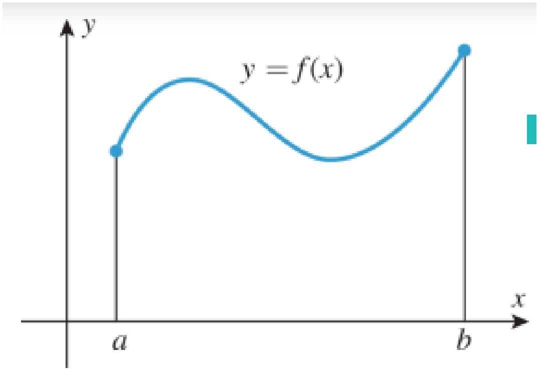
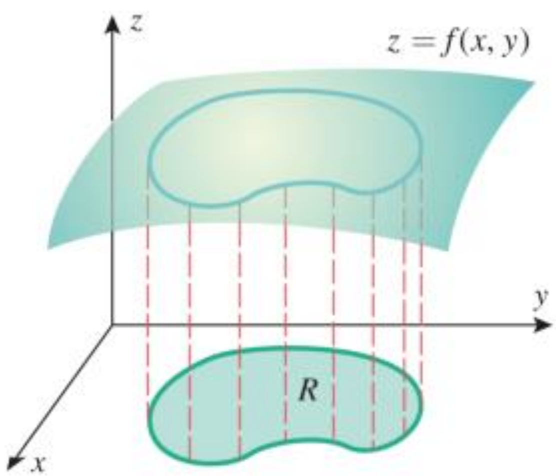
“Surface area over a region” is natural extension of the concept

“arc length over an interval”.

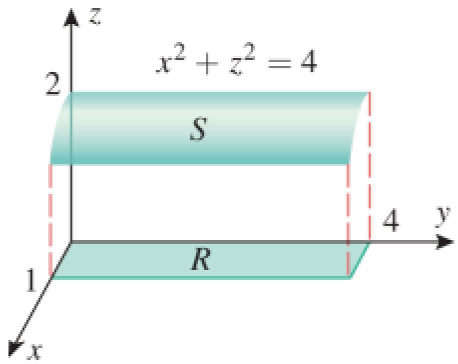



$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$




$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

**Example 1** Find the surface area of that portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle $R$ in the $xy$-plane whose coordinates satisfy $0 \le x \le 1$ and $0 \le y \le 4$.




$$z = \sqrt{4 - x^2}$$

$$\frac{\partial z}{\partial x} = z_x = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

$$\frac{\partial z}{\partial y} = z_y = 0$$

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{\left(\frac{-x}{\sqrt{4 - x^2}}\right)^2 + 0 + 1}$$

$$= \sqrt{\frac{x^2}{4 - x^2} + 1}$$

$$= \sqrt{\frac{x^2 + 4 - x^2}{4 - x^2}}$$

$$= \sqrt{\frac{4}{4 - x^2}} = \frac{2}{\sqrt{4 - x^2}}$$

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \; dA$$

$$S = \int_0^4 \int_0^1 \frac{2}{\sqrt{4-x^2}}\, dx\, dy$$

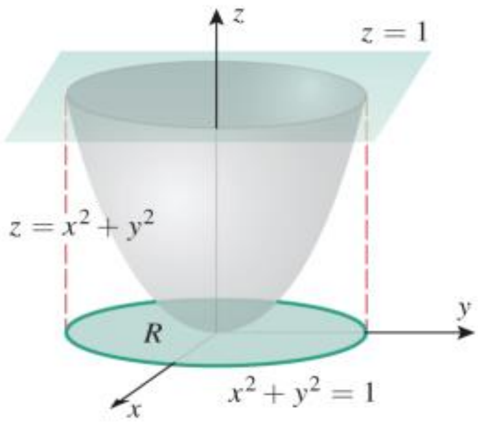
$$= 2\int_0^4 \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 dy \qquad \therefore \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= 2\int_0^4 \left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{0}{2}\right)\right] dy$$

$$= \cancel{2}^{1} \int_0^4 \frac{\pi}{\cancel{6}_{3}}\, dy = \frac{\pi}{3}\left[y\right]_0^4$$

$$S = \frac{4\pi}{3}$$

**Example 2** Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.




$$z = x^2 + y^2$$

$$\frac{\partial z}{\partial x} = 2x \quad , \quad \frac{\partial z}{\partial y} = 2y$$

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{(2x)^2 + (2y)^2 + 1}$$

$$= \sqrt{4x^2 + 4y^2 + 1}$$

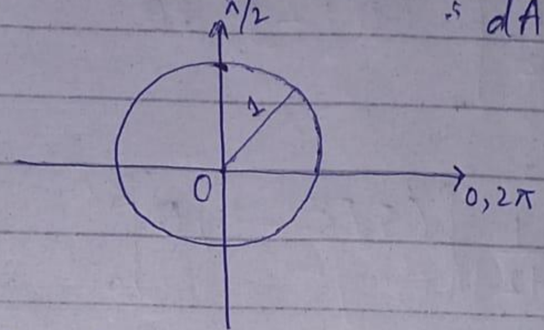
$$S = \iint_R \sqrt{4x^2 + 4y^2 + 1}$$

Since $z = 1$, the region $R$ is

enclosed by circle $x^2+y^2=1$.

In polar coordinates.

$\therefore dA = r\,dr\,d\theta$




$\theta = 0$, $\theta = 2\pi$ $\quad\therefore 0 \le \theta \le 2\pi$

$r = 0$, $r = 1$ $\quad\therefore r \geq 0$

$$S = \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (4r^2+1)^{1/2}\, r\,dr\,d\theta$$

let $u = 4r^2+1$, $du = 8r\,dr$

$$\Rightarrow \frac{du}{8} = r\,dr$$

$$S = \frac{1}{8}\int_0^{2\pi}\int_0^1 u^{1/2}\,du\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\left[\frac{2}{3}u^{3/2}\right]_0^1 d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}\left[\left(4r^2+1\right)^{3/2}\right]_0^1 d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}\left(5^{3/2}-1\right) d\theta$$

$$= \frac{1}{12}\left[\left(5\sqrt{5}-1\right)\theta\right]_0^{2\pi}$$

$$S = \frac{1}{12}\left(5\sqrt{5}-1\right)2\pi$$

$$\boxed{S = \frac{\pi}{6}\left(5\sqrt{5}-1\right)}$$

**1–4** Express the area of the given surface as an iterated double integral, and then find the surface area.

**2.** The portion of the plane $2x + 2y + z = 8$ in the first octant.

$$2x+2y+z=8$$

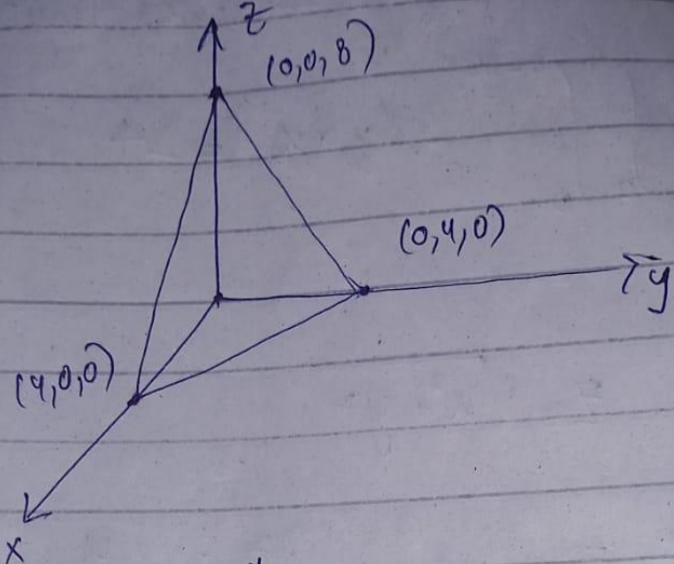
$$\Rightarrow z=8-2x-2y.$$

$$\frac{\partial z}{\partial x}=-2, \quad \frac{\partial z}{\partial y}=-2$$

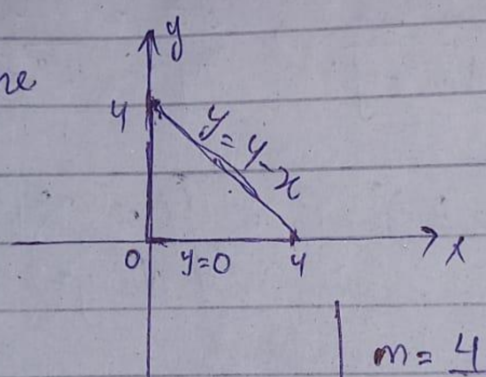
$$\sqrt{(z_x)^2+(z_y)^2+1}=\sqrt{(-2)^2+(-2)^2+1}$$

$$=\sqrt{4+4+1}=\sqrt{9}=3$$

For R.




xy-plane




using type-I

$x=0, \quad x=4$

$y=0, \quad y=4-x$

Other method.

$z=0 \Rightarrow 8-2x-2y=0$

$\Rightarrow 2y = 8-2x$

$y = 4-x \rightarrow$ upper limit

$y=0 \Rightarrow x=4$

$m = \frac{4-0}{0-4} = -1$

$y-y_0 = m(x-x_0)$

$y-0 = -1(x-4)$

$y = 4-x$

In first octant $x \geq 0, y \geq 0$.

$\Rightarrow$ $x = 0, y = 0 \rightarrow$ lower limit.

$$S = \int_0^4 \int_0^{4-x} 3\, dy\, dx = 24$$

**5–10** Express the area of the given surface as an iterated double integral in polar coordinates, and then find the surface area.

**5.** The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 2x$.

$$z = \sqrt{x^2 + y^2}$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{(z_x)^2 + (z_y)^2 + 1} = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 +}$$

$$= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}$$

$$= \sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}}$$

$$= \sqrt{\frac{2x^2+2y^2}{x^2+y^2}} = \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}$$

$$= \sqrt{2}$$

$$S = \iint_R \sqrt{2}\, dA$$

R in polar coordinates.

$x^2 + y^2 = 2x$ $\quad\therefore r^2 = x^2 + y^2$

$r^2 = 2r\cos\theta$ $\quad\therefore x = r\cos\theta$

$r = 2\cos\theta$

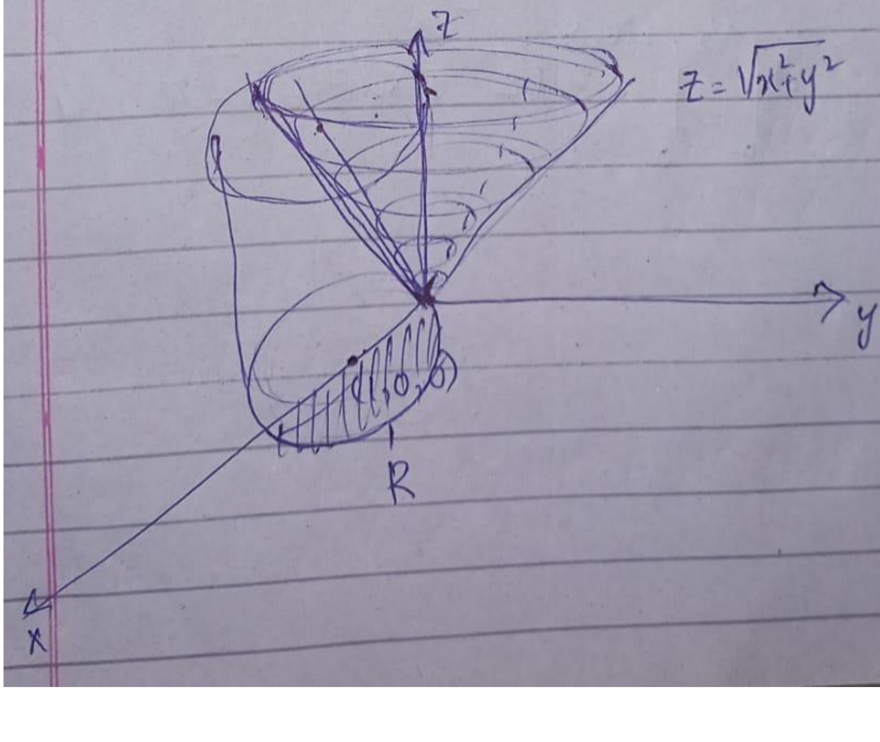
$r = 0$ , $r = 2\cos\theta$

$x^2 + y^2 = 2x$

$x^2 - 2x + 1 - 1 + y^2 = 0$

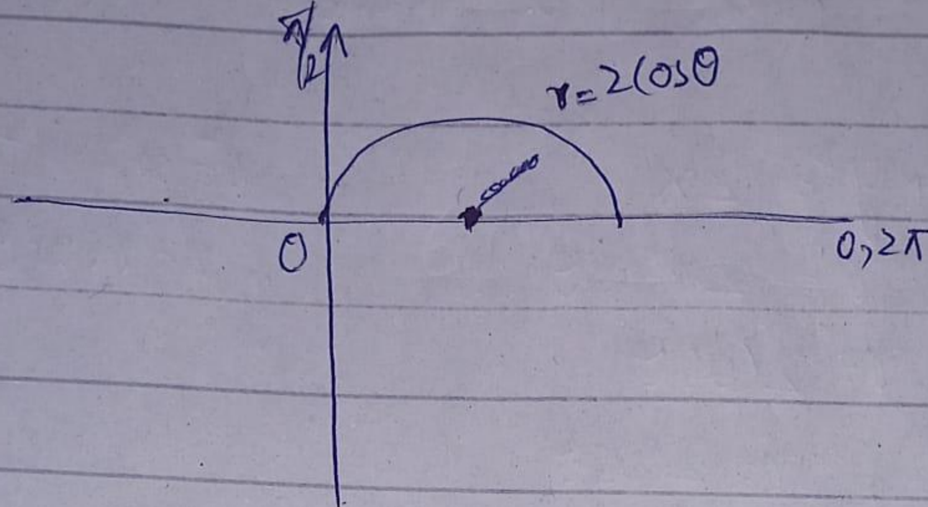
$(x-1)^2 + (y-0)^2 = 1$

cylinder with center $(1,0)$ and radius 1.



In xy-plane.

$$(x-1)^2+(y-0)^2=1$$




$$\theta=0, \quad \theta=\pi/2$$

$r=2\cos\theta$

$r=0 \Rightarrow \quad 0=2\cos\theta$

$$\theta=\cos^{-1}(0)$$

$$\theta=\frac{\pi}{2}$$

$$S=\int_0^{\pi/2}\int_0^{2\cos\theta}\sqrt{2}\, r\,dr\,d\theta=\sqrt{2}\,\pi$$