centroid of the region.

35. The region bounded by $y^2 = 4x$ and $y^2 = 8(x - 2)$.

36. The upper half of the ellipse $(x/a)^2 + (y/b)^2 = 1$.

37–38 Find the centroid of the solid.

37. The solid cone with vertex $(0, 0, h)$ and with base the disk $x^2 + y^2 \le a^2$ in the xy-plane.

38. The solid bounded by $y = x^2$, $z = 0$, and $y + z = 4$.

CHAPTER 14 MAKING CONNECTIONS

C CAS

1. The integral $\int_0^{+\infty} e^{-x^2}\, dx$, which arises in probability theory, can be evaluated using the following method. Let the value of the integral be I. Thus,

$$I = \int_0^{+\infty} e^{-x^2}\, dx = \int_0^{+\infty} e^{-y^2}\, dy$$

since the letter used for the variable of integration in a definite integral does not matter.

(a) Give a reasonable argument to show that

$$I^2 = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\, dx\, dy$$

(b) Evaluate the iterated integral in part (a) by converting to polar coordinates.

(c) Use the result in part (b) to show that $I = \sqrt{\pi}/2$.

2. Show that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(1 + x^2 + y^2)^2}\, dx\, dy = \frac{\pi}{4}$$

[*Hint:* See Exercise 1.]

C **3.** (a) Use the numerical integration capability of a CAS to approximate the value of the double integral

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)^2}\, dy\, dx$$

(b) Compare the approximation obtained in part (a) to the approximation that results if the integral is first converted to polar coordinates.

C **4.** (a) Find the region G over which the triple integral

$$\iiint_G (1 - x^2 - y^2 - z^2)\, dV$$

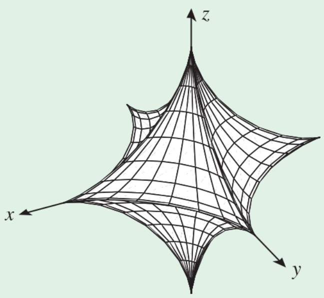
has its maximum value.

(b) Use the numerical triple integral operation of a CAS to approximate the maximum value.

(c) Find the exact maximum value.

5–6 The accompanying figure shows the graph of an ***astroidal sphere***

$$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$$



C **5.** (a) Show that the astroidal sphere can be represented parametrically as

$$\begin{aligned} x &= a(\sin u \cos v)^3 \\ y &= a(\sin u \sin v)^3 \qquad (0 \le u \le \pi,\ 0 \le v \le 2\pi) \\ z &= a(\cos u)^3 \end{aligned}$$

(b) Use a CAS to approximate the surface area in the case where $a = 1$.

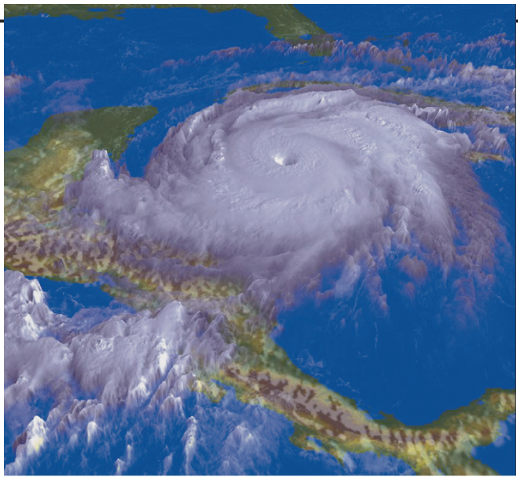
6. Find the volume of the astroidal sphere using a triple integral and the transformation

$$\begin{aligned} x &= \rho(\sin\phi\cos\theta)^3 \\ y &= \rho(\sin\phi\sin\theta)^3 \\ z &= \rho(\cos\phi)^3 \end{aligned}$$

for which $0 \le \rho \le a$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

15

TOPICS IN VECTOR CALCULUS



Hal Pierce/Laboratory for Atmospheres/NASA Goddard Space Flight Center

Results in this chapter provide tools for analyzing and understanding the behavior of hurricanes and other fluid flows.

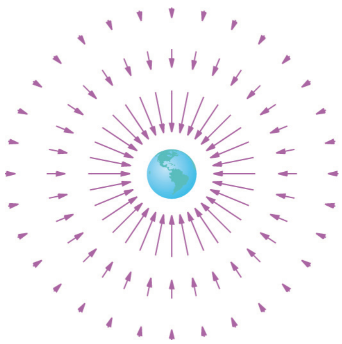
We begin this chapter by introducing the concept of a vector field, an important tool for the study of gravitational and electrostatic force fields, the flow of fluids, and conservation of energy. Next, we will introduce the "line integral," a new type of integral with a variety of applications to the analysis of vector fields. Finally, we conclude with three major theorems, Green's Theorem, the Divergence Theorem, and Stokes' Theorem. These theorems provide deep insight into the nature of vector fields and are the basis for many of the most important principles in physics and engineering.

15.1 VECTOR FIELDS

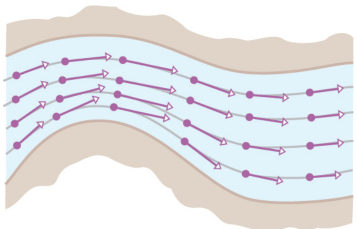
In this section we will consider functions that associate vectors with points in 2-space or 3-space. We will see that such functions play an important role in the study of fluid flow, gravitational force fields, electromagnetic force fields, and a wide range of other applied problems.

VECTOR FIELDS

Consider a *unit* point-mass located at any point in the Universe. According to Newton's Law of Universal Gravitation, the Earth exerts an attractive force on the mass that is directed toward the center of the Earth and has a magnitude that is inversely proportional to the square of the distance from the mass to the Earth's center (Figure 15.1.1). This association of force vectors with points in space is called the Earth's *gravitational field*. A similar association occurs in fluid flow. Imagine a stream in which the water flows horizontally at every level, and consider the layer of water at a specific depth. At each point of the layer, the water has a certain velocity, which we can represent by a vector at that point (Figure 15.1.2). This association of velocity vectors with points in the two-dimensional layer is called the *velocity field* at that layer. These ideas are captured in the following definition.



▲ **Figure 15.1.1**



▲ **Figure 15.1.2**

15.1.1 DEFINITION A ***vector field*** in a plane is a function that associates with each point P in the plane a unique vector $\mathbf{F}(P)$ parallel to the plane. Similarly, a vector field in 3-space is a function that associates with each point P in 3-space a unique vector $\mathbf{F}(P)$ in 3-space.

Observe that in this definition there is no reference to a coordinate system. However, for computational purposes it is useful to introduce a coordinate system so that vectors can be assigned components. Specifically, if $\mathbf{F}(P)$ is a vector field in an xy-coordinate system, then the point P will have some coordinates (x, y) and the associated vector will have components that are functions of x and y. Thus, the vector field $\mathbf{F}(P)$ can be expressed as

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

Notice that a vector field is really just a vector-valued function. The term "vector field" is commonly used in physics and engineering.

Similarly, in 3-space with an xyz-coordinate system, a vector field $\mathbf{F}(P)$ can be expressed as

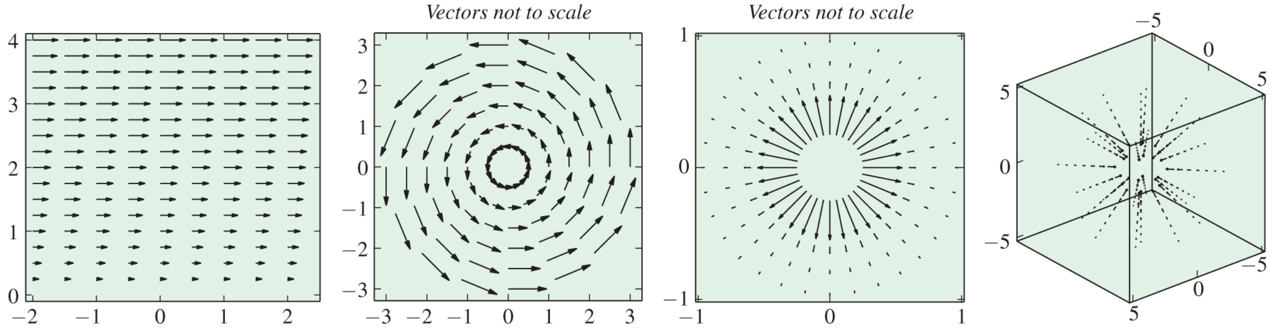
$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

GRAPHICAL REPRESENTATIONS OF VECTOR FIELDS

A vector field in 2-space can be pictured geometrically by drawing representative field vectors $\mathbf{F}(x, y)$ at some well-chosen points in the xy-plane. Such graphical representations can provide useful information about the general behavior of the field if the vectors are chosen appropriately. Figure 15.1.3 shows four computer-generated vector fields. The vector field in part (a) might describe the velocity of the current in a stream at various depths. At the bottom of the stream the velocity is zero, but the speed of the current increases as the depth decreases. Points at the same depth have the same speed. The vector field in part (b) might describe the velocity at points on a rotating wheel. At the center of the wheel the velocity is zero, but the speed increases with the distance from the center. Points at the same distance from the center have the same speed. The vector field in part (c) might describe the repulsive force of an electrical charge—the closer to the charge, the greater the force of repulsion. Part (d) shows a vector field in 3-space. Such pictures tend to be cluttered and hence are of lesser value than graphical representations of vector fields in 2-space. Note also that the vectors in parts (b) and (c) are not to scale—their lengths have been compressed for clarity. We will follow this procedure throughout this chapter.

TECHNOLOGY MASTERY

If you have a graphing utility that can generate vector fields, read the relevant documentation and try to make reasonable duplicates of parts (a) and (b) of Figure 15.1.3.



$\mathbf{F}(x, y) = \frac{1}{5}\sqrt{y}\,\mathbf{i}$ (a)

$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ (b)

$\mathbf{F}(x, y) = \dfrac{x\mathbf{i} + y\mathbf{j}}{10(x^2 + y^2)^{3/2}}$ (c)

$\mathbf{F}(x, y, z) = -\dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$ (d)

▲ **Figure 15.1.3**

A COMPACT NOTATION FOR VECTOR FIELDS

Sometimes it is helpful to denote the vector fields $\mathbf{F}(x, y)$ and $\mathbf{F}(x, y, z)$ entirely in vector notation by identifying (x, y) with the radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and (x, y, z) with the radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. With this notation a vector field in either 2-space or 3-space can be written as $\mathbf{F}(\mathbf{r})$. When no confusion is likely to arise, we will sometimes omit the $\mathbf{r}$ altogether and denote the vector field as $\mathbf{F}$.

INVERSE-SQUARE FIELDS

According to Newton's Law of Universal Gravitation, particles with masses m and M attract each other with a force $\mathbf{F}$ of magnitude

$$\|\mathbf{F}\| = \frac{GmM}{r^2} \tag{1}$$

where r is the distance between the particles and G is a constant. If we assume that the particle of mass M is located at the origin of an xyz-coordinate system and $\mathbf{r}$ is the radius vector to the particle of mass m, then $r = \|\mathbf{r}\|$, and the force $\mathbf{F}(\mathbf{r})$ exerted by the particle of mass M on the particle of mass m is in the direction of the unit vector $-\mathbf{r}/\|\mathbf{r}\|$. Thus, it follows from (1) that

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r} \tag{2}$$

If m and M are constant, and we let $c = -GmM$, then this formula can be expressed as

$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}$$

Vector fields of this form arise in electromagnetic as well as gravitational problems. Such fields are so important that they have their own terminology.

15.1.2 DEFINITION If $\mathbf{r}$ is a radius vector in 2-space or 3-space, and if c is a constant, then a vector field of the form

$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r} \tag{3}$$

is called an ***inverse-square field***.

Observe that if $c > 0$ in (3), then $\mathbf{F}(\mathbf{r})$ has the same direction as $\mathbf{r}$, so each vector in the field is directed away from the origin; and if $c < 0$, then $\mathbf{F}(\mathbf{r})$ is oppositely directed to $\mathbf{r}$, so each vector in the field is directed toward the origin. In either case the magnitude of $\mathbf{F}(\mathbf{r})$ is inversely proportional to the square of the distance from the terminal point of $\mathbf{r}$ to the origin, since

$$\|\mathbf{F}(\mathbf{r})\| = \frac{|c|}{\|\mathbf{r}\|^3}\|\mathbf{r}\| = \frac{|c|}{\|\mathbf{r}\|^2}$$

We leave it for you to verify that in 2-space Formula (3) can be written in component form as

$$\mathbf{F}(x, y) = \frac{c}{(x^2 + y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j}) \tag{4}$$

and in 3-space as

$$\mathbf{F}(x, y, z) = \frac{c}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{5}$$

[see parts (c) and (d) of Figure 15.1.3].

▶ **Example 1** ***Coulomb's law*** states that *the electrostatic force exerted by one charged particle on another is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.* This has the same form as Newton's Law of Universal Gravitation, so the electrostatic force field exerted by a charged particle is an inverse-square field. Specifically, if a particle of charge Q is at the origin of a coordinate system, and if $\mathbf{r}$ is the radius vector to a particle of charge q, then the force $\mathbf{F}(\mathbf{r})$ that the particle of charge Q exerts on the particle of charge q is of the form

$$\mathbf{F}(\mathbf{r}) = \frac{qQ}{4\pi\epsilon_0\|\mathbf{r}\|^3}\mathbf{r}$$

where ϵ_0 is a positive constant (called the ***permittivity constant***). This formula is of form (3) with $c = qQ/4\pi\epsilon_0$. ◀

GRADIENT FIELDS

An important class of vector fields comes from gradients. Recall that if ϕ is a function of three variables, then the gradient of ϕ is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

This formula defines a vector field in 3-space called the ***gradient field of*** ϕ. Similarly, the gradient of a function of two variables defines a gradient field in 2-space. At each point in a gradient field where the gradient is nonzero, the vector points in the direction in which the rate of increase of ϕ is maximum.

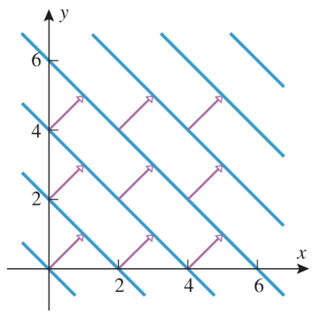


Figure 15.1.4

Example 2 Sketch the gradient field of $\phi(x, y) = x + y$.

Solution. The gradient of ϕ is

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = \mathbf{i} + \mathbf{j}$$

which is constant [i.e., is the same vector at each point (x, y)]. A portion of the vector field is sketched in Figure 15.1.4 together with some level curves of ϕ. Note that at each point, $\nabla\phi$ is normal to the level curve of ϕ through the point (Theorem 13.6.6). ◀

CONSERVATIVE FIELDS AND POTENTIAL FUNCTIONS

If $\mathbf{F}(\mathbf{r})$ is an arbitrary vector field in 2-space or 3-space, we can ask whether it is the gradient field of some function ϕ, and if so, how we can find ϕ. This is an important problem and we will study it in more detail later. However, there is some terminology for such fields that we will introduce now.

15.1.3 DEFINITION A vector field $\mathbf{F}$ in 2-space or 3-space is said to be ***conservative*** in a region if it is the gradient field for some function ϕ in that region, that is, if

$$\mathbf{F} = \nabla\phi$$

The function ϕ is called a ***potential function*** for $\mathbf{F}$ in the region.

Example 3 Inverse-square fields are conservative in any region that does not contain the origin. For example, in the two-dimensional case the function

$$\phi(x, y) = -\frac{c}{(x^2 + y^2)^{1/2}} \tag{6}$$

is a potential function for (4) in any region not containing the origin, since

$$\begin{aligned}
\nabla\phi(x, y) &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} \\
&= \frac{cx}{(x^2 + y^2)^{3/2}}\mathbf{i} + \frac{cy}{(x^2 + y^2)^{3/2}}\mathbf{j} \\
&= \frac{c}{(x^2 + y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j}) \\
&= \mathbf{F}(x, y)
\end{aligned}$$ ◀

DIVERGENCE AND CURL

We will now define two important operations on vector fields in 3-space—the *divergence* and the *curl* of the field. These names originate in the study of fluid flow, in which case the divergence relates to the way in which fluid flows toward or away from a point and the curl relates to the rotational properties of the fluid at a point. We will investigate the physical interpretations of these operations in more detail later, but for now we will focus only on their computation.

15.1.4 DEFINITION If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then we define the ***divergence of*** $\mathbf{F}$, written div $\mathbf{F}$, to be the function given by

$$\text{div}\,\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{7}$$

15.1.5 DEFINITION If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then we define the ***curl of* F**, written curl **F**, to be the vector field given by

$$\text{curl } \mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} \tag{8}$$

REMARK Observe that div **F** and curl **F** depend on the point at which they are computed, and hence are more properly written as div $\mathbf{F}(x, y, z)$ and curl $\mathbf{F}(x, y, z)$. However, even though these functions are expressed in terms of x, y, and z, it can be proved that their values at a fixed point depend only on the point and not on the coordinate system selected. This is important in applications, since it allows physicists and engineers to compute the curl and divergence in any convenient coordinate system.

Before proceeding to some examples, we note that div **F** has scalar values, whereas curl **F** has vector values (i.e., curl **F** is itself a vector field). Moreover, for computational purposes it is useful to note that the formula for the curl can be expressed in the determinant form

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{9}$$

You should verify that Formula (8) results if the determinant is computed by interpreting a "product" such as $(\partial/\partial x)(g)$ to mean $\partial g/\partial x$. Keep in mind, however, that (9) is just a mnemonic device and not a true determinant, since the entries in a determinant must be numbers, not vectors and partial derivative symbols.

► Example 4 Find the divergence and the curl of the vector field

$$\mathbf{F}(x, y, z) = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

Solution. From (7)

$$\begin{aligned} \text{div } \mathbf{F} &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z) \\ &= 2xy + 6y^2z + 3 \end{aligned}$$

and from (9)

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y^3z)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(3z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(2y^3z) - \frac{\partial}{\partial y}(x^2y)\right]\mathbf{k} \\ &= -2y^3\mathbf{i} - x^2\mathbf{k} \end{aligned}$$ ◄

TECHNOLOGY MASTERY

Most computer algebra systems can compute gradient fields, divergence, and curl. If you have a CAS with these capabilities, read the relevant documentation, and use your CAS to check the computations in Examples 2 and 4.

► Example 5 Show that the divergence of the inverse-square field

$$\mathbf{F}(x, y, z) = \frac{c}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

is zero.

Solution. The computations can be simplified by letting $r = (x^2 + y^2 + z^2)^{1/2}$, in which case $\mathbf{F}$ can be expressed as

$$\mathbf{F}(x, y, z) = \frac{cx\mathbf{i} + cy\mathbf{j} + cz\mathbf{k}}{r^3} = \frac{cx}{r^3}\mathbf{i} + \frac{cy}{r^3}\mathbf{j} + \frac{cz}{r^3}\mathbf{k}$$

We leave it for you to show that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Thus

$$\text{div } \mathbf{F} = c\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right] \tag{10}$$

But

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{r^3 - x(3r^2)(x/r)}{(r^3)^2} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{1}{r^3} - \frac{3y^2}{r^5}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{1}{r^3} - \frac{3z^2}{r^5}$$

Substituting these expressions in (10) yields

$$\text{div } \mathbf{F} = c\left[\frac{3}{r^3} - \frac{3x^2 + 3y^2 + 3z^2}{r^5}\right] = c\left[\frac{3}{r^3} - \frac{3r^2}{r^5}\right] = 0 \blacktriangleleft$$

THE ∇ OPERATOR

Thus far, the symbol ∇ that appears in the gradient expression $\nabla\phi$ has not been given a meaning of its own. However, it is often convenient to view ∇ as an operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{11}$$

which when applied to $\phi(x, y, z)$ produces the gradient

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

We call (11) the ***del operator***. This is analogous to the derivative operator d/dx, which when applied to $f(x)$ produces the derivative $f'(x)$.

The del operator allows us to express the divergence of a vector field

$$\mathbf{F} = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

in dot product notation as

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{12}$$

and the curl of this field in cross-product notation as

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{13}$$

THE LAPLACIAN ∇^2

The operator that results by taking the dot product of the del operator with itself is denoted by ∇^2 and is called the ***Laplacian operator***. This operator has the form

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{14}$$

When applied to $\phi(x, y, z)$ the Laplacian operator produces the function

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Note that $\nabla^2\phi$ can also be expressed as div $(\nabla\phi)$. The equation $\nabla^2\phi = 0$ or, equivalently,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

is known as ***Laplace's equation***. This partial differential equation plays an important role in a wide variety of applications, resulting from the fact that it is satisfied by the potential function for the inverse-square field.

✓QUICK CHECK EXERCISES 15.1 *(See page 979 for answers.)*

1. The function $\phi(x, y, z) = xy + yz + xz$ is a potential for the vector field $\mathbf{F} =$ ________.

2. The vector field $\mathbf{F}(x, y, z) =$ ________, defined for $(x, y, z) \neq (0, 0, 0)$, is always directed toward the origin and is of length equal to the distance from (x, y, z) to the origin.

3. An inverse-square field is one that can be written in the form $\mathbf{F}(\mathbf{r}) =$ ________.

4. The vector field

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + xy^2\mathbf{j} + yz^2\mathbf{k}$$

has divergence ________ and curl ________.

Pierre-Simon de Laplace (1749–1827) French mathematician and physicist. Laplace is sometimes referred to as the French Isaac Newton because of his work in celestial mechanics. In a five-volume treatise entitled *Traité de Mécanique Céleste*, he solved extremely difficult problems involving gravitational interactions between the planets. In particular, he was able to show that our solar system is stable and not prone to catastrophic collapse as a result of these interactions. This was an issue of major concern at the time because Jupiter's orbit appeared to be shrinking and Saturn's expanding; Laplace showed that these were expected periodic anomalies. In addition to his work in celestial mechanics, he founded modern probability theory, showed with Lavoisier that respiration is a form of combustion, and developed methods that fostered many new branches of pure mathematics.

Laplace was born to moderately successful parents in Normandy, his father being a farmer and cider merchant. He matriculated in the theology program at the University of Caen at age 16 but left for Paris at age 18 with a letter of introduction to the influential mathematician d'Alembert, who eventually helped him undertake a career in mathematics. Laplace was a prolific writer, and after his election to the Academy of Sciences in 1773, the secretary wrote that the Academy had never received so many important research papers by so young a person in such a short time. Laplace had little interest in pure mathematics—he regarded mathematics merely as a tool for solving applied problems. In his impatience with mathematical detail, he frequently omitted complicated arguments with the statement, "It is easy to show that. . . ." He admitted, however, that as time passed he often had trouble reconstructing the omitted details himself!

At the height of his fame, Laplace served on many government committees and held the posts of Minister of the Interior and Chancellor of the Senate. He barely escaped imprisonment and execution during the period of the Revolution, probably because he was able to convince each opposing party that he sided with them. Napoleon described him as a great mathematician but a poor administrator who "sought subtleties everywhere, had only doubtful ideas, and . . . carried the spirit of the infinitely small into administration." In spite of his genius, Laplace was both egotistic and insecure, attempting to ensure his place in history by conveniently failing to credit mathematicians whose work he used—an unnecessary pettiness since his own work was so brilliant. However, on the positive side he was supportive of young mathematicians, often treating them as his own children. Laplace ranks as one of the most influential mathematicians in history.

[*Image: georgios/Depositphotos*]

EXERCISE SET 15.1 Graphing Utility CAS

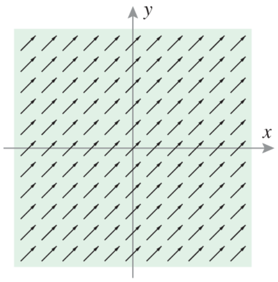
FOCUS ON CONCEPTS

1–2 Match the vector field $\mathbf{F}(x, y)$ with one of the plots, and explain your reasoning.

1. (a) $\mathbf{F}(x, y) = x\mathbf{i}$ (b) $\mathbf{F}(x, y) = \sin x\mathbf{i} + \mathbf{j}$

2. (a) $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

(b) $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$



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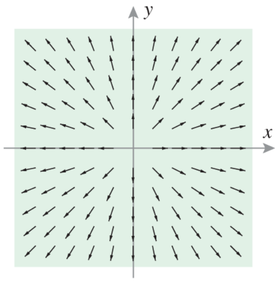

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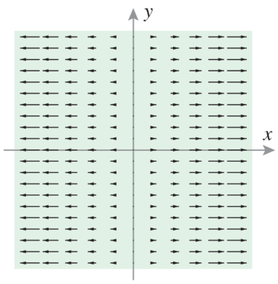
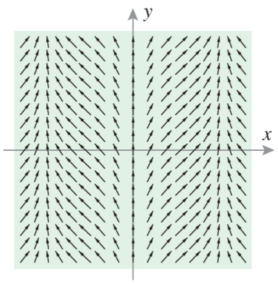


III



IV

3–4 Determine whether the statement about the vector field $\mathbf{F}(x, y)$ is true or false. If false, explain why.

3. $\mathbf{F}(x, y) = x^2\mathbf{i} - y\mathbf{j}$.

(a) $\|\mathbf{F}(x, y)\| \to 0$ as $(x, y) \to (0, 0)$.

(b) If (x, y) is on the positive y-axis, then the vector points in the negative y-direction.

(c) If (x, y) is in the first quadrant, then the vector points down and to the right.

4. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$.

(a) As (x, y) moves away from the origin, the lengths of the vectors decrease.

(b) If (x, y) is a point on the positive x-axis, then the vector points up.

(c) If (x, y) is a point on the positive y-axis, the vector points to the right.

5–8 Sketch the vector field by drawing some representative nonintersecting vectors. The vectors need not be drawn to scale, but they should be in reasonably correct proportion relative to each other.

5. $\mathbf{F}(x, y) = 2\mathbf{i} - \mathbf{j}$ **6.** $\mathbf{F}(x, y) = y\mathbf{j}, \quad y > 0$

7. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$. [*Note:* Each vector in the field is perpendicular to the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.]

8. $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$. [*Note:* Each vector in the field is a unit vector in the same direction as the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.]

9–10 Use a graphing utility to generate a plot of the vector field.

9. $\mathbf{F}(x, y) = \mathbf{i} + \cos y\mathbf{j}$ **10.** $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

11–14 True–False Determine whether the statement is true or false. Explain your answer.

11. The vector-valued function

$$\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j} + xy\mathbf{k}$$

is an example of a vector field in the xy-plane.

12. If $\mathbf{r}$ is a radius vector in 3-space, then a vector field of the form

$$\mathbf{F}(\mathbf{r}) = \frac{1}{\|\mathbf{r}\|^2}\mathbf{r}$$

is an example of an inverse-square field.

13. If $\mathbf{F}$ is a vector field, then so is $\nabla \times \mathbf{F}$.

14. If $\mathbf{F}$ is a vector field and $\nabla \cdot \mathbf{F} = \phi$, then ϕ is a potential function for $\mathbf{F}$.

15–16 Confirm that ϕ is a potential function for $\mathbf{F}(\mathbf{r})$ on some region, and state the region.

15. (a) $\phi(x, y) = \tan^{-1} xy$

$\mathbf{F}(x, y) = \dfrac{y}{1 + x^2y^2}\mathbf{i} + \dfrac{x}{1 + x^2y^2}\mathbf{j}$

(b) $\phi(x, y, z) = x^2 - 3y^2 + 4z^2$

$\mathbf{F}(x, y, z) = 2x\mathbf{i} - 6y\mathbf{j} + 8z\mathbf{k}$

16. (a) $\phi(x, y) = 2y^2 + 3x^2y - xy^3$

$\mathbf{F}(x, y) = (6xy - y^3)\mathbf{i} + (4y + 3x^2 - 3xy^2)\mathbf{j}$

(b) $\phi(x, y, z) = x \sin z + y \sin x + z \sin y$

$\mathbf{F}(x, y, z) = (\sin z + y \cos x)\mathbf{i} + (\sin x + z \cos y)\mathbf{j} + (\sin y + x \cos z)\mathbf{k}$

17–22 Find div $\mathbf{F}$ and curl $\mathbf{F}$.

17. $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2\mathbf{j} + yz\mathbf{k}$

18. $\mathbf{F}(x, y, z) = xz^3\mathbf{i} + 2y^4x^2\mathbf{j} + 5z^2y\mathbf{k}$

19. $\mathbf{F}(x, y, z) = 7y^3z^2\mathbf{i} - 8x^2z^5\mathbf{j} - 3xy^4\mathbf{k}$

20. $\mathbf{F}(x, y, z) = e^{xy}\mathbf{i} - \cos y\mathbf{j} + \sin^2 z\mathbf{k}$

21. $\mathbf{F}(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

22. $\mathbf{F}(x, y, z) = \ln x\mathbf{i} + e^{xyz}\mathbf{j} + \tan^{-1}(z/x)\mathbf{k}$

23–24 Find $\nabla \cdot (\mathbf{F} \times \mathbf{G})$.

23. $\mathbf{F}(x, y, z) = 2x\mathbf{i} + \mathbf{j} + 4y\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$

24. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$\mathbf{G}(x, y, z) = xy\mathbf{j} + xyz\mathbf{k}$

25–26 Find $\nabla \cdot (\nabla \times \mathbf{F})$.

25. $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos(x - y)\mathbf{j} + z\mathbf{k}$

26. $\mathbf{F}(x, y, z) = e^{xz}\mathbf{i} + 3xe^{y}\mathbf{j} - e^{yz}\mathbf{k}$

27–28 Find $\nabla \times (\nabla \times \mathbf{F})$.

27. $\mathbf{F}(x, y, z) = xy\mathbf{j} + xyz\mathbf{k}$

28. $\mathbf{F}(x, y, z) = y^2x\mathbf{i} - 3yz\mathbf{j} + xy\mathbf{k}$

C **29.** Use a CAS to check the calculations in Exercises 23, 25, and 27.

C **30.** Use a CAS to check the calculations in Exercises 24, 26, and 28.

31–38 Let k be a constant, $\mathbf{F} = \mathbf{F}(x, y, z)$, $\mathbf{G} = \mathbf{G}(x, y, z)$, and $\phi = \phi(x, y, z)$. Prove the following identities, assuming that all derivatives involved exist and are continuous.

31. $\operatorname{div}(k\mathbf{F}) = k \operatorname{div} \mathbf{F}$ **32.** $\operatorname{curl}(k\mathbf{F}) = k \operatorname{curl} \mathbf{F}$

33. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$

34. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$

35. $\operatorname{div}(\phi\mathbf{F}) = \phi \operatorname{div} \mathbf{F} + \nabla\phi \cdot \mathbf{F}$

36. $\operatorname{curl}(\phi\mathbf{F}) = \phi \operatorname{curl} \mathbf{F} + \nabla\phi \times \mathbf{F}$

37. $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ **38.** $\operatorname{curl}(\nabla\phi) = \mathbf{0}$

39. Rewrite the identities in Exercises 31, 33, 35, and 37 in an equivalent form using the notation $\nabla\cdot$ for divergence and $\nabla\times$ for curl.

40. Rewrite the identities in Exercises 32, 34, 36, and 38 in an equivalent form using the notation $\nabla\cdot$ for divergence and $\nabla\times$ for curl.

41–42 Verify that the radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has the stated property.

41. (a) $\operatorname{curl} \mathbf{r} = \mathbf{0}$ (b) $\nabla\|\mathbf{r}\| = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$

42. (a) $\operatorname{div} \mathbf{r} = 3$ (b) $\nabla\dfrac{1}{\|\mathbf{r}\|} = -\dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}$

43–44 Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, let $r = \|\mathbf{r}\|$, let f be a differentiable function of one variable, and let $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{r}$.

43. (a) Use the chain rule and Exercise 41(b) to show that

$$\nabla f(r) = \frac{f'(r)}{r}\mathbf{r}$$

(b) Use the result in part (a) and Exercises 35 and 42(a) to show that $\operatorname{div} \mathbf{F} = 3f(r) + rf'(r)$.

44. (a) Use part (a) of Exercise 43, Exercise 36, and Exercise 41(a) to show that $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

(b) Use the result in part (a) of Exercise 43 and Exercises 35 and 42(a) to show that

$$\nabla^2 f(r) = 2\frac{f'(r)}{r} + f''(r)$$

45. Use the result in Exercise 43(b) to show that the divergence of the inverse-square field $\mathbf{F} = \mathbf{r}/\|\mathbf{r}\|^3$ is zero.

46. Use the result of Exercise 43(b) to show that if $\mathbf{F}$ is a vector field of the form $\mathbf{F} = f(\|\mathbf{r}\|)\mathbf{r}$ and if $\operatorname{div} \mathbf{F} = 0$, then $\mathbf{F}$ is an inverse-square field. [*Suggestion:* Let $r = \|\mathbf{r}\|$ and multiply $3f(r) + rf'(r) = 0$ through by r^2. Then write the result as a derivative of a product.]

47. A curve C is called a ***flow line*** of a vector field $\mathbf{F}$ if $\mathbf{F}$ is a tangent vector to C at each point along C (see the accompanying figure).

(a) Let C be a flow line for $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$, and let (x, y) be a point on C for which $y \neq 0$. Show that the flow lines satisfy the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) Solve the differential equation in part (a) by separation of variables, and show that the flow lines are concentric circles centered at the origin.

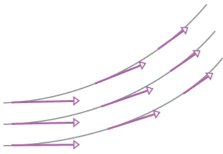


Figure Ex-47

48–50 Find a differential equation satisfied by the flow lines of $\mathbf{F}$ (see Exercise 47), and solve it to find equations for the flow lines of $\mathbf{F}$. Sketch some typical flow lines and tangent vectors.

48. $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ **49.** $\mathbf{F}(x, y) = x\mathbf{i} + \mathbf{j}, \quad x > 0$

50. $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}, \quad x > 0$ and $y > 0$

51. **Writing** Discuss the similarities and differences between the concepts "vector field" and "slope field."

52. **Writing** In physical applications it is often necessary to deal with vector quantities that depend not only on position in space but also on time. Give some examples and discuss how the concept of a vector field would need to be modified to apply to such situations.

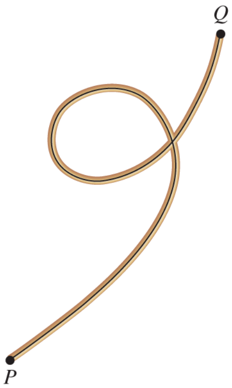
✓ QUICK CHECK ANSWERS 15.1 **1.** $(y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$ **2.** $-\mathbf{r} = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ **3.** $\dfrac{c}{\|\mathbf{r}\|^3}\mathbf{r}$ **4.** $2xy + 2yz;\ z^2\mathbf{i} + y\mathbf{j} + (y^2 - z)\mathbf{k}$

15.2 LINE INTEGRALS

In earlier chapters we considered three kinds of integrals in rectangular coordinates: single integrals over intervals, double integrals over two-dimensional regions, and triple integrals over three-dimensional regions. In this section we will discuss integrals along curves in two- or three-dimensional space.

LINE INTEGRALS

The first goal of this section is to define what it means to integrate a function along a curve. To motivate the definition we will consider the problem of finding the mass of a very thin wire whose linear density function (mass per unit length) is known. We assume that we can model the wire by a smooth curve C between two points P and Q in 3-space (Figure 15.2.1). Given any point (x, y, z) on C, we let $f(x, y, z)$ denote the corresponding value of the density function. To compute the mass of the wire, we proceed as follows:



▲ **Figure 15.2.1** A bent thin wire modeled by a smooth curve

- Divide C into n very small sections using a succession of distinct partition points

$$P = P_0, P_1, P_2, \dots, P_{n-1}, P_n = Q$$

as illustrated on the left side of Figure 15.2.2. Let ΔM_k be the mass of the kth section, and let Δs_k be the length of the arc between P_{k-1} and P_k.

- Choose an arbitrary sampling point $P_k^*(x_k^*, y_k^*, z_k^*)$ on the kth arc, as illustrated on the right side of Figure 15.2.2. If Δs_k is very small, the value of f will not vary much along the kth section and we can approximate f along this section by the value $f(x_k^*, y_k^*, z_k^*)$. It follows that the mass of the kth section can be approximated by

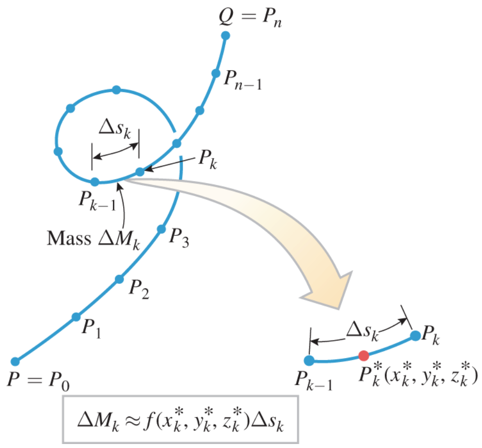
$$\Delta M_k \approx f(x_k^*, y_k^*, z_k^*)\Delta s_k$$

- The mass M of the entire wire can then be approximated by

$$M = \sum_{k=1}^{n} \Delta M_k \approx \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k \tag{1}$$

- We will use the expression $\max \Delta s_k \to 0$ to indicate the process of increasing n in such a way that the lengths of all the sections approach 0. It is plausible that the error in (1) will approach 0 as $\max \Delta s_k \to 0$ and the exact value of M will be given by

$$M = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k \tag{2}$$



▶ **Figure 15.2.2**

The limit in (2) is similar to the limit of Riemann sums used to define the definite integral of a function over an interval (Definition 4.5.1). With this similarity in mind, we make the following definition.

15.2.1 DEFINITION If C is a smooth curve in 2-space or 3-space, then the ***line integral of f with respect to s along C*** is

$$\int_C f(x, y)\, ds = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k \qquad \boxed{\text{2-space}} \tag{3}$$

or

$$\int_C f(x, y, z)\, ds = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k \qquad \boxed{\text{3-space}} \tag{4}$$

provided this limit exists and does not depend on the choice of partition or on the choice of sample points.

Although the term "curve integrals" is more descriptive, the integrals in Definition 15.2.1 are called "line integrals" for historical reasons.

It is usually impractical to evaluate line integrals directly from Definition 15.2.1. However, the definition is important in the application and interpretation of line integrals. For example:

- If C is a curve in 3-space that models a thin wire, and if $f(x, y, z)$ is the linear density function of the wire, then it follows from (2) and Definition 15.2.1 that the mass M of the wire is given by

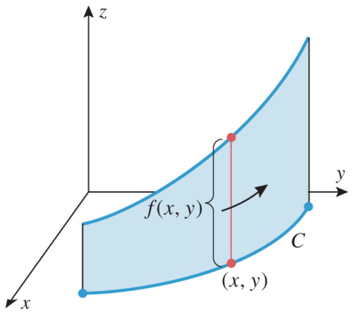
$$M = \int_C f(x, y, z)\, ds \tag{5}$$

  That is, to obtain the mass of a thin wire, we integrate the linear density function over the smooth curve that models the wire.
- If C is a smooth curve of arc length L, and f is identically 1, then it immediately follows from Definition 15.2.1 that

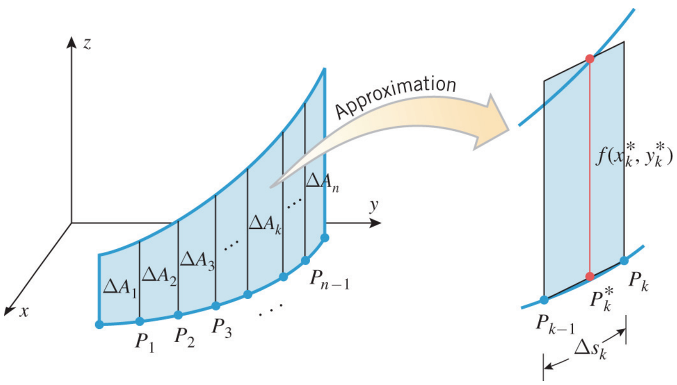
$$\int_C ds = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} \Delta s_k = \lim_{\max \Delta s_k \to 0} L = L \tag{6}$$

- If C is a curve in the xy-plane and $f(x, y)$ is a nonnegative continuous function defined on C, then $\int_C f(x, y)\, ds$ can be interpreted as the area A of the "sheet" that is swept out by a vertical line segment that extends upward from the point (x, y) to a height of $f(x, y)$ and moves along C from one endpoint to the other (Figure 15.2.3). To see why this is so, refer to Figure 15.2.4 in which $f(x_k^*, y_k^*)$ is the value of f at an arbitrary point P_k^* on the kth arc of the partition and note the approximation

$$\Delta A_k \approx f(x_k^*, y_k^*)\Delta s_k$$



▲ Figure 15.2.3



▶ Figure 15.2.4

It follows that

$$A = \sum_{k=1}^{n} \Delta A_k \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k$$

It is then plausible that

$$A = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k = \int_C f(x, y)\, ds \tag{7}$$

Since Definition 15.2.1 is closely modeled on Definition 4.5.1, line integrals share many of the common properties of ordinary definite integrals. For example, we have

$$\int_C [f(x, y) + g(x, y)]\, ds = \int_C f(x, y)\, ds + \int_C g(x, y)\, ds$$

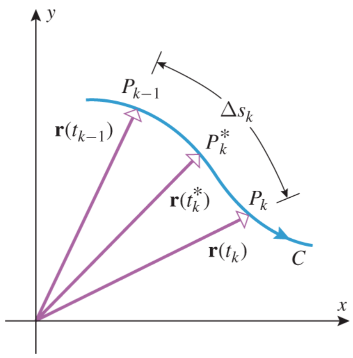
provided both line integrals on the right-hand side of this equation exist. Similarly, it can be shown that if f is continuous on C, then the line integral of f with respect to s along C exists.

EVALUATING LINE INTEGRALS

Except in simple cases, it will not be feasible to evaluate a line integral directly from (3) or (4). However, we will now show that it is possible to express a line integral as an ordinary definite integral, so that no special methods of evaluation are required. For example, suppose that C is a curve in the xy-plane that is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad (a \le t \le b)$$

Moreover, suppose that each partition point P_k of C corresponds to a parameter value of t_k in $[a, b]$. The arc length of C between points P_{k-1} and P_k is then given by



▲ **Figure 15.2.5**

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\mathbf{r}'(t)\|\, dt \tag{8}$$

(Theorem 12.3.1). If we let $\Delta t_k = t_k - t_{k-1}$, then it follows from (8) and the Mean-Value Theorem for Integrals (Theorem 4.6.2) that there exists a point t_k^* in $[t_{k-1}, t_k]$ such that

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\mathbf{r}'(t)\|\, dt = \|\mathbf{r}'(t_k^*)\|\Delta t_k$$

We let $P_k^*(x_k^*, y_k^*) = P_k^*(x(t_k^*), y(t_k^*))$ correspond to the parameter value t_k^* (Figure 15.2.5).

Since the parametrization of C is smooth, it can be shown that $\max \Delta s_k \to 0$ if and only if $\max \Delta t_k \to 0$ (Exercise 57). Furthermore, the composition $f(x(t), y(t))$ is a real-valued function defined on $[a, b]$ and we have

$$\begin{aligned}
\int_C f(x, y)\, ds &= \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta s_k && \boxed{\text{Definition 15.2.1}}\\
&= \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\|\mathbf{r}'(t_k^*)\|\Delta t_k && \boxed{\text{Substitution}}\\
&= \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\|\mathbf{r}'(t_k^*)\|\Delta t_k\\
&= \int_a^b f(x(t), y(t))\|\mathbf{r}'(t)\|\, dt && \boxed{\text{Definition 4.5.1}}
\end{aligned}$$

Therefore, if C is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad (a \le t \le b)$$

then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\|\mathbf{r}'(t)\|\, dt \tag{9}$$

Similarly, if C is a curve in 3-space that is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad (a \le t \le b)$$

Explain how Formulas (9) and (10) confirm Formula (6) for arc length.

then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\|\mathbf{r}'(t)\|\, dt \tag{10}$$

Example 1 Using the given parametrization, evaluate the line integral $\int_C (1 + xy^2)\, ds$.

(a) $C : \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} \quad (0 \le t \le 1)$ (see Figure 15.2.6*a*)

(b) $C : \mathbf{r}(t) = (1 - t)\mathbf{i} + (2 - 2t)\mathbf{j} \quad (0 \le t \le 1)$ (see Figure 15.2.6*b*)

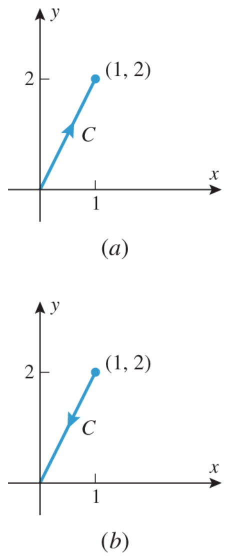


Figure 15.2.6

Solution (a). Since $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}$, we have $\|\mathbf{r}'(t)\| = \sqrt{5}$ and it follows from Formula (9) that

$$\begin{aligned}\int_C (1 + xy^2)\, ds &= \int_0^1 [1 + t(2t)^2]\sqrt{5}\, dt \\ &= \int_0^1 (1 + 4t^3)\sqrt{5}\, dt \\ &= \sqrt{5}\left[t + t^4\right]_0^1 = 2\sqrt{5}\end{aligned}$$

Solution (b). Since $\mathbf{r}'(t) = -\mathbf{i} - 2\mathbf{j}$, we have $\|\mathbf{r}'(t)\| = \sqrt{5}$ and it follows from Formula (9) that

$$\begin{aligned}\int_C (1 + xy^2)\, ds &= \int_0^1 [1 + (1 - t)(2 - 2t)^2]\sqrt{5}\, dt \\ &= \int_0^1 [1 + 4(1 - t)^3]\sqrt{5}\, dt \\ &= \sqrt{5}\left[t - (1 - t)^4\right]_0^1 = 2\sqrt{5}\end{aligned}$$ ◄

Note that the integrals in parts (a) and (b) of Example 1 agree, even though the corresponding parametrizations of C have opposite orientations. This illustrates the important result that the value of a line integral of f with respect to s along C *does not depend on an orientation of* C. (This is because Δs_k is always positive; therefore, it does not matter in which *direction* along C we list the partition points of the curve in Definition 15.2.1.) Later in this section we will discuss line integrals that are defined only for oriented curves.

Formula (9) has an alternative expression for a curve C in the xy-plane that is given by parametric equations

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

In this case, we write (9) in the expanded form

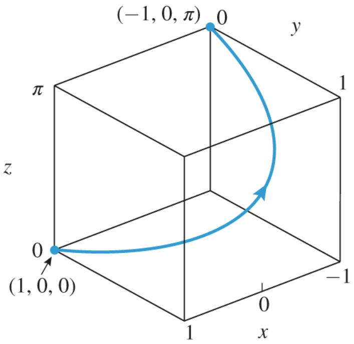
$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{11}$$

Similarly, if C is a curve in 3-space that is parametrized by

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (a \le t \le b)$$

then we write (10) in the form

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{12}$$



▲ **Figure 15.2.7**

▶ **Example 2** Evaluate the line integral $\int_C (xy + z^3)\,ds$ from $(1, 0, 0)$ to $(-1, 0, \pi)$ along the helix C that is represented by the parametric equations

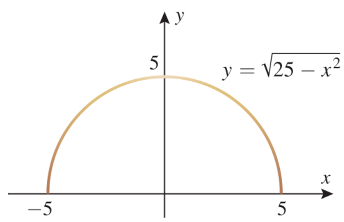
$$x = \cos t, \quad y = \sin t, \quad z = t \qquad (0 \le t \le \pi)$$

(Figure 15.2.7).

Solution. From (12)

$$\begin{aligned}
\int_C (xy + z^3)\,ds &= \int_0^{\pi} (\cos t \sin t + t^3)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\
&= \int_0^{\pi} (\cos t \sin t + t^3)\sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\,dt \\
&= \sqrt{2}\int_0^{\pi} (\cos t \sin t + t^3)\,dt \\
&= \sqrt{2}\left[\frac{\sin^2 t}{2} + \frac{t^4}{4}\right]_0^{\pi} = \frac{\sqrt{2}\pi^4}{4} \blacktriangleleft
\end{aligned}$$

If $\delta(x, y)$ is the linear density function of a wire that is modeled by a smooth curve C in the xy-plane, then an argument similar to the derivation of Formula (5) shows that the mass of the wire is given by $\int_C \delta(x, y)\,ds$.



▲ **Figure 15.2.8**

▶ **Example 3** Suppose that a semicircular wire has the equation $y = \sqrt{25 - x^2}$ and that its mass density is $\delta(x, y) = 15 - y$ (Figure 15.2.8). Physically, this means the wire has a maximum density of 15 units at the base ($y = 0$) and that the density of the wire decreases linearly with respect to y to a value of 10 units at the top ($y = 5$). Find the mass of the wire.

Solution. The mass M of the wire can be expressed as the line integral

$$M = \int_C \delta(x, y)\,ds = \int_C (15 - y)\,ds$$

along the semicircle C. To evaluate this integral we will express C parametrically as

$$x = 5\cos t, \quad y = 5\sin t \qquad (0 \le t \le \pi)$$

Thus, it follows from (11) that

$$\begin{aligned}
M = \int_C (15 - y)\,ds &= \int_0^{\pi} (15 - 5\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^{\pi} (15 - 5\sin t)\sqrt{(-5\sin t)^2 + (5\cos t)^2}\,dt \\
&= 5\int_0^{\pi} (15 - 5\sin t)\,dt \\
&= 5\left[15t + 5\cos t\right]_0^{\pi} \\
&= 75\pi - 50 \approx 185.6 \text{ units of mass} \blacktriangleleft
\end{aligned}$$

In the special case where t is an arc length parameter, say $t = s$, it follows from Formulas (20) and (21) in Section 12.3 that the radicals in (11) and (12) reduce to 1 and the equations simplify to

$$\int_C f(x, y)\,ds = \int_a^b f(x(s), y(s))\,ds \tag{13}$$

and

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(s), y(s), z(s))\,ds \tag{14}$$

respectively.

Example 4 Find the area of the surface extending upward from the circle $x^2 + y^2 = 1$ in the xy-plane to the parabolic cylinder $z = 1 - x^2$ (Figure 15.2.9).

Solution. It follows from (7) that the area A of the surface can be expressed as the line integral

$$A = \int_C (1 - x^2)\, ds \tag{15}$$

where C is the circle $x^2 + y^2 = 1$. This circle can be parametrized in terms of arc length as

$$x = \cos s, \quad y = \sin s \qquad (0 \le s \le 2\pi)$$

Thus, it follows from (13) and (15) that

$$\begin{aligned} A &= \int_C (1 - x^2)\, ds = \int_0^{2\pi} (1 - \cos^2 s)\, ds \\ &= \int_0^{2\pi} \sin^2 s\, ds = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2s)\, ds = \pi \end{aligned}$$

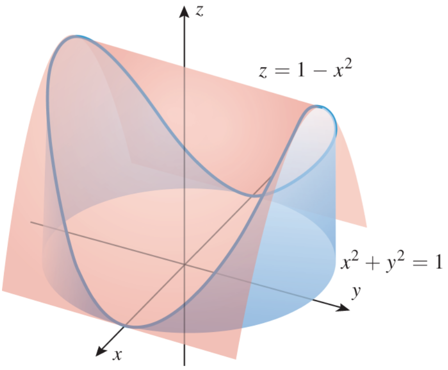


Figure 15.2.9

LINE INTEGRALS WITH RESPECT TO *x*, *y*, AND *z*

We now describe a second type of line integral in which we replace the "ds" in the integral by dx, dy, or dz. For example, suppose that f is a function defined on a smooth curve C in the xy-plane and that partition points of C are denoted by $P_k(x_k, y_k)$. Letting

$$\Delta x_k = x_k - x_{k-1} \quad \text{and} \quad \Delta y_k = y_k - y_{k-1}$$

we would like to define

$$\int_C f(x, y)\, dx = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta x_k \tag{16}$$

$$\int_C f(x, y)\, dy = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta y_k \tag{17}$$

However, unlike Δs_k, the values of Δx_k and Δy_k change sign if the order of the partition points along C is reversed. Therefore, in order to define the line integrals using Formulas (16) and (17), we must restrict ourselves to *oriented* curves C and to partitions of C in which the partition points are ordered in the direction of the curve. With this restriction, if the limit in (16) exists and does not depend on the choice of partition or sampling points, then we refer to (16) as the ***line integral of f with respect to x along C***. Similarly, (17) defines the ***line integral of f with respect to y along C***. If C is a smooth curve in 3-space, we can have ***line integrals of f with respect to x, y, and z along C***. For example,

> Explain why Formula (16) implies that $\int_C dx = x_f - x_i$, where x_f and x_i are the respective x-coordinates of the final and initial points of C. What about $\int_C dy$?

$$\int_C f(x, y, z)\, dx = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta x_k$$

Explain why Formula (16) implies that $\int_C f(x,y)\,dx = 0$ on any oriented segment parallel to the y-axis. What can you say about $\int_C f(x,y)\,dy$ on any oriented segment parallel to the x-axis?

and so forth. As was the case with line integrals with respect to s, line integrals of f with respect to x, y, and z exist if f is continuous on C.

The basic procedure for evaluating these line integrals is to find parametric equations for C, say

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (a \le t \le b)$$

in which the orientation of C is in the direction of increasing t, and then express the integrand in terms of t. For example,

$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))z'(t)\,dt$$

[Such a formula is easy to remember—just substitute for x, y, and z using the parametric equations and recall that $dz = z'(t)\,dt$.]

Example 5 Evaluate $\int_C 3xy\,dy$, where C is the line segment joining $(0,0)$ and $(1,2)$ with the given orientation.

(a) Oriented from $(0,0)$ to $(1,2)$ as in Figure 15.2.6a.

(b) Oriented from $(1,2)$ to $(0,0)$ as in Figure 15.2.6b.

Solution (a). Using the parametrization

$$x = t, \quad y = 2t \qquad (0 \le t \le 1)$$

we have

$$\int_C 3xy\,dy = \int_0^1 3(t)(2t)(2)\,dt = \int_0^1 12t^2\,dt = 4t^3\Big]_0^1 = 4$$

Solution (b). Using the parametrization

$$x = 1 - t, \quad y = 2 - 2t \qquad (0 \le t \le 1)$$

we have

$$\int_C 3xy\,dy = \int_0^1 3(1-t)(2-2t)(-2)\,dt = \int_0^1 -12(1-t)^2\,dt = 4(1-t)^3\Big]_0^1 = -4 \blacktriangleleft$$

In Example 5, note that reversing the orientation of the curve changed the sign of the line integral. This is because reversing the orientation of a curve changes the sign of Δx_k in definition (16). Thus, unlike line integrals of functions with respect to s along C, reversing the orientation of C changes the sign of a line integral with respect to x, y, and z. If C is a smooth oriented curve, we will let $-C$ denote the oriented curve consisting of the same points as C but with the opposite orientation (Figure 15.2.10). We then have

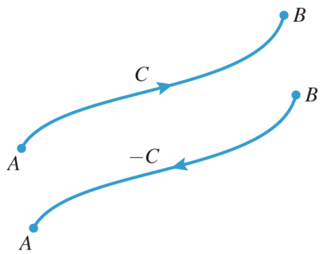


Figure 15.2.10

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \quad \text{and} \quad \int_{-C} g(x,y)\,dy = -\int_C g(x,y)\,dy \tag{18–19}$$

while

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds \tag{20}$$

Similar identities hold for line integrals in 3-space. Unless indicated otherwise, we will assume that parametric curves are oriented in the direction of increasing parameter.

Frequently, the line integrals with respect to x and y occur in combination, in which case we will dispense with one of the integral signs and write

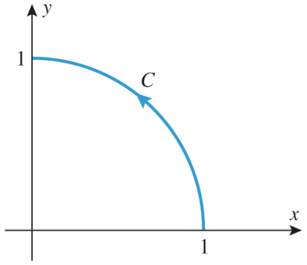
$$\int_C f(x,y)\,dx + g(x,y)\,dy = \int_C f(x,y)\,dx + \int_C g(x,y)\,dy \tag{21}$$

We will use a similar convention for combinations of line integrals with respect to x, y, and z along curves in 3-space.

Example 6 Evaluate

$$\int_C 2xy\,dx + (x^2 + y^2)\,dy$$

along the circular arc C given by $x = \cos t, y = \sin t\,(0 \le t \le \pi/2)$ (Figure 15.2.11).



$x = \cos t,\ y = \sin t\ (0 \le t \le \pi/2)$

Figure 15.2.11

Solution. We have

$$\int_C 2xy\,dx = \int_0^{\pi/2} (2\cos t \sin t)\left[\frac{d}{dt}(\cos t)\right] dt$$
$$= -2\int_0^{\pi/2} \sin^2 t \cos t\,dt = -\frac{2}{3}\sin^3 t\Big]_0^{\pi/2} = -\frac{2}{3}$$
$$\int_C (x^2 + y^2)\,dy = \int_0^{\pi/2} (\cos^2 t + \sin^2 t)\left[\frac{d}{dt}(\sin t)\right] dt$$
$$= \int_0^{\pi/2} \cos t\,dt = \sin t\Big]_0^{\pi/2} = 1$$

Thus, from (21)

$$\int_C 2xy\,dx + (x^2 + y^2)\,dy = \int_C 2xy\,dx + \int_C (x^2 + y^2)\,dy$$
$$= -\frac{2}{3} + 1 = \frac{1}{3} \blacktriangleleft$$

It can be shown that if f and g are continuous functions on C, then combinations of line integrals with respect to x and y can be expressed in terms of a limit and can be evaluated together in a single step. For example, we have

$$\int_C f(x, y)\,dx + g(x, y)\,dy = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} [f(x_k^*, y_k^*)\Delta x_k + g(x_k^*, y_k^*)\,\Delta y_k] \tag{22}$$

and

$$\int_C f(x, y)\,dx + g(x, y)\,dy = \int_a^b [f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t)]\,dt \tag{23}$$

Similar results hold for line integrals in 3-space. The evaluation of a line integral can sometimes be simplified by using Formula (23).

Example 7 Evaluate

$$\int_C (3x^2 + y^2)\,dx + 2xy\,dy$$

along the circular arc C given by $x = \cos t, y = \sin t\,(0 \le t \le \pi/2)$ (Figure 15.2.11).

Solution. From (23) we have

$$\int_C (3x^2 + y^2)\,dx + 2xy\,dy = \int_0^{\pi/2} [(3\cos^2 t + \sin^2 t)(-\sin t) + 2(\cos t)(\sin t)(\cos t)]\,dt$$
$$= \int_0^{\pi/2} (-3\cos^2 t \sin t - \sin^3 t + 2\cos^2 t \sin t)\,dt$$
$$= \int_0^{\pi/2} (-\cos^2 t - \sin^2 t)(\sin t)\,dt = \int_0^{\pi/2} -\sin t\,dt$$
$$= \cos t\Big]_0^{\pi/2} = -1 \blacktriangleleft$$

Compare the computations in Example 7 with those involved in computing

$$\int_C (3x^2 + y^2)\,dx + \int_C 2xy\,dy$$

It follows from (18) and (19) that

$$\int_{-C} f(x, y)\, dx + g(x, y)\, dy = -\int_C f(x, y)\, dx + g(x, y)\, dy \tag{24}$$

so that reversing the orientation of C changes the sign of a line integral in which x and y occur in combination. Similarly,

$$\begin{aligned}&\int_{-C} f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz \\ &\qquad = -\int_C f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz\end{aligned} \tag{25}$$

INTEGRATING A VECTOR FIELD ALONG A CURVE

There is an alternative notation for line integrals with respect to x, y, and z that is particularly appropriate for dealing with problems involving vector fields. We will interpret $d\mathbf{r}$ as

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} \quad \text{or} \quad d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

depending on whether C is in 2-space or 3-space. For an oriented curve C in 2-space and a vector field

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

we will write

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (f(x, y)\mathbf{i} + g(x, y)\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \int_C f(x, y)\, dx + g(x, y)\, dy \tag{26}$$

Similarily, for a curve C in 3-space and vector field

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

we will write

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_C f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz\end{aligned} \tag{27}$$

With these conventions, we are led to the following definition.

15.2.2 DEFINITION If $\mathbf{F}$ is a continuous vector field and C is a smooth oriented curve, then the ***line integral of* F *along C*** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} \tag{28}$$

The notation in Definition 15.2.2 makes it easy to remember the formula for evaluating the line integral of $\mathbf{F}$ along C. For example, suppose that C is an oriented curve in the plane given in vector form by

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad (a \le t \le b)$$

If we write

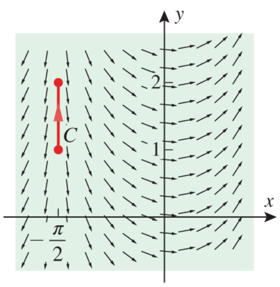
$$\mathbf{F}(\mathbf{r}(t)) = f(x(t), y(t))\mathbf{i} + g(x(t), y(t))\mathbf{j}$$

then

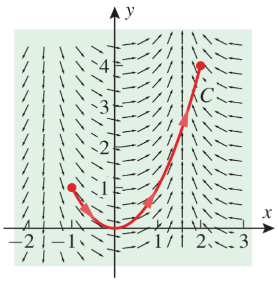
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \tag{29}$$

Formula (29) is also valid for oriented curves in 3-space.

Vectors not to scale



(a)



(b)

▲ **Figure 15.2.12**

▶ **Example 8** Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \cos x\mathbf{i} + \sin x\mathbf{j}$ and where C is the given oriented curve.

(a) $C : \mathbf{r}(t) = -\dfrac{\pi}{2}\mathbf{i} + t\mathbf{j} \quad (1 \le t \le 2)$ (see Figure 15.2.12*a*)

(b) $C : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \quad (-1 \le t \le 2)$ (see Figure 15.2.12*b*)

Solution (a). Using (29) we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_1^2 (-\mathbf{j}) \cdot \mathbf{j}\,dt = \int_1^2 (-1)\,dt = -1$$

Solution (b). Using (29) we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_{-1}^2 (\cos t\mathbf{i} + \sin t\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})\,dt$$

$$= \int_{-1}^2 (\cos t + 2t \sin t)\,dt = (-2t \cos t + 3 \sin t)\Big]_{-1}^2$$

$$= -2\cos 1 - 4\cos 2 + 3(\sin 1 + \sin 2) \approx 5.83629 \;◀$$

If we let t denote an arc length parameter, say $t = s$, with $\mathbf{T} = \mathbf{r}'(s)$ the unit tangent vector field along C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(s)) \cdot \mathbf{r}'(s)\,ds = \int_a^b \mathbf{F}(\mathbf{r}(s)) \cdot \mathbf{T}\,ds = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

which shows that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds \tag{30}$$

In words, the integral of a vector field along a curve has the same value as the integral of the tangential component of the vector field along the curve.

We can use (30) to interpret $\int_C \mathbf{F} \cdot d\mathbf{r}$ geometrically. If θ is the angle between $\mathbf{F}$ and $\mathbf{T}$ at a point on C, then at this point

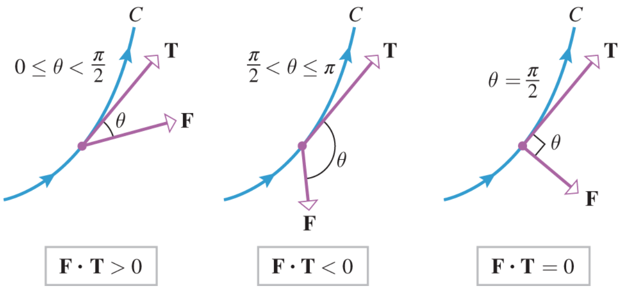
$$\mathbf{F} \cdot \mathbf{T} = \|\mathbf{F}\|\|\mathbf{T}\| \cos\theta \qquad \boxed{\text{Formula (4) in Section 11.3}}$$

$$= \|\mathbf{F}\| \cos\theta \qquad \boxed{\text{Since } \|\mathbf{T}\| = 1}$$

Thus,

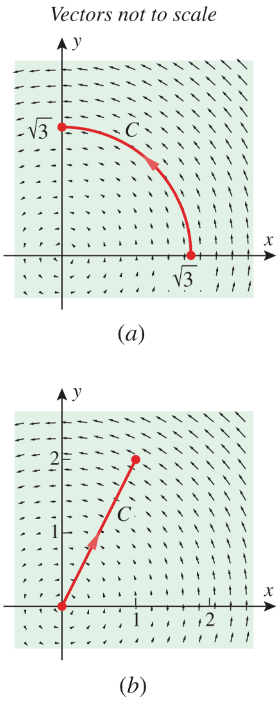
$$-\|\mathbf{F}\| \le \mathbf{F} \cdot \mathbf{T} \le \|\mathbf{F}\|$$

and if $\mathbf{F} \ne \mathbf{0}$, then the sign of $\mathbf{F} \cdot \mathbf{T}$ will depend on the angle between the direction of $\mathbf{F}$ and the direction of C (Figure 15.2.13). That is, $\mathbf{F} \cdot \mathbf{T}$ will be positive where $\mathbf{F}$ has the same general direction as C, it will be 0 if $\mathbf{F}$ is normal to C, and it will be negative where $\mathbf{F}$ and C have more or less opposite directions. The line integral of $\mathbf{F}$ along C can be interpreted



▶ **Figure 15.2.13**

as the accumulated effect of the magnitude of **F** along C, the extent to which **F** and C have the same direction, and the arc length of C.



▲ **Figure 15.2.14**

Refer to Figure 15.2.12 and explain the sign of each line integral in Example 8 geometrically. Exercises 5 and 6 take this geometric analysis further.

▶ **Example 9** Use (30) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ and where C is the given oriented curve.

(a) $C : x^2 + y^2 = 3 \quad (0 \le x, y$; oriented as in Figure 15.2.14a)

(b) $C : \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} \quad (0 \le t \le 1$; see Figure 15.2.14$b$)

Solution (a). At every point on C the direction of **F** and the direction of C are the same. (Why?) In addition, at every point on C

$$\|\mathbf{F}\| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = \sqrt{3}$$

Therefore, $\mathbf{F} \cdot \mathbf{T} = \|\mathbf{F}\| \cos(0) = \|\mathbf{F}\| = \sqrt{3}$, and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \sqrt{3}\, ds = \sqrt{3} \int_C ds = \frac{3\pi}{2}$$

Solution (b). The vector field **F** is normal to C at every point. (Why?) Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C 0\, ds = 0 \quad ◀$$

In light of (20) and (30), you might expect that reversing the orientation of C in $\int_C \mathbf{F} \cdot d\mathbf{r}$ would have no effect on the value of the line integral. However, reversing the orientation of C reverses the orientation of **T** in the integrand and hence reverses the sign of the integral; that is,

$$\int_{-C} \mathbf{F} \cdot \mathbf{T}\, ds = -\int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{31}$$

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r} \tag{32}$$

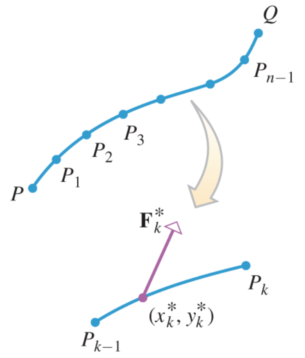
WORK AS A LINE INTEGRAL

An important application of line integrals with respect to x, y, and z is to the problem of defining the work performed by a variable force moving a particle along a curved path. In Section 5.6 we defined the work W performed by a force of constant magnitude acting on an object in the direction of motion (Definition 5.6.1), and later in that section we extended the definition to allow for a force of variable magnitude acting in the direction of motion (Definition 5.6.3). In Section 11.3 we took the concept of work a step further by defining the work W performed by a constant force **F** moving a particle in a straight line from point P to point Q. We defined the work to be

$$W = \mathbf{F} \cdot \overrightarrow{PQ} \tag{33}$$

[Formula (14) in Section 11.3]. Our next goal is to define a more general concept of work—the work performed by a variable force acting on a particle that moves along a curved path in 2-space or 3-space.

In many applications variable forces arise from force fields (gravitational fields, electromagnetic fields, and so forth), so we will consider the problem of work in that context. To motivate an appropriate definition for work performed by a force field, we will use a limit process, and since the procedure is the same for 2-space and 3-space, we will discuss it in detail for 2-space only. The idea is as follows:



▲ **Figure 15.2.15**

- Assume that a force field $\mathbf{F} = \mathbf{F}(x, y)$ moves a particle along a smooth curve C from a point P to a point Q. Divide C into n arcs using the partition points

$$P = P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_{n-1}(x_{n-1}, y_{n-1}), P_n(x_n, y_n) = Q$$

directed along C from P to Q, and denote the length of the kth arc by Δs_k. Let (x_k^*, y_k^*) be any point on the kth arc, and let

$$\mathbf{F}_k^* = \mathbf{F}(x_k^*, y_k^*) = f(x_k^*, y_k^*)\mathbf{i} + g(x_k^*, y_k^*)\mathbf{j}$$

be the force vector at this point (Figure 15.2.15).

- If the kth arc is small, then the force will not vary much, so we can approximate the force by the constant value $\mathbf{F}_k^*$ on this arc. Moreover, the direction of motion will not vary much over this small arc, so we can approximate the movement of the particle by the displacement vector

$$\overrightarrow{P_{k-1}P_k} = (\Delta x_k)\mathbf{i} + (\Delta y_k)\mathbf{j}$$

where $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$.

- Since the work done by a constant force $\mathbf{F}_k^*$ moving a particle along a straight line from P_{k-1} to P_k is

$$\begin{aligned}\mathbf{F}_k^* \cdot \overrightarrow{P_{k-1}P_k} &= (f(x_k^*, y_k^*)\mathbf{i} + g(x_k^*, y_k^*)\mathbf{j}) \cdot ((\Delta x_k)\mathbf{i} + (\Delta y_k)\mathbf{j}) \\ &= f(x_k^*, y_k^*)\Delta x_k + g(x_k^*, y_k^*)\Delta y_k\end{aligned}$$

[Formula (33)], the work ΔW_k performed by the force field along the kth arc of C can be approximated by

$$\Delta W_k \approx f(x_k^*, y_k^*)\Delta x_k + g(x_k^*, y_k^*)\Delta y_k$$

The total work W performed by the force moving the particle over the entire curve C can then be approximated as

$$W = \sum_{k=1}^{n} \Delta W_k \approx \sum_{k=1}^{n} [f(x_k^*, y_k^*)\Delta x_k + g(x_k^*, y_k^*)\Delta y_k]$$

- As $\max \Delta s_k \to 0$, it is plausible that the error in this approximation approaches 0 and the exact work performed by the force field is

$$\begin{aligned}W &= \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} [f(x_k^*, y_k^*)\Delta x_k + g(x_k^*, y_k^*)\Delta y_k] \\ &= \int_C f(x, y)\,dx + g(x, y)\,dy \qquad \boxed{\text{Formula (22)}} \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} \qquad \boxed{\text{Formula (26)}}\end{aligned}$$

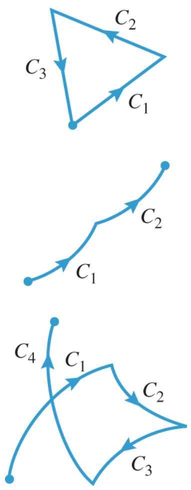
Thus, we are led to the following definition.

Note from Formula (30) that the work performed by a force field on a particle moving along a smooth curve C is obtained by integrating the scalar tangential component of force along C. This implies that the component of force orthogonal to the direction of motion of the particle has no effect on the work done.

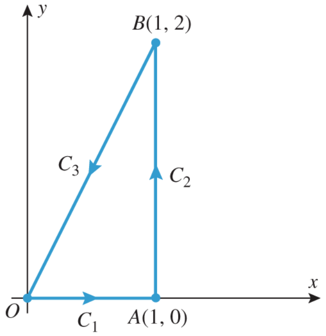
15.2.3 DEFINITION Suppose that under the influence of a continuous force field $\mathbf{F}$ a particle moves along a smooth curve C and that C is oriented in the direction of motion of the particle. Then the ***work performed by the force field*** on the particle is

$$\int_C \mathbf{F} \cdot d\mathbf{r} \tag{34}$$

For example, suppose that force is measured in pounds and distance is measured in feet. It follows from part (a) of Example 9 that the work done by a force $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ acting on a particle moving along the circle $x^2 + y^2 = 3$ from $(\sqrt{3}, 0)$ to $(0, \sqrt{3})$ is $3\pi/2$ foot-pounds.



▲ Figure 15.2.16



▲ Figure 15.2.17

LINE INTEGRALS ALONG PIECEWISE SMOOTH CURVES

Thus far, we have only considered line integrals along smooth curves. However, the notion of a line integral can be extended to curves formed from finitely many smooth curves $C_1, C_2, \ldots, C_n$ joined end to end. Such a curve is called ***piecewise smooth*** (Figure 15.2.16). We define a line integral along a piecewise smooth curve C to be the sum of the integrals along the sections:

$$\int_C = \int_{C_1} + \int_{C_2} + \cdots + \int_{C_n}$$

Example 10 Evaluate

$$\int_C x^2 y\,dx + x\,dy$$

where C is the triangular path shown in Figure 15.2.17.

Solution. We will integrate over C_1, C_2, and C_3 separately and add the results. For each of the three integrals we must find parametric equations that trace the path of integration in the correct direction. For this purpose recall from Formula (7) of Section 12.1 that the graph of the vector-valued function

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad (0 \le t \le 1)$$

is the line segment joining $\mathbf{r}_0$ and $\mathbf{r}_1$, oriented in the direction from $\mathbf{r}_0$ to $\mathbf{r}_1$. Thus, the line segments C_1, C_2, and C_3 can be represented in vector notation as

$$\begin{aligned}
C_1&: \mathbf{r}(t) = (1-t)\langle 0,0\rangle + t\langle 1,0\rangle = \langle t, 0\rangle \\
C_2&: \mathbf{r}(t) = (1-t)\langle 1,0\rangle + t\langle 1,2\rangle = \langle 1, 2t\rangle \\
C_3&: \mathbf{r}(t) = (1-t)\langle 1,2\rangle + t\langle 0,0\rangle = \langle 1-t, 2-2t\rangle
\end{aligned}$$

where t varies from 0 to 1 in each case. From these equations we obtain

$$\int_{C_1} x^2 y\,dx + x\,dy = \int_{C_1} x^2 y\,dx = \int_0^1 (t^2)(0)\frac{d}{dt}[t]\,dt = 0$$

$$\int_{C_2} x^2 y\,dx + x\,dy = \int_{C_2} x\,dy = \int_0^1 (1)\frac{d}{dt}[2t]\,dt = 2$$

$$\begin{aligned}
\int_{C_3} x^2 y\,dx + x\,dy &= \int_0^1 (1-t)^2(2-2t)\frac{d}{dt}[1-t]\,dt + \int_0^1 (1-t)\frac{d}{dt}[2-2t]\,dt \\
&= 2\int_0^1 (t-1)^3\,dt + 2\int_0^1 (t-1)\,dt = -\tfrac{1}{2} - 1 = -\tfrac{3}{2}
\end{aligned}$$

Thus,

$$\int_C x^2 y\,dx + x\,dy = 0 + 2 + \left(-\tfrac{3}{2}\right) = \tfrac{1}{2} \blacktriangleleft$$

✓ QUICK CHECK EXERCISES 15.2

(See page 995 for answers.)

1. The area of the surface extending upward from the line segment $y = x$ $(0 \le x \le 1)$ in the xy-plane to the plane $z = 2x + 1$ is ________.

2. Suppose that a wire has equation $y = 1 - x$ $(0 \le x \le 1)$ and that its mass density is $\delta(x, y) = 2 - x$. The mass of the wire is ________.

3. If C is the curve represented by the equations

$$x = \sin t, \quad y = \cos t, \quad z = t \qquad (0 \le t \le 2\pi)$$

then $\int_C y\,dx - x\,dy + dz =$ ________.

4. If C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise and $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \text{\_\_\_\_\_\_\_\_}$$

EXERCISE SET 15.2 Graphing Utility C CAS

FOCUS ON CONCEPTS

1. Let C be the line segment from $(0, 0)$ to $(0, 1)$. In each part, evaluate the line integral along C by inspection, and explain your reasoning.

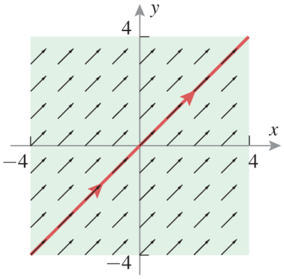
(a) $\int_C ds$ (b) $\int_C \sin xy\, dy$

2. Let C be the line segment from $(0, 2)$ to $(0, 4)$. In each part, evaluate the line integral along C by inspection, and explain your reasoning.

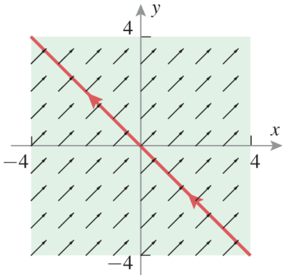
(a) $\int_C ds$ (b) $\int_C e^{xy}\, dx$

3–4 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by inspection for the force field $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$ and the curve C shown in the figure. Explain your reasoning. [*Note:* For clarity, the vectors in the force field are shown at less than true scale.]

3.



4.



5. Use (30) to explain why the line integral in part (a) of Example 8 can be found by multiplying the length of the line segment C by -1.

6. (a) Use (30) to explain why the line integral in part (b) of Example 8 should be close to, but somewhat less than, the length of the parabolic curve C.
(b) Verify the conclusion in part (a) of this exercise by computing the length of C and comparing the length with the value of the line integral.

7–10 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the line segment C from P to Q.

7. $\mathbf{F}(x, y) = 8\mathbf{i} + 8\mathbf{j}$; $P(-4, 4), Q(-4, 5)$

8. $\mathbf{F}(x, y) = 2\mathbf{i} + 5\mathbf{j}$; $P(1, -3), Q(4, -3)$

9. $\mathbf{F}(x, y) = 2x\mathbf{j}$; $P(-2, 4), Q(-2, 11)$

10. $\mathbf{F}(x, y) = -8x\mathbf{i} + 3y\mathbf{j}$; $P(-1, 0), Q(6, 0)$

11. Let C be the curve represented by the equations

$$x = 2t, \quad y = t^2 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\int_C (x - \sqrt{y})\, ds$ (b) $\int_C (x - \sqrt{y})\, dx$

(c) $\int_C (x - \sqrt{y})\, dy$

12. Let C be the curve represented by the equations

$$x = t, \quad y = 3t^2, \quad z = 6t^3 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\int_C xyz^2\, ds$ (b) $\int_C xyz^2\, dx$

(c) $\int_C xyz^2\, dy$ (d) $\int_C xyz^2\, dz$

13. In each part, evaluate the integral

$$\int_C (3x + 2y)\, dx + (2x - y)\, dy$$

along the stated curve.

(a) The line segment from $(0, 0)$ to $(1, 1)$.
(b) The parabolic arc $y = x^2$ from $(0, 0)$ to $(1, 1)$.
(c) The curve $y = \sin(\pi x/2)$ from $(0, 0)$ to $(1, 1)$.
(d) The curve $x = y^3$ from $(0, 0)$ to $(1, 1)$.

14. In each part, evaluate the integral

$$\int_C y\, dx + z\, dy - x\, dz$$

along the stated curve.

(a) The line segment from $(0, 0, 0)$ to $(1, 1, 1)$.
(b) The twisted cubic $x = t$, $y = t^2$, $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$.
(c) The helix $x = \cos \pi t$, $y = \sin \pi t$, $z = t$ from $(1, 0, 0)$ to $(-1, 0, 1)$.

15–18 True–False Determine whether the statement is true or false. Explain your answer.

15. If C is a smooth oriented curve in the xy-plane and $f(x, y)$ is a continuous function defined on C, then

$$\int_C f(x, y)\, ds = -\int_{-C} f(x, y)\, ds$$

16. The line integral of a continuous vector field along a smooth curve C is a vector.

17. If $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ along a smooth oriented curve C in the xy-plane, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f(x, y)\, dx + g(x, y)\, dy$$

18. If a smooth oriented curve C in the xy-plane is a contour for a differentiable function $f(x, y)$, then

$$\int_C \nabla f \cdot d\mathbf{r} = 0$$

19–22 Evaluate the line integral with respect to s along the curve C.

19. $\displaystyle\int_C \frac{1}{1 + x}\, ds$

$C : \mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{j} \quad (0 \le t \le 3)$

20. $\displaystyle\int_C \frac{x}{1 + y^2}\, ds$

$C : x = 1 + 2t, \; y = t \quad (0 \le t \le 1)$

21. $\displaystyle\int_C 3x^2yz\, ds$

$C : x = t, \; y = t^2, \; z = \frac{2}{3}t^3 \quad (0 \le t \le 1)$

22. $\displaystyle\int_C \frac{e^{-z}}{x^2+y^2}\,ds$
$C: \mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k} \quad (0 \le t \le 2\pi)$

23–30 Evaluate the line integral along the curve C.

23. $\displaystyle\int_C (x+2y)\,dx + (x-y)\,dy$
$C: x = 2\cos t,\ y = 4\sin t \quad (0 \le t \le \pi/4)$

24. $\displaystyle\int_C (x^2-y^2)\,dx + x\,dy$
$C: x = t^{2/3},\ y = t \quad (-1 \le t \le 1)$

25. $\displaystyle\int_C -y\,dx + x\,dy$
$C: y^2 = 3x$ from $(3, 3)$ to $(0, 0)$

26. $\displaystyle\int_C (y-x)\,dx + x^2y\,dy$
$C: y^2 = x^3$ from $(1, -1)$ to $(1, 1)$

27. $\displaystyle\int_C (x^2+y^2)\,dx - x\,dy$
$C: x^2 + y^2 = 1$, counterclockwise from $(1, 0)$ to $(0, 1)$

28. $\displaystyle\int_C (y-x)\,dx + xy\,dy$
C: the line segment from $(3, 4)$ to $(2, 1)$

29. $\displaystyle\int_C yz\,dx - xz\,dy + xy\,dz$
$C: x = e^t,\ y = e^{3t},\ z = e^{-t} \quad (0 \le t \le 1)$

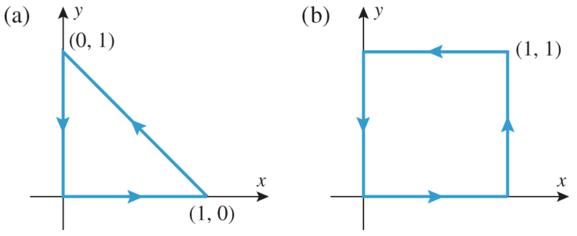
30. $\displaystyle\int_C x^2\,dx + xy\,dy + z^2\,dz$
$C: x = \sin t,\ y = \cos t,\ z = t^2 \quad (0 \le t \le \pi/2)$

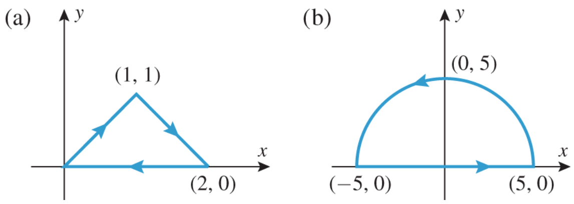
C **31–32** Use a CAS to evaluate the line integrals along the given curves.

31. (a) $\displaystyle\int_C (x^3+y^3)\,ds$
$C: \mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} \quad (0 \le t \le \ln 2)$
(b) $\displaystyle\int_C xe^z\,dx + (x-z)\,dy + (x^2+y^2+z^2)\,dz$
$C: x = \sin t,\ y = \cos t,\ z = t \quad (0 \le t \le \pi/2)$

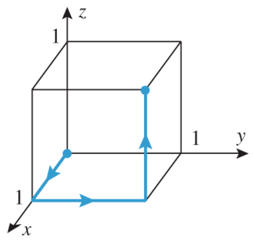
32. (a) $\displaystyle\int_C x^7y^3\,ds$
$C: x = \cos^3 t,\ y = \sin^3 t \quad (0 \le t \le \pi/2)$
(b) $\displaystyle\int_C x^5z\,dx + 7y\,dy + y^2z\,dz$
$C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k} \quad (1 \le t \le e)$

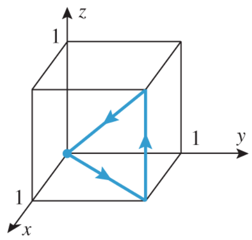
33–34 Evaluate $\int_C y\,dx - x\,dy$ along the curve C shown in the figure.

33.


34.


35–36 Evaluate $\int_C x^2z\,dx - yx^2\,dy + 3\,dz$ along the curve C shown in the figure.

35.


36.


37–40 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C.

37. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$
$C: \mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} \quad (0 \le t \le \pi)$

38. $\mathbf{F}(x, y) = x^2y\mathbf{i} + 4\mathbf{j}$
$C: \mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} \quad (0 \le t \le 1)$

39. $\mathbf{F}(x, y) = (x^2+y^2)^{-3/2}(x\mathbf{i} + y\mathbf{j})$
$C: \mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{j} \quad (0 \le t \le 1)$

40. $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
$C: \mathbf{r}(t) = \sin t\mathbf{i} + 3\sin t\mathbf{j} + \sin^2 t\mathbf{k} \quad (0 \le t \le \pi/2)$

41. Find the mass of a thin wire shaped in the form of the circular arc $y = \sqrt{9-x^2}$ $(0 \le x \le 3)$ if the density function is $\delta(x, y) = x\sqrt{y}$.

42. Find the mass of a thin wire shaped in the form of the curve $x = e^t\cos t$, $y = e^t\sin t$ $(0 \le t \le 1)$ if the density function δ is proportional to the distance from the origin.

43. Find the mass of a thin wire shaped in the form of the helix $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ $(0 \le t \le \pi/2)$ if the density function is $\delta = kx/(1+y^2)$ $(k > 0)$.

44. Find the mass of a thin wire shaped in the form of the curve $x = 2t$, $y = \ln t$, $z = 4\sqrt{t}$ $(1 \le t \le 4)$ if the density function is proportional to the distance above the xy-plane.

45–48 Find the work done by the force field $\mathbf{F}$ on a particle that moves along the curve C.

45. $\mathbf{F}(x, y) = xy\mathbf{i} + x^2\mathbf{j}$
$C: x = y^2$ from $(0, 0)$ to $(1, 1)$

46. $\mathbf{F}(x, y) = (x^2 + xy)\mathbf{i} + (y - x^2y)\mathbf{j}$
$C: x = t,\ y = 1/t \quad (1 \le t \le 3)$

47. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
$C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \quad (0 \le t \le 1)$

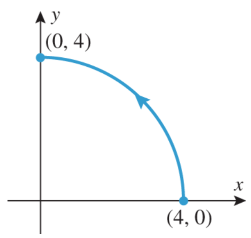
48. $\mathbf{F}(x, y, z) = (x+y)\mathbf{i} + xy\mathbf{j} - z^2\mathbf{k}$
C: along line segments from $(0, 0, 0)$ to $(1, 3, 1)$ to $(2, -1, 4)$

49–50 Find the work done by the force field

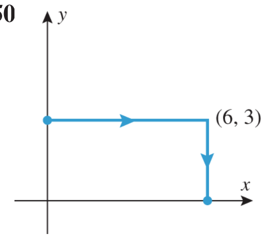
$$\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}\mathbf{i} + \frac{4}{x^2 + y^2}\mathbf{j}$$

on a particle that moves along the curve C shown in the figure.

49.



50



51–52 Use a line integral to find the area of the surface.

51. The surface that extends upward from the parabola $y = x^2$ $(0 \le x \le 2)$ in the xy-plane to the plane $z = 3x$.

52. The surface that extends upward from the semicircle $y = \sqrt{4 - x^2}$ in the xy-plane to the surface $z = x^2 y$.

53. As illustrated in the accompanying figure, a sinusoidal cut is made in the top of a cylindrical tin can. Suppose that the base is modeled by the parametric equations $x = \cos t$, $y = \sin t$, $z = 0$ $(0 \le t \le 2\pi)$, and the height of the cut as a function of t is $z = 2 + 0.5 \sin 3t$.

(a) Use a geometric argument to find the lateral surface area of the cut can.

(b) Write down a line integral for the surface area.

(c) Use the line integral to calculate the surface area.

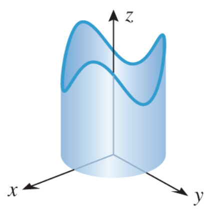


Figure Ex-53

54. Evaluate the integral $\displaystyle\int_{-C} \frac{x\,dy - y\,dx}{x^2 + y^2}$, where C is the circle $x^2 + y^2 = a^2$ traversed counterclockwise.

55. Suppose that a particle moves through the force field $\mathbf{F}(x, y) = xy\mathbf{i} + (x - y)\mathbf{j}$ from the point $(0, 0)$ to the point $(1, 0)$ along the curve $x = t$, $y = \lambda t(1 - t)$. For what value of λ will the work done by the force field be 1?

56. A farmer weighing 150 lb carries a sack of grain weighing 20 lb up a circular helical staircase around a silo of radius 25 ft. As the farmer climbs, grain leaks from the sack at a rate of 1 lb per 10 ft of ascent. How much work is performed by the farmer in climbing through a vertical distance of 60 ft in exactly four revolutions? [*Hint:* Find a vector field that represents the force exerted by the farmer in lifting his own weight plus the weight of the sack upward at each point along his path.]

57. Suppose that a curve C in the xy-plane is smoothly parametrized by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad (a \le t \le b)$$

In each part, refer to the notation used in the derivation of Formula (9).

(a) Let m and M denote the respective minimum and maximum values of $\|\mathbf{r}'(t)\|$ on $[a, b]$. Prove that

$$0 \le m\,(\max\, \Delta t_k) \le \max \Delta s_k \le M\,(\max\, \Delta t_k)$$

(b) Use part (a) to prove that $\max\, \Delta s_k \to 0$ if and only if $\max\, \Delta t_k \to 0$.

58. Writing Discuss the similarities and differences between the definition of a definite integral over an interval (Definition 4.5.1) and the definition of the line integral with respect to arc length along a curve (Definition 15.2.1).

59. Writing Describe the different types of line integrals, and discuss how they are related.

✓ QUICK CHECK ANSWERS 15.2 **1.** $2\sqrt{2}$ **2.** $\dfrac{3\sqrt{2}}{2}$ **3.** 4π **4.** 0

15.3 INDEPENDENCE OF PATH; CONSERVATIVE VECTOR FIELDS

In this section we will show that for certain kinds of vector fields **F** *the line integral of* **F** *along a curve depends only on the endpoints of the curve and not on the curve itself. Vector fields with this property, which include gravitational and electrostatic fields, are of special importance in physics and engineering.*

WORK INTEGRALS

We saw in the last section that if **F** is a force field in 2-space or 3-space, then the work performed by the field on a particle moving along a parametric curve C from an initial point P to a final point Q is given by the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{or equivalently} \quad \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Accordingly, we call an integral of this type a ***work integral***. Recall that a work integral can also be expressed in scalar form as

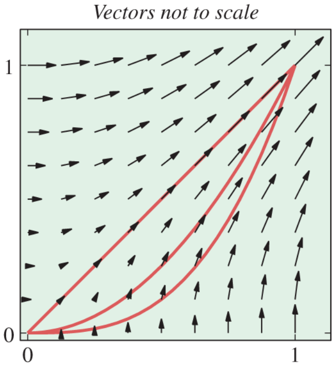
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f(x,y)\,dx + g(x,y)\,dy \qquad \boxed{\text{2-space}} \tag{1}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz \qquad \boxed{\text{3-space}} \tag{2}$$

where f, g, and h are the component functions of $\mathbf{F}$.

INDEPENDENCE OF PATH

The parametric curve C in a work integral is called the ***path of integration***. One of the important problems in applications is to determine how the path of integration affects the work performed by a force field on a particle that moves from a fixed point P to a fixed point Q. We will show shortly that if the force field $\mathbf{F}$ is conservative (i.e., is the gradient of some potential function ϕ), then the work that the field performs on a particle that moves from P to Q does not depend on the particular path C that the particle follows. This is illustrated in the following example.



▲ **Figure 15.3.1**

▶ Example 1 The force field $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j}$ is conservative since it is the gradient of $\phi(x,y) = xy$ (verify). Thus, the preceding discussion suggests that the work performed by the field on a particle that moves from the point $(0,0)$ to the point $(1,1)$ should be the same along different paths. Confirm that the value of the work integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

is the same along the following paths (Figure 15.3.1):

(a) The line segment $y = x$ from $(0,0)$ to $(1,1)$.
(b) The parabola $y = x^2$ from $(0,0)$ to $(1,1)$.
(c) The cubic $y = x^3$ from $(0,0)$ to $(1,1)$.

Solution (a). With $x = t$ as the parameter, the path of integration is given by

$$x = t, \quad y = t \qquad (0 \le t \le 1)$$

Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (y\mathbf{i} + x\mathbf{j})\cdot(dx\mathbf{i} + dy\mathbf{j}) = \int_C y\,dx + x\,dy$$
$$= \int_0^1 2t\,dt = 1$$

Solution (b). With $x = t$ as the parameter, the path of integration is given by

$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\,dx + x\,dy = \int_0^1 3t^2\,dt = 1$$

Solution (c). With $x = t$ as the parameter, the path of integration is given by

$$x = t, \quad y = t^3 \qquad (0 \le t \le 1)$$

Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\,dx + x\,dy = \int_0^1 4t^3\,dt = 1 \ ◀$$

THE FUNDAMENTAL THEOREM OF LINE INTEGRALS

Recall from the Fundamental Theorem of Calculus (Theorem 4.6.1) that if F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The following result is the analog of that theorem for line integrals in 2-space.

15.3.1 THEOREM (*The Fundamental Theorem of Line Integrals*) *Suppose that*

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

is a conservative vector field in some open region D containing the points (x_0, y_0) and (x_1, y_1) and that $f(x, y)$ and $g(x, y)$ are continuous in this region. If

$$\mathbf{F}(x, y) = \nabla\phi(x, y)$$

and if C is any piecewise smooth parametric curve that starts at (x_0, y_0), ends at (x_1, y_1), and lies in the region D, then

$$\int_C \mathbf{F}(x, y)\cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{3}$$

or, equivalently,

$$\int_C \nabla\phi\cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{4}$$

The value of

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

depends on the magnitude of $\mathbf{F}$ along C, the alignment of $\mathbf{F}$ with the direction of C at each point, and the length of C. If $\mathbf{F}$ is conservative, these various factors always "balance out" so that the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the initial and final points of C.

PROOF We will give the proof for a smooth curve C. The proof for a piecewise smooth curve, which is left as an exercise, can be obtained by applying the theorem to each individual smooth piece and adding the results. Suppose that C is given parametrically by $x = x(t), y = y(t)$ $(a \le t \le b)$, so that the initial and final points of the curve are

$$(x_0, y_0) = (x(a), y(a)) \quad \text{and} \quad (x_1, y_1) = (x(b), y(b))$$

Since $\mathbf{F}(x, y) = \nabla\phi$, it follows that

$$\mathbf{F}(x, y) = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j}$$

so

$$\begin{aligned}\int_C \mathbf{F}(x, y)\cdot d\mathbf{r} &= \int_C \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy = \int_a^b \left[\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}\right]dt\\ &= \int_a^b \frac{d}{dt}[\phi(x(t), y(t))]\,dt = \phi(x(t), y(t))\Big]_{t=a}^{b}\\ &= \phi(x(b), y(b)) - \phi(x(a), y(a))\\ &= \phi(x_1, y_1) - \phi(x_0, y_0) \quad \blacksquare\end{aligned}$$

Stated informally, this theorem shows that *the value of a line integral of a conservative vector field along a piecewise smooth path is* ***independent of the path***; that is, the value of the integral depends on the endpoints and not on the actual path C. Accordingly, for line integrals of conservative vector fields, it is common to express (3) and (4) as

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x_0,y_0)}^{(x_1,y_1)} \nabla\phi\cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{5}$$

If $\mathbf{F}$ is conservative, then you have a choice of methods for evaluating $\int_C \mathbf{F}\cdot d\mathbf{r}$. You can work directly with the curve C, you can replace C with another curve that has the same endpoints as C, or you can apply Formula (3).

▶ Example 2

We observed in Example 1 that the force field $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j}$ is the gradient of $\phi(x,y) = xy$. Use Formula (5) to evaluate

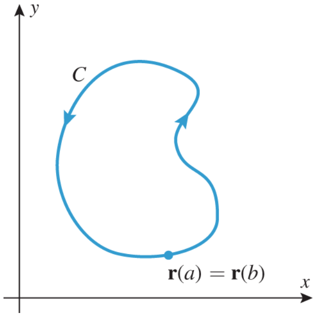
$$\int_{(0,0)}^{(1,1)} \mathbf{F}\cdot d\mathbf{r}$$

Solution. From (5) we obtain

$$\int_{(0,0)}^{(1,1)} \mathbf{F}\cdot d\mathbf{r} = \phi(1,1) - \phi(0,0) = 1 - 0 = 1$$

which agrees with the results obtained in Example 1 by integrating from $(0,0)$ to $(1,1)$ along specific paths. ◀

LINE INTEGRALS ALONG CLOSED PATHS



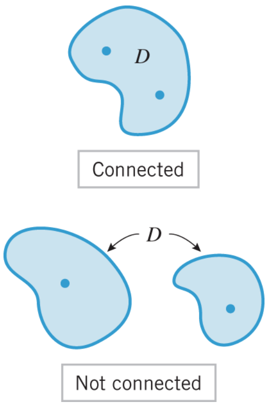
▲ Figure 15.3.2

Parametric curves that begin and end at the same point play an important role in the study of vector fields, so there is some special terminology associated with them. A parametric curve C that is represented by the vector-valued function $\mathbf{r}(t)$ for $a \le t \le b$ is said to be ***closed*** if the initial point $\mathbf{r}(a)$ and the terminal point $\mathbf{r}(b)$ coincide; that is, $\mathbf{r}(a) = \mathbf{r}(b)$ (Figure 15.3.2).

It follows from (5) that the line integral of a conservative vector field along a closed path C that begins and ends at (x_0, y_0) is zero. This is because the point (x_1, y_1) in (5) is the same as (x_0, y_0) and hence

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) = 0$$

Our next objective is to show that the converse of this result is also true. That is, we want to show that under appropriate conditions a vector field whose line integral is zero along *all* closed paths must be conservative. For this to be true we will need to require that the domain D of the vector field be ***connected***, by which we mean that any two points in D can be joined by some piecewise smooth curve that lies entirely in D. Stated informally, D is connected if it does not consist of two or more separate pieces (Figure 15.3.3).



▲ Figure 15.3.3

15.3.2 THEOREM *If $f(x,y)$ and $g(x,y)$ are continuous on some open connected region D, then the following statements are equivalent (all true or all false):*

(a) $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ *is a conservative vector field on the region D.*

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ *for every piecewise smooth closed curve C in D.*

(c) $\int_C \mathbf{F}\cdot d\mathbf{r}$ *is independent of the path from any point P in D to any point Q in D for every piecewise smooth curve C in D.*

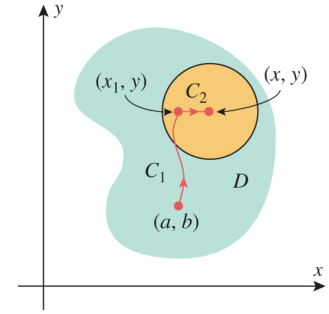
This theorem can be established by proving three implications: $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(c) \Rightarrow (a)$. Since we showed above that $(a) \Rightarrow (b)$, we need only prove the last two implications. We will prove $(c) \Rightarrow (a)$ and leave the other implication as an exercise.

PROOF $(c) \Rightarrow (a)$. We are assuming that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path for every piecewise smooth curve C in the region, and we want to show that there is a function $\phi = \phi(x,y)$ such that $\nabla\phi = \mathbf{F}(x,y)$ at each point of the region; that is,

$$\frac{\partial\phi}{\partial x} = f(x,y) \quad\text{and}\quad \frac{\partial\phi}{\partial y} = g(x,y) \tag{6}$$

Now choose a fixed point (a,b) in D, let (x,y) be any point in D, and define

$$\phi(x,y) = \int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r} \tag{7}$$



▲ **Figure 15.3.4**

This is an unambiguous definition because we have assumed that the integral is independent of the path. We will show that $\nabla\phi = \mathbf{F}$. Since D is open, we can find a circular disk centered at (x, y) whose points lie entirely in D. As shown in Figure 15.3.4, choose any point (x_1, y) in this disk that lies on the same horizontal line as (x, y) such that $x_1 < x$. Because the integral in (7) is independent of path, we can evaluate it by first integrating from (a, b) to (x_1, y) along an arbitrary piecewise smooth curve C_1 in D, and then continuing along the horizontal line segment C_2 from (x_1, y) to (x, y). This yields

$$\phi(x, y) = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{(a,b)}^{(x_1,y)} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

Since the first term does not depend on x, its partial derivative with respect to x is zero and hence

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \frac{\partial}{\partial x}\int_{C_2} f(x, y)\,dx + g(x, y)\,dy$$

However, the line integral with respect to y is zero along the horizontal line segment C_2, so this equation simplifies to

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\int_{C_2} f(x, y)\,dx \tag{8}$$

To evaluate the integral in this expression, we treat y as a constant and express the line C_2 parametrically as

$$x = t, \quad y = y \qquad (x_1 \le t \le x)$$

At the risk of confusion, but to avoid complicating the notation, we have used x both as the dependent variable in the parametric equations and as the endpoint of the line segment. With the latter interpretation of x, it follows that (8) can be expressed as

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\int_{x_1}^{x} f(t, y)\,dt$$

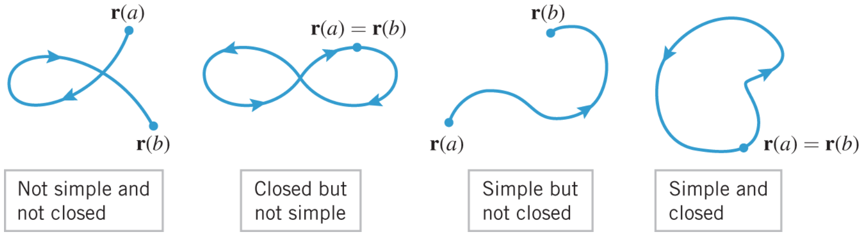
Now we apply Part 2 of the Fundamental Theorem of Calculus (Theorem 4.6.3), treating y as constant. This yields

$$\frac{\partial\phi}{\partial x} = f(x, y)$$

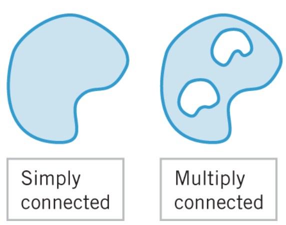
which proves the first part of (6). The proof that $\partial\phi/\partial y = g(x, y)$ can be obtained in a similar manner by joining (x, y) to a point (x, y_1) with a vertical line segment (Exercise 39). ■

A TEST FOR CONSERVATIVE VECTOR FIELDS

Although Theorem 15.3.2 is an important characterization of conservative vector fields, it is not an effective computational tool because it is usually not possible to evaluate the line integral over all possible piecewise smooth curves in D, as required in parts (b) and (c). To develop a method for determining whether a vector field is conservative, we will need to introduce some new concepts about parametric curves and connected sets. We will say that a parametric curve is ***simple*** if it does not intersect itself between its endpoints. A simple parametric curve may or may not be closed (Figure 15.3.5). In addition, we will say that a connected set D in 2-space is ***simply connected*** if no simple closed curve in D encloses points that are not in D. Stated informally, a connected set D is simply connected if it has no holes; a connected set with one or more holes is said to be ***multiply connected*** (Figure 15.3.6).



▲ **Figure 15.3.5**



▲ **Figure 15.3.6**

The following theorem is the primary tool for determining whether a vector field in 2-space is conservative.

WARNING

In (9), the $\mathbf{i}$-component of $\mathbf{F}$ is differentiated with respect to y and the $\mathbf{j}$-component with respect to x. It is easy to get this backwards by mistake.

15.3.3 THEOREM (*Conservative Field Test*) *If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open region D, and if the vector field $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is conservative on D, then*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \tag{9}$$

at each point in D. Conversely, if D is simply connected and (9) holds at each point in D, then $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is conservative.

A complete proof of this theorem requires results from advanced calculus and will be omitted. However, it is not hard to see why (9) must hold if $\mathbf{F}$ is conservative. For this purpose suppose that $\mathbf{F} = \nabla\phi$, in which case we can express the functions f and g as

$$\frac{\partial \phi}{\partial x} = f \quad \text{and} \quad \frac{\partial \phi}{\partial y} = g \tag{10}$$

Thus,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x} \quad \text{and} \quad \frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x \partial y}$$

But the mixed partial derivatives in these equations are equal (Theorem 13.3.2), so (9) follows.

► Example 3 Use Theorem 15.3.3 to determine whether the vector field

$$\mathbf{F}(x, y) = (y + x)\mathbf{i} + (y - x)\mathbf{j}$$

is conservative on some open set.

Solution. Let $f(x, y) = y + x$ and $g(x, y) = y - x$. Then

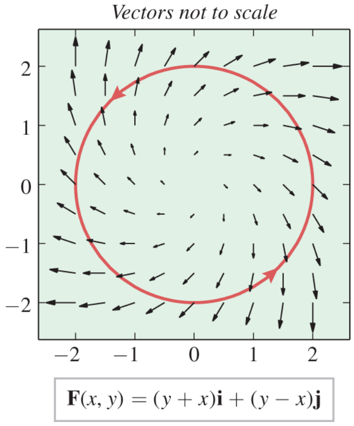
$$\frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = -1$$

Thus, there are no points in the xy-plane at which condition (9) holds, and hence $\mathbf{F}$ is not conservative on any open set. ◄

REMARK Since the vector field $\mathbf{F}$ in Example 3 is not conservative, it follows from Theorem 15.3.2 that there must exist piecewise smooth closed curves in every open connected set in the xy-plane on which

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \neq 0$$

One such curve is the oriented circle shown in Figure 15.3.7. The figure suggests that $\mathbf{F} \cdot \mathbf{T} < 0$ at each point of C (why?), so $\int_C \mathbf{F} \cdot \mathbf{T}\, ds < 0$.



▲ **Figure 15.3.7**

Once it is established that a vector field is conservative, a potential function for the field can be obtained by first integrating either of the equations in (10). This is illustrated in the following example.

► Example 4 Let $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + (1 + 3x^2y^2)\mathbf{j}$.

(a) Show that $\mathbf{F}$ is a conservative vector field on the entire xy-plane.

(b) Find ϕ by first integrating $\partial\phi/\partial x$.

(c) Find ϕ by first integrating $\partial\phi/\partial y$.

Solution (a). Since $f(x,y) = 2xy^3$ and $g(x,y) = 1 + 3x^2y^2$, we have

$$\frac{\partial f}{\partial y} = 6xy^2 = \frac{\partial g}{\partial x}$$

so (9) holds for all (x, y).

Solution (b). Since the field $\mathbf{F}$ is conservative, there is a potential function ϕ such that

$$\frac{\partial \phi}{\partial x} = 2xy^3 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 1 + 3x^2y^2 \tag{11}$$

Integrating the first of these equations with respect to x (and treating y as a constant) yields

$$\phi = \int 2xy^3\,dx = x^2y^3 + k(y) \tag{12}$$

where $k(y)$ represents the "constant" of integration. We are justified in treating the constant of integration as a function of y, since y is held constant in the integration process. To find $k(y)$ we differentiate (12) with respect to y and use the second equation in (11) to obtain

$$\frac{\partial \phi}{\partial y} = 3x^2y^2 + k'(y) = 1 + 3x^2y^2$$

from which it follows that $k'(y) = 1$. Thus,

$$k(y) = \int k'(y)\,dy = \int 1\,dy = y + K$$

where K is a (numerical) constant of integration. Substituting in (12) we obtain

$$\phi = x^2y^3 + y + K$$

The appearance of the arbitrary constant K tells us that ϕ is not unique. As a check on the computations, you may want to verify that $\nabla\phi = \mathbf{F}$.

> You can also use (7) to find a potential function for a conservative vector field. For example, find a potential function for the vector field in Example 4 by evaluating (7) on the line segment
>
> $\mathbf{r}(t) = t(x\mathbf{i}) + t(y\mathbf{j}) \quad (0 \le t \le 1)$
>
> from $(0, 0)$ to (x, y).

Solution (c). Integrating the second equation in (11) with respect to y (and treating x as a constant) yields

$$\phi = \int (1 + 3x^2y^2)\,dy = y + x^2y^3 + k(x) \tag{13}$$

where $k(x)$ is the "constant" of integration. Differentiating (13) with respect to x and using the first equation in (11) yields

$$\frac{\partial \phi}{\partial x} = 2xy^3 + k'(x) = 2xy^3$$

from which it follows that $k'(x) = 0$ and consequently that $k(x) = K$, where K is a numerical constant of integration. Substituting this in (13) yields

$$\phi = y + x^2y^3 + K$$

which agrees with the solution in part (b). ◀

▶ **Example 5** Use the potential function obtained in Example 4 to evaluate the integral

$$\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1 + 3x^2y^2)\,dy$$

In the solution to Example 5, note that the constant K drops out. In future integration problems of the type in this example, we will often omit K from the computations. See Exercise 7 for other ways to evaluate this integral.

Solution. The integrand can be expressed as $\mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field in Example 4. Thus, using Formula (3) and the potential function $\phi = y + x^2y^3 + K$ for $\mathbf{F}$, we obtain

$$\begin{aligned}\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1+3x^2y^2)\,dy &= \int_{(1,4)}^{(3,1)} \mathbf{F}\cdot d\mathbf{r} = \phi(3,1)-\phi(1,4)\\ &= (10+K)-(68+K) = -58\end{aligned}$$

Example 6 Let $\mathbf{F}(x,y) = e^y\mathbf{i} + xe^y\mathbf{j}$ denote a force field in the xy-plane.

(a) Verify that the force field $\mathbf{F}$ is conservative on the entire xy-plane.

(b) Find the work done by the field on a particle that moves from $(1,0)$ to $(-1,0)$ along the semicircular path C shown in Figure 15.3.8.

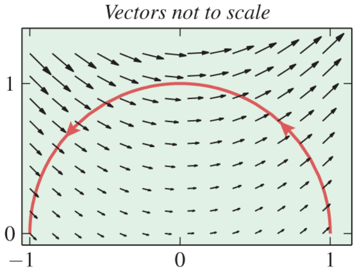


Figure 15.3.8

Solution (a). For the given field we have $f(x,y) = e^y$ and $g(x,y) = xe^y$. Thus,

$$\frac{\partial}{\partial y}(e^y) = e^y = \frac{\partial}{\partial x}(xe^y)$$

so (9) holds for all (x,y) and hence $\mathbf{F}$ is conservative on the entire xy-plane.

Solution (b). From Formula (34) of Section 15.2, the work done by the field is

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C e^y\,dx + xe^y\,dy \tag{14}$$

However, the calculations involved in integrating along C are tedious, so it is preferable to apply Theorem 15.3.1, taking advantage of the fact that the field is conservative and the integral is independent of path. Thus, we write (14) as

$$W = \int_{(1,0)}^{(-1,0)} e^y\,dx + xe^y\,dy = \phi(-1,0)-\phi(1,0) \tag{15}$$

As illustrated in Example 4, we can find ϕ by integrating either of the equations

$$\frac{\partial\phi}{\partial x} = e^y \quad\text{and}\quad \frac{\partial\phi}{\partial y} = xe^y \tag{16}$$

We will integrate the first. We obtain

$$\phi = \int e^y\,dx = xe^y + k(y) \tag{17}$$

Differentiating this equation with respect to y and using the second equation in (16) yields

$$\frac{\partial\phi}{\partial y} = xe^y + k'(y) = xe^y$$

from which it follows that $k'(y) = 0$ or $k(y) = K$. Thus, from (17)

$$\phi = xe^y + K$$

and hence from (15)

$$W = \phi(-1,0)-\phi(1,0) = (-1)e^0 - 1e^0 = -2$$

CONSERVATIVE VECTOR FIELDS IN 3-SPACE

All of the results in this section have analogs in 3-space: Theorems 15.3.1 and 15.3.2 can be extended to vector fields in 3-space simply by adding a third variable and modifying the hypotheses appropriately. For example, in 3-space, Formula (3) becomes

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \phi(x_1,y_1,z_1)-\phi(x_0,y_0,z_0) \tag{18}$$

Theorem 15.3.3 can also be extended to vector fields in 3-space. We leave it for the exercises to show that if $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$ is a conservative field, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \tag{19}$$

that is, curl $\mathbf{F} = \mathbf{0}$. Conversely, a vector field satisfying these conditions on a suitably restricted region is conservative on that region if f, g, and h are continuous and have continuous first partial derivatives in the region. Some problems involving Formulas (18) and (19) are given in the review exercises at the end of this chapter.

CONSERVATION OF ENERGY

If $\mathbf{F}(x, y, z)$ is a conservative force field with a potential function $\phi(x, y, z)$, then we call $V(x, y, z) = -\phi(x, y, z)$ the ***potential energy*** of the field at the point (x, y, z). Thus, it follows from the 3-space version of Theorem 15.3.1 that the work W done by $\mathbf{F}$ on a particle that moves along any path C from a point (x_0, y_0, z_0) to a point (x_1, y_1, z_1) is related to the potential energy by the equation

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0) = -[V(x_1, y_1, z_1) - V(x_0, y_0, z_0)] \tag{20}$$

That is, the work done by the field is the negative of the change in potential energy. In particular, it follows from the 3-space analog of Theorem 15.3.2 that if a particle traverses a piecewise smooth closed path in a conservative vector field, then the work done by the field is zero, and there is no change in potential energy. To take this a step further, suppose that a particle of mass m moves along any piecewise smooth curve (not necessarily closed) in a conservative force field $\mathbf{F}$, starting at (x_0, y_0, z_0) with speed v_i and ending at (x_1, y_1, z_1) with speed v_f. If $\mathbf{F}$ is the only force acting on the particle, then an argument similar to the derivation of Equation (6) in Section 5.6 shows that the work done on the particle by $\mathbf{F}$ is equal to the change in kinetic energy $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$ of the particle. (An argument for smooth curves appears in the Making Connections exercises.) If we let V_i denote the potential energy at the starting point and V_f the potential energy at the final point, then it follows from (20) that

$$\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = -[V_f - V_i]$$

which we can rewrite as

$$\tfrac{1}{2}mv_f^2 + V_f = \tfrac{1}{2}mv_i^2 + V_i$$

This equation states that the total energy of the particle (kinetic energy + potential energy) does not change as the particle moves along a path in a conservative vector field. This result, called the ***conservation of energy principle***, explains the origin of the term "conservative vector field."

✓ QUICK CHECK EXERCISES 15.3

(See page 1005 for answers.)

1. If C is a piecewise smooth curve from $(1, 2, 3)$ to $(4, 5, 6)$, then
$$\int_C dx + 2\,dy + 3\,dz = \_\_\_\_\_\_$$

2. If C is the portion of the circle $x^2 + y^2 = 1$ where $0 \le x$, oriented counterclockwise, and $f(x, y) = ye^x$, then
$$\int_C \nabla f \cdot d\mathbf{r} = \_\_\_\_\_\_$$

3. A potential function for the vector field
$$\mathbf{F}(x, y, z) = yz\mathbf{i} + (xz + z)\mathbf{j} + (xy + y + 1)\mathbf{k}$$
is $\phi(x, y, z) = \_\_\_\_\_\_$.

4. If a, b, and c are nonzero real numbers such that the vector field $x^5y^a\mathbf{i} + x^by^c\mathbf{j}$ is a conservative vector field, then
$$a = \_\_\_\_\_\_, \quad b = \_\_\_\_\_\_, \quad c = \_\_\_\_\_\_$$

EXERCISE SET 15.3 C CAS

1–6 Determine whether $\mathbf{F}$ is a conservative vector field. If so, find a potential function for it.

1. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$
2. $\mathbf{F}(x, y) = 3y^2\mathbf{i} + 6xy\mathbf{j}$
3. $\mathbf{F}(x, y) = x^2y\mathbf{i} + 5xy^2\mathbf{j}$
4. $\mathbf{F}(x, y) = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j}$
5. $\mathbf{F}(x, y) = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j}$
6. $\mathbf{F}(x, y) = x \ln y\mathbf{i} + y \ln x\mathbf{j}$
7. In each part, evaluate $\int_C 2xy^3\,dx + (1 + 3x^2y^2)\,dy$ over the curve C, and compare your answer with the result of Example 5.
   (a) C is the line segment from $(1, 4)$ to $(3, 1)$.
   (b) C consists of the line segment from $(1, 4)$ to $(1, 1)$, followed by the line segment from $(1, 1)$ to $(3, 1)$.
8. (a) Show that the line integral $\int_C y \sin x\,dx - \cos x\,dy$ is independent of the path.
   (b) Evaluate the integral in part (a) along the line segment from $(0, 1)$ to $(\pi, -1)$.
   (c) Evaluate the integral $\int_{(0,1)}^{(\pi,-1)} y \sin x\,dx - \cos x\,dy$ using Theorem 15.3.1, and confirm that the value is the same as that obtained in part (b).

9–14 Show that the integral is independent of the path, and use Theorem 15.3.1 to find its value.

9. $\displaystyle\int_{(1,2)}^{(4,0)} 3y\,dx + 3x\,dy$
10. $\displaystyle\int_{(0,0)}^{(1,\pi/2)} e^x \sin y\,dx + e^x \cos y\,dy$
11. $\displaystyle\int_{(0,0)}^{(3,2)} 2xe^y\,dx + x^2e^y\,dy$
12. $\displaystyle\int_{(-1,2)}^{(0,1)} (3x - y + 1)\,dx - (x + 4y + 2)\,dy$
13. $\displaystyle\int_{(2,-2)}^{(-1,0)} 2xy^3\,dx + 3y^2x^2\,dy$
14. $\displaystyle\int_{(1,1)}^{(3,3)} \left(e^x \ln y - \frac{e^y}{x}\right) dx + \left(\frac{e^x}{y} - e^y \ln x\right) dy$, where x and y are positive.

15–18 Confirm that the force field $\mathbf{F}$ is conservative in some open connected region containing the points P and Q, and then find the work done by the force field on a particle moving along an arbitrary smooth curve in the region from P to Q.

15. $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$; $P(1, 1), Q(0, 0)$
16. $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$; $P(-3, 0), Q(4, 1)$
17. $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$; $P(-1, 1), Q(2, 0)$
18. $\mathbf{F}(x, y) = e^{-y} \cos x\mathbf{i} - e^{-y} \sin x\mathbf{j}$; $P(\pi/2, 1), Q(-\pi/2, 0)$

19–22 True–False Determine whether the statement is true or false. Explain your answer.

19. If $\mathbf{F}$ is a vector field and there exists a closed curve C such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, then $\mathbf{F}$ is conservative.
20. If $\mathbf{F}(x, y) = ay\mathbf{i} + bx\mathbf{j}$ is a conservative vector field, then $a = b$.
21. If $\phi(x, y)$ is a potential function for a constant vector field, then the graph of $z = \phi(x, y)$ is a plane.
22. If $f(x, y)$ and $g(x, y)$ are differentiable functions defined on the xy-plane, and if $f_y(x, y) = g_x(x, y)$ for all (x, y), then there exists a function $\phi(x, y)$ such that $\phi_x(x, y) = f(x, y)$ and $\phi_y(x, y) = g(x, y)$.

23–24 Find the exact value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ using any method.

23. $\mathbf{F}(x, y) = (e^y + ye^x)\mathbf{i} + (xe^y + e^x)\mathbf{j}$
    $C: \mathbf{r}(t) = \sin(\pi t/2)\mathbf{i} + \ln t\,\mathbf{j} \quad (1 \le t \le 2)$
24. $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + \cos y)\mathbf{j}$
    $C: \mathbf{r}(t) = t\mathbf{i} + t \cos(t/3)\mathbf{j} \quad (0 \le t \le \pi)$

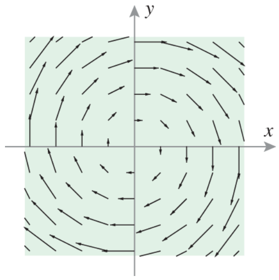
C 25. Use the numerical integration capability of a CAS or other calculating utility to approximate the value of the integral in Exercise 23 by direct integration. Confirm that the numerical approximation is consistent with the exact value.

C 26. Use the numerical integration capability of a CAS or other calculating utility to approximate the value of the integral in Exercise 24 by direct integration. Confirm that the numerical approximation is consistent with the exact value.

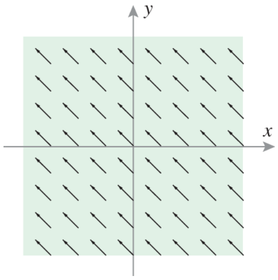
FOCUS ON CONCEPTS

27–28 Is the vector field conservative? Explain.

27.



28.



29. Suppose that C is a circle in the domain of a conservative nonzero vector field in the xy-plane whose component functions are continuous. Explain why there must be at least two points on C at which the vector field is normal to the circle.
30. Does the result in Exercise 29 remain true if the circle C is replaced by a square? Explain.

31. Prove: If
$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$
is a conservative field and f, g, and h are continuous and have continuous first partial derivatives in a region, then
$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$
in the region.

32. Use the result in Exercise 31 to show that the integral
$$\int_C yz\,dx + xz\,dy + yx^2\,dz$$
is not independent of the path.

33. Find a nonzero function h for which
$$\mathbf{F}(x, y) = h(x)[x \sin y + y \cos y]\mathbf{i} + h(x)[x \cos y - y \sin y]\mathbf{j}$$
is conservative.

34. (a) In Example 3 of Section 15.1 we showed that
$$\phi(x, y) = -\frac{c}{(x^2 + y^2)^{1/2}}$$
is a potential function for the two-dimensional inverse-square field
$$\mathbf{F}(x, y) = \frac{c}{(x^2 + y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j})$$
but we did not explain how the potential function $\phi(x, y)$ was obtained. Use Theorem 15.3.3 to show that the two-dimensional inverse-square field is conservative everywhere except at the origin, and then use the method of Example 4 to derive the formula for $\phi(x, y)$.

(b) Use an appropriate generalization of the method of Example 4 to derive the potential function
$$\phi(x, y, z) = -\frac{c}{(x^2 + y^2 + z^2)^{1/2}}$$
for the three-dimensional inverse-square field given by Formula (5) of Section 15.1.

35–36 Use the result in Exercise 34(b).

35. In each part, find the work done by the three-dimensional inverse-square field
$$\mathbf{F}(\mathbf{r}) = \frac{1}{\|\mathbf{r}\|^3}\mathbf{r}$$
on a particle that moves along the curve C.

(a) C is the line segment from $P(1, 1, 2)$ to $Q(3, 2, 1)$.

(b) C is the curve
$$\mathbf{r}(t) = (2t^2 + 1)\mathbf{i} + (t^3 + 1)\mathbf{j} + (2 - \sqrt{t})\mathbf{k}$$
where $0 \le t \le 1$.

(c) C is the circle in the xy-plane of radius 1 centered at $(2, 0, 0)$ traversed counterclockwise.

36. Let $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$.

(a) Show that
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \ne \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$
if C_1 and C_2 are the semicircular paths from $(1, 0)$ to $(-1, 0)$ given by
$$C_1 : x = \cos t, \quad y = \sin t \qquad (0 \le t \le \pi)$$
$$C_2 : x = \cos t, \quad y = -\sin t \qquad (0 \le t \le \pi)$$

(b) Show that the components of $\mathbf{F}$ satisfy Formula (9).

(c) Do the results in parts (a) and (b) contradict Theorem 15.3.3? Explain.

37. Prove Theorem 15.3.1 if C is a piecewise smooth curve composed of smooth curves $C_1, C_2, \ldots, C_n$.

38. Prove that (*b*) implies (*c*) in Theorem 15.3.2. [*Hint:* Consider any two piecewise smooth oriented curves C_1 and C_2 in the region from a point P to a point Q, and integrate around the closed curve consisting of C_1 and $-C_2$.]

39. Complete the proof of Theorem 15.3.2 by showing that $\partial\phi/\partial y = g(x, y)$, where $\phi(x, y)$ is the function in (7).

40. **Writing** Describe the different methods available for evaluating the integral of a conservative vector field over a smooth curve.

41. **Writing** Discuss some of the ways that you can show a vector field is *not* conservative.

✓ **QUICK CHECK ANSWERS 15.3** **1.** 18 **2.** 2 **3.** $xyz + yz + z$ **4.** 6, 6, 5

15.4 GREEN'S THEOREM

In this section we will discuss a remarkable and beautiful theorem that expresses a double integral over a plane region in terms of a line integral around its boundary.

GREEN'S THEOREM

15.4.1 THEOREM (*Green's Theorem*) *Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented counterclockwise. If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open set containing R, then*
$$\int_C f(x, y)\,dx + g(x, y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \tag{1}$$

PROOF For simplicity, we will prove the theorem for regions that are simultaneously type I and type II (see Definition 14.2.1). Such a region is shown in Figure 15.4.1. The crux of the proof is to show that

$$\int_C f(x,y)\,dx = -\iint_R \frac{\partial f}{\partial y}\,dA \quad \text{and} \quad \int_C g(x,y)\,dy = \iint_R \frac{\partial g}{\partial x}\,dA \tag{2–3}$$

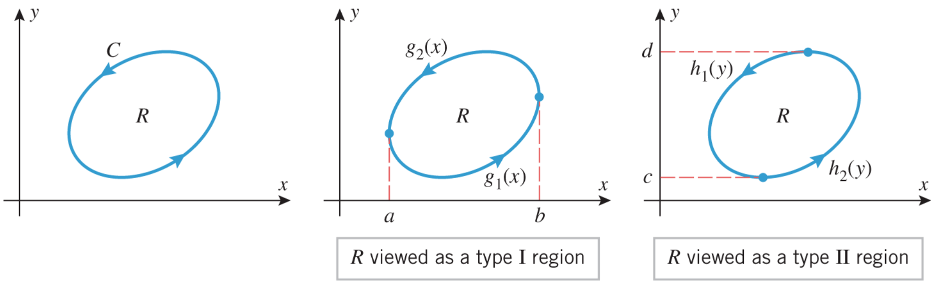


Figure 15.4.1

To prove (2), view R as a type I region and let C_1 and C_2 be the lower and upper boundary curves, oriented as in Figure 15.4.2. Then

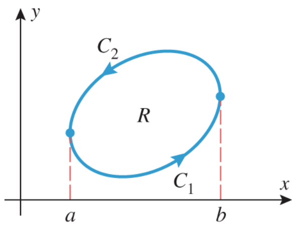


Figure 15.4.2

$$\int_C f(x,y)\,dx = \int_{C_1} f(x,y)\,dx + \int_{C_2} f(x,y)\,dx$$

or, equivalently,

$$\int_C f(x,y)\,dx = \int_{C_1} f(x,y)\,dx - \int_{-C_2} f(x,y)\,dx \tag{4}$$

(This step will help simplify our calculations since C_1 and $-C_2$ are then both oriented left to right.) The curves C_1 and $-C_2$ can be expressed parametrically as

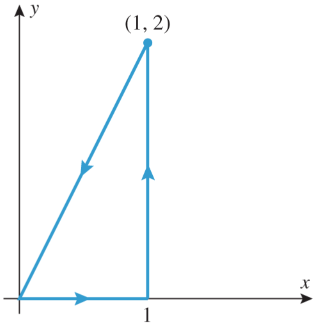
$$\begin{aligned} C_1 &: x = t, \quad y = g_1(t) \qquad (a \le t \le b) \\ -C_2 &: x = t, \quad y = g_2(t) \qquad (a \le t \le b) \end{aligned}$$

Thus, we can rewrite (4) as

$$\begin{aligned} \int_C f(x,y)\,dx &= \int_a^b f(t, g_1(t))x'(t)\,dt - \int_a^b f(t, g_2(t))x'(t)\,dt \\ &= \int_a^b f(t, g_1(t))\,dt - \int_a^b f(t, g_2(t))\,dt \\ &= -\int_a^b [f(t, g_2(t)) - f(t, g_1(t))]\,dt \\ &= -\int_a^b \Big[f(t,y)\Big]_{y=g_1(t)}^{y=g_2(t)} dt = -\int_a^b \left[\int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y}\,dy\right] dt \\ &= \underbrace{-\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial f}{\partial y}\,dy\,dx}_{\text{Since } x = t} = -\iint_R \frac{\partial f}{\partial y}\,dA \end{aligned}$$

Supply the details for the proof of (3).

The proof of (3) is obtained similarly by treating R as a type II region. We omit the details. ■



▲ **Figure 15.4.3**

▶ **Example 1** Use Green's Theorem to evaluate

$$\int_C x^2 y\,dx + x\,dy$$

along the triangular path shown in Figure 15.4.3.

Solution. Since $f(x,y) = x^2y$ and $g(x,y) = x$, it follows from (1) that

$$\begin{aligned}\int_C x^2y\,dx + x\,dy &= \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^2y)\right] dA = \int_0^1 \int_0^{2x} (1 - x^2)\,dy\,dx \\ &= \int_0^1 (2x - 2x^3)\,dx = \left[x^2 - \frac{x^4}{2}\right]_0^1 = \frac{1}{2}\end{aligned}$$

This agrees with the result obtained in Example 10 of Section 15.2, where we evaluated the line integral directly. Note how much simpler this solution is. ◀

A NOTATION FOR LINE INTEGRALS AROUND SIMPLE CLOSED CURVES

It is common practice to denote a line integral around a simple closed curve by an integral sign with a superimposed circle. With this notation Formula (1) would be written as

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

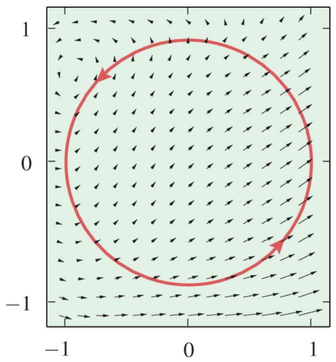
Sometimes a direction arrow is added to the circle to indicate whether the integration is clockwise or counterclockwise. Thus, if we wanted to emphasize the counterclockwise direction of integration required by Theorem 15.4.1, we could express (1) as

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \tag{5}$$

FINDING WORK USING GREEN'S THEOREM

It follows from Formula (26) of Section 15.2 that the integral on the left side of (5) is the work performed by the force field $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ on a particle moving counterclockwise around the simple closed curve C. In the case where this vector field is conservative, it follows from Theorem 15.3.2 that the integrand in the double integral on the right side of (5) is zero, so the work performed by the field is zero, as expected. For vector fields that are not conservative, it is often more efficient to calculate the work around simple closed curves by using Green's Theorem than by parametrizing the curve.

George Green (1793–1841) English mathematician and physicist. Green left school at an early age to work in his father's bakery and consequently had little early formal education. When his father opened a mill, the boy used the top room as a study in which he taught himself physics and mathematics from library books. In 1828 Green published his most important work, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Although Green's Theorem appeared in that paper, the result went virtually unnoticed because of the small press-run and local distribution. Following the death of his father in 1829, Green was urged by friends to seek a college education. In 1833, after four years of self-study to close the gaps in his elementary education, Green was admitted to Caius College, Cambridge. He graduated four years later, but with a disappointing performance on his final examinations—possibly because he was more interested in his own research. After a succession of works on light and sound, he was named to be Perse Fellow at Caius College. Two years later he died. In 1845, four years after his death, his paper of 1828 was published and the theories developed therein by this obscure, self-taught baker's son helped pave the way to the modern theories of electricity and magnetism.



▲ Figure 15.4.4

▶ **Example 2** Find the work done by the force field

$$\mathbf{F}(x, y) = (e^x - y^3)\mathbf{i} + (\cos y + x^3)\mathbf{j}$$

on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counterclockwise direction (Figure 15.4.4).

Solution. The work W performed by the field is

$$\begin{aligned} W &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (e^x - y^3)\,dx + (\cos y + x^3)\,dy \\ &= \iint_R \left[\frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3)\right] dA \qquad \boxed{\text{Green's Theorem}} \\ &= \iint_R (3x^2 + 3y^2)\,dA = 3\iint_R (x^2 + y^2)\,dA \\ &= 3\int_0^{2\pi}\int_0^1 (r^2) r\,dr\,d\theta = \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3\pi}{2} \blacktriangleleft \end{aligned}$$

We converted to polar coordinates.

FINDING AREAS USING GREEN'S THEOREM

Green's Theorem leads to some useful new formulas for the area A of a region R that satisfies the conditions of the theorem. Two such formulas can be obtained as follows:

$$A = \iint_R dA = \oint_C x\,dy \quad \text{and} \quad A = \iint_R dA = \oint_C (-y)\,dx$$

Set $f(x, y) = 0$ and $g(x, y) = x$ in (1).

Set $f(x, y) = -y$ and $g(x, y) = 0$ in (1).

A third formula can be obtained by adding these two equations together. Thus, we have the following three formulas that express the area A of a region R in terms of line integrals around the boundary:

Although the third formula in (6) looks more complicated than the other two, it often leads to simpler integrations. Each has advantages in certain situations.

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C -y\,dx + x\,dy \tag{6}$$

▶ **Example 3** Use a line integral to find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution. The ellipse, with counterclockwise orientation, can be represented parametrically by

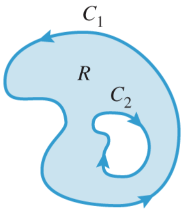
$$x = a\cos t, \quad y = b\sin t \qquad (0 \le t \le 2\pi)$$

If we denote this curve by C, then from the third formula in (6) the area A enclosed by the ellipse is

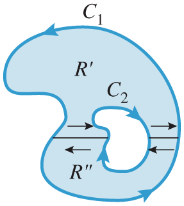
$$\begin{aligned} A &= \frac{1}{2}\oint_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} [(-b\sin t)(-a\sin t) + (a\cos t)(b\cos t)]\,dt \\ &= \frac{1}{2}ab\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \frac{1}{2}ab\int_0^{2\pi} dt = \pi ab \blacktriangleleft \end{aligned}$$

GREEN'S THEOREM FOR MULTIPLY CONNECTED REGIONS

Recall that a plane region is said to be simply connected if it has no holes and is said to be multiply connected if it has one or more holes (see Figure 15.3.6). At the beginning of this section we stated Green's Theorem for a counterclockwise integration around the boundary of a simply connected region R (Theorem 15.4.1). Our next goal is to extend this theorem to multiply connected regions. To make this extension we will need to assume that *the region lies on the left when any portion of the boundary is traversed in the direction of its orientation*. This implies that the outer boundary curve of the region is oriented counterclockwise and the boundary curves that enclose holes have clockwise orientation (Figure 15.4.5a). If all portions of the boundary of a multiply connected region R are oriented in this way, then we say that the boundary of R has ***positive orientation***.



(a)



(b)

▲ **Figure 15.4.5**

We will now derive a version of Green's Theorem that applies to multiply connected regions with positively oriented boundaries. For simplicity, we will consider a multiply connected region R with one hole, and we will assume that $f(x,y)$ and $g(x,y)$ have continuous first partial derivatives on some open set containing R. As shown in Figure 15.4.5b, let us divide R into two regions R' and R'' by introducing two "cuts" in R. The cuts are shown as line segments, but any piecewise smooth curves will suffice. If we assume that f and g satisfy the hypotheses of Green's Theorem on R (and hence on R' and R''), then we can apply this theorem to both R' and R'' to obtain

$$\begin{aligned}\iint\limits_R \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right) dA &= \iint\limits_{R'} \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right) dA + \iint\limits_{R''} \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right) dA \\ &= \oint\limits_{\substack{\text{Boundary}\\ \text{of } R'}} f(x,y)\,dx+g(x,y)\,dy + \oint\limits_{\substack{\text{Boundary}\\ \text{of } R''}} f(x,y)\,dx+g(x,y)\,dy\end{aligned}$$

However, the two line integrals are taken in opposite directions along the cuts, and hence cancel there, leaving only the contributions along C_1 and C_2. Thus,

$$\iint\limits_R \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right) dA = \oint_{C_1} f(x,y)\,dx+g(x,y)\,dy + \oint_{C_2} f(x,y)\,dx+g(x,y)\,dy \tag{7}$$

which is an extension of Green's Theorem to a multiply connected region with one hole. Observe that the integral around the outer boundary is taken counterclockwise and the integral around the hole is taken clockwise. More generally, if R is a multiply connected region with n holes, then the analog of (7) involves a sum of $n+1$ integrals, one taken counterclockwise around the outer boundary of R and the rest taken clockwise around the holes.

► **Example 4** Evaluate the integral

$$\oint_C \frac{-y\,dx+x\,dy}{x^2+y^2}$$

if C is a piecewise smooth simple closed curve oriented counterclockwise such that (a) C does not enclose the origin and (b) C encloses the origin.

Solution (a). Let

$$f(x,y)=-\frac{y}{x^2+y^2}, \quad g(x,y)=\frac{x}{x^2+y^2} \tag{8}$$

so that

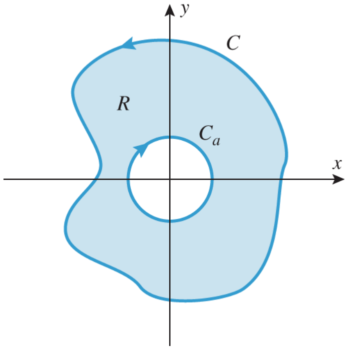
$$\frac{\partial g}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}=\frac{\partial f}{\partial y}$$

if x and y are not both zero. Thus, if C does not enclose the origin, we have

$$\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}=0 \tag{9}$$

on the simply connected region enclosed by C, and hence the given integral is zero by Green's Theorem.

Solution (b). Unlike the situation in part (a), we cannot apply Green's Theorem directly because the functions $f(x, y)$ and $g(x, y)$ in (8) are discontinuous at the origin. Our problems are further compounded by the fact that we do not have a specific curve C that we can parametrize to evaluate the integral. Our strategy for circumventing these problems will be to replace C with a specific curve that produces the same value for the integral and then use that curve for the evaluation. To obtain such a curve, we will apply Green's Theorem for multiply connected regions to a region that does not contain the origin. For this purpose we construct a circle C_a with *clockwise* orientation, centered at the origin, and with sufficiently small radius a that it lies inside the region enclosed by C (Figure 15.4.6). This creates a multiply connected region R whose boundary curves C and C_a have the orientations required by Formula (7) and such that within R the functions $f(x, y)$ and $g(x, y)$ in (8) satisfy the hypotheses of Green's Theorem (the origin does not belong to R). Thus, it follows from (7) and (9) that



▲ **Figure 15.4.6**

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} + \oint_{C_a} \frac{-y\,dx + x\,dy}{x^2+y^2} = \iint_R 0\,dA = 0$$

It follows from this equation that

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = -\oint_{C_a} \frac{-y\,dx + x\,dy}{x^2+y^2}$$

which we can rewrite as

$$\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \oint_{-C_a} \frac{-y\,dx + x\,dy}{x^2+y^2}$$

Reversing the orientation of C_a reverses the sign of the integral.

But C_a has clockwise orientation, so $-C_a$ has counterclockwise orientation. Thus, we have shown that the original integral can be evaluated by integrating counterclockwise around a circle of radius a that is centered at the origin and lies within the region enclosed by C. Such a circle can be expressed parametrically as $x = a\cos t$, $y = a\sin t$ $(0 \le t \le 2\pi)$; and hence

$$\begin{aligned}\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} &= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t)\,dt + (a\cos t)(a\cos t)\,dt}{(a\cos t)^2 + (a\sin t)^2} \\ &= \int_0^{2\pi} 1\,dt = 2\pi \quad ◀\end{aligned}$$

✓QUICK CHECK EXERCISES 15.4 *(See page 1013 for answers.)*

1. If C is the square with vertices $(\pm 1, \pm 1)$ oriented counterclockwise, then

$$\int_C -y\,dx + x\,dy = \underline{\qquad\qquad}$$

2. If C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$ oriented counterclockwise, then

$$\int_C 2xy\,dx + (x^2 + x)\,dy = \underline{\qquad\qquad}$$

3. If C is the unit circle centered at the origin and oriented counterclockwise, then

$$\int_C (y^3 - y - x)\,dx + (x^3 + x + y)\,dy = \underline{\qquad\qquad}$$

4. What region R and choice of functions $f(x, y)$ and $g(x, y)$ allow us to use Formula (1) of Theorem 15.4.1 to claim that

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (2x + 2y)\,dy\,dx = \int_0^{\pi/2} (\sin^3 t + \cos^3 t)\,dt?$$

EXERCISE SET 15.4 C CAS

1–2 Evaluate the line integral using Green's Theorem and check the answer by evaluating it directly.

1. $\oint_C y^2\,dx + x^2\,dy$, where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ oriented counterclockwise.

2. $\oint_C y\,dx + x\,dy$, where C is the unit circle oriented counterclockwise.

3–13 Use Green's Theorem to evaluate the integral. In each exercise, assume that the curve C is oriented counterclockwise.

3. $\oint_C 3xy\,dx + 2xy\,dy$, where C is the rectangle bounded by $x=-2$, $x=4$, $y=1$, and $y=2$.

4. $\oint_C (x^2-y^2)\,dx + x\,dy$, where C is the circle $x^2+y^2=9$.

5. $\oint_C x\cos y\,dx - y\sin x\,dy$, where C is the square with vertices $(0,0)$, $(\pi/2,0)$, $(\pi/2,\pi/2)$, and $(0,\pi/2)$.

6. $\oint_C y\tan^2 x\,dx + \tan x\,dy$, where C is the circle $x^2+(y+1)^2=1$.

7. $\oint_C (x^2-y)\,dx + x\,dy$, where C is the circle $x^2+y^2=4$.

8. $\oint_C (e^x+y^2)\,dx + (e^y+x^2)\,dy$, where C is the boundary of the region between $y=x^2$ and $y=x$.

9. $\oint_C \ln(1+y)\,dx - \dfrac{xy}{1+y}\,dy$, where C is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,4)$.

10. $\oint_C x^2y\,dx - y^2x\,dy$, where C is the boundary of the region in the first quadrant, enclosed between the coordinate axes and the circle $x^2+y^2=16$.

11. $\oint_C \tan^{-1}y\,dx - \dfrac{y^2x}{1+y^2}\,dy$, where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

12. $\oint_C \cos x\sin y\,dx + \sin x\cos y\,dy$, where C is the triangle with vertices $(0,0)$, $(3,3)$, and $(0,3)$.

13. $\oint_C x^2y\,dx + (y+xy^2)\,dy$, where C is the boundary of the region enclosed by $y=x^2$ and $x=y^2$.

14. Let C be the boundary of the region enclosed between $y=x^2$ and $y=2x$. Assuming that C is oriented counterclockwise, evaluate the following integrals by Green's Theorem:

(a) $\oint_C (6xy-y^2)\,dx$ (b) $\oint_C (6xy-y^2)\,dy$.

15–18 True–False Determine whether the statement is true or false. Explain your answer. (In Exercises 16–18, assume that C is a simple, smooth, closed curve, oriented counterclockwise.)

15. Green's Theorem allows us to replace any line integral by a double integral.

16. If

$$\int_C f(x,y)\,dx + g(x,y)\,dy = 0$$

then $\partial g/\partial x = \partial f/\partial y$ at all points in the region bounded by C.

17. It must be the case that

$$\int_C x\,dy > 0$$

18. It must be the case that

$$\int_C e^{x^2}\,dx + \sin y^3\,dy = 0$$

C **19.** Use a CAS to check Green's Theorem by evaluating both integrals in the equation

$$\oint_C e^y\,dx + ye^x\,dy = \iint_R \left[\frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}(e^y)\right] dA$$

where

(a) C is the circle $x^2+y^2=1$

(b) C is the boundary of the region enclosed by $y=x^2$ and $x=y^2$.

20. In Example 3, we used Green's Theorem to obtain the area of an ellipse. Obtain this area using the first and then the second formula in (6).

21. Use a line integral to find the area of the region enclosed by the astroid

$$x = a\cos^3\phi, \quad y = a\sin^3\phi \qquad (0 \le \phi \le 2\pi)$$

22. Use a line integral to find the area of the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$, where $a>0$ and $b>0$.

23. Use the formula

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

to find the area of the region swept out by the line from the origin to the ellipse $x=a\cos t$, $y=b\sin t$ if t varies from $t=0$ to $t=t_0$ $(0 \le t_0 \le 2\pi)$.

24. Use the formula

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

to find the area of the region swept out by the line from the origin to the hyperbola $x=a\cosh t$, $y=b\sinh t$ if t varies from $t=0$ to $t=t_0$ $(t_0 \ge 0)$.

FOCUS ON CONCEPTS

25. Suppose that $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is a vector field whose component functions f and g have continuous first partial derivatives. Let C denote a simple, closed, piecewise smooth curve oriented counterclockwise that bounds a region R contained in the domain of $\mathbf{F}$. We can think of $\mathbf{F}$ as a vector field in 3-space by writing it as

$$\mathbf{F}(x, y, z) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j} + 0\mathbf{k}$$

With this convention, explain why

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA$$

26. Suppose that $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is a vector field on the xy-plane and that f and g have continuous first partial derivatives with $f_y = g_x$ everywhere. Use Green's Theorem to explain why

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

where C_1 and C_2 are the oriented curves in the accompanying figure. [*Note:* Compare this result with Theorems 15.3.2 and 15.3.3.]

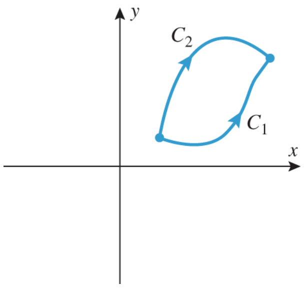


Figure Ex-26

27. Suppose that $f(x)$ and $g(x)$ are continuous functions with $g(x) \le f(x)$. Let R denote the region bounded by the graph of f, the graph of g, and the vertical lines $x = a$ and $x = b$. Let C denote the boundary of R oriented counterclockwise. What familiar formula results from applying Green's Theorem to $\int_C (-y)\, dx$?

28. In the accompanying figure, C is a smooth oriented curve from $P(x_0, y_0)$ to $Q(x_1, y_1)$ that is contained inside the rectangle with corners at the origin and Q and outside the rectangle with corners at the origin and P.

(a) What region in the figure has area $\int_C x\, dy$?

(b) What region in the figure has area $\int_C y\, dx$?

(c) Express $\int_C x\, dy + \int_C y\, dx$ in terms of the coordinates of P and Q.

(d) Interpret the result of part (c) in terms of the Fundamental Theorem of Line Integrals.

(e) Interpret the result in part (c) in terms of integration by parts.

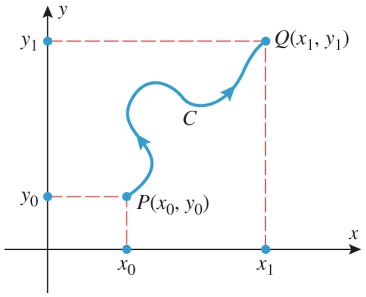


Figure Ex-28

29–30 Use Green's Theorem to find the work done by the force field $\mathbf{F}$ on a particle that moves along the stated path.

29. $\mathbf{F}(x, y) = xy\mathbf{i} + \left(\frac{1}{2}x^2 + xy\right)\mathbf{j}$; the particle starts at $(5, 0)$, traverses the upper semicircle $x^2 + y^2 = 25$, and returns to its starting point along the x-axis.

30. $\mathbf{F}(x, y) = \sqrt{y}\,\mathbf{i} + \sqrt{x}\,\mathbf{j}$; the particle moves counterclockwise one time around the closed curve given by the equations $y = 0$, $x = 2$, and $y = x^3/4$.

31. Evaluate $\oint_C y\, dx - x\, dy$, where C is the cardioid

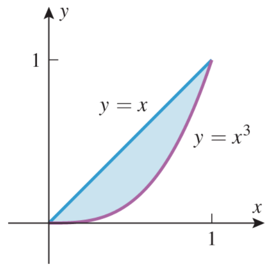
$$r = a(1 + \cos\theta) \quad (0 \le \theta \le 2\pi)$$

32. Let R be a plane region with area A whose boundary is a piecewise smooth, simple, closed curve C. Use Green's Theorem to prove that the centroid $(\bar{x}, \bar{y})$ of R is given by

$$\bar{x} = \frac{1}{2A}\oint_C x^2\, dy, \quad \bar{y} = -\frac{1}{2A}\oint_C y^2\, dx$$

33–36 Use the result in Exercise 32 to find the centroid of the region.

33.



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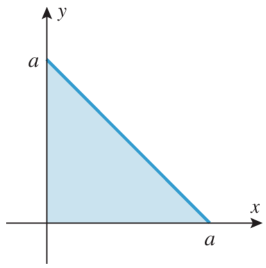

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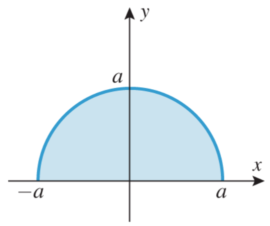

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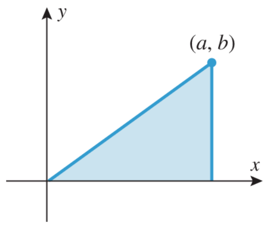


37. Find a simple closed curve C with counterclockwise orientation that maximizes the value of

$$\oint_C \tfrac{1}{3}y^3\, dx + \left(x - \tfrac{1}{3}x^3\right) dy$$

and explain your reasoning.

38. (a) Let C be the line segment from a point (a, b) to a point (c, d). Show that
$$\int_C -y\,dx + x\,dy = ad - bc$$
(b) Use the result in part (a) to show that the area A of a triangle with successive vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) going counterclockwise is
$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$$
(c) Find a formula for the area of a polygon with successive vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ going counterclockwise.
(d) Use the result in part (c) to find the area of a quadrilateral with vertices $(0, 0)$, $(3, 4)$, $(-2, 2)$, $(-1, 0)$.

39–40 Evaluate the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the boundary of the region R and C is oriented so that the region is on the left when the boundary is traversed in the direction of its orientation. ■

39. $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + (4x - \cos y)\mathbf{j}$; C is the boundary of the region R that is inside the square with vertices $(0, 0)$, $(5, 0)$, $(5, 5)$, $(0, 5)$ but is outside the rectangle with vertices $(1, 1)$, $(3, 1)$, $(3, 2)$, $(1, 2)$.

40. $\mathbf{F}(x, y) = (e^{-x} + 3y)\mathbf{i} + x\mathbf{j}$; C is the boundary of the region R inside the circle $x^2 + y^2 = 16$ and outside the circle $x^2 - 2x + y^2 = 3$.

41. Writing Discuss the role of the Fundamental Theorem of Calculus in the proof of Green's Theorem.

42. Writing Use the Internet or other sources to find information about "planimeters," and then write a paragraph that describes the relationship between these devices and Green's Theorem.

✓**QUICK CHECK ANSWERS 15.4** **1.** 8 **2.** $\frac{1}{2}$ **3.** 2π **4.** R is the region $x^2 + y^2 \le 1$ $(0 \le x, 0 \le y)$ and $f(x, y) = -y^2$, $g(x, y) = x^2$.

15.5 SURFACE INTEGRALS

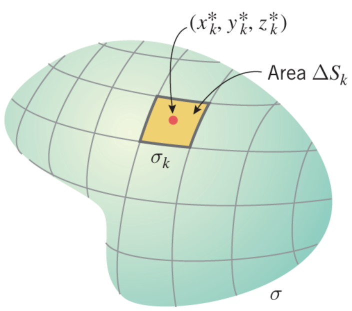
In this section we will discuss integrals over surfaces in three-dimensional space. Such integrals occur in problems involving fluid and heat flow, electricity, magnetism, mass, and center of gravity.

DEFINITION OF A SURFACE INTEGRAL

In this section we will define what it means to integrate a function $f(x, y, z)$ over a smooth parametric surface σ. To motivate the definition we will consider the problem of finding the mass of a curved lamina whose density function (mass per unit area) is known. Recall that in Section 5.7 we defined a *lamina* to be an idealized flat object that is thin enough to be viewed as a plane region. Analogously, a ***curved lamina*** is an idealized object that is thin enough to be viewed as a surface in 3-space. A curved lamina may look like a bent plate, as in Figure 15.5.1, or it may enclose a region in 3-space, like the shell of an egg. We will model the lamina by a smooth parametric surface σ. Given any point (x, y, z) on σ, we let $f(x, y, z)$ denote the corresponding value of the density function. To compute the mass of the lamina, we proceed as follows:

The thickness of a curved lamina is negligible.

▲ **Figure 15.5.1**



▲ **Figure 15.5.2**

- As shown in Figure 15.5.2, we divide σ into n very small patches $\sigma_1, \sigma_2, \ldots, \sigma_n$ with areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$, respectively. Let (x_k^*, y_k^*, z_k^*) be a sample point in the kth patch with ΔM_k the mass of the corresponding section.
- If the dimensions of σ_k are very small, the value of f will not vary much along the kth section and we can approximate f along this section by the value $f(x_k^*, y_k^*, z_k^*)$. It follows that the mass of the kth section can be approximated by
$$\Delta M_k \approx f(x_k^*, y_k^*, z_k^*)\Delta S_k$$
- The mass M of the entire lamina can then be approximated by
$$M = \sum_{k=1}^{n} \Delta M_k \approx \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{1}$$

- We will use the expression $n \to \infty$ to indicate the process of increasing n in such a way that the maximum dimension of each patch approaches 0. It is plausible that the error in (1) will approach 0 as $n \to \infty$ and the exact value of M will be given by

$$M = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{2}$$

The limit in (2) is very similar to the limit used to find the mass of a thin wire [Formula (2) in Section 15.2]. By analogy to Definition 15.2.1, we make the following definition.

15.5.1 DEFINITION If σ is a smooth parametric surface, then the ***surface integral*** of $f(x, y, z)$ over σ is

$$\iint_{\sigma} f(x, y, z)\, dS = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{3}$$

provided this limit exists and does not depend on the way the subdivisions of σ are made or how the sample points (x_k^*, y_k^*, z_k^*) are chosen.

It can be shown that the integral of f over σ exists if f is continuous on σ.

We see from (2) and Definition 15.5.1 that if σ models a lamina and if $f(x, y, z)$ is the density function of the lamina, then the mass M of the lamina is given by

$$M = \iint_{\sigma} f(x, y, z)\, dS \tag{4}$$

That is, to obtain the mass of a lamina, we integrate the density function over the smooth surface that models the lamina.

Note that if σ is a smooth surface of surface area S, and f is identically 1, then it immediately follows from Definition 15.5.1 that

$$\iint_{\sigma} dS = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta S_k = \lim_{n \to \infty} S = S \tag{5}$$

EVALUATING SURFACE INTEGRALS

There are various procedures for evaluating surface integrals that depend on how the surface σ is represented. The following theorem provides a method for evaluating a surface integral when σ is represented parametrically.

15.5.2 THEOREM *Let σ be a smooth parametric surface whose vector equation is*

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies over a region R in the uv-plane. If $f(x, y, z)$ is continuous on σ, then

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \tag{6}$$

Explain how to use Formula (6) to confirm Formula (5).

To motivate this result, suppose that the parameter domain R is subdivided as in Figure 14.4.14, and suppose that the point (x_k^*, y_k^*, z_k^*) in (3) corresponds to parameter values of u_k^* and v_k^*. If we use Formula (11) of Section 14.4 to approximate ΔS_k, and if we assume that the errors in the approximations approach zero as $n \to +\infty$, then it follows from (3) that

$$\iint_{\sigma} f(x, y, z)\, dS = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x(u_k^*, v_k^*), y(u_k^*, v_k^*), z(u_k^*, v_k^*)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A_k$$

which suggests Formula (6).

Although Theorem 15.5.2 is stated for *smooth* parametric surfaces, Formula (6) remains valid even if $\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v$ is allowed to equal $\mathbf{0}$ on the boundary of R.

▶ **Example 1** Evaluate the surface integral $\iint_{\sigma} x^2\, dS$ over the sphere $x^2 + y^2 + z^2 = 1$.

Solution. As in Example 11 of Section 14.4 (with $a = 1$), the sphere is the graph of the vector-valued function

$$\mathbf{r}(\phi, \theta) = \sin\phi \cos\theta \mathbf{i} + \sin\phi \sin\theta \mathbf{j} + \cos\phi \mathbf{k} \quad (0 \le \phi \le \pi,\ \ 0 \le \theta \le 2\pi) \tag{7}$$

and

$$\left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \sin\phi$$

Explain why the function $\mathbf{r}(\phi, \theta)$ given in (7) fails to be smooth on its domain.

From the $\mathbf{i}$-component of $\mathbf{r}$, the integrand in the surface integral can be expressed in terms of ϕ and θ as $x^2 = \sin^2\phi \cos^2\theta$. Thus, it follows from (6) with ϕ and θ in place of u and v and R as the rectangular region in the $\phi\theta$-plane determined by the inequalities in (7) that

$$\begin{aligned}
\iint_{\sigma} x^2\, dS &= \iint_{R} (\sin^2\phi \cos^2\theta) \left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| dA \\
&= \int_0^{2\pi} \int_0^{\pi} \sin^3\phi \cos^2\theta \, d\phi \, d\theta \\
&= \int_0^{2\pi} \left[\int_0^{\pi} \sin^3\phi \, d\phi \right] \cos^2\theta \, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{3}\cos^3\phi - \cos\phi \right]_0^{\pi} \cos^2\theta \, d\theta && \boxed{\text{Formula (11), Section 7.3}} \\
&= \frac{4}{3} \int_0^{2\pi} \cos^2\theta \, d\theta \\
&= \frac{4}{3} \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \frac{4\pi}{3} && \boxed{\text{Formula (8), Section 7.3}} \ ◀
\end{aligned}$$

SURFACE INTEGRALS OVER *z* = *g*(*x*, *y*), *y* = *g*(*x*, *z*), AND *x* = *g*(*y*, *z*)

In the case where σ is a surface of the form $z = g(x, y)$, we can take $x = u$ and $y = v$ as parameters and express the equation of the surface as

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + g(u, v)\mathbf{k}$$

in which case we obtain

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

(verify). Thus, it follows from (6) that

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Note that in this formula the region R lies in the xy-plane because the parameters are x and y. Geometrically, this region is the projection of σ on the xy-plane. The following theorem summarizes this result and gives analogous formulas for surface integrals over surfaces of the form $y = g(x, z)$ and $x = g(y, z)$.

15.5.3 THEOREM

(*a*) *Let σ be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \qquad (8)$$

(*b*) *Let σ be a surface with equation $y = g(x, z)$ and let R be its projection on the xz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, g(x, z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA \qquad (9)$$

(*c*) *Let σ be a surface with equation $x = g(y, z)$ and let R be its projection on the yz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(g(y, z), y, z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\, dA \qquad (10)$$

Formulas (9) and (10) can be recovered from Formula (8). Explain how.

▶ Example 2 Evaluate the surface integral

$$\iint_{\sigma} xz\, dS$$

where σ is the part of the plane $x + y + z = 1$ that lies in the first octant.

Solution. The equation of the plane can be written as

$$z = 1 - x - y$$

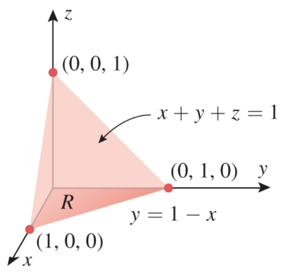
Consequently, we can apply Formula (8) with $z = g(x, y) = 1 - x - y$ and $f(x, y, z) = xz$. We have

$$\frac{\partial z}{\partial x} = -1 \quad \text{and} \quad \frac{\partial z}{\partial y} = -1$$

so (8) becomes

$$\iint_{\sigma} xz\, dS = \iint_{R} x(1 - x - y)\sqrt{(-1)^2 + (-1)^2 + 1}\, dA \qquad (11)$$

where R is the projection of σ on the xy-plane (Figure 15.5.3). Rewriting the double integral in (11) as an iterated integral yields



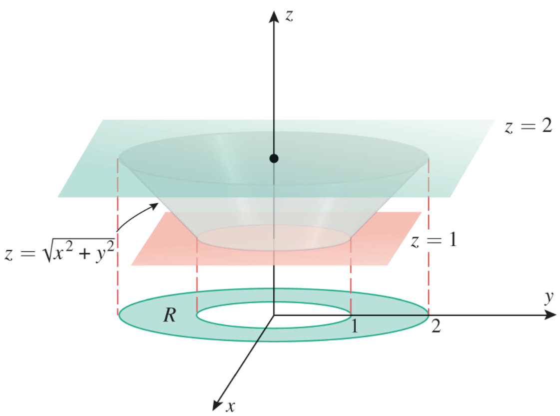
▲ **Figure 15.5.3**

$$\begin{aligned}\iint_\sigma xz\, dS &= \sqrt{3}\int_0^1 \int_0^{1-x} (x - x^2 - xy)\, dy\, dx \\ &= \sqrt{3}\int_0^1 \left[xy - x^2y - \frac{xy^2}{2}\right]_{y=0}^{1-x} dx \\ &= \sqrt{3}\int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right) dx \\ &= \sqrt{3}\left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24}\end{aligned}$$ ◀

▶ **Example 3** Evaluate the surface integral

$$\iint_\sigma y^2z^2\, dS$$

where σ is the part of the cone $z = \sqrt{x^2+y^2}$ that lies between the planes $z = 1$ and $z = 2$ (Figure 15.5.4).



▶ **Figure 15.5.4**

Solution. We will apply Formula (8) with

$$z = g(x, y) = \sqrt{x^2+y^2} \quad \text{and} \quad f(x, y, z) = y^2z^2$$

Thus,

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

so

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{2}$$

(verify), and (8) yields

$$\iint_\sigma y^2z^2\, dS = \iint_R y^2\left(\sqrt{x^2+y^2}\right)^2 \sqrt{2}\, dA = \sqrt{2}\iint_R y^2(x^2+y^2)\, dA$$

where R is the annulus enclosed between $x^2+y^2=1$ and $x^2+y^2=4$ (Figure 15.5.4). Using polar coordinates to evaluate this double integral over the annulus R yields

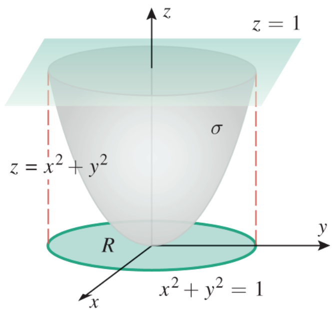
$$\begin{aligned}\iint_\sigma y^2z^2\,dS &= \sqrt{2}\int_0^{2\pi}\int_1^2 (r\sin\theta)^2(r^2)r\,dr\,d\theta\\ &= \sqrt{2}\int_0^{2\pi}\int_1^2 r^5\sin^2\theta\,dr\,d\theta\\ &= \sqrt{2}\int_0^{2\pi}\left[\frac{r^6}{6}\sin^2\theta\right]_{r=1}^2 d\theta = \frac{21}{\sqrt{2}}\int_0^{2\pi}\sin^2\theta\,d\theta\\ &= \frac{21}{\sqrt{2}}\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{21\pi}{\sqrt{2}} \quad \boxed{\text{Formula (7), Section 7.3}} \blacktriangleleft\end{aligned}$$

Evaluate the integral in Example 3 with the help of Formula (6) and the parametrization

$\mathbf{r} = \langle r\cos\theta, r\sin\theta, r\rangle$

$(1\le r\le 2, 0\le\theta\le 2\pi)$

▶ Example 4 Suppose that a curved lamina σ with constant density $\delta(x,y,z)=\delta_0$ is the portion of the paraboloid $z=x^2+y^2$ below the plane $z=1$ (Figure 15.5.5). Find the mass of the lamina.



▲ **Figure 15.5.5**

Solution. Since $z=g(x,y)=x^2+y^2$, it follows that

$$\frac{\partial z}{\partial x}=2x \quad\text{and}\quad \frac{\partial z}{\partial y}=2y$$

Therefore,

$$M=\iint_\sigma \delta_0\,dS=\iint_R \delta_0\sqrt{(2x)^2+(2y)^2+1}\,dA=\delta_0\iint_R\sqrt{4x^2+4y^2+1}\,dA \tag{12}$$

where R is the circular region enclosed by $x^2+y^2=1$. To evaluate (12) we use polar coordinates:

$$\begin{aligned}M &= \delta_0\int_0^{2\pi}\int_0^1\sqrt{4r^2+1}\,r\,dr\,d\theta = \frac{\delta_0}{12}\int_0^{2\pi}(4r^2+1)^{3/2}\bigg]_{r=0}^1 d\theta\\ &= \frac{\delta_0}{12}\int_0^{2\pi}(5^{3/2}-1)\,d\theta = \frac{\pi\delta_0}{6}(5\sqrt{5}-1) \blacktriangleleft\end{aligned}$$

✓ QUICK CHECK EXERCISES 15.5

(See page 1021 for answers.)

1. Consider the surface integral $\iint_\sigma f(x,y,z)\,dS$.
   (a) If σ is a parametric surface whose vector equation is
   $$\mathbf{r}=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}$$
   to evaluate the integral replace dS by ________.
   (b) If σ is the graph of a function $z=g(x,y)$ with continuous first partial derivatives, to evaluate the integral replace dS by ________.
2. If σ is the triangular region with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, then
   $$\iint_\sigma (x+y+z)\,dS = \text{\_\_\_\_\_\_\_\_}$$
3. If σ is the sphere of radius 2 centered at the origin, then
   $$\iint_\sigma (x^2+y^2+z^2)\,dS = \text{\_\_\_\_\_\_\_\_}$$
4. If $f(x,y,z)$ is the mass density function of a curved lamina σ, then the mass of σ is given by the integral ________.

EXERCISE SET 15.5 C CAS

1–8 Evaluate the surface integral

$$\iint_{\sigma} f(x, y, z)\, dS$$

1. $f(x, y, z) = z^2$; σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.
2. $f(x, y, z) = xy$; σ is the portion of the plane $x + y + z = 1$ lying in the first octant.
3. $f(x, y, z) = x^2y$; σ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$, $y = 1$, and above the xy-plane.
4. $f(x, y, z) = (x^2 + y^2)z$; σ is the portion of the sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$.
5. $f(x, y, z) = x - y - z$; σ is the portion of the plane $x + y = 1$ in the first octant between $z = 0$ and $z = 1$.
6. $f(x, y, z) = x + y$; σ is the portion of the plane $z = 6 - 2x - 3y$ in the first octant.
7. $f(x, y, z) = x + y + z$; σ is the surface of the cube defined by the inequalities $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. [*Hint:* Integrate over each face separately.]
8. $f(x, y, z) = x^2 + y^2$; σ is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

9–12 True–False Determine whether the statement is true or false. Explain your answer.

9. If $f(x, y, z) \ge 0$ on σ, then
$$\iint_{\sigma} f(x, y, z)\, dS \ge 0$$
10. If σ has surface area S, and if
$$\iint_{\sigma} f(x, y, z)\, dS = S$$
then $f(x, y, z)$ is equal to 1 identically on σ.
11. If σ models a curved lamina, and if $f(x, y, z)$ is the density function of the lamina, then
$$\iint_{\sigma} f(x, y, z)\, dS$$
represents the total density of the lamina.
12. If σ is the portion of a plane $z = c$ over a region R in the xy-plane, then
$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, c)\, dA$$
for every continuous function f on σ.

13–14 Sometimes evaluating a surface integral results in an improper integral. When this happens, one can either attempt to determine the value of the integral using an appropriate limit or one can try another method. These exercises explore both approaches.

13. Consider the integral of $f(x, y, z) = z + 1$ over the upper hemisphere σ: $z = \sqrt{1 - x^2 - y^2}$ $(0 \le x^2 + y^2 \le 1)$.
 (a) Explain why evaluating this surface integral using (8) results in an improper integral.
 (b) Use (8) to evaluate the integral of f over the surface σ_r : $z = \sqrt{1 - x^2 - y^2}$ $(0 \le x^2 + y^2 \le r^2 < 1)$. Take the limit of this result as $r \to 1^-$ to determine the integral of f over σ.
 (c) Parametrize σ using spherical coordinates and evaluate the integral of f over σ using (6). Verify that your answer agrees with the result in part (b).
14. Consider the integral of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ over the cone $\sigma : z = \sqrt{x^2 + y^2}$ $(0 \le z \le 1)$.
 (a) Explain why evaluating this surface integral using (8) results in an improper integral.
 (b) Use (8) to evaluate the integral of f over the surface $\sigma_r : z = \sqrt{x^2 + y^2}$ $(0 < r^2 \le x^2 + y^2 \le 1)$. Take the limit of this result as $r \to 0^+$ to determine the integral of f over σ.
 (c) Parametrize σ using spherical coordinates and evaluate the integral of f over σ using (6). Verify that your answer agrees with the result in part (b).

FOCUS ON CONCEPTS

15–18 In some cases it is possible to use Definition 15.5.1 along with symmetry considerations to evaluate a surface integral without reference to a parametrization of the surface. In these exercises, σ denotes the unit sphere centered at the origin.

15. (a) Explain why it is possible to subdivide σ into patches and choose corresponding sample points (x_k^*, y_k^*, z_k^*) such that (i) the dimensions of each patch are as small as desired and (ii) for each sample point (x_k^*, y_k^*, z_k^*), there exists a sample point (x_j^*, y_j^*, z_j^*) with
$$x_k = -x_j, \quad y_k = y_j, \quad z_k = z_j$$
and with $\Delta S_k = \Delta S_j$.
 (b) Use Definition 15.5.1, the result in part (a), and the fact that surface integrals exist for continuous functions to prove that $\iint_{\sigma} x^n\, dS = 0$ for n an odd positive integer.
16. Use the argument in Exercise 15 to prove that if $f(x)$ is a continuous odd function of x, and if $g(y, z)$ is a continuous function, then
$$\iint_{\sigma} f(x)g(y, z)\, dS = 0$$
17. (a) Explain why
$$\iint_{\sigma} x^2\, dS = \iint_{\sigma} y^2\, dS = \iint_{\sigma} z^2\, dS$$
 (b) Conclude from part (a) that
$$\iint_{\sigma} x^2\, dS = \frac{1}{3}\left[\iint_{\sigma} x^2\, dS + \iint_{\sigma} y^2\, dS + \iint_{\sigma} y^2\, dS\right]$$

(cont.)

(c) Use part (b) to evaluate

$$\iint_\sigma x^2\,dS$$

without performing an integration.

18. Use the results of Exercises 16 and 17 to evaluate

$$\iint_\sigma (x-y)^2\,dS$$

without performing an integration.

19–20 Set up, but do not evaluate, an iterated integral equal to the given surface integral by projecting σ on (a) the xy-plane, (b) the yz-plane, and (c) the xz-plane.

19. $\displaystyle\iint_\sigma xyz\,dS$, where σ is the portion of the plane $2x+3y+4z=12$ in the first octant.

20. $\displaystyle\iint_\sigma xz\,dS$, where σ is the portion of the sphere $x^2+y^2+z^2=a^2$ in the first octant.

C **21.** Use a CAS to confirm that the three integrals you obtained in Exercise 19 are equal, and find the exact value of the surface integral.

C **22.** Try to confirm with a CAS that the three integrals you obtained in Exercise 20 are equal. If you did not succeed, what was the difficulty?

23–24 Set up, but do not evaluate, two different iterated integrals equal to the given integral.

23. $\displaystyle\iint_\sigma xyz\,dS$, where σ is the portion of the surface $y^2=x$ between the planes $z=0$, $z=4$, $y=1$, and $y=2$.

24. $\displaystyle\iint_\sigma x^2y\,dS$, where σ is the portion of the cylinder $y^2+z^2=a^2$ in the first octant between the planes $x=0$, $x=9$, $z=y$, and $z=2y$.

C **25.** Use a CAS to confirm that the two integrals you obtained in Exercise 23 are equal, and find the exact value of the surface integral.

C **26.** Use a CAS to find the value of the surface integral

$$\iint_\sigma x^2yz\,dS$$

where the surface σ is the portion of the elliptic paraboloid $z=5-3x^2-2y^2$ that lies above the xy-plane.

27–28 Find the mass of the lamina with constant density δ_0.

27. The lamina that is the portion of the circular cylinder $x^2+z^2=4$ that lies directly above the rectangle $R=\{(x,y): 0\le x\le 1, 0\le y\le 4\}$ in the xy-plane.

28. The lamina that is the portion of the paraboloid $2z=x^2+y^2$ inside the cylinder $x^2+y^2=8$.

29. Find the mass of the lamina that is the portion of the surface $y^2=4-z$ between the planes $x=0$, $x=3$, $y=0$, and $y=3$ if the density is $\delta(x,y,z)=y$.

30. Find the mass of the lamina that is the portion of the cone $z=\sqrt{x^2+y^2}$ between $z=1$ and $z=4$ if the density is $\delta(x,y,z)=x^2z$.

31. If a curved lamina has constant density δ_0, what relationship must exist between its mass and surface area? Explain your reasoning.

32. Show that if the density of the lamina $x^2+y^2+z^2=a^2$ at each point is equal to the distance between that point and the xy-plane, then the mass of the lamina is $2\pi a^3$.

33–34 The centroid of a surface σ is defined by

$$\bar{x}=\frac{\displaystyle\iint_\sigma x\,dS}{\text{area of }\sigma},\quad \bar{y}=\frac{\displaystyle\iint_\sigma y\,dS}{\text{area of }\sigma},\quad \bar{z}=\frac{\displaystyle\iint_\sigma z\,dS}{\text{area of }\sigma}$$

Find the centroid of the surface.

33. The portion of the paraboloid $z=\frac{1}{2}(x^2+y^2)$ below the plane $z=4$.

34. The portion of the sphere $x^2+y^2+z^2=4$ above the plane $z=1$.

35–38 Evaluate the integral $\iint_\sigma f(x,y,z)\,dS$ over the surface σ represented by the vector-valued function $\mathbf{r}(u,v)$.

35. $f(x,y,z)=xyz$; $\mathbf{r}(u,v)=u\cos v\,\mathbf{i}+u\sin v\,\mathbf{j}+3u\mathbf{k}$
$(1\le u\le 2, 0\le v\le \pi/2)$

36. $f(x,y,z)=\dfrac{x^2+z^2}{y}$; $\mathbf{r}(u,v)=2\cos v\,\mathbf{i}+u\mathbf{j}+2\sin v\,\mathbf{k}$
$(1\le u\le 3, 0\le v\le 2\pi)$

37. $f(x,y,z)=\dfrac{1}{\sqrt{1+4x^2+4y^2}}$;
$\mathbf{r}(u,v)=u\cos v\,\mathbf{i}+u\sin v\,\mathbf{j}+u^2\mathbf{k}$
$(0\le u\le \sin v, 0\le v\le \pi)$

38. $f(x,y,z)=e^{-z}$;
$\mathbf{r}(u,v)=2\sin u\cos v\,\mathbf{i}+2\sin u\sin v\,\mathbf{j}+2\cos u\,\mathbf{k}$
$(0\le u\le \pi/2, 0\le v\le 2\pi)$

C **39.** Use a CAS to approximate the mass of the curved lamina $z=e^{-x^2-y^2}$ that lies above the region in the xy-plane enclosed by $x^2+y^2=9$ given that the density function is $\delta(x,y,z)=\sqrt{x^2+y^2}$.

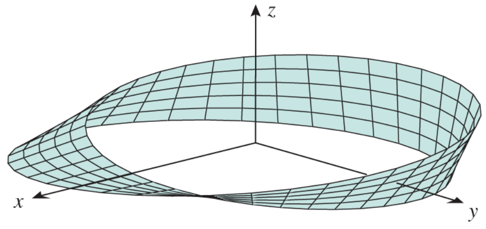
C **40.** The surface σ shown in the accompanying figure on the next page, called a ***Möbius strip***, is represented by the parametric equations

$$x=(5+u\cos(v/2))\cos v$$
$$y=(5+u\cos(v/2))\sin v$$
$$z=u\sin(v/2)$$

where $-1\le u\le 1$ and $0\le v\le 2\pi$.

(a) Use a CAS to generate a reasonable facsimile of this surface.

(b) Use a CAS to approximate the location of the centroid of σ (see the definition preceding Exercise 33).



▲ **Figure Ex-40**

41. **Writing** Discuss the similarities and differences between the definition of a surface integral and the definition of a double integral.

42. **Writing** Suppose that a surface σ in 3-space and a function $f(x, y, z)$ are described geometrically. For example, σ might be the sphere of radius 1 centered at the origin and $f(x, y, z)$ might be the distance from the point (x, y, z) to the z-axis. How would you explain to a classmate a procedure for evaluating the surface integral of f over σ?

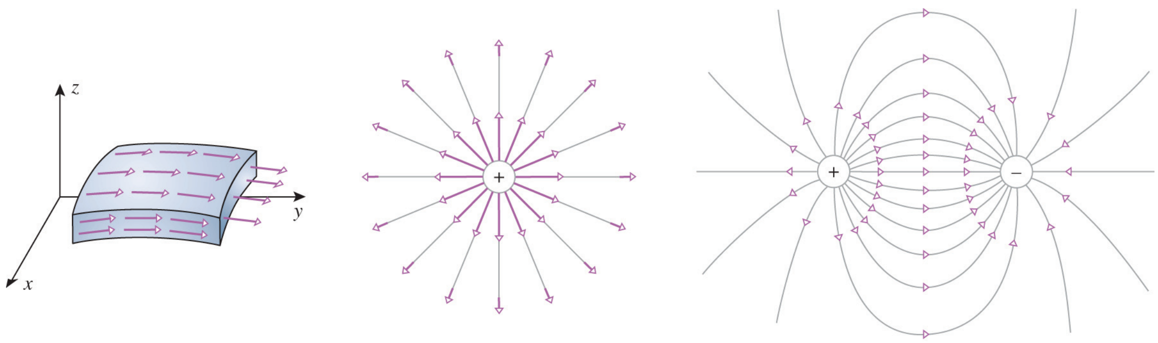
✓**QUICK CHECK ANSWERS 15.5** **1.** (a) $\left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\| dA$ (b) $\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$ **2.** $\frac{\sqrt{3}}{2}$ **3.** 64π **4.** $\iint_{\sigma} f(x, y, z)\, dS$

15.6 APPLICATIONS OF SURFACE INTEGRALS; FLUX

In this section we will discuss applications of surface integrals to vector fields associated with fluid flow and electrostatic forces. However, the ideas that we will develop will be general in nature and applicable to other kinds of vector fields as well.

FLOW FIELDS

We will be concerned in this section with vector fields in 3-space that involve some type of "flow"—the flow of a fluid or the flow of charged particles in an electrostatic field, for example. In the case of fluid flow, the vector field $\mathbf{F}(x, y, z)$ represents the velocity of a fluid particle at the point (x, y, z), and the fluid particles flow along "streamlines" that are tangential to the velocity vectors (Figure 15.6.1*a*). In the case of an electrostatic field, $\mathbf{F}(x, y, z)$ is the force that the field exerts on a small unit of positive charge at the point (x, y, z), and such charges have acceleration in the directions of "electric lines" that are tangential to the force vectors (Figures 15.6.1*b* and 15.6.1*c*).



The velocity vectors of the fluid particles are tangent to the streamlines.

By Coulomb's law the electrostatic field resulting from a single positive charge is an inverse-square field in which **F** is the repulsive force on a small unit positive charge.

The electrostatic field **F** that results from two charges of equal strength but opposite polarity.

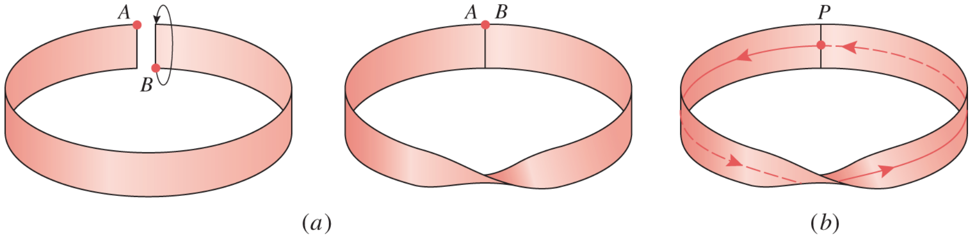
(*a*) (*b*) (*c*)

▲ **Figure 15.6.1**

ORIENTED SURFACES

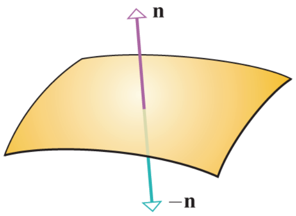
Our main goal in this section is to study flows of vector fields through permeable surfaces placed in the field. For this purpose we will need to consider some basic ideas about surfaces. Most surfaces that we encounter in applications have two sides—a sphere has an inside and an outside, and an infinite horizontal plane has a top side and a bottom side, for example. However, there exist mathematical surfaces with only one side. For example, Figure 15.6.2*a* shows the construction of a surface called a ***Möbius strip*** [in honor of the German mathematician August Möbius (1790–1868)]. The Möbius strip has only one side in the sense that a bug can traverse the *entire* surface without crossing an edge (Figure 15.6.2*b*). In contrast, a sphere is two-sided in the sense that a bug walking on the sphere can traverse the inside surface or the outside surface but cannot traverse both without somehow passing through the sphere. A two-sided surface is said to be ***orientable***, and a one-sided surface is said to be ***nonorientable***. In the rest of this text we will only be concerned with orientable surfaces.

If a bug starts at P with its back facing you and makes one circuit around the strip, then its back will face away from you when it returns to P.

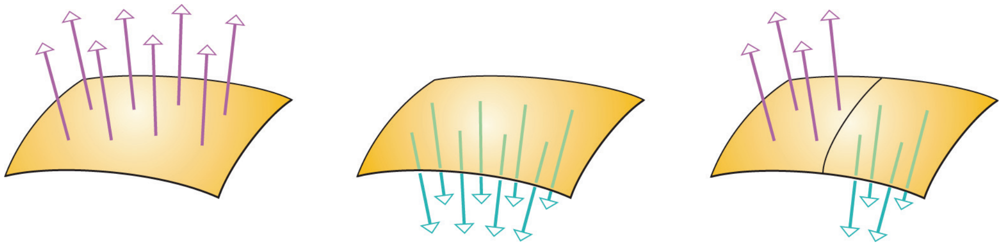


▲ **Figure 15.6.2**

In applications, it is important to have some way of distinguishing between the two sides of an orientable surface. For this purpose let us suppose that σ is an orientable surface that has a unit normal vector $\mathbf{n}$ at each point. As illustrated in Figure 15.6.3, the vectors $\mathbf{n}$ and $-\mathbf{n}$ point to opposite sides of the surface and hence serve to distinguish between the two sides. It can be proved that if σ is a smooth orientable surface, then it is always possible to choose the direction of $\mathbf{n}$ at each point so that $\mathbf{n} = \mathbf{n}(x, y, z)$ varies continuously over the surface. These unit vectors are then said to form an ***orientation*** of the surface. It can also be proved that a smooth orientable surface has only two possible orientations. For example, the surface in Figure 15.6.4 is oriented up by the purple vectors and down by the green vectors. However, we cannot create a third orientation by mixing the two since this produces points on the surface at which there is an abrupt change in direction (across the black curve in the figure, for example).



▲ **Figure 15.6.3**

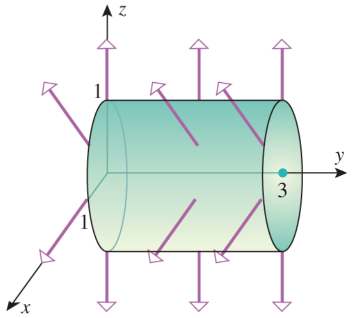


▲ **Figure 15.6.4**

ORIENTATION OF A SMOOTH PARAMETRIC SURFACE

When a surface is expressed parametrically, the parametric equations create a natural orientation of the surface. To see why this is so, recall from Section 14.4 that if a smooth parametric surface σ is given by the vector equation

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$



▲ **Figure 15.6.5**

See if you can find a parametrization of the cylinder in which the positive direction is inward.

then the unit normal

$$\mathbf{n} = \mathbf{n}(u, v) = \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|} \tag{1}$$

is a continuous vector-valued function of u and v. Thus, Formula (1) defines an orientation of the surface; we call this the ***positive orientation*** of the parametric surface and we say that $\mathbf{n}$ points in the ***positive direction*** from the surface. The orientation determined by $-\mathbf{n}$ is called the ***negative orientation*** of the surface and we say that $-\mathbf{n}$ points in the ***negative direction*** from the surface. For example, consider the cylinder that is represented parametrically by the vector equation

$$\mathbf{r}(u, v) = \cos u\mathbf{i} + v\mathbf{j} - \sin u\mathbf{k} \qquad (0 \le u \le 2\pi,\ \ 0 \le v \le 3)$$

Then

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \cos u\mathbf{i} - \sin u\mathbf{k}$$

has unit length, so that Formula (1) becomes

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \cos u\mathbf{i} - \sin u\mathbf{k}$$

Since $\mathbf{n}$ has the same $\mathbf{i}$- and $\mathbf{k}$-components as $\mathbf{r}$, the positive orientation of the cylinder is *outward* and the negative orientation is *inward* (Figure 15.6.5).

FLUX

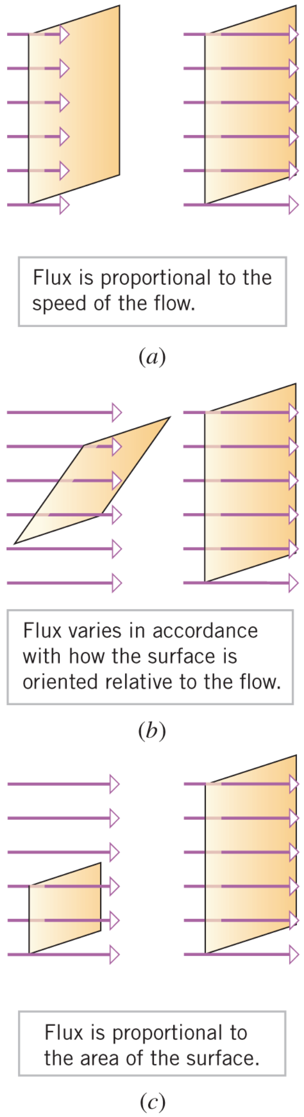
In physics, the term *fluid* is used to describe both liquids and gases. Liquids are usually regarded to be ***incompressible***, meaning that the liquid has a uniform density (mass per unit volume) that cannot be altered by compressive forces. Gases are regarded to be ***compressible***, meaning that the density may vary from point to point and can be altered by compressive forces. In this text we will be concerned primarily with incompressible fluids. Moreover, we will assume that the velocity of the fluid at a fixed point does not vary with time. Fluid flows with this property are said to be in a ***steady state***.

Our next goal in this section is to define a fundamental concept of physics known as *flux* (from the Latin word *fluxus*, meaning "flowing"). This concept is applicable to any vector field, but we will motivate it in the context of steady-state incompressible fluid flow. Imagine that fluid is flowing freely through a permeable surface, from one side of the surface to the other. In this context, you can think of flux as:

The volume of fluid that passes through the surface in one unit of time.

This idea is illustrated in Figure 15.6.6, which suggests that the volume of fluid that flows through a portion of the surface depends on three factors:

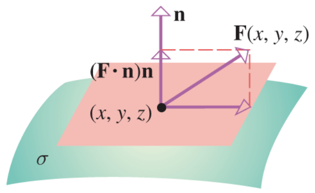
- The speed of the fluid; the greater the speed, the greater the volume (Figure 15.6.6*a*).
- How the surface is oriented relative to the flow; the more nearly orthogonal the flow is to the surface, the greater the volume (Figure 15.6.6*b*).
- The area of the portion of the surface; the greater the area, the greater the volume (Figure 15.6.6*c*).



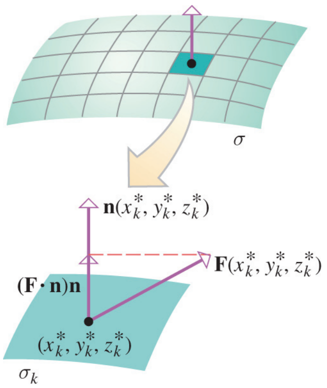
▲ **Figure 15.6.6**

These ideas lead us to the following problem.

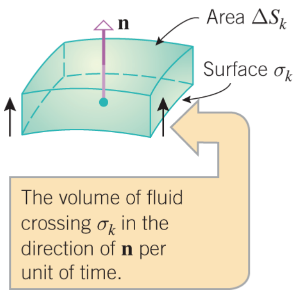
15.6.1 PROBLEM Suppose that an oriented surface σ is immersed in an incompressible, steady-state fluid flow and that the surface is permeable so that the fluid can flow through it freely in either direction. Find the net volume of fluid Φ that passes through the surface per unit of time, where the net volume is interpreted to mean the volume that passes through the surface in the positive direction minus the volume that passes through the surface in the negative direction.



▲ Figure 15.6.7



▲ Figure 15.6.8



▲ Figure 15.6.9

To solve this problem, suppose that the velocity of the fluid at a point (x, y, z) on the surface σ is given by

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

Let **n** be the unit normal toward the positive side of σ at the point (x, y, z). As illustrated in Figure 15.6.7, the velocity vector **F** can be resolved into two orthogonal components—a component $(\mathbf{F} \cdot \mathbf{n})\mathbf{n}$ that is perpendicular to the surface σ and a second component that is along the "face" of σ. The component of velocity along the face of the surface does not contribute to the flow through σ and hence can be ignored in our computations. Moreover, observe that the sign of $\mathbf{F} \cdot \mathbf{n}$ determines the direction of flow—a positive value means the flow is in the direction of **n** and a negative value means that it is opposite to **n**.

To solve Problem 15.6.1, we subdivide σ into n patches $\sigma_1, \sigma_2, \ldots, \sigma_n$ with areas

$$\Delta S_1, \Delta S_2, \ldots, \Delta S_n$$

If the patches are small and the flow is not too erratic, it is reasonable to assume that the velocity does not vary much on each patch. Thus, if (x_k^*, y_k^*, z_k^*) is any point in the kth patch, we can assume that $\mathbf{F}(x, y, z)$ is constant and equal to $\mathbf{F}(x_k^*, y_k^*, z_k^*)$ throughout the patch and that the component of velocity across the surface σ_k is

$$\mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n}(x_k^*, y_k^*, z_k^*) \tag{2}$$

(Figure 15.6.8). Thus, we can interpret

$$\mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k$$

as the approximate volume of fluid crossing the patch σ_k in the direction of **n** per unit of time (Figure 15.6.9). For example, if the component of velocity in the direction of **n** is $\mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n} = 25$ cm/s, and the area of the patch is $\Delta S_k = 2$ cm^2, then the volume of fluid ΔV_k crossing the patch in the direction of **n** per unit of time is approximately

$$\Delta V_k \approx \mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k = 25 \text{ cm/s} \cdot 2 \text{ cm}^2 = 50 \text{ cm}^3/\text{s}$$

In the case where the velocity component $\mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n}(x_k^*, y_k^*, z_k^*)$ is negative, the flow is in the direction opposite to **n**, so that $-\mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k$ is the approximate volume of fluid crossing the patch σ_k in the direction opposite to **n** per unit time. Thus, the sum

$$\sum_{k=1}^{n} \mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k$$

measures the approximate net volume of fluid that crosses the surface σ in the direction of its orientation **n** per unit of time.

If we now increase n in such a way that the maximum dimension of each patch approaches zero, then it is plausible that the errors in the approximations approach zero, and the limit

$$\Phi = \lim_{n \to +\infty} \sum_{k=1}^{n} \mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{3}$$

represents the exact net volume of fluid that crosses the surface σ in the direction of its orientation **n** per unit of time. The quantity Φ defined by Equation (3) is called the ***flux of F across σ***. The flux can also be expressed as the surface integral

$$\Phi = \iint_{\sigma} \mathbf{F}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS \tag{4}$$

If the fluid has mass density δ, then $\Phi\delta$ (volume/time × density) represents the net mass of fluid that passes through σ per unit of time.

A positive flux means that in one unit of time a greater volume of fluid passes through σ in the positive direction than in the negative direction, a negative flux means that a greater volume passes through the surface in the negative direction than in the positive direction, and a zero flux means that the same volume passes through the surface in each direction. Integrals of form (4) arise in other contexts as well and are called ***flux integrals***.

EVALUATING FLUX INTEGRALS

An effective formula for evaluating flux integrals can be obtained by applying Theorem 15.5.2 and using Formula (1) for $\mathbf{n}$. This yields

$$\iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{R} \mathbf{F}\cdot\mathbf{n}\left\|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right\| dA$$

$$= \iint_{R} \mathbf{F}\cdot\frac{\dfrac{\partial\mathbf{r}}{\partial u}\times\dfrac{\partial\mathbf{r}}{\partial v}}{\left\|\dfrac{\partial\mathbf{r}}{\partial u}\times\dfrac{\partial\mathbf{r}}{\partial v}\right\|}\left\|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right\| dA$$

$$= \iint_{R} \mathbf{F}\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right) dA$$

In summary, we have the following result.

15.6.2 THEOREM *Let σ be a smooth parametric surface represented by the vector equation $\mathbf{r}=\mathbf{r}(u,v)$ in which (u,v) varies over a region R in the uv-plane. If the component functions of the vector field $\mathbf{F}$ are continuous on σ, and if $\mathbf{n}$ determines the positive orientation of σ, then*

$$\Phi = \iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{R} \mathbf{F}\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right) dA \tag{5}$$

where it is understood that the integrand on the right side of the equation is expressed in terms of u and v.

Although Theorem 15.6.2 was derived for smooth parametric surfaces, Formula (5) is valid more generally. For example, as long as σ has a continuous normal vector field $\mathbf{n}$ and the component functions of $\mathbf{r}(u,v)$ have continuous first partial derivatives, Formula (5) can be applied whenever $\partial\mathbf{r}/\partial u\times\partial\mathbf{r}/\partial v$ is a positive multiple of $\mathbf{n}$ in the *interior* of R. (That is, $\partial\mathbf{r}/\partial u\times\partial\mathbf{r}/\partial v$ is allowed to equal $\mathbf{0}$ on the boundary of R.)

▶ Example 1 Find the flux of the vector field $\mathbf{F}(x,y,z)=z\mathbf{k}$ across the outward-oriented sphere $x^2+y^2+z^2=a^2$.

Solution. The sphere with outward positive orientation can be represented by the vector-valued function

$$\mathbf{r}(\phi,\theta)=a\sin\phi\cos\theta\mathbf{i}+a\sin\phi\sin\theta\mathbf{j}+a\cos\phi\mathbf{k}\qquad(0\le\phi\le\pi,\ \ 0\le\theta\le 2\pi)$$

From this formula we obtain (see Example 11 of Section 14.4 for the computations)

$$\frac{\partial\mathbf{r}}{\partial\phi}\times\frac{\partial\mathbf{r}}{\partial\theta}=a^2\sin^2\phi\cos\theta\mathbf{i}+a^2\sin^2\phi\sin\theta\mathbf{j}+a^2\sin\phi\cos\phi\mathbf{k}$$

Moreover, for points on the sphere we have $\mathbf{F}=z\mathbf{k}=a\cos\phi\mathbf{k}$; hence,

$$\mathbf{F}\cdot\left(\frac{\partial\mathbf{r}}{\partial\phi}\times\frac{\partial\mathbf{r}}{\partial\theta}\right)=a^3\sin\phi\cos^2\phi$$

Solve Example 1 using symmetry: First argue that the vector fields $x\mathbf{i}$, $y\mathbf{j}$, and $z\mathbf{k}$ will have the same flux across the sphere. Then define

$$\mathbf{H} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and explain why

$$\mathbf{H} \cdot \mathbf{n} = a$$

Use this to compute Φ.

Thus, it follows from (5) with the parameters u and v replaced by ϕ and θ that

$$\begin{aligned}
\Phi &= \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS \\
&= \iint_{R} \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\right) dA \\
&= \int_0^{2\pi} \int_0^{\pi} a^3 \sin\phi \cos^2\phi \, d\phi \, d\theta \\
&= a^3 \int_0^{2\pi} \left[-\frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta \\
&= \frac{2a^3}{3} \int_0^{2\pi} d\theta = \frac{4\pi a^3}{3} \blacktriangleleft
\end{aligned}$$

REMARK Reversing the orientation of the surface σ in (5) reverses the sign of $\mathbf{n}$, hence the sign of $\mathbf{F} \cdot \mathbf{n}$, and hence reverses the sign of Φ. This can also be seen physically by interpreting the flux integral as the volume of fluid per unit time that crosses σ in the positive direction minus the volume per unit time that crosses in the negative direction—reversing the orientation of σ changes the sign of the difference. Thus, in Example 1 an inward orientation of the sphere would produce a flux of $-4\pi a^3/3$.

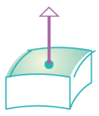
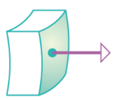
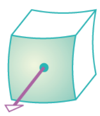
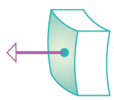
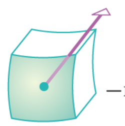
ORIENTATION OF NONPARAMETRIC SURFACES

Nonparametric surfaces of the form $z = g(x, y)$, $y = g(z, x)$, and $x = g(y, z)$ can be expressed parametrically using the independent variables as parameters. More precisely, these surfaces can be represented by the vector equations

$$\underbrace{\mathbf{r} = u\mathbf{i} + v\mathbf{j} + g(u, v)\mathbf{k}}_{\boxed{z = g(x, y)}}, \quad \underbrace{\mathbf{r} = v\mathbf{i} + g(u, v)\mathbf{j} + u\mathbf{k}}_{\boxed{y = g(z, x)}}, \quad \underbrace{\mathbf{r} = g(u, v)\mathbf{i} + u\mathbf{j} + v\mathbf{k}}_{\boxed{x = g(y, z)}} \qquad (6\text{–}8)$$

These representations impose positive and negative orientations on the surfaces in accordance with Formula (1). We leave it as an exercise to calculate $\mathbf{n}$ and $-\mathbf{n}$ in each case and to show that the positive and negative orientations are as shown in Table 15.6.1. (To assist with perspective, each graph is pictured as a portion of the surface of a small solid region.)

Table 15.6.1

$z = g(x, y)$	$y = g(z, x)$	$x = g(y, z)$
 $\mathbf{n} = \dfrac{-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$ Positive **k**-component; Positive orientation	 $\mathbf{n} = \dfrac{-\frac{\partial y}{\partial x}\mathbf{i} + \mathbf{j} - \frac{\partial y}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}}$ Positive **j**-component; Positive orientation	 $\mathbf{n} = \dfrac{\mathbf{i} - \frac{\partial x}{\partial y}\mathbf{j} - \frac{\partial x}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}}$ Positive **i**-component; Positive orientation
 $-\mathbf{n} = \dfrac{\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$ Negative **k**-component; Negative orientation	 $-\mathbf{n} = \dfrac{\frac{\partial y}{\partial x}\mathbf{i} - \mathbf{j} + \frac{\partial y}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}}$ Negative **j**-component; Negative orientation	 $-\mathbf{n} = \dfrac{-\mathbf{i} + \frac{\partial x}{\partial y}\mathbf{j} + \frac{\partial x}{\partial z}\mathbf{k}}{\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}}$ Negative **i**-component; Negative orientation

The dependent variable will increase as you move away from a surface

$$z = g(x, y), \quad y = g(z, x)$$

or

$$x = g(y, z)$$

in the direction of positive orientation.

The results in Table 15.6.1 can also be obtained using gradients. To see how this can be done, rewrite the equations of the surfaces as

$$z - g(x, y) = 0, \quad y - g(z, x) = 0, \quad x - g(y, z) = 0$$

Each of these equations has the form $G(x, y, z) = 0$ and hence can be viewed as a level surface of a function $G(x, y, z)$. Since the gradient of G is normal to the level surface, it follows that the unit normal $\mathbf{n}$ is either $\nabla G/\|\nabla G\|$ or $-\nabla G/\|\nabla G\|$. However,if $G(x, y, z) = z - g(x, y)$, then ∇G has a $\mathbf{k}$-component of 1; if $G(x, y, z) = y - g(z, x)$, then ∇G has a $\mathbf{j}$-component of 1; and if $G(x, y, z) = x - g(y, z)$, then ∇G has an $\mathbf{i}$-component of 1. Thus, it is evident from Table 15.6.1 that in all three cases we have

$$\mathbf{n} = \frac{\nabla G}{\|\nabla G\|} \tag{9}$$

Moreover, we leave it as an exercise to show that if the surfaces $z = g(x, y)$, $y = g(z, x)$, and $x = g(y, z)$ are expressed in vector forms (6), (7), and (8), then

$$\nabla G = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \tag{10}$$

[compare (1) and (9)]. Thus, we are led to the following version of Theorem 15.6.2 for non-parametric surfaces.

15.6.3 THEOREM *Let σ be a smooth surface of the form $z = g(x, y)$, $y = g(z, x)$, or $x = g(y, z)$, and suppose that the component functions of the vector field $\mathbf{F}$ are continuous on σ. Suppose also that the equation for σ is rewritten as $G(x, y, z) = 0$ by taking g to the left side of the equation, and let R be the projection of σ on the coordinate plane determined by the independent variables of g. If σ has positive orientation, then*

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{R} \mathbf{F} \cdot \nabla G\, dA \tag{11}$$

Formula (11) can either be used directly for computations or to derive some more specific formulas for each of the three surface types. For example, if $z = g(x, y)$, then we have $G(x, y, z) = z - g(x, y)$, so

$$\nabla G = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} = -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}$$

Substituting this expression for ∇G in (11) and taking R to be the projection of the surface $z = g(x, y)$ on the xy-plane yields

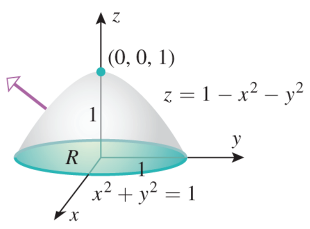
$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{R} \mathbf{F} \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA \tag{12}$$

σ of the form $z = g(x, y)$ and oriented up

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{R} \mathbf{F} \cdot \left(\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}\right) dA \tag{13}$$

σ of the form $z = g(x, y)$ and oriented down

The derivations of the corresponding formulas when $y = g(z, x)$ and $x = g(y, z)$ are left as exercises.



▲ **Figure 15.6.10**

▶ **Example 2** Let σ be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane, and suppose that σ is oriented up, as shown in Figure 15.6.10. Find the flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across σ.

Solution. From (12) the flux Φ is given by

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{R} \mathbf{F} \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \iint_{R} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\, dA$$

$$= \iint_{R} (x^2 + y^2 + 1)\, dA \qquad \boxed{\text{Since } z = 1 - x^2 - y^2 \text{ on the surface}}$$

$$= \int_0^{2\pi} \int_0^1 (r^2 + 1) r\, dr\, d\theta \qquad \boxed{\text{Using polar coordinates to evaluate the integral}}$$

$$= \int_0^{2\pi} \left(\frac{3}{4}\right) d\theta = \frac{3\pi}{2} \blacktriangleleft$$

✓ QUICK CHECK EXERCISES 15.6 *(See page 1030 for answers.)*

In these exercises, $\mathbf{F}(x, y, z)$ denotes a vector field defined on a surface σ oriented by a unit normal vector field $\mathbf{n}(x, y, z)$, and Φ denotes the flux of $\mathbf{F}$ across σ.

1. (a) Φ is the value of the surface integral ________.

(b) If σ is the unit sphere and $\mathbf{n}$ is the outward unit normal, then the flux of

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

across σ is $\Phi =$ ________.

2. (a) Assume that σ is parametrized by a vector-valued function $\mathbf{r}(u, v)$ whose domain is a region R in the uv-plane and that $\mathbf{n}$ is a positive multiple of

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

Then the double integral over R whose value is Φ is ________.

(b) Suppose that σ is the parametric surface

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u + v)\mathbf{k} \qquad (0 \le u^2 + v^2 \le 1)$$

and that $\mathbf{n}$ is a positive multiple of

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

Then the flux of $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across σ is $\Phi =$ ________.

3. (a) Assume that σ is the graph of a function $z = g(x, y)$ over a region R in the xy-plane and that $\mathbf{n}$ has a positive $\mathbf{k}$-component for every point on σ. Then a double integral over R whose value is Φ is ________.

(b) Suppose that σ is the triangular region with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ with upward orientation. Then the flux of

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

across σ is $\Phi =$ ________.

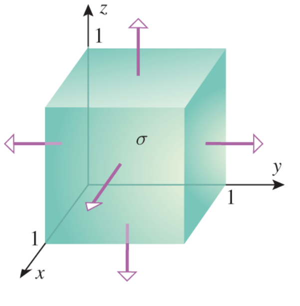
4. In the case of steady-state incompressible fluid flow, with $\mathbf{F}(x, y, z)$ the fluid velocity at (x, y, z) on σ, Φ can be interpreted as ________.

EXERCISE SET 15.6 [C] CAS

FOCUS ON CONCEPTS

1. Suppose that the surface σ of the unit cube in the accompanying figure has an outward orientation. In each part, determine whether the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{j}$ across the specified face is positive, negative, or zero.

(a) The face $x = 1$ (b) The face $x = 0$
(c) The face $y = 1$ (d) The face $y = 0$
(e) The face $z = 1$ (f) The face $z = 0$



◀ Figure Ex-1

2. Find the flux of the constant vector field $\mathbf{F}(x, y, z) = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ across the entire surface σ in Figure Ex-1. Explain your reasoning.
3. Find the flux of $\mathbf{F}(x, y, z) = x\mathbf{i}$ through a square of side 4 in the plane $x = -5$ oriented in the positive x-direction.
4. Find the flux of $\mathbf{F}(x, y, z) = (y+1)\mathbf{j}$ through a square of side 5 in the xz-plane oriented in the negative y-direction.
5. Find the flux of $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + (z^2+4)\mathbf{k}$ through a 2×3 rectangle in the plane $z = 1$ oriented in the positive z-direction.
6. Find the flux of $\mathbf{F}(x, y, z) = 2\mathbf{i} + 3\mathbf{j}$ through a disk of radius 5 in the plane $y = 3$ oriented in the direction of increasing y.
7. Find the flux of $\mathbf{F}(x, y, z) = 9\mathbf{j} + 8\mathbf{k}$ through a disk of radius 5 in the plane $z = 2$ oriented in the upward direction.
8. Let σ be the cylindrical surface that is represented by the vector-valued function $\mathbf{r}(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k}$ with $0 \le u \le 1$ and $0 \le v \le 2\pi$.
   (a) Find the unit normal $\mathbf{n} = \mathbf{n}(u, v)$ that defines the positive orientation of σ.
   (b) Is the positive orientation inward or outward? Justify your answer.

9–16 Find the flux of the vector field $\mathbf{F}$ across σ.

9. $\mathbf{F}(x, y, z) = x\mathbf{k}$; σ is the square $0 \le x \le 2, 0 \le y \le 2$ in the xy-plane, oriented in the upward direction.
10. $\mathbf{F}(x, y, z) = 5z\mathbf{i} + y\mathbf{j} + 2x\mathbf{k}$; σ is the rectangle $0 \le x \le 2$, $0 \le y \le 3$ in the plane $z = 2$, oriented in the positive z-direction.
11. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$; σ is the portion of the surface $z = 1 - x^2 - y^2$ above the xy-plane, oriented by upward normals.
12. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + (x + e^y)\mathbf{j} - \mathbf{k}$; σ is the vertical rectangle $0 \le x \le 2, 0 \le z \le 4$ in the plane $y = -1$, oriented in the negative y-direction.
13. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$; σ is the portion of the cone $z^2 = x^2 + y^2$ between the planes $z = 1$ and $z = 2$, oriented by upward unit normals.
14. $\mathbf{F}(x, y, z) = y\mathbf{j} + \mathbf{k}$; σ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$, oriented by downward unit normals.
15. $\mathbf{F}(x, y, z) = x\mathbf{k}$; the surface σ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = y$, oriented by downward unit normals.
16. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + yx\mathbf{j} + zx\mathbf{k}$; σ is the portion of the plane $6x + 3y + 2z = 6$ in the first octant, oriented by unit normals with positive components.

17–20 Find the flux of the vector field $\mathbf{F}$ across σ in the direction of positive orientation.

17. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$; σ is the portion of the paraboloid
$$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + (1 - u^2)\mathbf{k}$$
with $1 \le u \le 2, 0 \le v \le 2\pi$.
18. $\mathbf{F}(x, y, z) = e^{-y}\mathbf{i} - y\mathbf{j} + x\sin z\mathbf{k}$; σ is the portion of the elliptic cylinder
$$\mathbf{r}(u, v) = 2\cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k}$$
with $0 \le u \le 5, 0 \le v \le 2\pi$.
19. $\mathbf{F}(x, y, z) = \sqrt{x^2+y^2}\,\mathbf{k}$; σ is the portion of the cone
$$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + 2u\mathbf{k}$$
with $0 \le u \le \sin v, 0 \le v \le \pi$.
20. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the portion of the sphere
$$\mathbf{r}(u, v) = 2\sin u\cos v\mathbf{i} + 2\sin u\sin v\mathbf{j} + 2\cos u\mathbf{k}$$
with $0 \le u \le \pi/3, 0 \le v \le 2\pi$.
21. Let σ be the surface of the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$, oriented by outward unit normals. In each part, find the flux of $\mathbf{F}$ across σ.
   (a) $\mathbf{F}(x, y, z) = x\mathbf{i}$
   (b) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
   (c) $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
22. Let σ be the closed surface consisting of the portion of the paraboloid $z = x^2 + y^2$ for which $0 \le z \le 1$ and capped by the disk $x^2 + y^2 \le 1$ in the plane $z = 1$. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{j} - y\mathbf{k}$ in the outward direction across σ.

23–26 True–False Determine whether the statement is true or false. Explain your answer.

23. The Möbius strip is a surface that has two orientations.
24. The flux of a vector field is another vector field.
25. If the net volume of fluid that passes through a surface per unit time in the positive direction is zero, then the velocity of the fluid is everywhere tangent to the surface.
26. If a surface σ is oriented by a unit normal vector field $\mathbf{n}$, the flux of $\mathbf{n}$ across σ is numerically equal to the surface area of σ.

27–28 Find the flux of $\mathbf{F}$ across the surface σ by expressing σ parametrically.

27. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$; the surface σ is the portion of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$, oriented by downward unit normals.
28. $\mathbf{F}(x, y, z) = 3\mathbf{i} - 7\mathbf{j} + z\mathbf{k}$; σ is the portion of the cylinder $x^2 + y^2 = 16$ between the planes $z = -2$ and $z = 2$, oriented by outward unit normals.
29. Let x, y, and z be measured in meters, and suppose that $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 3y\mathbf{j} + z\mathbf{k}$ is the velocity vector (in m/s) of a fluid particle at the point (x, y, z) in a steady-state incompressible fluid flow.
   (a) Find the net volume of fluid that passes in the upward direction through the portion of the plane $x + y + z = 1$ in the first octant in 1 s.
   (b) Assuming that the fluid has a mass density of 806 kg/m^3, find the net mass of fluid that passes in the upward direction through the surface in part (a) in 1 s.
30. Let x, y, and z be measured in meters, and suppose that $\mathbf{F}(x, y, z) = -y\mathbf{i} + z\mathbf{j} + 3x\mathbf{k}$ is the velocity vector (in m/s) of a fluid particle at the point (x, y, z) in a steady-state incompressible fluid flow.

(a) Find the net volume of fluid that passes in the upward direction through the hemisphere $z = \sqrt{9 - x^2 - y^2}$ in 1 s.
(b) Assuming that the fluid has a mass density of 1060 kg/m^3, find the net mass of fluid that passes in the upward direction through the surface in part (a) in 1 s.

31. (a) Derive the analogs of Formulas (12) and (13) for surfaces of the form $x = g(y, z)$.
(b) Let σ be the portion of the paraboloid $x = y^2 + z^2$ for $x \leq 1$ and $z \geq 0$ oriented by unit normals with negative x-components. Use the result in part (a) to find the flux of
$$\mathbf{F}(x, y, z) = y\mathbf{i} - z\mathbf{j} + 8\mathbf{k}$$
across σ.

32. (a) Derive the analogs of Formulas (12) and (13) for surfaces of the form $y = g(z, x)$.
(b) Let σ be the portion of the paraboloid $y = z^2 + x^2$ for $y \leq 1$ and $z \geq 0$ oriented by unit normals with positive y-components. Use the result in part (a) to find the flux of
$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
across σ.

33. Let $\mathbf{F} = \|\mathbf{r}\|^k\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and k is a constant. (Note that if $k = -3$, this is an inverse-square field.) Let σ be the sphere of radius a centered at the origin and oriented by the outward normal $\mathbf{n} = \mathbf{r}/\|\mathbf{r}\| = \mathbf{r}/a$.
(a) Find the flux of $\mathbf{F}$ across σ without performing any integrations. [*Hint:* The surface area of a sphere of radius a is $4\pi a^2$.]
(b) For what value of k is the flux independent of the radius of the sphere?

C **34.** Let
$$\mathbf{F}(x, y, z) = a^2x\mathbf{i} + (y/a)\mathbf{j} + az^2\mathbf{k}$$
and let σ be the sphere of radius 1 centered at the origin and oriented outward. Use a CAS to find all values of a such that the flux of $\mathbf{F}$ across σ is 3π.

C **35.** Let
$$\mathbf{F}(x, y, z) = \left(\frac{6}{a} + 1\right)x\mathbf{i} - 4ay\mathbf{j} + a^2z\mathbf{k}$$
and let σ be the sphere of radius a centered at the origin and oriented outward. Use a CAS to find all values of a such that the flux of $\mathbf{F}$ across σ is zero.

36. **Writing** Discuss the similarities and differences between the flux of a vector field across a surface and the line integral of a vector field along a curve.

37. **Writing** Write a paragraph explaining the concept of flux to someone unfamiliar with its meaning.

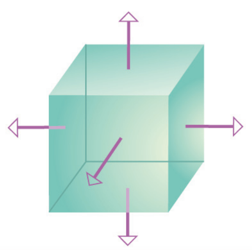
✓ QUICK CHECK ANSWERS 15.6 **1.** (a) $\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS$ (b) 4π **2.** (a) $\iint_R \mathbf{F}\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)dA$ (b) 0 **3.** (a) $\iint_R \mathbf{F}\cdot\left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right)dA$ (b) $\frac{1}{2}$ **4.** the net volume of fluid crossing σ in the positive direction per unit time

15.7 THE DIVERGENCE THEOREM

In this section we will be concerned with flux across surfaces, such as spheres, that "enclose" a region of space. We will show that the flux across such surfaces can be expressed in terms of the divergence of the vector field, and we will use this result to give a physical interpretation of the concept of divergence.

ORIENTATION OF PIECEWISE SMOOTH CLOSED SURFACES

In the last section we studied flux across general surfaces. Here we will be concerned exclusively with surfaces that are boundaries of finite solids—the surface of a solid sphere, the surface of a solid box, or the surface of a solid cylinder, for example. Such surfaces are said to be ***closed***. A closed surface may or may not be smooth, but most of the surfaces that arise in applications are ***piecewise smooth***; that is, they consist of finitely many smooth surfaces joined together at the edges (a box, for example). We will limit our discussion to piecewise smooth surfaces that can be assigned an ***inward orientation*** (toward the interior of the solid) and an ***outward orientation*** (away from the interior). It is difficult to make this concept mathematically precise, but the basic idea is that each piece of the surface is orientable, and oriented pieces fit together in such a way that the entire surface can be assigned an orientation (Figure 15.7.1).



Box with outward orientation

▲ **Figure 15.7.1**

THE DIVERGENCE THEOREM

In Section 15.1 we defined the divergence of a vector field
$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

as

$$\operatorname{div} \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

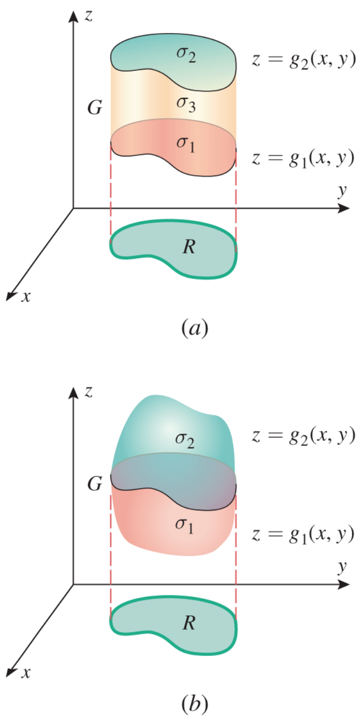
but we did not attempt to give a physical explanation of its meaning at that time. The following result, known as the ***Divergence Theorem*** or ***Gauss's Theorem***, will provide us with a physical interpretation of divergence in the context of fluid flow.

15.7.1 THEOREM (*The Divergence Theorem*) *Let G be a solid whose surface σ is oriented outward. If*

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

where f, g, and h have continuous first partial derivatives on some open set containing G, and if $\mathbf{n}$ is the outward unit normal on σ, then

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_G \operatorname{div} \mathbf{F}\, dV \tag{1}$$



▲ **Figure 15.7.2**

We will give a proof for the special case where G is simultaneously a simple xy-solid, a simple yz-solid, and a simple zx-solid (see Figure 14.5.3 and the related discussion for terminology).

PROOF Formula (1) can be expressed as

$$\iint_\sigma [f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}]\cdot\mathbf{n}\, dS = \iiint_G \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dV$$

so it suffices to prove the three equalities

$$\iint_\sigma f(x, y, z)\mathbf{i}\cdot\mathbf{n}\, dS = \iiint_G \frac{\partial f}{\partial x}\, dV \tag{2a}$$

$$\iint_\sigma g(x, y, z)\mathbf{j}\cdot\mathbf{n}\, dS = \iiint_G \frac{\partial g}{\partial y}\, dV \tag{2b}$$

$$\iint_\sigma h(x, y, z)\mathbf{k}\cdot\mathbf{n}\, dS = \iiint_G \frac{\partial h}{\partial z}\, dV \tag{2c}$$

Since the proofs of all three equalities are similar, we will prove only the third.

Suppose that G has upper surface $z = g_2(x, y)$, lower surface $z = g_1(x, y)$, and projection R on the xy-plane. Let σ_1 denote the lower surface, σ_2 the upper surface, and σ_3 the lateral surface (Figure 15.7.2*a*). If the upper surface and lower surface meet as in Figure 15.7.2*b*, then there is no lateral surface σ_3. Our proof will allow for both cases shown in those figures.

It follows from Theorem 14.5.2 that

$$\iiint_G \frac{\partial h}{\partial z}\, dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} \frac{\partial h}{\partial z}\, dz\right] dA = \iint_R \Big[h(x, y, z)\Big]_{z=g_1(x,y)}^{g_2(x,y)} dA$$

so

$$\iiint_G \frac{\partial h}{\partial z}\, dV = \iint_R [h(x, y, g_2(x, y)) - h(x, y, g_1(x, y))]\, dA \tag{3}$$

Next we will evaluate the surface integral in (2c) by integrating over each surface of G separately. If there is a lateral surface σ_3, then at each point of this surface $\mathbf{k} \cdot \mathbf{n} = 0$ since $\mathbf{n}$ is horizontal and $\mathbf{k}$ is vertical. Thus,

$$\iint_{\sigma_3} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = 0$$

Therefore, regardless of whether G has a lateral surface, we can write

$$\iint_{\sigma} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{\sigma_1} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS + \iint_{\sigma_2} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS \tag{4}$$

Carl Friedrich Gauss (1777–1855) German mathematician and scientist. Sometimes called the "prince of mathematicians," Gauss ranks with Newton and Archimedes as one of the three greatest mathematicians who ever lived. His father, a laborer, was an uncouth but honest man who would have liked Gauss to take up a trade such as gardening or bricklaying; but the boy's genius for mathematics was not to be denied. In the entire history of mathematics there may never have been a child so precocious as Gauss—by his own account he worked out the rudiments of arithmetic before he could talk. One day, before he was even three years old, his genius became apparent to his parents in a very dramatic way. His father was preparing the weekly payroll for the laborers under his charge while the boy watched quietly from a corner. At the end of the long and tedious calculation, Gauss informed his father that there was an error in the result and stated the answer, which he had worked out in his head. To the astonishment of his parents, a check of the computations showed Gauss to be correct!

For his elementary education Gauss was enrolled in a squalid school run by a man named Büttner whose main teaching technique was thrashing. Büttner was in the habit of assigning long addition problems which, unknown to his students, were arithmetic progressions that he could sum up using formulas. On the first day that Gauss entered the arithmetic class, the students were asked to sum the numbers from 1 to 100. But no sooner had Büttner stated the problem than Gauss turned over his slate and exclaimed in his peasant dialect, "Ligget se'." (Here it lies.) For nearly an hour Büttner glared at Gauss, who sat with folded hands while his classmates toiled away. When Büttner examined the slates at the end of the period, Gauss's slate contained a single number, 5050—the only correct solution in the class. To his credit, Büttner recognized the genius of Gauss and with the help of his assistant, John Bartels, had him brought to the attention of Karl Wilhelm Ferdinand, Duke of Brunswick. The shy and awkward boy, who was then fourteen, so captivated the Duke that he subsidized him through preparatory school, college, and the early part of his career.

From 1795 to 1798 Gauss studied mathematics at the University of Göttingen, receiving his degree in absentia from the University of Helmstadt. For his dissertation, he gave the first complete proof of the fundamental theorem of algebra, which states that every polynomial equation has as many solutions as its degree. At age 19 he solved a problem that baffled Euclid, inscribing a regular polygon of 17 sides in a circle using straightedge and compass; and in 1801, at age 24, he published his first masterpiece, *Disquisitiones Arithmeticae*, considered by many to be one of the most brilliant achievements in mathematics. In that book Gauss systematized the study of number theory (properties of the integers) and formulated the basic concepts that form the foundation of that subject.

In the same year that the *Disquisitiones* was published, Gauss again applied his phenomenal computational skills in a dramatic way. The astronomer Giuseppi Piazzi had observed the asteroid Ceres for $\frac{1}{40}$ of its orbit, but lost it in the Sun. Using only three observations and the "method of least squares" that he had developed in 1795, Gauss computed the orbit with such accuracy that astronomers had no trouble relocating it the following year. This achievement brought him instant recognition as the premier mathematician in Europe, and in 1807 he was made Professor of Astronomy and head of the astronomical observatory at Göttingen.

In the years that followed, Gauss revolutionized mathematics by bringing to it standards of precision and rigor undreamed of by his predecessors. He had a passion for perfection that drove him to polish and rework his papers rather than publish less finished work in greater numbers—his favorite saying was "Pauca, sed matura" (Few, but ripe). As a result, many of his important discoveries were squirreled away in diaries that remained unpublished until years after his death.

Among his myriad achievements, Gauss discovered the Gaussian or "bell-shaped" error curve fundamental in probability, gave the first geometric interpretation of complex numbers and established their fundamental role in mathematics, developed methods of characterizing surfaces intrinsically by means of the curves that they contain, developed the theory of conformal (angle-preserving) maps, and discovered non-Euclidean geometry 30 years before the ideas were published by others. In physics he made major contributions to the theory of lenses and capillary action, and with Wilhelm Weber he did fundamental work in electromagnetism. Gauss invented the heliotrope, bifilar magnetometer, and an electrotelegraph.

Gauss was deeply religious and aristocratic in demeanor. He mastered foreign languages with ease, read extensively, and enjoyed mineralogy and botany as hobbies. He disliked teaching and was usually cool and discouraging to other mathematicians, possibly because he had already anticipated their work. It has been said that if Gauss had published all of his discoveries, the current state of mathematics would be advanced by 50 years. He was without a doubt the greatest mathematician of the modern era.

On the upper surface σ_2, the outer normal is an upward normal, and on the lower surface σ_1, the outer normal is a downward normal. Thus, Formulas (12) and (13) of Section 15.6 imply that

$$\iint_{\sigma_2} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_R h(x, y, g_2(x, y))\mathbf{k} \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA$$
$$= \iint_R h(x, y, g_2(x, y))\, dA \tag{5}$$

and

$$\iint_{\sigma_1} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_R h(x, y, g_1(x, y))\mathbf{k} \cdot \left(\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}\right) dA$$
$$= -\iint_R h(x, y, g_1(x, y))\, dA \tag{6}$$

Substituting (5) and (6) into (4) and combining the terms into a single integral yields

$$\iint_{\sigma} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_R [h(x, y, g_2(x, y)) - h(x, y, g_1(x, y))]\, dA \tag{7}$$

Equation (2c) now follows from (3) and (7). ■

Explain how the derivation of (2c) should be modified to yield a proof of (2a) or (2b).

The flux of a vector field across a closed surface with outward orientation is sometimes called the ***outward flux*** across the surface. In words, the Divergence Theorem states:

The outward flux of a vector field across a closed surface is equal to the triple integral of the divergence over the region enclosed by the surface.

USING THE DIVERGENCE THEOREM TO FIND FLUX

Sometimes it is easier to find the flux across a closed surface by using the Divergence Theorem than by evaluating the flux integral directly.

▶ **Example 1** Use the Divergence Theorem to find the outward flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$.

Solution. Let σ denote the outward-oriented spherical surface and G the region that it encloses. The divergence of the vector field is

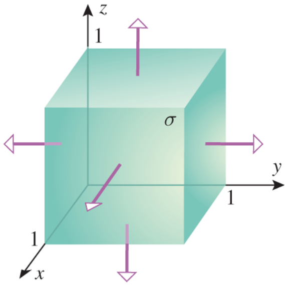
$$\text{div } \mathbf{F} = \frac{\partial z}{\partial z} = 1$$

so from (1) the flux across σ is

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_G dV = \text{volume of } G = \frac{4\pi a^3}{3}$$

Note how much simpler this calculation is than that in Example 1 of Section 15.6. ◀

The Divergence Theorem is usually the method of choice for finding the flux across closed piecewise smooth surfaces with multiple sections, since it eliminates the need for a separate integral evaluation over each section. This is illustrated in the next three examples.



▲ **Figure 15.7.3**

▶ **Example 2** Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3y\mathbf{j} + z^2\mathbf{k}$$

across the unit cube in Figure 15.7.3.

Solution. Let σ denote the outward-oriented surface of the cube and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = 5 + 2z$$

so from (1) the flux across σ is

$$\Phi = \iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_G (5+2z)\,dV = \int_0^1\int_0^1\int_0^1 (5+2z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left[5z + z^2\right]_{z=0}^1 dy\,dx = \int_0^1\int_0^1 6\,dy\,dx = 6 \blacktriangleleft$$

Let $\mathbf{F}(x, y, z)$ be the vector field in Example 2 and show that $\mathbf{F}\cdot\mathbf{n}$ is constant on each of the six faces of the cube in Figure 15.7.3. Use your computations to confirm the result in Example 2.

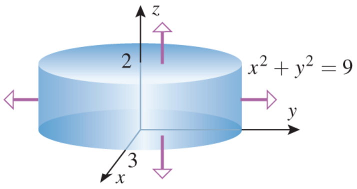


Figure 15.7.4

Example 3 Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k}$$

across the surface of the region that is enclosed by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$ (Figure 15.7.4).

Solution. Let σ denote the outward-oriented surface and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^2) = 3x^2 + 3y^2 + 2z$$

so from (1) the flux across σ is

$$\Phi = \iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_G (3x^2 + 3y^2 + 2z)\,dV$$

$$= \int_0^{2\pi}\int_0^3\int_0^2 (3r^2 + 2z)r\,dz\,dr\,d\theta$$

Using cylindrical coordinates

$$= \int_0^{2\pi}\int_0^3 \left[3r^3z + z^2r\right]_{z=0}^2 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (6r^3 + 4r)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^4}{2} + 2r^2\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{279}{2}\,d\theta = 279\pi \blacktriangleleft$$

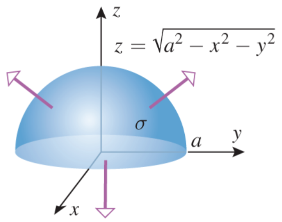


Figure 15.7.5

Example 4 Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$$

across the surface of the region that is enclosed by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and the plane $z = 0$ (Figure 15.7.5).

Solution. Let σ denote the outward-oriented surface and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

so from (1) the flux across σ is

$$\begin{aligned}
\Phi = \iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_{G} (3x^2+3y^2+3z^2)\,dV \\
&= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} (3\rho^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \qquad \boxed{\text{Using spherical coordinates}} \\
&= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho^4\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 3\int_0^{2\pi}\int_0^{\pi/2}\left[\frac{\rho^5}{5}\sin\phi\right]_{\rho=0}^{a} d\phi\,d\theta \\
&= \frac{3a^5}{5}\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\,d\phi\,d\theta \\
&= \frac{3a^5}{5}\int_0^{2\pi}[-\cos\phi]_0^{\pi/2}\,d\theta \\
&= \frac{3a^5}{5}\int_0^{2\pi}d\theta = \frac{6\pi a^5}{5} \blacktriangleleft
\end{aligned}$$

DIVERGENCE VIEWED AS FLUX DENSITY

The Divergence Theorem provides a way of interpreting the divergence of a vector field $\mathbf{F}$. Suppose that G is a *small* spherical region centered at the point P_0 and that its surface, denoted by $\sigma(G)$, is oriented outward. Denote the volume of the region by vol(G) and the flux of $\mathbf{F}$ across $\sigma(G)$ by $\Phi(G)$. If div $\mathbf{F}$ is continuous on G, then across the small region G the value of div $\mathbf{F}$ will not vary much from its value div $\mathbf{F}(P_0)$ at the center, and we can reasonably approximate div $\mathbf{F}$ by the constant div $\mathbf{F}(P_0)$ on G. Thus, the Divergence Theorem implies that the flux $\Phi(G)$ of $\mathbf{F}$ across $\sigma(G)$ can be approximated by

$$\Phi(G) = \iint_{\sigma(G)} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{G} \operatorname{div}\mathbf{F}\,dV \approx \operatorname{div}\mathbf{F}(P_0)\iiint_{G} dV = \operatorname{div}\mathbf{F}(P_0)\operatorname{vol}(G)$$

from which we obtain the approximation

$$\operatorname{div}\mathbf{F}(P_0) \approx \frac{\Phi(G)}{\operatorname{vol}(G)} \tag{8}$$

The expression on the right side of (8) is called the ***outward flux density of* F *across G***. If we now let the radius of the sphere approach zero [so that vol(G) approaches zero], then it is plausible that the error in this approximation will approach zero, and the divergence of $\mathbf{F}$ at the point P_0 will be given exactly by

$$\operatorname{div}\mathbf{F}(P_0) = \lim_{\operatorname{vol}(G)\to 0}\frac{\Phi(G)}{\operatorname{vol}(G)}$$

which we can express as

$$\operatorname{div}\mathbf{F}(P_0) = \lim_{\operatorname{vol}(G)\to 0}\frac{1}{\operatorname{vol}(G)}\iint_{\sigma(G)} \mathbf{F}\cdot\mathbf{n}\,dS \tag{9}$$

Formula (9) is sometimes taken as the definition of divergence. This is a useful alternative to Definition 15.1.4 because it does not require a coordinate system.

This limit, which is called the ***outward flux density of* F *at* P_0**, tells us that if $\mathbf{F}$ denotes the velocity field of a fluid, then *in a steady-state fluid flow* div $\mathbf{F}$ *can be interpreted as the limiting flux per unit volume at a point*. Moreover, it follows from (8) that for a small spherical region G centered at a point P_0 in the flow, the outward flux across the surface of G can be approximated by

$$\Phi(G) \approx (\operatorname{div}\mathbf{F}(P_0))(\operatorname{vol}(G)) \tag{10}$$

SOURCES AND SINKS

If P_0 is a point in an incompressible fluid at which div $\mathbf{F}(P_0) > 0$, then it follows from (8) that $\Phi(G) > 0$ for a sufficiently small sphere G centered at P_0. Thus, there is a greater volume of fluid going out through the surface of G than coming in. But this can only happen if there is some point *inside* the sphere at which fluid is entering the flow (say by condensation, melting of a solid, or a chemical reaction); otherwise the net outward flow through the surface would result in a decrease in density within the sphere, contradicting the incompressibility assumption. Similarly, if div $\mathbf{F}(P_0) < 0$, there would have to be a point *inside* the sphere at which fluid is leaving the flow (say by evaporation); otherwise the net inward flow through the surface would result in an increase in density within the sphere. In an incompressible fluid, points at which div $\mathbf{F}(P_0) > 0$ are called ***sources*** and points at which div $\mathbf{F}(P_0) < 0$ are called ***sinks***. Fluid enters the flow at a source and drains out at a sink. In an incompressible fluid without sources or sinks we must have

$$\operatorname{div} \mathbf{F}(P) = 0$$

at every point P. In hydrodynamics this is called the ***continuity equation for incompressible fluids*** and is sometimes taken as the defining characteristic of an incompressible fluid.

GAUSS'S LAW FOR INVERSE-SQUARE FIELDS

The Divergence Theorem applied to inverse-square fields (see Definition 15.1.2) produces a result called ***Gauss's Law for Inverse-Square Fields***. This result is the basis for many important principles in physics.

15.7.2 GAUSS'S LAW FOR INVERSE-SQUARE FIELDS If

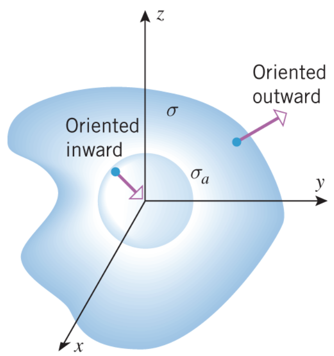
$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}$$

is an inverse-square field in 3-space, and if σ is a closed orientable surface that surrounds the origin, then the outward flux of $\mathbf{F}$ across σ is

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi c \tag{11}$$

To derive this result, recall from Formula (5) of Section 15.1 that $\mathbf{F}$ can be expressed in component form as

$$\mathbf{F}(x, y, z) = \frac{c}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{12}$$



▲ Figure 15.7.6

Since the components of $\mathbf{F}$ are not continuous at the origin, we cannot apply the Divergence Theorem across the solid enclosed by σ. However, we can circumvent this difficulty by constructing a sphere of radius a centered at the origin, where the radius is sufficiently small that the sphere lies entirely within the region enclosed by σ (Figure 15.7.6). We will denote the surface of this sphere by σ_a. The solid G enclosed between σ_a and σ is an example of a three-dimensional solid with an internal "cavity." Just as we were able to extend Green's Theorem to multiply connected regions in the plane (regions with holes), so it is possible to extend the Divergence Theorem to solids in 3-space with internal cavities, provided the surface integral in the theorem is taken over the *entire* boundary with the outside boundary of the solid oriented outward and the boundaries of the cavities oriented inward. Thus, if $\mathbf{F}$ is the inverse-square field in (12), and if σ_a is oriented inward, then the Divergence Theorem yields

$$\iiint_G \operatorname{div} \mathbf{F}\, dV = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{\sigma_a} \mathbf{F} \cdot \mathbf{n}\, dS \tag{13}$$

But we showed in Example 5 of Section 15.1 that div $\mathbf{F} = 0$, so (13) yields

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = -\iint_{\sigma_a} \mathbf{F} \cdot \mathbf{n}\, dS \tag{14}$$

We can evaluate the surface integral over σ_a by expressing the integrand in terms of components; however, it is easier to leave it in vector form. At each point on the sphere the unit normal $\mathbf{n}$ points inward along a radius from the origin, and hence $\mathbf{n} = -\mathbf{r}/\|\mathbf{r}\|$. Thus, (14) yields

$$\begin{aligned}
\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS &= -\iint_{\sigma_a} \frac{c}{\|\mathbf{r}\|^3}\mathbf{r} \cdot \left(-\frac{\mathbf{r}}{\|\mathbf{r}\|}\right) dS \\
&= \iint_{\sigma_a} \frac{c}{\|\mathbf{r}\|^4}(\mathbf{r} \cdot \mathbf{r})\, dS \\
&= \iint_{\sigma_a} \frac{c}{\|\mathbf{r}\|^2}\, dS \\
&= \frac{c}{a^2} \iint_{\sigma_a} dS \qquad \boxed{\|\mathbf{r}\| = a \text{ on } \sigma_a} \\
&= \frac{c}{a^2}(4\pi a^2) \qquad \boxed{\text{The integral is the surface area of the sphere.}} \\
&= 4\pi c
\end{aligned}$$

which establishes (11).

GAUSS'S LAW IN ELECTROSTATICS

It follows from Example 1 of Section 15.1 with $q = 1$ that a single charged particle of charge Q located at the origin creates an inverse-square field

$$\mathbf{F}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0\|\mathbf{r}\|^3}\mathbf{r}$$

in which $\mathbf{F}(\mathbf{r})$ is the electrical force exerted by Q on a unit positive charge ($q = 1$) located at the point with position vector $\mathbf{r}$. In this case Gauss's law (15.7.2) states that the outward flux Φ across any closed orientable surface σ that surrounds Q is

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi\left(\frac{Q}{4\pi\epsilon_0}\right) = \frac{Q}{\epsilon_0}$$

This result, which is called ***Gauss's Law for Electric Fields***, can be extended to more than one charge. It is one of the fundamental laws in electricity and magnetism.

✓QUICK CHECK EXERCISES 15.7

(See page 1039 for answers.)

1. Let G be a solid whose surface σ is oriented outward by the unit normal $\mathbf{n}$, and let $\mathbf{F}(x, y, z)$ denote a vector field whose component functions have continuous first partial derivatives on some open set containing G. The Divergence Theorem states that the surface integral ________ and the triple integral ________ have the same value.
2. The outward flux of $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across any unit cube is ________.
3. If $\mathbf{F}(x, y, z)$ is the velocity vector field for a steady-state incompressible fluid flow, then a point at which div $\mathbf{F}$ is positive is called a ________ and a point at which div $\mathbf{F}$ is negative is called a ________. The continuity equation for an incompressible fluid states that ________.
4. If
$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}$$
is an inverse-square field, and if σ is a closed orientable surface that surrounds the origin, then Gauss's law states that the outward flux of $\mathbf{F}$ across σ is ________. On the other hand, if σ does not surround the origin, then it follows from the Divergence Theorem that the outward flux of $\mathbf{F}$ across σ is ________.

EXERCISE SET 15.7 C CAS

1–4 Verify Formula (1) in the Divergence Theorem by evaluating the surface integral and the triple integral.

1. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
2. $\mathbf{F}(x, y, z) = 5\mathbf{j} + 7\mathbf{k}$; σ is the sphere $x^2 + y^2 + z^2 = 1$.
3. $\mathbf{F}(x, y, z) = 2x\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k}$; the surface σ is the paraboloid $z = x^2 + y^2$ capped by the disk $x^2 + y^2 \leq 1$ in the plane $z = 1$.
4. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$; σ is the surface of the cube bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 2$, $z = 0$, $z = 2$.

5–8 True–False Determine whether the statement is true or false. Explain your answer.

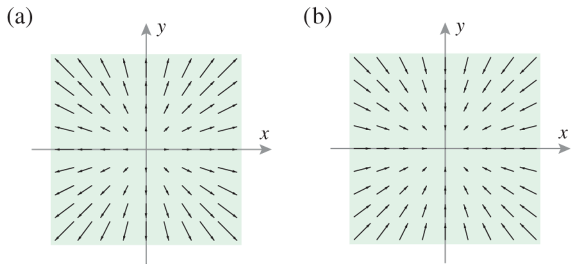
5. The Divergence Theorem equates a surface integral and a line integral.
6. If G is a solid whose surface σ is oriented outward, and if div $\mathbf{F} > 0$ at all points of G, then the flux of $\mathbf{F}$ across σ is positive.
7. The continuity equation for incompressible fluids states that the divergence of the velocity vector field of the fluid is zero.
8. Since the divergence of an inverse-square field is zero, the flux of an inverse-square field across any closed orientable surface must be zero as well.

9–19 Use the Divergence Theorem to find the flux of $\mathbf{F}$ across the surface σ with outward orientation.

9. $\mathbf{F}(x, y, z) = (x^2 + y)\mathbf{i} + z^2\mathbf{j} + (e^y - z)\mathbf{k}$; σ is the surface of the rectangular solid bounded by the coordinate planes and the planes $x = 3$, $y = 1$, and $z = 2$.
10. $\mathbf{F}(x, y, z) = z^3\mathbf{i} - x^3\mathbf{j} + y^3\mathbf{k}$, where σ is the spherical surface $x^2 + y^2 + z^2 = a^2$.
11. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - x)\mathbf{j} + (z - y)\mathbf{k}$; σ is the surface of the cylindrical solid bounded by $x^2 + y^2 = a^2$, $z = 0$, and $z = 1$.
12. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane.
13. $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$; σ is the surface of the cylindrical solid bounded by $x^2 + y^2 = 4$, $z = 0$, and $z = 3$.
14. $\mathbf{F}(x, y, z) = (x^2 + y)\mathbf{i} + xy\mathbf{j} - (2xz + y)\mathbf{k}$; σ is the surface of the tetrahedron in the first octant bounded by $x + y + z = 1$ and the coordinate planes.
15. $\mathbf{F}(x, y, z) = (x^3 - e^y)\mathbf{i} + (y^3 + \sin z)\mathbf{j} + (z^3 - xy)\mathbf{k}$, where σ is the surface of the solid bounded above by $z = \sqrt{4 - x^2 - y^2}$ and below by the xy-plane. [*Hint:* Use spherical coordinates.]
16. $\mathbf{F}(x, y, z) = 2xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$, where σ is the surface of the solid bounded above by $z = \sqrt{a^2 - x^2 - y^2}$ and below by the xy-plane.
17. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; σ is the surface of the conical solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = 1$.
18. $\mathbf{F}(x, y, z) = x^2y\mathbf{i} - xy^2\mathbf{j} + (z + 2)\mathbf{k}$; σ is the surface of the solid bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$.
19. $\mathbf{F}(x, y, z) = x^3\mathbf{i} + x^2y\mathbf{j} + xy\mathbf{k}$; σ is the surface of the solid bounded by $z = 4 - x^2$, $y + z = 5$, $z = 0$, and $y = 0$.
20. Prove that if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and σ is the surface of a solid G oriented by outward unit normals, then
$$\text{vol}(G) = \frac{1}{3}\iint_{\sigma} \mathbf{r} \cdot \mathbf{n}\, dS$$
where vol(G) is the volume of G.
21. Use the result in Exercise 20 to find the outward flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the surface σ of the cylindrical solid bounded by $x^2 + 4x + y^2 = 5$, $z = -1$, and $z = 4$.

FOCUS ON CONCEPTS

22. Let $\mathbf{F}(x, y, z) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a constant vector field and let σ be the surface of a solid G. Use the Divergence Theorem to show that the flux of $\mathbf{F}$ across σ is zero. Give an informal physical explanation of this result.
23. Find a vector field $\mathbf{F}(x, y, z)$ that has
(a) positive divergence everywhere
(b) negative divergence everywhere.
24. In each part, the figure shows a horizontal layer of the vector field of a fluid flow in which the flow is parallel to the xy-plane at every point and is identical in each layer (i.e., is independent of z). For each flow, what can you say about the sign of the divergence at the origin? Explain your reasoning.



25. Let $\mathbf{F}(x, y, z)$ be a nonzero vector field in 3-space whose component functions have continuous first partial derivatives, and assume that div $\mathbf{F} = 0$ everywhere. If σ is any sphere in 3-space, explain why there are infinitely many points on σ at which $\mathbf{F}$ is tangent to the sphere.
26. Does the result in Exercise 25 remain true if the sphere σ is replaced by a cube? Explain.

27–31 Prove the identity, assuming that $\mathbf{F}$, σ, and G satisfy the hypotheses of the Divergence Theorem and that all necessary differentiability requirements for the functions $f(x, y, z)$ and $g(x, y, z)$ are met.

27. $\displaystyle\iint_\sigma \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = 0$ [*Hint:* See Exercise 37, Section 15.1.]

28. $\displaystyle\iint_\sigma \nabla f\cdot\mathbf{n}\,dS = \iiint_G \nabla^2 f\,dV$

$$\left(\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

29. $\displaystyle\iint_\sigma (f\nabla g)\cdot\mathbf{n}\,dS = \iiint_G (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV$

30. $\displaystyle\iint_\sigma (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS = \iiint_G (f\nabla^2 g - g\nabla^2 f)\,dV$
[*Hint:* Interchange f and g in Exercise 29.]

31. $\displaystyle\iint_\sigma (f\mathbf{n})\cdot\mathbf{v}\,dS = \iiint_G \nabla f\cdot\mathbf{v}\,dV$ ($\mathbf{v}$ a fixed vector)

32. Use the Divergence Theorem to find all positive values of k such that

$$\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{\|\mathbf{r}\|^k}$$

satisfies the condition div $\mathbf{F} = 0$ when $\mathbf{r} \neq \mathbf{0}$.
[*Hint:* Modify the proof of (11).]

33–36 Determine whether the vector field $\mathbf{F}(x, y, z)$ is free of sources and sinks. If it is not, locate them.

33. $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} - xz^3\mathbf{j} + (x^2 \sin y)\mathbf{k}$

34. $\mathbf{F}(x, y, z) = xy\mathbf{i} - xy\mathbf{j} + y^2\mathbf{k}$

35. $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$

36. $\mathbf{F}(x, y, z) = (x^3 - x)\mathbf{i} + (y^3 - y)\mathbf{j} + (z^3 - z)\mathbf{k}$

C 37. Let σ be the surface of the solid G that is enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$. Use a CAS to verify Formula (1) in the Divergence Theorem for the vector field

$$\mathbf{F} = (x^2y - z^2)\mathbf{i} + (y^3 - x)\mathbf{j} + (2x + 3z - 1)\mathbf{k}$$

by evaluating the surface integral and the triple integral.

38. **Writing** Discuss what it means to say that the divergence of a vector field is independent of a coordinate system. Explain how we know this to be true.

39. **Writing** Describe some geometrical and physical applications of the Divergence Theorem.

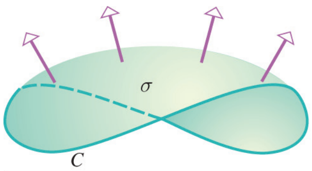
✓ **QUICK CHECK ANSWERS 15.7** 1. $\displaystyle\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS$; $\displaystyle\iiint_G \operatorname{div}\mathbf{F}\,dV$ 2. 3 3. source; sink; div $\mathbf{F} = 0$ 4. $4\pi c$; 0

15.8 STOKES' THEOREM

In this section we will discuss a generalization of Green's Theorem to three dimensions that has important applications in the study of vector fields, particularly in the analysis of rotational motion of fluids. This theorem will also provide us with a physical interpretation of the curl of a vector field.

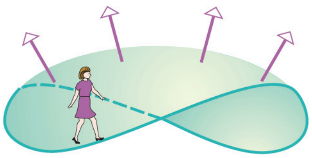
RELATIVE ORIENTATION OF CURVES AND SURFACES

We will be concerned in this section with oriented surfaces in 3-space that are bounded by simple closed parametric curves (Figure 15.8.1a). If σ is an oriented surface bounded by a simple closed parametric curve C, then there are two possible relationships between the orientations of σ and C, which can be described as follows. Imagine a person walking along the curve C with his or her head in the direction of the orientation of σ. The person is said to be walking in the ***positive direction*** of C relative to the orientation of σ if the surface is on the person's left (Figure 15.8.1b), and the person is said to be walking in the ***negative direction*** of C relative to the orientation of σ if the surface is on the person's right (Figure 15.8.1c). The positive direction of C establishes a right-hand relationship between



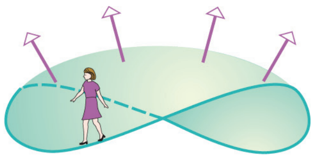
The oriented surface σ is bounded by the simple closed curve C.

(a)



The positive direction of C relative to the orientation of σ.

(b)



The negative direction of C relative to the orientation of σ.

(c)

▲ **Figure 15.8.1**

the orientations of σ and C in the sense that if the fingers of the right hand are curled from the direction of C toward σ, then the thumb points (roughly) in the direction of the orientation of σ.

STOKES' THEOREM

In Section 15.1 we defined the curl of a vector field

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

as

$$\text{curl }\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{1}$$

but we did not attempt to give a physical explanation of its meaning at that time. The following result, known as ***Stokes' Theorem***, will provide us with a physical interpretation of the curl in the context of fluid flow.

15.8.1 THEOREM (*Stokes' Theorem*) *Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. If the components of the vector field*

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

are continuous and have continuous first partial derivatives on some open set containing σ, and if $\mathbf{T}$ is the unit tangent vector to C, then

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_\sigma (\text{curl }\mathbf{F}) \cdot \mathbf{n}\, dS \tag{2}$$

The proof of this theorem is beyond the scope of this text, so we will focus on its applications.

Recall from Formulas (30) and (34) in Section 15.2 that if $\mathbf{F}$ is a force field, the integral on the left side of (2) represents the work performed by the force field on a particle that traverses the curve C. Thus, loosely phrased, Stokes' Theorem states:

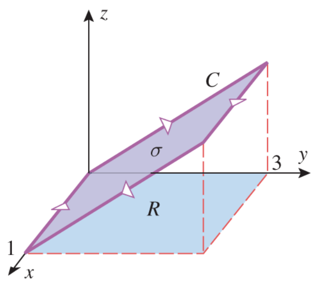
The work performed by a force field on a particle that traverses a simple, closed, piecewise smooth curve C in the positive direction can be obtained by integrating the normal component of the curl over an oriented surface σ bounded by C.

USING STOKES' THEOREM TO CALCULATE WORK

For computational purposes it is usually preferable to use Formula (30) in Section 15.2 to rewrite the formula in Stokes' Theorem as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\sigma (\text{curl }\mathbf{F}) \cdot \mathbf{n}\, dS \tag{3}$$

Stokes' Theorem is usually the method of choice for calculating work around piecewise smooth curves with multiple sections, since it eliminates the need for a separate integral evaluation over each section.



▲ Figure 15.8.2

▶ Example 1 Find the work performed by the force field

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + 4xy^3\mathbf{j} + y^2x\mathbf{k}$$

on a particle that traverses the rectangle C in the plane $z = y$ shown in Figure 15.8.2.

Solution. The work performed by the field is

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

However, to evaluate this integral directly would require four separate integrations, one over each side of the rectangle. Instead, we will use Formula (3) to express the work as the surface integral

$$W = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

in which the plane surface σ enclosed by C is assigned a *downward* orientation to make the orientation of C positive, as required by Stokes' Theorem.

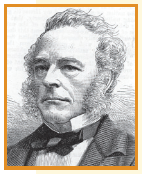
Since the surface σ has equation $z = y$ and

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & 4xy^3 & xy^2 \end{vmatrix} = 2xy\mathbf{i} - y^2\mathbf{j} + 4y^3\mathbf{k}$$

it follows from Formula (13) of Section 15.6 with curl $\mathbf{F}$ replacing $\mathbf{F}$ that

$$\begin{aligned} W = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS &= \iint_R (\text{curl } \mathbf{F}) \cdot \left(\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}\right) dA \\ &= \iint_R (2xy\mathbf{i} - y^2\mathbf{j} + 4y^3\mathbf{k}) \cdot (0\mathbf{i} + \mathbf{j} - \mathbf{k})\, dA \\ &= \int_0^1 \int_0^3 (-y^2 - 4y^3)\, dy\, dx \\ &= -\int_0^1 \left[\frac{y^3}{3} + y^4\right]_{y=0}^3 dx \\ &= -\int_0^1 90\, dx = -90 \end{aligned}$$

Explain how the result in Example 1 shows that the given force field is not conservative.

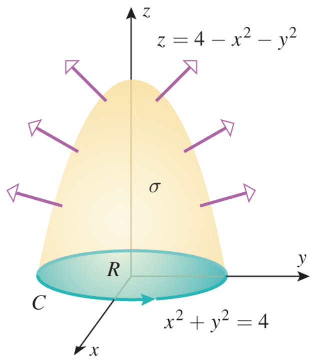


George Gabriel Stokes (1819–1903) Irish mathematician and physicist. Born in Skreen, Ireland, Stokes came from a family deeply rooted in the Church of Ireland. His father was a rector, his mother the daughter of a rector, and three of his brothers took holy orders. He received his early education from his father and a local parish clerk. In 1837, he entered Pembroke College and after graduating with top honors accepted a fellowship at the college. In 1847 he was appointed Lucasian professor of mathematics at Cambridge, a position once held by Isaac Newton (and now held by the British physicist, Stephen Hawking), but one that had lost its esteem through the years. By virtue of his accomplishments, Stokes ultimately restored the position to the eminence it once held. Unfortunately, the position paid very little and Stokes was forced to teach at the Government School of Mines during the 1850s to supplement his income.

Stokes was one of several outstanding nineteenth century scientists who helped turn the physical sciences in a more empirical direction. He systematically studied hydrodynamics, elasticity of solids, behavior of waves in elastic solids, and diffraction of light. For Stokes, mathematics was a tool for his physical studies. He wrote classic papers on the motion of viscous fluids that laid the foundation for modern hydrodynamics; he elaborated on the wave theory of light; and he wrote papers on gravitational variation that established him as a founder of the modern science of geodesy.

Stokes was honored in his later years with degrees, medals, and memberships in foreign societies. He was knighted in 1889. Throughout his life, Stokes gave generously of his time to learned societies and readily assisted those who sought his help in solving problems. He was deeply religious and vitally concerned with the relationship between science and religion.

[*Image: photos.com/Getty Images*]



▲ **Figure 15.8.3**

▶ **Example 2** Verify Stokes' Theorem for the vector field $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, taking σ to be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation, and C to be the positively oriented circle $x^2 + y^2 = 4$ that forms the boundary of σ in the xy-plane (Figure 15.8.3).

Solution. We will verify Formula (3). Since σ is oriented up, the positive orientation of C is counterclockwise looking down the positive z-axis. Thus, C can be represented parametrically (with positive orientation) by

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 0 \qquad (0 \leq t \leq 2\pi) \tag{4}$$

Therefore,

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C 2z\,dx + 3x\,dy + 5y\,dz \\ &= \int_0^{2\pi} [0 + (6\cos t)(2\cos t) + 0]\,dt \\ &= \int_0^{2\pi} 12\cos^2 t\,dt = 12\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 12\pi\end{aligned}$$

To evaluate the right side of (3), we start by finding curl $\mathbf{F}$. We obtain

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

Since σ is oriented up and is expressed in the form $z = g(x, y) = 4 - x^2 - y^2$, it follows from Formula (12) of Section 15.6 with curl $\mathbf{F}$ replacing $\mathbf{F}$ that

$$\begin{aligned}\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\,dS &= \iint_R (\text{curl } \mathbf{F}) \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA \\ &= \iint_R (5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA \\ &= \iint_R (10x + 4y + 3)\,dA \\ &= \int_0^{2\pi} \int_0^2 (10r\cos\theta + 4r\sin\theta + 3)r\,dr\,d\theta \\ &= \int_0^{2\pi} \left[\frac{10r^3}{3}\cos\theta + \frac{4r^3}{3}\sin\theta + \frac{3r^2}{2}\right]_{r=0}^2 d\theta \\ &= \int_0^{2\pi} \left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right) d\theta \\ &= \left[\frac{80}{3}\sin\theta - \frac{32}{3}\cos\theta + 6\theta\right]_0^{2\pi} = 12\pi\end{aligned}$$

As guaranteed by Stokes' Theorem, the value of this surface integral is the same as the value obtained for the line integral. Note, however, that the line integral was simpler to evaluate and hence would be the method of choice in this case. ◀

REMARK Observe that in Formula (3) the only relationships required between σ and C are that C be the boundary of σ and that C be positively oriented relative to the orientation of σ. Thus, if σ_1 and σ_2 are *different* oriented surfaces but have the *same* positively oriented boundary curve C, then it follows from (3) that

$$\iint_{\sigma_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\sigma_2} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

For example, the parabolic surface in Example 2 has the same positively oriented boundary C as the disk R in Figure 15.8.3 with upper orientation. Thus, the value of the surface integral in that example would not change if σ is replaced by R (or by any other oriented surface that has the positively oriented circle C as its boundary). This can be useful in computations because it is sometimes possible to circumvent a difficult integration by changing the surface of integration.

RELATIONSHIP BETWEEN GREEN'S THEOREM AND STOKES' THEOREM

It is sometimes convenient to regard a vector field

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

in 2-space as a vector field in 3-space by expressing it as

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j} + 0\mathbf{k} \tag{5}$$

If R is a region in the xy-plane enclosed by a simple, closed, piecewise smooth curve C, then we can treat R as a *flat* surface, and we can treat a surface integral over R as an ordinary double integral over R. Thus, if we orient R up and C counterclockwise looking down the positive z-axis, then Formula (3) applied to (5) yields

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA \tag{6}$$

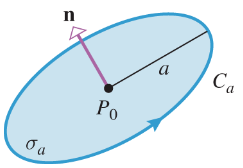
But

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix} = -\frac{\partial g}{\partial z}\mathbf{i} + \frac{\partial f}{\partial z}\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

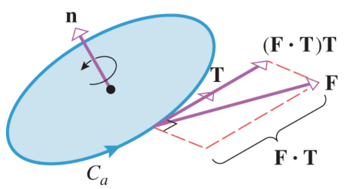
since $\partial g/\partial z = \partial f/\partial z = 0$. Substituting this expression in (6) and expressing the integrals in terms of components yields

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

which is Green's Theorem [Formula (1) of Section 15.4]. Thus, we have shown that Green's Theorem can be viewed as a special case of Stokes' Theorem.



▲ Figure 15.8.4



▲ Figure 15.8.5

CURL VIEWED AS CIRCULATION

Stokes' Theorem provides a way of interpreting the curl of a vector field $\mathbf{F}$ in the context of fluid flow. For this purpose let σ_a be a small oriented disk of radius a centered at a point P_0 in a steady-state fluid flow, and let $\mathbf{n}$ be a unit normal vector at the center of the disk that points in the direction of orientation. Let us assume that the flow of liquid past the disk causes it to spin around the axis through $\mathbf{n}$, and let us try to find the direction of $\mathbf{n}$ that will produce the maximum rotation rate in the positive direction of the boundary curve C_a (Figure 15.8.4). For convenience, we will denote the area of the disk σ_a by $A(\sigma_a)$; that is, $A(\sigma_a) = \pi a^2$.

If the direction of $\mathbf{n}$ is fixed, then at each point of C_a the only component of $\mathbf{F}$ that contributes to the rotation of the disk about $\mathbf{n}$ is the component $\mathbf{F} \cdot \mathbf{T}$ tangent to C_a (Figure 15.8.5). Thus, for a fixed $\mathbf{n}$ the integral

$$\oint_{C_a} \mathbf{F} \cdot \mathbf{T}\, ds \tag{7}$$

can be viewed as a measure of the tendency for the fluid to flow in the positive direction around C_a. Accordingly, (7) is called the ***circulation of* F *around* C_a**. For example, in the extreme case where the flow is normal to the circle at each point, the circulation around C_a is zero, since $\mathbf{F}\cdot\mathbf{T}=0$ at each point. The more closely that **F** aligns with **T** along the circle, the larger the value of $\mathbf{F}\cdot\mathbf{T}$ and the larger the value of the circulation.

To see the relationship between circulation and curl, suppose that curl **F** is continuous on σ_a, so that when σ_a is small the value of curl **F** at any point of σ_a will not vary much from the value of curl $\mathbf{F}(P_0)$ at the center. Thus, for a small disk σ_a we can reasonably approximate curl **F** on σ_a by the constant value curl $\mathbf{F}(P_0)$. Moreover, because the surface σ_a is flat, the unit normal vectors that orient σ_a are all equal. Thus, the vector quantity **n** in Formula (3) can be treated as a constant, and we can write

$$\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_{\sigma_a} (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS \approx \text{curl}\,\mathbf{F}(P_0)\cdot\mathbf{n}\iint_{\sigma_a} dS$$

where the line integral is taken in the positive direction of C_a. But the last double integral in this equation represents the surface area of σ_a, so

$$\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \approx [\text{curl}\,\mathbf{F}(P_0)\cdot\mathbf{n}]A(\sigma_a)$$

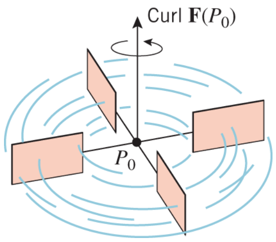
from which we obtain

$$\text{curl}\,\mathbf{F}(P_0)\cdot\mathbf{n} \approx \frac{1}{A(\sigma_a)}\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \tag{8}$$

Formula (9) is sometimes taken as a definition of curl. This is a useful alternative to Definition 15.1.5 because it does not require a coordinate system.

The quantity on the right side of (8) is called the ***circulation density of* F *around* C_a**. If we now let the radius a of the disk approach zero (with **n** fixed), then it is plausible that the error in this approximation will approach zero and the exact value of curl $\mathbf{F}(P_0)\cdot\mathbf{n}$ will be given by

$$\text{curl}\,\mathbf{F}(P_0)\cdot\mathbf{n} = \lim_{a\to 0}\frac{1}{A(\sigma_a)}\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \tag{9}$$



▲ **Figure 15.8.6**

We call curl $\mathbf{F}(P_0)\cdot\mathbf{n}$ the ***circulation density of* F *at* P_0 *in the direction of* n**. This quantity has its maximum value when **n** is in the same direction as curl $\mathbf{F}(P_0)$; this tells us that *at each point in a steady-state fluid flow the maximum circulation density occurs in the direction of the curl*. Physically, this means that if a small paddle wheel is immersed in the fluid so that the pivot point is at P_0, then the paddles will turn most rapidly when the spindle is aligned with curl $\mathbf{F}(P_0)$ (Figure 15.8.6). If curl $\mathbf{F}=\mathbf{0}$ at each point of a region, then **F** is said to be ***irrotational*** in that region, since no circulation occurs about any point of the region.

✓QUICK CHECK EXERCISES 15.8 *(See page 1046 for answers.)*

1. Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. If the component functions of the vector field $\mathbf{F}(x,y,z)$ have continuous first partial derivatives on some open set containing σ, and if **T** is the unit tangent vector to C, then Stokes' Theorem states that the line integral ________ and the surface integral ________ are equal.

2. We showed in Example 2 that the vector field

$$\mathbf{F}(x,y,z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$$

satisfies the equation curl $\mathbf{F} = 5\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. It follows from Stokes' Theorem that if C is any circle of radius a in the xy-plane that is oriented counterclockwise when viewed from the positive z-axis, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \underline{\qquad\qquad}$$

where **T** denotes the unit tangent vector to C.

3. (a) If σ_1 and σ_2 are two oriented surfaces that have the same positively oriented boundary curve C, and if the vector field $\mathbf{F}(x,y,z)$ has continuous first partial derivatives on some open set containing σ_1 and σ_2, then it follows from Stokes' Theorem that the surface integrals ________ and ________ are equal.

(b) Let $\mathbf{F}(x,y,z) = 2z\mathbf{i}+3x\mathbf{j}+5y\mathbf{k}$, let a be a positive number, and let σ be the portion of the paraboloid

$z = a^2 - x^2 - y^2$ for which $z \geq 0$ with upward orientation. Using part (a) and Quick Check Exercise 2, it follows that

$$\iint_{\sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \underline{\hspace{2cm}}$$

4. For steady-state flow, the maximum circulation density occurs in the direction of the ________ of the velocity vector field for the flow.

EXERCISE SET 15.8 C CAS

1–4 Verify Formula (2) in Stokes' Theorem by evaluating the line integral and the surface integral. Assume that the surface has an upward orientation.

1. $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$; σ is the portion of the plane $x + y + z = 1$ in the first octant.
2. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 1$.
3. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the upper hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.
4. $\mathbf{F}(x, y, z) = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$; σ is the portion of the paraboloid $z = 9 - x^2 - y^2$ above the xy-plane.

5–12 Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

5. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2x\mathbf{j} - y^3\mathbf{k}$; C is the circle $x^2 + y^2 = 1$ in the xy-plane with counterclockwise orientation looking down the positive z-axis.
6. $\mathbf{F}(x, y, z) = xz\mathbf{i} + 3x^2y^2\mathbf{j} + yx\mathbf{k}$; C is the rectangle in the plane $z = y$ shown in Figure 15.8.2.
7. $\mathbf{F}(x, y, z) = 3z\mathbf{i} + 4x\mathbf{j} + 2y\mathbf{k}$; C is the boundary of the paraboloid shown in Figure 15.8.3.
8. $\mathbf{F}(x, y, z) = -3y^2\mathbf{i} + 4z\mathbf{j} + 6x\mathbf{k}$; C is the triangle in the plane $z = \frac{1}{2}y$ with vertices $(2, 0, 0)$, $(0, 2, 1)$, and $(0, 0, 0)$ with a counterclockwise orientation looking down the positive z-axis.
9. $\mathbf{F}(x, y, z) = xy\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$; C is the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = y$ with a counterclockwise orientation looking down the positive z-axis.
10. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$; C is the triangle in the plane $x + y + z = 1$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ with a counterclockwise orientation looking from the first octant toward the origin.
11. $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$; C is the circle $x^2 + y^2 = a^2$ in the xy-plane with counterclockwise orientation looking down the positive z-axis.
12. $\mathbf{F}(x, y, z) = (z + \sin x)\mathbf{i} + (x + y^2)\mathbf{j} + (y + e^z)\mathbf{k}$; C is the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$ with counterclockwise orientation looking down the positive z-axis.

13–16 True–False Determine whether the statement is true or false. Explain your answer.

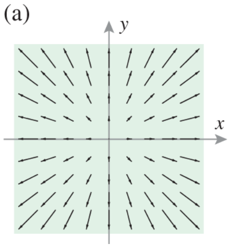
13. Stokes' Theorem equates a line integral and a surface integral.
14. Stokes' Theorem is a special case of Green's Theorem.
15. The circulation of a vector field $\mathbf{F}$ around a closed curve C is defined to be
$$\int_C (\text{curl } \mathbf{F}) \cdot \mathbf{T}\, ds$$
16. If $\mathbf{F}(x, y, z)$ is defined everywhere in 3-space, and if curl $\mathbf{F}$ has no $\mathbf{k}$-component at any point in the xy-plane, then
$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$$
for every smooth, simple, closed curve in the xy-plane.
17. Consider the vector field given by the formula
$$\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - x)\mathbf{j} + (z - xy)\mathbf{k}$$
   (a) Use Stokes' Theorem to find the circulation around the triangle with vertices $A(1, 0, 0)$, $B(0, 2, 0)$, and $C(0, 0, 1)$ oriented counterclockwise looking from the origin toward the first octant.
   (b) Find the circulation density of $\mathbf{F}$ at the origin in the direction of $\mathbf{k}$.
   (c) Find the unit vector $\mathbf{n}$ such that the circulation density of $\mathbf{F}$ at the origin is maximum in the direction of $\mathbf{n}$.

FOCUS ON CONCEPTS

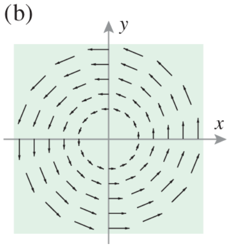
18. (a) Let σ denote the surface of a solid G with $\mathbf{n}$ the outward unit normal vector field to σ. Assume that $\mathbf{F}$ is a vector field with continuous first-order partial derivatives on σ. Prove that
$$\iint_{\sigma} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 0$$
[*Hint:* Let C denote a simple closed curve on σ that separates the surface into two subsurfaces σ_1 and σ_2 that share C as their common boundary. Apply Stokes' Theorem to σ_1 and to σ_2 and add the results.]
   (b) The vector field curl($\mathbf{F}$) is called the ***curl field*** of $\mathbf{F}$. In words, interpret the formula in part (a) as a statement about the flux of the curl field.

19–20 The figures in these exercises show a horizontal layer of the vector field of a fluid flow in which the flow is parallel to the xy-plane at every point and is identical in each layer (i.e., is independent of z). For each flow, state whether you believe that the curl is nonzero at the origin, and explain your reasoning. If you believe that it is nonzero, then state whether it points in the positive or negative z-direction.

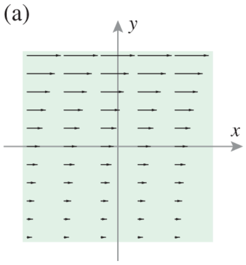
19. (a)



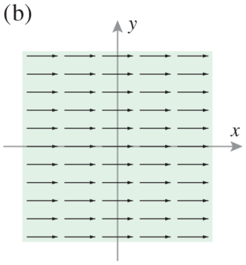
(b)



20. (a)



(b)



21. Let $\mathbf{F}(x, y, z)$ be a conservative vector field in 3-space whose component functions have continuous first partial derivatives. Explain how to use Formula (9) to prove that curl $\mathbf{F} = \mathbf{0}$.

22. In 1831 the physicist Michael Faraday discovered that an electric current can be produced by varying the magnetic flux through a conducting loop. His experiments showed that the electromotive force $\mathbf{E}$ is related to the magnetic induction $\mathbf{B}$ by the equation

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\iint_\sigma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS$$

Use this result to make a conjecture about the relationship between curl $\mathbf{E}$ and $\mathbf{B}$, and explain your reasoning.

C **23.** Let σ be the portion of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$, and let C be the circle $x^2 + y^2 = 1$ that forms the boundary of σ in the xy-plane. Assuming that σ is oriented up, use a CAS to verify Formula (2) in Stokes' Theorem for the vector field

$$\mathbf{F} = (x^2y - z^2)\mathbf{i} + (y^3 - x)\mathbf{j} + (2x + 3z - 1)\mathbf{k}$$

by evaluating the line integral and the surface integral.

24. **Writing** Discuss what it means to say that the curl of a vector field is independent of a coordinate system. Explain how we know this to be true.

25. **Writing** Compare and contrast the Fundamental Theorem of Line Integrals, the Divergence Theorem, and Stokes' Theorem.

✓ **QUICK CHECK ANSWERS 15.8** **1.** $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$; $\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$ **2.** $3\pi a^2$ **3.** (a) $\iint_{\sigma_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$; $\iint_{\sigma_2} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$ (b) $3\pi a^2$ **4.** curl

CHAPTER 15 REVIEW EXERCISES

1. In words, what is a vector field? Give some physical examples of vector fields.

2. (a) Give a physical example of an inverse-square field $\mathbf{F}(\mathbf{r})$ in 3-space.
(b) Write a formula for a general inverse-square field $\mathbf{F}(\mathbf{r})$ in terms of the radius vector $\mathbf{r}$.
(c) Write a formula for a general inverse-square field $\mathbf{F}(x, y, z)$ in 3-space using rectangular coordinates.

3. Find an explicit coordinate expression for the vector field $\mathbf{F}(x, y)$ that at every point $(x, y) \neq (1, 2)$ is the unit vector directed from (x, y) to $(1, 2)$.

4. Find $\nabla\left(\dfrac{x+y}{x-y}\right)$.

5. Find curl$(z\mathbf{i} + x\mathbf{j} + y\mathbf{k})$.

6. Let

$$\mathbf{F}(x, y, z) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j} + \frac{z}{x^2 + y^2}\mathbf{k}$$

Sketch the level surface div $\mathbf{F} = 1$.

7. Assume that C is the parametric curve $x = x(t)$, $y = y(t)$, where t varies from a to b. In each part, express the line integral as a definite integral with variable of integration t.

(a) $\displaystyle\int_C f(x, y)\, dx + g(x, y)\, dy$ (b) $\displaystyle\int_C f(x, y)\, ds$

8. (a) Express the mass M of a thin wire in 3-space as a line integral.
(b) Express the length of a curve as a line integral.

9. Give a physical interpretation of $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

10. State some alternative notations for $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

11–13 Evaluate the line integral. ■

11. $\displaystyle\int_C (x - y)\, ds$; $C : x^2 + y^2 = 1$

12. $\displaystyle\int_C x\, dx + z\, dy - 2y^2\, dz$;
$C : x = \cos t,\ y = \sin t,\ z = t \quad (0 \leq t \leq 2\pi)$

13. $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = (x/y)\mathbf{i} - (y/x)\mathbf{j}$;
$\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} \quad (1 \le t \le 2)$

14. Find the work done by the force field

$$\mathbf{F}(x, y) = y^2\mathbf{i} + xy\mathbf{j}$$

moving a particle from $(0, 0)$ to $(1, 1)$ along the parabola $y = x^2$.

15. State the Fundamental Theorem of Line Integrals, including all required hypotheses.

16. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ where $f(x, y, z) = xy^2z^3$ and

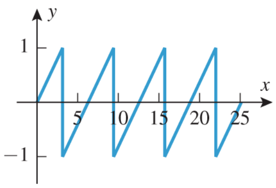
$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + t)\mathbf{j} + \sin(3\pi t/2)\mathbf{k} \quad (0 \le t \le 1)$$

17. Let $\mathbf{F}(x, y) = y\mathbf{i} - 2x\mathbf{j}$.
(a) Find a nonzero function $h(x)$ such that $h(x)\mathbf{F}(x, y)$ is a conservative vector field.
(b) Find a nonzero function $g(y)$ such that $g(y)\mathbf{F}(x, y)$ is a conservative vector field.

18. Let $\mathbf{F}(x, y) = (ye^{xy} - 1)\mathbf{i} + xe^{xy}\mathbf{j}$.
(a) Show that $\mathbf{F}$ is a conservative vector field.
(b) Find a potential function for $\mathbf{F}$.
(c) Find the work performed by the force field on a particle that moves along the sawtooth curve represented by the parametric equations

$$x = t + \sin^{-1}(\sin t), \quad y = (2/\pi)\sin^{-1}(\sin t) \qquad (0 \le t \le 8\pi)$$

(see the accompanying figure).



◀ **Figure Ex-18**

19. State Green's Theorem, including all of the required hypotheses.

20. Express the area of a plane region as a line integral.

21. Let α and β denote angles that satisfy $0 < \beta - \alpha \le 2\pi$ and assume that $r = f(\theta)$ is a smooth polar curve with $f(\theta) > 0$ on the interval $[\alpha, \beta]$. Use the formula

$$A = \frac{1}{2}\int_C -y\,dx + x\,dy$$

to find the area of the region R enclosed by the curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$.

22. (a) Use Green's Theorem to prove that

$$\int_C f(x)\,dx + g(y)\,dy = 0$$

if f and g are differentiable functions and C is a simple, closed, piecewise smooth curve.
(b) What does this tell you about the vector field $\mathbf{F}(x, y) = f(x)\mathbf{i} + g(y)\mathbf{j}$?

23. Assume that σ is the parametric surface

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

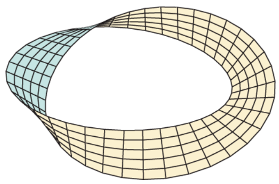
where (u, v) varies over a region R. Express the surface integral

$$\iint_\sigma f(x, y, z)\,dS$$

as a double integral with variables of integration u and v.

24. Evaluate $\iint_\sigma z\,dS$; $\sigma : x^2 + y^2 = 1\ (0 \le z \le 1)$.

25. Do you think that the surface in the accompanying figure is orientable? Explain your reasoning.



◀ **Figure Ex-25**

26. Give a physical interpretation of $\iint_\sigma \mathbf{F} \cdot \mathbf{n}\,dS$.

27. Find the flux of $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ through the portion of the paraboloid $z = 1 - x^2 - y^2$ that is on or above the xy-plane, with upward orientation.

28. Find the flux of $\mathbf{F}(x, y, z) = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$ through the unit sphere centered at the origin with outward orientation.

29. State the Divergence Theorem and Stokes' Theorem, including all required hypotheses.

30. Let G be a solid with the surface σ oriented by outward unit normals, suppose that ϕ has continuous first and second partial derivatives in some open set containing G, and let $D_\mathbf{n}\phi$ be the directional derivative of ϕ, where $\mathbf{n}$ is an outward unit normal to σ. Show that

$$\iint_\sigma D_\mathbf{n}\phi\,dS = \iiint_G \left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right] dV$$

31. Let σ be the sphere $x^2 + y^2 + z^2 = 1$, let $\mathbf{n}$ be an inward unit normal, and let $D_\mathbf{n} f$ be the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$. Use the result in Exercise 30 to evaluate the surface integral

$$\iint_\sigma D_\mathbf{n} f\,dS$$

32. Use Stokes' Theorem to evaluate $\iint_\sigma \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\,dS$ where $\mathbf{F}(x, y, z) = (z - y)\mathbf{i} + (x + z)\mathbf{j} - (x + y)\mathbf{k}$ and σ is the portion of the paraboloid $z = 2 - x^2 - y^2$ on or above the plane $z = 1$, with upward orientation.

33. Let $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$ and suppose that f, g, and h are continuous and have continuous first partial derivatives in a region. It was shown in Exercise 31 of Section 15.3 that if $\mathbf{F}$ is conservative in the region, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

there. Use this result to show that if **F** is conservative in an open spherical region, then curl $\mathbf{F} = \mathbf{0}$ in that region.

34–35 With the aid of Exercise 33, determine whether **F** is conservative.

34. (a) $\mathbf{F}(x, y, z) = z^2\mathbf{i} + e^{-y}\mathbf{j} + 2xz\mathbf{k}$
(b) $\mathbf{F}(x, y, z) = xy\mathbf{i} + x^2\mathbf{j} + \sin z\mathbf{k}$

35. (a) $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$
(b) $\mathbf{F}(x, y, z) = z\mathbf{i} + 2yz\mathbf{j} + y^2\mathbf{k}$

36. As discussed in Example 1 of Section 15.1, *Coulomb's law* states that the electrostatic force $\mathbf{F}(\mathbf{r})$ that a particle of charge Q exerts on a particle of charge q is given by the formula

$$\mathbf{F}(\mathbf{r}) = \frac{qQ}{4\pi\epsilon_0\|\mathbf{r}\|^3}\mathbf{r}$$

where $\mathbf{r}$ is the radius vector from Q to q and ϵ_0 is the permittivity constant.
(a) Express the vector field $\mathbf{F}(\mathbf{r})$ in coordinate form $\mathbf{F}(x, y, z)$ with Q at the origin.
(b) Find the work performed by the force field **F** on a charge q that moves along a straight line from $(3, 0, 0)$ to $(3, 1, 5)$.

CHAPTER 15 MAKING CONNECTIONS

Assume that the motion of a particle of mass m is described by a smooth vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad (a \le t \le b)$$

where t denotes time. Let C denote the graph of the vector-valued function and let $v(t)$ and $\mathbf{a}(t)$ denote the respective speed and acceleration of the particle at time t.

1. We will say that the particle is moving "freely" under the influence of a force field $\mathbf{F}(x, y, z)$, provided **F** is the *only* force acting on the particle. In this case Newton's Second Law of Motion becomes

$$\mathbf{F}(x(t), y(t), z(t)) = m\mathbf{a}(t)$$

Use Theorem 12.6.2 to prove that when the particle is moving freely

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_C m\left(\frac{dv}{dt}\right)ds = m\int_a^b v(t)\left(\frac{dv}{dt}\right)dt\\ &= m\int_a^b \frac{d}{dt}\left(\frac{1}{2}[v(t)]^2\right)dt\\ &= \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2\end{aligned}$$

This tells us that the work performed by **F** on the particle is equal to the change in kinetic energy of the particle.

2. Suppose that the particle moves along a *prescribed* curve C under the influence of a force field $\mathbf{F}(x, y, z)$. In addition to **F**, the particle will experience a concurrent "support force" $\mathbf{S}(x(t), y(t), z(t))$ from the curve. (Imagine a roller-coaster car falling without friction under the influence of the gravitational force **F**. The tracks of the coaster provide the support force **S**.) In this case we will say that the particle has a "constrained" motion under the influence of **F**. Prove that the work performed by **F** on a particle with constrained motion is also equal to the change in kinetic energy of the particle. [*Hint:* Apply the argument of Exercise 1 to the resultant force $\mathbf{F} + \mathbf{S}$ on the particle. Use the fact that at each point on C, **S** will be normal to the curve.]

3. Suppose that **F** is a conservative force field. Use Exercises 1 and 2, along with the discussion in Section 15.3, to develop the conservation of energy principle for both free and constrained motion under **F**.

4. As shown in the accompanying figure, a girl with mass m is sliding down a smooth (frictionless) playground slide that is inclined at an angle of θ with the horizontal. If the acceleration due to gravity is g and the length of the slide is l, prove that the speed of the child when she reaches the base of the slide is $v = \sqrt{2gl\sin\theta}$. Assume that she starts from rest at the top of the slide.

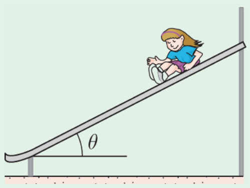


Figure Ex-4

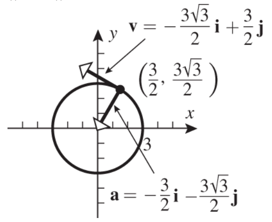
EXPANDING THE CALCULUS HORIZON

To learn how the topics in this chapter can be used to model hurricane behavior, see the module entitled **Hurricane Modeling** at:

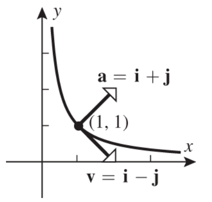
www.wiley.com/college/anton

► Exercise Set 12.6 (Page 790)

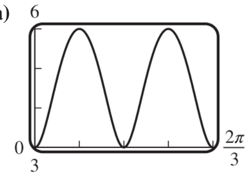
1. $\mathbf{v}(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j}$
$\mathbf{a}(t) = -3\cos t\mathbf{i} - 3\sin t\mathbf{j}$
$\|\mathbf{v}(t)\| = 3$



3. $\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$
$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$
$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$



5. $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \|\mathbf{v}\| = \sqrt{3}, \mathbf{a} = \mathbf{j} + 2\mathbf{k}$
7. $\mathbf{v} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}, \|\mathbf{v}\| = \sqrt{5}, \mathbf{a} = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$
13. minimum speed $3\sqrt{2}$ when $\mathbf{r} = 24\mathbf{i} + 8\mathbf{j}$
15. **(a)**



(b) maximum speed $= 6$, minimum speed $= 3$
(d) The maximum speed first occurs when $t = \pi/6$.

17. $\mathbf{v}(t) = (1 - \sin t)\mathbf{i} + (\cos t - 1)\mathbf{j}$;
$\mathbf{r}(t) = (t + \cos t - 1)\mathbf{i} + (\sin t - t + 1)\mathbf{j}$
19. $\mathbf{v}(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j} + e^t\mathbf{k}$;
$\mathbf{r}(t) = (t - \sin t - 1)\mathbf{i} + (1 - \cos t)\mathbf{j} + e^t\mathbf{k}$
21. $15°$ 23. **(a)** $0.7\mathbf{i} + 2.7\mathbf{j} - 3.4\mathbf{k}$ **(b)** $\mathbf{r}_0 = -0.7\mathbf{i} - 2.9\mathbf{j} + 4.8\mathbf{k}$
25. $\Delta\mathbf{r} = 8\mathbf{i} + \frac{26}{3}\mathbf{j}, s = (13\sqrt{13} - 5\sqrt{5})/3$
27. $\Delta\mathbf{r} = 2\mathbf{i} - \frac{2}{3}\mathbf{j} + \sqrt{2}\ln 3\mathbf{k}; s = \frac{8}{3}$
31. **(a)** $a_T = 0, a_N = \sqrt{2}$ **(b)** $a_T\mathbf{T} = \mathbf{0}, a_N\mathbf{N} = \mathbf{i} + \mathbf{j}$ **(c)** $1/\sqrt{2}$
33. **(a)** $a_T = 2\sqrt{5}, a_N = 2\sqrt{5}$ **(b)** $a_T\mathbf{T} = 2\mathbf{i} + 4\mathbf{j}, a_N\mathbf{N} = 4\mathbf{i} - 2\mathbf{j}$
(c) $2/\sqrt{5}$
35. **(a)** $a_T = -7/\sqrt{6}, a_N = \sqrt{53/6}$
(b) $a_T\mathbf{T} = -\frac{7}{6}(\mathbf{i} - 2\mathbf{j} + \mathbf{k}), a_N\mathbf{N} = \frac{13}{6}\mathbf{i} + \frac{5}{3}\mathbf{j} + \frac{7}{6}\mathbf{k}$ **(c)** $\dfrac{\sqrt{53}}{6\sqrt{6}}$
37. $a_T = -3, a_N = 2, \mathbf{T} = -\mathbf{j}, \mathbf{N} = \mathbf{i}$ 39. $-3/2$
41. $a_N = 8.41 \times 10^{10}$ km/s^2
43. $a_N = 18/(1 + 4x^2)^{3/2}$ 45. $a_N = 0$

Responses to True–False questions may be abridged to save space.

47. True; the velocity and unit tangent vectors have the same direction, so are parallel.
49. False; in this case the velocity and acceleration vectors will be parallel, but they may have opposite direction.
53. ≈ 257.20 N
55. $40\sqrt{3}$ ft 57. 800 ft/s 59. $15°$ or $75°$ 61. **(c)** ≈ 14.942 ft
63. **(a)** $\rho \approx 176.78$ m **(b)** $\frac{125}{4}$ m
65. **(b)** R is maximum when $\alpha = 45°$, maximum value v_0^2/g
67. **(a)** 2.62 s **(b)** 181.5 ft
69. **(a)** $v_0 \approx 83$ ft/s, $\alpha \approx 8°$ **(b)** 268.76 ft

► Exercise Set 12.7 (Page 799)

7. 7.75 km/s 9. 10.88 km/s
11. **(a)** minimum distance $= 220{,}680$ mi,
maximum distance $= 246{,}960$ mi **(b)** 27.5 days
13. **(a)** 17,224 mi/h **(b)** $e \approx 0.071$, apogee altitude $= 819$ mi

► Chapter 12 Review Exercises (Page 801)

3. the circle of radius 3 in the xy-plane, with center at the origin
5. a parabola in the plane $x = -2$, vertex at $(-2, 0, -1)$, opening upward
11. $x = 1 + t, y = -t, z = t$ 13. $(\sin t)\mathbf{i} - (\cos t)\mathbf{j} + \mathbf{C}$
15. $y(t) = \left(\frac{1}{3}t^3 + 1\right)\mathbf{i} + (t^2 + 1)\mathbf{j}$ 17. $15/4$
19. $\mathbf{r}(s) = \dfrac{s-3}{3}\mathbf{i} + \dfrac{12-2s}{3}\mathbf{j} + \dfrac{9+2s}{3}\mathbf{k}$ 25. $3/5$ 27. 0
29. **(a)** speed **(b)** distance traveled
(c) distance of the particle from the origin
33. **(a)** $\mathbf{r}(t) = \left(\frac{1}{6}t^4 + t\right)\mathbf{i} + \left(\frac{1}{2}t^2 + 2t\right)\mathbf{j} - \left(\frac{1}{4}\cos 2t + t - \frac{1}{4}\right)\mathbf{k}$
(b) 3.475 35. 10.65 km/s 37. 24.78 ft

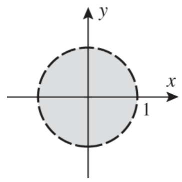
► Chapter 12 Making Connections (Page 802)

Where correct answers to a Making Connections exercise may vary, no answer is listed. Sample answers for these questions are available on the Book Companion Site.

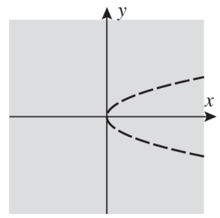
1. **(c)** (i) $\mathbf{N} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$ (ii) $\mathbf{N} = -\mathbf{j}$
2. **(b)** (i) $\mathbf{N} = -\sin t\mathbf{i} - \cos t\mathbf{j}$
(ii) $\mathbf{N} = \dfrac{-(4t + 18t^3)\mathbf{i} + (2 - 18t^4)\mathbf{j} + (6t + 12t^3)\mathbf{k}}{2\sqrt{81t^8 + 117t^6 + 54t^4 + 13t^2 + 1}}$
3. **(c)** $\kappa(s) \to +\infty$, so the spiral winds ever tighter.
4. semicircle: 53.479 ft; quarter-circle: 60.976 ft; point: 64.001 ft

► Exercise Set 13.1 (Page 812)

1. **(a)** 5 **(b)** 3 **(c)** 1 **(d)** -2 **(e)** $9a^3 + 1$ **(f)** $a^3b^2 - a^2b^3 + 1$
3. **(a)** $x^2 - y^2 + 3$ **(b)** $3x^3y^4 + 3$ 5. $x^3e^{x^3(3y+1)}$
7. **(a)** $t^2 + 3t^{10}$ **(b)** 0 **(c)** 3076
9. **(a)** 2.5 mg/L **(b)** $C(100, t) = 20(e^{-0.2t} - e^{-t})$
(c) $C(x, 1) = 0.2x(e^{-0.2} - e^{-1}) \approx 0.09x$
11. **(a)** WCI $= 17.8°$F **(b)** WCI $= 22.6°$F 13. **(a)** $30°$F
(b) $22.5°$F
15. **(a)** 66% **(b)** 73.5% **(c)** 60.6%
17. **(a)** 19 **(b)** -9 **(c)** 3 **(d)** $a^6 + 3$ **(e)** $-t^8 + 3$ **(f)** $(a + b)(a - b)^2b^3 + 3$
19. $(y + 1)e^{x^2(y+1)z^2}$ 21. **(a)** $80\sqrt{\pi}$ **(b)** $n(n + 1)/2$
23.



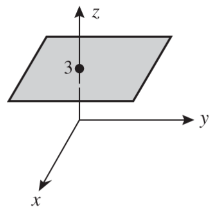
25.



27. **(a)** all points above or on the line $y = -2$ **(b)** all points on or within the sphere $x^2 + y^2 + z^2 = 25$ **(c)** all points in 3-space

Responses to True–False questions may be abridged to save space.

29. True; the interval [0, 1] is the intersection of the domains of $\sin^{-1} t$ and $\sqrt{t}$.
31. False; the natural domain is an infinite solid cylinder.
33.



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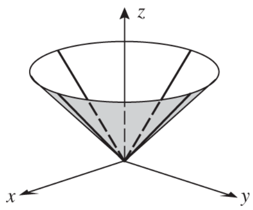

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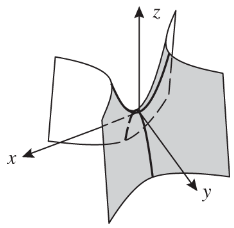

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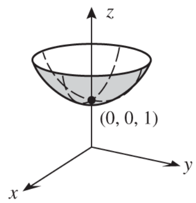

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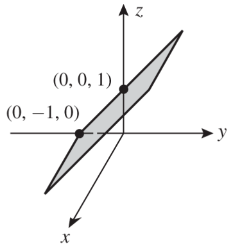


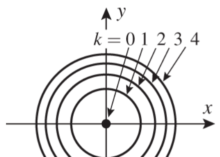
43. **(a)** hyperbolas
(b) parabolas
(c) noncircular ellipses
(d) lines

45. **(a)** \$130
(b) \$275

47. **(a)** $1 - x^2 - y^2$
(b) $\sqrt{x^2 + y^2}$
(c) $x^2 + y^2$

49. **(a)** A **(b)** B **(c)** increase **(d)** decrease **(e)** increase
(f) decrease

51.



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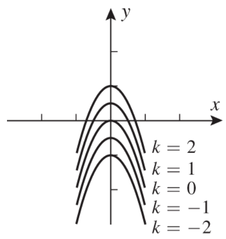

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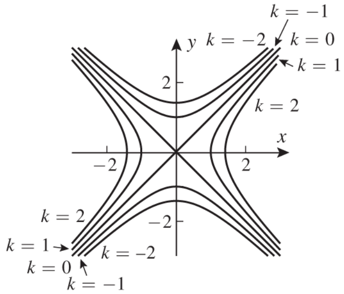

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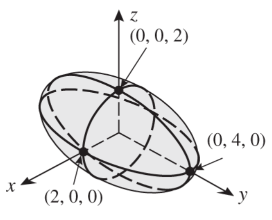

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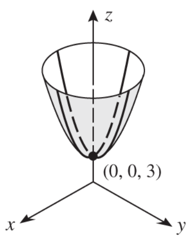


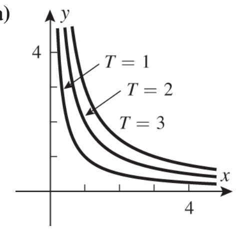
61. concentric spheres, common center at (2, 0, 0)

63. concentric cylinders, common axis the y-axis

65. **(a)** $x^2 - 2x^3 + 3xy = 0$ **(b)** $x^2 - 2x^3 + 3xy = 0$
(c) $x^2 - 2x^3 + 3xy = -18$

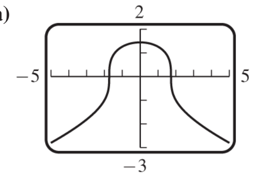
67. **(a)** $x^2 + y^2 - z = 5$ **(b)** $x^2 + y^2 - z = -2$ **(c)** $x^2 + y^2 - z = 0$

69. **(a)**

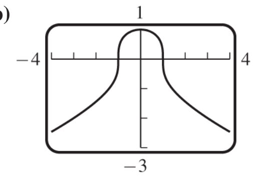


(b) the path $xy = 4$

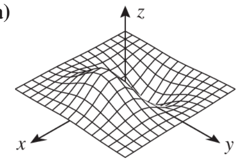
71. **(a)**



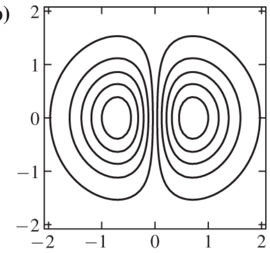
(b)



73. **(a)**



(b)



75. **(a)** The graph of g is the graph of f shifted one unit in the positive x-direction.
(b) The graph of g is the graph of f shifted one unit up the z-axis.
(c) The graph of g is the graph of f shifted one unit down the y-axis and then inverted with respect to the plane $z = 0$.

▶ Exercise Set 13.2 (Page 823)

1. 35 **3.** −8 **5.** 0

7. **(a)** along $x = 0$ limit does not exist
(b) along $x = 0$ limit does not exist

9. 1 **11.** 0 **13.** 0 **15.** limit does not exist **17.** $\frac{8}{3}$ **19.** 0

21. limit does not exist **23.** 0 **25.** 0 **27.** 0

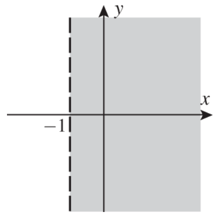
Responses to True–False questions may be abridged to save space.

29. True; by the definition of open set.

31. False; let $f(x, y) = \begin{cases} 1, & x \le 0 \\ -1, & x > 0 \end{cases}$ and let $g(x, y) = -f(x, y)$.

33. **(a)** no **(d)** no; yes **37.** $-\pi/2$ **39.** no

41.



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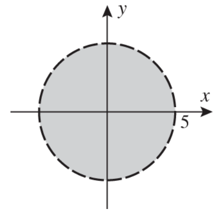

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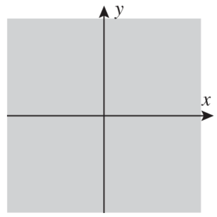

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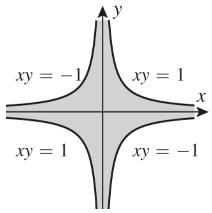


49. all of 3-space **51.** all points not on the cylinder $x^2 + z^2 = 1$

▶ Exercise Set 13.3 (Page 833)

1. **(a)** $9x^2y^2$ **(b)** $6x^3y$ **(c)** $9y^2$ **(d)** $9x^2$ **(e)** $6y$ **(f)** $6x^3$ **(g)** 36 **(h)** 12

3. $18xy - 15x^4y,\ 9x^2 - 3x^5$

5. $(16x + 40)(x^2 + 5x - 2y)^7,\ -16(x^2 + 5x - 2y)^7$

7. $-\dfrac{7}{q}e^{-7p/q},\ \dfrac{7p}{q^2}e^{-7p/q}$

9. $(15x^2y + 7y^2)\cos(5x^3y + 7xy^2),\ (5x^3 + 14xy)\cos(5x^3y + 7xy^2)$

11. **(a)** $\frac{3}{8}$ **(b)** $\frac{1}{4}$ **13.** **(a)** $-4\cos 7$ **(b)** $2\cos 7$

15. $\partial z/\partial x = -4;\ \partial z/\partial y = \frac{1}{2}$ **17.** **(a)** 4.9 **(b)** 1.2

19. $z = f(x, y)$ has II as its graph, f_x has I as its graph, and f_y has III as its graph.

Responses to True–False questions may be abridged to save space.

21. True; on $y = 2$, $f(x, 2) = c$ is a constant function of x.

23. True; z must be a linear function of x and y.

25. $8xy^3e^{x^2y^3},\ 12x^2y^2e^{x^2y^3}$

27. $x^3/(y^{3/5} + x) + 3x^2\ln(1 + xy^{-3/5}),\ -\frac{3}{5}x^4/(y^{8/5} + xy)$

29. $-\dfrac{y(x^2 - y^2)}{(x^2 + y^2)^2},\ \dfrac{x(x^2 - y^2)}{(x^2 + y^2)^2}$

31. $(3/2)x^2y(5x^2 - 7)(3x^5y - 7x^3y)^{-1/2}$

$(1/2)x^3(3x^2-7)(3x^5y-7x^3y)^{-1/2}$

33. $\dfrac{y^{-1/2}}{y^2+x^2}, -\dfrac{xy^{-3/2}}{y^2+x^2} - \dfrac{3}{2}y^{-5/2}\tan^{-1}\left(\dfrac{x}{y}\right)$

35. $-\frac{4}{3}y^2\sec^2 x(y^2\tan x)^{-7/3}, -\frac{8}{3}y\tan x(y^2\tan x)^{-7/3}$

37. $-6, -21$ **39.** $1/\sqrt{17}, 8/\sqrt{17}$

41. **(a)** $2xy^4z^3+y$ **(b)** $4x^2y^3z^3+x$ **(c)** $3x^2y^4z^2+2z$ **(d)** $2y^4z^3+y$ **(e)** $32z^3+1$ **(f)** 438

43. $2z/x, z/y, \ln(x^2y\cos z) - z\tan z$

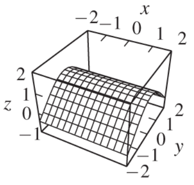
45. $-y^2z^3/(1+x^2y^4z^6), -2xyz^3/(1+x^2y^4z^6), -3xy^2z^2/(1+x^2y^4z^6)$

47. $yze^z\cos(xz), e^z\sin(xz), ye^z(\sin(xz)+x\cos(xz))$

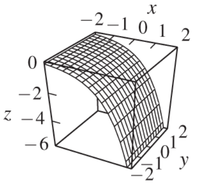
49. $x/\sqrt{x^2+y^2+z^2}, y/\sqrt{x^2+y^2+z^2}, z/\sqrt{x^2+y^2+z^2}$

51. **(a)** e **(b)** $2e$ **(c)** e

53. **(a)**



(b)



55. 4 **57.** -2

59. **(a)** $\partial V/\partial r = 2\pi rh$ **(b)** $\partial V/\partial h = \pi r^2$ **(c)** 48π **(d)** 64π

61. **(a)** $\dfrac{1}{5}\dfrac{\text{lb}}{\text{in}^2\cdot\text{K}}$ **(b)** $-\dfrac{25}{8}\dfrac{\text{in}^5}{\text{lb}}$

63. **(a)** $\dfrac{\partial V}{\partial l}=6$ **(b)** $\dfrac{\partial V}{\partial w}=15$ **(c)** $\dfrac{\partial V}{\partial h}=10$

67. **(a)** $\pm\sqrt{6}/4$ **69.** $-x/z, -y/z$

71. $-\dfrac{2x+yz^2\cos(xyz)}{xyz\cos(xyz)+\sin(xyz)}; -\dfrac{xz^2\cos(xyz)}{xyz\cos(xyz)+\sin(xyz)}$

73. $-x/w, -y/w, -z/w$

75. $-\dfrac{yzw\cos(xyz)}{2w+\sin(xyz)}, -\dfrac{xzw\cos(xyz)}{2w+\sin(xyz)}, -\dfrac{xyw\cos(xyz)}{2w+\sin(xyz)}$

77. $e^{x^2}, -e^{y^2}$

79. $f_x(x,y) = 2xy^3\sin(x^6y^9), f_y(x,y) = 3x^2y^2\sin(x^6y^9)$

81. **(a)** $-\dfrac{\cos y}{4\sqrt{x^3}}$ **(b)** $-\sqrt{x}\cos y$ **(c)** $-\dfrac{1}{2\sqrt{x}}\sin y$ **(d)** $-\dfrac{1}{2\sqrt{x}}\sin y$

83. **(a)** $6\cos(3x^2+6y^2) - 36x^2\sin(3x^2+6y^2)$
(b) $12\cos(3x^2+6y^2) - 144y^2\sin(3x^2+6y^2)$
(c) $-72xy\sin(3x^2+6y^2)$ **(d)** $-72xy\sin(3x^2+6y^2)$

85. $-32y^3$ **87.** $-e^x\sin y$ **89.** $\dfrac{20}{(4x-5y)^2}$ **91.** $\dfrac{2(x-y)}{(x+y)^3}$

93. **(a)** $\dfrac{\partial^3 f}{\partial x^3}$ **(b)** $\dfrac{\partial^3 f}{\partial y^2\partial x}$ **(c)** $\dfrac{\partial^4 f}{\partial x^2\partial y^2}$ **(d)** $\dfrac{\partial^4 f}{\partial y^3\partial x}$

95. **(a)** $30xy^4-4$ **(b)** $60x^2y^3$ **(c)** $60x^3y^2$

97. **(a)** -30 **(b)** -125 **(c)** 150

99. **(a)** $15x^2y^4z^7+2y$ **(b)** $35x^3y^4z^6+3y^2$ **(c)** $21x^2y^5z^6$ **(d)** $42x^3y^5z^5$ **(e)** $140x^3y^3z^6+6y$ **(f)** $30xy^4z^7$ **(g)** $105x^2y^4z^6$ **(h)** $210xy^4z^6$

107. $\dfrac{\partial f}{\partial v} = 8vw^3x^4y^5, \dfrac{\partial f}{\partial w} = 12v^2w^2x^4y^5, \dfrac{\partial f}{\partial x} = 16v^2w^3x^3y^5,$
$\dfrac{\partial f}{\partial y} = 20v^2w^3x^4y^4$

109. $\dfrac{\partial f}{\partial v_1} = \dfrac{2v_1}{v_3^2+v_4^2}, \dfrac{\partial f}{\partial v_2} = \dfrac{-2v_2}{v_3^2+v_4^2}, \dfrac{\partial f}{\partial v_3} = \dfrac{-2v_3(v_1^2-v_2^2)}{(v_3^2+v_4^2)^2},$
$\dfrac{\partial f}{\partial v_4} = \dfrac{-2v_4(v_1^2-v_2^2)}{(v_3^2+v_4^2)^2}$

111. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 0 **(e)** $2(1+yw)e^{yw}\sin z\cos z$ **(f)** $2xw(2+yw)e^{yw}\sin z\cos z$

113. $-i\sin(x_1+2x_2+\cdots+nx_n)$

115. **(a)** xy-plane, $12x^2+6x$ **(b)** $y\neq 0, -3x^2/y^2$

117. $f_x(2,-1)=11, f_y(2,-1)=-8$

119. **(b)** does not exist if $y\neq 0$ and $x=-y$

► Exercise Set 13.4 (Page 843)

1. 5.04 **3.** 4.14 **9.** $dz = 7\,dx - 2\,dy$ **11.** $dz = 3x^2y^2\,dx + 2x^3y\,dy$

13. $dz = \dfrac{y}{1+x^2y^2}dx + \dfrac{x}{1+x^2y^2}dy$ **15.** $dw = 8\,dx - 3\,dy + 4\,dz$

17. $dw = 3x^2y^2z\,dx + 2x^3yz\,dy + x^3y^2\,dz$

19. $dw = \dfrac{yz}{1+x^2y^2z^2}dx + \dfrac{xz}{1+x^2y^2z^2}dy + \dfrac{xy}{1+x^2y^2z^2}dz$

21. $df = 0.10, \Delta f = 0.1009$ **23.** $df = 0.03, \Delta f \approx 0.029412$

25. $df = 0.96, \Delta f \approx 0.97929$

Responses to True–False questions may be abridged to save space.

27. False; see the discussion at the beginning of this section.

29. True; see Theorems 13.4.3 and 13.4.4.

31. The increase in the area of the rectangle is given by the sum of the areas of the three small rectangles, and the total differential is given by the sum of the areas of the upper left and lower right rectangles.

33. **(a)** $L = \frac{1}{5} - \frac{4}{125}(x-4) - \frac{3}{125}(y-3)$ **(b)** 0.000176603

35. **(a)** $L=0$ **(b)** 0.0024

37. **(a)** $L = 6+6(x-1)+3(y-2)+2(z-3)$ **(b)** -0.00481

39. **(a)** $L = e+e(x-1)-e(y+1)-e(z+1)$ **(b)** 0.01554

45. 0.5 **47.** $1, 1, -1, 2$ **49.** $(-1, 1)$ **51.** $(1, 0, 1)$ **53.** 8%

55. 0.3% **57.** 1.2%

59. **(a)** $(r+s)\%$ **(b)** $(r+s)\%$ **(c)** $(2r+3s)\%$ **(d)** $\left(3r+\dfrac{s}{2}\right)\%$

61. $\approx 39\text{ ft}^2$

► Exercise Set 13.5 (Page 852)

1. $42t^{13}$ **3.** $3t^{-2}\sin(1/t)$ **5.** $-\frac{10}{3}t^{7/3}e^{1-t^{10/3}}$ **7.** $\dfrac{dw}{dt} = 165t^{32}$

9. $-2t\cos t^2$ **11.** 3264 **13.** 0

17. $24u^2v^2-16uv^3-2v+3, 16u^3v-24u^2v^2-2u-3$

19. $-\dfrac{2\sin u}{3\sin v}, -\dfrac{2\cos u\cos v}{3\sin^2 v}$ **21.** $e^u, 0$

23. $3r^2\sin\theta\cos^2\theta - 4r^3\sin^3\theta\cos\theta,$
$-2r^3\sin^2\theta\cos\theta + r^4\sin^4\theta + r^3\cos^3\theta - 3r^4\sin^2\theta\cos^2\theta$

25. $\dfrac{x^2+y^2}{4x^2y^3}, \dfrac{y^2-3x^2}{4xy^4}$ **27.** $\dfrac{\partial z}{\partial r} = \dfrac{2r\cos^2\theta}{r^2\cos^2\theta+1}, \dfrac{\partial z}{\partial\theta} = \dfrac{-2r^2\cos\theta\sin\theta}{r^2\cos^2\theta+1}$

29. $\dfrac{dw}{d\rho} = 2\rho(4\sin^2\phi+\cos^2\phi), \dfrac{\partial w}{\partial\phi} = 6\rho^2\sin\phi\cos\phi, \dfrac{dw}{d\theta} = 0$

31. $-\pi$ **33.** $\sqrt{3}e^{\sqrt{3}}, (2-4\sqrt{3})e^{\sqrt{3}}$ **35.** -0.779 rad/s

Responses to True–False questions may be abridged to save space.

37. False; the symbols ∂z and ∂x have no individual meaning.

39. False; consider $z = xy, x = t, y = t$.

41. $-\dfrac{2xy^3}{3x^2y^2-\sin y}$

43. $-\dfrac{ye^{xy}}{xe^{xy}+ye^y+e^y}$ **45.** $\dfrac{2x+yz}{6yz-xy}, \dfrac{xz-3z^2}{6yz-xy}$

47. $\dfrac{ye^x}{15\cos 3z+3}, \dfrac{e^x}{15\cos 3z+3}$

61. $\dfrac{\partial w}{\partial\rho} = (\sin\phi\cos\theta)\dfrac{\partial w}{\partial x} + (\sin\phi\sin\theta)\dfrac{\partial w}{\partial y} + (\cos\phi)\dfrac{\partial w}{\partial z},$
$\dfrac{\partial w}{\partial\phi} = (\rho\cos\phi\cos\theta)\dfrac{\partial w}{\partial x} + (\rho\cos\phi\sin\theta)\dfrac{\partial w}{\partial y} - (\rho\sin\phi)\dfrac{\partial w}{\partial z},$
$\dfrac{\partial w}{\partial\theta} = -(\rho\sin\phi\sin\theta)\dfrac{\partial w}{\partial x} + (\rho\sin\phi\cos\theta)\dfrac{\partial w}{\partial y}$

65. **(a)** $\dfrac{dw}{dt} = \sum_{i=1}^{4}\dfrac{\partial w}{\partial x_i}\dfrac{dx_i}{dt}$ **(b)** $\dfrac{\partial w}{\partial v_j} = \sum_{i=1}^{4}\dfrac{\partial w}{\partial x_i}\dfrac{\partial x_i}{\partial v_j}, j = 1, 2, 3$

► Exercise Set 13.6 (Page 863)

1. $6\sqrt{2}$ **3.** $-3/\sqrt{10}$ **5.** -320 **7.** $-314/741$ **9.** 0 **11.** $-8\sqrt{2}$

13. $\sqrt{2}/4$ **15.** $5/\sqrt{3}$ **17.** $-8/63$ **19.** $1/2+\sqrt{3}/8$ **21.** $2\sqrt{2}$

23. $1/\sqrt{5}$ **25.** $-\frac{3}{2}e$ **27.** $3/\sqrt{11}$ **29.** **(a)** 5 **(b)** 10 **(c)** $-5\sqrt{5}$

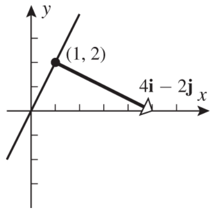
31. III **33.** $\cos(7y^2-7xy)(-7y\mathbf{i}+(14y-7x)\mathbf{j})$

35. $\left(\dfrac{-84y}{(6x-7y)^2}\right)\mathbf{i} + \left(\dfrac{84x}{(6x-7y)^2}\right)\mathbf{j}$ **37.** $-9x^8\mathbf{i} - 3y^2\mathbf{j} + 12z^{11}\mathbf{k}$

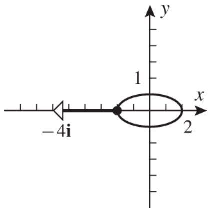
39. $\nabla w = \dfrac{x}{x^2+y^2+z^2}\mathbf{i} + \dfrac{y}{x^2+y^2+z^2}\mathbf{j} + \dfrac{z}{x^2+y^2+z^2}\mathbf{k}$

41. $40\mathbf{i} + 32\mathbf{j}$ **43.** $-36\mathbf{i} - 12\mathbf{j}$ **45.** $4(\mathbf{i} + \mathbf{j} + \mathbf{k})$

47.



49.



51. $\pm(-4\mathbf{i} + \mathbf{j})/\sqrt{17}$ **53.** $\mathbf{u} = (3\mathbf{i} - 2\mathbf{j})/\sqrt{13}$, $\|\nabla f(-1, 1)\| = 4\sqrt{13}$

55. $\mathbf{u} = (4\mathbf{i} - 3\mathbf{j})/5$, $\|\nabla f(4, -3)\| = 1$ **57.** $\dfrac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$, $3\sqrt{2}$

59. $\dfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$, $\dfrac{1}{\sqrt{2}}$

61. $\mathbf{u} = -(\mathbf{i} + 3\mathbf{j})/\sqrt{10}$, $-\|\nabla f(-1, -3)\| = -2\sqrt{10}$

63. $\mathbf{u} = (3\mathbf{i} - \mathbf{j})/\sqrt{10}$, $-\|\nabla f(\pi/6, \pi/4)\| = -\sqrt{5}$

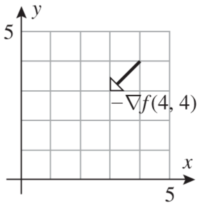
65. $(\mathbf{i} - 11\mathbf{j} + 12\mathbf{k})/\sqrt{266}$, $-\sqrt{266}$

Responses to True–False questions may be abridged to save space.

67. False; they are equal. **69.** False; let $\mathbf{u} = \mathbf{i}$ and let $f(x, y) = y$.

71. $8/\sqrt{29}$

73. **(a)** $\approx 1/\sqrt{2}$
(b)



75. $9x^2 + y^2 = 9$

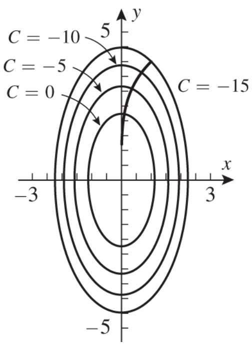
77. $36/\sqrt{17}$

79. **(a)** $2e^{-\pi/2}\mathbf{i}$

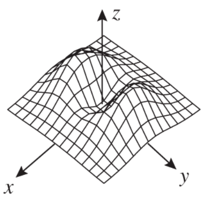
81. $-\frac{5}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

87. $x(t) = e^{-8t}$, $y(t) = 4e^{-2t}$

89.



91. **(a)**



(c) $\nabla f = [2x - 2x(x^2 + 3y^2)]e^{-(x^2+y^2)}\mathbf{i} + [6y - 2y(x^2 + 3y^2)]e^{-(x^2+y^2)}\mathbf{j}$
(d) $x = y = 0$ or $x = 0, y = \pm 1$ or $x = \pm 1, y = 0$

► Exercise Set 13.7 (Page 870)

1. **(a)** $x + y + 2z = 6$ **(b)** $x = 2 + t, y = 2 + t, z = 1 + 2t$
(c) $35.26°$

3. tangent plane: $3x - 4z = -25$;
normal line: $x = -3 + (3t/4), y = 0, z = 4 - t$

5. tangent plane: $9x - 4y - 10z = -76$;
normal line: $x = -4 + 9t, y = 5 - 4t, z = 2 - 10t$

7. tangent plane: $48x - 14y - z = 64$;
normal line: $x = 1 + 48t, y = -2 - 14t, z = 12 - t$

9. tangent plane: $x - y - z = 0$;
normal line: $x = 1 + t, y = -t, z = 1 - t$

11. tangent plane: $3y - z = -1$;
normal line: $x = \pi/6, y = 3t, z = 1 - t$

13. **(a)** all points on the x-axis or y-axis **(b)** $(0, -2, -4)$

15. $\left(\frac{1}{2}, -2, -\frac{3}{4}\right)$ **17.** **(a)** $(-2, 1, 5), (0, 3, 9)$ **(b)** $\dfrac{4}{3\sqrt{14}}, \dfrac{4}{\sqrt{222}}$

Responses to True–False questions may be abridged to save space.

19. False; they need only be parallel.

21. True; see Formula (15) of Section 13.4.

23. $\pm\dfrac{1}{\sqrt{227}}(\mathbf{i} - \mathbf{j} - 15\mathbf{k})$ **27.** $(1, 2/3, 2/3), (-1, -2/3, -2/3)$

29. $x = 1 + 8t, y = -1 + 5t, z = 2 + 6t$

31. $x = 3 + 4t, y = -3 - 4t, z = 4 - 3t$

► Exercise Set 13.8 (Page 879)

1. **(a)** minimum at $(2, -1)$, no maxima
(b) maximum at $(0, 0)$, no minima **(c)** no maxima or minima

3. minimum at $(3, -2)$, no maxima **5.** relative minimum at $(0, 0)$

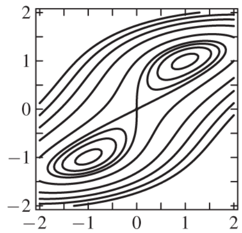
7. relative minimum at $(0, 0)$; saddle points at $(\pm 2, 1)$

9. saddle point at $(1, -2)$ **11.** relative minimum at $(2, -1)$

13. relative minima at $(-1, -1)$ and $(1, 1)$ **15.** saddle point at $(0, 0)$

17. no critical points **19.** relative maximum at $(-1, 0)$

21. saddle point at $(0, 0)$; relative minima at $(1, 1)$ and $(-1, -1)$



Responses to True–False questions may be abridged to save space.

23. False; let $f(x, y) = y$.

25. True; this follows from Theorem 13.8.6.

27. **(b)** relative minimum at $(0, 0)$

31. absolute maximum 0,
absolute minimum -12

33. absolute maximum 3,
absolute minimum -1

35. absolute maximum $\frac{33}{4}$,
absolute minimum $-\frac{1}{4}$

37. 16, 16, 16

39. maximum at $(1, 2, 2)$

41. $2a/\sqrt{3}, 2a/\sqrt{3}, 2a/\sqrt{3}$

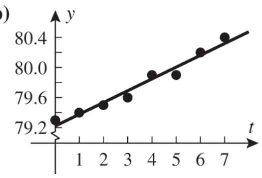
43. length and width 2 ft, height 4 ft

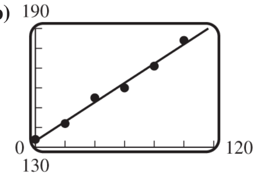
45. **(a)** $x = 0$: minimum -3, maximum 0;
$x = 1$: minimum 3, maximum $13/3$;
$y = 0$: minimum 0, maximum 4;
$y = 1$: minimum -3, maximum 3
(b) $y = x$: minimum 0, maximum 3;
$y = 1 - x$: maximum 4, minimum -3
(c) minimum -3, maximum $13/3$

47. length and width $\sqrt[3]{2V}$, height $\sqrt[3]{2V}/2$ **51.** $y = \frac{3}{4}x + \frac{19}{12}$

53. $y = 0.5x + 0.8$

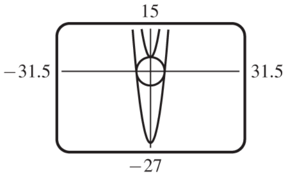
55. **(a)** $y = 79.22 + 0.1571t$ **(b)**
(c) about 81.6 years



57. **(a)** $P = \dfrac{2798}{21} + \dfrac{171}{350}T$ **(b)**


(c) $T \approx -272.7096°\,\text{C}$

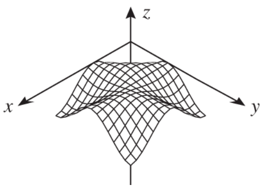
▶ Exercise Set 13.9 (Page 889)

1. **(a)** 4 **3.** **(a)**


(c) maximum $\frac{101}{4}$, minimum -5

5. maximum $\sqrt{2}$ at $(-\sqrt{2}, -1)$ and $(\sqrt{2}, 1)$, minimum $-\sqrt{2}$ at $(-\sqrt{2}, 1)$ and $(\sqrt{2}, -1)$
7. maximum $\sqrt{2}$ at $(1/\sqrt{2}, 0)$, minimum $-\sqrt{2}$ at $(-1/\sqrt{2}, 0)$
9. maximum 6 at $\left(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3}\right)$, minimum -6 at $\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)$
11. maximum is $1/(3\sqrt{3})$ at $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $(1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, and $(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$; minimum is $-1/(3\sqrt{3})$ at $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, and $(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$

Responses to True–False questions may be abridged to save space.

13. False; a Lagrange multiplier is a scalar.
15. False; we must solve three equations in three unknowns.
17. $\left(\frac{3}{10}, -\frac{3}{5}\right)$ **19.** $\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{6}\right)$
21. $(3, 6)$ is closest and $(-3, -6)$ is farthest **23.** $5(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$
25. 9, 9, 9 **27.** $(\pm\sqrt{5}, 0, 0)$ **29.** length and width 2 ft, height 4 ft
33. **(a)** $\alpha = \beta = \gamma = \pi/3$, maximum $1/8$
(b)


▶ Chapter 13 Review Exercises (Page 891)

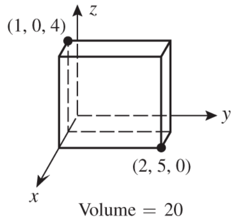
1. **(a)** xy **(b)** $e^{r+s}\ln(rs)$
5. **(a)** not defined on line $y = x$ **(b)** not continuous
9. **(a)** 12 Pa/min **(b)** 240 Pa/min
15. df (the differential of f) is an approximation for Δf (the change in f)
17. $dV = -0.06667\ \text{m}^3$; $\Delta V = -0.07267\ \text{m}^3$ **19.** 2
21. $\dfrac{-f_y^2 f_{xx} + 2f_x f_y f_{xy} - f_x^2 f_{yy}}{f_y^3}$ **25.** $\dfrac{7}{2} + \dfrac{4}{5}\ln 2$ **27.** $-7/\sqrt{5}$
29. $(0, 0, 2), (1, 1, 1), (-1, -1, 1)$ **31.** $\left(-\frac{1}{3}, -\frac{1}{2}, 2\right)$
33. relative minimum at $(15, -8)$
35. saddle point at $(0, 0)$, relative minimum at $(3, 9)$
37. absolute maximum of 4 at $(\pm 1, \pm 2)$, absolute minimum of 0 at $(\pm\sqrt{2}, 0)$ and $(0, \pm 2\sqrt{2})$
39. $I_1 : I_2 : I_3 = \dfrac{1}{R_1} : \dfrac{1}{R_2} : \dfrac{1}{R_3}$
41. **(a)** $\partial P/\partial L = c\alpha L^{\alpha-1}K^{\beta}, \partial P/\partial K = c\beta L^{\alpha}K^{\beta-1}$

▶ Chapter 13 Making Connections (Page 893)

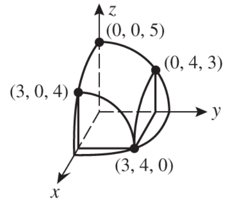
Answers are provided in the Student Solutions Manual.

▶ Exercise Set 14.1 (Page 900)

1. 7 **3.** 2 **5.** 2 **7.** 3 **9.** $1 - \ln 2$ **11.** $\dfrac{1 - \ln 2}{2}$ **13.** 0 **15.** $\frac{1}{3}$
17. **(a)** $37/4$ **(b)** exact value = $28/3$; differ by $1/12$

19.


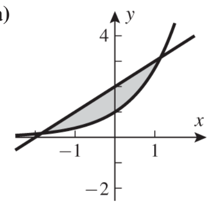
Volume = 20

21.


Responses to True–False questions may be abridged to save space.

23. False; ΔA_k is the area of such a rectangular region.
25. False; $\displaystyle\iint_R f(x, y)\,dA = \int_1^5 \int_2^4 f(x, y)\,dy\,dx.$
29. 19 **31.** 8 **33.** $\dfrac{1}{3\pi}$ **35.** 48 **37.** $1 - \dfrac{2}{\pi}$ **39.** $\frac{14}{3}°\,\text{C}$
41. 1.381737122 **43.** first integral equals $\frac{1}{2}$, second equals $-\frac{1}{2}$; no

▶ Exercise Set 14.2 (Page 908)

1. $\frac{1}{40}$ **3.** 9 **5.** $\dfrac{\pi}{2}$ **7.** $\frac{1}{12}$
9. **(a)** $\displaystyle\int_0^2 \int_0^{x^2} f(x, y)\,dy\,dx$ **(b)** $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 f(x, y)\,dx\,dy$
11. **(a)** $\displaystyle\int_1^2 \int_{-2x+5}^3 f(x, y)\,dy\,dx + \int_2^4 \int_1^3 f(x, y)\,dy\,dx + \int_4^5 \int_{2x-7}^3 f(x, y)\,dy\,dx$
(b) $\displaystyle\int_1^3 \int_{(5-y)/2}^{(y+7)/2} f(x, y)\,dx\,dy$
13. **(a)** $\frac{16}{3}$ **(b)** 38 **15.** 576 **17.** 0 **19.** $\dfrac{\sqrt{17} - 1}{2}$ **21.** $\frac{50}{3}$
23. $-\frac{7}{60}$ **25.** $\dfrac{1 - \cos 8}{3}$
27. **(a)**


(b) $(-1.8414, 0.1586), (1.1462, 3.1462)$
(c) -0.4044
(d) -0.4044

29. $\sqrt{2} - 1$ **31.** 32

Responses to True–False questions may be abridged to save space.

33. False; $\displaystyle\int_0^1 \int_{x^2}^{2x} f(x, y)\,dy\,dx$ integrates $f(x, y)$ over the region between the graphs of $y = x^2$ and $y = 2x$ for $0 \le x \le 1$ and results in a number, but $\displaystyle\int_{x^2}^{2x} \int_0^1 f(x, y)\,dx\,dy$ produces an expression involving x.
35. False; although R is symmetric across the x-axis, the integrand may not be.
37. 12 **39.** 27π **41.** 170 **43.** $\dfrac{27\pi}{2}$ **45.** $\dfrac{\pi}{2}$
47. $\displaystyle\int_0^{\sqrt{2}} \int_{y^2}^2 f(x, y)\,dx\,dy$ **49.** $\displaystyle\int_1^{e^2} \int_{\ln x}^2 f(x, y)\,dy\,dx$
51. $\displaystyle\int_0^{\pi/2} \int_0^{\sin x} f(x, y)\,dy\,dx$ **53.** $\dfrac{1 - e^{-16}}{8}$ **55.** $\dfrac{e^8 - 1}{3}$
57. **(a)** 0 **(b)** tan 1 **59.** 0 **61.** $\dfrac{\pi}{2} - \ln 2$ **63.** $\frac{2}{3}°\,\text{C}$ **65.** 0.676089

▶ Exercise Set 14.3 (Page 916)

1. $\frac{1}{6}$ **3.** $\frac{2}{9}a^3$ **5.** 0 **7.** $\dfrac{3\pi}{2}$ **9.** $\dfrac{\pi}{16}$ **11.** $\displaystyle\int_{\pi/6}^{5\pi/6} \int_2^{4\sin\theta} f(r, \theta)\,r\,dr\,d\theta$
13. $8\displaystyle\int_0^{\pi/2} \int_1^3 r\sqrt{9 - r^2}\,dr\,d\theta$ **15.** $2\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} (1 - r^2)\,r\,dr\,d\theta$
17. $\dfrac{64\sqrt{2}}{3}\pi$ **19.** $\dfrac{5\pi}{32}$ **21.** $\dfrac{27\pi}{16}$ **23.** $(1 - \cos 9)\pi$ **25.** $\dfrac{\pi}{8}\ln 5$ **27.** $\dfrac{\pi}{8}$
29. $\frac{16}{9}$ **31.** $\dfrac{\pi}{2}\left(1 - \dfrac{1}{\sqrt{1 + a^2}}\right)$ **33.** $\dfrac{\pi}{4}(\sqrt{5} - 1)$

Responses to True–False questions may be abridged to save space.

35. True; the disk is given in polar coordinates by $0 \le r \le 2, 0 \le \theta \le 2\pi$.

37. False; the integrand is missing a factor of r:

$$\iint_R f(r, \theta)\, dA = \int_0^{\pi/2} \int_1^2 f(r, \theta) r\, dr\, d\theta$$

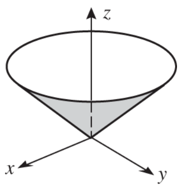
39. $\dfrac{1}{5} + \dfrac{\pi}{2}$

41. **(a)** $\dfrac{4}{3}\pi a^2 c$ **(b)** $\approx 1.0831682 \times 10^{21}$ m^3 **43.** $2a^2$

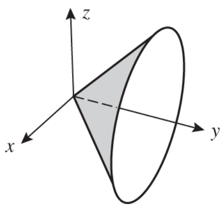
▶ Exercise Set 14.4 (Page 927)

1. 6π **3.** $\dfrac{\sqrt{5}}{6}$ **5.** $\sqrt{2}\pi$ **7.** $\dfrac{(10\sqrt{10} - 1)\pi}{18}$ **9.** 8π

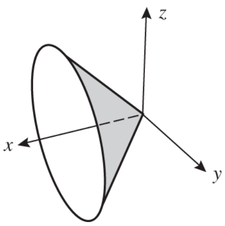
11. **(a)**



(b)



(c)



13. **(a)** $x = u, y = v, z = \frac{5}{2} + \frac{3}{2}u - 2v$ **(b)** $x = u, y = v, z = u^2$

15. **(a)** $x = \sqrt{5}\cos u, y = \sqrt{5}\sin u, z = v; 0 \le u \le 2\pi, 0 \le v \le 1$
(b) $x = 2\cos u, y = v, z = 2\sin u; 0 \le u \le 2\pi, 1 \le v \le 3$

17. $x = u, y = \sin u \cos v, z = \sin u \sin v$

19. $x = r\cos\theta, y = r\sin\theta, z = \dfrac{1}{1 + r^2}$

21. $x = r\cos\theta, y = r\sin\theta, z = 2r^2\cos\theta\sin\theta$

23. $x = r\cos\theta, y = r\sin\theta, z = \sqrt{9 - r^2}; r \le \sqrt{5}$

25. $x = \dfrac{1}{2}\rho\cos\theta, y = \dfrac{1}{2}\rho\sin\theta, z = \dfrac{\sqrt{3}}{2}\rho$ **27.** $z = x - 2y$; a plane

29. $(x/3)^2 + (y/2)^2 = 1; 2 \le z \le 4$; part of an elliptic cylinder

31. $(x/3)^2 + (y/4)^2 = z^2; 0 \le z \le 1$; part of an elliptic cone

33. **(a)** $x = r\cos\theta, y = r\sin\theta, z = r, 0 \le r \le 2$;
$x = u, y = v, z = \sqrt{u^2 + v^2}, 0 \le u^2 + v^2 \le 4$

35. **(a)** $0 \le u \le 3, 0 \le v \le \pi$ **(b)** $0 \le u \le 4, -\pi/2 \le v \le \pi/2$

37. **(a)** $0 \le \phi \le \pi/2, 0 \le \theta \le 2\pi$ **(b)** $0 \le \phi \le \pi, 0 \le \theta \le \pi$

39. $2x + 4y - z = 5$ **41.** $z = 0$ **43.** $x - y + \dfrac{\sqrt{2}}{2}z = \dfrac{\pi\sqrt{2}}{8}$

45. $\dfrac{(17\sqrt{17} - 5\sqrt{5})\pi}{6}$

Responses to True–False questions may be abridged to save space.

47. False; the surface area is $S = \displaystyle\iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$.

49. True; see the discussion preceding Definition 14.4.1.

51. $4\pi a^2$ **55.** $4\pi^2 ab$ **57.** 9.099

59. $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$; ellipsoid

61. $(x/a)^2 + (y/b)^2 - (z/c)^2 = -1$; hyperboloid of two sheets

▶ Exercise Set 14.5 (Page 936)

1. 8 **3.** $\frac{47}{3}$ **5.** $\frac{81}{5}$ **7.** $\frac{128}{15}$ **9.** $\pi(\pi - 3)/2$ **11.** $\frac{1}{6}$ **13.** 9.425

15. 4 **17.** $\frac{256}{15}$

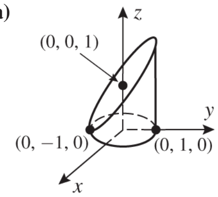
19. **(a)** $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{4x^2+y^2}^{4-3y^2} f(x, y, z)\, dz\, dy\, dx$

(b) $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{4x^2+y^2}^{4-3y^2} f(x, y, z)\, dz\, dx\, dy$

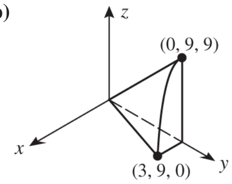
21. $4\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{4x^2+y^2}^{4-3y^2} dz\, dy\, dx$

23. $2\displaystyle\int_{-3}^{3} \int_0^{\frac{1}{3}\sqrt{9-x^2}} \int_0^{x+3} dz\, dy\, dx$

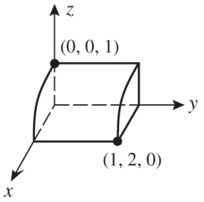
25. **(a)**



(b)



(c)



Responses to True–False questions may be abridged to save space.

27. True; apply Fubini's Theorem (Theorem 14.5.1).

29. False;

$$\iiint_G f(x, y, z)\, dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x, y, z)\, dz\, dy\, dx.$$

33. $\frac{3}{4}$ **35.** 3.291

37. **(a)** $\displaystyle\int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dz\, dy\, dx$ is one example.

39. **(a)** $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^5 f(x, y, z)\, dz\, dy\, dx$

(b) $\displaystyle\int_0^9 \int_0^{3-\sqrt{x}} \int_y^{3-\sqrt{x}} f(x, y, z)\, dz\, dy\, dx$

(c) $\displaystyle\int_0^2 \int_0^{4-x^2} \int_y^{8-y} f(x, y, z)\, dz\, dy\, dx$

▶ Exercise Set 14.6 (Page 946)

1. $\dfrac{\pi}{4}$ **3.** $\dfrac{\pi}{16}$

5. The region is bounded by the xy-plane and the upper half of a sphere of radius 1 centered at the origin; $f(r, \theta, z) = z$.

7. The region is the portion of the first octant inside a sphere of radius 1 centered at the origin; $f(\rho, \theta, \phi) = \rho\cos\phi$.

9. $\dfrac{81\pi}{2}$ **11.** $\frac{152}{3}\pi$ **13.** $\dfrac{64\pi}{3}$ **15.** $\dfrac{11\pi a^3}{3}$ **17.** $\dfrac{\pi a^6}{48}$

19. $\dfrac{32(2\sqrt{2} - 1)\pi}{15}$

Responses to True–False questions may be abridged to save space.

21. False; the factor r^2 should be r [Formula (6)]:

$$\iiint_G f(x, y, z)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta.$$

23. True; G is the spherical wedge bounded by the spheres $\rho = 1$ and $\rho = 3$, the half-planes $\theta = 0$ and $\theta = 2\pi$, and above the cone $\phi = \pi/4$, so

$$(\text{volume of } G) = \iiint_G dV = \int_0^{\pi/4} \int_0^{2\pi} \int_1^3 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

25. **(a)** $\frac{5}{2}(-8 + 3\ln 3)\ln(\sqrt{5} - 2)$ **(b)** $f(x, y, z) = \dfrac{y^3}{x^3\sqrt{1 + z^2}}$;
G is the cylindrical wedge $1 \le r \le 4, \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{3}, -2 \le z \le 2$

27. $\dfrac{4\pi a^3}{3}$ **29.** $\dfrac{2(\sqrt{3} - 1)\pi}{3}$

▶ Exercise Set 14.7 (Page 957)

1. -17 **3.** $\cos(u-v)$ **5.** $x=\frac{2}{9}u+\frac{5}{9}v,\ y=-\frac{1}{9}u+\frac{2}{9}v;\ \frac{1}{9}$

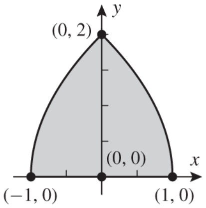
7. $x=\dfrac{\sqrt{u+v}}{\sqrt{2}},\ y=\dfrac{\sqrt{v-u}}{\sqrt{2}};\ \dfrac{1}{4\sqrt{v^2-u^2}}$ **9.** 5 **11.** $\dfrac{1}{v}$

Responses to True–False questions may be abridged to save space.

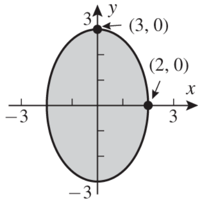
13. False; $|\partial(x,y)/\partial(u,v)| = \|\partial\mathbf{r}/\partial u\times\partial\mathbf{r}/\partial v\|$; evaluating this at (u_0,v_0) gives the area of the indicated parallelogram.

15. False; $\partial(x,y)/\partial(r,\theta)=r$.

17.



19.



21. $\frac{3}{2}\ln 3$ **23.** $1-\frac{1}{2}\sin 2$ **25.** 96π **27.** $\dfrac{\pi}{24}(1-\cos 1)$ **29.** $\frac{192}{5}\pi$

31. $u=\begin{cases}\cot^{-1}(x/y), & y\neq 0\\ 0, & y=0 \text{ and } x>0\\ \pi, & y=0 \text{ and } x<0\end{cases}$

$v=\sqrt{x^2+y^2}$; other answers possible

33. $u=(3/7)x-(2/7)y,\ v=(-1/7)x+(3/7)y$; other answers possible

35. $\frac{1}{4}\ln\frac{5}{2}$

37. $\dfrac{1}{2}\left[\ln(\sqrt{2}+1)-\dfrac{\pi}{4}\right]$ **39.** $\frac{35}{256}$ **41.** $2\ln 3$ **45.** $21/8$

▶ Exercise Set 14.8 (Page 965)

1. $M=\frac{13}{20}$, center of gravity $\left(\frac{190}{273},\frac{6}{13}\right)$

3. $M=a^4/8$, center of gravity $(8a/15,8a/15)$

5. $\left(\frac{1}{2},\frac{1}{2}\right)$ **7.** $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

Responses to True–False questions may be abridged to save space.

9. True; recall this from Section 6.7.

11. False; the center of gravity of the lamina is $(\bar{x},\bar{y})=(M_y/M, M_x/M)$, where M_y and M_x are the lamina's first moments about the y- and x-axes, respectively, and M is the mass of the lamina.

15. $\left(\dfrac{128}{105\pi},\dfrac{128}{105\pi}\right)$ **17.** $\left(\dfrac{4a}{3\pi},0\right)$ **19.** $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ **21.** $\left(\frac{1}{2},0,\frac{3}{5}\right)$

23. $(3a/8,3a/8,3a/8)$

25. $M=a^4/2$, center of gravity $(a/3,a/2,a/2)$

27. $M=\frac{1}{6}$, center of gravity $\left(0,\frac{16}{35},\frac{1}{2}\right)$ **29.** **(a)** $\left(\frac{5}{8},\frac{5}{8}\right)$ **(b)** $\left(\frac{2}{3},\frac{1}{2}\right)$

31. $(1.177406, 0.353554, 0.231557)$

33. $\dfrac{27\pi}{4}$ **35.** $\pi k a^4$ **37.** $\left(0,0,\frac{7}{16\sqrt{2}-14}\right)$ **39.** $\left(\frac{4}{3},0,\frac{10}{9}\right)$

41. $(3a/8,3a/8,3a/8)$ **43.** $(2-\sqrt{2})\pi/4$ **45.** $(0,0,8/15)$

47. $(0,195/152,0)$ **51.** $\frac{1}{2}\delta\pi a^4 h$ **53.** $\frac{1}{2}\delta\pi h(a_2^4-a_1^4)$ **57.** $2\pi^2 abk$

59. $(a/3,b/3)$

▶ Chapter 14 Review Exercises (Page 968)

3. **(a)** $\displaystyle\iint_R dA$ **(b)** $\displaystyle\iiint_G dV$ **(c)** $\displaystyle\iint_R\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$

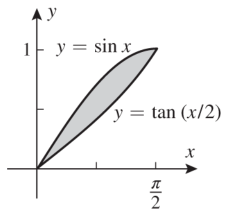
5. $\displaystyle\int_0^1\int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy$

7. **(a)** $a=2, b=1, c=1, d=2$ or $a=1, b=2, c=2, d=1$ **(b)** 3

9. $-\dfrac{1}{\sqrt{2\pi}}$

11. $\displaystyle\int_0^1\int_{2y}^2 e^x e^y\,dx\,dy$

13.



15. $\frac{1}{3}(1-\cos 64)$ **17.** a^2 **19.** $\frac{3}{2}$ **21.** 32π

23. **(a)** $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^a \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$

(b) $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}a/2}\int_{r/\sqrt{3}}^{\sqrt{a^2-r^2}} r^3\,dz\,dr\,d\theta$

(c) $\displaystyle\int_{-\sqrt{3}a/2}^{\sqrt{3}a/2}\int_{-\sqrt{(3a^2/4)-x^2}}^{\sqrt{(3a^2/4)-x^2}}\int_{\sqrt{x^2+y^2}/\sqrt{3}}^{\sqrt{a^2-x^2-y^2}}(x^2+y^2)\,dz\,dy\,dx$

25. $\dfrac{\pi a^3}{9}$ **27.** $\dfrac{1}{24}(26^{3/2}-10^{3/2})\approx 4.20632$ **29.** $2x+4y-z=5$

33. **(a)** $\dfrac{1}{2(u+w)}$ **(b)** $\frac{1}{2}(7\ln 7-5\ln 5-3\ln 3)$ **35.** $\left(\frac{8}{5},0\right)$

37. $(0,0,h/4)$

▶ Chapter 14 Making Connections (Page 970)

Where correct answers to a Making Connections exercise may vary, no answer is listed. Sample answers for these questions are available on the Book Companion Site.

1. **(b)** $\dfrac{\pi}{4}$ **3.** **(a)** 1.173108605 **(b)** 1.173108605

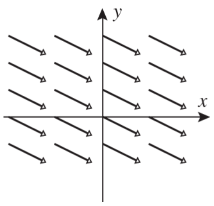
4. **(a)** the sphere $0\le x^2+y^2+z^2\le 1$ **(b)** 4.934802202 **(c)** $\pi^2/2$

5. **(b)** 4.4506 **6.** $\frac{4}{35}\pi a^3$

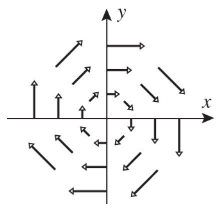
▶ Exercise Set 15.1 (Page 978)

1. **(a)** III **(b)** IV **3.** **(a)** true **(b)** true **(c)** true

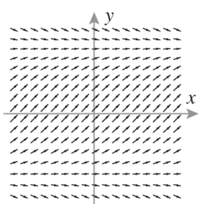
5.



7.



9.



Responses to True–False questions may be abridged to save space.

11. False; the vector field has a nonzero $\mathbf{k}$-component.

13. True; this is the curl of $\mathbf{F}$.

15. **(a)** all x, y **(b)** all x, y **17.** $\operatorname{div}\mathbf{F}=2x+y$, $\operatorname{curl}\mathbf{F}=z\mathbf{i}$

19. $\operatorname{div}\mathbf{F}=0$, $\operatorname{curl}\mathbf{F}=(40x^2z^4-12xy^3)\mathbf{i}+(14y^3z+3y^4)\mathbf{j}-(16xz^5+21y^2z^2)\mathbf{k}$

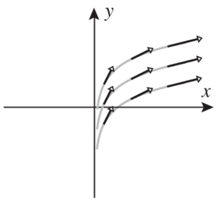
21. $\operatorname{div}\mathbf{F}=\dfrac{2}{\sqrt{x^2+y^2+z^2}}$, $\operatorname{curl}\mathbf{F}=\mathbf{0}$ **23.** $4x$ **25.** 0

27. $(1+y)\mathbf{i}+x\mathbf{j}$

39. $\nabla\cdot(k\mathbf{F})=k\nabla\cdot\mathbf{F}$, $\nabla\cdot(\mathbf{F}+\mathbf{G})=\nabla\cdot\mathbf{F}+\nabla\cdot\mathbf{G}$, $\nabla\cdot(\phi\mathbf{F})=\phi\nabla\cdot\mathbf{F}+\nabla\phi\cdot\mathbf{F}$, $\nabla\cdot(\nabla\times\mathbf{F})=0$

47. **(b)** $x^2+y^2=K$

49. $\dfrac{dy}{dx} = \dfrac{1}{x}, y = \ln x + K$



▶ Exercise Set 15.2 (Page 993)

1. **(a)** 1 **(b)** 0 **3.** 16 **7.** 8 **9.** -28

11. **(a)** $\frac{4\sqrt{2}-2}{3}$ **(b)** 1 **(c)** $\frac{2}{3}$

13. **(a)** 3 **(b)** 3 **(c)** 3 **(d)** 3

Responses to True–False questions may be abridged to save space.

15. False; line integrals of functions are independent of the orientation of the curve. **17.** True; this is Equation (26).

19. 2 **21.** $\frac{13}{20}$ **23.** $1-\pi$ **25.** 3 **27.** $-1-(\pi/4)$ **29.** $1-e^3$

31. **(a)** $\dfrac{63\sqrt{17}}{64} + \dfrac{1}{4}\ln(4+\sqrt{17}) - \dfrac{1}{8}\ln\dfrac{\sqrt{17}+1}{\sqrt{17}-1} - \dfrac{1}{4}\ln(\sqrt{2}+1) + \dfrac{1}{8}\ln\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ **(b)** $\dfrac{\pi^3}{24} + \dfrac{e^{\pi/2}}{5} + \dfrac{\pi}{4} + \dfrac{6}{5}$

33. **(a)** -1 **(b)** -2 **35.** $\frac{5}{2}$ **37.** 0 **39.** $1-e^{-1}$ **41.** $6\sqrt{3}$

43. $5k\tan^{-1}3$ **45.** $\frac{3}{5}$ **47.** $\frac{27}{28}$ **49.** $\frac{3}{4}$ **51.** $\dfrac{17\sqrt{17}-1}{4}$

53. **(b)** $S = \displaystyle\int_C z(t)\,dt$ **(c)** 4π **55.** $\lambda = -12$

▶ Exercise Set 15.3 (Page 1004)

1. conservative, $\phi = \dfrac{x^2}{2} + \dfrac{y^2}{2} + K$ **3.** not conservative

5. conservative, $\phi = x\cos y + y\sin x + K$

9. -6 **11.** $9e^2$ **13.** 32 **15.** $W = -\frac{1}{2}$ **17.** $W = 1-e^{-1}$

Responses to True–False questions may be abridged to save space.

19. False; the integral must be 0 for *all* closed curves C.

21. True; if $\boldsymbol{\nabla}\phi$ is constant, then ϕ must be a linear function.

23. $\ln 2 - 1$ **25.** ≈ -0.307 **27.** no **33.** $h(x) = Ce^x$

35. **(a)** $W = -\dfrac{1}{\sqrt{14}} + \dfrac{1}{\sqrt{6}}$ **(b)** $W = -\dfrac{1}{\sqrt{14}} + \dfrac{1}{\sqrt{6}}$ **(c)** $W = 0$

▶ Exercise Set 15.4 (Page 1011)

1. 0 **3.** 0 **5.** 0 **7.** 8π **9.** -4 **11.** -1 **13.** 0

Responses to True–False questions may be abridged to save space.

15. False; Green's Theorem applies to closed curves.

17. True; the integral is the area of the region bounded by C.

19. **(a)** ≈ -3.550999378 **(b)** ≈ -0.269616482 **21.** $\frac{3}{8}a^2\pi$ **23.** $\frac{1}{2}abt_0$

27. Formula (1) of Section 6.1 **29.** $\frac{250}{3}$ **31.** $-3\pi a^2$ **33.** $\left(\frac{8}{15}, \frac{8}{21}\right)$

35. $\left(0, \dfrac{4a}{3\pi}\right)$ **37.** the circle $x^2+y^2=1$ **39.** 69

▶ Exercise Set 15.5 (Page 1019)

1. $\dfrac{15}{2}\pi\sqrt{2}$ **3.** $\dfrac{\pi}{4}$ **5.** $-\dfrac{\sqrt{2}}{2}$ **7.** 9

Responses to True–False questions may be abridged to save space.

9. True; this follows from the definition.

11. False; the integral is the total mass of the lamina.

13. **(b)** $2\pi\left[1-\sqrt{1-r^2}+\dfrac{r^2}{2}\right] \to 3\pi$ as $r \to 1^-$

(c) $\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}$, $0 \le \theta \le 2\pi, 0 \le \phi \le \pi/2$;

$$\iint (1+z)\,dS = \int_0^{2\pi}\int_0^{\pi/2}(1+\cos\phi)\sin\phi\,d\phi\,d\theta = 3\pi$$

17. **(c)** $4\pi/3$

19. **(a)** $\dfrac{\sqrt{29}}{16}\displaystyle\int_0^6\int_0^{(12-2x)/3} xy(12-2x-3y)\,dy\,dx$

(b) $\dfrac{\sqrt{29}}{4}\displaystyle\int_0^3\int_0^{(12-4z)/3} yz(12-3y-4z)\,dy\,dz$

(c) $\dfrac{\sqrt{29}}{9}\displaystyle\int_0^3\int_0^{6-2z} xz(12-2x-4z)\,dx\,dz$

21. $\dfrac{18\sqrt{29}}{5}$

23. $\displaystyle\int_0^4\int_1^2 y^3z\sqrt{4y^2+1}\,dy\,dz; \ \frac{1}{2}\int_0^4\int_1^4 xz\sqrt{1+4x}\,dx\,dz$

25. $\dfrac{391\sqrt{17}}{15} - \dfrac{5\sqrt{5}}{3}$ **27.** $\frac{4}{3}\pi\delta_0$ **29.** $\frac{1}{4}(37\sqrt{37}-1)$ **31.** $M = \delta_0 S$

33. $(0, 0, 149/65)$ **35.** $\dfrac{93}{\sqrt{10}}$ **37.** $\dfrac{\pi}{4}$ **39.** 57.895751

▶ Exercise Set 15.6 (Page 1028)

1. **(a)** zero **(b)** zero **(c)** positive **(d)** negative **(e)** zero **(f)** zero

3. -80 **5.** 30 **7.** 200π **9.** 4 **11.** 2π **13.** $\dfrac{14\pi}{3}$ **15.** 0

17. 18π **19.** $\frac{4}{9}$ **21.** **(a)** 8 **(b)** 24 **(c)** 0

Responses to True–False questions may be abridged to save space.

23. False; the Möbius strip has no orientation.

25. False; the net volume can be zero because as much fluid passes through the surface in the negative direction as in the positive direction.

27. -3π **29.** **(a)** $0\ \mathrm{m^3/s}$ **(b)** $0\ \mathrm{kg/s}$ **31.** **(b)** $32/3$

33. **(a)** $4\pi a^{k+3}$ **(b)** $k=-3$ **35.** $a = 2, 3$

▶ Exercise Set 15.7 (Page 1038)

1. 3 **3.** $\dfrac{4\pi}{3}$

Responses to True–False questions may be abridged to save space.

5. False; it equates a surface integral and a triple integral.

7. True; see subsection entitled Sources and Sinks.

9. 12 **11.** $3\pi a^2$ **13.** 180π **15.** $\dfrac{192\pi}{5}$ **17.** $\dfrac{\pi}{2}$ **19.** $\dfrac{4608}{35}$

21. 135π **23.** **(a)** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ **(b)** $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

33. no sources or sinks

35. sources at all points except the origin, no sinks **37.** $\dfrac{7\pi}{4}$

▶ Exercise Set 15.8 (Page 1045)

1. $\frac{3}{2}$ **3.** 0 **5.** 2π **7.** 16π **9.** 0 **11.** πa^2

Responses to True–False questions may be abridged to save space.

13. True; see Theorem 15.8.1. **15.** False; the circulation is $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$.

17. **(a)** $\frac{3}{2}$ **(b)** -1 **(c)** $-\dfrac{1}{\sqrt{2}}\mathbf{j} - \dfrac{1}{\sqrt{2}}\mathbf{k}$ **23.** $-\dfrac{5\pi}{4}$

▶ Chapter 15 Review Exercises (Page 1046)

3. $\dfrac{1-x}{\sqrt{(1-x)^2+(2-y)^2}}\mathbf{i} + \dfrac{2-y}{\sqrt{(1-x)^2+(2-y)^2}}\mathbf{j}$ **5.** $\mathbf{i}+\mathbf{j}+\mathbf{k}$

7. **(a)** $\displaystyle\int_a^b\left[f(x(t),y(t))\frac{dx}{dt} + g(x(t),y(t))\frac{dy}{dt}\right]dt$

(b) $\displaystyle\int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$

11. 0 **13.** $-7/2$ **17.** **(a)** $h(x) = Cx^{-3/2}$ **(b)** $g(y) = C/y^3$

21. $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$

23. $\displaystyle\iint_R f(x(u,v),y(u,v),z(u,v))\|r_u \times r_v\|\,du\,dv$ **25.** yes **27.** 2π

31. -8π **35.** **(a)** conservative **(b)** not conservative

▶ Chapter 15 Making Connections (Page 1048)

Answers are provided in the Student Solutions Manual.