FTA S. NO 62

Name: Muhammad Ali Hadi Class: 2B - BCS.

Subject: Multivariable Calculus Sheet No. —

Roll No.: 23K-0663 Day: Wednesday Date: 06-03-2024

Assignment: 1

Question 1.

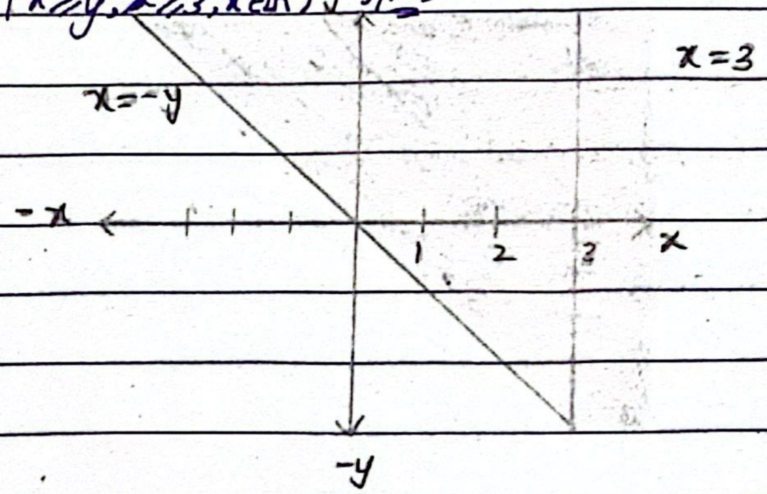
(a) Find and sketch the domain of the given functions.

(i) $f(x,y) = \sqrt{x+y} - \sqrt{x-3}$

$x+y \geq 0 \Rightarrow x \geq -y.$

$x-3 \geq 0 \Rightarrow x \geq 3.$

Df: $\{(x,y) \mid x \geq y, x \geq 3; x \in \mathbb{R}\}$ Ans!



(ii) $f(x,y) = \sqrt{\frac{1}{x^2} - \frac{1}{y^2}}$

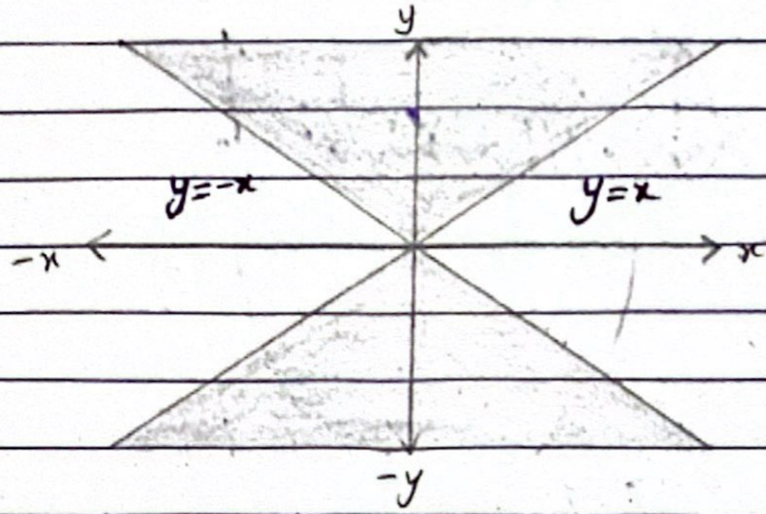
$$\frac{1}{x^2} - \frac{1}{y^2} \geq 0$$

$$\frac{y^2 - x^2}{x^2 - y^2} \geq 0$$

$y^2 - x^2 \geq 0$

$y^2 \geq x^2$

$\pm y \geq \pm x$



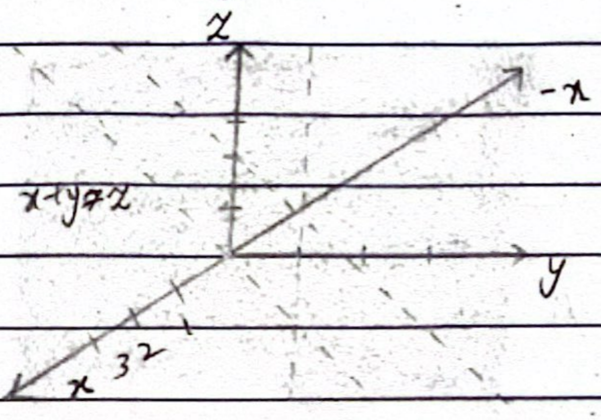
DF: $\{(x,y) \mid y \geq x,\ y \geq -x;\ x, y \in \mathbb{R}\}$ Ans!

(iii) $f(x,y,z) = \dfrac{1}{x+1} + \dfrac{1}{y-1} + \dfrac{1}{x+y-z}$

$x+1 \neq 0 \Rightarrow x \neq -1$

$y-1 \neq 0 \Rightarrow y \neq 1$

$x+y-z \neq 0 \Rightarrow x+y \neq z$



DF: $\{(x,y,z) \mid x+1 \neq 0,\ y \neq 1,\ x+y \neq z;\ x, y, z \in \mathbb{R}\}$ Ans!

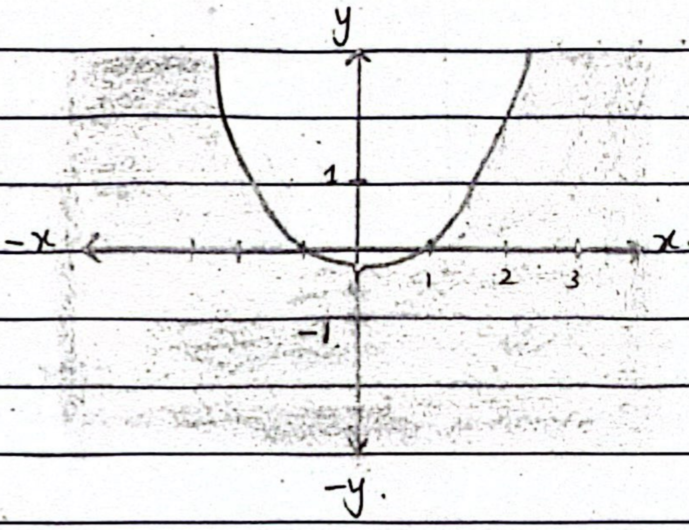
(iv) $f(x,y) = \ln(x^2 - 8y)$.

$x^2 - 8y \geq 1$.

$x^2 - 1 \geq 8y$

$8y \leq x^2 - 1$.

$y \leq \frac{x^2-1}{8}$.



DF: $\{(x,y) \mid y \leq \frac{x^2-1}{8} ; x,y \in \mathbb{R}\}$ Ans!

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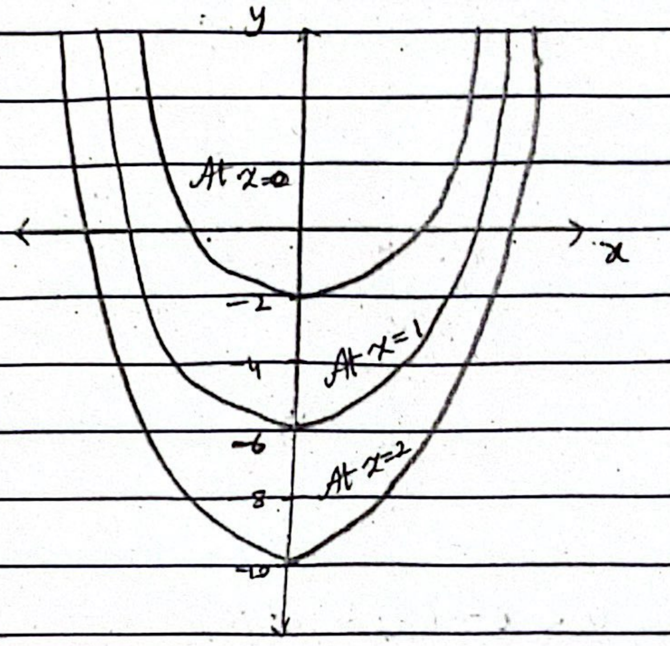
(b) Identify and sketch the level curves (or contours) for the given function.

(i) $x^2 - 4z - y = 2.$

$x^2 - y - 2 = 4z.$

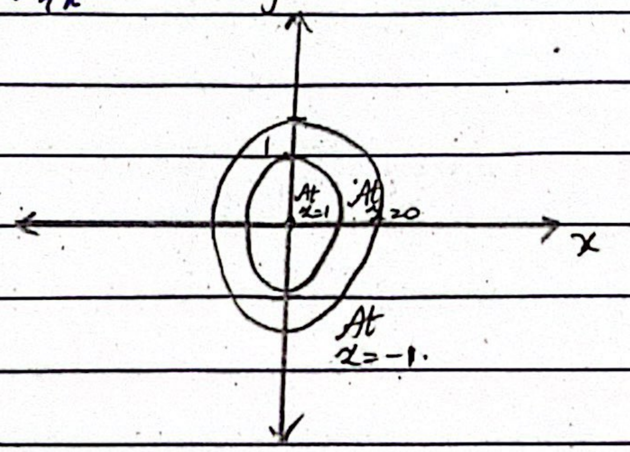
$z = \frac{1}{4}(x^2 - y - 2).$



(ii) ~~$z^2 + 4x$~~ $z^2 + 4x^2 = 1 - y^2$

$z^2 = 1 - y^2 - 4x^2.$

$z = \sqrt{1 - y^2 - 4x^2}$



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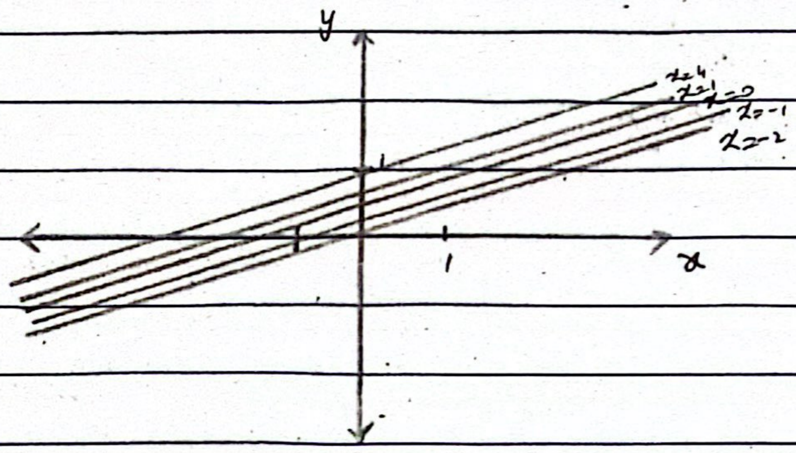
(iii) $2x-6y+z=-2$.

$z=6y-2x-2$.



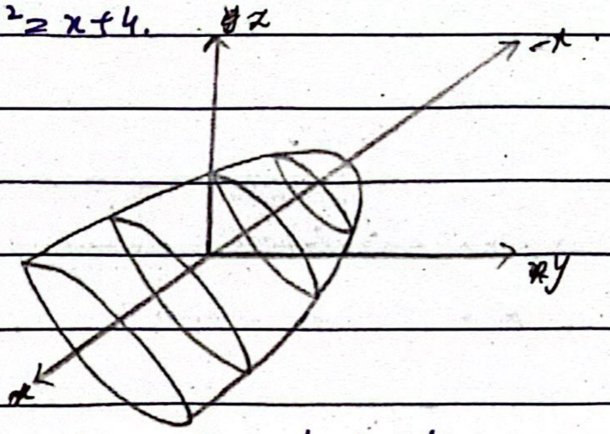
(c) Identify and sketch the Level ~~curves~~ surfaces (or contours) for the given functions at the specified value of 'k'.

(i) $f(x,y,z)=x-y^2-z^2+1, k=-3$,

$-3=x-y^2-z^2+1$

$-4=x-y^2-z^2$.

$y^2+z^2=x+4$.



Circular Paraboloid.

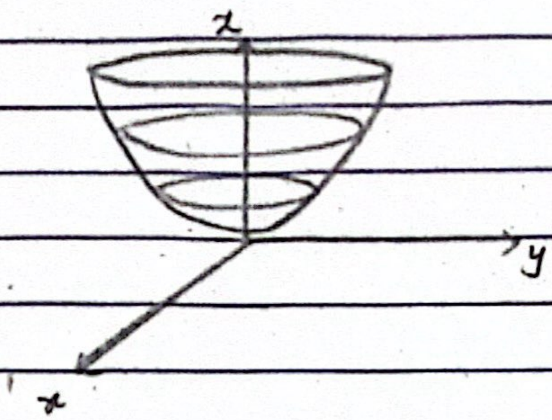
~~(iii)~~ $\cancel{\frac{3x^2+y^2}{z^2} = f(x,y,z)}$ $\cancel{k=9}$

$\cancel{9 = 3x^2}$

(ii) $f(x,y,z) = \frac{3x^2+y^2}{z^2}$, $k=9$.

$$9 = \frac{3x^2+y^2}{z^2}$$

$$9z^2 = 3x^2+y^2$$

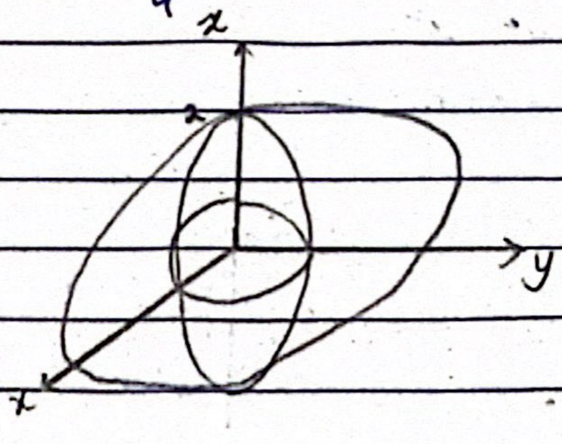


Elliptic Paraboloid.

(iii) $f(x,y,z) = 9x^2+4y^2+z^2$, $k=4$.

$$4 = 9x^2+4y^2+z^2$$

$$1 = \frac{9x^2}{4} + y^2 + \frac{z^2}{4}$$



Ellipsoid.

Question 2.

(a) Examine whether the following limits exists and find values if they exist.

(i) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4+y^2}$

Using Polar Coordinates.

$x = r\cos\theta$. $(x,y)\to(0,0)$ as $r\to 0^+$

$y = r\sin\theta$.

Hence,

$$\lim_{r\to 0^+} \frac{(r\cos\theta)^3(r\sin\theta)}{(r\cos\theta)^4+(r\sin\theta)^2}$$

$$\lim_{r\to 0^+} \frac{r^4\sin\theta\cos^3\theta}{r^4\cos^4\theta+r^2\sin^2\theta}$$

$$\lim_{r\to 0^+} \frac{r^{2}\sin\theta\cos^3\theta}{(\sin^2\theta+r^2\cos^4\theta)}$$

Applying limit.

0 (Hence, the limit exists!)

Ans!

(ii) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$.

Using Polar Coordinates.

$r^2 = x^2+y^2$.

$x = r\cos\theta$.

$y = r\sin\theta$

$(x,y)\to(0,0)$ as $r\to 0^+$.

Hence,

$$\lim_{r\to 0^+} \frac{(r\cos\theta)^3-(r\sin\theta)^3}{r^2}$$

$$\lim_{r\to 0^+} \frac{r^{\cancel{3}}(\cos^3\theta-\sin^3\theta)}{\cancel{r^2}}$$

Applying limit.

0

Hence, the limit exists!

Ans!

(iii) $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$

Let,

$r = x^2+y^2$

$(x,y)\to(0,0) \Rightarrow r\to 0^+$.

Hence,

$$\lim_{r\to 0^+} \frac{1-\cos r}{r^2}$$

Applying L-Hopital Rule.

$$\lim_{r\to 0^+} \frac{\sin r}{2r}$$

Again Applying L-Hopital Rule.

$$\lim_{r\to 0^+} \frac{\cos r}{2}$$

Applying limit.

$\frac{1}{2}$

Hence, the limit exists!

Ans!

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(b) Examine the Following functions for continuity at point (0,0), where $f(0,0)=0$ and $f(x,y)$ for $f(x,y)\neq(0,0)$ is given by,

i) $\dfrac{xy}{\sqrt{x^2+y^2}}$

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

$$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}$$

Using Polar Coordinates.

$r^2=x^2+y^2$. $\quad (x,y)\to(0,0)=r\to0^+$

$x=r\cos\theta$.

$y=r\sin\theta$.

Hence,

$$\lim_{r\to0^+}\frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2}}$$

$$\lim_{r\to0^+}\frac{r^{\cancel{2}}\cos\theta\sin\theta}{\cancel{r}}$$

Applying limit.

0

Since, the function at point (0,0) and $\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}$ are equal, Hence, the function is continuous.

Am!

(ii) $\frac{xy}{x^2+y^2}$ $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0). \\ 0 & , (x,y) = (0,0) \end{cases}$

Using Polar Coordinates

$x = r\cos\theta.$

$y = r\sin\theta.$ $(x,y) \rightarrow (0,0) = r \rightarrow 0^+.$

$r^2 = x^2 + y^2$

Hence,

$\lim_{r\to 0^+} \frac{(r\sin\theta)(r\cos\theta)}{r^2}$

$\lim_{r\to 0^+} \frac{\cancel{r^2}\sin\theta\cos\theta}{\cancel{r^2}}$

Applying limit.

$\sin\theta\cos\theta.$

Since, the function at point $(0,0)$ and $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

are different, hence, the function is discontinuous.

Ans

(iii) $\frac{x^4-y^2}{x^4+y^2}.$

$f(x,y) = \begin{cases} \frac{x^4-y^2}{x^4+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Using Polar Coordinates.

$x = r\cos\theta.$

$y = r\sin\theta.$

$(x,y) \rightarrow (0,0) = r \rightarrow 0^+.$

Hence,

$$\cancel{\lim_{r\to 0^+} \frac{x^4-y^2}{x^4+y^2}}$$

$$\lim_{r\to 0^+} \frac{(r\cos\theta)^4-(r\sin\theta)^2}{(r\cos\theta)^4+(r\sin\theta)^2}$$

$$\lim_{r\to 0^+} \frac{r^4\cos^4\theta - r^2\sin^2\theta}{r^4\cos^4\theta + r^2\sin^2\theta}$$

$$\lim_{r\to 0^+} \frac{\cancel{r^2}(r^2\cos^4\theta - \sin^2\theta)}{\cancel{r^2}(r^2\cos^4\theta - \sin^2\theta)}$$

Applying limit.

$$\frac{-\sin\theta}{+\sin\theta}$$

$$1.$$

Since, the function at point $(0,0)$ and $\lim_{(x,y)\to(0,0)} \cancel{\frac{xy}{x^2+y^2}}$

$\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$ are different, hence, the function

is discontinuous. ~~at p~~

Ans!

(iv) $\frac{x^2y}{x^4+y^2}$ $f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Using Polar Coordinates

$x=r\cos\theta.$

$y=r\sin\theta.$

$(x,y)\to(0,0)=r\to0^+.$

Hence,

$$\lim_{r\to0^+}\frac{(r\cos\theta)^2(r\sin\theta)}{(r\cos\theta)^4+(r\sin\theta)^2}$$

$$\lim_{r\to0^+}\frac{r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta+r^2\sin^2\theta}$$

$$\lim_{r\to0^+}\frac{\cancel{r^3}\cos^4\theta\sin\theta}{\cancel{r^2}(r^2\cos^4\theta+\sin^2\theta)}$$

Applying limit.

0.

Since, the function at point (0,0) and $\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$ are same, hence, the function is continuous.

Ans.

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Question 3

(I) Let $f(x,y) = xy\dfrac{x^2-y^2}{x^2+y^2}$, if $(x,y) \neq (0,0)$ and 0, otherwise. Prove that

$$f(x,y) = \begin{cases} (xy)\dfrac{x^2-y^2}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

(a) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all 'x' and 'y';

Using first Principle.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y) - f(0,y)}{h}$$

$$f_x(0,y) = \lim_{h\to 0} \frac{1}{h}\left[hy\frac{h^2-y^2}{h^2+y^2} - 0\right].$$

$$f_x(0,y) = \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\cancel{h}y\frac{h^2-y^2}{h^2+y^2}\right)$$

Applying limit.

$$f_x(0,y) = y\frac{(0)^2-y^2}{(0)^2+y^2}.$$

$$f_x(0,y) = -y$$

Hence, Proved! Ans!

(b) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ ~~and~~

Since,

$f_x(0,y) = -y$.

Applying Partial Differentiation w.r.t y.

$f_{xy}(0,y) = -1$

~~Hence,~~ Hence,

~~$f(0,0) =$~~ $f_{xy}(0,0) = -1$

Hence Proved!

$$f_y(x,0) = \lim_{h \to 0} \frac{f(x,0+h) - f(x,0)}{h}$$

$$f_y(x,0) = \lim_{h \to 0} \frac{1}{h}\left[\frac{xh(x^2-h^2)}{x^2-h^2} - 0\right]$$

$$f_y(x,0) = \lim_{h \to 0} \frac{1}{\cancel{h}}\left(\frac{x\cancel{h}(x^2-h^2)}{x^2-h^2}\right)$$

Applying Limit.

$$f_y(x,0) = \frac{x(\cancel{x^2})}{\cancel{x^2}}$$

$f_y(x,0) = x$.

Applying Partial Differentiation w.r.t x.

$f_{yx}(x,0) = 1$.

Hence,

$f_{yx}(0,0) = 1$

Hence Proved!

(c) $f(x,y)$ is differentiable at $(0,0)$.

Using Polar Coordinates.

$x = r\cos\theta$.

$y = r\sin\theta$.

$r^2 = x^2 + y^2$.

$(x,y) \to (0,0) = r \to 0^+$.

Hence,

$$\lim_{r\to 0^+} (r\cos\theta)(r\sin\theta)\frac{[(r\cos\theta)^2 - (r\sin\theta)^2]}{[(r\cos\theta)^2 + (r\sin\theta)^2]}.$$

$$\lim_{r\to 0^+} r^2\cos\theta\sin\theta\frac{(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta)}.$$

$$\lim_{r\to 0^+} \cos\theta\sin\theta \cdot r^2(\cos^2\theta - \sin^2\theta) \quad \because \sin^2\theta + \cos^2\theta = 1.$$

Applying limit.

0.

~~Since~~ Hence, the function is differentiable.

Ans!

(II) Suppose f is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y) Then show that $f(x,y) = c$, a constant.

Given that:

$f_x(x,y) = 0$.

Integrating w.r.t "x".

$\int \frac{\partial f}{\partial x} dx = \int 0 \, dx$.

$f(x,y) = c \longrightarrow$ eq (A).

$f_y(x,y) = 0$.

Integrating w.r.t "y".

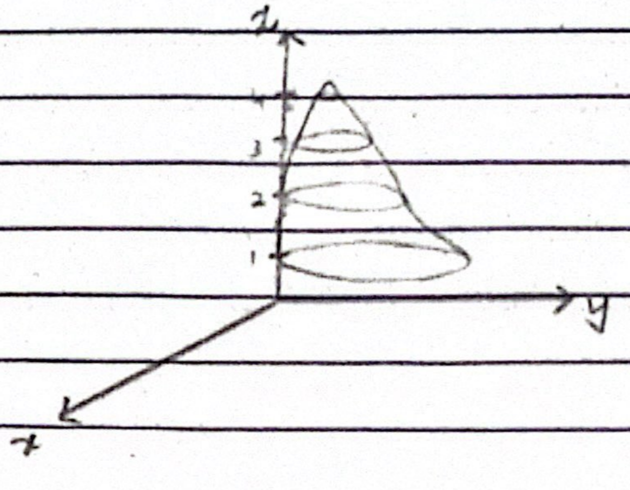
$\int \frac{\partial f}{\partial y} dy = \int 0 \, dy$.

$f(x,y) = c \longrightarrow$ eq (B).

From eq (A) and eq (B), the given condition satisfies that $f(x,y) = c$

Ans!

(III) A contour map of a function "f" is shown. Use it to make a rough sketch of the graph of "f".



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Question 4.

(a) The directional derivatives of a differentiable function $f: \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions $(1,2)$ and $(2,1)$ are 1 and 2 respectively. Find $f_x(0,0)$ and $f_y(0,0)$

~~(a) Suppose $z = f(x,y)$, where~~

~~$D_u f(x,y) =$~~ Using,

$D_u f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \hat{u}$.

$D_{(1,2)} f(0,0) = \nabla f(0,0) \cdot (1,2)$.

$D_{(1,2)} f(0,0) = 1$.

Similarly,

$D_{(2,1)} f(0,0) = \cancel{2}. \nabla f(0,0) \cdot (2,1)$.

$D_{(2,1)} f(0,0) = 2$.

Now,

~~∇f~~ $1 = \langle f_x(0,0), f_y(0,0) \rangle \cdot \langle 1,2 \rangle$.

$2 = \langle f_x(0,0), f_y(0,0) \rangle \cdot \langle 2,1 \rangle$.

$f_x + 2f_y = 1. \longrightarrow eq①$

$+\ 2f_x + f_y = 2. \longrightarrow eq②$

$3f_x + 3f_y = 3.$

$f_x + f_y = 2. \longrightarrow eq(A)$.

~~Since~~ Hence,

$\nabla f(x_0,y_0) = 1$

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$\cancel{\nabla f(x_0,y_0) = f_x(x_0,y_0)\hat{i} + f_y(x_0,y_0)\hat{j}}$

$\cancel{1 = f_x(0,0) + f_y(0,0)}$

Multiply eq① by 2.

$2f_x + 4f_y = 2.$

$\underline{-\;2f_x \mp f_y = \mp 2}$

$3f_y = 0.$

$f_y = 0. \longrightarrow$ eq Ⓐ. Ans!

Put f_y in eq ①

$2f_x + 0 = 2.$

$f_x = 1 \longrightarrow$ eq Ⓑ

Ans!.

~~(b) If $z = y + f(x^2 - y^2)$, where f is differentiable, show that~~

~~$y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x$~~

(b) Suppose $z = f(x,y)$, where $x = g(s,t)$, $y = h(s,t)$, $g(1,2) = 3$, $g_s(1,2) = -1$, $g_t(1,2) = 4$, $h(1,2) = 6$, $h_s(1,2) = -5$, $h_t(1,2) = 10$, $f_x(3,6) = 7$, and $f_y(3,6) = 8$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $s = 1$ and $t = 2$

Using Chain Rule,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Now.

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$$\frac{\partial z}{\partial s} = f_x(3,6) \cdot g_s(1,2) + f_y(3,6) \cdot h_s(1,2).$$

$$\frac{\partial z}{\partial s} = (7)(-1) + (8)(-5).$$

$$\frac{\partial z}{\partial s} = -7 - 40$$

$$\frac{\partial z}{\partial s} = -47$$

Ans!

Again from Chain Rule.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Now,

$$\frac{\partial z}{\partial t} = f_x(3,6) \cdot g_t(1,2) + f_y(3,6) \cdot h_t(1,2).$$

$$\frac{\partial z}{\partial t} = (7)(4) + (8)(10).$$

$$\frac{\partial z}{\partial t} = 28 + 80$$

$$\frac{\partial z}{\partial t} = 108$$

Ans!

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If $z = y + f(x^2 - y^2)$, where 'f' is differentiable, show that

$$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = x.$$

Taking Partial Differentiation w.r.t 'x'.

$$\frac{\partial z}{\partial x} = 2x f'(x^2 - y^2)$$

~~Again~~ Taking Partial Differentiation w.r.t 'y'.

$$\frac{\partial z}{\partial y} = 1 - 2yf'(x^2 - y^2).$$

According to Given Condition

$$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = x.$$

Substitute values of '$\partial z/\partial x$' and '$\partial z/\partial y$':

$$y\,(2x f'(x^2 - y^2)) + x(1 - 2y f'(x^2 - y^2)) = x.$$

$$2xyf'\cancel{(x^2 - y^2)} + x - 2xyf'\cancel{(x^2 - y^2)} = x$$

$$x = x.$$

Hence, the condition satisfied!

$F(2,3,4) = (2)^2 (3) + (4)^2 = $

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Question 5

(a) Find the linear approximation of the function $f(x,y,z) = x^3\sqrt{y^2+z^2}$ at the point $(2,3,4)$ and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2+(3.97)^2}$.

$f(x,y,z) = x^3\sqrt{y^2+z^2}$.

Applying Partial Differentiation w.r.t x.

$f_x(x,y,z) = 3x^2\sqrt{y^2+z^2}$

Applying Partial Differentiation w.r.t y.

$f_y(x,y,z) = \dfrac{\cancel{2}x^3y}{\cancel{2}\sqrt{y^2+z^2}} = \dfrac{x^3y}{\sqrt{y^2+z^2}}$

Applying Partial Differentiation w.r.t z.

~~Therefore,~~

$f_z(x,y,z) = \dfrac{x^3z}{\sqrt{y^2+z^2}}$

Therefore,

$f_x(2,3,4) = 3(2)^2\sqrt{(3)^2+(4)^2}$

$f_x(2,3,4) = 60.$

~~$f_y(x,y,z) =$~~ and,

$f_y(2,3,4) = \dfrac{(2)^3(3)}{\sqrt{3^2+4^2}} = \dfrac{24}{5}.$

and,

$f_z(2,3,4) = \dfrac{(2)^3(4)}{\sqrt{3^2+4^2}} = \dfrac{32}{5}$

and Lastly,

$f(2,3,4) = (2)^3\sqrt{(3)^2+(4)^2} = 40$

Moreover,

$x-x_0$

$\cancel{\Delta x} = dx = 1.98 - 2 = -0.02$.

$y-y_0$

$\Delta y = dy = 3.01 - 3 = 0.01$.

$z-z_0$

$\Delta z = dz = 3.97 - 4 = -0.03$.

Using Linear Approximation

$L(x,y,z) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$.

$L(x,y,z) = f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0)$
$+ f_z(x_0,y_0,z_0)(z-z_0)$.

Substitute values

$L(1.98, 3.01, 3.97) = (60)(-0.02) + \left(\frac{24}{5}\right)(0.01) + \left(\frac{32}{5}\right)(-0.03)$

$L(1.98, 3.01, 3.97) = -1.344$

Moreover,

$x - x_0 = 1.98 - 2 = -0.02$

$y - y_0 = 3.01 - 3 = 0.01$

$z - z_0 = 3.97 - 4 = -0.03$.

Using Linear Approximation:

$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0) + f_y(x_0,y_0,z_0)$
$+ f_z(x_0,y_0,z_0)$.

~~L(2, 3,~~ Substituting Values.

$L(1.98, 3.01, 3.97) = 40 + (60)(-0.02) + \left(\frac{24}{5}\right)(0.01) + \left(\frac{32}{5}\right)(-0.03)$

$L(1.98, 3.01, 2.97) = 38.656$.

Ans!

(b). A metal plate is situated in the xy-pl[ane] and occupies the rectangle $0 \leq x \leq 10; 0 \leq y \leq$ [...] where, 'x' and 'y' are measured in meters. The temperature at the point (x,y) in the plate is T(x,y), where 'T' is measured in degrees célcius. Temperatures at equally spaced points were measured and recorde[d] in the table.

(a) Estimate the values of the partial derivatives $T_x(6,4)$ and $T_y(6,4)$. What are the units?

~~$T_x(6,4)$~~ Using First Principle.

$$F_x(x,y) = \lim_{h\to 0} \frac{F(x+h,y) - F(x,y)}{h}$$

Therefore,

$$T_x(6,4) = \lim_{h\to 0} \frac{T(6+h,4) - T(6,4)}{h}$$

~~$T_x(6,4) =$~~ At $h = -2$:- (Left Hand Estimate).

$$T_x(6,4) = \lim_{h\to 0} \frac{T(4,4) - T(6,4)}{-2}$$

$$T_x(6,4) = \lim_{h\to 0} \frac{72-80}{-2} = \frac{\cancel{-8}^{4}}{\cancel{-2}_{1}} = 4. \longrightarrow eq(1)$$

At $h=2$:- (Right Hand Estimate).

$$T_x(6,4) = \lim_{h\to 0} \frac{T(8,4) - T(6,4)}{h}$$

$$T_x(6,4) = \lim_{h\to 0} \frac{86-80}{2} = \frac{6}{2} = 3. \longrightarrow eq(2)$$

Hence,

$$T_x(6,4) \approx \frac{3+4}{2}$$

$$T_x(6,4) \approx \frac{7}{2} = 3.5 \ ^\circ C/m.$$

Ans

~~(b) Estimate the values of $D_u T(6,4)$, where $u = (\hat{i}+\hat{j})/\sqrt{2}$. Interpret your result.~~

~~Using,~~

~~$D_u f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \hat{u}$.~~

~~$D_u T(6,4) = \nabla T(6,4) \cdot \hat{u}$.~~

Using First Principle.

$$F_y(x,y) = \lim_{h \to 0} \frac{F(x,y+h) - F(x,y)}{h}$$

Therefore,

~~$T_x(6,4) = \lim_{h\to 0} T(6,4+$~~ At $h = -2$:- (Left Hand Estimate).

$$T_y(6,4) = \lim_{h \to 0} \frac{T(6,4-2) - T(6,4)}{h}$$

$$T_y(6,4) = \lim_{h \to 0} \frac{87-80}{-2} = -\frac{7}{2} = -3.5 \longrightarrow eq(3)$$

At $h = 2$:- (Right Hand Estimate).

$$T_y(6,4) = \lim_{h \to 0} \frac{T(6,4+2) - T(6,4)}{h}$$

$$T_y(6,4) = \lim_{h \to 0} \frac{75-80}{2} = -\frac{5}{2} = -2.5 \longrightarrow eq(4)$$

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Hence,

$$T_y(6,4) \approx \frac{-3.5-2.5}{2}$$

$$T_y(6,4) \approx -3 \ {}^\circ C/m.$$

Ans!

(b) Estimate the value of $D_uT(6,4)$, where $\hat{u} = (i+j)/\sqrt{2}$. Interpret your result.

Using,

$$D_u f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \hat{u}.$$

Therefore,

$$D_uT(6,4) = \nabla T(6,4) \cdot \hat{u}. \longrightarrow eq\ A.$$

$$\nabla T(6,4) = 3.5\hat{i} - 3\hat{j}$$

Hence, eq A is.

$$D_uT(6,4) = (3.5\hat{i} - 3\hat{j}) \cdot \left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$D_uT(6,4) = 2.47 - 2.14$$

$$D_uT(6,4) = \cancel{2.12}\ 0.33$$

Ans!

(c) Estimate the value of $T_{xy}(6,4)$.

$T_{xy}(x,y) = \frac{\partial}{\partial y}[T_x(x,y)]$

$T_x(x,y) = \lim_{h\to0} \frac{T(x+h,y)-T(x,y)}{h}$, so, $T_{xy}(6,4) = \lim_{h\to0} \frac{T_x(6,4+h)-T_x(6,4)}{h}$

At $h=-2$:- (Left Hand Estimate)

$T_x(6,6) = \frac{T(4,6)-T(6,6)}{h}$

$T_x(6,6) = \frac{68-75}{-2} = \frac{7}{2} = 3.5$

At $h=2$:- (Right Hand Estimate).

$T_x(6,6) = \frac{T(8,6)-T(6,6)}{h}$

$T_x(6,6) = \frac{80-75}{2} = \frac{5}{2} = 2.5$

Hence,

$T_x(6,6) = \frac{3.5+2.5}{2} = 3.$

Similarly;

$T_x(6,2) = \frac{T(4,2)-T(6,2)}{h}$ (Left Hand Estimate).

$T_x(6,2) = \frac{74-87}{-2} = \frac{-13}{-2} = 6.5$

$T_x(6,2) = T(8,2)-T(6,2)/h$ (Right Hand Estimate)

$T_x(6,2) = \frac{90-87}{2} = \frac{3}{2} = 1.5$

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Hence,

$$T_x(6,4) \approx \frac{6.5+1.5}{2} = \frac{8}{2} = 4$$

Finally,

$$T_{xy} = \lim_{h\to 0} \frac{T_x(6,4+h) - T_x(6,4)}{h}$$

~~$T_{xy} =$~~ At $h=-2$:- (Left Hand Estimate).

$$T_{xy} = \lim_{h\to 0} \frac{T_x(6,2) - T_x(6,4)}{h}$$

$$T_{xy} = \lim_{h\to 0} \frac{4-3.5}{-2} = -\frac{1}{4} = -0.25 \rightarrow \text{eq(1)}$$

At $h=2$:- (Right Hand Estimate).

$$\cancel{T_{xy} = \lim_{h\to 0} \frac{3-3.5}{-2} = -\frac{1}{4} = -0.25.}$$

$$T_{xy} = \lim_{h\to 0} \frac{T_x(6,6) - T_x(6,4)}{h}$$

$$T_{xy} = \lim_{h\to 0} \frac{3-3.5}{2} = -\frac{1}{4} = -0.25$$

Hence,

$$T_{xy}(6,4) = \frac{-0.25-0.25}{2} = -0.25$$

Ans!

(d) Find a Linear Approximation to the temperature function $T(x,y)$ near the point $(6,4)$. Then use it to estimate the temperature at the point $(5,3.8)$.

According to Linear Approximation

$L(x,y) = T(x_0,y_0) + T_x(x_0,y_0)(x-x_0) + T_y(x_0,y_0)(y-y_0) \rightarrow$ eq (A).

Now,

$T(6,4) = 80$.

$T_x(6,4) = 3.5$

$T_y(6,4) = -3$

Hence, eq (A).

$L(x,y) = 80 + (3.5)(x-6) + (-3)(y-4)$.

$L(x,y) = 80 + 3.5x - \cancel{39}21 - 3y + 12$

$L(x,y) = \cancel{53}71 + 3.5x - 3y$. Ans!

At Point $(5,3.8)$:-

$L(5,3.8) = \cancel{53}71 + 3.5(5) - 3(3.8)$.

$L(5,3.8) = 77.1$

Ans!

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Question 6.

(a) The two legs of a right triangle a[re] measured as 5m and 12 m with a possi[ble] error in measurement of atmost 0.2 cm in each. Use differentials to estimate t[he] maximum error in the calculated value of (a) the area of the triangle and (b) the leng[th] of the hypotnuse.

(a). According to Chain Rule.

$$dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy.$$

$$dA = \frac{1}{2} y \, dx + \frac{1}{2} x \, dy.$$

~~$dx \approx \Delta x =$~~ $dA = \frac{1}{\cancel{2}} (\cancel{12}^{6}) dx + \frac{5}{2} dy.$

$$dA = 6dx + \frac{5}{2} dy. \longrightarrow eq(A).$$

Where,

$dx \approx \Delta x = 0.2cm = 0.2/100 = 0.002m.$

$dy \approx dy = 0.2 cm = 0.2/100 = 0.002m.$

Hence eq (A).

$dA = 6(0.002) + \frac{5}{2}(0.002) = \cancel{0.0012}$ 0.017

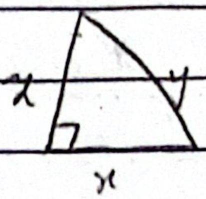
Ans!

(b). Using Pythagorus Theorem

$$Hyp^2 = Base^2 + Perp^2.$$

$$z^2 = x^2 + y^2.$$

$$z = \sqrt{x^2+y^2} \longrightarrow eq(A)$$



Applying Partial Differentiation w.r.t x:

$$\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{\sqrt{x^2+y^2}}$$

Similarly,

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

Hence, the Differential Equation.

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

$$dz = \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy.$$

Since,

$$dx = 0.002$$

$$dy = 0.002$$

Therefore,

$$dz = \frac{5}{\sqrt{5^2+12^2}}(0.002) + \frac{12}{\sqrt{5^2+12^2}}(0.002).$$

$$dz = 0.00261 \text{ m}$$

Ans!

The length 'x' of a side of a triangle is increasing at a rate of 3 in/sec, the length 'y' of another side is decreasing at a rate of 2 in/sec, and the contained angle 'θ' is increasing at a rate of 0.05 rad/sec. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \frac{\pi}{6}$?

As we know that.

$$A = \frac{1}{2} ab \sin\theta. \Rightarrow A = \frac{1}{2} xy \sin\theta.$$

According to Chain Rule.

$$\frac{dA}{dt} = \frac{1}{2} y \sin\theta \, dx + \frac{1}{2} x \sin\theta \, dy + \frac{1}{2} xy \cos\theta . d\theta.$$

Substituting values.

$$\frac{dA}{dt} = \frac{1}{2}(50)\sin\frac{\pi}{6}(3) + \frac{1}{2}(40)\sin\frac{\pi}{6}(2) + \frac{1}{2}(40)(50)\cos\frac{\pi}{6}(0.05).$$

$$\frac{dA}{dt} = 60.4 \text{ in}^2/\text{sec}$$

Ans!

Question 7.

(a) Find a direction in which $f(x,y,z) = ze^{xy}$ increases most rapidly at the point $(0,1,2)$. What is the maximum rate of increase?

Applying partial differentiation w.r.t x.

$f_x = yze^{xy}$.

Applying Partial differentiation w.r.t y.

$f_y = xz\,e^{xy}$.

Applying Partial differentiation w.r.t z.

$f_y = e^{xy}$

So,

$f_x(0,1,2) = (1)(2)e^{(0)(1)} = 2$

$f_y(0,1,2) = (0)(2)e^{(0)(1)} = 0.$

$f_z(0,1,2) = e^{(0)(1)} = 1.$

Now,

$\nabla f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\hat{i} + f_y(x_0,y_0,z_0)\hat{j} + f_z(x_0,y_0,z_0)\hat{k}.$

$\nabla f(0,1,2) = 2\hat{i} + \hat{k}.$ $\xrightarrow{\text{Ans!}}$ Direction.

~~Direction:~~

Rate:-

$\|\nabla f(0,1,2)\| = \sqrt{(2)^2+(0)^2+(1)^2}$

$\|\nabla f(0,1,2)\| = \sqrt{5}$

Ans!

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(b). Let $f(x,y,z) = |r|^{-n}$ where $r = x\hat{i} + y\hat{j} + z\hat{k}$.

Show that.

$$\nabla f = \frac{-nr}{|r|^{n+2}}$$

$\nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$. $\longrightarrow$ eq (A).

Applying Partial Differentiation w.r.t "x"

~~f(x,y,z) = |x~~ $f_x = \frac{\partial}{\partial x} |r|^{-n} \frac{\partial r}{\partial x}$.

~~$f_x = \frac{\partial}{\partial x} \frac{-n|r|^{-n-1}}{|r|^{n+2}}$~~

~~Applying Partial differentiation w.r.t y.~~

~~$f_y = -$~~

$$f_x = -nx|r|^{-n-1} \frac{\partial r}{\partial x} = \frac{-nx}{|r|^{n+2}}$$

Similarly,

$$\frac{\partial f}{\partial y} = \frac{-ny}{|r|^{n+2}}$$

and,

$$\frac{\partial f}{\partial z} = \frac{-nz}{|r|^{n+2}}$$

Hence, eq (A) is.

$$\nabla f = \frac{-nx}{|r|^{n+2}}\hat{i} - \frac{-ny}{|r|^{n+2}}\hat{j} - \frac{nz}{|r|^{n+2}}\hat{k}.$$

Therefore, the ~~General form~~ will be written as:-

$$\nabla f = \frac{-n}{|r|^{n+2}}(x\hat{i}+y\hat{j}+z\hat{k}).$$

Since,

$$r = x\hat{i}+y\hat{j}+z\hat{k}.$$

Therefore,

$$\nabla f = \frac{-nr}{|r|^{n+2}}.$$

Hence, the condition satisfied!

(c) Find the gradient of the function $f(x,y,z) = x^2 e^{yz^2}$.

(i) When the directional derivative of f is maximum?

$\nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$.

$\nabla f = 2xe^{yz^2} + x^2z^2e^{yz^2} + 2x^2yz\, e^{yz^2}$

Ans!

Due to the same direction of $\hat{u}$ the gradient vector, the value of directional derivative will be maximum.

Ans!

As,

$D_uf = \nabla f \cdot \hat{u}$.

Due to the dot product, $\cos 0° = 1$, which will always give maximum value for directional derivative.

Ans!

(ii) When it is minimum?

Due to the opposite direction of unit vector ($\hat{u}$) from the gradient vector, the value of directional derivative will be minimum.

$D_uf = \nabla f \cdot \hat{u}$.

Due to the dot product, $\cos 180° = \vec{0}$, which will alway give minimum value for the directional derivative.

Ans!

(iii). When it is zero.
Due to the perpendicular direction of unit vector ($\hat{u}$) from the gradient vector the directional derivative will always be equal to zero.

$$D_u f = \nabla f \cdot \hat{u}.$$

Due to the dot product. $\cos 90° = 0$. which will always generate the directional derivative equal to zero.

Am!

(iv) When it is half of its maximum value
At $\theta = 60°$, because of the dot product between unit vector ($\hat{u}$) and gradient vector. the directional derivative generated will always be half of its maximum value.

Am!