Partial Derivatives

Exercise Set 13.1

1. **(a)** $f(2,1) = (2)^2(1) + 1 = 5$. **(b)** $f(1,2) = (1)^2(2) + 1 = 3$. **(c)** $f(0,0) = (0)^2(0) + 1 = 1$.

(d) $f(1,-3) = (1)^2(-3) + 1 = -2$. **(e)** $f(3a,a) = (3a)^2(a) + 1 = 9a^3 + 1$.

(f) $f(ab,\, a-b) = (ab)^2(a-b) + 1 = a^3b^2 - a^2b^3 + 1$.

3. **(a)** $f(x+y, x-y) = (x+y)(x-y) + 3 = x^2 - y^2 + 3$. **(b)** $f\left(xy, 3x^2y^3\right) = (xy)\left(3x^2y^3\right) + 3 = 3x^3y^4 + 3$.

5. $F(g(x), h(y)) = F\left(x^3, 3y+1\right) = x^3 e^{x^3(3y+1)}$.

7. **(a)** $t^2 + 3t^{10}$ **(b)** 0 **(c)** 3076

9. **(a)** 2.50 mg/L. **(b)** $C(100,t) = 20(e^{-0.2t} - e^{-t})$. **(c)** $C(x,1) = 0.2x(e^{-0.2} - e^{-1})$.

11. **(a)** $v = 7$ lies between $v = 5$ and $v = 15$, and $7 = 5+2 = 5+\dfrac{2}{10}(15-5)$, so $WCI \approx 19+\dfrac{2}{10}(13-19) = 19-1.2 =$ $17.8°$F.

(b) $T = 28$ lies between $T = 25$ and $T = 30$, and $28 = 25 + \dfrac{3}{5}(30-25)$, so $WCI \approx 19 + \dfrac{3}{5}(25-19) = 19+3.6 =$ $22.6°$F.

13. **(a)** At $v = 25$, $WCI = 16$, so $T = 30°$F.

(b) At $v = 25$, $WCI = 6 = 3 + \dfrac{1}{2}(9-3)$, so $T \approx 20 + \dfrac{1}{2}(25-20) = 22.5°$F.

15. **(a)** The depression is $20 - 16 = 4$, so the relative humidity is 66%.

(b) The relative humidity $\approx 77 - (1/2)7 = 73.5\%$.

(c) The relative humidity $\approx 59 + (2/5)4 = 60.6\%$.

17. **(a)** 19 **(b)** -9 **(c)** 3 **(d)** $a^6 + 3$ **(e)** $-t^8 + 3$ **(f)** $(a+b)(a-b)^2b^3 + 3$

19. $F\left(x^2, y+1, z^2\right) = (y+1)e^{x^2(y+1)z^2}$.

21. **(a)** $f(\sqrt{5}, 2, \pi, -3\pi) = 80\sqrt{\pi}$. **(b)** $f(1,1,\ldots,1) = \displaystyle\sum_{k=1}^{n} k = n(n+1)/2$.

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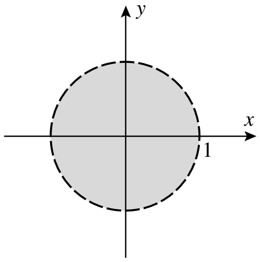

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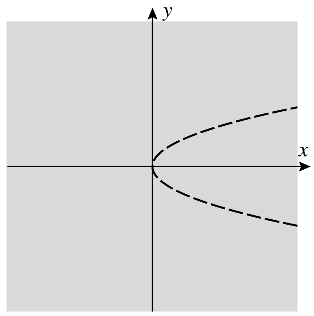


27. **(a)** All points in 2-space above or on the line $y = -2$.

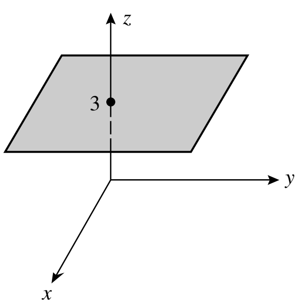
(b) All points in 3-space on or within the sphere $x^2 + y^2 + z^2 = 25$.

(c) All points in 3-space.

29. True; it is the intersection of the domain $[-1, 1]$ of $\sin^{-1} t$ and the domain $[0, +\infty)$ of $\sqrt{t}$.

31. False; z has no constraints so the domain is an infinite solid circular cylinder.

33.



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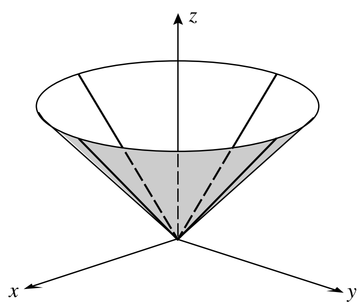

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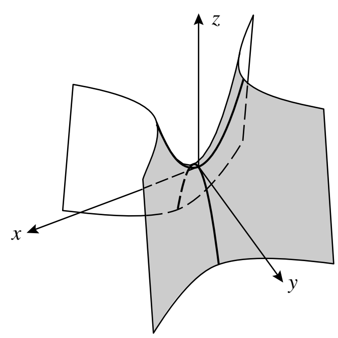

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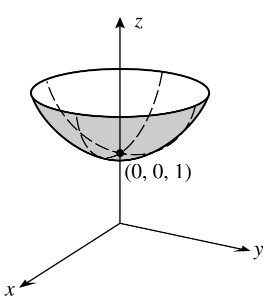

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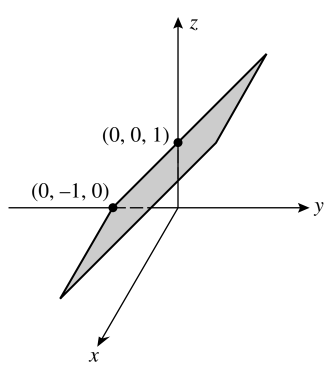


43. **(a)** Hyperbolas. **(b)** Parabolas. **(c)** Noncircular ellipses. **(d)** Lines.

45. **(a)** $\approx$ \$130. **(b)** $\approx$ \$275 more.

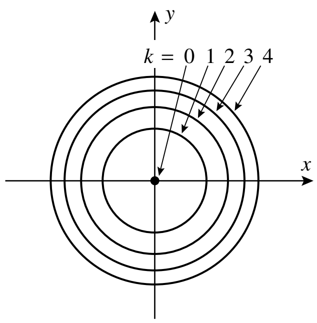
47. **(a)** $f(x, y) = 1 - x^2 - y^2$, because $f = c$ is a circle of radius $\sqrt{1 - c}$ (provided $c \leq 1$), and the radii in (a) decrease as c increases.

(b) $f(x, y) = \sqrt{x^2 + y^2}$ because $f = c$ is a circle of radius c, and the radii increase uniformly.

(c) $f(x, y) = x^2 + y^2$ because $f = c$ is a circle of radius $\sqrt{c}$ and the radii in the plot grow like the square root function.

49. **(a)** A **(b)** B **(c)** Increase. **(d)** Decrease. **(e)** Increase. **(f)** Decrease.

51.



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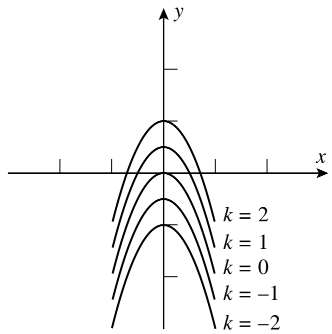

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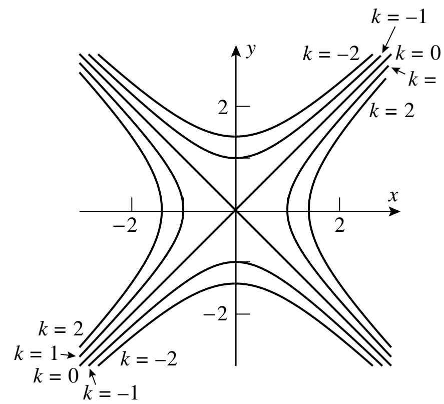

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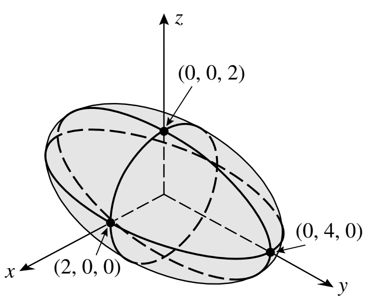

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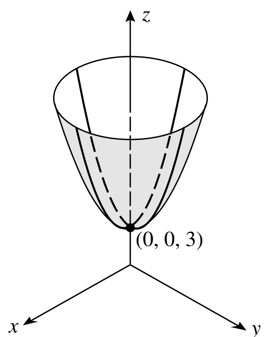


61. Concentric spheres, common center at (2,0,0).

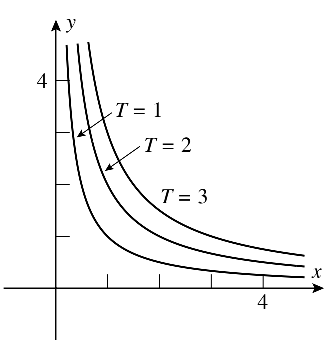
63. Concentric cylinders, common axis the y-axis.

65. **(a)** $f(-1,1) = 0$; $x^2 - 2x^3 + 3xy = 0$. **(b)** $f(0,0) = 0$; $x^2 - 2x^3 + 3xy = 0$.

(c) $f(2,-1) = -18$; $x^2 - 2x^3 + 3xy = -18$.

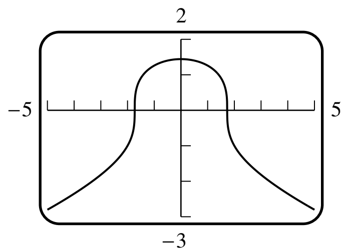
67. **(a)** $f(1,-2,0) = 5$; $x^2 + y^2 - z = 5$. **(b)** $f(1,0,3) = -2$; $x^2 + y^2 - z = -2$. **(c)** $f(0,0,0) = 0$; $x^2 + y^2 - z = 0$.

69. **(a)**



(b) At $(1,4)$ the temperature is $T(1,4) = 4$ so the temperature will remain constant along the path $xy = 4$.

71. **(a)**



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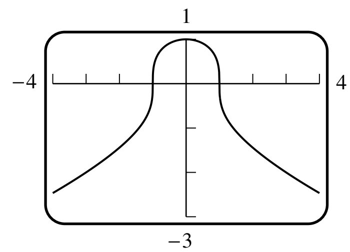
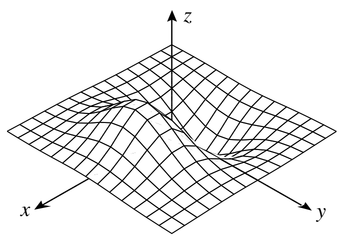


73. **(a)**



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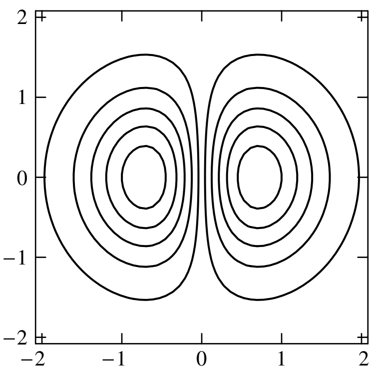


75. **(a)** The graph of g is the graph of f shifted one unit in the positive x-direction.

(b) The graph of g is the graph of f shifted one unit up the z-axis.

(c) The graph of g is the graph of f shifted one unit down the y-axis and then inverted with respect to the plane $z = 0$.

Exercise Set 13.2

1. $\lim\limits_{(x,y)\to(1,3)} (4xy^2 - x) = 4 \cdot 1 \cdot 3^2 - 1 = 35.$

3. $\lim\limits_{(x,y)\to(-1,2)} \dfrac{xy^3}{x+y} = \dfrac{-1 \cdot 2^3}{-1+2} = -8.$

5. $\lim\limits_{(x,y)\to(0,0)} \ln(1 + x^2y^3) = \ln(1 + 0^2 \cdot 0^3) = 0.$

7. **(a)** Along $x = 0$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{3}{x^2+2y^2} = \lim\limits_{y\to 0} \dfrac{3}{2y^2}$ does not exist.

(b) Along $x = 0$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{2x^2+y^2} = \lim\limits_{y\to 0} \dfrac{1}{y}$ does not exist.

9. Let $z = x^2 + y^2$, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim\limits_{z\to 0^+} \dfrac{\sin z}{z} = 1.$

11. Let $z = x^2 + y^2$, then $\lim\limits_{(x,y)\to(0,0)} e^{-1/\left(x^2+y^2\right)} = \lim\limits_{z\to 0^+} e^{-1/z} = 0.$

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\left(x^2+y^2\right)\left(x^2-y^2\right)}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \left(x^2-y^2\right) = 0.$

15. Along $y = 0$: $\lim\limits_{x\to 0} \dfrac{0}{3x^2} = \lim\limits_{x\to 0} 0 = 0$; along $y = x$: $\lim\limits_{x\to 0} \dfrac{x^2}{5x^2} = \lim\limits_{x\to 0} 1/5 = 1/5$, so the limit does not exist.

17. $\displaystyle\lim_{(x,y,z)\to(2,-1,2)} \frac{xz^2}{\sqrt{x^2+y^2+z^2}} = \frac{2\cdot 2^2}{\sqrt{2^2+(-1)^2+2^2}} = \frac{8}{3}.$

19. Let $t = \sqrt{x^2+y^2+z^2}$, then $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{\sin\left(x^2+y^2+z^2\right)}{\sqrt{x^2+y^2+z^2}} = \lim_{t\to 0^+} \frac{\sin\left(t^2\right)}{t} = 0.$

21. $\displaystyle\frac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}} = \frac{e^\rho}{\rho}$, so $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}} = \lim_{\rho\to 0^+} \frac{e^\rho}{\rho}$ does not exist.

23. $\displaystyle\lim_{r\to 0} r\ln r^2 = \lim_{r\to 0}(2\ln r)/(1/r) = \lim_{r\to 0}(2/r)/(-1/r^2) = \lim_{r\to 0}(-2r) = 0.$

25. $\displaystyle\frac{x^2y^2}{\sqrt{x^2+y^2}} = \frac{(r^2\cos^2\theta)(r^2\sin^2\theta)}{r} = r^3\cos^2\theta\sin^2\theta$, so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}} = 0.$

27. $\displaystyle\left|\frac{\rho^3\sin^2\phi\cos\phi\sin\theta\cos\theta}{\rho^2}\right| \le \rho$, so $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0.$

29. True: contains no boundary points, therefore each point of D is an interior point.

31. False: let $f(x,y) = -1$ for $x < 0$ and $f(x,y) = 1$ for $x \ge 0$ and let $g(x.y) = -f(x,y)$.

33. **(a)** No, since there seem to be points near $(0,0)$ with $z = 0$ and other points near $(0,0)$ with $z \approx 1/2$.

(b) $\displaystyle\lim_{x\to 0} \frac{mx^3}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0.$

(c) $\displaystyle\lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} 1/2 = 1/2.$

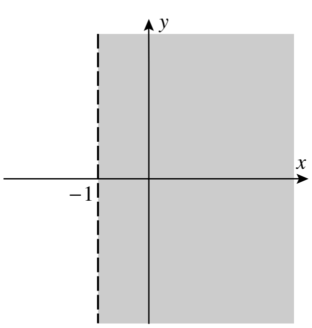
(d) A limit must be unique if it exists, so $f(x,y)$ cannot have a limit as $(x,y)\to(0,0)$.

35. **(a)** We may assume that $a^2+b^2+c^2 > 0$, since we are dealing with a line (not just the point $(0,0,0)$). Assume first that $a \ne 0$. Then $\displaystyle\lim_{t\to 0} \frac{abct^3}{a^2t^2+b^4t^4+c^4t^4} = \lim_{t\to 0} \frac{abct}{a^2+b^4t^2+c^4t^2} = 0$. If, on the other hand, $a = 0$, the result is trivial, as the quotient is then zero.

(b) $\displaystyle\lim_{t\to 0} \frac{t^4}{t^4+t^4+t^4} = \lim_{t\to 0} 1/3 = 1/3.$

37. $-\pi/2$ because $\displaystyle\frac{x^2-1}{x^2+(y-1)^2} \to -\infty$ as $(x,y)\to(0,1)$.

39. The required limit does not exist, so the singularity is not removable.



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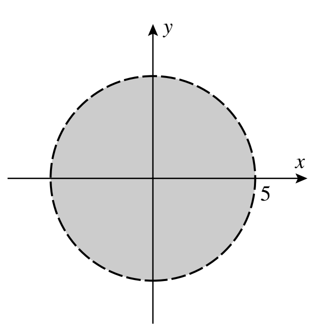

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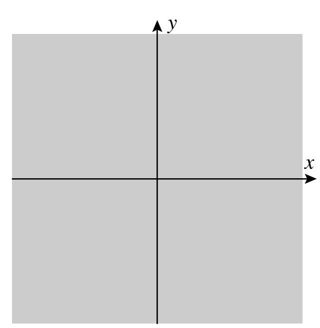

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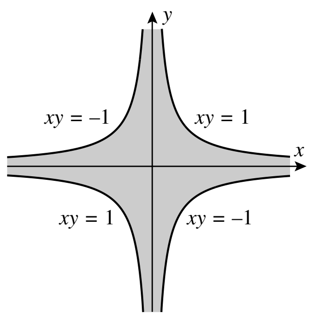
47. 

49. All of 3-space.

51. All points not on the cylinder $x^2 + z^2 = 1$.

Exercise Set 13.3

1. **(a)** $9x^2y^2$ **(b)** $6x^3y$ **(c)** $9y^2$ **(d)** $9x^2$ **(e)** $6y$ **(f)** $6x^3$ **(g)** 36 **(h)** 12

3. $\dfrac{\partial z}{\partial x} = 18xy - 15x^4y$, $\dfrac{\partial z}{\partial y} = 9x^2 - 3x^5$.

5. $\dfrac{\partial z}{\partial x} = 8(x^2+5x-2y)^7(2x+5)$, $\dfrac{\partial z}{\partial y} = -16(x^2+5x-2y)^7$.

7. $\dfrac{\partial}{\partial p}(e^{-7p/q}) = -7e^{-7p/q}/q$, $\dfrac{\partial}{\partial q}(e^{-7p/q}) = 7pe^{-7p/q}/q^2$.

9. $\dfrac{\partial z}{\partial x} = (15x^2y + 7y^2)\cos(5x^3y + 7xy^2)$, $\dfrac{\partial z}{\partial y} = (5x^3 + 14xy)\cos(5x^3y + 7xy^2)$.

11. **(a)** $\dfrac{\partial z}{\partial x} = \dfrac{3}{2\sqrt{3x+2y}}$; slope $= \dfrac{3}{8}$. **(b)** $\dfrac{\partial z}{\partial y} = \dfrac{1}{\sqrt{3x+2y}}$; slope $= \dfrac{1}{4}$.

13. **(a)** $\dfrac{\partial z}{\partial x} = -4\cos(y^2 - 4x)$; rate of change $= -4\cos 7$. **(b)** $\dfrac{\partial z}{\partial y} = 2y\cos(y^2 - 4x)$; rate of change $= 2\cos 7$.

15. $\partial z/\partial x =$ slope of line parallel to xz-plane $= -4$; $\partial z/\partial y =$ slope of line parallel to yz-plane $= 1/2$.

17. **(a)** The right-hand estimate is $\partial r/\partial v \approx (222-197)/(85-80) = 5$; the left-hand estimate is $\partial r/\partial v \approx (197 - 173)/(80-75) = 4.8$; the average is $\partial r/\partial v \approx 4.9$.

(b) The right-hand estimate is $\partial r/\partial \theta \approx (200-197)/(45-40) = 0.6$; the left-hand estimate is $\partial r/\partial \theta \approx (197 - 188)/(40-35) = 1.8$; the average is $\partial r/\partial \theta \approx 1.2$.

19. III is a plane, and its partial derivatives are constants, so III cannot be $f(x,y)$. If I is the graph of $z = f(x,y)$ then (by inspection) f_y is constant as y varies, but neither II nor III is constant as y varies. Hence $z = f(x,y)$ has II as its graph, and as II seems to be an odd function of x and an even function of y, f_x has I as its graph and f_y has III as its graph.

21. True: f is constant along the line $y = 2$ so $f_x(4,2) = 0$.

23. True; z is a linear function of both x and y.

25. $\partial z/\partial x = 8xy^3e^{x^2y^3}$, $\partial z/\partial y = 12x^2y^2e^{x^2y^3}$.

27. $\partial z/\partial x = x^3/(y^{3/5}+x) + 3x^2\ln(1+xy^{-3/5}),\ \partial z/\partial y = -(3/5)x^4/(y^{8/5}+xy)$.

29. $\dfrac{\partial z}{\partial x} = -\dfrac{y(x^2-y^2)}{(x^2+y^2)^2},\ \dfrac{\partial z}{\partial y} = \dfrac{x(x^2-y^2)}{(x^2+y^2)^2}$.

31. $f_x(x,y) = (3/2)x^2y\left(5x^2-7\right)\left(3x^5y-7x^3y\right)^{-1/2},\ f_y(x,y) = (1/2)x^3\left(3x^2-7\right)\left(3x^5y-7x^3y\right)^{-1/2}$.

33. $f_x(x,y) = \dfrac{y^{-1/2}}{y^2+x^2},\ f_y(x,y) = -\dfrac{xy^{-3/2}}{y^2+x^2} - \dfrac{3}{2}y^{-5/2}\tan^{-1}(x/y)$.

35. $f_x(x,y) = -(4/3)y^2\sec^2 x\left(y^2\tan x\right)^{-7/3},\ f_y(x,y) = -(8/3)y\tan x\left(y^2\tan x\right)^{-7/3}$.

37. $f_x(x,y) = -2x,\ f_x(3,1) = -6;\ f_y(x,y) = -21y^2,\ f_y(3,1) = -21$.

39. $\partial z/\partial x = x(x^2+4y^2)^{-1/2},\ \partial z/\partial x\,\big|_{(1,2)} = 1/\sqrt{17}\ ;\ \partial z/\partial y = 4y(x^2+4y^2)^{-1/2},\ \partial z/\partial y\,\big|_{(1,2)} = 8/\sqrt{17}$.

41. **(a)** $2xy^4z^3+y$ **(b)** $4x^2y^3z^3+x$ **(c)** $3x^2y^4z^2+2z$ **(d)** $2y^4z^3+y$ **(e)** $32z^3+1$ **(f)** 438

43. $f_x = 2z/x,\ f_y = z/y,\ f_z = \ln(x^2y\cos z) - z\tan z$.

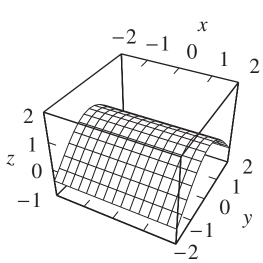
45. $f_x = -y^2z^3/\left(1+x^2y^4z^6\right),\ f_y = -2xyz^3/\left(1+x^2y^4z^6\right),\ f_z = -3xy^2z^2/\left(1+x^2y^4z^6\right)$.

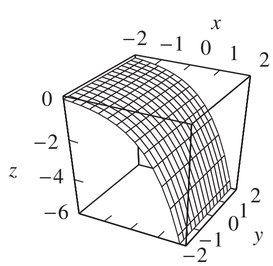
47. $\partial w/\partial x = yze^z\cos xz,\ \partial w/\partial y = e^z\sin xz,\ \partial w/\partial z = ye^z(\sin xz + x\cos xz)$.

49. $\partial w/\partial x = x/\sqrt{x^2+y^2+z^2},\ \partial w/\partial y = y/\sqrt{x^2+y^2+z^2},\ \partial w/\partial z = z/\sqrt{x^2+y^2+z^2}$.

51. **(a)** e **(b)** $2e$ **(c)** e

53.





55. $\partial z/\partial x = 2x + 6y(\partial y/\partial x) = 2x,\ \partial z/\partial x\big|_{(2,1)} = 4$.

57. $\partial z/\partial x = -x\left(29-x^2-y^2\right)^{-1/2},\ \partial z/\partial x|_{(4,3)} = -2$.

59. **(a)** $\partial V/\partial r = 2\pi rh$. **(b)** $\partial V/\partial h = \pi r^2$. **(c)** $\partial V/\partial r|_{r=6,\ h=4} = 48\pi$. **(d)** $\partial V/\partial h|_{r=8,\ h=10} = 64\pi$.

61. **(a)** $P = 10T/V,\ \partial P/\partial T = 10/V,\ \partial P/\partial T|_{T=80,\ V=50} = 1/5$ lb/(in^{2}K).

(b) $V = 10T/P, \partial V/\partial P = -10T/P^2$, if $V = 50$ and $T = 80$, then $P = 10(80)/(50) = 16$, $\partial V/\partial P|_{T=80,\ P=16} = -25/8$(in^5/lb).

63. **(a)** $V = lwh, \partial V/\partial l = wh = 6$. **(b)** $\partial V/\partial w = lh = 15$. **(c)** $\partial V/\partial h = lw = 10$.

65. $\partial V/\partial r = \dfrac{2}{3}\pi rh = \dfrac{2}{r}(\dfrac{1}{3}\pi r^2h) = 2V/r$.

67. **(a)** $2x - 2z(\partial z/\partial x) = 0,\ \partial z/\partial x = x/z = \pm 3/(2\sqrt{6}) = \pm\sqrt{6}/4.$

(b) $z = \pm\sqrt{x^2 + y^2 - 1},\ \partial z/\partial x = \pm x/\sqrt{x^2 + y^2 - 1} = \pm\sqrt{6}/4.$

69. $\dfrac{3}{2}\left(x^2 + y^2 + z^2\right)^{1/2}\left(2x + 2z\dfrac{\partial z}{\partial x}\right) = 0$, $\partial z/\partial x = -x/z$; similarly, $\partial z/\partial y = -y/z$.

71. $2x + z\left(xy\dfrac{\partial z}{\partial x} + yz\right)\cos xyz + \dfrac{\partial z}{\partial x}\sin xyz = 0,\ \dfrac{\partial z}{\partial x} = -\dfrac{2x + yz^2\cos xyz}{xyz\cos xyz + \sin xyz}$;
$z\left(xy\dfrac{\partial z}{\partial y} + xz\right)\cos xyz + \dfrac{\partial z}{\partial y}\sin xyz = 0,\ \dfrac{\partial z}{\partial y} = -\dfrac{xz^2\cos xyz}{xyz\cos xyz + \sin xyz}.$

73. $(3/2)\left(x^2 + y^2 + z^2 + w^2\right)^{1/2}\left(2x + 2w\dfrac{\partial w}{\partial x}\right) = 0$, $\partial w/\partial x = -x/w$; similarly, $\partial w/\partial y = -y/w$ and $\partial w/\partial z = -z/w$.

75. $\dfrac{\partial w}{\partial x} = -\dfrac{yzw\cos xyz}{2w + \sin xyz},\ \dfrac{\partial w}{\partial y} = -\dfrac{xzw\cos xyz}{2w + \sin xyz},\ \dfrac{\partial w}{\partial z} = -\dfrac{xyw\cos xyz}{2w + \sin xyz}.$

77. $f_x = e^{x^2},\ f_y = -e^{y^2}.$

79. $f_x = 2xy^3\sin x^6y^9, f_y = 3x^2y^2\sin x^6y^9.$

81. **(a)** $-\dfrac{1}{4x^{3/2}}\cos y$ **(b)** $-\sqrt{x}\cos y$ **(c)** $-\dfrac{\sin y}{2\sqrt{x}}$ **(d)** $-\dfrac{\sin y}{2\sqrt{x}}$

83. **(a)** $6\cos(3x^2 + 6y^2) - 36x^2\sin(3x^2 + 6y^2)$ **(b)** $12\cos(3x^2 + 6y^2) - 144y^2\sin(3x^2 + 6y^2)$

(c) $-72xy\sin(3x^2 + 6y^2)$ **(d)** $-72xy\sin(3x^2 + 6y^2)$

85. $f_x = 8x - 8y^4,\ f_y = -32xy^3 + 35y^4,\ f_{xy} = f_{yx} = -32y^3.$

87. $f_x = e^x\cos y,\ f_y = -e^x\sin y,\ f_{xy} = f_{yx} = -e^x\sin y.$

89. $f_x = 4/(4x - 5y),\ f_y = -5/(4x - 5y),\ f_{xy} = f_{yx} = 20/(4x - 5y)^2.$

91. $f_x = 2y/(x + y)^2, f_y = -2x/(x + y)^2, f_{xy} = f_{yx} = 2(x - y)/(x + y)^3.$

93. **(a)** $\dfrac{\partial^3 f}{\partial x^3}$ **(b)** $\dfrac{\partial^3 f}{\partial y^2\partial x}$ **(c)** $\dfrac{\partial^4 f}{\partial x^2\partial y^2}$ **(d)** $\dfrac{\partial^4 f}{\partial y^3\partial x}$

95. **(a)** $30xy^4 - 4$ **(b)** $60x^2y^3$ **(c)** $60x^3y^2$

97. **(a)** $f_{xyy}(0, 1) = -30$ **(b)** $f_{xxx}(0, 1) = -125$ **(c)** $f_{yyxx}(0, 1) = 150$

99. **(a)** $f_{xy} = 15x^2y^4z^7 + 2y.$ **(b)** $f_{yz} = 35x^3y^4z^6 + 3y^2.$ **(c)** $f_{xz} = 21x^2y^5z^6.$

(d) $f_{zz} = 42x^3y^5z^5.$ **(e)** $f_{zyy} = 140x^3y^3z^6 + 6y.$ **(f)** $f_{xxy} = 30xy^4z^7.$

(g) $f_{zyx} = 105x^2y^4z^6.$ **(h)** $f_{xxyz} = 210xy^4z^6.$

101. **(a)** $z_x = 2x + 2y,\ z_{xx} = 2,\ z_y = -2y + 2x,\ z_{yy} = -2;\ z_{xx} + z_{yy} = 2 - 2 = 0.$

(b) $z_x = e^x\sin y - e^y\sin x,\ z_{xx} = e^x\sin y - e^y\cos x,\ z_y = e^x\cos y + e^y\cos x,\ z_{yy} = -e^x\sin y + e^y\cos x;\ z_{xx} + z_{yy} = e^x\sin y - e^y\cos x - e^x\sin y + e^y\cos x = 0.$

(c) $z_x = \dfrac{2x}{x^2+y^2} - 2\dfrac{y}{x^2}\dfrac{1}{1+(y/x)^2} = \dfrac{2x-2y}{x^2+y^2}$, $z_{xx} = -2\dfrac{x^2-y^2-2xy}{(x^2+y^2)^2}$, $z_y = \dfrac{2y}{x^2+y^2} + 2\dfrac{1}{x}\dfrac{1}{1+(y/x)^2} = \dfrac{2y+2x}{x^2+y^2}$, $z_{yy} = -2\dfrac{y^2-x^2+2xy}{(x^2+y^2)^2}$; $z_{xx}+z_{yy} = -2\dfrac{x^2-y^2-2xy}{(x^2+y^2)^2} - 2\dfrac{y^2-x^2+2xy}{(x^2+y^2)^2} = 0$.

103. $u_x = \omega \sin c\omega t \cos \omega x$, $u_{xx} = -\omega^2 \sin c\omega t \sin \omega x$, $u_t = c\omega \cos c\omega t \sin \omega x$, $u_{tt} = -c^2\omega^2 \sin c\omega t \sin \omega x$; $u_{xx} - \dfrac{1}{c^2}u_{tt} = -\omega^2 \sin c\omega t \sin \omega x - \dfrac{1}{c^2}(-c^2)\omega^2 \sin c\omega t \sin \omega x = 0$.

105. $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ so $\partial^2 u/\partial x^2 = \partial^2 v/\partial x\partial y$, and $\partial^2 u/\partial y^2 = -\partial^2 v/\partial y\partial x$, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \partial^2 v/\partial x\partial y - \partial^2 v/\partial y\partial x$, if $\partial^2 v/\partial x\partial y = \partial^2 v/\partial y\partial x$ then $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$; thus u satisfies Laplace's equation. The proof that v satisfies Laplace's equation is similar. Adding Laplace's equations for u and v gives Laplaces' equation for $u+v$.

107. $\partial f/\partial v = 8vw^3x^4y^5$, $\partial f/\partial w = 12v^2w^2x^4y^5$, $\partial f/\partial x = 16v^2w^3x^3y^5$, $\partial f/\partial y = 20v^2w^3x^4y^4$.

109. $\partial f/\partial v_1 = 2v_1/\left(v_3^2+v_4^2\right)$, $\partial f/\partial v_2 = -2v_2/\left(v_3^2+v_4^2\right)$, $\partial f/\partial v_3 = -2v_3\left(v_1^2-v_2^2\right)/\left(v_3^2+v_4^2\right)^2$, $\partial f/\partial v_4 = -2v_4\left(v_1^2-v_2^2\right)/\left(v_3^2+v_4^2\right)^2$.

111. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 0 **(e)** $2(1+yw)e^{yw}\sin z\cos z$ **(f)** $2xw(2+yw)e^{yw}\sin z\cos z$

113. $\partial w/\partial x_i = -i\sin(x_1+2x_2+\ldots+nx_n)$.

115. **(a)** xy-plane, $f_x = 12x^2y+6xy$, $f_y = 4x^3+3x^2$, $f_{xy} = f_{yx} = 12x^2+6x$.

(b) $y \neq 0, f_x = 3x^2/y$, $f_y = -x^3/y^2$, $f_{xy} = f_{yx} = -3x^2/y^2$.

117. $f_x(2,-1) = \lim\limits_{x\to 2}\dfrac{f(x,-1)-f(2,-1)}{x-2} = \lim\limits_{x\to 2}\dfrac{2x^2+3x+1-15}{x-2} = \lim\limits_{x\to 2}(2x+7) = 11$ and
$f_y(2,-1) = \lim\limits_{y\to -1}\dfrac{f(2,y)-f(2,-1)}{y+1} = \lim\limits_{y\to -1}\dfrac{8-6y+y^2-15}{y+1} = \lim\limits_{y\to -1} y-7 = -8$.

119. **(a)** $f_y(0,0) = \dfrac{d}{dy}[f(0,y)]\Big|_{y=0} = \dfrac{d}{dy}[y]\Big|_{y=0} = 1$.

(b) If $(x,y)\neq(0,0)$, then $f_y(x,y) = \dfrac{1}{3}(x^3+y^3)^{-2/3}(3y^2) = \dfrac{y^2}{(x^3+y^3)^{2/3}}$; $f_y(x,y)$ does not exist when $y \neq 0$ and $y=-x$.

Exercise Set 13.4

1. $f(x,y) \approx f(3,4) + f_x(x-3) + f_y(y-4) = 5 + 2(x-3) - (y-4)$ and $f(3.01, 3.98) \approx 5 + 2(0.01) - (-0.02) = 5.04$.

3. $L(x,y,z) = f(1,2,3) + (x-1) + 2(y-2) + 3(z-3)$, $f(1.01, 2.02, 3.03) \approx 4 + 0.01 + 2(0.02) + 3(0.03) = 4.14$.

5. Suppose $f(x,y) = c$ for all (x,y). Then at (x_0,y_0) we have $\dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} = 0$ and hence $f_x(x_0,y_0)$ exists and is equal to 0 (Definition 13.3.1). A similar result holds for f_y. From equation (2), it follows that $\Delta f = 0$, and then by Definition 13.4.1 we see that f is differentiable at (x_0,y_0). An analogous result holds for functions $f(x,y,z)$ of three variables.

7. $f_x = 2x, f_y = 2y, f_z = 2z$ so $L(x,y,z) = 0$, $E = f - L = x^2+y^2+z^2$, and $\lim\limits_{(x,y,z)\to(0,0,0)}\dfrac{E(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \lim\limits_{(x,y,z)\to(0,0,0)}\sqrt{x^2+y^2+z^2} = 0$, so f is differentiable at $(0,0,0)$.

9. $dz = 7dx - 2dy$.

11. $dz = 3x^2y^2dx + 2x^3ydy$.

13. $dz = \left[y/\left(1+x^2y^2\right)\right]dx + \left[x/\left(1+x^2y^2\right)\right]dy$.

15. $dw = 8dx - 3dy + 4dz$.

17. $dw = 3x^2y^2zdx + 2x^3yzdy + x^3y^2dz$.

19. $dw = \dfrac{yz}{1+x^2y^2z^2}dx + \dfrac{xz}{1+x^2y^2z^2}dy + \dfrac{xy}{1+x^2y^2z^2}dz$.

21. $df = (2x+2y-4)dx + 2xdy$; $x = 1$, $y = 2$, $dx = 0.01$, $dy = 0.04$ so $df = 0.10$ and $\Delta f = 0.1009$.

23. $df = -x^{-2}dx - y^{-2}dy$; $x = -1$, $y = -2$, $dx = -0.02$, $dy = -0.04$ so $df = 0.03$ and $\Delta f \approx 0.029412$.

25. $df = 2y^2z^3dx + 4xyz^3dy + 6xy^2z^2dz, x = 1, y = -1, z = 2, dx = -0.01, dy = -0.02, dz = 0.02$ so $df = 0.96$ and $\Delta f \approx 0.97929$.

27. False: Example 9, Section 13.3 gives such a function which is not even continuous at (x_0, y_0), let alone differentiable.

29. True; indeed, by Theorem 13.4.4, f is differentiable.

31. Label the four smaller rectangles A, B, C, D starting with the lower left and going clockwise. Then the increase in the area of the rectangle is represented by B, C and D; and the portions B and D represent the approximation of the increase in area given by the total differential.

33. (a) $f(P) = 1/5, f_x(P) = -x/(x^2+y^2)^{-3/2}\Big|_{(x,y)=(4,3)} = -4/125,\ f_y(P) = -y/(x^2+y^2)^{-3/2}\Big|_{(x,y)=(4,3)} = -3/125, L(x,y) = \dfrac{1}{5} - \dfrac{4}{125}(x-4) - \dfrac{3}{125}(y-3)$.

(b) $L(Q) - f(Q) = \dfrac{1}{5} - \dfrac{4}{125}(-0.08) - \dfrac{3}{125}(0.01) - 0.2023342382 \approx -0.0000142382$, $|PQ| = \sqrt{0.08^2 + 0.01^2} \approx 0.08062257748, |L(Q) - f(Q)|/|PQ| \approx 0.000176603$.

35. (a) $f(P) = 0, f_x(P) = 0, f_y(P) = 0, L(x,y) = 0$.

(b) $L(Q) - f(Q) = -0.003\sin(0.004) \approx -0.000012, |PQ| = \sqrt{0.003^2 + 0.004^2} = 0.005$, $|L(Q) - f(Q)|/|PQ| \approx 0.0024$.

37. (a) $f(P) = 6, f_x(P) = 6, f_y(P) = 3, f_z(P) = 2, L(x,y) = 6 + 6(x-1) + 3(y-2) + 2(z-3)$.

(b) $L(Q) - f(Q) = 6 + 6(0.001) + 3(0.002) + 2(0.003) - 6.018018006 = -.000018006$,
$|PQ| = \sqrt{0.001^2 + 0.002^2 + 0.003^2} \approx .0003741657387$; $|L(Q) - f(Q)|/|PQ| \approx -0.000481$.

39. (a) $f(P) = e, f_x(P) = e, f_y(P) = -e, f_z(P) = -e, L(x,y) = e + e(x-1) - e(y+1) - e(z+1)$.

(b) $L(Q) - f(Q) = e - 0.01e + 0.01e - 0.01e - 0.99e^{0.9999} = 0.99(e - e^{0.9999})$, $|PQ| = \sqrt{0.01^2 + 0.01^2 + 0.01^2} \approx 0.01732, |L(Q) - f(Q)|/|PQ| \approx 0.01554$.

41. (a) Let $f(x,y) = e^x \sin y$; $f(0,0) = 0, f_x(0,0) = 0, f_y(0,0) = 1$, so $e^x \sin y \approx y$.

(b) Let $f(x,y) = \dfrac{2x+1}{y+1}$; $f(0,0) = 1, f_x(0,0) = 2, f_y(0,0) = -1$, so $\dfrac{2x+1}{y+1} \approx 1 + 2x - y$.

43. **(a)** Let $f(x,y,z) = xyz + 2$, then $f_x = f_y = f_z = 1$ at $x = y = z = 1$, and $L(x,y,z) = f(1,1,1) + f_x(x-1) + f_y(y-1) + f_z(z-1) = 3 + x - 1 + y - 1 + z - 1 = x + y + z$.

(b) Let $f(x,y,z) = \dfrac{4x}{y+z}$, then $f_x = 2, f_y = -1, f_z = -1$ at $x = y = z = 1$, and $L(x,y,z) = f(1,1,1) + f_x(x-1) + f_y(y-1) + f_z(z-1) = 2 + 2(x-1) - (y-1) - (z-1) = 2x - y - z + 2$.

45. $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$ and $L(1.1, 0.9) = 3.15 = 3 + 2(0.1) + f_y(1,1)(-0.1)$ so $f_y(1,1) = -0.05/(-0.1) = 0.5$.

47. $x - y + 2z - 2 = L(x,y,z) = f(3,2,1) + f_x(3,2,1)(x-3) + f_y(3,2,1)(y-2) + f_z(3,2,1)(z-1)$, so $f_x(3,2,1) = 1, f_y(3,2,1) = -1, f_z(3,2,1) = 2$ and $f(3,2,1) = L(3,2,1) = 1$.

49. $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0), 2y - 2x - 2 = x_0^2 + y_0^2 + 2x_0(x-x_0) + 2y_0(y-y_0)$, from which it follows that $x_0 = -1, y_0 = 1$.

51. $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$, $y + 2z - 1 = x_0y_0 + z_0^2 + y_0(x-x_0) + x_0(y-y_0) + 2z_0(z-z_0)$, so that $x_0 = 1, y_0 = 0, z_0 = 1$.

53. $A = xy$, $dA = y dx + x dy$, $dA/A = dx/x + dy/y$, $|dx/x| \le 0.03$ and $|dy/y| \le 0.05$, $|dA/A| \le |dx/x| + |dy/y| \le 0.08 = 8\%$.

55. $dT = \dfrac{\pi}{g\sqrt{L/g}}dL - \dfrac{\pi L}{g^2\sqrt{L/g}}dg$, $\dfrac{dT}{T} = \dfrac{1}{2}\dfrac{dL}{L} - \dfrac{1}{2}\dfrac{dg}{g}$; $|dL/L| \le 0.005$ and $|dg/g| \le 0.001$ so $|dT/T| \le (1/2)(0.005) + (1/2)(0.001) = 0.003 = 0.3\%$.

57. $E = kq/r^2$, thus $dE = kr^{-2}dq - 2kqr^{-3}dr$, and then $dE/E = dq/q - 2dr/r$. We are given that $|dq/q| \le 0.002$ and $|dr/r| \le 0.005$, so $|dE/E| \le 0.002 + 2(0.005) = 0.012 = 1.2\%$.

59. **(a)** $\left|\dfrac{d(xy)}{xy}\right| = \left|\dfrac{y\,dx + x\,dy}{xy}\right| = \left|\dfrac{dx}{x} + \dfrac{dy}{y}\right| \le \left|\dfrac{dx}{x}\right| + \left|\dfrac{dy}{y}\right| \le \dfrac{r}{100} + \dfrac{s}{100}$; $(r+s)\%$.

(b) $\left|\dfrac{d(x/y)}{x/y}\right| = \left|\dfrac{y\,dx - x\,dy}{xy}\right| = \left|\dfrac{dx}{x} - \dfrac{dy}{y}\right| \le \left|\dfrac{dx}{x}\right| + \left|\dfrac{dy}{y}\right| \le \dfrac{r}{100} + \dfrac{s}{100}$; $(r+s)\%$.

(c) $\left|\dfrac{d(x^2y^3)}{x^2y^3}\right| = \left|\dfrac{2xy^3\,dx + 3x^2y^2\,dy}{x^2y^3}\right| = \left|2\dfrac{dx}{x} + 3\dfrac{dy}{y}\right| \le 2\left|\dfrac{dx}{x}\right| + 3\left|\dfrac{dy}{y}\right| \le 2\dfrac{r}{100} + 3\dfrac{s}{100}$; $(2r+3s)\%$.

(d) $\left|\dfrac{d(x^3y^{1/2})}{x^3y^{1/2}}\right| = \left|\dfrac{3x^2y^{1/2}\,dx + (1/2)x^3y^{-1/2}\,dy}{x^3y^{1/2}}\right| = \left|3\dfrac{dx}{x} + \dfrac{1}{2}\dfrac{dy}{y}\right| \le 3\left|\dfrac{dx}{x}\right| + \dfrac{1}{2}\left|\dfrac{dy}{y}\right| \le 3\dfrac{r}{100} + \dfrac{1}{2}\dfrac{s}{100}$; $(3r + \frac{1}{2}s)\%$.

61. $dA = \dfrac{1}{2}b\sin\theta da + \dfrac{1}{2}a\sin\theta db + \dfrac{1}{2}ab\cos\theta d\theta$, $|dA| \le \dfrac{1}{2}b\sin\theta|da| + \dfrac{1}{2}a\sin\theta|db| + \dfrac{1}{2}ab\cos\theta|d\theta| \le \dfrac{1}{2}(50)(1/2)(1/2) + \dfrac{1}{2}(40)(1/2)(1/4) + \dfrac{1}{2}(40)(50)\left(\sqrt{3}/2\right)(\pi/90) = 35/4 + 50\pi\sqrt{3}/9 \approx 39\text{ ft}^2$.

63. $f_x = 2x\sin y, f_y = x^2\cos y$ are both continuous everywhere, so f is differentiable everywhere.

65. That f is differentiable means that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{E_f(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$, where $E_f(x,y) = f(x,y) - L_f(x,y)$; here $L_f(x,y)$ is the linear approximation to f at (x_0,y_0). Let f_x and f_y denote $f_x(x_0,y_0), f_y(x_0,y_0)$ respectively. Then $g(x,y,z) = z - f(x,y), L_f(x,y) = f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0), L_g(x,y,z) = g(x_0,y_0,z_0) + g_x(x-x_0) + g_y(y-y_0) + g_z(z-z_0) = 0 - f_x(x-x_0) - f_y(y-y_0) + (z-z_0)$, and $E_g(x,y,z) = g(x,y,z) - L_g(x,y,z) = (z - f(x,y)) + f_x(x-x_0) + f_y(y-y_0) - (z-z_0) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - f(x,y) = -E_f(x,y)$. Thus $\dfrac{|E_g(x,y,z)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \le \dfrac{|E_f(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$, so

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E_g(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0 \text{ and } g \text{ is differentiable at } (x_0,y_0,z_0).$$

Exercise Set 13.5

1. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = 42t^{13}$.

3. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = 3t^{-2}\sin(1/t)$.

5. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = -\dfrac{10}{3}t^{7/3}e^{1-t^{10/3}}$.

7. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = 165t^{32}$.

9. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = -2t\cos\left(t^2\right)$.

11. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = 3264$.

13. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = 3(2t)_{t=2} - (3t^2)_{t=2} = 12 - 12 = 0$.

15. Let $z = xy$, and let $x = f(t)$ and $y = g(t)$. Then $z = f(t)g(t)$ and $(f(t)g(t))' = \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = y\dfrac{dx}{dt} + x\dfrac{dy}{dt} = g(t)f'(t) + f(t)g'(t)$.

17. $\partial z/\partial u = 24u^2v^2 - 16uv^3 - 2v + 3$, $\partial z/\partial v = 16u^3v - 24u^2v^2 - 2u - 3$.

19. $\partial z/\partial u = -\dfrac{2\sin u}{3\sin v}$, $\partial z/\partial v = -\dfrac{2\cos u\cos v}{3\sin^2 v}$.

21. $\partial z/\partial u = e^u$, $\partial z/\partial v = 0$.

23. $\partial T/\partial r = 3r^2\sin\theta\cos^2\theta - 4r^3\sin^3\theta\cos\theta$, $\partial T/\partial\theta = -2r^3\sin^2\theta\cos\theta + r^4\sin^4\theta + r^3\cos^3\theta - 3r^4\sin^2\theta\cos^2\theta$.

25. $\partial t/\partial x = \left(x^2+y^2\right)/\left(4x^2y^3\right)$, $\partial t/\partial y = \left(y^2-3x^2\right)/\left(4xy^4\right)$.

27. $\partial z/\partial r = (dz/dx)(\partial x/\partial r) = 2r\cos^2\theta/\left(r^2\cos^2\theta+1\right)$, $\partial z/\partial\theta = (dz/dx)(\partial x/\partial\theta) = -2r^2\sin\theta\cos\theta/\left(r^2\cos^2\theta+1\right)$.

29. $\partial w/\partial\rho = 2\rho\left(4\sin^2\phi + \cos^2\phi\right)$, $\partial w/\partial\phi = 6\rho^2\sin\phi\cos\phi$, $\partial w/\partial\theta = 0$.

31. $-\pi$.

33. $\sqrt{3}e^{\sqrt{3}}$, $\left(2-4\sqrt{3}\right)e^{\sqrt{3}}$.

35. $A = \dfrac{1}{2}ab\sin\theta$, so $\dfrac{dA}{dt} = \dfrac{\partial A}{\partial a}\dfrac{da}{dt} + \dfrac{\partial A}{\partial b}\dfrac{db}{dt} + \dfrac{\partial A}{\partial\theta}\dfrac{d\theta}{dt}$. This gives us $0 = \dfrac{dA}{dt} = \dfrac{1}{2}b\sin\theta\dfrac{da}{dt} + \dfrac{1}{2}a\sin\theta\dfrac{db}{dt} + \dfrac{1}{2}ab\cos\theta\dfrac{d\theta}{dt}$. From here, $\dfrac{d\theta}{dt} = -(b\sin\theta\dfrac{da}{dt} + a\sin\theta\dfrac{db}{dt})/(ab\cos\theta)$, and with the given values, $\dfrac{d\theta}{dt} = -\dfrac{9\sqrt{3}}{20} \approx -0.779423$ rad/s.

37. False; by themselves they have no meaning.

39. False; consider $z = xy, x = t, y = t$; then $z = t^2$.

41. $F(x, y) = x^2y^3 + \cos y$, $\dfrac{dy}{dx} = -\dfrac{2xy^3}{3x^2y^2 - \sin y}$.

43. $F(x, y) = e^{xy} + ye^y - 1$, $\dfrac{dy}{dx} = -\dfrac{ye^{xy}}{xe^{xy} + ye^y + e^y}$.

45. $\dfrac{\partial z}{\partial x} = \dfrac{2x + yz}{6yz - xy}$, $\dfrac{\partial z}{\partial y} = \dfrac{xz - 3z^2}{6yz - xy}$.

47. $ye^x - 5\sin 3z - 3z = 0$; $\dfrac{\partial z}{\partial x} = -\dfrac{ye^x}{-15\cos 3z - 3} = \dfrac{ye^x}{15\cos 3z + 3}$, $\dfrac{\partial z}{\partial y} = \dfrac{e^x}{15\cos 3z + 3}$.

49. **(a)** $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x}$, $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\dfrac{\partial u}{\partial y}$.

(b) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}\left(\dfrac{dz}{du}\right)\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{d^2 z}{du^2}\left(\dfrac{\partial u}{\partial x}\right)^2$;

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial}{\partial y}\left(\dfrac{dz}{du}\right)\dfrac{\partial u}{\partial y} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y^2} + \dfrac{d^2 z}{du^2}\left(\dfrac{\partial u}{\partial y}\right)^2$; $\dfrac{\partial^2 z}{\partial y\partial x} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y\partial x} + \dfrac{\partial}{\partial y}\left(\dfrac{dz}{du}\right)\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}\dfrac{\partial^2 u}{\partial y\partial x} + \dfrac{d^2 z}{du^2}\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}$.

51. Let $z = f(u)$ where $u = x + 2y$; then $\partial z/\partial x = (dz/du)(\partial u/\partial x) = dz/du$, $\partial z/\partial y = (dz/du)(\partial u/\partial y) = 2dz/du$ so $2\partial z/\partial x - \partial z/\partial y = 2dz/du - 2dz/du = 0$.

53. $\dfrac{\partial w}{\partial x} = \dfrac{dw}{du}\dfrac{\partial u}{\partial x} = \dfrac{dw}{du}$, $\dfrac{\partial w}{\partial y} = \dfrac{dw}{du}\dfrac{\partial u}{\partial y} = 2\dfrac{dw}{du}$, $\dfrac{\partial w}{\partial z} = \dfrac{dw}{du}\dfrac{\partial u}{\partial z} = 3\dfrac{dw}{du}$, so $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} + \dfrac{\partial w}{\partial z} = 6\dfrac{dw}{du}$.

55. $z = f(u, v)$ where $u = x - y$ and $v = y - x$, $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$ and $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = -\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$ so $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

57. **(a)** $1 = -r\sin\theta\dfrac{\partial\theta}{\partial x} + \cos\theta\dfrac{\partial r}{\partial x}$ and $0 = r\cos\theta\dfrac{\partial\theta}{\partial x} + \sin\theta\dfrac{\partial r}{\partial x}$; solve for $\partial r/\partial x$ and $\partial\theta/\partial x$.

(b) $0 = -r\sin\theta\dfrac{\partial\theta}{\partial y} + \cos\theta\dfrac{\partial r}{\partial y}$ and $1 = r\cos\theta\dfrac{\partial\theta}{\partial y} + \sin\theta\dfrac{\partial r}{\partial y}$; solve for $\partial r/\partial y$ and $\partial\theta/\partial y$.

(c) $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial x} = \dfrac{\partial z}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\sin\theta$, $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial y} = \dfrac{\partial z}{\partial r}\sin\theta + \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\cos\theta$.

(d) Square and add the results of parts (a) and (b).

(e) From part (c), $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial r}\left(\dfrac{\partial z}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\sin\theta\right)\dfrac{\partial r}{\partial x} + \dfrac{\partial}{\partial\theta}\left(\dfrac{\partial z}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\sin\theta\right)\dfrac{\partial\theta}{\partial x} =$

$= \left(\dfrac{\partial^2 z}{\partial r^2}\cos\theta + \dfrac{1}{r^2}\dfrac{\partial z}{\partial\theta}\sin\theta - \dfrac{1}{r}\dfrac{\partial^2 z}{\partial r\partial\theta}\sin\theta\right)\cos\theta + \left(\dfrac{\partial^2 z}{\partial\theta\partial r}\cos\theta - \dfrac{\partial z}{\partial r}\sin\theta - \dfrac{1}{r}\dfrac{\partial^2 z}{\partial\theta^2}\sin\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\cos\theta\right)\left(-\dfrac{\sin\theta}{r}\right) =$
$\dfrac{\partial^2 z}{\partial r^2}\cos^2\theta + \dfrac{2}{r^2}\dfrac{\partial z}{\partial\theta}\sin\theta\cos\theta - \dfrac{2}{r}\dfrac{\partial^2 z}{\partial\theta\partial r}\sin\theta\cos\theta + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial\theta^2}\sin^2\theta + \dfrac{1}{r}\dfrac{\partial z}{\partial r}\sin^2\theta$.

Similarly, from part (c), $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2}\sin^2\theta - \dfrac{2}{r^2}\dfrac{\partial z}{\partial\theta}\sin\theta\cos\theta + \dfrac{2}{r}\dfrac{\partial^2 z}{\partial\theta\partial r}\sin\theta\cos\theta + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial\theta^2}\cos^2\theta + \dfrac{1}{r}\dfrac{\partial z}{\partial r}\cos^2\theta$.

Add these to get $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r}$.

59. **(a)** By the chain rule, $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and $\dfrac{\partial v}{\partial \theta} = -\dfrac{\partial v}{\partial x} r\sin\theta + \dfrac{\partial v}{\partial y} r\cos\theta$, use the Cauchy-Riemann conditions $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ in the equation for $\dfrac{\partial u}{\partial r}$ to get $\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial y}\cos\theta - \dfrac{\partial v}{\partial x}\sin\theta$ and compare to $\dfrac{\partial v}{\partial \theta}$ to see that $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$. The result $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ can be obtained by considering $\dfrac{\partial v}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$.

(b) $u_x = \dfrac{2x}{x^2+y^2}$, $v_y = 2\dfrac{1}{x}\dfrac{1}{1+(y/x)^2} = \dfrac{2x}{x^2+y^2} = u_x; u_y = \dfrac{2y}{x^2+y^2}$, $v_x = -2\dfrac{y}{x^2}\dfrac{1}{1+(y/x)^2} = -\dfrac{2y}{x^2+y^2} = -u_y; u = \ln r^2, v = 2\theta, u_r = 2/r, v_\theta = 2$, so $u_r = \dfrac{1}{r}v_\theta$, $u_\theta = 0, v_r = 0$, so $v_r = -\dfrac{1}{r}u_\theta$.

61. $\partial w/\partial\rho = (\sin\phi\cos\theta)\partial w/\partial x + (\sin\phi\sin\theta)\partial w/\partial y + (\cos\phi)\,\partial w/\partial z$,
$\partial w/\partial\phi = (\rho\cos\phi\cos\theta)\partial w/\partial x + (\rho\cos\phi\sin\theta)\partial w/\partial y - (\rho\sin\phi)\partial w/\partial z$,
$\partial w/\partial\theta = -(\rho\sin\phi\sin\theta)\partial w/\partial x + (\rho\sin\phi\cos\theta)\partial w/\partial y$.

63. $w_r = e^r/(e^r+e^s+e^t+e^u)$, $w_{rs} = -e^re^s/(e^r+e^s+e^t+e^u)^2$, $w_{rst} = 2e^re^se^t/(e^r+e^s+e^t+e^u)^3$, $w_{rstu} = -6e^re^se^te^u/\left(e^r+e^s+e^t+e^u\right)^4 = -6e^{r+s+t+u}/e^{4w} = -6e^{r+s+t+u-4w}$.

65. **(a)** $dw/dt = \displaystyle\sum_{i=1}^{4}(\partial w/\partial x_i)\,(dx_i/dt)$. **(b)** $\partial w/\partial v_j = \displaystyle\sum_{i=1}^{4}(\partial w/\partial x_i)\,(\partial x_i/\partial v_j)$ for $j = 1, 2, 3$.

67. $dF/dx = (\partial F/\partial u)(du/dx) + (\partial F/\partial v)(dv/dx) = f(u)g'(x) - f(v)h'(x) = f(g(x))g'(x) - f(h(x))h'(x)$.

69. Let (a,b) be any point in the region, if (x,y) is in the region then by the result of Exercise 74 $f(x,y) - f(a,b) = f_x(x^*,y^*)(x-a) + f_y(x^*,y^*)(y-b)$, where (x^*,y^*) is on the line segment joining (a,b) and (x,y). If $f_x(x,y) = f_y(x,y) = 0$ throughout the region then $f(x,y) - f(a,b) = (0)(x-a) + (0)(y-b) = 0$, $f(x,y) = f(a,b)$ so $f(x,y)$ is constant on the region.

Exercise Set 13.6

1. $\nabla f(x,y) = (3y/2)(1+xy)^{1/2}\mathbf{i} + (3x/2)(1+xy)^{1/2}\mathbf{j}$, $\nabla f(3,1) = 3\mathbf{i} + 9\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u} = 12/\sqrt{2} = 6\sqrt{2}$.

3. $\nabla f(x,y) = \left[2x/\left(1+x^2+y\right)\right]\mathbf{i} + \left[1/\left(1+x^2+y\right)\right]\mathbf{j}$, $\nabla f(0,0) = \mathbf{j}$, $D_{\mathbf{u}}f = -3/\sqrt{10}$.

5. $\nabla f(x,y,z) = 20x^4y^2z^3\mathbf{i} + 8x^5yz^3\mathbf{j} + 12x^5y^2z^2\mathbf{k}$, $\nabla f(2,-1,1) = 320\mathbf{i} - 256\mathbf{j} + 384\mathbf{k}$, $D_{\mathbf{u}}f = -320$.

7. $\nabla f(x,y,z) = \dfrac{2x}{x^2+2y^2+3z^2}\mathbf{i} + \dfrac{4y}{x^2+2y^2+3z^2}\mathbf{j} + \dfrac{6z}{x^2+2y^2+3z^2}\mathbf{k}$, $\nabla f(-1,2,4) = (-2/57)\mathbf{i} + (8/57)\mathbf{j} + (24/57)\mathbf{k}$, $D_{\mathbf{u}}f = -314/741$.

9. $\nabla f(x,y) = 12x^2y^2\mathbf{i} + 8x^3y\mathbf{j}$, $\nabla f(2,1) = 48\mathbf{i} + 64\mathbf{j}$, $\mathbf{u} = (4/5)\mathbf{i} - (3/5)\mathbf{j}$, $D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u} = 0$.

11. $\nabla f(x,y) = \left(y^2/x\right)\mathbf{i} + 2y\ln x\mathbf{j}$, $\nabla f(1,4) = 16\mathbf{i}$, $\mathbf{u} = (-\mathbf{i}+\mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = -8\sqrt{2}$.

13. $\nabla f(x,y) = -\left[y/\left(x^2+y^2\right)\right]\mathbf{i} + \left[x/\left(x^2+y^2\right)\right]\mathbf{j}, \nabla f(-2,2) = -(\mathbf{i}+\mathbf{j})/4$, $\mathbf{u} = -(\mathbf{i}+\mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = \sqrt{2}/4$.

15. $\nabla f(x,y,z) = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, $\nabla f(-3,0,4) = -3\mathbf{j} + 8\mathbf{k}$, $\mathbf{u} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$, $D_{\mathbf{u}}f = 5/\sqrt{3}$.

17. $\nabla f(x,y,z) = -\dfrac{1}{z+y}\mathbf{i} - \dfrac{z-x}{(z+y)^2}\mathbf{j} + \dfrac{y+x}{(z+y)^2}\mathbf{k}$, $\nabla f(1,0,-3) = (1/3)\mathbf{i} + (4/9)\mathbf{j} + (1/9)\mathbf{k}$, $\mathbf{u} = (-6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})/7$, $D_{\mathbf{u}}f = -8/63$.

19. $\nabla f(x,y) = (y/2)(xy)^{-1/2}\mathbf{i} + (x/2)(xy)^{-1/2}\mathbf{j}$, $\nabla f(1,4) = \mathbf{i} + (1/4)\mathbf{j}$, $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} = (1/2)\mathbf{i} + \left(\sqrt{3}/2\right)\mathbf{j}$, $D_{\mathbf{u}}f = 1/2 + \sqrt{3}/8$.

21. $\nabla f(x,y) = 2\sec^2(2x+y)\mathbf{i} + \sec^2(2x+y)\mathbf{j}$, $\nabla f(\pi/6,\pi/3) = 8\mathbf{i}+4\mathbf{j}$, $\mathbf{u} = (\mathbf{i}-\mathbf{j})/\sqrt{2}$, $D_{\mathbf{u}}f = 2\sqrt{2}$.

23. $\nabla f(x,y) = y(x+y)^{-2}\mathbf{i} - x(x+y)^{-2}\mathbf{j}$, $\nabla f(1,0) = -\mathbf{j}$, $\overrightarrow{PQ} = -2\mathbf{i}-\mathbf{j}$, $\mathbf{u} = (-2\mathbf{i}-\mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}f = 1/\sqrt{5}$.

25. $\nabla f(x,y) = \dfrac{ye^y}{2\sqrt{xy}}\mathbf{i} + \left(\sqrt{xy}e^y + \dfrac{xe^y}{2\sqrt{xy}}\right)\mathbf{j}$, $\nabla f(1,1) = (e/2)(\mathbf{i}+3\mathbf{j})$, $\mathbf{u} = -\mathbf{j}$, $D_{\mathbf{u}}f = -3e/2$.

27. $\nabla f(2,1,-1) = -\mathbf{i}+\mathbf{j}-\mathbf{k}$. $\overrightarrow{PQ} = -3\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{u} = (-3\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{11}$, $D_{\mathbf{u}}f = 3/\sqrt{11}$.

29. Solve the system $(3/5)f_x(1,2) - (4/5)f_y(1,2) = -5$, $(4/5)f_x(1,2) + (3/5)f_y(1,2) = 10$ for

(a) $f_x(1,2) = 5$. **(b)** $f_y(1,2) = 10$. **(c)** $\nabla f(1,2) = 5\mathbf{i}+10\mathbf{j}$, $\mathbf{u} = (-\mathbf{i}-2\mathbf{j})/\sqrt{5}$, $D_{\mathbf{u}}f = -5\sqrt{5}$.

31. f increases the most in the direction of III.

33. $\nabla z = -7y\cos(7y^2-7xy)\mathbf{i} + (14y-7x)\cos(7y^2-7xy)\mathbf{j}$.

35. $\nabla z = -\dfrac{84y}{(6x-7y)^2}\mathbf{i} + \dfrac{84x}{(6x-7y)^2}\mathbf{j}$.

37. $\nabla w = -9x^8\mathbf{i} - 3y^2\mathbf{j} + 12z^{11}\mathbf{k}$.

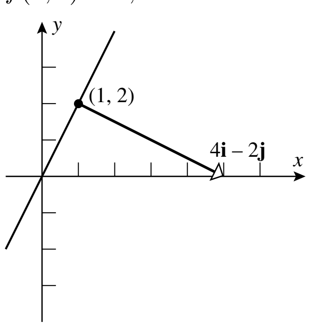
39. $\nabla w = \dfrac{x}{x^2+y^2+z^2}\mathbf{i} + \dfrac{y}{x^2+y^2+z^2}\mathbf{j} + \dfrac{z}{x^2+y^2+z^2}\mathbf{k}$.

41. $\nabla f(x,y) = 10x\mathbf{i} + 4y^3\mathbf{j}$, $\nabla f(4,2) = 40\mathbf{i}+32\mathbf{j}$.

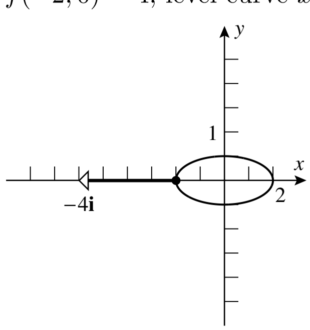
43. $\nabla f(x,y) = 3(2x+y)\left(x^2+xy\right)^2\mathbf{i} + 3x\left(x^2+xy\right)^2\mathbf{j}$, $\nabla f(-1,-1) = -36\mathbf{i}-12\mathbf{j}$.

45. $\nabla f(x,y,z) = [y/(x+y+z)]\mathbf{i} + [y/(x+y+z) + \ln(x+y+z)]\mathbf{j} + [y/(x+y+z)]\mathbf{k}$, $\nabla f(-3,4,0) = 4\mathbf{i}+4\mathbf{j}+4\mathbf{k}$.

47. $f(1,2) = 3$, level curve $4x-2y+3 = 3$, $2x-y = 0$; $\nabla f(x,y) = 4\mathbf{i}-2\mathbf{j}$, $\nabla f(1,2) = 4\mathbf{i}-2\mathbf{j}$.



49. $f(-2,0) = 4$, level curve $x^2+4y^2 = 4$, $x^2/4+y^2 = 1$. $\nabla f(x,y) = 2x\mathbf{i}+8y\mathbf{j}$, $\nabla f(-2,0) = -4\mathbf{i}$.



51. $\nabla f(x,y) = 8xy\mathbf{i} + 4x^2\mathbf{j}$, $\nabla f(1,-2) = -16\mathbf{i} + 4\mathbf{j}$ is normal to the level curve through P so $\mathbf{u} = \pm(-4\mathbf{i} + \mathbf{j})/\sqrt{17}$.

53. $\nabla f(x,y) = 12x^2y^2\mathbf{i} + 8x^3y\mathbf{j}$, $\nabla f(-1,1) = 12\mathbf{i} - 8\mathbf{j}$, $\mathbf{u} = (3\mathbf{i} - 2\mathbf{j})/\sqrt{13}$, $\|\nabla f(-1,1)\| = 4\sqrt{13}$.

55. $\nabla f(x,y) = x\left(x^2+y^2\right)^{-1/2}\mathbf{i} + y\left(x^2+y^2\right)^{-1/2}\mathbf{j}$, $\nabla f(4,-3) = (4\mathbf{i} - 3\mathbf{j})/5$, $\mathbf{u} = (4\mathbf{i} - 3\mathbf{j})/5$, $\|\nabla f(4,-3)\| = 1$.

57. $\nabla f(1,1,-1) = 3\mathbf{i} - 3\mathbf{j}$, $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, $\|\nabla f(1,1,-1)\| = 3\sqrt{2}$.

59. $\nabla f(1,2,-2) = (-\mathbf{i} + \mathbf{j})/2$, $\mathbf{u} = (-\mathbf{i} + \mathbf{j})/\sqrt{2}$, $\|\nabla f(1,2,-2)\| = 1/\sqrt{2}$.

61. $\nabla f(x,y) = -2x\mathbf{i} - 2y\mathbf{j}$, $\nabla f(-1,-3) = 2\mathbf{i} + 6\mathbf{j}$, $\mathbf{u} = -(\mathbf{i} + 3\mathbf{j})/\sqrt{10}$, $-\|\nabla f(-1,-3)\| = -2\sqrt{10}$.

63. $\nabla f(x,y) = -3\sin(3x-y)\mathbf{i} + \sin(3x-y)\mathbf{j}$, $\nabla f(\pi/6,\pi/4) = (-3\mathbf{i} + \mathbf{j})/\sqrt{2}$, $\mathbf{u} = (3\mathbf{i} - \mathbf{j})/\sqrt{10}$, $-\|\nabla f(\pi/6,\pi/4)\| = -\sqrt{5}$.

65. $\nabla f(5,7,6) = -\mathbf{i} + 11\mathbf{j} - 12\mathbf{k}$, $\mathbf{u} = (\mathbf{i} - 11\mathbf{j} + 12\mathbf{k})/\sqrt{266}$, $-\|\nabla f(5,7,6)\| = -\sqrt{266}$.

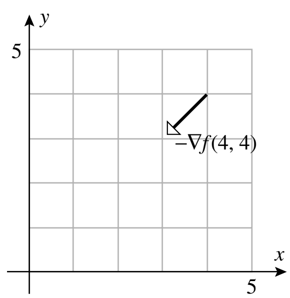
67. False; actually they are equal: $D_{\mathbf{v}}(f) = \nabla f \cdot \mathbf{v}/\|\mathbf{v}\| = \nabla f \cdot 2\|\mathbf{u}\|/2 = D_{\mathbf{u}}(f)$.

69. False; $f(x,y) = x$ and $\mathbf{u} = \mathbf{j}$.

71. $\nabla f(4,-5) = 2\mathbf{i} - \mathbf{j}$, $\mathbf{u} = (5\mathbf{i} + 2\mathbf{j})/\sqrt{29}$, $D_{\mathbf{u}}f = 8/\sqrt{29}$.

73. **(a)** At $(1,2)$ the steepest ascent seems to be in the direction $\mathbf{i} + \mathbf{j}$ and the slope in that direction seems to be $0.5/(\sqrt{2}/2) = 1/\sqrt{2}$, so $\nabla f \approx \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$, which has the required direction and magnitude.

(b) The direction of $-\nabla f(4,4)$ appears to be $-\mathbf{i} - \mathbf{j}$ and its magnitude appears to be $1/0.8 = 5/4$.



75. $\nabla z = 6x\mathbf{i} - 2y\mathbf{j}$, $\|\nabla z\| = \sqrt{36x^2 + 4y^2} = 6$ if $36x^2 + 4y^2 = 36$; all points on the ellipse $9x^2 + y^2 = 9$.

77. $\mathbf{r} = t\mathbf{i} - t^2\mathbf{j}$, $d\mathbf{r}/dt = \mathbf{i} - 2t\mathbf{j} = \mathbf{i} - 4\mathbf{j}$ at the point $(2,-4)$, $\mathbf{u} = (\mathbf{i} - 4\mathbf{j})/\sqrt{17}$; $\nabla z = 2x\mathbf{i} + 2y\mathbf{j} = 4\mathbf{i} - 8\mathbf{j}$ at $(2,-4)$, hence $dz/ds = D_{\mathbf{u}}z = \nabla z \cdot \mathbf{u} = 36/\sqrt{17}$.

79. **(a)** $\nabla V(x,y) = -2e^{-2x}\cos 2y\mathbf{i} - 2e^{-2x}\sin 2y\mathbf{j}$, $\mathbf{E} = -\nabla V(\pi/4,0) = 2e^{-\pi/2}\mathbf{i}$.

(b) $V(x,y)$ decreases most rapidly in the direction of $-\nabla V(x,y)$ which is $\mathbf{E}$.

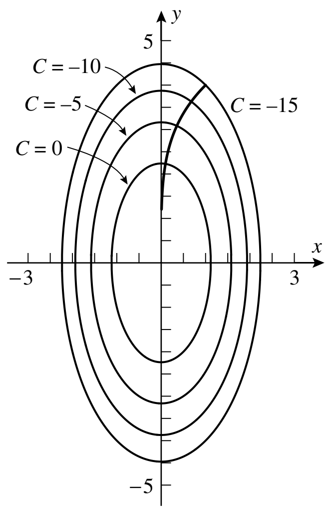
81. Let $\mathbf{u}$ be the unit vector in the direction of $\mathbf{a}$, then $D_{\mathbf{u}}f(3,-2,1) = \nabla f(3,-2,1) \cdot \mathbf{u} = \|\nabla f(3,-2,1)\|\cos\theta = 5\cos\theta = -5$, $\cos\theta = -1$, $\theta = \pi$ so $\nabla f(3,-2,1)$ is oppositely directed to $\mathbf{u}$; $\nabla f(3,-2,1) = -5\mathbf{u} = -10/3\mathbf{i} + 5/3\mathbf{j} + 10/3\mathbf{k}$.

83. **(a)** $\nabla r = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j} = \mathbf{r}/r$.

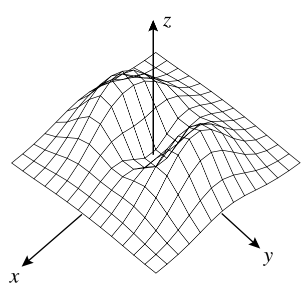
(b) $\nabla f(r) = \dfrac{\partial f(r)}{\partial x}\mathbf{i} + \dfrac{\partial f(r)}{\partial y}\mathbf{j} = f'(r)\dfrac{\partial r}{\partial x}\mathbf{i} + f'(r)\dfrac{\partial r}{\partial y}\mathbf{j} = f'(r)\nabla r$.

85. $\mathbf{u}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $\mathbf{u}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$, $\nabla z = \dfrac{\partial z}{\partial x}\mathbf{i} + \dfrac{\partial z}{\partial y}\mathbf{j} = \left(\dfrac{\partial z}{\partial r}\cos\theta - \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\sin\theta\right)\mathbf{i} + \left(\dfrac{\partial z}{\partial r}\sin\theta + \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\cos\theta\right)\mathbf{j} = \dfrac{\partial z}{\partial r}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dfrac{\partial z}{\partial r}\mathbf{u}_r + \dfrac{1}{r}\dfrac{\partial z}{\partial\theta}\mathbf{u}_\theta$.

87. $\mathbf{r}'(t) = \mathbf{v}(t) = k(x,y)\nabla\mathbf{T} = -8k(x,y)x\mathbf{i} - 2k(x,y)y\mathbf{j}$; $\dfrac{dx}{dt} = -8kx$, $\dfrac{dy}{dt} = -2ky$. Divide and solve to get $y^4 = 256x$; one parametrization is $x(t) = e^{-8t},\ y(t) = 4e^{-2t}$.



89.



91. (a)

(c) $\nabla f = [2x - 2x(x^2+3y^2)]e^{-(x^2+y^2)}\mathbf{i} + [6y - 2y(x^2+3y^2)]e^{-(x^2+y^2)}\mathbf{j}$.

(d) $\nabla f = \mathbf{0}$ if $x = y = 0$ or $x = 0, y = \pm 1$ or $x = \pm 1, y = 0$.

93. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$, if $\nabla f(x,y) = 0$ throughout the region then $f_x(x,y) = f_y(x,y) = 0$ throughout the region, the result follows from Exercise 69, Section 13.5.

95. $\nabla f(u,v,w) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k} = \left(\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x}\right)\mathbf{i} + \left(\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial y}\right)\mathbf{j} + \left(\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial z} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial z} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial z}\right)\mathbf{k} = \dfrac{\partial f}{\partial u}\nabla u + \dfrac{\partial f}{\partial v}\nabla v + \dfrac{\partial f}{\partial w}\nabla w$.

Exercise Set 13.7

1. (a) $f(x,y,z) = x^2 + y^2 + 4z^2$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$, $\nabla f(2,2,1) = 4\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$, $\mathbf{n} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $x + y + 2z = 6$.

(b) $\mathbf{r}(t) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, $x(t) = 2 + t$, $y(t) = 2 + t$, $z(t) = 1 + 2t$.

(c) $\cos\theta = \dfrac{\mathbf{n}\cdot\mathbf{k}}{\|\mathbf{n}\|} = \dfrac{\sqrt{2}}{\sqrt{3}}$, $\theta \approx 35.26^\circ$.

3. $\nabla F = \langle 2x, 2y, 2z\rangle$, so $\mathbf{n} = \langle -6, 0, 8\rangle$, so the tangent plane is given by $-6(x+3)+8(z-4) = 0$ or $3x - 4z = -25$, normal line $x = -3 - 6t$, $y = 0$, $z = 4 + 8t$.

5. $\nabla F = \langle 2x - yz, -xz, -xy\rangle$, so $\mathbf{n} = \langle -18, 8, 20\rangle$, so the tangent plane is given by $-18x + 8y + 20z = 152$, normal line $x = -4 - 18t$, $y = 5 + 8t$, $z = 2 + 20t$.

7. At P, $\partial z/\partial x = 48$ and $\partial z/\partial y = -14$, tangent plane $48x - 14y - z = 64$, normal line $x = 1 + 48t$, $y = -2 - 14t$, $z = 12 - t$.

9. At P, $\partial z/\partial x = 1$ and $\partial z/\partial y = -1$, tangent plane $x - y - z = 0$, normal line $x = 1 + t$, $y = -t$, $z = 1 - t$.

11. At P, $\partial z/\partial x = 0$ and $\partial z/\partial y = 3$, tangent plane $3y - z = -1$, normal line $x = \pi/6$, $y = 3t$, $z = 1 - t$.

13. The tangent plane is horizontal if the normal $\partial z/\partial x\mathbf{i} + \partial z/\partial y\mathbf{j} - \mathbf{k}$ is parallel to $\mathbf{k}$ which occurs when $\partial z/\partial x = \partial z/\partial y = 0$.

(a) $\partial z/\partial x = 3x^2y^2$, $\partial z/\partial y = 2x^3y$; $3x^2y^2 = 0$ and $2x^3y = 0$ for all (x, y) on the x-axis or y-axis, and $z = 0$ for these points, the tangent plane is horizontal at all points on the x-axis or y-axis.

(b) $\partial z/\partial x = 2x - y - 2$, $\partial z/\partial y = -x + 2y + 4$; solve the system $2x - y - 2 = 0$, $-x + 2y + 4 = 0$, to get $x = 0$, $y = -2$. $z = -4$ at $(0, -2)$, the tangent plane is horizontal at $(0, -2, -4)$.

15. $\partial z/\partial x = -6x$, $\partial z/\partial y = -4y$ so $-6x_0\mathbf{i} - 4y_0\mathbf{j} - \mathbf{k}$ is normal to the surface at a point (x_0, y_0, z_0) on the surface. This normal must be parallel to the given line and hence to the vector $-3\mathbf{i} + 8\mathbf{j} - \mathbf{k}$ which is parallel to the line so $-6x_0 = -3$, $x_0 = 1/2$ and $-4y_0 = 8$, $y_0 = -2$. $z = -3/4$ at $(1/2, -2)$. The point on the surface is $(1/2, -2, -3/4)$.

17. **(a)** $2t + 7 = (-1 + t)^2 + (2 + t)^2$, $t^2 = 1$, $t = \pm 1$ so the points of intersection are $(-2, 1, 5)$ and $(0, 3, 9)$.

(b) $\partial z/\partial x = 2x$, $\partial z/\partial y = 2y$ so at $(-2, 1, 5)$ the vector $\mathbf{n} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is normal to the surface. $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is parallel to the line; $\mathbf{n} \cdot \mathbf{v} = -4$ so the cosine of the acute angle is $[\mathbf{n} \cdot (-\mathbf{v})]/(\|\mathbf{n}\|\,\| - \mathbf{v}\|) = 4/\left(\sqrt{21}\sqrt{6}\right) = 4/\left(3\sqrt{14}\right)$. Similarly, at (0,3,9) the vector $\mathbf{n} = 6\mathbf{j} - \mathbf{k}$ is normal to the surface, $\mathbf{n} \cdot \mathbf{v} = 4$ so the cosine of the acute angle is $4/\left(\sqrt{37}\sqrt{6}\right) = 4/\sqrt{222}$.

19. False, they only need to be parallel.

21. True, see Section 13.4 equation (15).

23. Set $f(x, y, z) = z + x - z^4(y - 1)$, then $f(x, y, z) = 0, \mathbf{n} = \pm\nabla f(3, 5, 1) = \pm(\mathbf{i} - \mathbf{j} - 15\mathbf{k})$, unit vectors $\pm\dfrac{1}{\sqrt{227}}(\mathbf{i} - \mathbf{j} - 15\mathbf{k})$.

25. $f(x, y, z) = x^2 + y^2 + z^2$, if (x_0, y_0, z_0) is on the sphere then $\nabla f(x_0, y_0, z_0) = 2(x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k})$ is normal to the sphere at (x_0, y_0, z_0), the normal line is $x = x_0 + x_0t$, $y = y_0 + y_0t$, $z = z_0 + z_0t$ which passes through the origin when $t = -1$.

27. $f(x, y, z) = x^2 + y^2 - z^2$, if (x_0, y_0, z_0) is on the surface then $\nabla f(x_0, y_0, z_0) = 2(x_0\mathbf{i} + y_0\mathbf{j} - z_0\mathbf{k})$ is normal there and hence so is $\mathbf{n}_1 = x_0\mathbf{i} + y_0\mathbf{j} - z_0\mathbf{k}$; $\mathbf{n}_1$ must be parallel to $\overrightarrow{PQ} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ so $\mathbf{n}_1 = c\,\overrightarrow{PQ}$ for some constant c. Equate components to get $x_0 = 3c$, $y_0 = 2c$ and $z_0 = 2c$ which when substituted into the equation of the surface yields $9c^2 + 4c^2 - 4c^2 = 1$, $c^2 = 1/9$, $c = \pm 1/3$ so the points are $(1, 2/3, 2/3)$ and $(-1, -2/3, -2/3)$.

29. $\mathbf{n}_1 = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}, \mathbf{n}_2 = 2\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}, \mathbf{n}_1 \times \mathbf{n}_2 = -16\mathbf{i} - 10\mathbf{j} - 12\mathbf{k}$ is tangent to the line, so $x(t) = 1 + 8t, y(t) = -1 + 5t, z(t) = 2 + 6t$.

31. $f(x, y, z) = x^2 + z^2 - 25,\ g(x, y, z) = y^2 + z^2 - 25, \mathbf{n}_1 = \nabla f(3, -3, 4) = 6\mathbf{i} + 8\mathbf{k}, \mathbf{n}_2 = \nabla g(3, -3, 4) = -6\mathbf{j} + 8\mathbf{k}, \mathbf{n}_1 \times \mathbf{n}_2 = 48\mathbf{i} - 48\mathbf{j} - 36\mathbf{k}$ is tangent to the line, $x(t) = 3 + 4t, y(t) = -3 - 4t, z(t) = 4 - 3t$. The point $(3, -3, 4)$ lies on both surfaces.

33. Use implicit differentiation to get $\partial z/\partial x = -c^2x/\left(a^2z\right)$, $\partial z/\partial y = -c^2y/\left(b^2z\right)$. At (x_0, y_0, z_0), $z_0 \neq 0$, a normal to the surface is $-\left[c^2x_0/\left(a^2z_0\right)\right]\mathbf{i} - \left[c^2y_0/\left(b^2z_0\right)\right]\mathbf{j} - \mathbf{k}$ so the tangent plane is $-\dfrac{c^2x_0}{a^2z_0}x - \dfrac{c^2y_0}{b^2z_0}y - z = -\dfrac{c^2x_0^2}{a^2z_0} - \dfrac{c^2y_0^2}{b^2z_0} - z_0$, $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} + \dfrac{z_0z}{c^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2} = 1$.

35. $\mathbf{n}_1 = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = g_x(x_0, y_0)\mathbf{i} + g_y(x_0, y_0)\mathbf{j} - \mathbf{k}$ are normal, respectively, to $z = f(x, y)$ and $z = g(x, y)$ at P; $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$, $f_x(x_0, y_0)\,g_x(x_0, y_0) + f_y(x_0, y_0)\,g_y(x_0, y_0) + 1 = 0$, $f_x(x_0, y_0)\,g_x(x_0, y_0) + f_y(x_0, y_0)\,g_y(x_0, y_0) = -1$.

37. $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ and $\nabla g = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k}$ evaluated at (x_0, y_0, z_0) are normal, respectively, to the surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ at (x_0, y_0, z_0). The surfaces are orthogonal at (x_0, y_0, z_0) if and only if $\nabla f \cdot \nabla g = 0$ so $f_xg_x + f_yg_y + f_zg_z = 0$.

39. $z = \dfrac{k}{xy}$; at a point $\left(a, b, \dfrac{k}{ab}\right)$ on the surface, $\left\langle -\dfrac{k}{a^2b}, -\dfrac{k}{ab^2}, -1\right\rangle$ and hence $\langle bk, ak, a^2b^2\rangle$ is normal to the surface so the tangent plane is $bkx + aky + a^2b^2z = 3abk$. The plane cuts the x, y, and z-axes at the points $3a$, $3b$, and $\dfrac{3k}{ab}$, respectively, so the volume of the tetrahedron that is formed is $V = \dfrac{1}{3}\left(\dfrac{3k}{ab}\right)\left[\dfrac{1}{2}(3a)(3b)\right] = \dfrac{9}{2}k$, which does not depend on a and b.

Exercise Set 13.8

1. **(a)** Minimum at $(2, -1)$, no maxima. **(b)** Maximum at $(0, 0)$, no minima. **(c)** No maxima or minima.

3. $f(x, y) = (x-3)^2 + (y+2)^2$, minimum at $(3, -2)$, no maxima.

5. $f_x = 6x + 2y = 0$, $f_y = 2x + 2y = 0$; critical point (0,0); $D = 8 > 0$ and $f_{xx} = 6 > 0$ at (0,0), relative minimum.

7. $f_x = 2x - 2xy = 0$, $f_y = 4y - x^2 = 0$; critical points (0,0) and $(\pm 2, 1)$; $D = 8 > 0$ and $f_{xx} = 2 > 0$ at (0,0), relative minimum; $D = -16 < 0$ at $(\pm 2, 1)$, saddle points.

9. $f_x = y + 2 = 0$, $f_y = 2y + x + 3 = 0$; critical point $(1, -2)$; $D = -1 < 0$ at $(1, -2)$, saddle point.

11. $f_x = 2x + y - 3 = 0$, $f_y = x + 2y = 0$; critical point $(2, -1)$; $D = 3 > 0$ and $f_{xx} = 2 > 0$ at $(2, -1)$, relative minimum.

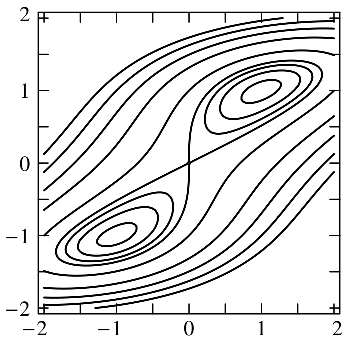
13. $f_x = 2x - 2/\left(x^2y\right) = 0$, $f_y = 2y - 2/\left(xy^2\right) = 0$; critical points $(-1, -1)$ and $(1, 1)$; $D = 32 > 0$ and $f_{xx} = 6 > 0$ at $(-1, -1)$ and $(1, 1)$, relative minima.

15. $f_x = 2x = 0$, $f_y = 1 - e^y = 0$; critical point $(0, 0)$; $D = -2 < 0$ at $(0, 0)$, saddle point.

17. $f_x = e^x \sin y = 0$, $f_y = e^x \cos y = 0$, $\sin y = \cos y = 0$ is impossible, no critical points.

19. $f_x = -2(x+1)e^{-\left(x^2+y^2+2x\right)} = 0$, $f_y = -2ye^{-\left(x^2+y^2+2x\right)} = 0$; critical point $(-1, 0)$; $D = 4e^2 > 0$ and $f_{xx} = -2e < 0$ at $(-1, 0)$, relative maximum.

21. $\nabla f = (4x - 4y)\mathbf{i} - (4x - 4y^3)\mathbf{j} = \mathbf{0}$ when $x = y, x = y^3$, so $x = y = 0$ or $x = y = \pm 1$. At $(0, 0), D = -16$, a saddle point; at $(1, 1)$ and $(-1, -1), D = 32 > 0, f_{xx} = 4$, a relative minimum.



23. False, e.g. $f(x, y) = x$.

25. True, Theorem 13.8.6.

27. **(a)** Critical point (0,0); $D = 0$.

(b) $f(0, 0) = 0$, $x^4 + y^4 \geq 0$ so $f(x, y) \geq f(0, 0)$, relative minimum.

29. **(a)** $f_x = 3e^y - 3x^2 = 3\left(e^y - x^2\right) = 0$, $f_y = 3xe^y - 3e^{3y} = 3e^y\left(x - e^{2y}\right) = 0$, $e^y = x^2$ and $e^{2y} = x$, $x^4 = x$, $x\left(x^3 - 1\right) = 0$ so $x = 0, 1$; critical point $(1, 0)$; $D = 27 > 0$ and $f_{xx} = -6 < 0$ at $(1, 0)$, relative maximum.

(b) $\lim_{x \to -\infty} f(x, 0) = \lim_{x \to -\infty} \left(3x - x^3 - 1\right) = +\infty$ so no absolute maximum.

31. $f_x = y - 1 = 0$, $f_y = x - 3 = 0$; critical point $(3, 1)$. Along $y = 0$: $u(x) = -x$; no critical points, along $x = 0$: $v(y) = -3y$; no critical points, along $y = -\dfrac{4}{5}x + 4$: $w(x) = -\dfrac{4}{5}x^2 + \dfrac{27}{5}x - 12$; critical point $(27/8, 13/10)$.

(x, y)	$(3, 1)$	$(0, 0)$	$(5, 0)$	$(0, 4)$	$(27/8, 13/10)$
$f(x, y)$	-3	0	-5	-12	$-231/80$

Absolute maximum value is 0, absolute minimum value is -12.

33. $f_x = 2x - 2 = 0$, $f_y = -6y + 6 = 0$; critical point $(1,1)$. Along $y = 0$: $u_1(x) = x^2 - 2x$; critical point $(1, 0)$, along $y = 2$: $u_2(x) = x^2 - 2x$; critical point $(1, 2)$, along $x = 0$: $v_1(y) = -3y^2 + 6y$; critical point $(0, 1)$, along $x = 2$: $v_2(y) = -3y^2 + 6y$; critical point $(2, 1)$.

(x, y)	$(1, 1)$	$(1, 0)$	$(1, 2)$	$(0, 1)$	$(2, 1)$	$(0, 0)$	$(0, 2)$	$(2, 0)$	$(2, 2)$
$f(x, y)$	2	-1	-1	3	3	0	0	0	0

Absolute maximum value is 3, absolute minimum value is -1.

35. $f_x = 2x - 1 = 0$, $f_y = 4y = 0$; critical point $(1/2, 0)$. Along $x^2 + y^2 = 4$: $y^2 = 4 - x^2$, $u(x) = 8 - x - x^2$ for $-2 \leq x \leq 2$; critical points $(-1/2, \pm\sqrt{15}/2)$.

(x, y)	$(1/2, 0)$	$\left(-1/2, \sqrt{15}/2\right)$	$\left(-1/2, -\sqrt{15}/2\right)$	$(-2, 0)$	$(2, 0)$
$f(x, y)$	$-1/4$	$33/4$	$33/4$	6	2

Absolute maximum value is $33/4$, absolute minimum value is $-1/4$.

37. Maximize $P = xyz$ subject to $x + y + z = 48$, $x > 0$, $y > 0$, $z > 0$. $z = 48 - x - y$ so $P = xy(48 - x - y) = 48xy - x^2y - xy^2$, $P_x = 48y - 2xy - y^2 = 0$, $P_y = 48x - x^2 - 2xy = 0$. But $x \neq 0$ and $y \neq 0$ so $48 - 2x - y = 0$ and $48 - x - 2y = 0$; critical point (16,16). $P_{xx}P_{yy} - P_{xy}^2 > 0$ and $P_{xx} < 0$ at $(16, 16)$, relative maximum. $z = 16$ when $x = y = 16$, the product is maximum for the numbers $16, 16, 16$.

39. Maximize $w = xy^2z^2$ subject to $x+y+z=5$, $x>0$, $y>0$, $z>0$. $x = 5-y-z$ so $w = (5-y-z)y^2z^2 = 5y^2z^2 - y^3z^2 - y^2z^3$, $w_y = 10yz^2 - 3y^2z^2 - 2yz^3 = yz^2(10-3y-2z) = 0$, $w_z = 10y^2z - 2y^3z - 3y^2z^2 = y^2z(10-2y-3z) = 0$, $10-3y-2z=0$ and $10-2y-3z=0$; critical point when $y=z=2$; $w_{yy}w_{zz} - w_{yz}^2 = 320 > 0$ and $w_{yy} = -24 < 0$ when $y=z=2$, relative maximum. $x=1$ when $y=z=2$, xy^2z^2 is maximum at $(1,2,2)$.

41. The diagonal of the box must equal the diameter of the sphere, thus we maximize $V = xyz$ or, for convenience, $w = V^2 = x^2y^2z^2$ subject to $x^2+y^2+z^2 = 4a^2$, $x>0$, $y>0$, $z>0$; $z^2 = 4a^2 - x^2 - y^2$ hence $w = 4a^2x^2y^2 - x^4y^2 - x^2y^4$, $w_x = 2xy^2(4a^2 - 2x^2 - y^2) = 0, w_y = 2x^2y\left(4a^2 - x^2 - 2y^2\right) = 0, 4a^2 - 2x^2 - y^2 = 0$ and $4a^2 - x^2 - 2y^2 = 0$; critical point $\left(2a/\sqrt{3}, 2a/\sqrt{3}\right)$; $w_{xx}w_{yy} - w_{xy}^2 = \dfrac{4096}{27}a^8 > 0$ and $w_{xx} = -\dfrac{128}{9}a^4 < 0$ at $\left(2a/\sqrt{3}, 2a/\sqrt{3}\right)$, relative maximum. $z = 2a/\sqrt{3}$ when $x = y = 2a/\sqrt{3}$, the dimensions of the box of maximum volume are $2a/\sqrt{3}, 2a/\sqrt{3}, 2a/\sqrt{3}$.

43. Let x, y, and z be, respectively, the length, width, and height of the box. Minimize $C = 10(2xy) + 5(2xz+2yz) = 10(2xy + xz + yz)$ subject to $xyz = 16$. $z = 16/(xy)$, so $C = 20(xy + 8/y + 8/x)$, $C_x = 20(y - 8/x^2) = 0$, $C_y = 20(x - 8/y^2) = 0$; critical point (2,2); $C_{xx}C_{yy} - C_{xy}^2 = 1200 > 0$ and $C_{xx} = 40 > 0$ at (2,2), relative minimum. $z = 4$ when $x = y = 2$. The cost of materials is minimum if the length and width are 2 ft and the height is 4 ft.

45. (a) $x = 0 : f(0,y) = -3y^2$, minimum -3, maximum $0; x = 1, f(1,y) = 4 - 3y^2 + 2y, \dfrac{\partial f}{\partial y}(1,y) = -6y + 2 = 0$ at $y = 1/3$, minimum 3, maximum 13/3; $y = 0, f(x,0) = 4x^2$, minimum 0, maximum 4; $y = 1, f(x,1) = 4x^2 + 2x - 3, \dfrac{\partial f}{\partial x}(x,1) = 8x + 2 \neq 0$ for $0 < x < 1$, minimum -3, maximum 3.

(b) $f(x,x) = 3x^2$, minimum 0, maximum 3; $f(x,1-x) = -x^2+8x-3, \dfrac{d}{dx}f(x,1-x) = -2x+8 \neq 0$ for $0 < x < 1$, maximum 4, minimum -3.

(c) $f_x(x,y) = 8x+2y = 0, f_y(x,y) = -6y+2x = 0$, solution is $(0,0)$, which is not an interior point of the square, so check the sides: minimum -3, maximum $13/3$.

47. Minimize $S = xy+2xz+2yz$ subject to $xyz = V$, $x>0$, $y>0$, $z>0$ where x, y, and z are, respectively, the length, width, and height of the box. $z = V/(xy)$ so $S = xy + 2V/y + 2V/x$, $S_x = y - 2V/x^2 = 0$, $S_y = x - 2V/y^2 = 0$; critical point $(\sqrt[3]{2V}, \sqrt[3]{2V})$; $S_{xx}S_{yy} - S_{xy}^2 = 3 > 0$ and $S_{xx} = 2 > 0$ at this point so there is a relative minimum there. The length and width are each $\sqrt[3]{2V}$, the height is $z = \sqrt[3]{2V}/2$.

49. (a) $\dfrac{\partial g}{\partial m} = \displaystyle\sum_{i=1}^{n} 2\left(mx_i + b - y_i\right)x_i = 2\left(m\sum_{i=1}^{n}x_i^2 + b\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}x_iy_i\right) = 0$ if $\displaystyle\left(\sum_{i=1}^{n}x_i^2\right)m + \left(\sum_{i=1}^{n}x_i\right)b = \sum_{i=1}^{n}x_iy_i$, $\dfrac{\partial g}{\partial b} = \displaystyle\sum_{i=1}^{n}2\left(mx_i + b - y_i\right) = 2\left(m\sum_{i=1}^{n}x_i + bn - \sum_{i=1}^{n}y_i\right) = 0$ if $\displaystyle\left(\sum_{i=1}^{n}x_i\right)m + nb = \sum_{i=1}^{n}y_i$.

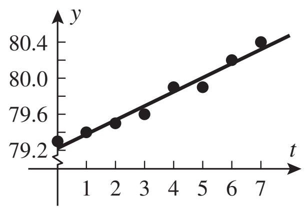
(b) $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}\left(x_i^2 - 2\bar{x}x_i + \bar{x}^2\right) = \sum_{i=1}^{n}x_i^2 - 2\bar{x}\sum_{i=1}^{n}x_i + n\bar{x}^2 = \sum_{i=1}^{n}x_i^2 - \frac{2}{n}\left(\sum_{i=1}^{n}x_i\right)^2 + \frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2 = \sum_{i=1}^{n}x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2 \geq 0$ so $\displaystyle n\sum_{i=1}^{n}x_i^2 - \left(\sum_{i=1}^{n}x_i\right)^2 \geq 0$. This is an equality if and only if $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2 = 0$, which means $x_i = \bar{x}$ for each i.

(c) The system of equations $Am+Bb = C, Dm+Eb = F$ in the unknowns m and b has a unique solution provided $AE \neq BD$, and if so the solution is $m = \dfrac{CE-BF}{AE-BD}, b = \dfrac{F-Dm}{E}$, which after the appropriate substitution yields the desired result.

51. $n = 3, \sum_{i=1}^{3} x_i = 3, \sum_{i=1}^{3} y_i = 7, \sum_{i=1}^{3} x_i y_i = 13, \sum_{i=1}^{3} x_i^2 = 11, y = \frac{3}{4}x + \frac{19}{12}$.

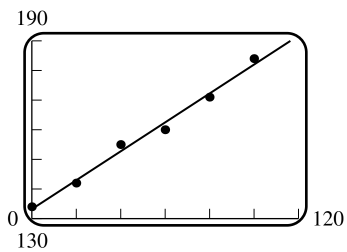
53. $\sum_{i=1}^{4} x_i = 10, \sum_{i=1}^{4} y_i = 8.2, \sum_{i=1}^{4} x_i^2 = 30, \sum_{i=1}^{4} x_i y_i = 23, n = 4;\ m = 0.5,\ b = 0.8,\ y = 0.5x + 0.8.$

55. **(a)** $y \approx 79.225 + 0.1571t$. **(b)**



(c) $y \approx 81.6$.

57. **(a)** $P = \frac{2798}{21} + \frac{171}{350}T$. **(b)**



(c) $T \approx -272.7096°$ C.

59. $f(x_0, y_0) \geq f(x, y)$ for all (x, y) inside a circle centered at (x_0, y_0) by virtue of Definition 14.8.1. If r is the radius of the circle, then in particular $f(x_0, y_0) \geq f(x, y_0)$ for all x satisfying $|x - x_0| < r$ so $f(x, y_0)$ has a relative maximum at x_0. The proof is similar for the function $f(x_0, y)$.

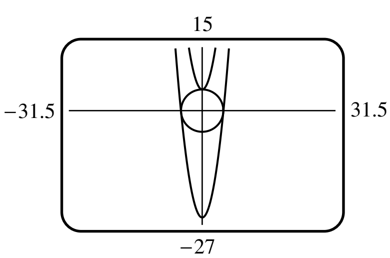
Exercise Set 13.9

1. **(a)** $xy = 4$ is tangent to the line, so the maximum value of f is 4.

(b) $xy = 2$ intersects the curve and so gives a smaller value of f.

(c) Maximize $f(x, y) = xy$ subject to the constraint $g(x, y) = x + y - 4 = 0, \nabla f = \lambda \nabla g, y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$, so solve the equations $y = \lambda, x = \lambda$ with solution $x = y = \lambda$, but $x + y = 4$, so $x = y = 2$, and the maximum value of f is $f = xy = 4$.

3. **(a)**



(b) One extremum at $(0, 5)$ and one at approximately $(\pm 5, 0)$, so minimum value -5, maximum value ≈ 25.

(c) Find the minimum and maximum values of $f(x, y) = x^2 - y$ subject to the constraint $g(x, y) = x^2 + y^2 - 25 = 0, \nabla f = \lambda \nabla g, 2x\mathbf{i} - \mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, so solve $2x = 2\lambda x, -1 = 2\lambda y, x^2 + y^2 - 25 = 0$. If $x = 0$ then $y = \pm 5, f = \mp 5$, and if $x \neq 0$ then $\lambda = 1, y = -1/2, x^2 = 25 - 1/4 = 99/4, f = 99/4 + 1/2 = 101/4$, so the maximum value of f is $101/4$ at $(\pm 3\sqrt{11}/2, -1/2)$ and the minimum value of f is -5 at $(0, 5)$.

5. $y = 8x\lambda$, $x = 16y\lambda$; $y/(8x) = x/(16y)$, $x^2 = 2y^2$ so $4\left(2y^2\right) + 8y^2 = 16$, $y^2 = 1$, $y = \pm 1$. Test $\left(\pm\sqrt{2}, -1\right)$ and $(\pm\sqrt{2}, 1)$. $f\left(-\sqrt{2}, -1\right) = f\left(\sqrt{2}, 1\right) = \sqrt{2}$, $f\left(-\sqrt{2}, 1\right) = f\left(\sqrt{2}, -1\right) = -\sqrt{2}$. Maximum $\sqrt{2}$ at $\left(-\sqrt{2}, -1\right)$ and $\left(\sqrt{2}, 1\right)$, minimum $-\sqrt{2}$ at $\left(-\sqrt{2}, 1\right)$ and $\left(\sqrt{2}, -1\right)$.

7. $12x^2 = 4x\lambda$, $2y = 2y\lambda$. If $y \neq 0$ then $\lambda = 1$ and $12x^2 = 4x$, $12x(x - 1/3) = 0$, $x = 0$ or $x = 1/3$ so from $2x^2 + y^2 = 1$ we find that $y = \pm 1$ when $x = 0$, $y = \pm\sqrt{7}/3$ when $x = 1/3$. If $y = 0$ then $2x^2 + (0)^2 = 1$, $x = \pm 1/\sqrt{2}$. Test $(0, \pm 1)$, $\left(1/3, \pm\sqrt{7}/3\right)$, and $\left(\pm 1/\sqrt{2}, 0\right)$. $f(0, \pm 1) = 1$, $f\left(1/3, \pm\sqrt{7}/3\right) = 25/27$, $f\left(1/\sqrt{2}, 0\right) = \sqrt{2}$, $f\left(-1/\sqrt{2}, 0\right) = -\sqrt{2}$. Maximum $\sqrt{2}$ at $\left(1/\sqrt{2}, 0\right)$, minimum $-\sqrt{2}$ at $\left(-1/\sqrt{2}, 0\right)$.

9. $2 = 2x\lambda, 1 = 2y\lambda, -2 = 2z\lambda$; $1/x = 1/(2y) = -1/z$ thus $x = 2y$, $z = -2y$ so $(2y)^2 + y^2 + (-2y)^2 = 4$, $y^2 = 4/9$, $y = \pm 2/3$. Test $(-4/3, -2/3, 4/3)$ and $(4/3, 2/3, -4/3)$. $f(-4/3, -2/3, 4/3) = -6$, $f(4/3, 2/3, -4/3) = 6$. Maximum 6 at $(4/3, 2/3, -4/3)$, minimum -6 at $(-4/3, -2/3, 4/3)$.

11. $yz = 2x\lambda$, $xz = 2y\lambda$, $xy = 2z\lambda$; $yz/(2x) = xz/(2y) = xy/(2z)$ thus $y^2 = x^2$, $z^2 = x^2$ so $x^2 + x^2 + x^2 = 1$, $x = \pm 1/\sqrt{3}$. Test the eight possibilities with $x = \pm 1/\sqrt{3}$, $y = \pm 1/\sqrt{3}$, and $z = \pm 1/\sqrt{3}$ to find the maximum is $1/\left(3\sqrt{3}\right)$ at $\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$, $\left(1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}\right)$, $\left(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\right)$, and $\left(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\right)$; the minimum is $-1/\left(3\sqrt{3}\right)$ at $\left(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\right)$, $\left(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\right)$, $\left(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$, and $\left(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}\right)$.

13. False, it is a scalar.

15. False, there are three equations in three unknowns.

17. $f(x, y) = x^2 + y^2$; $2x = 2\lambda$, $2y = -4\lambda$; $y = -2x$ so $2x - 4(-2x) = 3$, $x = 3/10$. The point is $(3/10, -3/5)$.

19. $f(x, y, z) = x^2 + y^2 + z^2$; $2x = \lambda$, $2y = 2\lambda$, $2z = \lambda$; $y = 2x$, $z = x$ so $x + 2(2x) + x = 1$, $x = 1/6$. The point is $(1/6, 1/3, 1/6)$.

21. $f(x, y) = (x - 1)^2 + (y - 2)^2$; $2(x - 1) = 2x\lambda$, $2(y - 2) = 2y\lambda$; $(x - 1)/x = (y - 2)/y$, $y = 2x$ so $x^2 + (2x)^2 = 45$, $x = \pm 3$. $f(-3, -6) = 80$ and $f(3, 6) = 20$ so $(3, 6)$ is closest and $(-3, -6)$ is farthest.

23. $f(x, y, z) = x + y + z$, $x^2 + y^2 + z^2 = 25$ where x, y, and z are the components of the vector; $1 = 2x\lambda$, $1 = 2y\lambda$, $1 = 2z\lambda$; $1/(2x) = 1/(2y) = 1/(2z)$; $y = x$, $z = x$ so $x^2 + x^2 + x^2 = 25$, $x = \pm 5/\sqrt{3}$. $f\left(-5/\sqrt{3}, -5/\sqrt{3}, -5/\sqrt{3}\right) = -5\sqrt{3}$ and $f\left(5/\sqrt{3}, 5/\sqrt{3}, 5/\sqrt{3}\right) = 5\sqrt{3}$ so the vector is $5(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$.

25. Minimize $f = x^2 + y^2 + z^2$ subject to $g(x, y, z) = x + y + z - 27 = 0$. $\nabla f = \lambda \nabla g$, $2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda\mathbf{i} + \lambda\mathbf{j} + \lambda\mathbf{k}$, solution $x = y = z = 9$, minimum value 243.

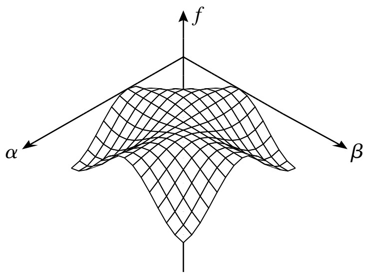
27. Minimize $f = x^2 + y^2 + z^2$ subject to $x^2 - yz = 5, \nabla f = \lambda \nabla g, 2x = 2x\lambda, 2y = -z\lambda, 2z = -y\lambda$. If $\lambda \neq \pm 2$, then $y = z = 0, x = \pm\sqrt{5}, f = 5$; if $\lambda = \pm 2$ then $x = 0$, and since $-yz = 5$, $y = -z = \pm\sqrt{5}, f = 10$, thus the minimum value is 5 at $(\pm\sqrt{5}, 0, 0)$.

29. Let x, y, and z be, respectively, the length, width, and height of the box. Minimize $f(x, y, z) = 10(2xy) + 5(2xz + 2yz) = 10(2xy + xz + yz)$ subject to $g(x, y, z) = xyz - 16 = 0$, $\nabla f = \lambda \nabla g, 20y + 10z = \lambda yz, 20x + 10z = \lambda xz, 10x + 10y = \lambda xy$. Since $V = xyz = 16, x, y, z \neq 0$, thus $\lambda z = 20 + 10(z/y) = 20 + 10(z/x)$, so $x = y$. From this and $10x + 10y = \lambda xy$ it follows that $20 = \lambda x$, so $10z = 20x, z = 2x = 2y, V = 2x^3 = 16$ and thus $x = y = 2$ ft, $z = 4$ ft, $f(2, 2, 4) = 240$ cents.

31. Maximize $A(a, b, \alpha) = ab \sin\alpha$ subject to $g(a, b, \alpha) = 2a + 2b - \ell = 0, \nabla_{(a,b,\alpha)} A = \lambda \nabla_{(a,b,\alpha)} g$, $b \sin\alpha = 2\lambda, a \sin\alpha = 2\lambda, ab\cos\alpha = 0$ with solution $a = b\ (= \ell/4), \alpha = \pi/2$ maximum value if parallelogram is a square.

33. **(a)** Maximize $f(\alpha, \beta, \gamma) = \cos\alpha\cos\beta\cos\gamma$ subject to $g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - \pi = 0, \nabla f = \lambda\nabla g, -\sin\alpha\cos\beta\cos\gamma = \lambda, -\cos\alpha\sin\beta\cos\gamma = \lambda, -\cos\alpha\cos\beta\sin\gamma = \lambda$ with solution $\alpha = \beta = \gamma = \pi/3$, maximum value $1/8$.

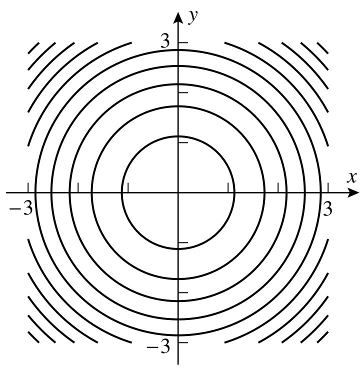
(b) For example, $f(\alpha, \beta) = \cos\alpha\cos\beta\cos(\pi - \alpha - \beta)$.



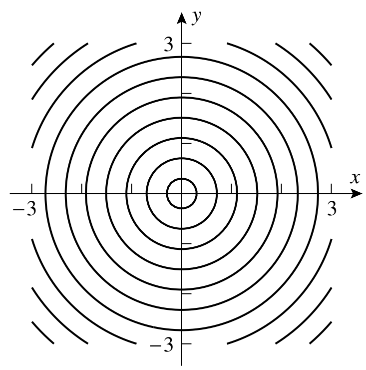
Chapter 13 Review Exercises

1. **(a)** $f(\ln y, e^x) = e^{\ln y} \ln e^x = xy$. **(b)** $f(r+s, rs) = e^{r+s} \ln(rs)$.

3. $z = \sqrt{x^2+y^2} = c$ implies $x^2+y^2 = c^2$, which is the equation of a circle; $x^2+y^2 = c$ is also the equation of a circle (for $c > 0$).



$z = x^2 + y^2$



$z = \sqrt{x^2 + y^2}$

5. $x^4 - x + y - x^3y = (x^3-1)(x-y)$, limit $= -1$, not defined on the line $y = x$ so not continuous at $(0,0)$.

7. **(a)** They approximate the profit per unit of any additional sales of the standard or high-resolution monitors, respectively.

(b) The rates of change with respect to the two directions x and y, and with respect to time.

9. **(a)** $P = \dfrac{10T}{V}$, $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial P}{\partial V}\dfrac{dV}{dt} = \dfrac{10}{V} \cdot 3 - \dfrac{10T}{V^2} \cdot 0 = \dfrac{30}{V} = \dfrac{30}{2.5} = 12 \text{ N/(m}^2\text{min)} = 12 \text{ Pa/min}$.

(b) $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial P}{\partial V}\dfrac{dV}{dt} = \dfrac{10}{V} \cdot 0 - \dfrac{10T}{V^2} \cdot (-3) = \dfrac{30T}{V^2} = \dfrac{30 \cdot 50}{(2.5)^2} = 240 \text{ Pa/min}$.

11. $w_x = 2x\sec^2(x^2+y^2) + \sqrt{y}$, $w_{xy} = 8xy\sec^2(x^2+y^2)\tan(x^2+y^2) + \dfrac{1}{2}y^{-1/2}$, $w_y = 2y\sec^2(x^2+y^2) + \dfrac{1}{2}xy^{-1/2}$, $w_{yx} = 8xy\sec^2(x^2+y^2)\tan(x^2+y^2) + \dfrac{1}{2}y^{-1/2}$.

13. $F_x = -6xz, F_{xx} = -6z, F_y = -6yz, F_{yy} = -6z, F_z = 6z^2 - 3x^2 - 3y^2$, $F_{zz} = 12z, F_{xx} + F_{yy} + F_{zz} = -6z - 6z + 12z = 0$.

17. $dV = \dfrac{2}{3}xh\,dx + \dfrac{1}{3}x^2 dh = \dfrac{2}{3}2(-0.1) + \dfrac{1}{3}(0.2) = -0.06667 \text{ m}^3; \Delta V = -0.07267 \text{ m}^3$.

19. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$, so when $t = 0$, $4\left(-\dfrac{1}{2}\right) + 2\dfrac{dy}{dt} = 2$. Solve to obtain $\left.\dfrac{dy}{dt}\right|_{t=0} = 2$.

21. $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$, $\dfrac{d^2y}{dx^2} = -\dfrac{f_y(d/dx)f_x - f_x(d/dx)f_y}{f_y^2} = -\dfrac{f_y(f_{xx} + f_{xy}(dy/dx)) - f_x(f_{xy} + f_{yy}(dy/dx))}{f_y^2} =$
$= -\dfrac{f_y(f_{xx} + f_{xy}(-f_x/f_y)) - f_x(f_{xy} + f_{yy}(-f_x/f_y))}{f_y^2} = \dfrac{-f_y^2 f_{xx} + 2f_x f_y f_{xy} - f_x^2 f_{yy}}{f_y^3}$.

25. $\nabla f = \dfrac{y}{x+y}\mathbf{i} + \left(\ln(x+y) + \dfrac{y}{x+y}\right)\mathbf{j}$, so when $(x,y) = (-3,5)$, $\dfrac{\partial f}{\partial u} = \nabla f \cdot \mathbf{u} = \left[\dfrac{5}{2}\mathbf{i} + \left(\ln 2 + \dfrac{5}{2}\right)\mathbf{j}\right] \cdot \left[\dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}\right] = \dfrac{3}{2} + 2 + \dfrac{4}{5}\ln 2 = \dfrac{7}{2} + \dfrac{4}{5}\ln 2$.

27. Use the unit vectors $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$, $\mathbf{v} = \langle 0, -1 \rangle$, $\mathbf{w} = \langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle = -\frac{\sqrt{2}}{\sqrt{5}}\mathbf{u} + \frac{1}{\sqrt{5}}\mathbf{v}$, so that $D_{\mathbf{w}}f = -\frac{\sqrt{2}}{\sqrt{5}}D_{\mathbf{u}}f + \frac{1}{\sqrt{5}}D_{\mathbf{v}}f = -\frac{\sqrt{2}}{\sqrt{5}}2\sqrt{2} + \frac{1}{\sqrt{5}}(-3) = -\frac{7}{\sqrt{5}}$.

29. The origin is not such a point, so assume that the normal line at $(x_0, y_0, z_0) \neq (0,0,0)$ passes through the origin, then $\mathbf{n} = z_x\mathbf{i} + z_y\mathbf{j} - \mathbf{k} = -y_0\mathbf{i} - x_0\mathbf{j} - \mathbf{k}$; the line passes through the origin and is normal to the surface if it has the form $\mathbf{r}(t) = -y_0 t\mathbf{i} - x_0 t\mathbf{j} - t\mathbf{k}$ and $(x_0, y_0, z_0) = (x_0, y_0, 2 - x_0 y_0)$ lies on the line if $-y_0 t = x_0, -x_0 t = y_0, -t = 2 - x_0 y_0$, with solutions $x_0 = y_0 = -1$, $x_0 = y_0 = 1, x_0 = y_0 = 0$; thus the points are $(0,0,2), (1,1,1), (-1,-1,1)$.

31. The line is tangent to $6\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, a normal to the surface is $\mathbf{n} = 18x\mathbf{i} + 8y\mathbf{j} - \mathbf{k}$, so solve $18x = 6k, 8y = 4k, -1 = k$; $k = -1, x = -1/3, y = -1/2, z = 2$.

33. $\nabla f = (2x + 3y - 6)\mathbf{i} + (3x + 6y + 3)\mathbf{j} = \mathbf{0}$ if $2x + 3y = 6, x + 2y = -1, x = 15, y = -8, D = 3 > 0, f_{xx} = 2 > 0$, so f has a relative minimum at $(15, -8)$.

35. $\nabla f = (3x^2 - 3y)\mathbf{i} - (3x - y)\mathbf{j} = \mathbf{0}$ if $y = x^2, 3x = y$, so $x = y = 0$ or $x = 3, y = 9$; at $x = y = 0, D = -9$, saddle point; at $x = 3, y = 9, D = 9, f_{xx} = 18 > 0$, relative minimum.

37. **(a)** $y^2 = 8 - 4x^2$, find extrema of $f(x) = x^2(8 - 4x^2) = -4x^4 + 8x^2$ defined for $-\sqrt{2} \leq x \leq \sqrt{2}$. Then $f'(x) = -16x^3 + 16x = 0$ when $x = 0, \pm 1, f''(x) = -48x^2 + 16$, so f has a relative maximum at $x = \pm 1, y = \pm 2$ and a relative minimum at $x = 0, y = \pm 2\sqrt{2}$. At the endpoints $x = \pm\sqrt{2}, y = 0$ we obtain the minimum $f = 0$ again.

(b) $f(x,y) = x^2y^2, g(x,y) = 4x^2 + y^2 - 8 = 0, \nabla f = 2xy^2\mathbf{i} + 2x^2y\mathbf{j} = \lambda\nabla g = 8\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, so solve $2xy^2 = \lambda 8x, 2x^2y = \lambda 2y$. If $x = 0$ then $y = \pm 2\sqrt{2}$, and if $y = 0$ then $x = \pm\sqrt{2}$. In either case f has a relative and absolute minimum. Assume $x, y \neq 0$, then $y^2 = 4\lambda, x^2 = \lambda$, use $g = 0$ to obtain $x^2 = 1, x = \pm 1, y = \pm 2$, and $f = 4$ is a relative and absolute maximum at $(\pm 1, \pm 2)$.

39. Denote the currents I_1, I_2, I_3 by x, y, z respectively. Then minimize $F(x,y,z) = x^2R_1 + y^2R_2 + z^2R_3$ subject to $g(x,y,z) = x + y + z - I = 0$, so solve $\nabla F = \lambda\nabla g, 2xR_1\mathbf{i} + 2yR_2\mathbf{j} + 2zR_3\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\lambda = 2xR_1 = 2yR_2 = 2zR_3$, so the minimum value of F occurs when $I_1 : I_2 : I_3 = \dfrac{1}{R_1} : \dfrac{1}{R_2} : \dfrac{1}{R_3}$.

41. **(a)** $\partial P/\partial L = c\alpha L^{\alpha-1}K^{\beta}, \partial P/\partial K = c\beta L^{\alpha}K^{\beta-1}$.

(b) The rates of change of output with respect to labor and capital equipment, respectively.

(c) $K(\partial P/\partial K) + L(\partial P/\partial L) = c\beta L^{\alpha}K^{\beta} + c\alpha L^{\alpha}K^{\beta} = (\alpha + \beta)P = P$.

Chapter 13 Making Connections

1. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = \cos\theta\dfrac{\partial z}{\partial x} + \sin\theta\dfrac{\partial z}{\partial y}$, multiply by r to get the first equation. $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta} = -r\sin\theta\dfrac{\partial z}{\partial x} + r\cos\theta\dfrac{\partial z}{\partial y} = -y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y}$.

3. Suppose $g(\theta)$ exists such that $f(x,y) = r^n g(\theta)$ is homogeneous of degree n. Then $f(tx,ty) = (tr)^n g(\theta) = t^n[r^n g(\theta)] = t^n f(x,y)$. Conversely if $f(x,y)$ is homogeneous of degree n then let $g(\theta) = f(\cos\theta, \sin\theta)$. Then $f(x,y) = f(r\cos\theta, r\sin\theta) = r^n f(\cos\theta,\sin\theta) = r^n g(\theta)$; moreover, $g(\theta)$ has period 2π.

5. Write $f(x,y) = z(r,\theta)$ in polar form. From the hypotheses and Exercise 1 of this section we see that $r\dfrac{\partial z}{\partial r} - nz = 0$. Divide by r^{n+1} to obtain $r^{-n}\dfrac{\partial z}{\partial r} - nr^{-n-1}z = 0, \dfrac{\partial}{\partial r}(r^{-n}z) = 0$. Thus $r^{-n}z$ is independent of r, say $r^{-n}z = g(\theta), z = r^n g(\theta)$. From Exercise 3 it follows that f is homogeneous of degree n provided that g is 2π periodic; but this follows from the fact that z is defined in terms of sines and cosines.