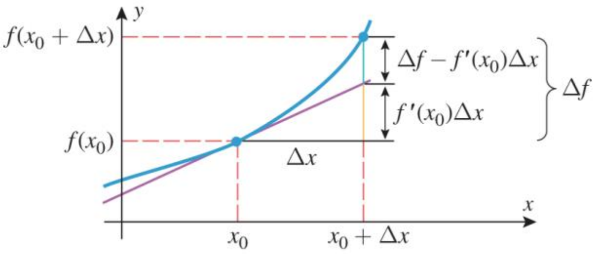
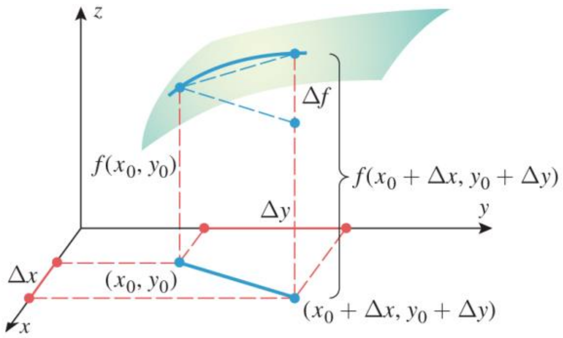
13.4 DIFFERENTIABILITY, DIFFERENTIALS, AND LOCAL LINEARITY



▲ **Figure 13.4.2**



▲ **Figure 13.4.3**

Based on these ideas, we can now give our definition of differentiability for functions of two variables.

13.4.1 DEFINITION A function f of two variables is said to be ***differentiable*** at (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 \tag{4}$$

▶ **Example 1** Use Definition 13.4.1 to prove that $f(x, y) = x^2 + y^2$ is differentiable at $(0, 0)$.

Solution. The increment is

$$\Delta f = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = (\Delta x)^2 + (\Delta y)^2$$

Since $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, we have $f_x(0, 0) = f_y(0, 0) = 0$, and (4) becomes

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{(\Delta x)^2 + (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \to (0,0)} \sqrt{(\Delta x)^2 + (\Delta y)^2} = 0$$

Therefore, f is differentiable at $(0, 0)$. ◀

We now derive an important consequence of limit (4). Define a function

$$\epsilon = \epsilon(\Delta x, \Delta y) = \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \quad \text{for } (\Delta x, \Delta y) \neq (0, 0)$$

and define $\epsilon(0, 0)$ to be 0. Equation (4) then implies that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \epsilon(\Delta x, \Delta y) = 0$$

Furthermore, it immediately follows from the definition of ϵ that

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{5}$$

In other words, if f is differentiable at (x_0, y_0), then Δf may be expressed as shown in (5), where $\epsilon \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ and where $\epsilon = 0$ if $(\Delta x, \Delta y) = (0, 0)$.

For functions of three variables we have an analogous definition of differentiability in terms of the increment

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0)$$

13.4.2 DEFINITION A function f of three variables is said to be ***differentiable*** at (x_0, y_0, z_0) provided $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, and $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z) \to (0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0 \quad (6)$$

DIFFERENTIABILITY AND CONTINUITY

Recall that we want a function to be continuous at every point at which it is differentiable. The next result shows this to be the case.

13.4.3 THEOREM *If a function is differentiable at a point, then it is continuous at that point.*

It can be difficult to verify that a function is differentiable at a point directly from the definition. The next theorem, whose proof is usually studied in more advanced courses, provides simple conditions for a function to be differentiable at a point.

13.4.4 THEOREM *If all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.*

For example, consider the function

$$f(x, y, z) = x + yz$$

Since $f_x(x, y, z) = 1$, $f_y(x, y, z) = z$, and $f_z(x, y, z) = y$ are defined and continuous everywhere, we conclude from Theorem 13.4.4 that f is differentiable everywhere.

DIFFERENTIALS

As with the one-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for a function of two variables and the approximation

$$\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z \tag{7}$$

for a function of three variables have a convenient formulation in the language of differentials. If $z = f(x, y)$ is differentiable at a point (x_0, y_0), we let

$$dz = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy \tag{8}$$

denote a new function with dependent variable dz and independent variables dx and dy. We refer to this function (also denoted df) as the ***total differential of*** z at (x_0, y_0) or as the ***total differential of*** f at (x_0, y_0). Similarly, for a function $w = f(x, y, z)$ of three variables we have the ***total differential of*** w at (x_0, y_0, z_0),

$$dw = f_x(x_0, y_0, z_0)\, dx + f_y(x_0, y_0, z_0)\, dy + f_z(x_0, y_0, z_0)\, dz \tag{9}$$

► **Example 2** Use (13) to approximate the change in $z = xy^2$ from its value at $(0.5, 1.0)$ to its value at $(0.503, 1.004)$. Compare the magnitude of the error in this approximation with the distance between the points $(0.5, 1.0)$ and $(0.503, 1.004)$.

Solution. For $z = xy^2$ we have $dz = y^2\, dx + 2xy\, dy$. Evaluating this differential at $(x, y) = (0.5, 1.0)$, $dx = \Delta x = 0.503 - 0.5 = 0.003$, and $dy = \Delta y = 1.004 - 1.0 = 0.004$ yields

$$dz = 1.0^2(0.003) + 2(0.5)(1.0)(0.004) = 0.007$$

Since $z = 0.5$ at $(x, y) = (0.5, 1.0)$ and $z = 0.507032048$ at $(x, y) = (0.503, 1.004)$, we have

$$\Delta z = 0.507032048 - 0.5 = 0.007032048$$

and the error in approximating Δz by dz has magnitude

$$|dz - \Delta z| = |0.007 - 0.007032048| = 0.000032048$$

Since the distance between $(0.5, 1.0)$ and $(0.503, 1.004) = (0.5 + \Delta x, 1.0 + \Delta y)$ is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.003)^2 + (0.004)^2} = \sqrt{0.000025} = 0.005$$

we have

$$\frac{|dz - \Delta z|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{0.000032048}{0.005} = 0.0064096 < \frac{1}{150}$$

Thus, the magnitude of the error in our approximation is less than $\frac{1}{150}$ of the distance between the two points. ◄

LOCAL LINEAR APPROXIMATIONS

We now show that if a function f is differentiable at a point, then it can be very closely approximated by a linear function near that point. For example, suppose that $f(x, y)$ is differentiable at the point (x_0, y_0). Then approximation (3) can be written in the form

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{15}$$

and refer to $L(x, y)$ as the ***local linear approximation to f at*** $(\boldsymbol{x_0}, \boldsymbol{y_0})$.

▶ Example 4 Let $L(x, y)$ denote the local linear approximation to $f(x, y) = \sqrt{x^2 + y^2}$ at the point $(3, 4)$. Compare the error in approximating

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$$

by $L(3.04, 3.98)$ with the distance between the points $(3, 4)$ and $(3.04, 3.98)$.

Solution. We have

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

with $f_x(3, 4) = \frac{3}{5}$ and $f_y(3, 4) = \frac{4}{5}$. Therefore, the local linear approximation to f at $(3, 4)$ is given by

$$L(x, y) = 5 + \tfrac{3}{5}(x - 3) + \tfrac{4}{5}(y - 4)$$

Consequently,

$$f(3.04, 3.98) \approx L(3.04, 3.98) = 5 + \tfrac{3}{5}(0.04) + \tfrac{4}{5}(-0.02) = 5.008$$

Since

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$$

the error in the approximation is about $5.00819 - 5.008 = 0.00019$. This is less than $\frac{1}{200}$ of the distance

$$\sqrt{(3.04 - 3)^2 + (3.98 - 4)^2} \approx 0.045$$

between the points $(3, 4)$ and $(3.04, 3.98)$. ◀

For a function $f(x, y, z)$ that is differentiable at (x_0, y_0, z_0), the local linear approximation is

$$\begin{aligned} L(x, y, z) = f(x_0, y_0, z_0) &+ f_x(x_0, y_0, z_0)(x - x_0) \\ &+ f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \end{aligned} \tag{16}$$

We have formulated our definitions in this section in such a way that continuity and local linearity are consequences of differentiability. In Section 13.7 we will show that if a function $f(x, y)$ is differentiable at a point (x_0, y_0), then the graph of $L(x, y)$ is a nonvertical tangent plane to the graph of f at the point $(x_0, y_0, f(x_0, y_0))$.